HAESE MATHEMATICS

Mathematics

Core topics HL

1

for use with
Mathematics: Analysis and approaches HL
Mathematics: Applications and interpretation HL

Michael Haese
Mark Humphries
Chris Sangwin
Ngoc Vo

for use with

IB Diploma Programme

HAESE MATHEMATICS

Specialists in mathematics education

Mathematics

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IB Diploma
Programme

MATHEMATICS: CORE TOPICS HL

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FOREWORD

Mathematics: Core Topics HL has been written for the International Baccalaureate Diploma Programme courses *Mathematics: Analysis and Approaches HL*, and *Mathematics: Applications and Interpretation HL*, for first teaching in August 2019, and first assessment in May 2021.

HL Mathematics

This is the first of two books students will require for the completion of their HL Mathematics course, and it contains the content that is common to both courses. Upon the completion of this book, students progress to the particular HL textbook for their course. This is expected to occur approximately 6-7 months into the two year course.

We have chosen to write in this way so that:

- The courses can be thoroughly covered while keeping the books reasonably sized.
- All HL students can start together, and potentially delay making a final choice of course.

There is some material in this book that is only relevant for *Mathematics: Analysis and Approaches HL* students, and some material that is only relevant for *Mathematics: Applications and Interpretation HL* students. This material is indicated in the table of contents. This material is included in the Mathematics: Core Topics HL book because it is more sensible to include the material under the appropriate chapter headings here, rather than making smaller stand-alone sections in the particular HL textbooks.

When students complete this book and move on to the second book, it is important that they retain this book, as it will be essential for exam revision.

Each chapter begins with an Opening Problem, offering an insight into the application of the mathematics that will be studied in the chapter. Important information and key notes are highlighted, while worked examples provide step-by-step instructions with concise and relevant explanations. Discussions, Activities, Investigations, and Research exercises are used throughout the chapters to develop understanding, problem solving, and reasoning.

In this changing world of mathematics education, we believe that the contextual approach shown in this book, with the associated use of technology, will enhance the students' understanding, knowledge and appreciation of mathematics, and its universal application.

We welcome your feedback.

Email: info@haesemathematics.com   Web: www.haesemathematics.com

PMH, MAH, CS, NV

ACKNOWLEDGEMENTS

The photo of Dr Jonathon Hare and Dr Ellen McCallie on page 385 was reproduced from www.creative-science.org.uk/parabola.html with permission.

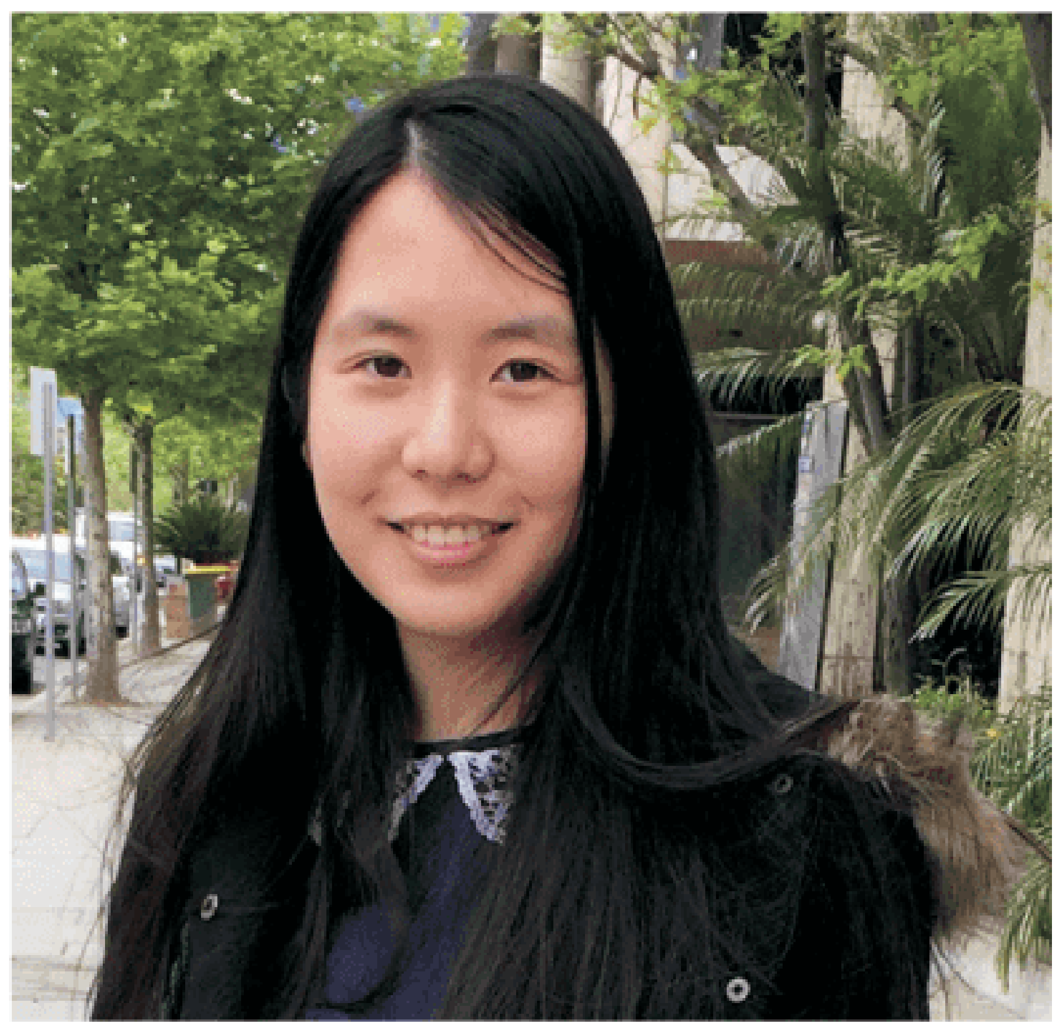
ABOUT THE AUTHORS

Michael Haese completed a BSc at the University of Adelaide, majoring in Infection and Immunity, and Applied Mathematics. He completed Honours in Applied Mathematics, and a PhD in high speed fluid flows. Michael has a keen interest in education and a desire to see mathematics come alive in the classroom through its history and relationship with other subject areas. He is passionate about girls' education and ensuring they have the same access and opportunities that boys do. His other interests are wide-ranging, including show jumping, cycling, and agriculture. He has been the principal editor for Haese Mathematics since 2008.

Mark Humphries completed a degree in Mathematical and Computer Science, and an Economics degree at the University of Adelaide. He then completed an Honours degree in Pure Mathematics. His mathematical interests include public key cryptography, elliptic curves, and number theory. Mark enjoys the challenge of piquing students' curiosity in mathematics, and encouraging students to think about mathematics in different ways. He has been working at Haese Mathematics since 2006, and is currently the writing manager.

Chris Sangwin completed a BA in Mathematics at the University of Oxford, and an MSc and PhD in Mathematics at the University of Bath. He spent thirteen years in the Mathematics Department at the University of Birmingham, and from 2000 - 2011 was seconded half time to the UK Higher Education Academy "Maths Stats and OR Network" to promote learning and teaching of university mathematics. He was awarded a National Teaching Fellowship in 2006, and is now Professor of Technology Enhanced Science Education at the University of Edinburgh.

His research interests focus on technology and mathematics education and include automatic assessment of mathematics using computer algebra, and problem solving using the Moore method and similar student-centred approaches.



Ngoc Vo completed a BMaSc at the University of Adelaide, majoring in Statistics and Applied Mathematics. Her mathematical interests include regression analysis, Bayesian statistics, and statistical computing. Ngoc has been working at Haese Mathematics as a proof reader and writer since 2016.

Dedicated to Mrs Ruba Tayoun, friend and inspiration,
so passionate about teaching her girls.

Michael

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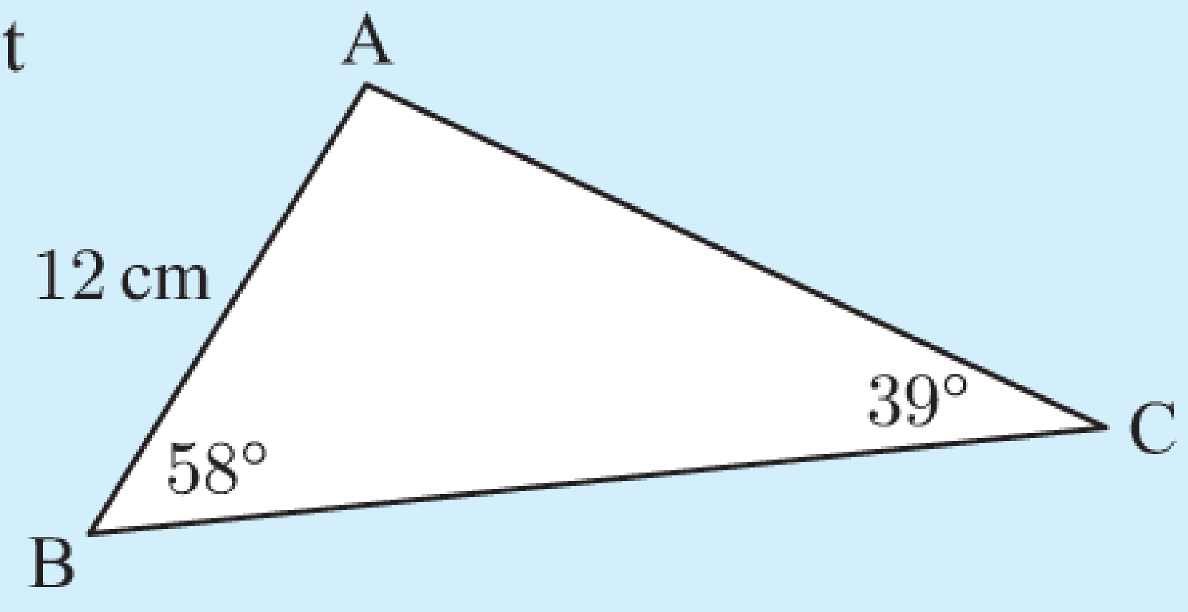
SELF TUTOR

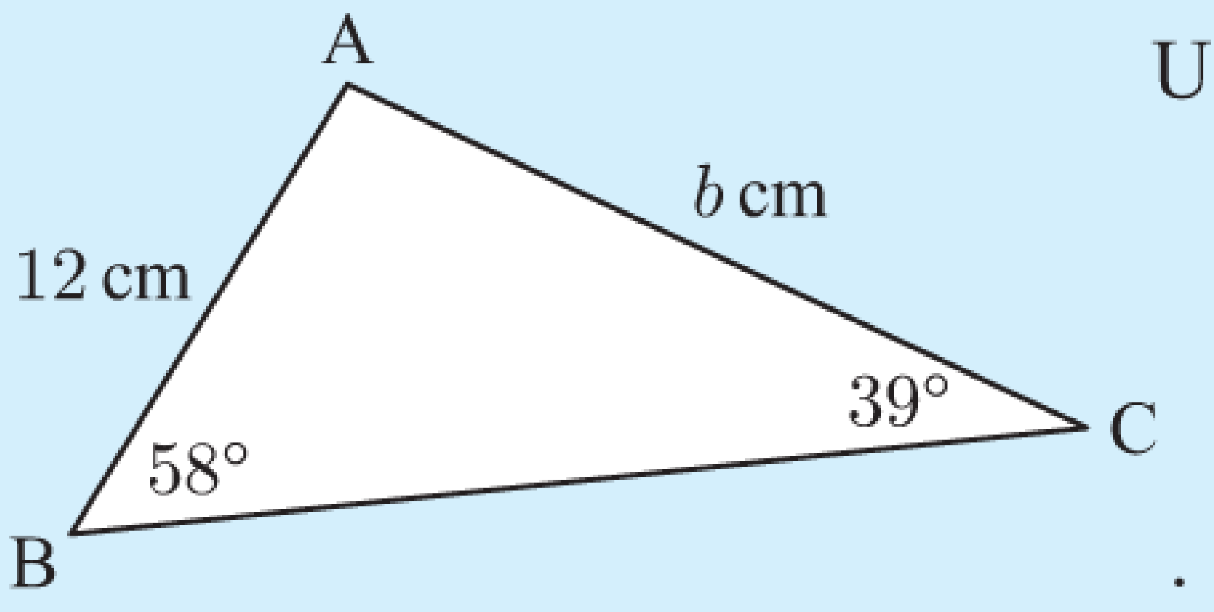
Simply 'click' on the **Self Tutor** (or anywhere in the example box) to access the worked example, with a teacher's voice explaining each step necessary to reach the answer.

Play any line as often as you like. See how the basic processes come alive using movement and colour on the screen.

Example 5 **Self Tutor**

Find the length of [AC] correct to 2 decimal places.





Using the sine rule, $\dfrac{b}{\sin 58^\circ} = \dfrac{12}{\sin 39^\circ}$

$$\therefore \quad b = \frac{12 \times \sin 58^\circ}{\sin 39^\circ}$$

$$\therefore \quad b \approx 16.17$$

$\therefore$ [AC] is about 16.17 cm long.

See **Chapter 9**, **Non-right angled triangle trigonometry**, p. 217

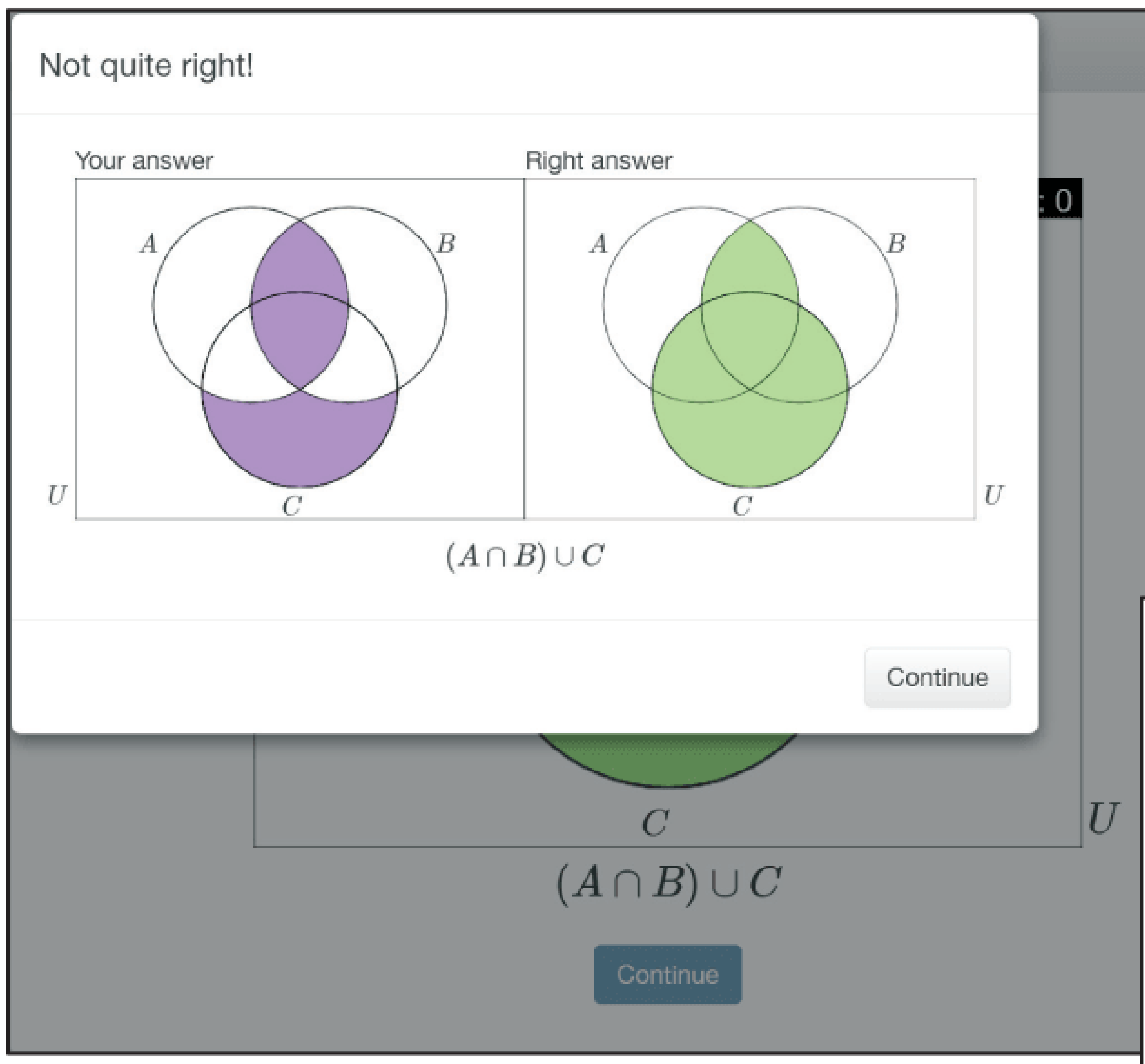
INTERACTIVE LINKS

Interactive links to in-browser tools which complement the text are included to assist teaching and learning.

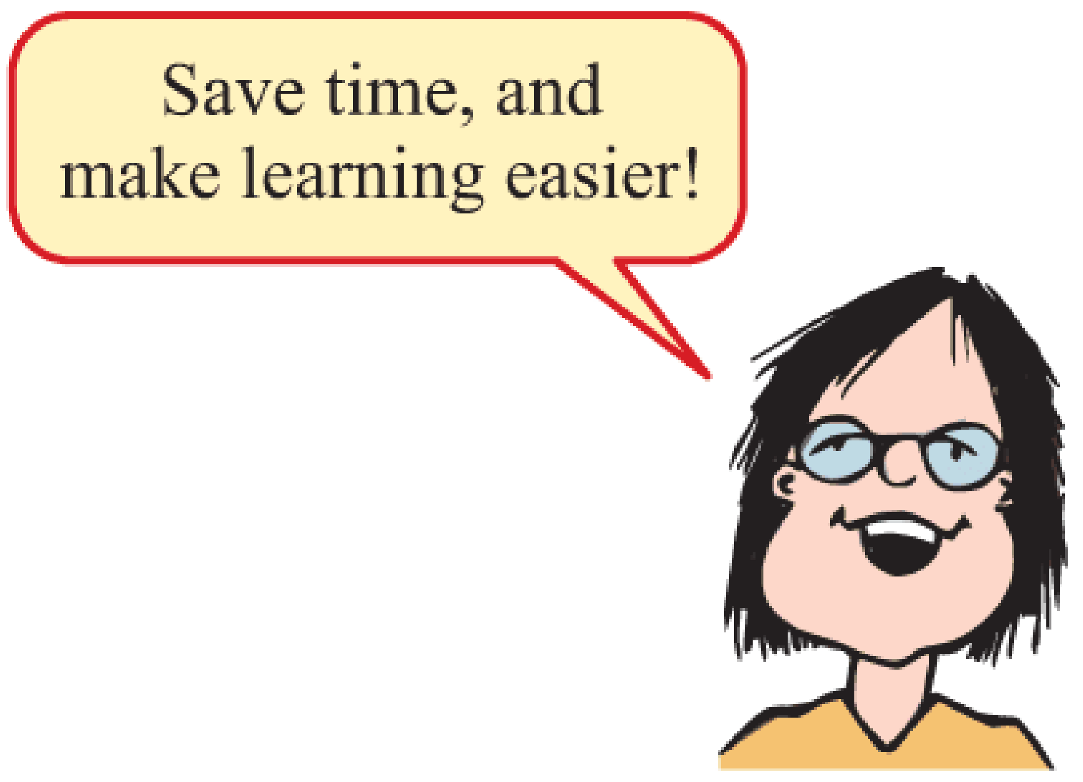
ICON

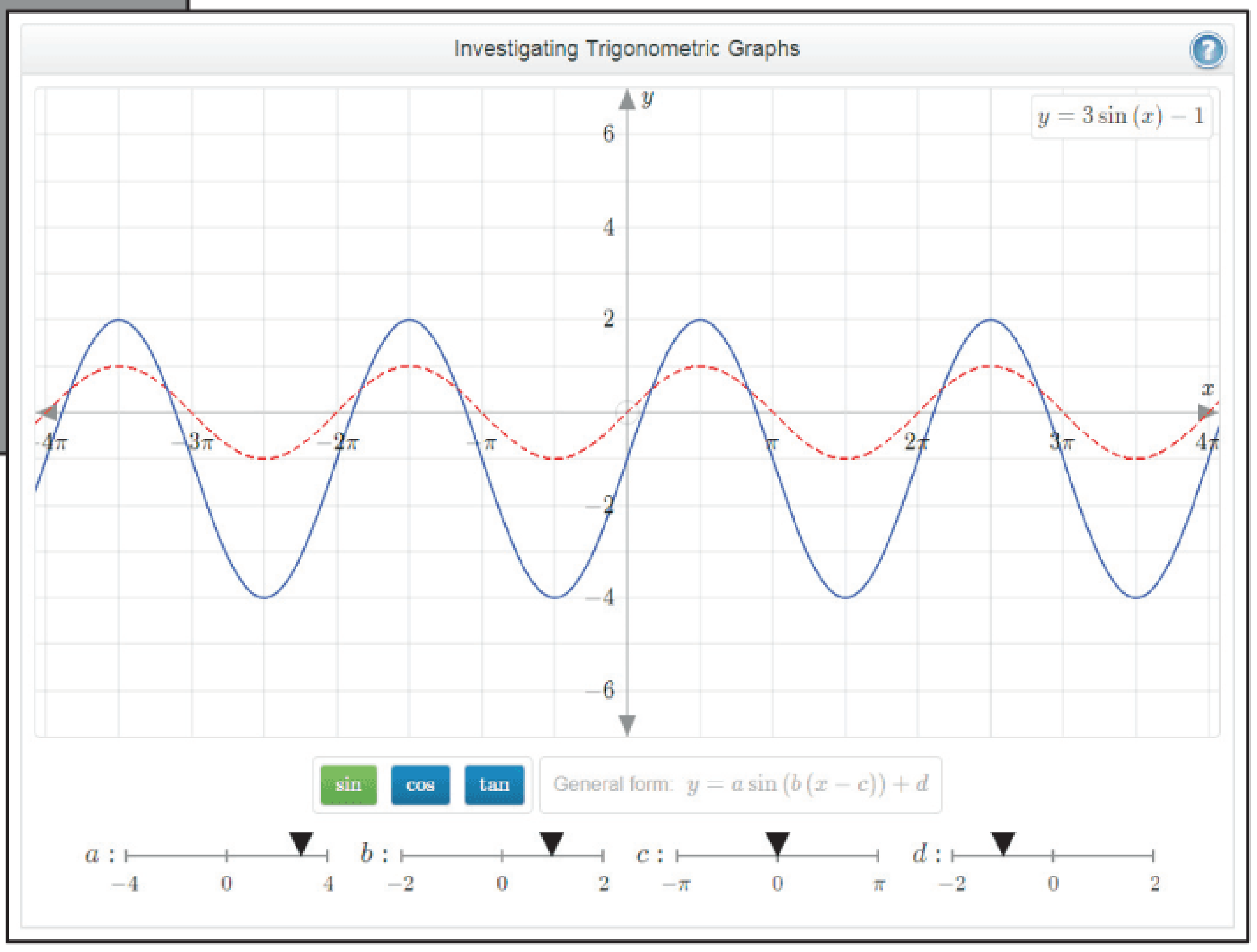
Icons like this will direct you to:

- interactive demonstrations to illustrate and animate concepts
- games and other tools for practising your skills
- graphing and statistics packages which are fast, powerful alternatives to using a graphics calculator
- printable pages to save class time.



See **Chapter 2**, **Sets and Venn diagrams**, p. 46





See **Chapter 17**, **Trigonometric functions**, p. 455

GRAPHICS CALCULATOR INSTRUCTIONS

Graphics calculator instruction booklets are available for the **Casio fx-CG50**, **TI-84 Plus CE**, **TI-*n*spire**, and the **HP Prime**. Click on the relevant icon below.

CASIO fx-CG50

TI-84 Plus CE

TI-*n*spire

HP Prime

When additional calculator help may be needed, specific instructions are available from icons within the text.

GRAPHICS CALCULATOR INSTRUCTIONS

TABLE OF CONTENTS

Sections which are shaded green are only required for *Mathematics: Analysis and Approaches HL* students.
Sections which are shaded purple are only required for *Mathematics: Applications and Interpretation HL* students.
Headings for these sections and their related exercise and review set questions are also shaded.

10 POINTS IN SPACE 233

11 PROBABILITY 247

12 SAMPLING AND DATA 289

13 STATISTICS 315

14 QUADRATIC FUNCTIONS 359

15 FUNCTIONS 393

16 TRANSFORMATIONS OF FUNCTIONS 425

17 TRIGONOMETRIC FUNCTIONS 447

ANSWERS 483

INDEX 546

SYMBOLS AND NOTATION USED IN THIS COURSE

$\mathbb{N}$	the set of positive integers and zero, $\{0, 1, 2, 3, ...\}$
$\mathbb{Z}$	the set of integers, $\{0, \pm 1, \pm 2, \pm 3, ...\}$
$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, ...\}$
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}'$	the set of irrational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers $\{a + bi \mid a, b \in \mathbb{R}\}$
i	$\sqrt{-1}$
z	a complex number
z^*	the complex conjugate of z
$\lvert z \rvert$	the modulus of z
arg z	the argument of z
$\mathscr{R}e\,(z)$	the real part of z
$\mathscr{I}m\,(z)$	the imaginary part of z
$\{x_1, x_2, ...\}$	the set with elements $x_1, x_2, ...$
$n(A)$	the number of elements in the finite set A
$\{x \mid ....$	the set of all x such that
$\in$	is an element of
$\notin$	is not an element of
$\varnothing$ or $\{\ \}$	the empty (null) set
U	the universal set
$\cup$	union
$\cap$	intersection
$\subset$	is a proper subset of
$\subseteq$	is a subset of
A'	the complement of the set A
$a^{\frac{1}{n}}, \sqrt[n]{a}$	a to the power of $\dfrac{1}{n}$, nth root of a
$a^{\frac{1}{2}}, \sqrt{a}$	a to the power $\frac{1}{2}$, square root of a
$\lvert x \rvert$	the modulus or absolute value of x, that is $\begin{cases} x & \text{for } x \geqslant 0, \quad x \in \mathbb{R} \\ -x & \text{for } x < 0, \quad x \in \mathbb{R} \end{cases}$
$\equiv$	identity or is equivalent to
$\approx$	is approximately equal to
$>$	is greater than
$\geq$ or $\geqslant$	is greater than or equal to
$<$	is less than
$\leq$ or $\leqslant$	is less than or equal to
$\ngtr$	is not greater than
$\nless$	is not less than

$[a, b]$	the closed interval $a \leqslant x \leqslant b$
$]a, b[$	the open interval $a < x < b$
u_n	the nth term of a sequence or series
d	the common difference of an arithmetic sequence
r	the common ratio of a geometric sequence
S_n	the sum of the first n terms of a sequence, $u_1 + u_2 + ... + u_n$
S_∞ or S	the sum to infinity of a sequence, $u_1 + u_2 + ...$
$\sum\limits_{i=1}^{n} u_i$	$u_1 + u_2 + ... + u_n$
$\prod\limits_{i=1}^{n} u_i$	$u_1 \times u_2 \times .... \times u_n$
$n!$	$n \times (n-1) \times (n-2) \times .... \times 3 \times 2 \times 1$
$\binom{n}{r}$ or ${}^nC_r$	the rth binomial coefficient, $r = 0, 1, 2, ...$ in the expansion of $(a+b)^n$
$f : A \to B$	f is a function under which each element of set A has an image in set B
$f : x \mapsto y$	f is a function under which x is mapped to y
$f(x)$	the image of x under the function f
f^{-1}	the inverse function of the function f
$f \circ g$	the composite function of f and g
$\lim\limits_{x \to a} f(x)$	the limit of $f(x)$ as x tends to a
$\dfrac{dy}{dx}$	the derivative of y with respect to x
$\dfrac{d}{dx}(....)$	the derivative of .... with respect to x
$f'(x)$	the derivative of $f(x)$ with respect to x
$\dfrac{d^2y}{dx^2}$	the second derivative of y with respect to x
$f''(x)$	the second derivative of $f(x)$ with respect to x
$\dfrac{d^ny}{dx^n}$	the nth derivative of y with respect to x
$f^{(n)}(x)$	the nth derivative of $f(x)$ with respect to x
$\dot{x}$	the first derivative of $x(t)$ with respect to time (t)
$\ddot{x}$	the second derivative of $x(t)$ with respect to time (t)

$M_n(x)$	the nth Maclaurin polynomial
$R_n(x)$	the remainder between a function and its nth order Maclaurin polynomial
$\int y\,dx$	the indefinite integral of y with respect to x
$\int_a^b y\,dx$	the definite integral of y with respect to x between the limits $x = a$ and $x = b$
e^x	exponential function of x
$\log_a x$	the logarithm in base a of x
$\ln x$	the natural logarithm of x, $\log_e x$
sin, cos, tan	the circular functions
arcsin, arccos, arctan	the inverse circular functions
cosec, sec, cot	the reciprocal circular functions
$\text{cis}\,\theta$	$\cos\theta + i\sin\theta$
A$(x,\, y)$	the point A in the plane with Cartesian coordinates x and y
[AB]	the line segment with end points A and B
AB	the length of [AB]
(AB)	the line containing points A and B
PB(A, B)	the perpendicular bisector of [AB]
$\widehat{A}$	the angle at A
$\text{C}\widehat{\text{A}}\text{B}$	the angle between [CA] and [AB]
$\triangle$ABC	the triangle whose vertices are A, B, and C
$\parallel$	is parallel to
$\perp$	is perpendicular to
$\mathbf{v}$	the vector $\mathbf{v}$
$\overrightarrow{\text{AB}}$	the displacement vector represented in magnitude and direction by the directed line segment from A to B
$\mathbf{a}$	the position vector $\overrightarrow{\text{OA}}$
$\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$	unit vectors in the directions of the Cartesian coordinate axes
$\lvert\, \mathbf{a} \,\rvert$	the magnitude of vector $\mathbf{a}$
$\lvert\, \overrightarrow{\text{AB}} \,\rvert$	the magnitude of $\overrightarrow{\text{AB}}$
$\mathbf{a} \bullet \mathbf{b}$	the scalar product of $\mathbf{a}$ and $\mathbf{b}$
$\mathbf{a} \times \mathbf{b}$	the vector product of $\mathbf{a}$ and $\mathbf{b}$
a_{ij}	the element in the ith row and jth column of matrix $\mathbf{A}$
$\mathbf{A}^{-1}$	the inverse of the non-singular matrix $\mathbf{A}$
$\det \mathbf{A}$, $\lvert\, \mathbf{A} \,\rvert$	the determinant of the square matrix $\mathbf{A}$

$\mathbf{I}$	the identity matrix
$\mathbf{O}$	the zero matrix
$\mathbf{A}_G$	the adjacency matrix of a graph G
k_n	a complete graph with n vertices
$p(\lambda) = \lvert\, \lambda\mathbf{I} \ - \mathbf{A} \,\rvert$	the characteristic polynomial of matrix $\mathbf{A}$
$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$	the diagonalisation of matrix $\mathbf{A}$
$\mathbf{T}$	a transition matrix
$\mathbf{s}_0$	the initial state matrix
$\mathbf{s}_n$	the state matrix at time n
$\mathbf{s}$	the steady state matrix
$\text{P}(A)$	probability of event A
$\text{P}(A')$	probability of the event "not A"
$\text{P}(A \mid B)$	probability of the event A given B
$x_1,\, x_2,\, ...$	observations of a variable
$f_1,\, f_2,\, ...$	frequencies with which the observations $x_1,\, x_2,\, x_3,\, ...$ occur
$p_1,\, p_2,\, ...$	probabilities with which the observations $x_1,\, x_2,\, x_3,\, ...$ occur
$\text{P}(X = x)$, $P(x)$	probability mass function of the discrete random variable X
$f(x)$	probability density function of the continuous random variable X
$\text{E}(X)$	the expected value of the random variable X
$\text{Var}\,(X)$	the variance of the random variable X
μ	population mean
σ	population standard deviation
σ^2	population variance
$\overline{x}$	sample mean
s^2	sample variance
s	sample standard deviation
$\overline{X}_n$	the sample mean random variable for a sample of size n
$S_{n-1}^{\,2}$	the sample variance random variable for a sample of size n
$\text{B}(n,\, p)$	binomial distribution with n trials and probability of success p
$\text{Po}(\lambda)$	Poisson distribution with rate λ
$\text{N}(\mu,\, \sigma^2)$	normal distribution with mean μ and variance σ^2
$\sim$	is distributed as

z	standardised normal z-score, $z = \dfrac{x - \mu}{\sigma}$
r	Pearson's product-moment correlation coefficient
ρ	the population product-moment correlation coefficient
H_0	the null hypothesis
H_1	the alternative hypothesis
$\mathcal{C}$	the critical region of a hypothesis test
c	the critical value of a hypothesis test
$\mathcal{A}$	the acceptance region of a hypothesis test
α	the significance level of a hypothesis test
β	the probability of a Type II error
df, ν	degrees of freedom
$T \sim t_{n-1}$	the random variable T has the Student's t distribution with $n - 1$ degrees of freedom
$\{X_1, ..., X_n\}$	a random sample of size n
$a_1 X_1 + ... + a_n X_n$	a linear combination of random variables
z_α	the value a such that $\mathrm{P}(Z \geqslant a) = \alpha$, where $Z \sim \mathrm{N}(0, 1^2)$
$t_{\alpha,\, \nu}$	the value a such that $\mathrm{P}(T \geqslant a) = \alpha$, where $T \sim t_\nu$
χ^2	chi-squared
χ^2_{calc}	calculated chi-squared value
χ^2_{crit}	critical value of the chi-squared distribution
f_{obs}	observed frequency
f_{exp}	expected frequency

THEORY OF KNOWLEDGE

Theory of Knowledge is a Core requirement in the International Baccalaureate Diploma Programme.

Students are encouraged to think critically and challenge the assumptions of knowledge. Students should be able to analyse different ways of knowing and areas of knowledge, while considering different cultural and emotional perceptions, fostering an international understanding.

The activities and discussion topics in the below table aim to help students discover and express their views on knowledge issues.

Chapter 2: Sets and Venn diagrams p. 39	**MATHEMATICAL PROOF**
Chapter 5: Sequences and series p. 118	**HOW MANY TERMS DO WE NEED TO CONSIDER BEFORE A RESULT IS PROVEN?**
Chapter 5: Sequences and series p. 125	**THE NATURE OF INFINITY**
Chapter 7: Right angled triangle trigonometry p. 158	**OBSERVATION AND BELIEF**
Chapter 8: The unit circle and radian measure p. 189	**MEASURES OF ANGLE**
Chapter 9: Non-right angled triangle trigonometry p. 226	**MATHEMATICS IN SOCIETY**
Chapter 10: Points in space p. 243	**EUCLID'S POSTULATES**
Chapter 11: Probability p. 283	**APPLICATIONS OF PROBABILITY**
Chapter 12: Sampling and data p. 299	**CLINICAL TRIALS**
Chapter 13: Statistics p. 322	**MEASURES OF CENTRE**
Chapter 15: Functions p. 420	**THE SYNTAX OF MATHEMATICS**

THEORY OF KNOWLEDGE

The study of celestial objects such as the sun, moon, stars, and planets is called **astronomy**. It has been important to civilisations all over the world for thousands of years, not only because it allowed them to navigate at night, but because the celestial objects feature in so many of their myths and beliefs.

To create an accurate star map, astronomers measure the angles between objects in the sky. The oldest known star map was found in the Silk Road town of Dunhuang in 1907. It was made in the 7th century AD, presumably from the Imperial Observatory in either Chang'an (present day Xi'an) or Luoyang. A possible author of the map was the mathematician and astronomer **Li Chunfeng** (602 - 670). The map shows 1339 stars in 257 star groups recorded with great precision on 12 charts, each covering approximately 30 degree sections of the night sky.[1]

1. How much of what we *believe* comes from what we *observe*? Is it necessary to *understand* something, in order to *believe* it? How much of what we *study* is a quest to *understand* what we *observe*, and *prove* what we *believe*?
2. How much of what we want to know is a common desire of people and cultures all over the world?
3. How did ancient people calculate with such accuracy before computer technology?

[1] "The Dunhuang Chinese Sky: A comprehensive study of the oldest known star atlas", J-M Bonnet-Bidaud, F. Praderie, S. Whitfield, *J. Astronomical History and Heritage*, 12(1), 39-59 (2009).

See **Chapter 7**, **Right angled triangle trigonometry**, p. 158

WRITING A MATHEMATICAL EXPLORATION

In addition to sitting examination papers, students are also required to complete a **mathematical exploration**. This is a short report written by the student, based on a topic of his or her choice, and should focus on the mathematics of that topic. The mathematical exploration comprises 20% of the final mark.

The exploration should be approximately 12-20 pages long, and should be written at a level which is accessible to an audience of your peers. The exploration should also include a bibliography.

Group work should not be used for explorations. Each student's exploration is an individual piece of work.

When deciding how to structure your exploration, you may wish to include the following sections:

Introduction: This section can be used to explain why the topic has been chosen, and to include any relevant background information.

Aim: A clear statement of intent should be given to provide perspective and direction to your exploration. This should be a short paragraph which outlines the problem or scenario under investigation.

Method and Results: This section can be used to describe the process which was followed to investigate the problem, as well as recording the unprocessed results of your investigations, in the form of a table, for example.

Analysis of Results: In this section, you should use graphs, diagrams, and calculations to analyse your results. Any graphs and diagrams should be included in the appropriate place in the report, and not attached as appendices at the end. You should also form some conjectures based on your analysis.

Conclusion: You should summarise your investigation, giving a clear response to your aim. You should also reflect on your exploration. Limitations and sources of error could be discussed, as well as potential for further exploration.

Click on the icon to view some examples of mathematical explorations from previous IB students. Our sincere thanks goes to these students for allowing us to reproduce their work.

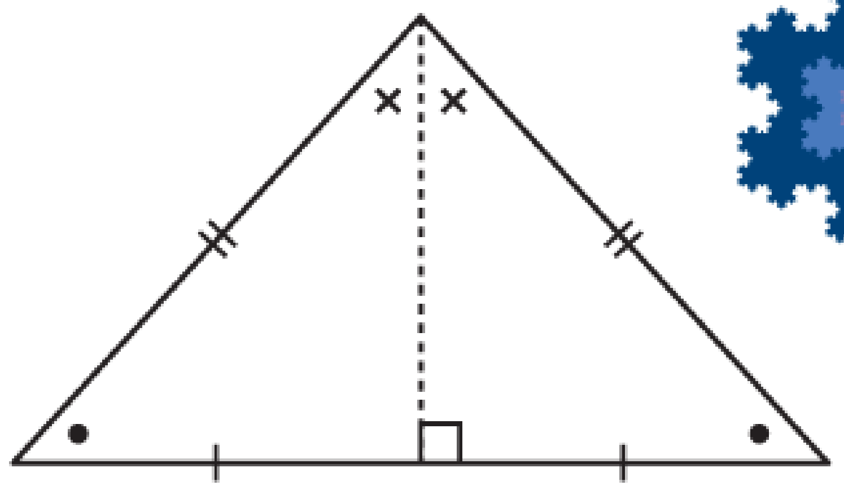
MATHEMATICAL
EXPLORATIONS

GEOMETRIC FACTS

TRIANGLE FACTS

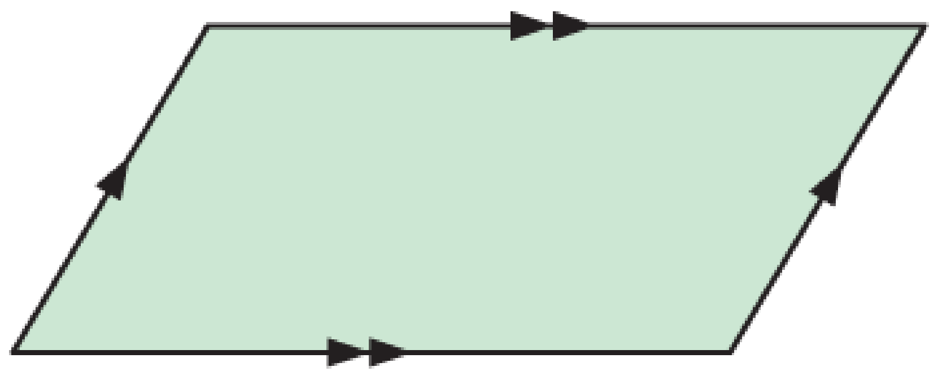
GEOMETRY PACKAGE

- The sum of the interior angles of a triangle is 180°.
- In any isosceles triangle:
  - the base angles are equal
  - the line joining the apex to the midpoint of the base bisects the vertical angle and meets the base at right angles.



QUADRILATERAL FACTS

- The sum of the interior angles of a quadrilateral is 360°.
- A **parallelogram** is a quadrilateral which has opposite sides parallel.

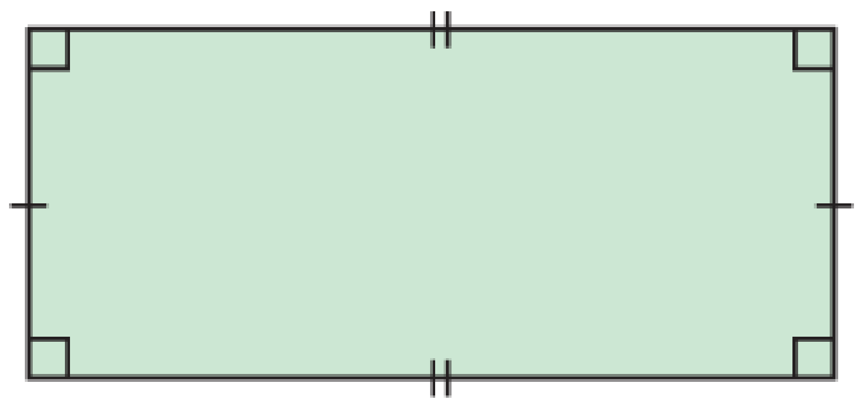


Properties:

- opposite sides are equal in length
- opposite angles are equal in size
- diagonals bisect each other.

GEOMETRY PACKAGE

- A **rectangle** is a parallelogram with four equal angles of 90°.

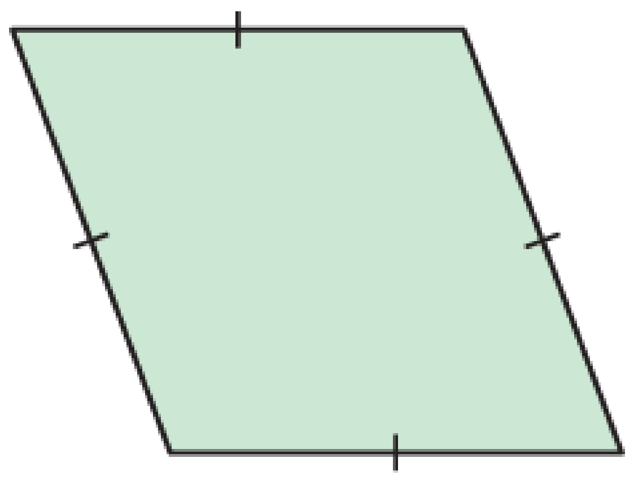


Properties:

- opposite sides are parallel and equal
- diagonals bisect each other
- diagonals are equal in length.

GEOMETRY PACKAGE

- A **rhombus** is a parallelogram in which all sides are equal in length.

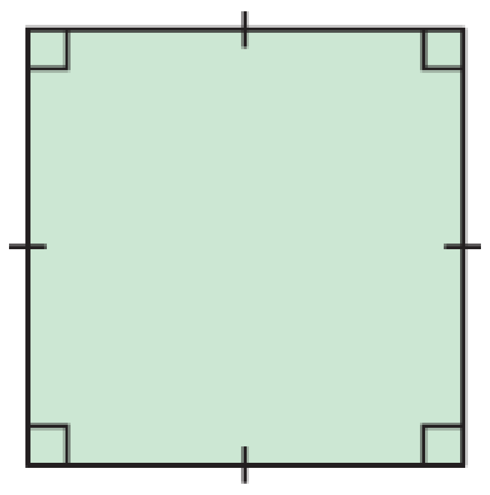


Properties:

- opposite sides are parallel
- opposite angles are equal in size
- diagonals bisect each other at right angles
- diagonals bisect the angles at each vertex.

GEOMETRY PACKAGE

- A **square** is a rhombus with four equal angles of 90°.

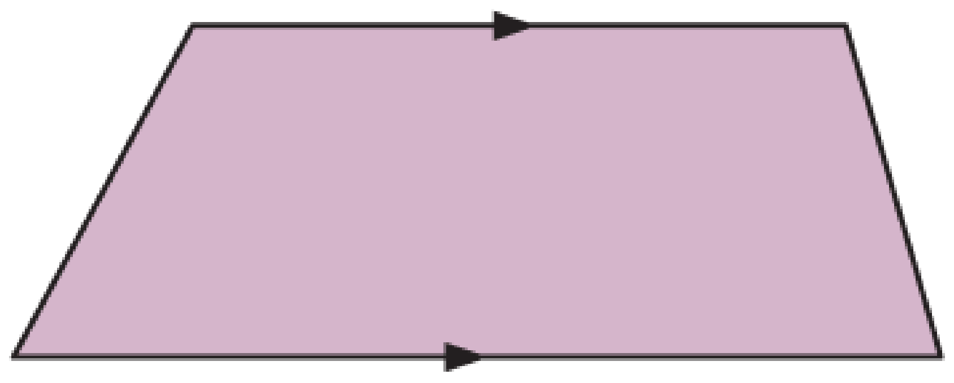


Properties:

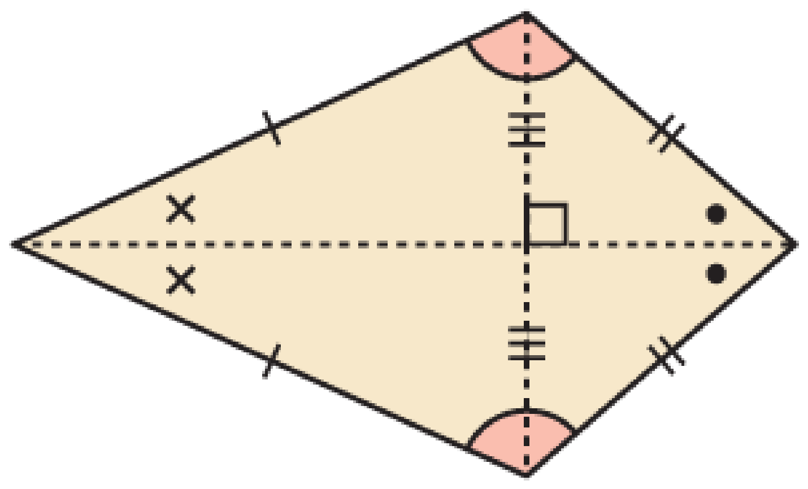
- opposite sides are parallel
- diagonals bisect each other at right angles
- diagonals bisect the angles at each vertex
- diagonals are equal in length.

GEOMETRY PACKAGE

- A **trapezium** is a quadrilateral which has a pair of parallel opposite sides.



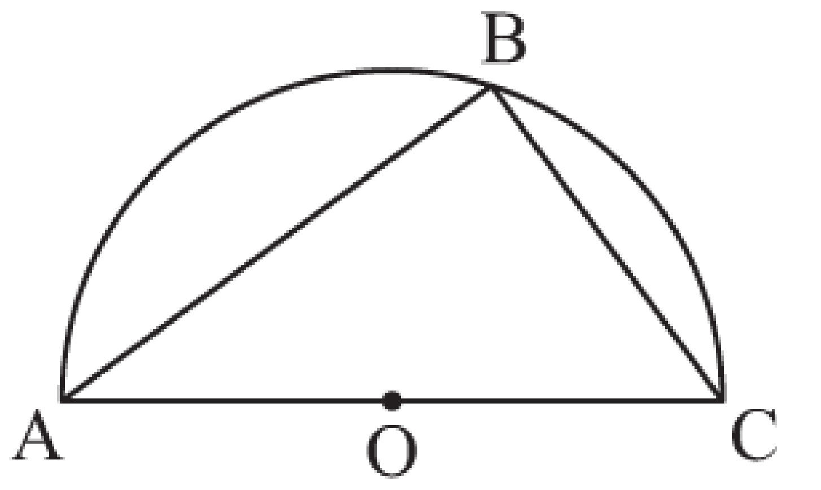
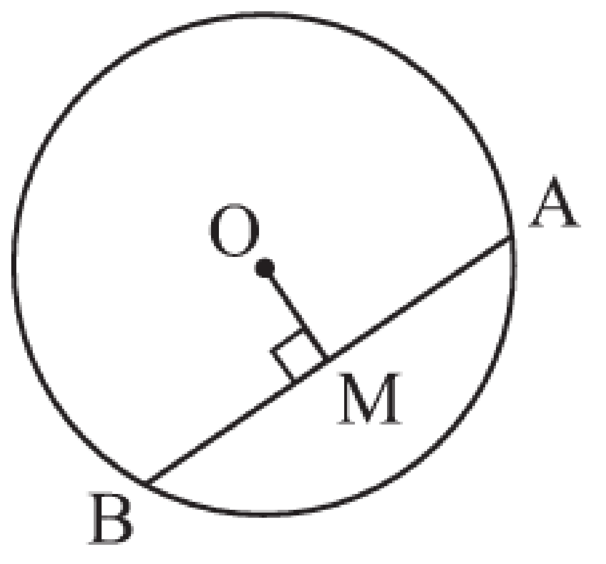
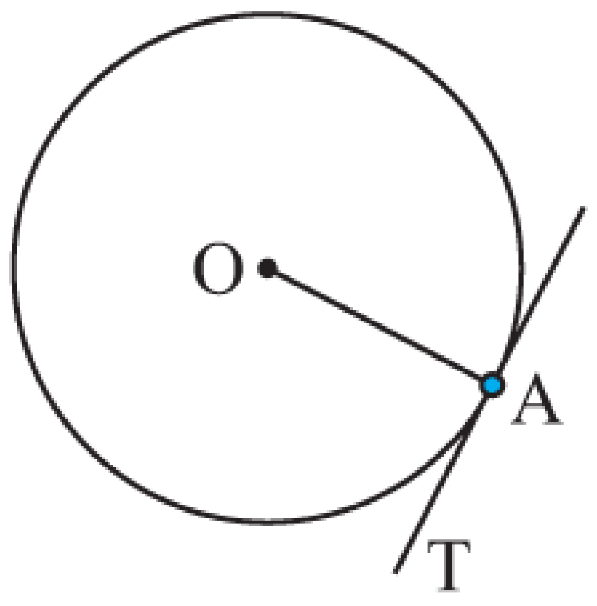
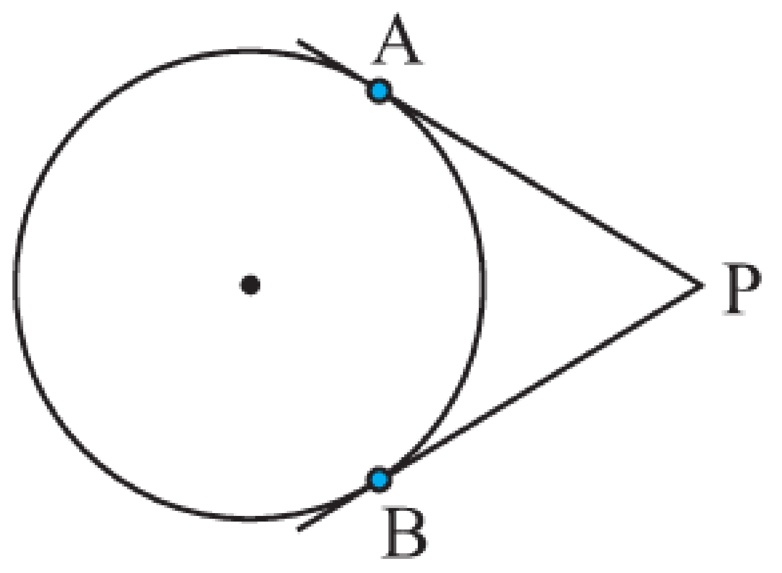
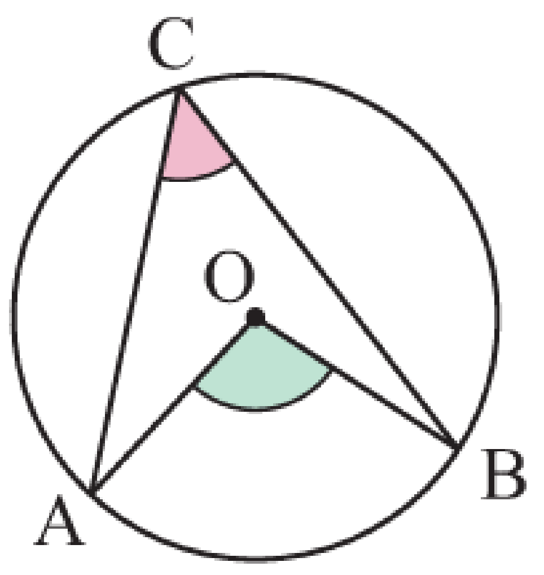
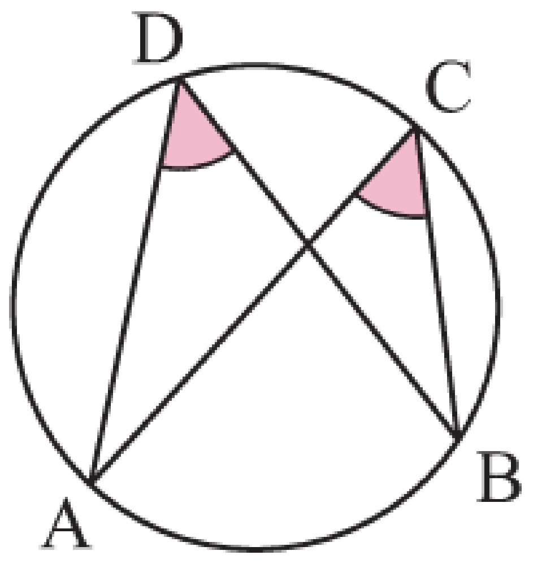
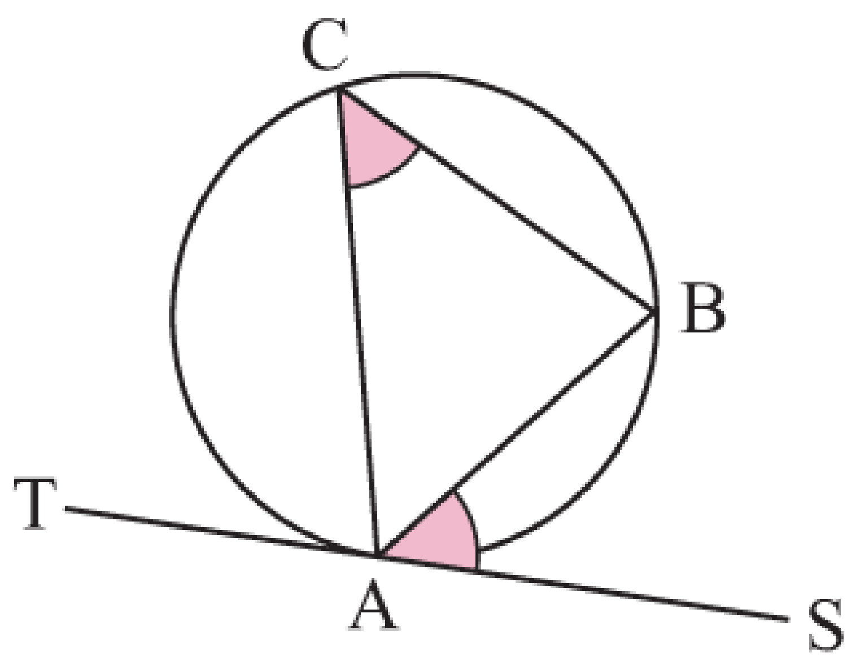
- A **kite** is a quadrilateral which has two pairs of adjacent sides equal in length.



Properties:

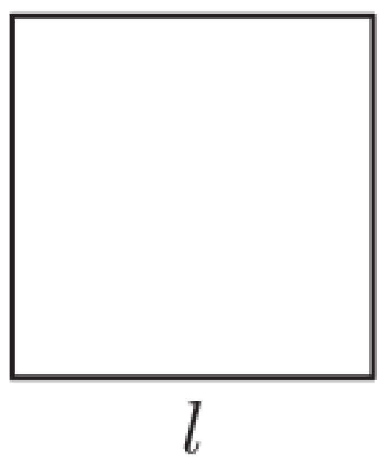
- one diagonal is a line of symmetry
- one pair of opposite angles are equal
- diagonals cut each other at right angles
- **one** diagonal bisects **one** pair of angles at the vertices
- one of the diagonals bisects the other.

CIRCLE FACTS

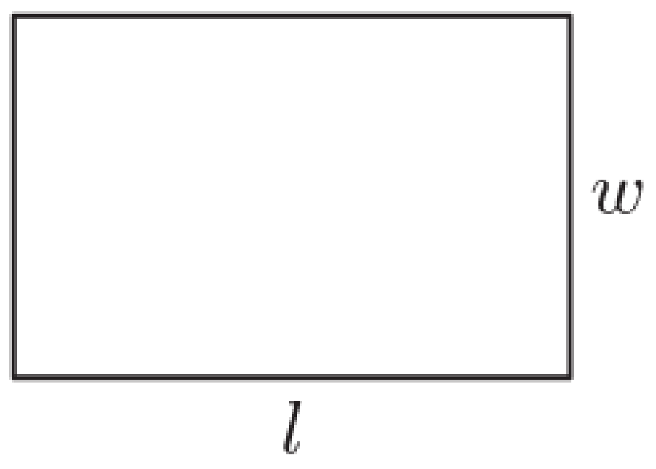
Name of theorem	*Statement*	*Diagram*
Angle in a semi-circle	The angle in a semi-circle is a right angle.	$A\hat{B}C = 90°$ GEOMETRY PACKAGE 
Chords of a circle	The perpendicular from the centre of a circle to a chord bisects the chord.	$AM = BM$ GEOMETRY PACKAGE 
Radius-tangent	The tangent to a circle is perpendicular to the radius at the point of contact.	$O\hat{A}T = 90°$ GEOMETRY PACKAGE 
Tangents from an external point	Tangents from an external point are equal in length.	$AP = BP$ GEOMETRY PACKAGE 
Angle at the centre	The angle at the centre of a circle is twice the angle on the circle subtended by the same arc.	$A\hat{O}B = 2 \times A\hat{C}B$ GEOMETRY PACKAGE 
Angles subtended by the same arc	Angles subtended by an arc on the circle are equal in size.	$A\hat{D}B = A\hat{C}B$ GEOMETRY PACKAGE 
Angle between a tangent and a chord	The angle between a tangent and a chord at the point of contact is equal to the angle subtended by the chord in the alternate segment.	$B\hat{A}S = A\hat{C}B$ GEOMETRY PACKAGE 

USEFUL FORMULAE

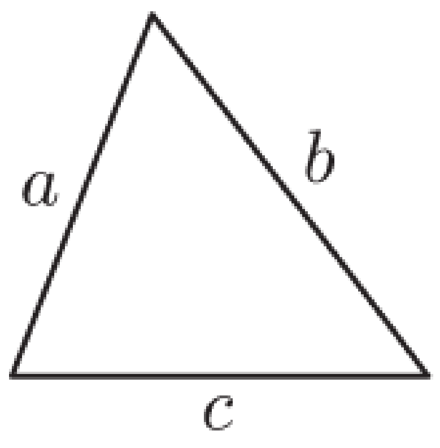
PERIMETER FORMULAE



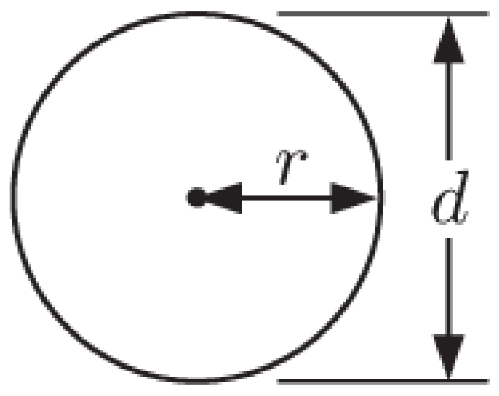
square
$P = 4l$



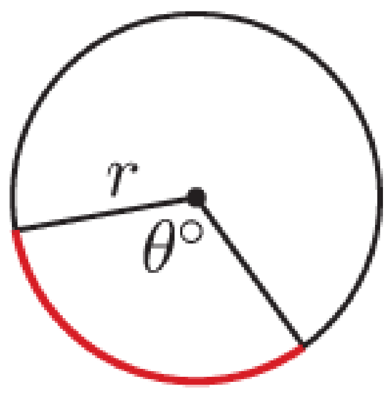
rectangle
$P = 2(l + w)$



triangle
$P = a + b + c$

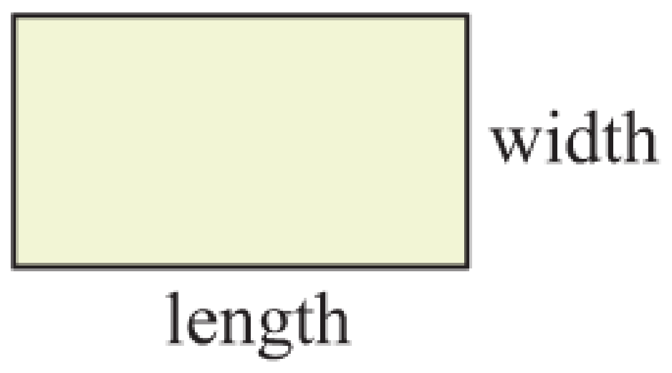
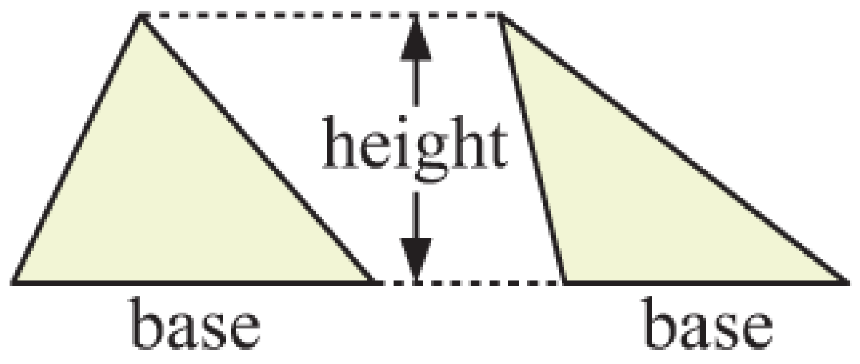
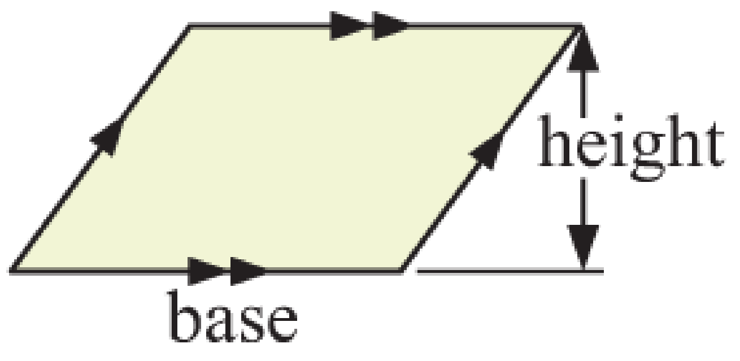
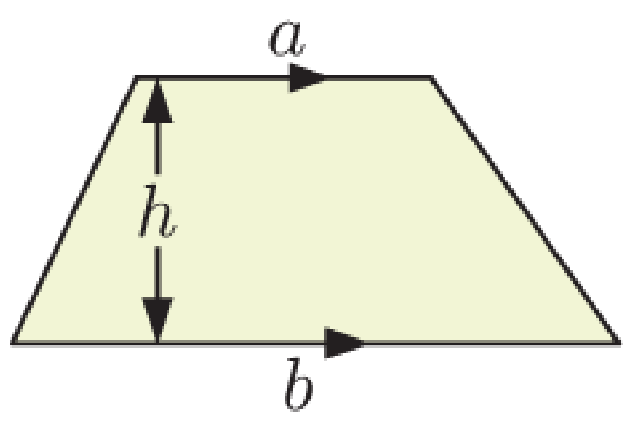
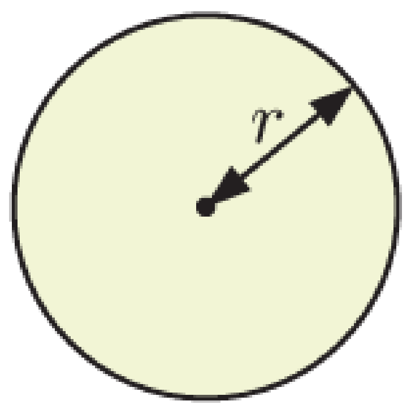
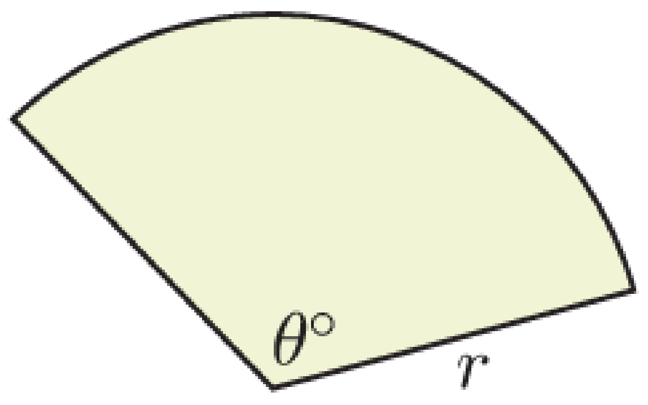


circle
$C = 2\pi r$
or $C = \pi d$



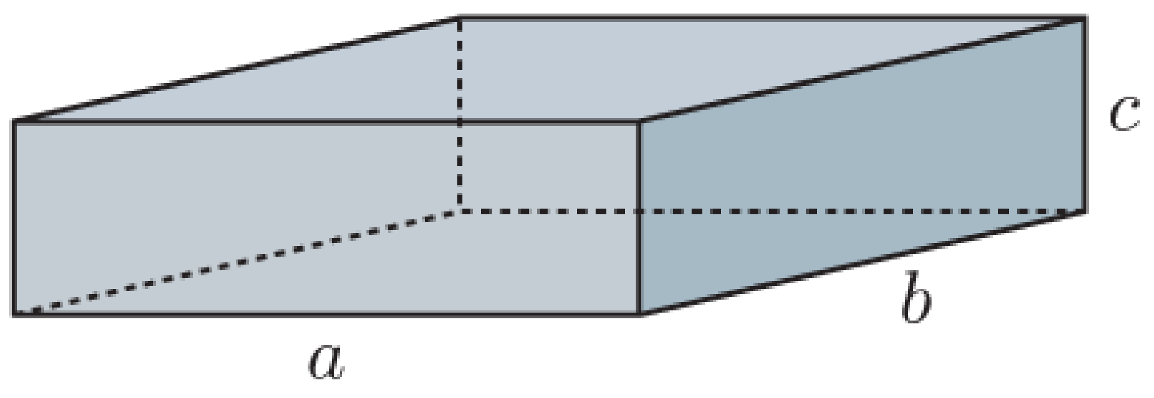
arc
$l = \left(\frac{\theta}{360}\right) 2\pi r$

AREA FORMULAE

Shape	*Diagram*	*Formula*
Rectangle		$A = \text{length} \times \text{width}$
Triangle		$A = \frac{1}{2} \times \text{base} \times \text{height}$
Parallelogram		$A = \text{base} \times \text{height}$
Trapezium or **Trapezoid**		$A = \left(\frac{a+b}{2}\right) \times h$
Circle		$A = \pi r^2$
Sector		$A = \left(\frac{\theta}{360}\right) \times \pi r^2$

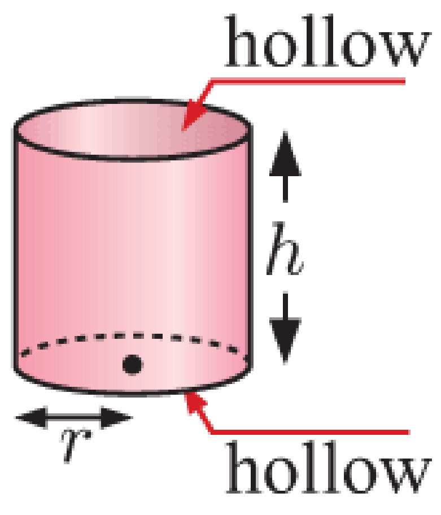
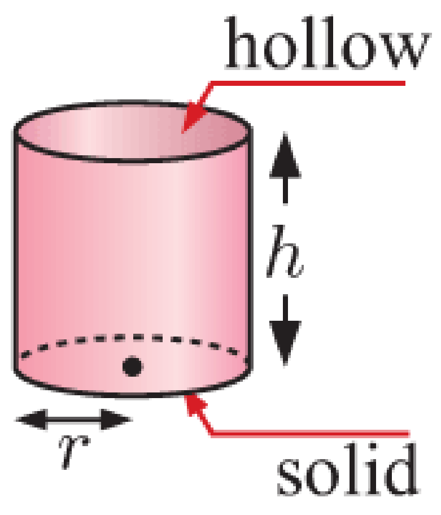
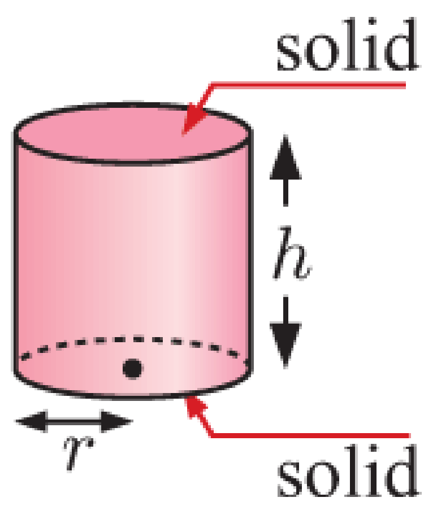
SURFACE AREA FORMULAE

RECTANGULAR PRISM

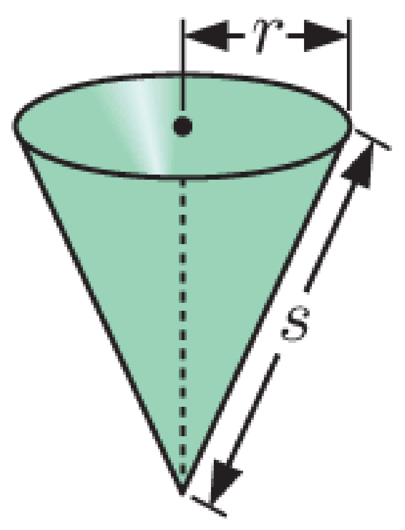
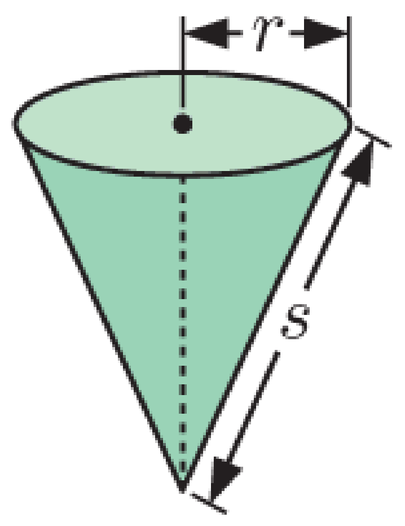


$$A = 2(ab + bc + ac)$$

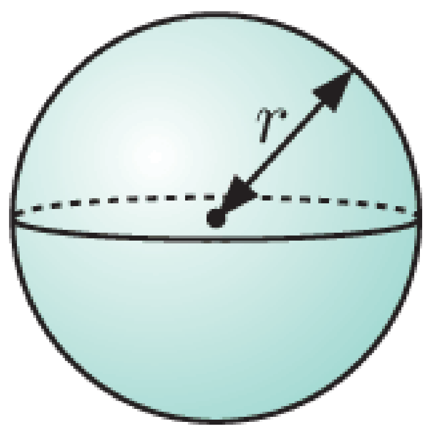
CYLINDER

Object	*Outer surface area*
Hollow cylinder  	$A = 2\pi rh$ (no ends)
Open cylinder  	$A = 2\pi rh + \pi r^2$ (one end)
Solid cylinder  	$A = 2\pi rh + 2\pi r^2$ (two ends)

CONE

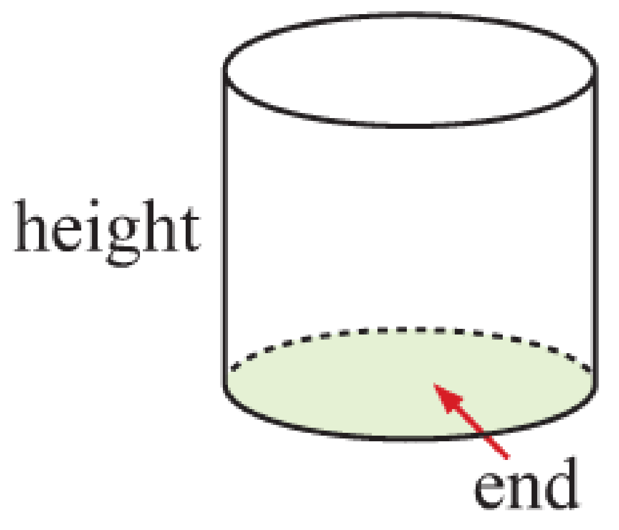
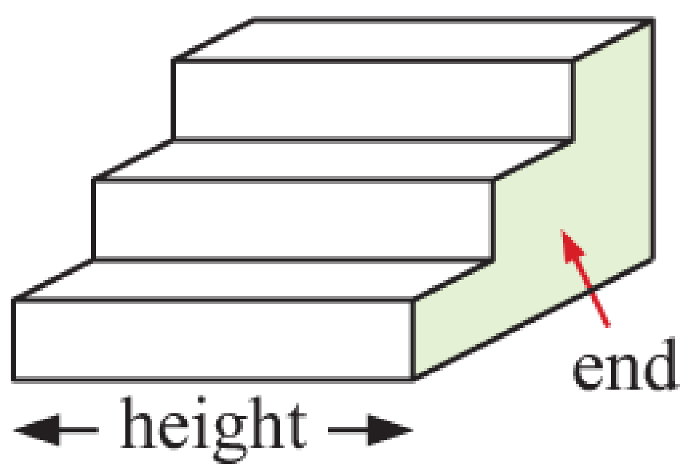
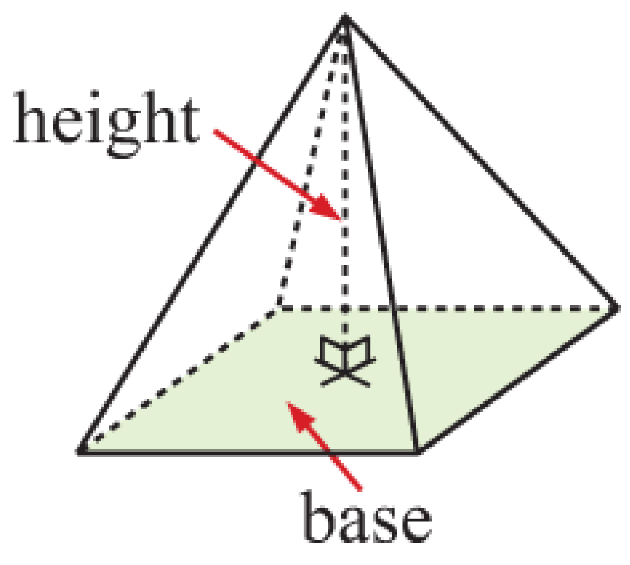
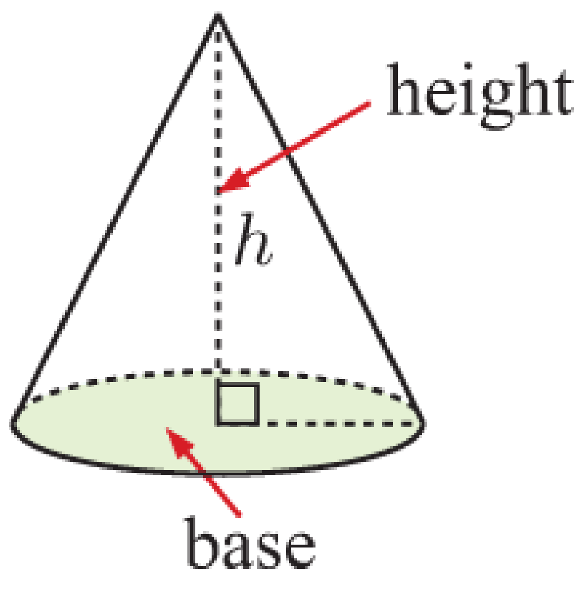
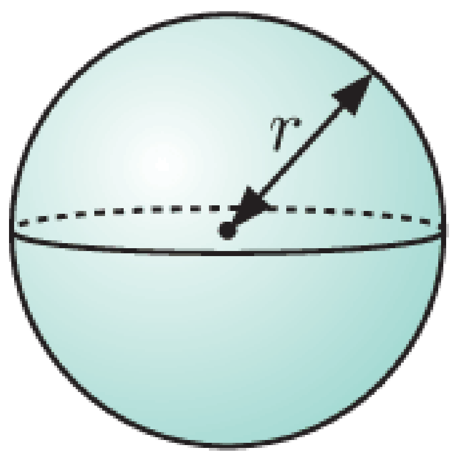
Object	*Outer surface area*
Open cone  	$A = \pi rs$ (no base)
Solid cone  	$A = \pi rs + \pi r^2$ (solid)

SPHERE



$A = 4\pi r^2$

VOLUME FORMULAE

Object	*Diagram*	*Volume*
Solids of uniform cross-section	  	$V =$ **area of end** $\times$ **length**
Pyramids and cones	  	$V = \frac{1}{3}$(**area of base** $\times$ **height**)
Spheres	 	$V = \frac{4}{3}\pi r^3$

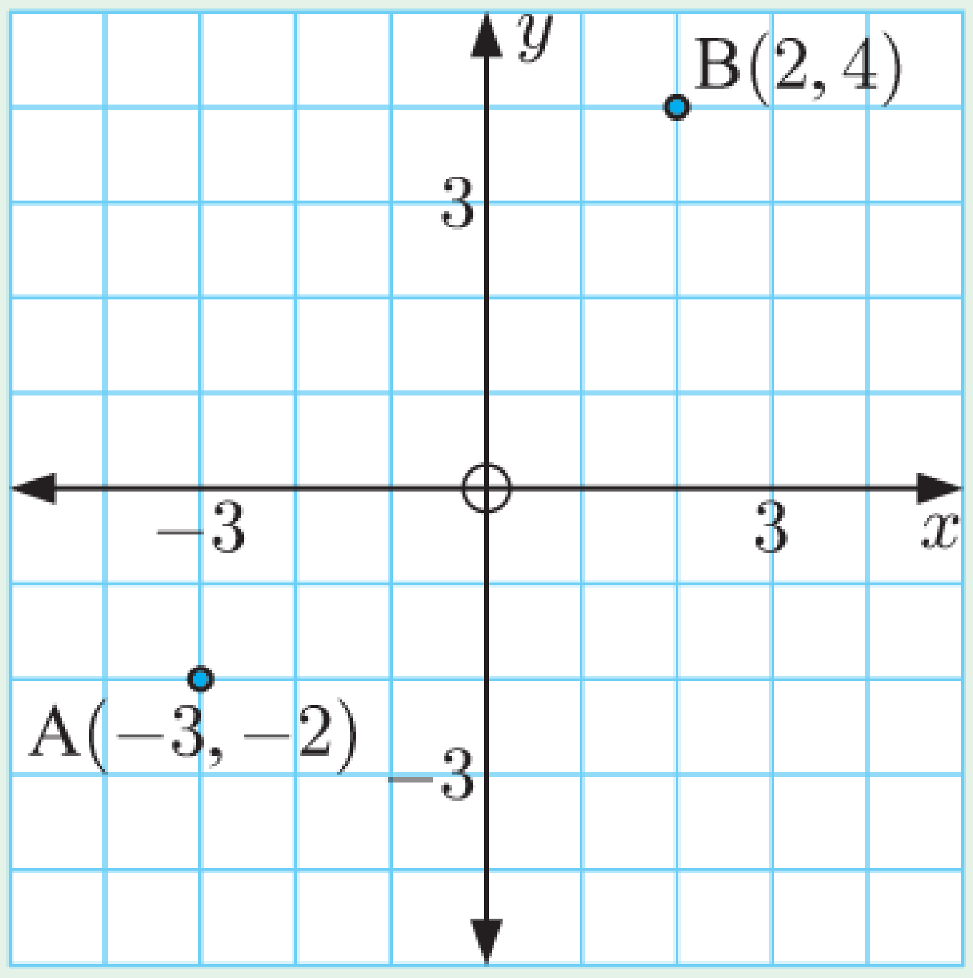
Chapter 1

Straight lines

Contents:

OPENING PROBLEM

A town has two hospitals located at $A(-3,\ -2)$ and $B(2,\ 4)$. The grid units are kilometres. In an emergency, an ambulance crew will be sent from the nearest hospital.



Things to think about:

- **a** What is the midpoint between the hospitals?
- **b** How can we tell which hospital an ambulance crew should be sent from?

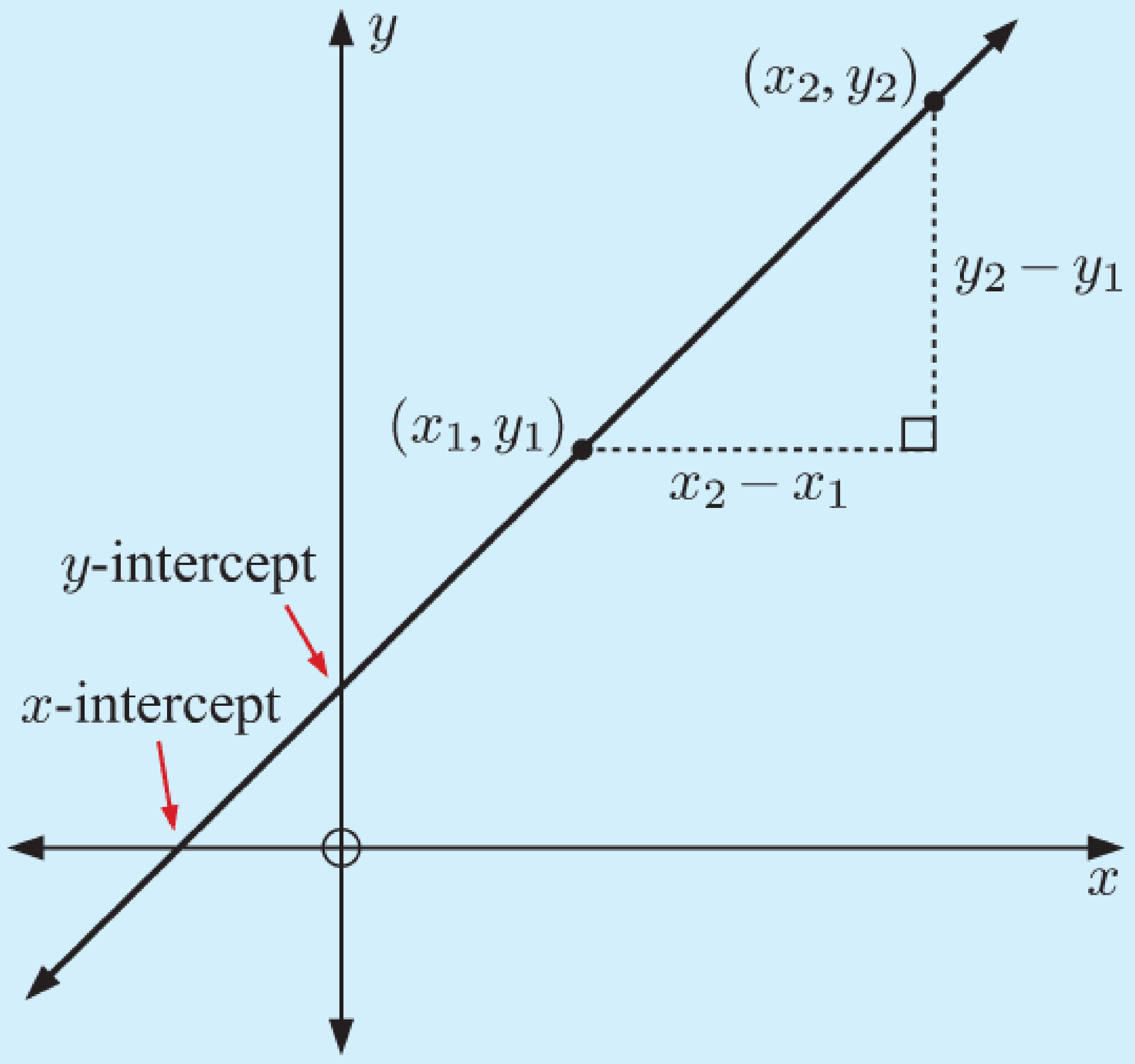
In this Chapter we study the equations and graphs of straight lines in the Cartesian plane. We will consider **perpendicular bisectors** which define the set of points equidistant from two locations, and consider simultaneous linear equations corresponding to the intersection of lines.

A LINES IN THE CARTESIAN PLANE

In previous years you should have seen that:

- The **x-intercept** of a line is the value of x where the line cuts the x-axis.
- The **y-intercept** of a line is the value of y where the line cuts the y-axis.
- The **gradient** of a line is a measure of its steepness.

  The gradient of the line passing through $(x_1,\ y_1)$ and $(x_2,\ y_2)$ is $\dfrac{y\text{-step}}{x\text{-step}} = \dfrac{y_2 - y_1}{x_2 - x_1}$.
- Two lines are **parallel** if their gradients are equal.
- Two lines are **perpendicular** if their gradients are negative reciprocals of one another.



THE EQUATION OF A LINE

The **equation of a line** is an equation which connects the x and y values for every point on the line.

Using the gradient formula, the position of a general point $(x,\ y)$ on a line with gradient m passing through $(x_1,\ y_1)$, is given by $\dfrac{y - y_1}{x - x_1} = m$.

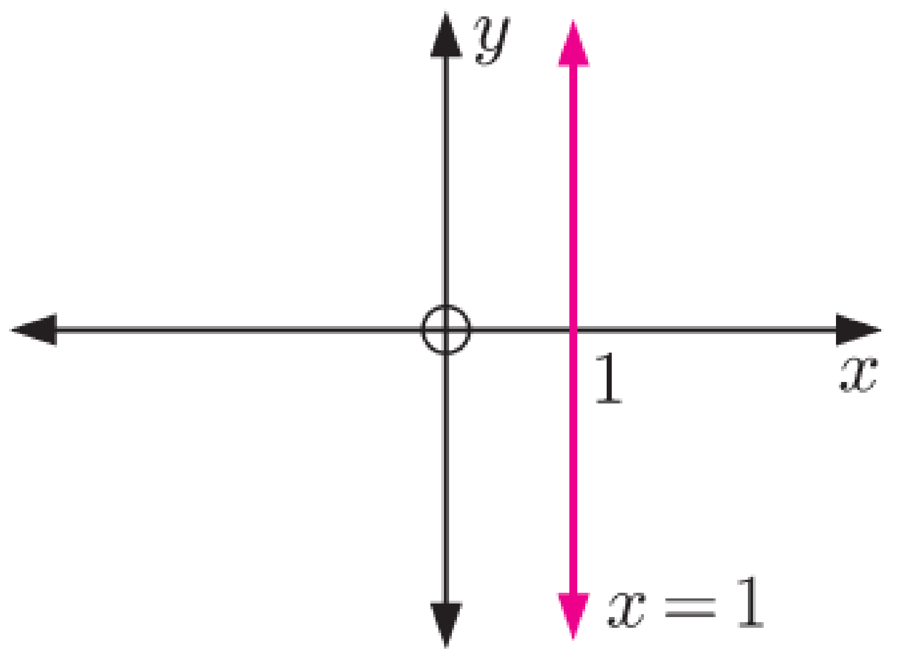
Rearranging, we find the **equation of the line** is $y - y_1 = m(x - x_1)$.

We call this **point-gradient** form. It allows us to quickly write down the equation of a line given its gradient and any point on it.

We can then rearrange the equation into other forms:

- In **gradient-intercept form**, the equation of the line with gradient m and y-intercept c is $y = mx + c$.
- In **general form**, the equation of a line is $ax + by = d$ where a, b, d are constants.

The general form allows us to write the equations of vertical lines, for which the gradient is undefined.
For the line $x = 1$ we let $a = 1$, $b = 0$, and $d = 1$.



In examinations you may also be asked to write the equation of a line in the form $ax + by + d = 0$.

Example 1 **Self Tutor**

Find, in gradient-intercept form, the equation of the line with gradient -3 that passes through $(4, -5)$.

The equation of the line is $y - (-5) = -3(x - 4)$
$\therefore\ y + 5 = -3x + 12$
$\therefore\ y = -3x + 7$

We can find a line's equation given the gradient and a point which lies on the line.

EXERCISE 1A

1 State the gradient and y-intercept of the line with equation:

a $y = 3x + 7$ **b** $y = -2x - 5$ **c** $y = \frac{2}{3}x - \frac{1}{3}$

d $y = \dfrac{7x + 2}{9}$ **e** $y = \dfrac{2x - 3}{6}$ **f** $y = \dfrac{3 - 5x}{8}$

2 Find, in gradient-intercept form, the equation of the line which has:

a gradient 3 and passes through $(4, 1)$ **b** gradient -2 and passes through $(-3, 5)$

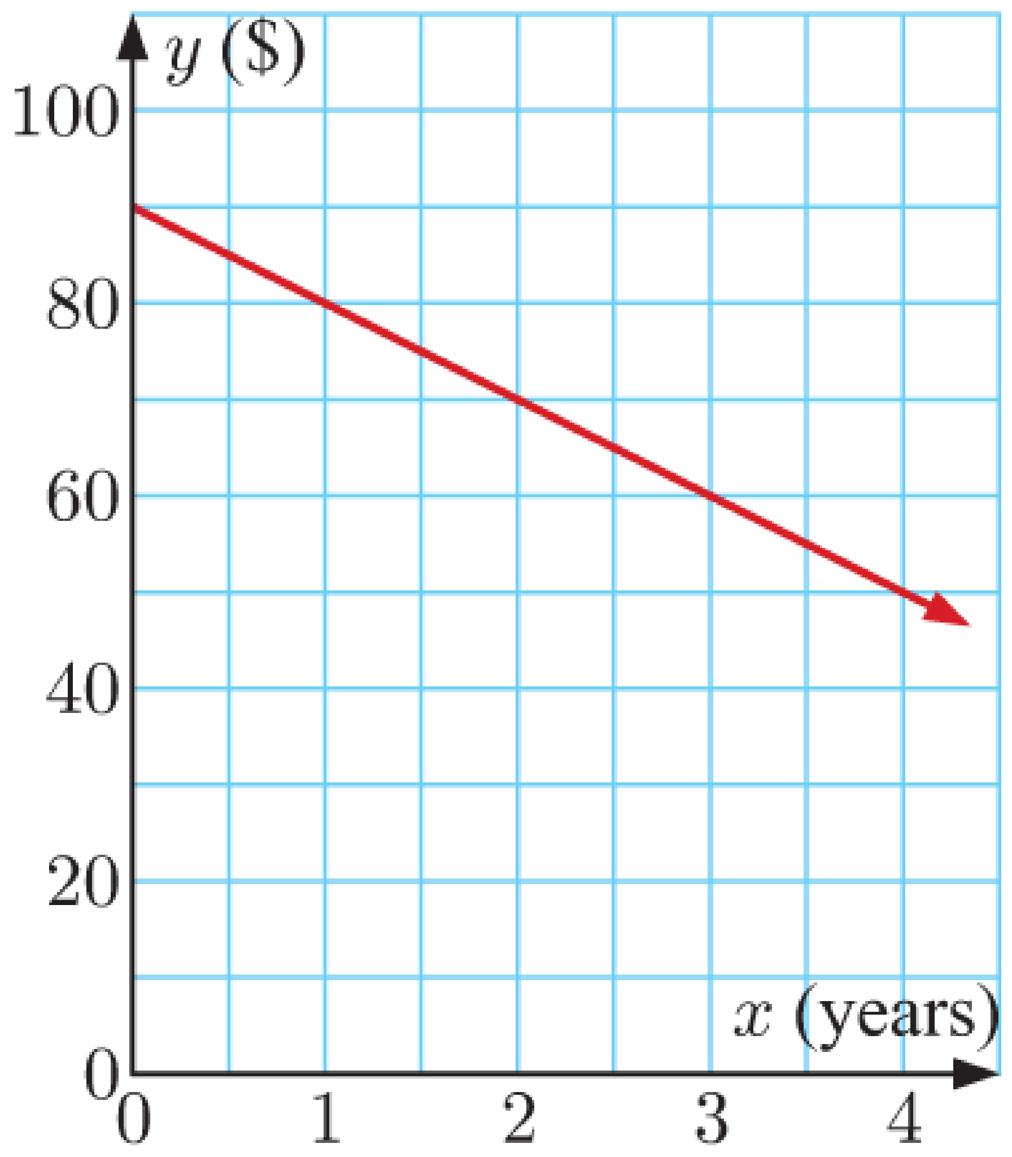
c gradient $\frac{1}{4}$ and passes through $(4, -3)$ **d** gradient $-\frac{3}{4}$ and y-intercept 4.

3 An unused bank account is charged a yearly fee. The graph alongside shows the balance of the account after x years.

a Find the gradient and y-intercept of the line, and interpret your answers.

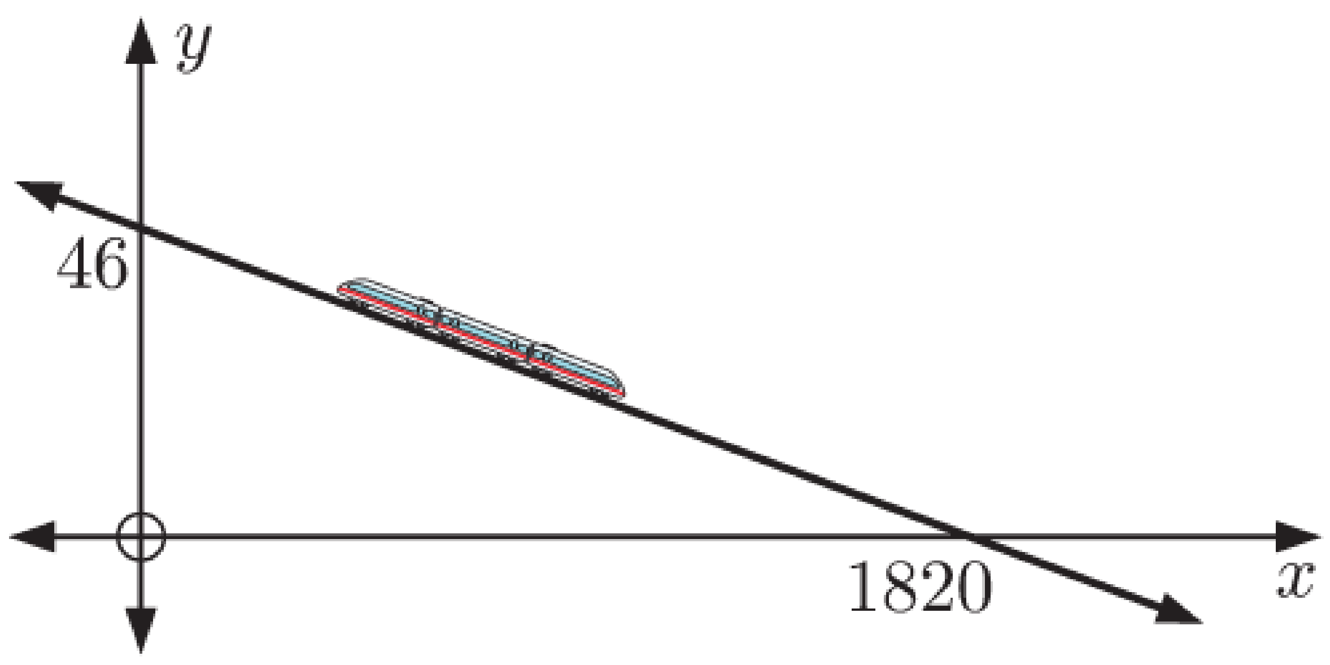
b Find the equation of the line.

c How long will it take for the account to run out of money?



4 The graph alongside shows the descent of a train down a hill. The units are metres.

- **a** Calculate the gradient of the train's descent.
- **b** Find the equation of the train line.



5 The height of a helicopter above sea level t minutes after taking off is $H = 150 + 120t$ metres.

- **a** What height above sea level did the helicopter take off from?
- **b** Interpret the value 120 in the equation.
- **c** Find the height of the helicopter above sea level after 2 minutes.
- **d** How long will it take for the helicopter to be 650 m above sea level?

Example 2 **Self Tutor**

Write the equation:

- **a** $y = -\frac{2}{3}x + 2$ in general form
- **b** $3x - 4y = -2$ in gradient-intercept form.

a

$$\begin{aligned} y &= -\tfrac{2}{3}x + 2 \\ \therefore\ 3y &= -2x + 6 \\ \therefore\ 2x + 3y &= 6 \end{aligned}$$

b

$$\begin{aligned} 3x - 4y &= -2 \\ \therefore\ -4y &= -3x - 2 \\ \therefore\ y &= \tfrac{3}{4}x + \tfrac{1}{2} \end{aligned}$$

6 Write in general form:

- **a** $y = -4x + 6$
- **b** $y = 5x - 3$
- **c** $y = -\frac{3}{4}x + \frac{5}{4}$
- **d** $y = \frac{3}{5}x - \frac{1}{5}$

7 Write in gradient-intercept form:

- **a** $5x + y = 2$
- **b** $3x + 7y = -2$
- **c** $2x - y = 6$
- **d** $3x - 13y = -4$

8 Explain why the gradient of the line with general form $ax + by = d$ is $-\dfrac{a}{b}$.

Example 3 **Self Tutor**

Find, in general form, the equation of the line with gradient $\frac{2}{3}$ that passes through $(-2, -1)$.

Since the line has gradient $\frac{2}{3}$, the general form of its equation is $2x - 3y = d$

Using the point $(-2, -1)$, the equation is $2x - 3y = 2(-2) - 3(-1)$
which is $2x - 3y = -1$

9 Find, in general form, the equation of the line which has:

- **a** gradient -4 and passes through $(1, 2)$
- **b** gradient $\frac{1}{2}$ and passes through $(3, -5)$
- **c** gradient $-\frac{5}{3}$ and passes through $(-2, 6)$
- **d** gradient $\frac{7}{6}$ and passes through $(-1, -4)$.

Example 4

Self Tutor

Find, in gradient-intercept form, the equation of the line which passes through A(3, 2) and B(5, −1).

The line has gradient $= \dfrac{-1-2}{5-3} = \dfrac{-3}{2} = -\dfrac{3}{2}$,

and passes through the point A(3, 2).

∴ the equation of the line is

$$y - 2 = -\tfrac{3}{2}(x-3)$$
$$\therefore\ y - 2 = -\tfrac{3}{2}x + \tfrac{9}{2}$$
$$\therefore\ y = -\tfrac{3}{2}x + \tfrac{13}{2}$$

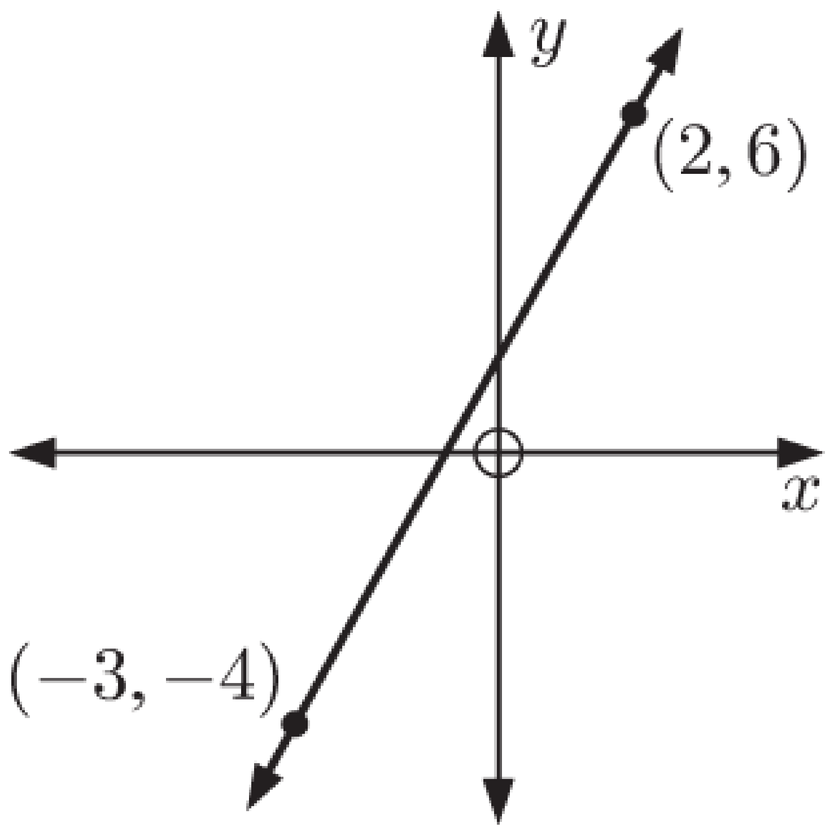
We could use *either* A or B as the point which lies on the line.

10 Find, in gradient-intercept form, the equation of the line which passes through:

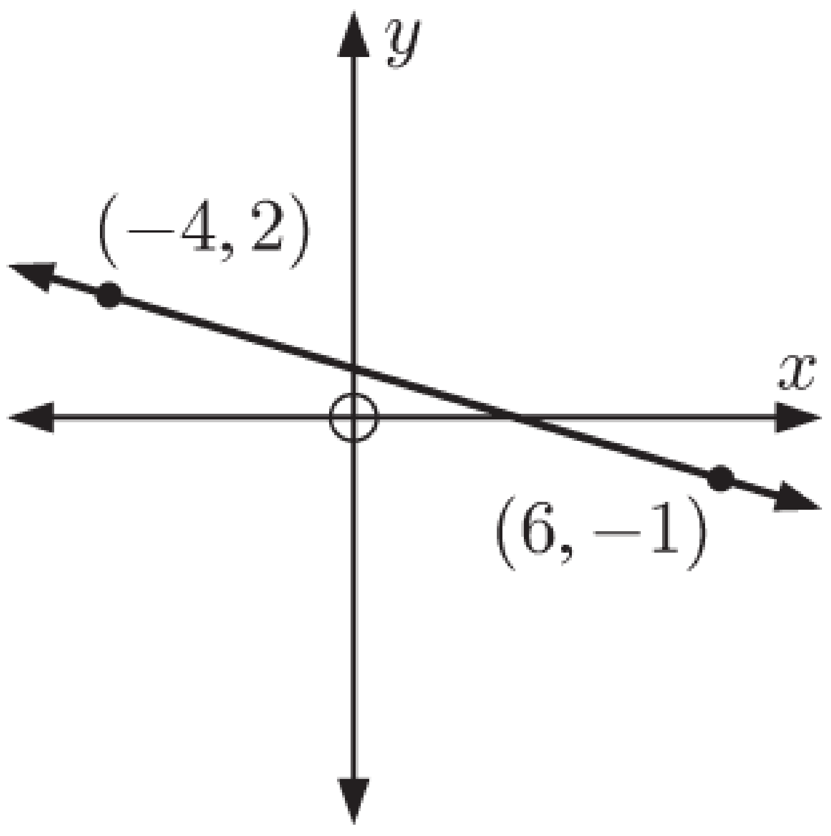
a A(−2, 1) and B(3, 11)
b A(7, 2) and B(4, 5)
c M(−2, −5) and N(3, 2)
d R(5, −1) and S(−7, 9).

11 Find, in general form, the equation of each line:

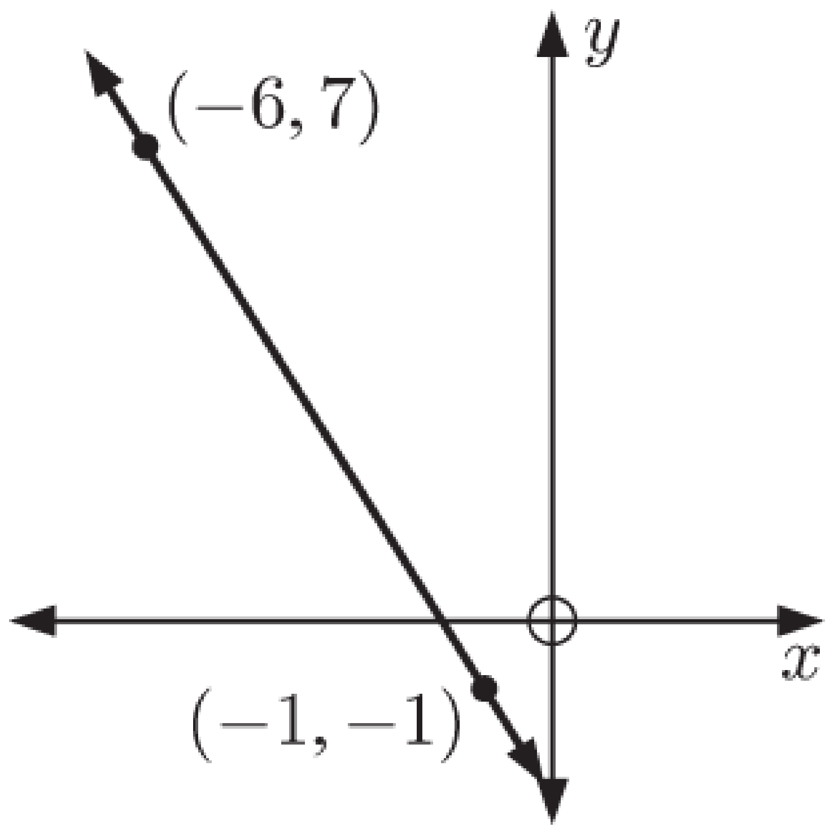
a



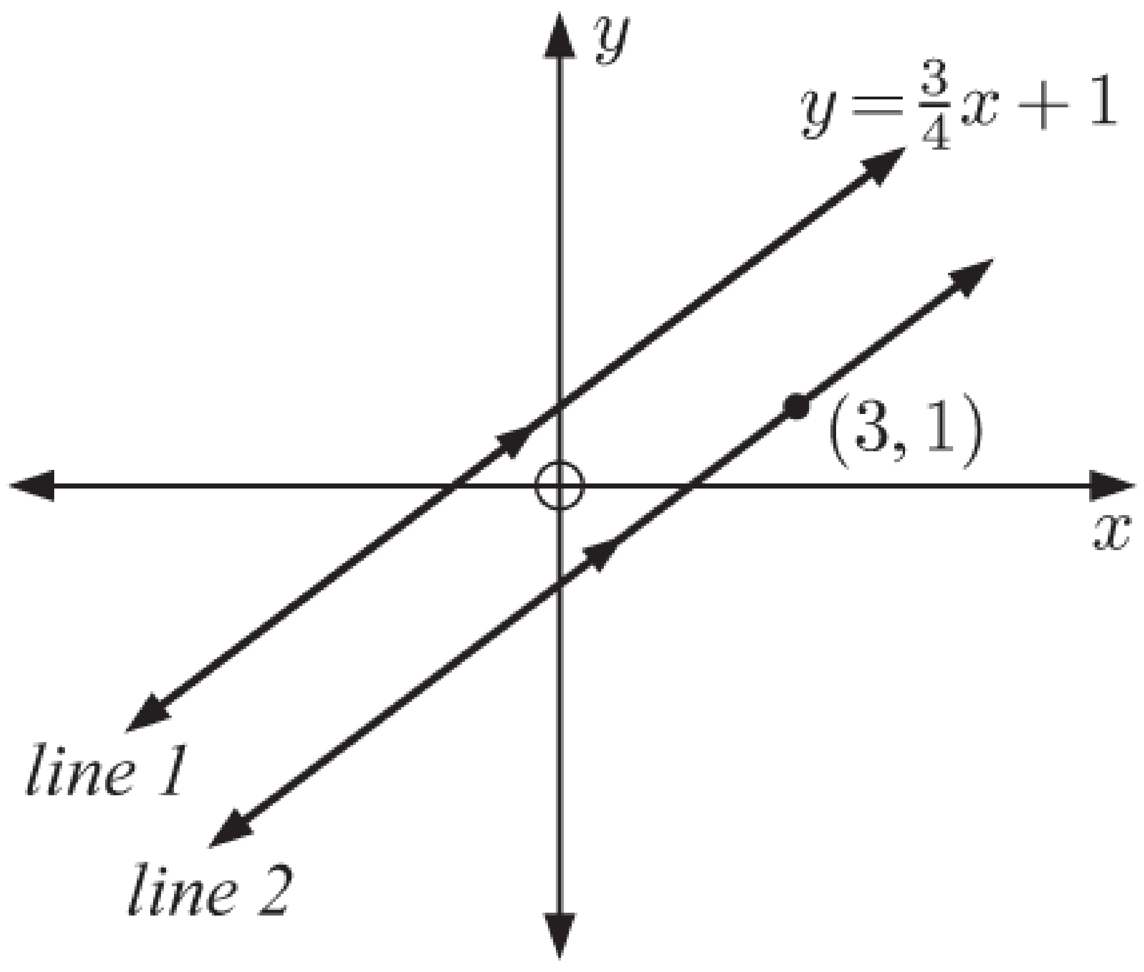
b



c



12 **a** Find, in gradient-intercept form, the equation of *line 2*.
b Hence find the y-intercept of *line 2*.



13 Find the equation of the line which is:

a parallel to $y = 3x - 2$ and passes through (1, 4)
b parallel to $2x - y = -3$ and passes through (3, −1)
c perpendicular to $y = -2x + 1$ and passes through (−1, 5)
d perpendicular to $x + 2y = 6$ and passes through (−2, −1).

14 *Line 1* passes through A(−2, −1) and B(4, 3). *Line 2* is perpendicular to *line 1* and passes through A. Find the equation of each line.

Example 5 **Self Tutor**

a Find m given that $(-2, 3)$ lies on the line with equation $y = mx + 7$.

b Find k given that $(3, k)$ lies on the line with equation $x + 4y = -9$.

a Substituting $x = -2$ and $y = 3$ into the equation gives

$$3 = m(-2) + 7$$
$$\therefore \ 2m = 4$$
$$\therefore \ m = 2$$

b Substituting $x = 3$ and $y = k$ into the equation gives

$$3 + 4k = -9$$
$$\therefore \ 4k = -12$$
$$\therefore \ k = -3$$

15 Determine whether:

a $(3, 11)$ lies on the line with equation $y = 4x - 1$

b $(-6, -2)$ lies on the line with equation $y = \frac{2}{3}x - 6$

c $(-4, -8)$ lies on the line with equation $7x - 3y = -4$

d $(-\frac{1}{2}, 2)$ lies on the line with equation $6x + 10y = 17$.

16 **a** Find c given that $(2, 15)$ lies on the line with equation $y = 4x + c$.

b Find m given that $(\frac{1}{2}, 3)$ lies on the line with equation $y = mx - \frac{5}{2}$.

c Find t given that $(t, 4)$ lies on the line with equation $y = \frac{2}{3}x - \frac{4}{3}$.

17 Find k given that:

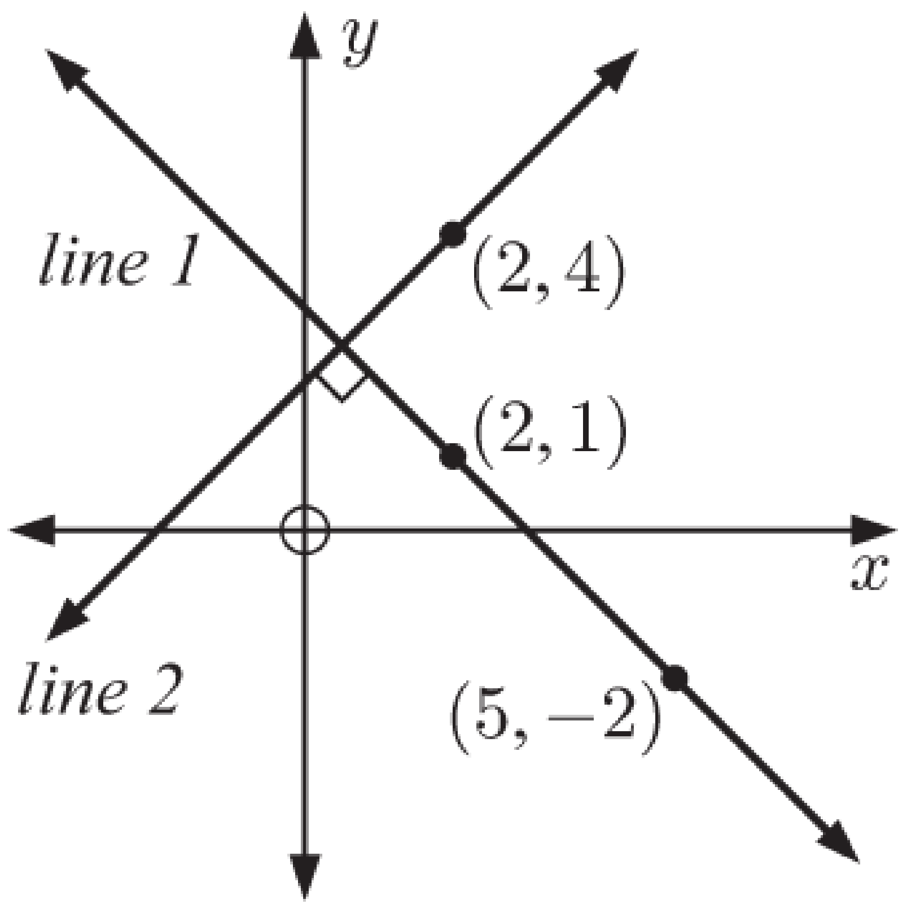
a $(6, -3)$ lies on the line with equation $2x + 5y = k$

b $(-8, -5)$ lies on the line with equation $7x - y = k$

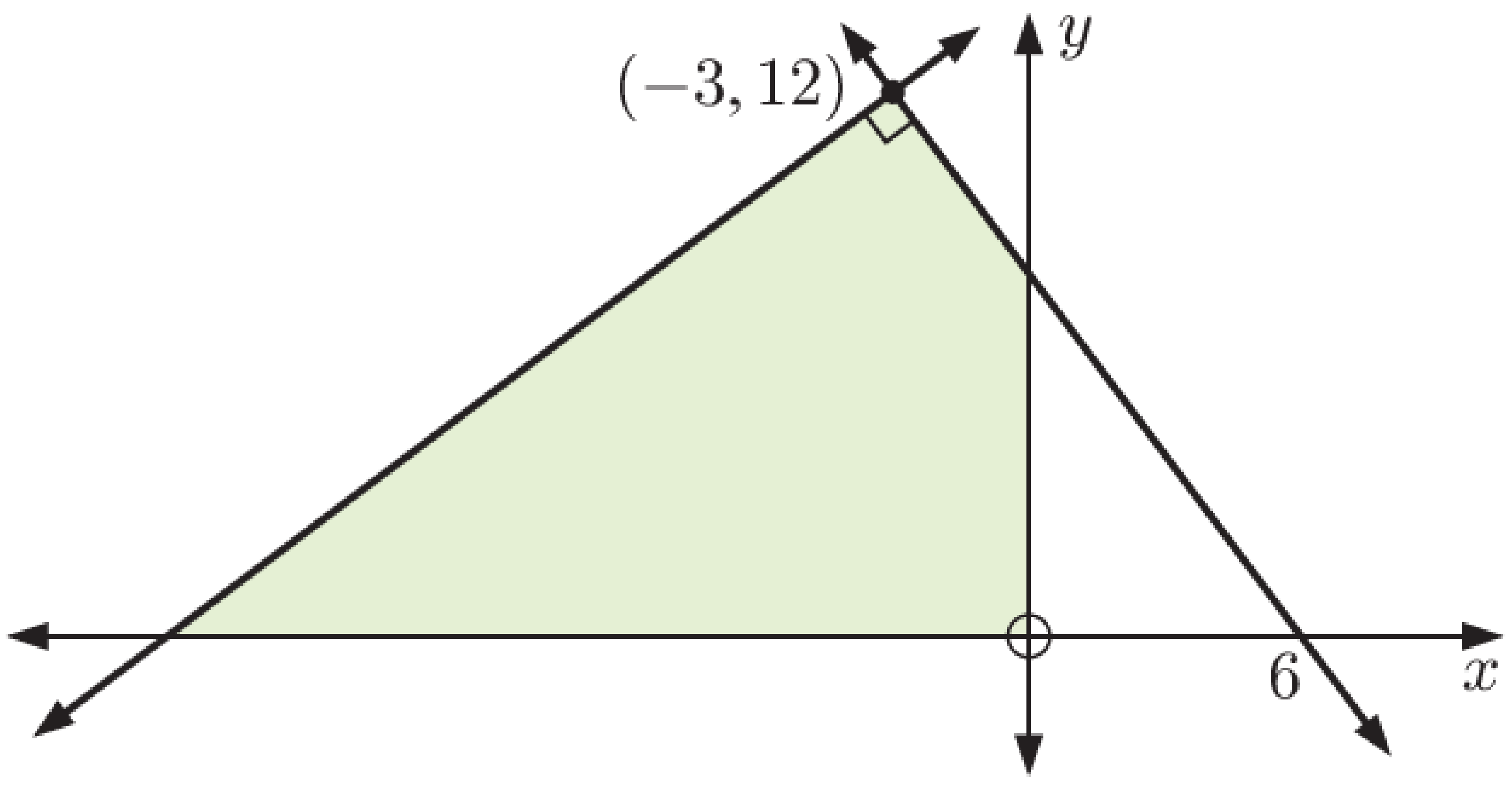
c $(k, 0)$ lies on the line with equation $3x - 4y = -36$.

18 **a** Find the equation of *line 2*. Write your answer in the form $ax + by + d = 0$.

b Find the x-intercept of *line 2*.



19 Find the shaded area:

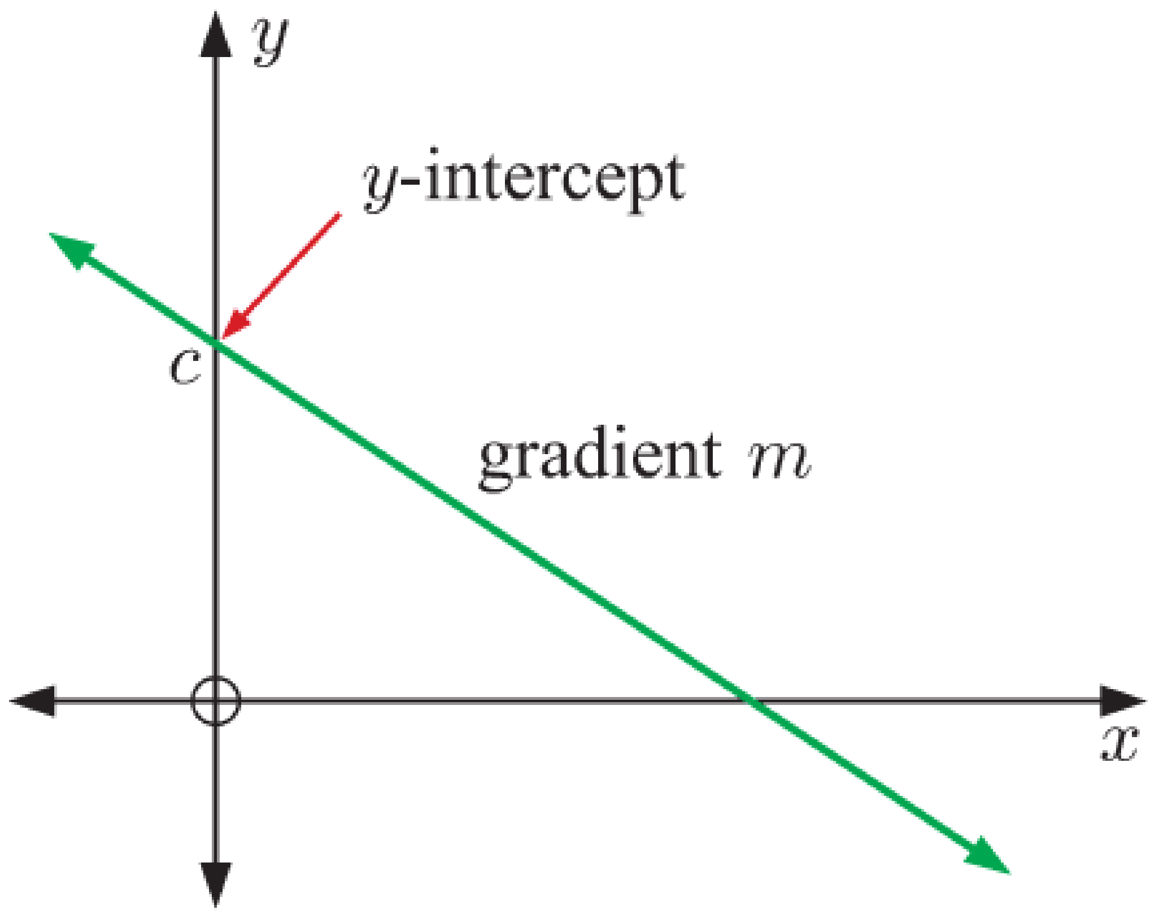


B GRAPHING A STRAIGHT LINE

LINES IN GRADIENT-INTERCEPT FORM

To draw the graph of $y = mx + c$ we:

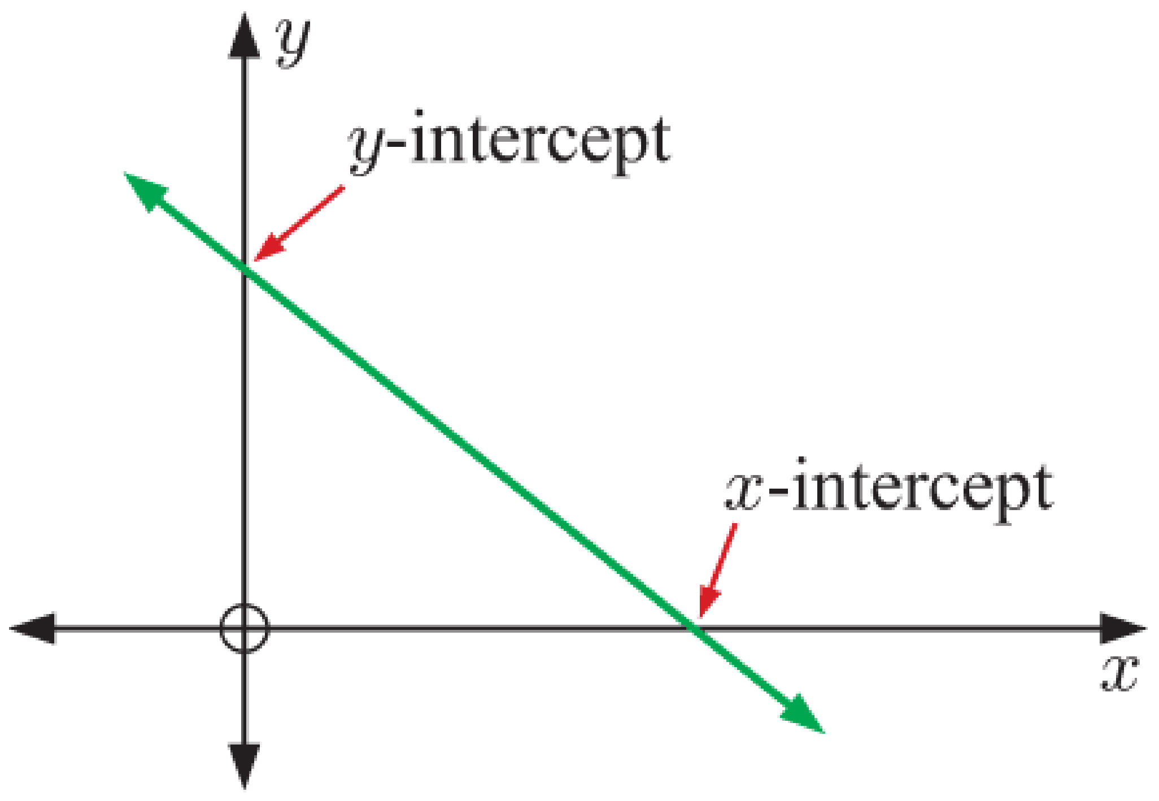
- Use the y-intercept c to plot the point $(0,\ c)$.
- Use x and y-steps from the gradient m to locate another point on the line.
- Join the two points and extend the line in either direction.



LINES IN GENERAL FORM

To draw the graph of $ax + by = d$ we:

- Find the y-intercept by letting $x = 0$.
- Find the x-intercept by letting $y = 0$.
- Join the points where the line cuts the axes and extend the line in either direction.



If $d = 0$ then the graph passes through the origin. In this case we plot $y = -\frac{a}{b}x$ using its gradient.

Example 6 **Self Tutor**

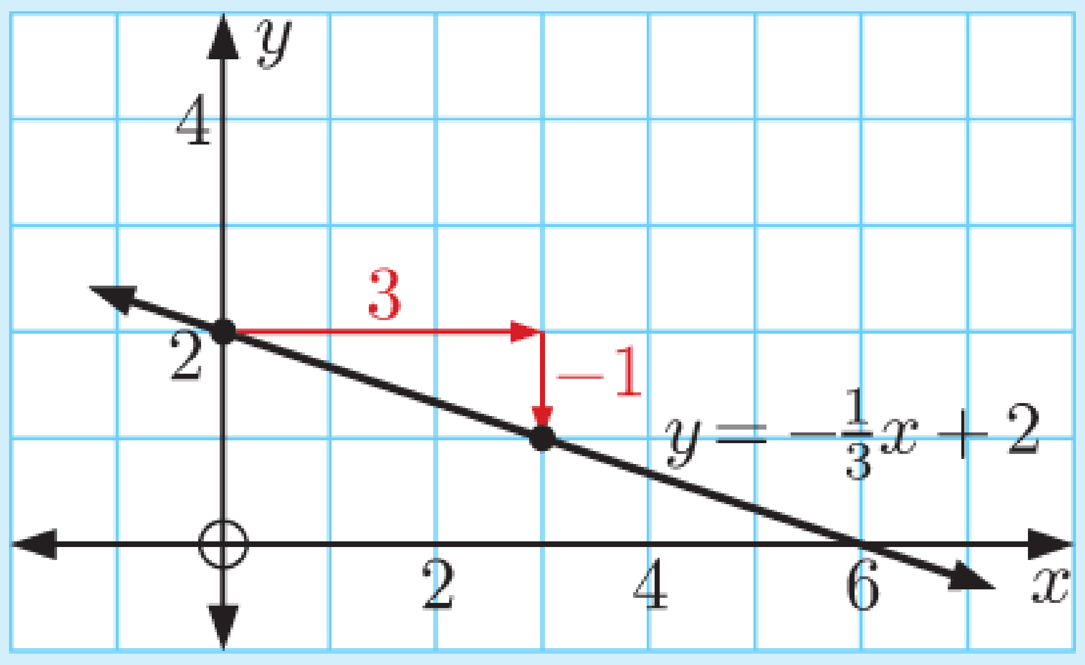
Draw the graph of:

a $y = -\frac{1}{3}x + 2$ **b** $2x - 4y = 12$

a For $y = -\frac{1}{3}x + 2$:

- the y-intercept is $c = 2$
- the gradient is

$$m = \frac{-1}{3} \begin{matrix} \leftarrow y\text{-step} \\ \leftarrow x\text{-step} \end{matrix}$$



b When $x = 0$, $-4y = 12$

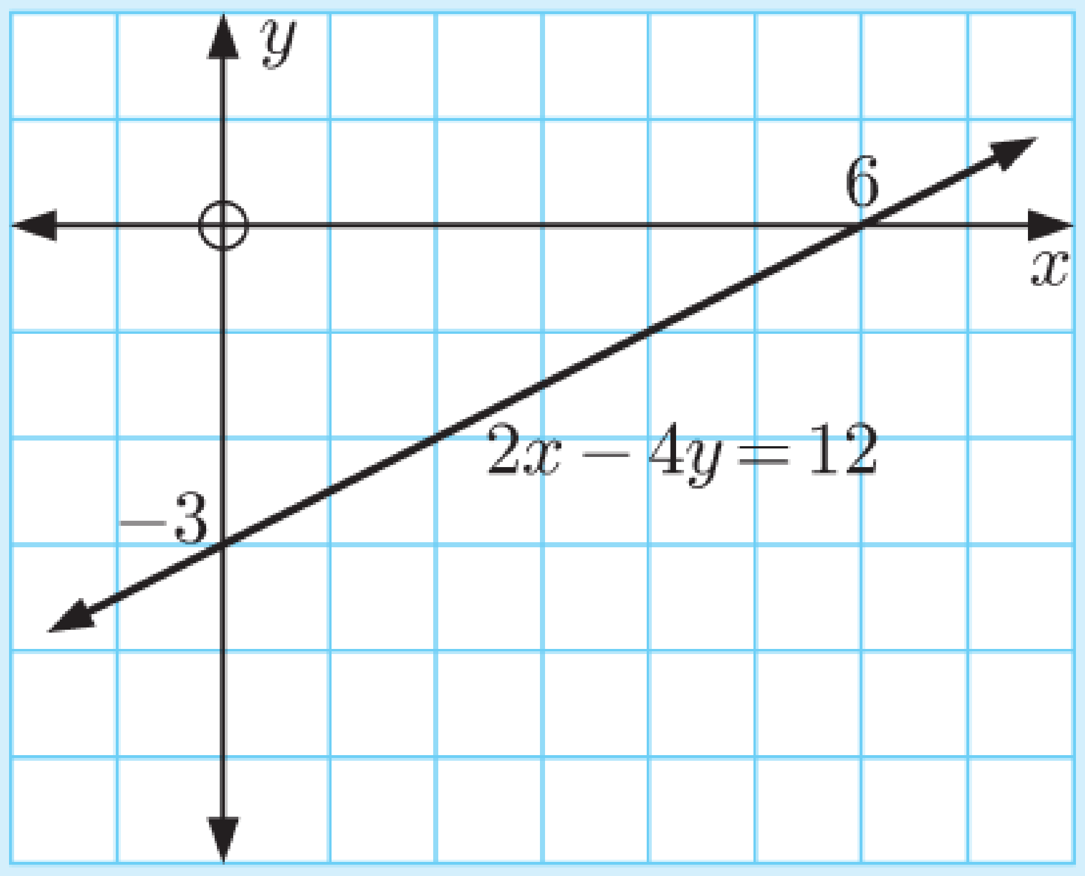
$\therefore\ y = -3$

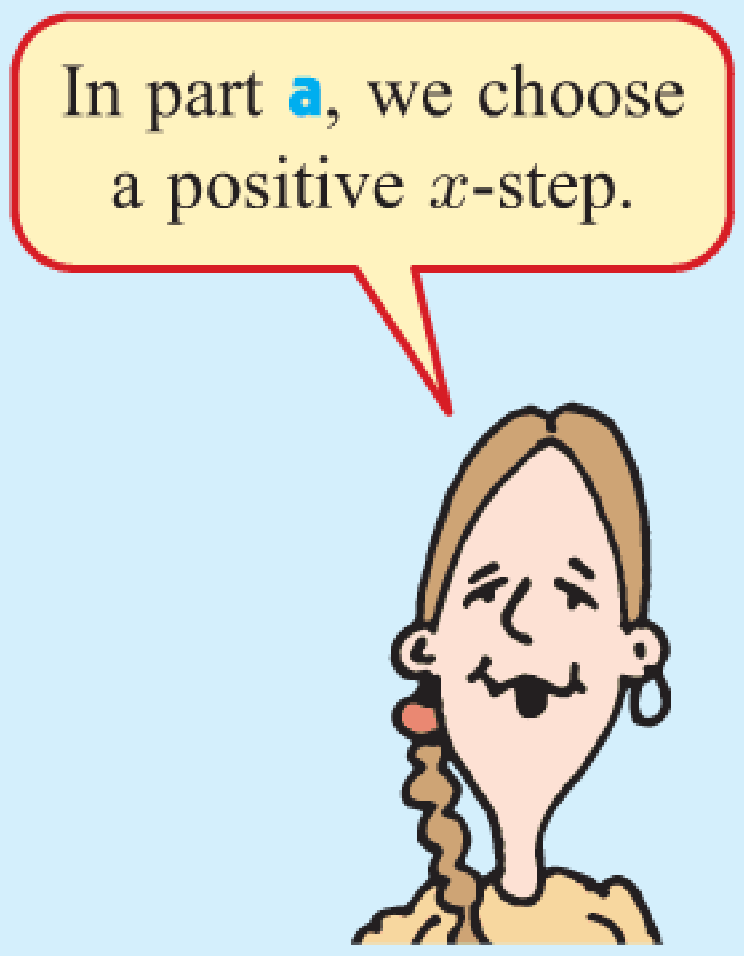
So, the y-intercept is -3.

When $y = 0$, $2x = 12$

$\therefore\ x = 6$

So, the x-intercept is 6.





EXERCISE 1B

1 Draw the graph of:

a $y = \frac{1}{2}x + 1$ **b** $y = 3x - 2$ **c** $y = -\frac{3}{2}x + 4$

d $y = -4x$ **e** $y = \frac{6}{5}x - 3$ **f** $y = -\frac{5}{3}x - 1$

Check your answers using technology.

GRAPHICS CALCULATOR INSTRUCTIONS

GRAPHING PACKAGE

2 Draw the graph of:

a $3x + 2y = 12$ **b** $4x - y = 8$ **c** $3x - 4y = -24$

d $2x + 5y = 15$ **e** $6x + 4y = -36$ **f** $7x + 4y = 42$

3 Consider the line with equation $y = -\frac{3}{4}x + 2$.

a Find the gradient and y-intercept of the line.

b Determine whether the following points lie on the line:

i $(8, -4)$ **ii** $(1, 3)$ **iii** $(-2, \frac{7}{2})$

c Draw the graph of the line, showing the results you have found.

4 Consider the line with equation $2x - 3y = 18$.

a Find the axes intercepts of the line.

b Determine whether the following points lie on the line:

i $(3, -4)$ **ii** $(7, -2)$

c Find c such that $(-3, c)$ lies on the line.

d Draw the graph of the line, showing the results you have found.

5 At a sushi restaurant, *nigiri* costs \$4.50 per serve and *sashimi* costs \$9.00 per serve. Hiroko spent a total of \$45 buying x serves of *nigiri* and y serves of *sashimi*.

a Explain why $4.5x + 9y = 45$.

b If Hiroko bought 4 serves of *nigiri*, how much *sashimi* did she buy?

c If Hiroko bought 1 serve of *sashimi*, how much *nigiri* did she buy?

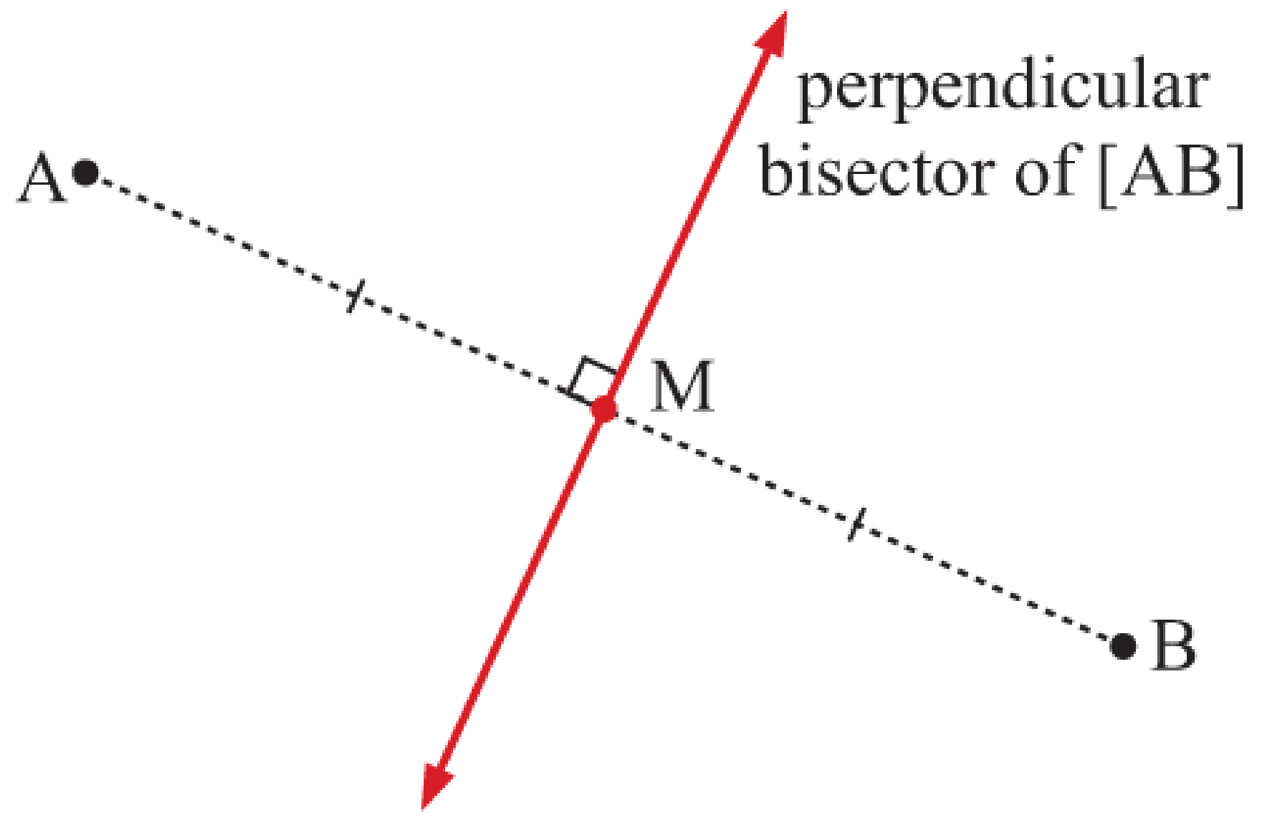
d Draw the graph of $4.5x + 9y = 45$. Mark two points on your graph to indicate your answers to **b** and **c**.

C PERPENDICULAR BISECTORS

The **perpendicular bisector** of a line segment [AB] is the line perpendicular to [AB] which passes through its midpoint.

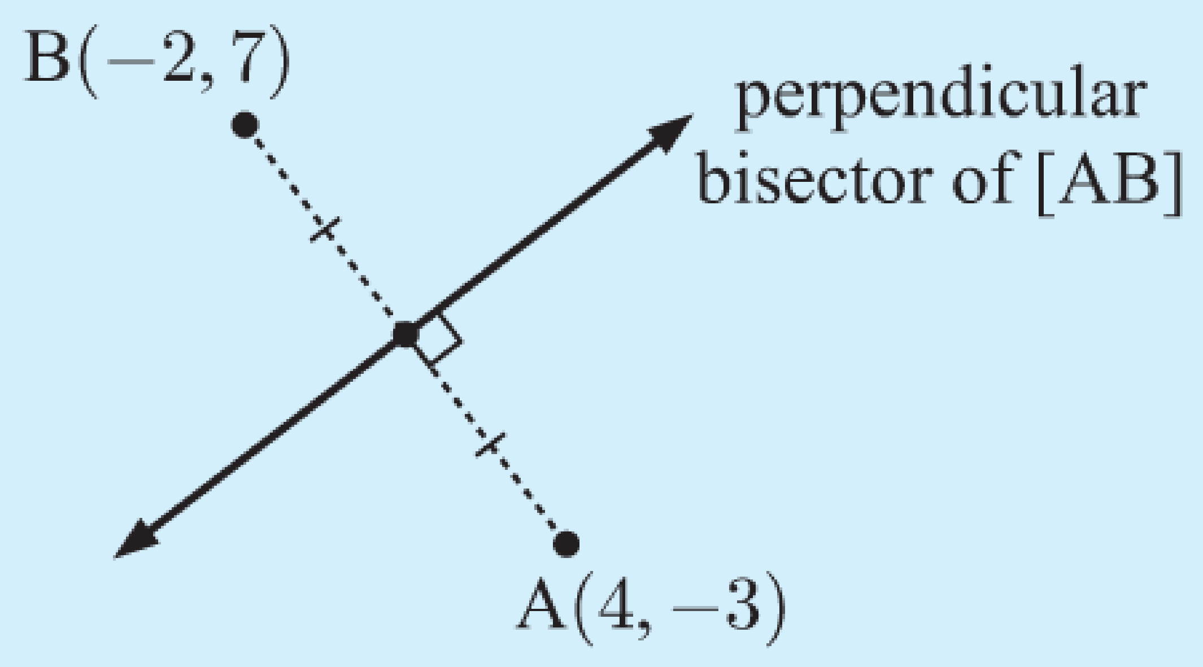
Notice that:

- Points on the perpendicular bisector are equidistant from A and B.
- The perpendicular bisector divides the number plane into two regions. On one side of the line are points that are closer to A than to B, and on the other side are points that are closer to B than to A.



Example 7 **Self Tutor**

Given A(4, −3) and B(−2, 7), find the equation of the perpendicular bisector of [AB].



The midpoint M of [AB] is $\left(\frac{4+-2}{2}, \frac{-3+7}{2}\right)$

or (1, 2).

The gradient of [AB] is $\frac{7--3}{-2-4} = \frac{10}{-6} = -\frac{5}{3}$

$\therefore$ the gradient of the perpendicular bisector is $\frac{3}{5}$.

$\therefore$ the equation of the perpendicular bisector is $3x - 5y = 3(1) - 5(2)$

which is $3x - 5y = -7$.

EXERCISE 1C

1 Consider the points A(3, 1) and B(5, 7).

- **a** Find the midpoint of [AB].
- **b** Find the gradient of [AB].
- **c** Hence state the gradient of the perpendicular bisector.
- **d** Find the equation of the perpendicular bisector.

2 Find the equation of the perpendicular bisector of:

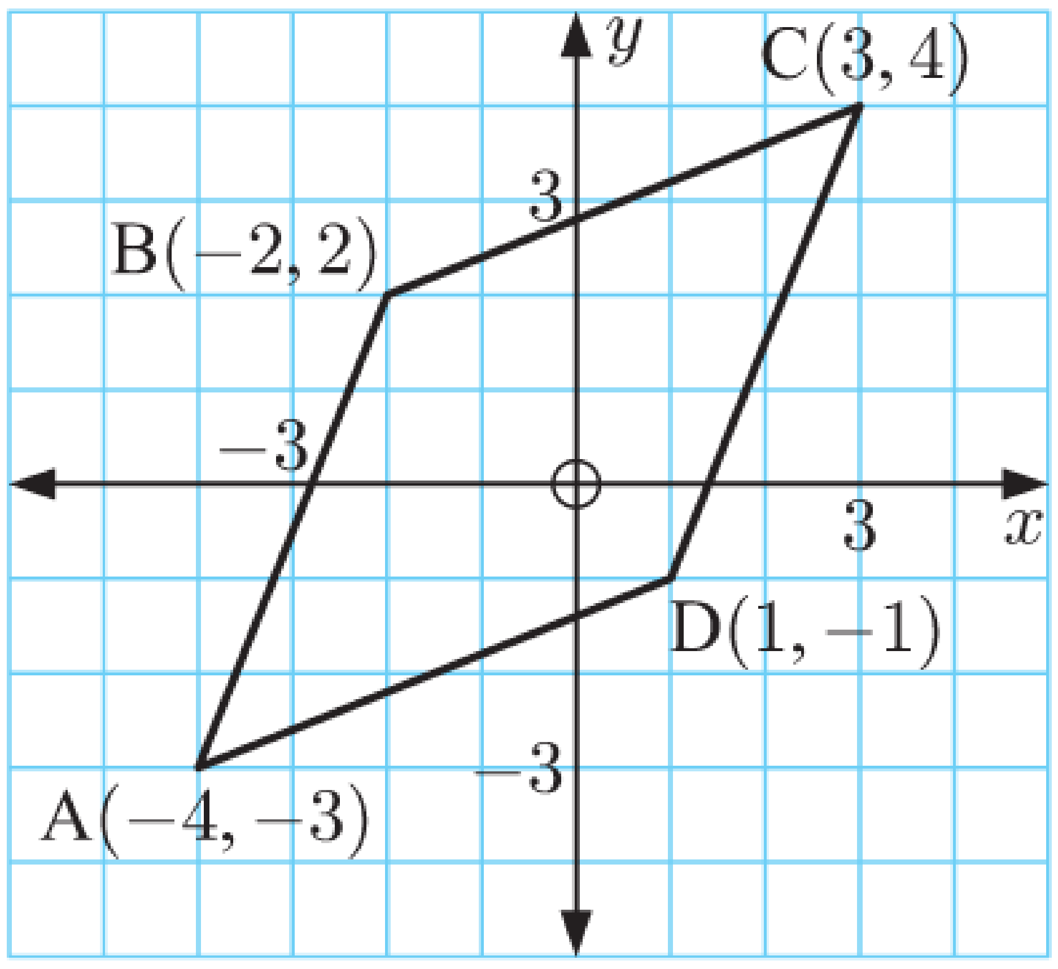
- **a** A(5, 2) and B(1, 4)
- **b** A(−1, 5) and B(5, 3)
- **c** M(6, −3) and N(2, 1)
- **d** M(7, 2) and N(−1, 6)
- **e** O(0, 0) and P(9, 0)
- **f** A(3, 6) and B(−1, 3).

3 Suppose P is (6, −1) and Q is (2, 5).

- **a** Find the equation of the perpendicular bisector of [PQ].
- **b** Show that R(1, 0) lies on the perpendicular bisector.
- **c** Show that R is equidistant from P and Q.

4 Consider the quadrilateral ABCD.

- **a** Use side lengths to show that ABCD is a rhombus.
- **b** Find the equation of the perpendicular bisector of [AC].
- **c** Show that B and D both lie on this perpendicular bisector.



5 A line segment has equation $3x - 2y + 1 = 0$. Its midpoint is (3, 5).

- **a** State the gradient of:
  - **i** the line segment
  - **ii** its perpendicular bisector.
- **b** State the equation of the perpendicular bisector. Write your answer in the form $ax + by + d = 0$.

6 Answer the **Opening Problem** on page **20**.

7 Consider the points $A(x_1, y_1)$ and $B(x_2, y_2)$.

- **a** Show that the equation of the perpendicular bisector of [AB] is

$$(x_2 - x_1)x + (y_2 - y_1)y = \frac{({x_2}^2 + {y_2}^2) - ({x_1}^2 + {y_1}^2)}{2}.$$

- **b** Suggest one advantage of writing this equation in general form.

8 Consider three points $A(1, 2)$, $B(4, 5)$, and $C(2, -1)$.

- **a** Find the equation of the perpendicular bisector of: **i** [AB] **ii** [AC] **iii** [BC].
- **b** Graph the three perpendicular bisectors on the same set of axes. Discuss your observations.
- **c** Describe how to find the centre of the circle which passes through three non-collinear points.

9 Three post offices are located in a small city at $P(-8, -6)$, $Q(1, 5)$, and $R(4, -2)$.

- **a** Find the equation of the perpendicular bisector of: **i** [PQ] **ii** [PR] **iii** [QR].
- **b** Graph the three post offices and the three perpendicular bisectors on the same set of axes. Use these lines to locate the point that is equidistant from all three post offices. Shade regions of your graph in different colours according to their closest post office.

D SIMULTANEOUS EQUATIONS

In previous years you should have seen how the intersection of two straight lines corresponds to the simultaneous solution of their equations.

A system of two equations in two unknowns can be solved by:

- **graphing** the straight lines on the same set of axes
- algebra using **substitution** or **elimination**
- technology.

To use the **equation solver function** on your calculator, you will need to write each equation in the form $ax + by = d$.

GRAPHING PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

Example 8 **Self Tutor**

Solve simultaneously: $\begin{cases} y = x - 3 \\ 2x + 3y = 16 \end{cases}$

Illustrate your answer.

$y = x - 3$ .... (1)

$2x + 3y = 16$ .... (2)

Substituting (1) into (2) gives $2x + 3(x - 3) = 16$

$\therefore\ 2x + 3x - 9 = 16$

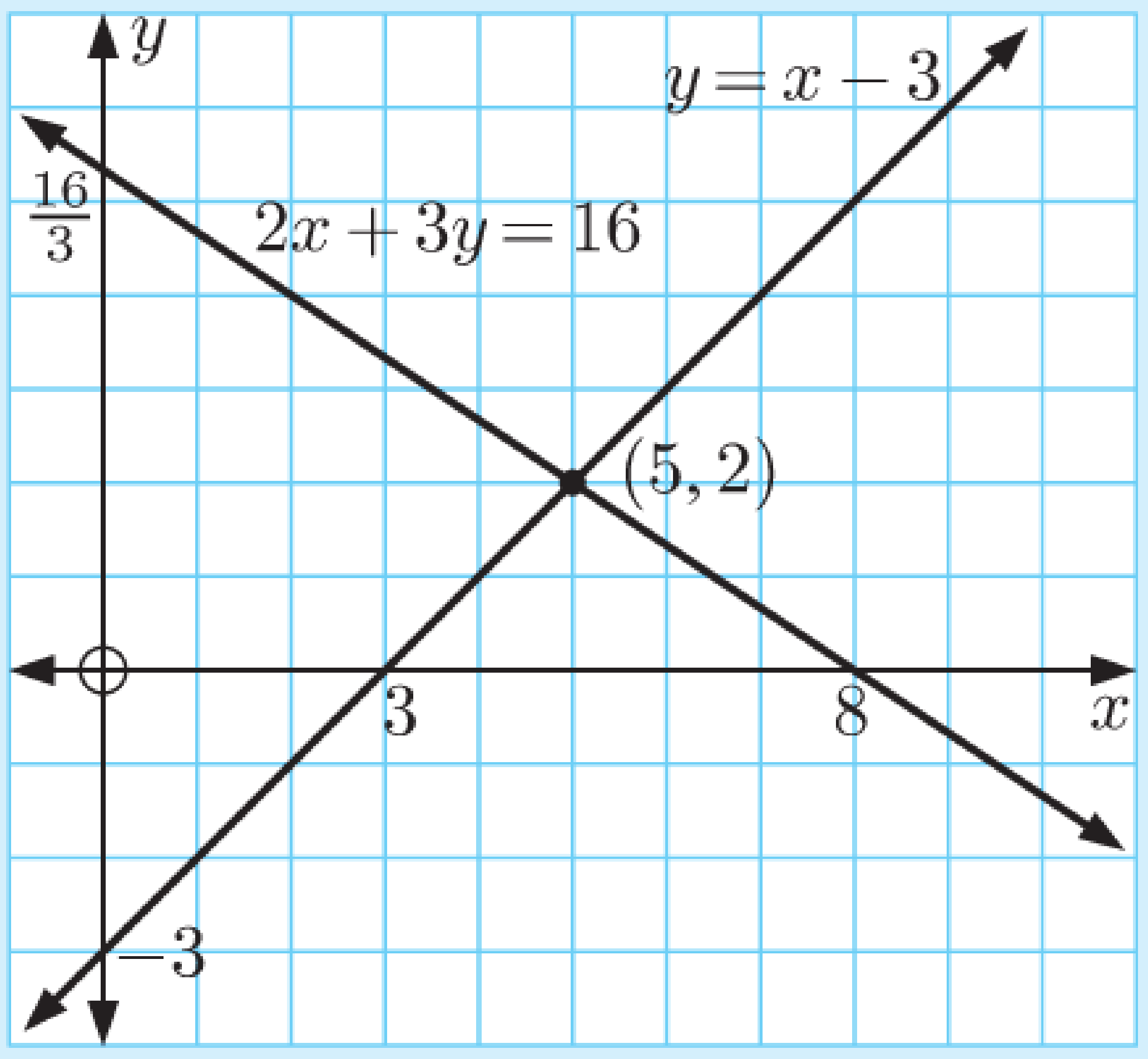
$\therefore\ 5x = 25$

$\therefore\ x = 5$

Substituting $x = 5$ into (1) gives $y = 5 - 3$

$\therefore\ y = 2$

The solution is $x = 5$, $y = 2$.



Check: Substituting into (2), $2(5) + 3(2) = 10 + 6 = 16$ ✓

EXERCISE 1D

1 Solve the following simultaneous equations graphically:

a $\begin{cases} y = 3x + 2 \\ y = x - 2 \end{cases}$ **b** $\begin{cases} y = -4x + 1 \\ y = 3x - 6 \end{cases}$ **c** $\begin{cases} y = 2x - 5 \\ y = \frac{1}{2}x + 4 \end{cases}$

d $\begin{cases} y = x - 1 \\ 2x + 3y = 12 \end{cases}$ **e** $\begin{cases} x + 3y = 9 \\ x - 2y = 4 \end{cases}$ **f** $\begin{cases} 3x - 2y = 30 \\ 4x + y = -4 \end{cases}$

2 Solve by substitution:

a $\begin{cases} y = x + 2 \\ 2x + 3y = 21 \end{cases}$ **b** $\begin{cases} y = 2x - 3 \\ 4x - 3y = 7 \end{cases}$ **c** $\begin{cases} 5x + 3y = 19 \\ y = 6 - 2x \end{cases}$

d $\begin{cases} x = y - 3 \\ 5x - 2y = 9 \end{cases}$ **e** $\begin{cases} 3x + 4y = -13 \\ x = 8y - 2 \end{cases}$ **f** $\begin{cases} x = -5y - 2 \\ 7x + 4y = -10 \end{cases}$

g $\begin{cases} y = \frac{1}{2}x + 5 \\ 3x + 4y = 5 \end{cases}$ **h** $\begin{cases} x = -\frac{3}{4}y \\ 4x - 5y = -24 \end{cases}$ **i** $\begin{cases} 3x + 7y = 6 \\ x = \frac{5}{3}y - 1 \end{cases}$

Example 9

Self Tutor

Solve by elimination: $\begin{cases} 3x + 4y = 2 \\ 2x - 3y = 7 \end{cases}$

$3x + 4y = 2 \quad .... \ (1)$
$2x - 3y = 7 \quad .... \ (2)$

To make the coefficients of y the same size but opposite in sign, we multiply (1) by 3 and (2) by 4.

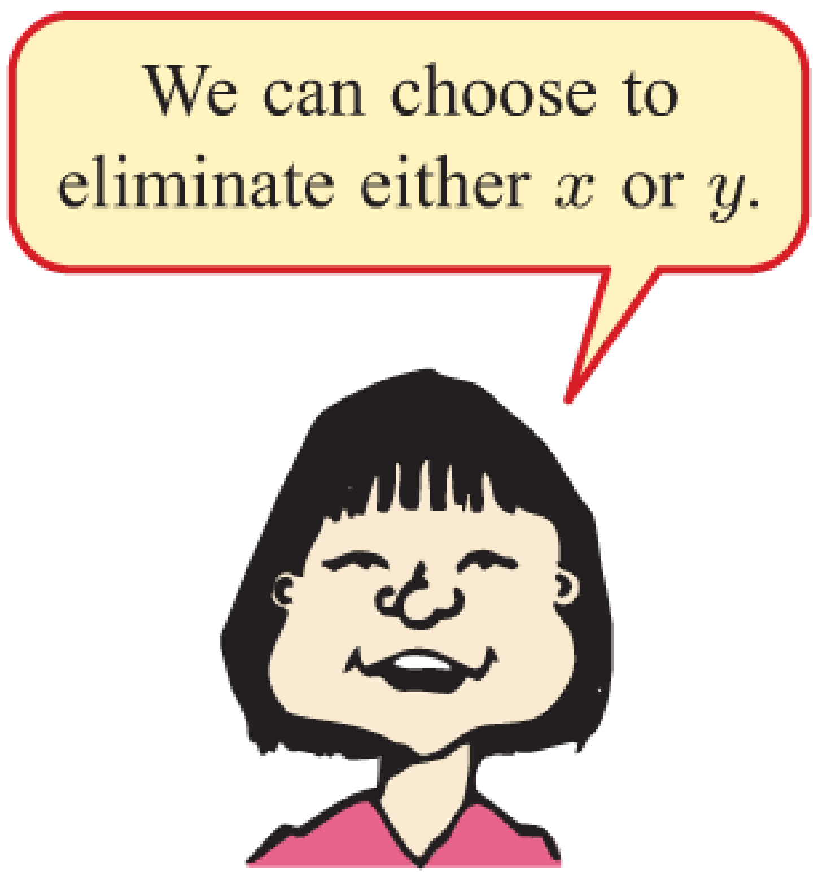
$$\begin{array}{lrl} \therefore & 9x + 12y = 6 & \{(1) \times 3\} \\ & 8x - 12y = 28 & \{(2) \times 4\} \\ \hline \text{Adding,} & 17x \qquad = 34 & \\ & \therefore \ x = 2 & \end{array}$$

Substituting $x = 2$ into (1) gives $3(2) + 4y = 2$

$$\therefore \ 6 + 4y = 2$$
$$\therefore \ 4y = -4$$
$$\therefore \ y = -1$$

The solution is $x = 2$, $y = -1$.

Check: In (2): $2(2) - 3(-1) = 4 + 3 = 7$ ✓



3 Solve by elimination:

a $\begin{cases} 3x - y = 5 \\ 4x + y = 9 \end{cases}$ **b** $\begin{cases} 5x - 2y = 17 \\ 3x + 2y = 7 \end{cases}$ **c** $\begin{cases} 3x + y = 16 \\ 7x - 2y = 7 \end{cases}$

d $\begin{cases} 3x - 7y = -27 \\ -6x + 5y = 18 \end{cases}$ **e** $\begin{cases} 3x - 7y = -8 \\ 9x + 11y = 16 \end{cases}$ **f** $\begin{cases} 4x + 3y = 14 \\ 3x - 4y = 23 \end{cases}$

g $\begin{cases} 2x - 3y = 6 \\ 5x - 4y = 1 \end{cases}$ **h** $\begin{cases} 4x + 2y = -23 \\ 5x - 7y = -5 \end{cases}$ **i** $\begin{cases} 4x - 7y = 9 \\ 5x - 8y = -2 \end{cases}$

4 Find the area of the triangle defined by:

- **a** $y = x + 2$, $x + y = 9$, and $y = 2$
- **b** $5x - 2y = 18$, $2x + 5y = 13$, and $8x - 9y = 11.4$

5 Consider the simultaneous equations $\begin{cases} y = 4x + 7 \\ 2y - 8x = 1 \end{cases}$.

- **a** Graph each line on the same set of axes. What do you notice?
- **b** Try to solve the simultaneous equations using:
  - **i** substitution
  - **ii** elimination
  - **iii** technology.
- **c** How many solutions does this system of simultaneous equations have?

6 Consider the simultaneous equations $\begin{cases} y = -2x + 5 \\ 4x + 2y = 10 \end{cases}$.

- **a** Graph each line on the same set of axes. What do you notice?
- **b** Try to solve the simultaneous equations using:
  - **i** substitution
  - **ii** elimination
  - **iii** technology.
- **c** How many solutions does this system of simultaneous equations have?

7 Consider the system of simultaneous equations $\begin{cases} 3x - 2y = 12 \\ y = mx - 6 \end{cases}$.

- **a** Find the gradient of each line.
- **b** Hence determine the value of m for which the simultaneous equations do *not* have a unique solution. Explain what is happening in this case.
- **c** For any other value of m, the system has a unique solution. Find this solution.

8 Consider the system of simultaneous equations $\begin{cases} 12 = 4x - cy \\ 2x + 6 = 3y \end{cases}$.

- **a** Find the gradient of each line.
- **b** Hence determine the value of c for which the simultaneous equations do *not* have a unique solution. Explain what is happening in this case.
- **c** For any other value of c, the system has a unique solution. Find this solution in terms of c.

REVIEW SET 1A

1 Consider the table of values alongside.

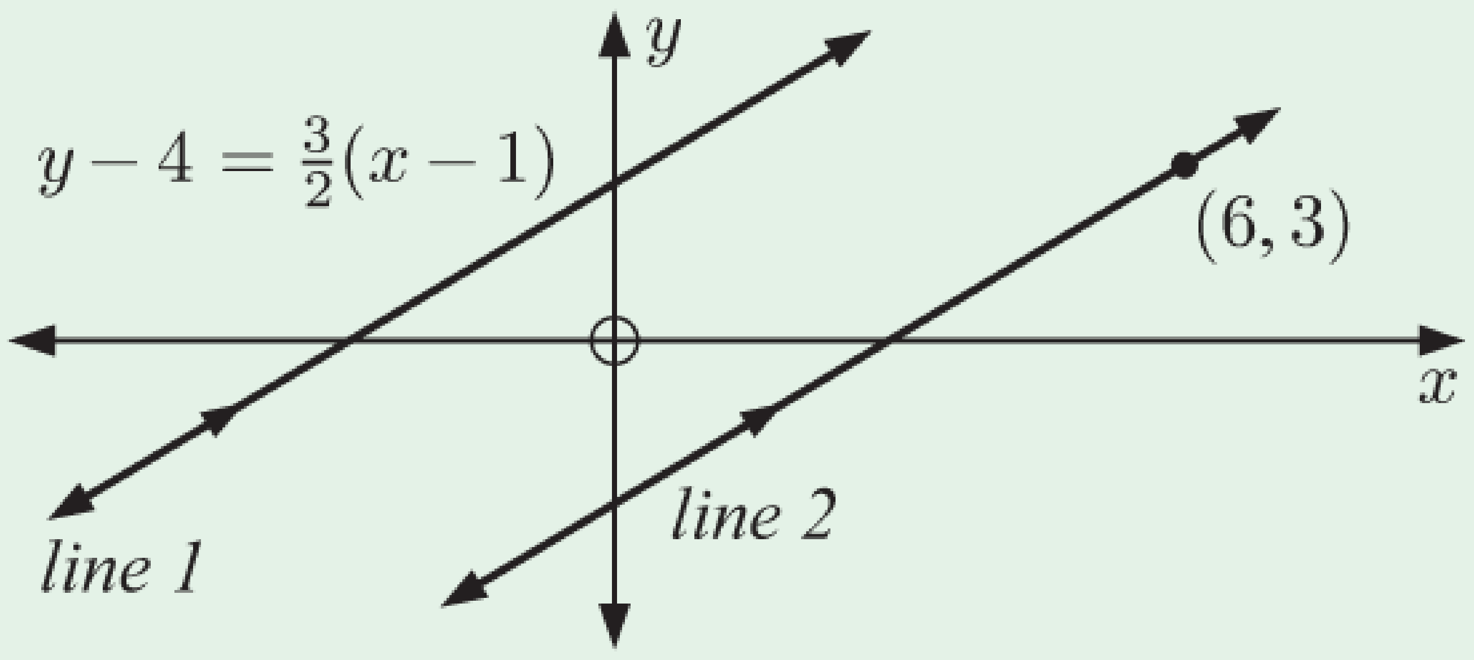
x	0	1	2	3	4
y	20	17	14	11	8

- **a** Draw a graph of y against x.
- **b** Are the variables linearly related? Explain your answer.
- **c** Find the gradient and y-intercept of the graph.
- **d** Find the equation connecting x and y.
- **e** Find the value of y when $x = 7$.

2
- **a** Find, in gradient-intercept form, the equation of the line which has gradient $-\frac{1}{3}$ and passes through $(6, 2)$.
- **b** Write the equation of the line in the form $ax + by + d = 0$.

3 **a** Find, in general form, the equation of *line 2*.

b Hence find the x-intercept of *line 2*.



4 Determine whether:

a $(5, -2)$ lies on the line with equation $y=-x+3$

b $(-3, \frac{1}{2})$ lies on the line with equation $3x+8y=-5$.

5 Draw the graph of:

a $y=-\frac{3}{4}x+1$ **b** $3x-4y=72$ **c** $2x+5y=-20$

6 Find the equation of the perpendicular bisector of:

a A(5, 2) and B(5, −4) **b** A(8, 1) and B(2, 5).

7 Quadrilateral ABCD has vertices A(3, 2), B(2, −4), C(−4, −3), and D(−3, 3).

a Find the equation of the perpendicular bisector of: **i** [AC] **ii** [BD].

b Classify quadrilateral ABCD.

8 Solve graphically: **a** $\begin{cases} y=3x+1 \\ x-y=3 \end{cases}$ **b** $\begin{cases} 2x+y=6 \\ x-2y=8 \end{cases}$

9 Solve by substitution: **a** $\begin{cases} y=3x+4 \\ 2x-y=-5 \end{cases}$ **b** $\begin{cases} x=2y-5 \\ 3x+4y=5 \end{cases}$

10 Solve by elimination: **a** $\begin{cases} 3x+2y=7 \\ 5x-2y=17 \end{cases}$ **b** $\begin{cases} 2x+7y=13 \\ -4x+3y=25 \end{cases}$

11 Consider the system of simultaneous equations $\begin{cases} x=k-2y \\ y=-\frac{1}{2}x+2 \end{cases}$.

a Find the gradient of each line.

b Find the value(s) of k such that the system has:

i no solutions **ii** infinitely many solutions.

Interpret these cases geometrically.

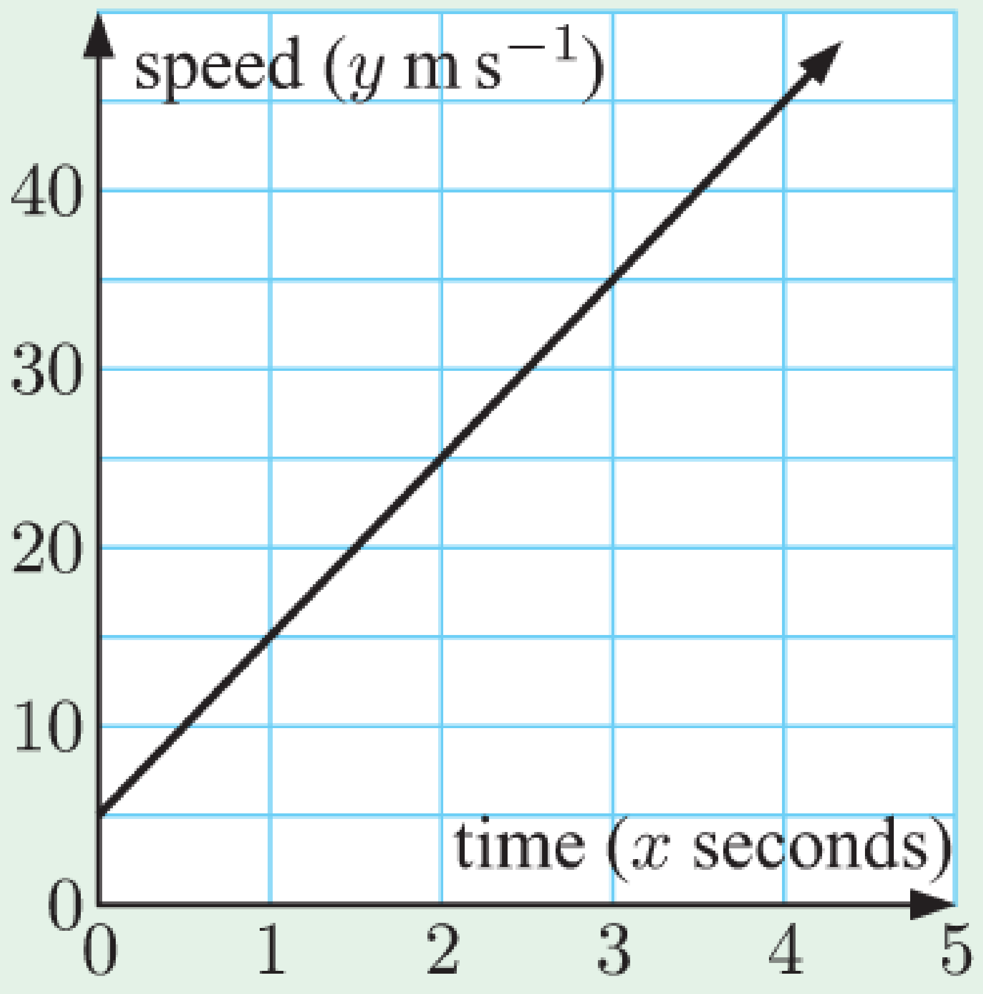
REVIEW SET 1B

1 The speed of a pebble thrown from the top of a cliff is shown alongside.

a Find the gradient and y-intercept of the line, and explain what these values mean.

b Find the equation of the line.

c Find the speed of the pebble after 8 seconds.



2 Find the equation of the line which is:

- **a** parallel to $y = 3x - 8$ and passes through $(2, 7)$
- **b** perpendicular to $2x + 5y = 7$ and passes through $(-1, -1)$.

3 Find k given that:

- **a** $(2, k)$ lies on the line with equation $y = 5x - 3$
- **b** $(\frac{1}{2}, -\frac{3}{2})$ lies on the line with equation $5x + 9y = k$.

4 Draw the graph of:

a $3x + 2y = 30$ **b** $y = -2x + 5$ **c** $y = -\frac{1}{3}x + \frac{4}{3}$

5 Consider the line with equation $y = \frac{2}{3}x - \frac{8}{3}$.

- **a** Find the gradient of the line.
- **b** Determine whether: **i** $(-2, -4)$ **ii** $(4, 5)$ lie on the line.
- **c** Draw the graph of the line, showing your results from **a** and **b**.

6 *Line 1* has equation $2x + 3y = -24$. *Line 2* is perpendicular to *line 1*, and meets *line 1* at R$(3, -10)$. *Line 1* and *line 2* meet the x-axis at P and Q respectively. Find the area of triangle PQR.

7 A line segment has equation $x - 5y + 6 = 0$. Its midpoint is $(4, 2)$.

- **a** State the gradient of:
  - **i** the line segment **ii** its perpendicular bisector.
- **b** State the equation of the perpendicular bisector.

8 Triangle ABC has vertices A$(3, 6)$, B$(-1, 4)$, and C$(1, 0)$.

- **a** Find the equation of the perpendicular bisector of:
  - **i** [AB] **ii** [AC] **iii** [BC].
- **b** Graph the three perpendicular bisectors on the same set of axes. Discuss your observations.

9 Solve by substitution: **a** $\begin{cases} y = 6x + 2 \\ 3x - 2y = -7 \end{cases}$ **b** $\begin{cases} y = \frac{1}{2}x + 5 \\ 4x + 3y = 4 \end{cases}$

10 Solve by elimination: **a** $\begin{cases} 3x + 2y = 8 \\ 5x - 4y = 17 \end{cases}$ **b** $\begin{cases} 4x + 6y = -15 \\ 3x - 5y = 22 \end{cases}$

11 Consider the system of simultaneous equations $\begin{cases} ax + 4y = 6 \\ x - 2y = -2 \end{cases}$.

- **a** Find the gradient of each line.
- **b** Hence determine the value of a for which there is *not* a unique solution. Explain what is happening in this case.
- **c** For any other value of a, the system has a unique solution. Find this solution in terms of a.

Chapter 2

Sets and Venn diagrams

Contents:

OPENING PROBLEM

The Human Development Index (HDI) is a composite statistic of life expectancy, education, and per capita income indicators used to rank nations on their levels of human development.

Suppose L represents the nations which have a life expectancy of more than 75 years, S represents the nations where the mean years of schooling is greater than 10, and I represents the nations where the gross national income (GNI) is more than \$18 000 USD per capita.

In 2016, of the 100 nations with the highest HDI:

- 69 were in L
- 50 were in L and I
- 37 were in L, S, and I
- 58 were in S
- 44 were in S and I
- 61 were in I
- 43 were in L and S

Things to think about:

a How can we display this information on a diagram?

b Can we find how many nations out of the 100 with the highest HDI were not in any of L, S, or I?

c How many nations were in:

i S only **ii** L or I but not S **iii** S and I but not L?

d Do you think that life expectancy, mean years of schooling, and gross national income are linked in any way? You may wish to visit the link alongside to consider data for nations ranked outside of the top 100 for HDI.

LINK

A SETS

A **set** is a collection of distinct numbers or objects.

Each object is called an **element** or **member** of the set.

When we record a set, we write its members within curly brackets, with commas between them.

We often use a capital letter to represent a set so that we can refer to it easily.

SET NOTATION

$\in$ means "is an element of" or "is in"

$\notin$ means "is not an element of" or "is not in"

$n(A)$ means "the number of elements in set A".

For example, if $P = \{\text{prime numbers less than } 20\} = \{2, 3, 5, 7, 11, 13, 17, 19\}$ then $11 \in P$, $15 \notin P$, and $n(P) = 8$.

SET DEFINITIONS

- Two sets are **equal** if they contain exactly the same elements.
- Set A is a **subset** of set B if every element of A is also an element of B. We write $A \subseteq B$.
  $A \subseteq B$ can be interpreted as: "For all $x \in A$, we know that $x \in B$".
- A is a **proper subset** of B if every element of A is also an element of B, but $A \neq B$.
  We write $A \subset B$.
- The **empty set** $\varnothing$ or $\{\ \}$ is a set which contains no elements.
- Set A is a **finite set** if $n(A)$ has a particular defined value.
  If A has an endless number of elements, we say it is an **infinite set**.

Example 1 **Self Tutor**

Let P be the set of letters in the word AMATEUR and Q be the set of letters in the word TEAM.

a List the elements of P and Q.

b Find $n(P)$ and $n(Q)$.

c Decide whether the statement is true or false:

i $\text{U} \in Q$ **ii** $\text{R} \notin Q$ **iii** $P \subseteq Q$ **iv** $Q \subset P$

a $P = \{\text{A, M, T, E, U, R}\}$, $Q = \{\text{T, E, A, M}\}$

b $n(P) = 6$, $n(Q) = 4$

c **i** U is not in the word TEAM, so $\text{U} \in Q$ is false.

ii R is not in the word TEAM, so $\text{R} \notin Q$ is true.

iii $\text{U} \in P$ but $\text{U} \notin Q$, so $P \subseteq Q$ is false.

iv Every element in Q is also an element of P, so $Q \subseteq P$.
However, $\text{U} \in P$ but $\text{U} \notin Q$, so $Q \neq P$.
$\therefore$ $Q \subset P$ is true.

We do not include duplicates when listing elements of a set. So, A appears only once in P.

EXERCISE 2A

1 For each set A, list the elements of A and hence state $n(A)$:

a {factors of 8} **b** {composite numbers less than 20}

c {letters in the word AARDVARK} **d** {prime numbers between 40 and 50}

2 Decide whether each set is finite or infinite:

a {factors of 10} **b** {multiples of 10} **c** {perfect squares}

3 Let $S = \{1, 2, 4, 5, 9, 12\}$ and $T = \{2, 5, 9\}$.

a Find: **i** $n(S)$ **ii** $n(T)$.

b Decide whether the statement is true or false:

i $4 \in S$ **ii** $4 \in T$ **iii** $1 \notin T$ **iv** $T \subseteq S$ **v** $T \subset S$

4 Let $S = \{1, 2\}$ and $T = \{1, 2, 3\}$.

a List *all* the subsets of S and T. **b** Is every subset of S a subset of T?

c What fraction of the subsets of T are also subsets of S?

5 State the number of subsets of $\{p, q, r, s\}$. Explain your answer.

6 Consider two sets A and B such that $A \subseteq B$. Find the possible values of x if $A = \{2, 4, 6, x\}$ and $B = \{2, 3, 5, 6, x + 1\}$.

7 Suppose $A \subseteq B$ and $B \subseteq A$. Show that $A = B$.

B INTERSECTION AND UNION

INTERSECTION

The **intersection** of two sets A and B is the set of elements that are in **both** set A **and** set B.

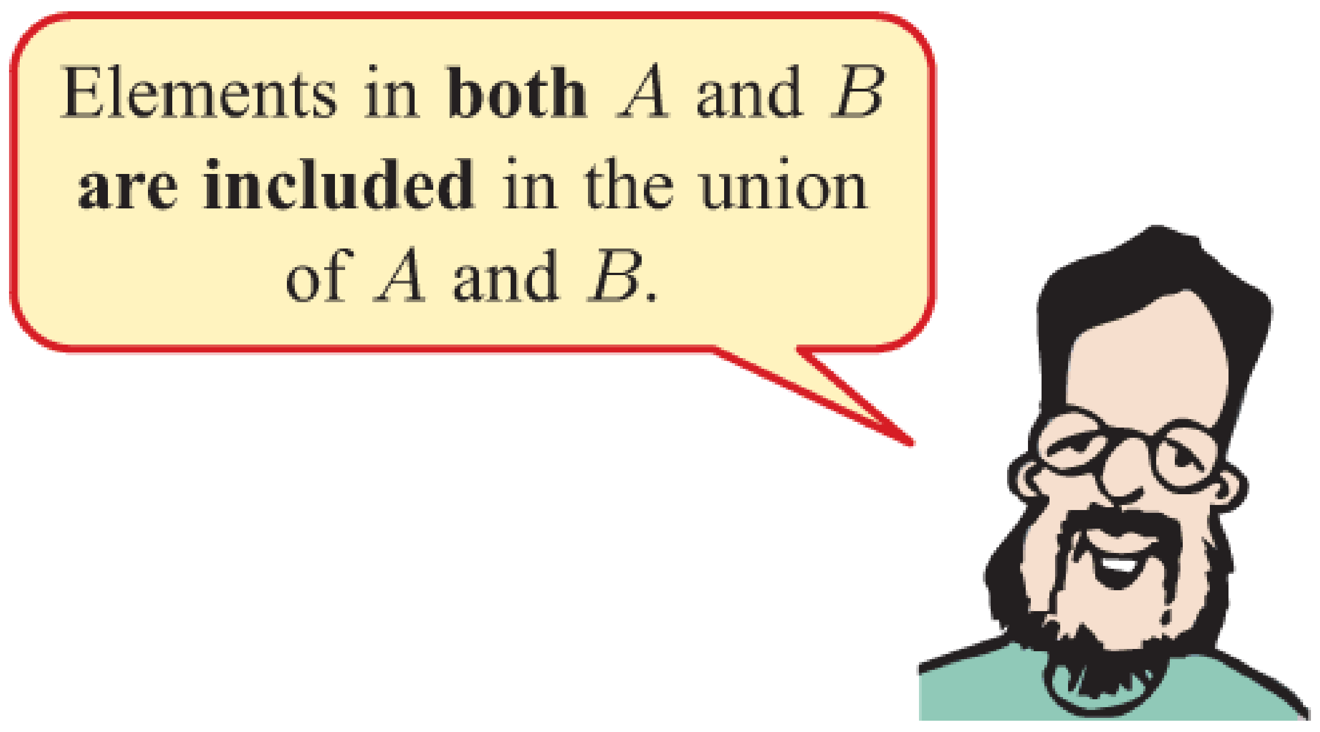
The intersection of sets A and B is written $\boldsymbol{A \cap B}$.

Two sets A and B are **disjoint** or **mutually exclusive** if they have no elements in common. In this case $A \cap B = \varnothing$.

UNION

The **union** of two sets A and B is the set of elements that are in **either** set A **or** set B.

The union of sets A and B is written $\boldsymbol{A \cup B}$.



Example 2 Self Tutor

$M = \{2, 3, 5, 7, 8, 9\}$ and $N = \{3, 4, 6, 9, 10\}$.

List the sets: **a** $M \cap N$ **b** $M \cup N$.

a $M \cap N = \{3, 9\}$ since 3 and 9 are elements of both sets.

b Every element which is in either M or N is in the union of M and N.

$\therefore \quad M \cup N = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$

EXERCISE 2B

1 Find **i** $A \cap B$ **ii** $A \cup B$ for:

- **a** $A = \{6, 7, 9, 11, 12\}$ and $B = \{5, 8, 10, 13, 9\}$
- **b** $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8\}$
- **c** $A = \{1, 3, 5, 7\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- **d** $A = \{0, 3, 5, 8, 14\}$ and $B = \{1, 4, 5, 8, 11, 13\}$

2 State whether the sets are disjoint:

- **a** $A = \{3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8\}$
- **b** $P = \{3, 5, 6, 7, 8, 10\}$ and $Q = \{4, 9, 10\}$

3 Determine whether each statement is true or false. Explain your answers.

- **a** If R and S are two non-empty sets and $R \cap S = \varnothing$ then R and S are disjoint.
- **b** For any sets A and B, $n(A \cap B) \leqslant n(A)$ and $n(A \cap B) \leqslant n(B)$.
- **c** If $A \cap B = A \cup B$ then $A = B$.
- **d** If A and B are infinite sets, then $A \cap B$ is also an infinite set.

4 Suppose A and B are disjoint sets, and B and C are disjoint sets. Are A and C disjoint sets? Explain your answer.

5 Suppose $n(A) = 8$ and $n(B) = 11$. Find the possible values of:

- **a** $n(A \cap B)$
- **b** $n(A \cup B)$

6 Show that $n(A \cup B) \leqslant n(A) + n(B)$.

C COMPLEMENT OF A SET

The **universal set** U is the set of all elements we are considering.

For example, if we are considering the digits we write whole numbers with, the universal set is $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

From this universal set we can define subsets of U, such as $C = \{\text{composites}\} = \{4, 6, 8, 9\}$ and $P = \{\text{primes}\} = \{2, 3, 5, 7\}$.

The **complement** of a set A is the set of all elements of U that are *not* elements of A.
The complement of A is written A'.

For example, if $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $A = \{1, 3, 5, 7, 8\}$, then $A' = \{2, 4, 6\}$.

We can make three immediate observations about complementary sets:

For any set A with complement A':

- $A \cap A' = \varnothing$ as A' and A have no common members.
- $A \cup A' = U$ as all elements of A and A' combined make up U.
- $n(A) + n(A') = n(U)$ provided U is finite.

EXERCISE 2C

1 Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

- **a** Find the complement of $A = \{2, 3, 6, 7, 8\}$.
- **b** If $P = \{\text{prime numbers}\}$, is $P' = \{\text{composite numbers}\}$? Explain your answer.

2 Let $U = \{\text{whole numbers between 10 and 20 inclusive}\}$, $A = \{\text{factors of 120}\}$, and $B = \{\text{multiples of 3}\}$.
List the elements of:

- **a** A
- **b** B
- **c** A'
- **d** B'
- **e** $A \cap B$
- **f** $A \cup B$
- **g** $A' \cap B$
- **h** $A' \cup B$
- **i** $A \cap B'$
- **j** $A \cup B'$
- **k** $A' \cap B'$
- **l** $A' \cup B'$

3 **a** Suppose $U = \{2, 3, 4, 5, 6, 7, 8\}$, $A = \{3, 5, 7\}$, and $B = \{2, 4, 7, 8\}$. Find:

  i $n(U)$ **ii** $n(A)$ **iii** $n(A')$ **iv** $n(B)$ **v** $n(B')$.

  b Copy and complete: For any set S within a universal set U, $n(S) + n(S') = ...$

4 Suppose P and Q' are subsets of U. $n(U) = 15$, $n(P) = 6$, and $n(Q') = 4$. Find:

  a $n(P')$ **b** $n(Q)$.

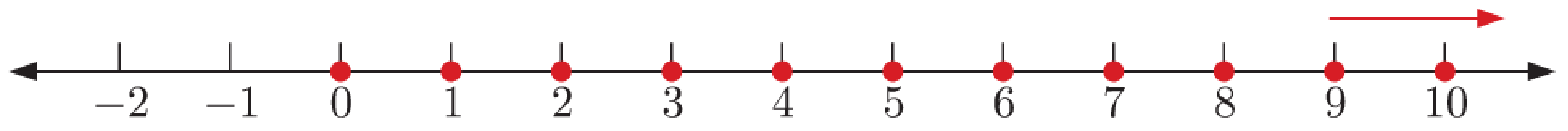
5 Given that $P \subseteq Q$, show that $Q' \subseteq P'$.

6 Suppose U and P are infinite sets. Provide examples to show that P' may be finite or infinite.

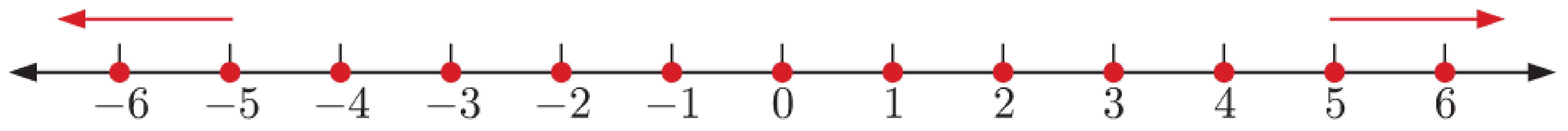
D SPECIAL NUMBER SETS

Following is a list of some special number sets you should be familiar with. They are all endless, so they are infinite sets.

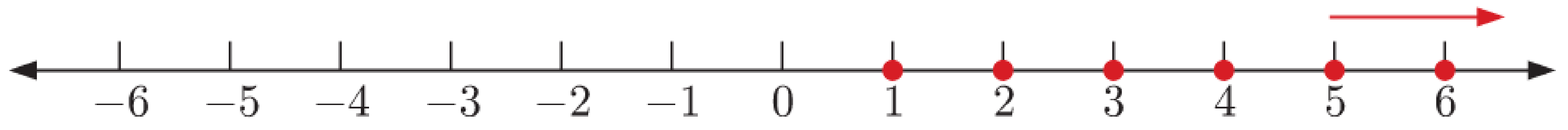
- $\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, ...\}$ is the set of all **natural** or **counting numbers**.



- $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \pm 4, ...\}$ is the set of all **integers**.



- $\mathbb{Z}^+ = \{1, 2, 3, 4, 5, 6, 7, ...\}$ is the set of all **positive integers**.



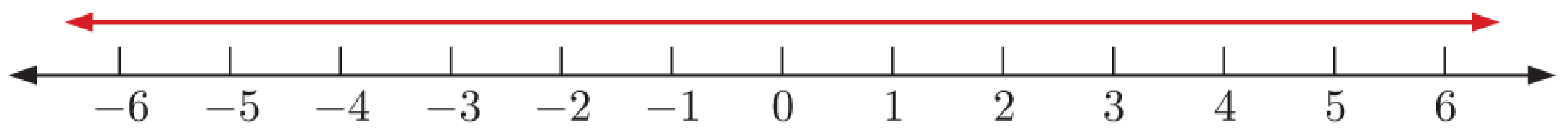
- $\mathbb{Q}$ is the set of all **rational numbers**, or numbers which can be written in the form $\frac{p}{q}$ where p and q are integers, $q \neq 0$.

  For example:
  - $\frac{3}{8}$ and $\frac{-7}{5}$ are rational
  - $-3\frac{2}{3}$ is rational as $-3\frac{2}{3} = \frac{-11}{3}$
  - $0.\overline{3}$ is rational as $0.\overline{3} = \frac{3}{9} = \frac{1}{3}$
  - $\sqrt{16}$ is rational as $\sqrt{16} = \frac{4}{1}$
  - All decimal numbers that terminate or recur are rational numbers.

- $\mathbb{Q}'$ is the set of all **irrational numbers**, or numbers which cannot be written in rational form.

  For example: $\sqrt{3}$ and π are irrational.

- $\mathbb{R}$ is the set of all **real numbers**, which are all numbers which can be placed on the number line.



Notice that $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}'$.

$\frac{1}{0}$ and $\sqrt{-2}$ cannot be placed on a number line, and so are not real.

EXERCISE 2D

1 Copy and complete:

Number	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
6	✓	✓	✓	✓
$-\frac{3}{8}$				
1.8				
$1.\overline{8}$				
-17				
$\sqrt{64}$				
$\frac{\pi}{2}$				
$\sqrt{-3}$				
$-\sqrt{3}$				

2 Determine whether each statement is true or false:

a $-7 \in \mathbb{Z}^+$ **b** $\frac{2}{3} \notin \mathbb{Z}$ **c** $\sqrt{3} \in \mathbb{Q}$ **d** $\frac{7}{9} \in \mathbb{Q}$

e $0.201 \in \mathbb{Z}$ **f** $\dfrac{7}{0.31} \in \mathbb{Q}$ **g** $\sqrt{|-1|} \in \mathbb{R}$ **h** $\sqrt{-9} \in \mathbb{R}$

3 Determine whether each statement is true or false:

a $\mathbb{Z}^+ \subseteq \mathbb{N}$ **b** $\mathbb{N} \subset \mathbb{Z}$ **c** $\mathbb{N} = \mathbb{Z}^+$ **d** $\mathbb{Z}^- \subseteq \mathbb{Z}$

e $\mathbb{Q} \subset \mathbb{Z}$ **f** $\{0\} \subseteq \mathbb{Z}$ **g** $\mathbb{Z} \subseteq \mathbb{Q}$ **h** $\mathbb{Z}^+ \cup \mathbb{Z}^- = \mathbb{Z}$

4 Describe the following sets as either finite or infinite:

a the set of integers between 10 and 20

b the set of integers greater than 5

c the set of all rational numbers between 0 and 1

d the set of all irrational numbers between 0 and 1.

5 If $U = \mathbb{Z}$, find the complement of $\mathbb{Z}^+$.

THEORY OF KNOWLEDGE

A number is *rational* if and only if its decimal expansion eventually terminates or recurs.

Equivalently, a number is *irrational* if and only if its decimal expansion never terminates nor recurs.

If we begin to write the decimal expansion of $\sqrt{2}$, there is no indication that it will terminate or recur, and we might therefore suspect that $\sqrt{2}$ is irrational.

$$1.414\,213\,562\,373\,095\,048\,801\,688\,724\,209\,698\,078\,569\,671\,875\,376\,948\,073\,....$$

However, we cannot *prove* that $\sqrt{2}$ is irrational by writing out its decimal expansion, as we would have to write an infinite number of decimal places. We might therefore *believe* that $\sqrt{2}$ is irrational, but it may also seem impossible to *prove* it.

1 If something has not yet been proven, does that make it untrue?

2 Is the state of an idea being true or false dependent on our ability to prove it?

In fact, we can prove that $\sqrt{2}$ is irrational using a method called **proof by contradiction**. In this method we suppose that the opposite of what we want to show is true, and follow a series of logical steps until we arrive at a contradiction. The contradiction confirms that our original supposition must be false. Proof by contradiction is an **indirect** proof.

Proof: Suppose $\sqrt{2}$ is rational, so $\sqrt{2} = \frac{p}{q}$ for some (positive) integers p and q, $q \neq 0$.

We assume this fraction has been written in lowest terms, so p and q have no common factors.

Squaring both sides, $2 = \frac{p^2}{q^2}$ and so $p^2 = 2q^2$ .... (1)

$\therefore$ p^2 is even, and so p must be even.

$\therefore$ $p = 2k$ for some $k \in \mathbb{Z}^+$.

Substituting into (1), $4k^2 = 2q^2$

$\therefore$ $q^2 = 2k^2$

$\therefore$ q^2 is even, and so q must be even.

Here we have a contradiction, as p and q have no common factors.

$\therefore$ our original supposition is false, and $\sqrt{2}$ is irrational.

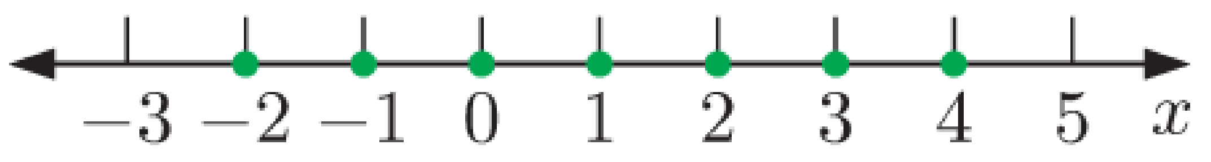
3 Is proof by contradiction unique to mathematics, or do we use it elsewhere?

4 Is it useful to be able to prove something in more than one way?

5 In what other fields of human endeavour is it necessary to establish truth?

E INTERVAL NOTATION

To describe the set of all integers between -3 and 5, we can list the set as $\{-2, -1, 0, 1, 2, 3, 4\}$ or illustrate the set as points on a number line.



Alternatively, we can write the set using **interval notation** as $\{x \in \mathbb{Z} \mid -3 < x < 5\}$.

the set of all such that

We read this as "the set of all integers x such that x lies between -3 and 5".

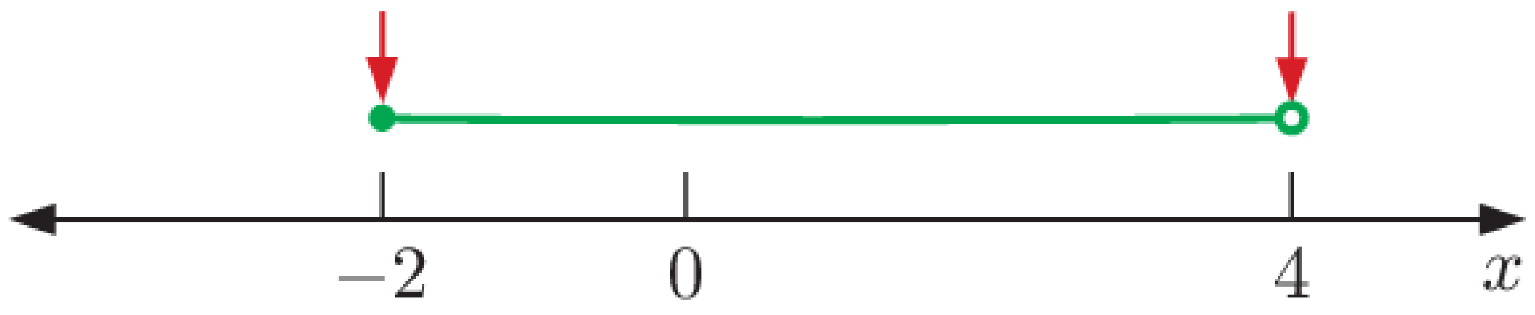
Interval notation is very useful if the set contains a large or infinite number of elements and listing them would be time consuming or impossible.

For example: $\{x \in \mathbb{R} \mid -2 \leqslant x < 4\}$ reads "the set of all real x such that x is greater than or equal to -2 and less than 4".

We represent this set on a number line as:

a filled in circle indicates -2 is included

an open circle indicates 4 is not included



We commonly write $\{x \mid -2 \leqslant x < 4\}$ in which case we *assume* that $x \in \mathbb{R}$.

Example 3 **Self Tutor**

Suppose $A = \{x \in \mathbb{Z} \mid 3 < x \leqslant 10\}$.

a Write down the meaning of the interval notation.

b List the elements of set A.

c Find $n(A)$.

a The set of all x such that x is an integer between 3 and 10, including 10.

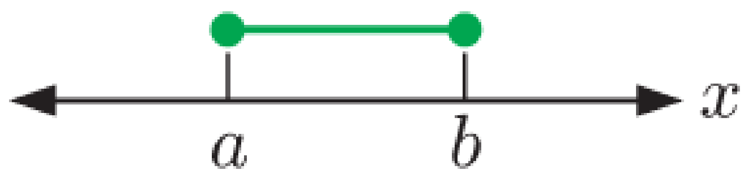
b $A = \{4, 5, 6, 7, 8, 9, 10\}$

c $n(A) = 7$

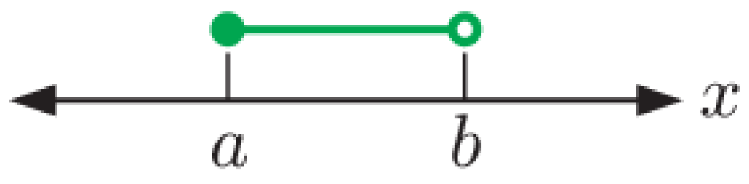
BRACKET NOTATION

For $x \in \mathbb{R}$, we can use the following notation to concisely write intervals:

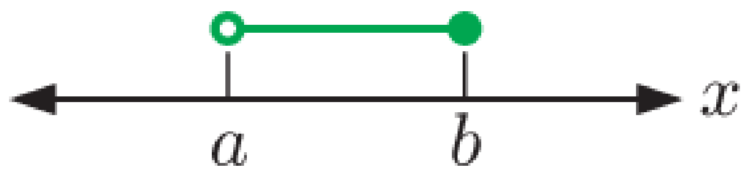
$[a, b]$ represents the closed interval $\{x \mid a \leqslant x \leqslant b\}$



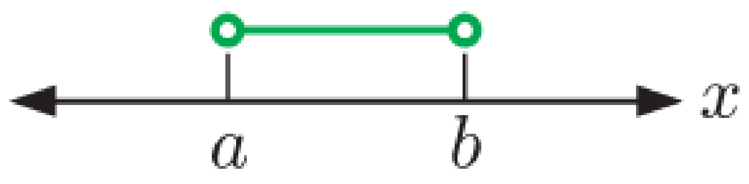
$[a, b[$ represents the interval $\{x \mid a \leqslant x < b\}$



$]a, b]$ represents the interval $\{x \mid a < x \leqslant b\}$



$]a, b[$ represents the open interval $\{x \mid a < x < b\}$



An interval which extends to infinity has no defined endpoint. So, for $\{x \mid x \geqslant a\}$ we write $[a, \infty[$.

EXERCISE 2E

1 For the following sets:

i Write down the meaning of the interval notation.

ii If possible, list the elements of A.

iii Find $n(A)$.

a $A = \{x \in \mathbb{Z} \mid -1 \leqslant x \leqslant 7\}$

b $A = \{x \in \mathbb{N} \mid -2 < x < 8\}$

c $A = \{x \mid 0 \leqslant x \leqslant 1\}$

d $A = \{x \in \mathbb{Q} \mid 5 \leqslant x \leqslant 6\}$

2 Represent on a number line:

a $\{x \in \mathbb{N} \mid x < 5\}$

b $\{x \in \mathbb{Z}^+ \mid 3 < x \leqslant 7\}$

c $\{x \in \mathbb{R} \mid x \geqslant 3\}$

d $\{x \in \mathbb{R} \mid x < 6\}$

e $\{x \in \mathbb{R} \mid 2 \leqslant x < 6\}$

f $\{x \mid 3.6 \leqslant x \leqslant 10.2\}$

g $x \in [5, 9]$

h $x \in]-1, 6]$

i $x \in [2, \infty[$

j $x \in]-\infty, -3[$

k $\{x \mid x < 3\} \cup \{x \mid x > 6\}$

l $\{x \mid x \leqslant 2\} \cup \{x \mid x > 4\}$

m $\{x \mid x < 3\} \cup \{x \mid 7 < x < 12\}$

n $\{x \in \mathbb{Z}^+ \mid x \leqslant 6\} \cup \{x \in \mathbb{Z}^+ \mid 8 \leqslant x \leqslant 11\}$

o $x \in]-\infty, 0] \cup x \in [4, \infty[$

p $x \in]-\infty, 1[\cup x \in]5, 8]$

3 Write in interval notation:

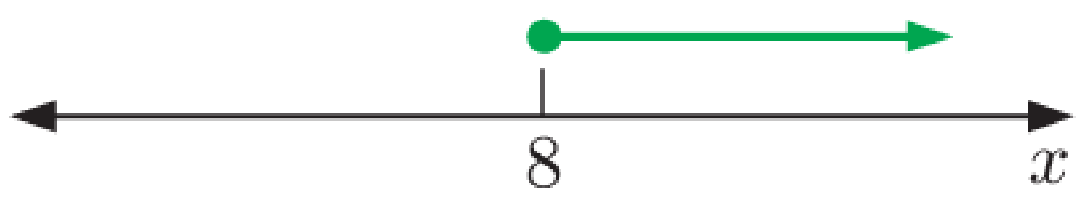
a the set of all integers between -100 and 100

b the set of all real numbers greater than 1000

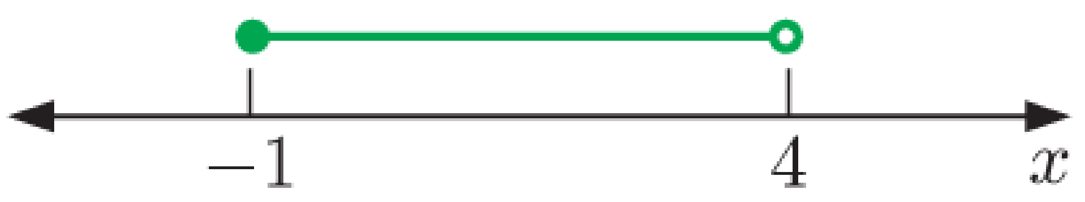
c the set of all rational numbers between 2 and 3 inclusive.

4 Write in interval notation:

a



b



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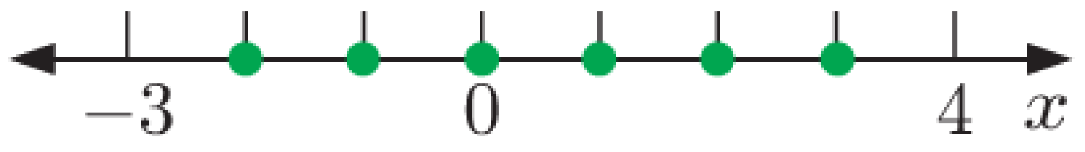

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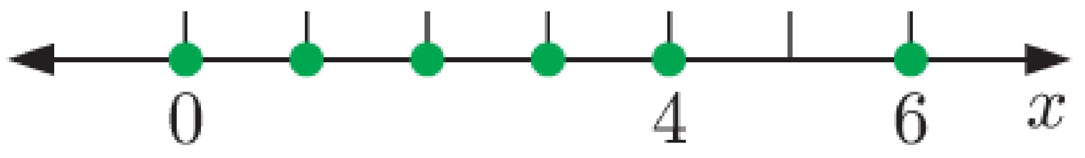
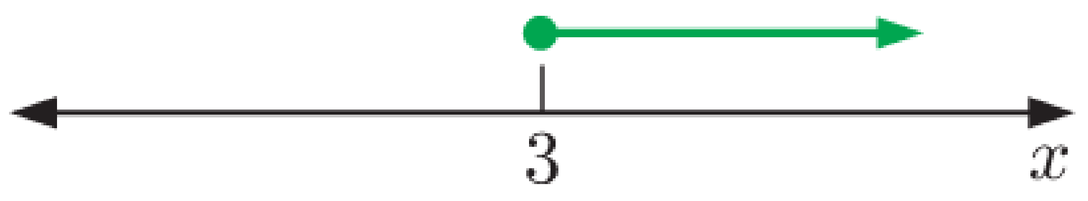


5 Write in bracket notation:

a

b



c

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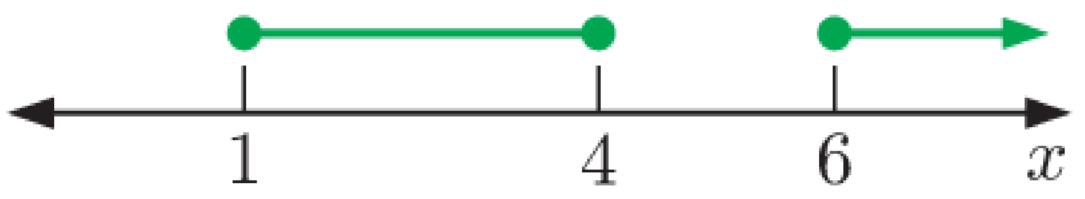


6 State whether $A \subseteq B$:

a $A = \varnothing,\ B = \{2, 5, 7, 9\}$
b $A = \{2, 5, 8, 9\},\ B = \{8, 9\}$
c $A = \{x \in \mathbb{R} \mid 2 \leqslant x \leqslant 3\},\ B = \{x \in \mathbb{R}\}$
d $A = \{x \in \mathbb{Q} \mid 3 \leqslant x \leqslant 9\},\ B = \{x \in \mathbb{R} \mid 0 \leqslant x \leqslant 10\}$
e $A = \{x \in \mathbb{Z} \mid -10 \leqslant x \leqslant 10\},\ B = \{z \in \mathbb{Z} \mid 0 \leqslant z \leqslant 5\}$
f $A = \{x \in \mathbb{Q} \mid 0 \leqslant x \leqslant 1\},\ B = \{y \in \mathbb{Q} \mid 0 < y \leqslant 2\}$

Example 4 **Self Tutor**

Suppose $U = \{x \in \mathbb{Z} \mid -5 \leqslant x \leqslant 5\}$, $A = \{x \in \mathbb{Z} \mid 1 \leqslant x \leqslant 4\}$, and $B = \{x \in \mathbb{Z} \mid -3 \leqslant x < 2\}$. List the elements of:

a A **b** B **c** A' **d** B'
e $A \cap B$ **f** $A \cup B$ **g** $A' \cap B$ **h** $A' \cup B'$

a $A = \{1, 2, 3, 4\}$
b $B = \{-3, -2, -1, 0, 1\}$
c $A' = \{-5, -4, -3, -2, -1, 0, 5\}$
d $B' = \{-5, -4, 2, 3, 4, 5\}$
e $A \cap B = \{1\}$
f $A \cup B = \{-3, -2, -1, 0, 1, 2, 3, 4\}$
g $A' \cap B = \{-3, -2, -1, 0\}$
h $A' \cup B' = \{-5, -4, -3, -2, -1, 0, 2, 3, 4, 5\}$

7 Suppose $U = \{x \in \mathbb{Z} \mid 0 \leqslant x \leqslant 8\}$, $A = \{x \in \mathbb{Z} \mid 2 \leqslant x \leqslant 7\}$, and $B = \{x \in \mathbb{Z} \mid 5 \leqslant x \leqslant 8\}$. List the elements of:

a A **b** A' **c** B **d** B'
e $A \cap B$ **f** $A \cup B$ **g** $A \cap B'$

8 Suppose $U = \{x \in \mathbb{Z} \mid 0 \leqslant x \leqslant 40\}$, $P = \{\text{factors of } 28\}$, and $Q = \{\text{factors of } 40\}$.

a List P and Q. **b** Find $P \cap Q$. **c** Find $P \cup Q$.
d Show that $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$.

9 Suppose $U = \mathbb{Z}$, $C = \{y \in \mathbb{Z} \mid -4 \leqslant y \leqslant -1\}$, and $D = \{y \in \mathbb{Z} \mid -7 \leqslant y < 0\}$.

a List C and D. **b** Find $C \cap D$. **c** Find $C \cup D$.
d Show that $n(C \cup D) = n(C) + n(D) - n(C \cap D)$.

10 Suppose $U = \{x \in \mathbb{Z}^+ \mid x < 31\}$, $A = \{\text{multiples of 6 less than 31}\}$, $B = \{\text{factors of 30}\}$, and $C = \{\text{primes} < 30\}$.

a List the sets A, B, and C.

b Find:

i $A \cap B$
ii $B \cap C$
iii $A \cap C$
iv $A \cap B \cap C$
v $A \cup B \cup C$

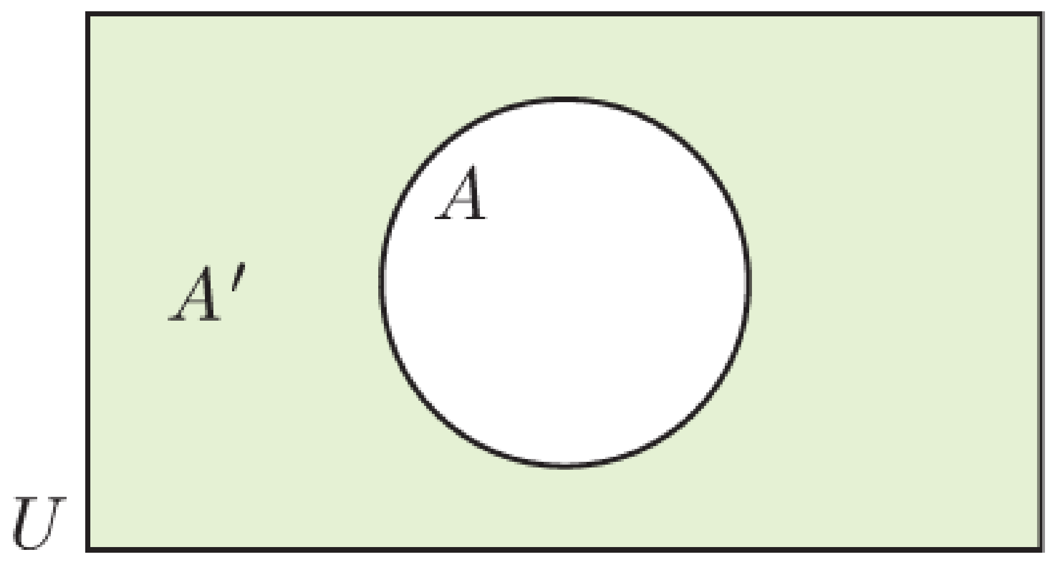
c Show that $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$.

F VENN DIAGRAMS

A **Venn diagram** consists of a universal set U represented by a rectangle, and subsets within it that are generally represented by circles.

The Venn diagram alongside shows set A within the universal set U.

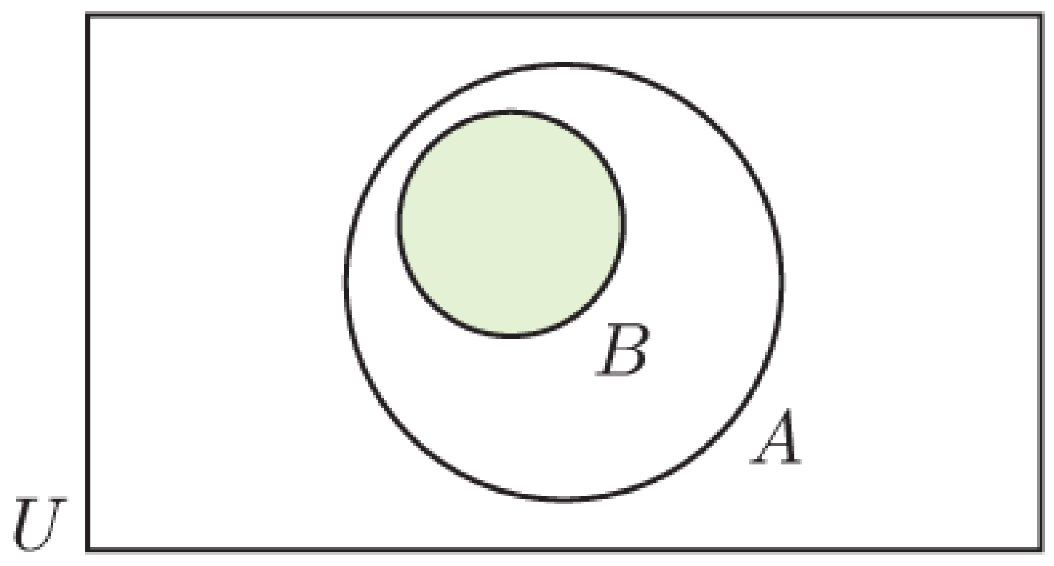
The **complement** of A is the shaded region outside the circle.



SUBSETS

If $B \subseteq A$ then every element of B is also in A.

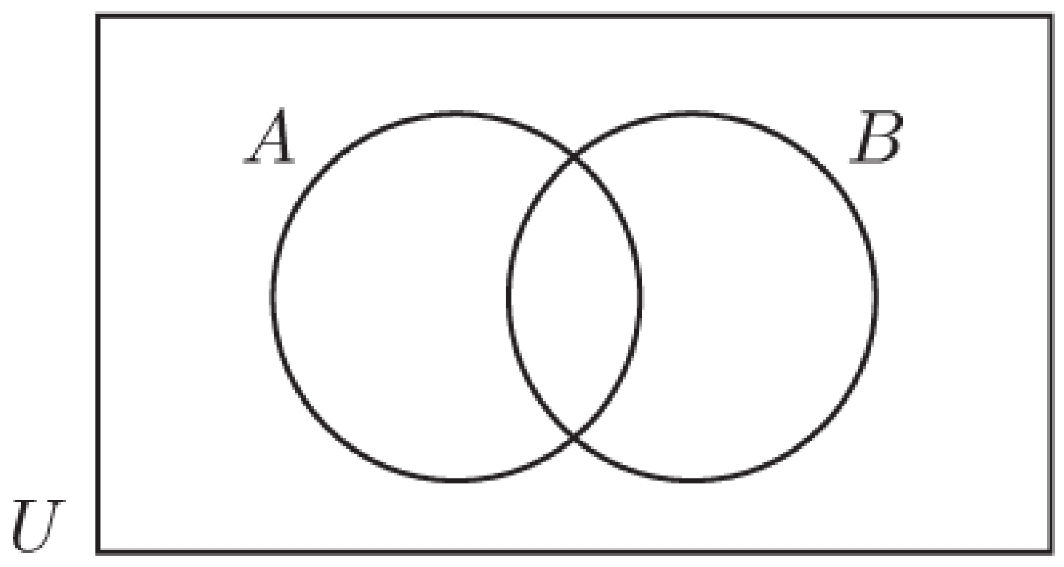
The circle representing B is placed within the circle representing A.



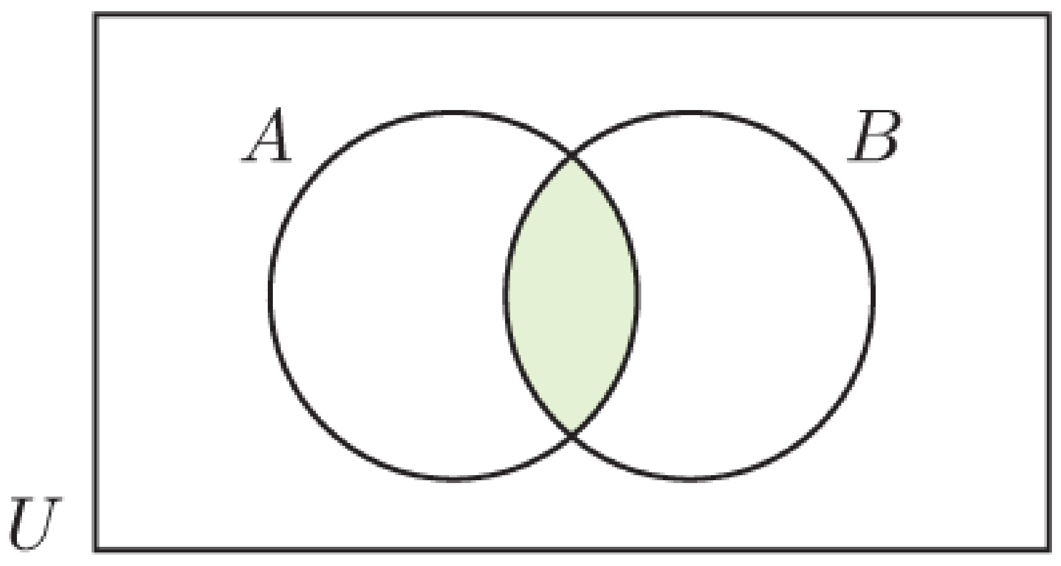
$B \subseteq A$

INTERSECTING SETS

For two sets A and B which have some elements in common, but where neither is a subset of the other, we draw the circles overlapping.

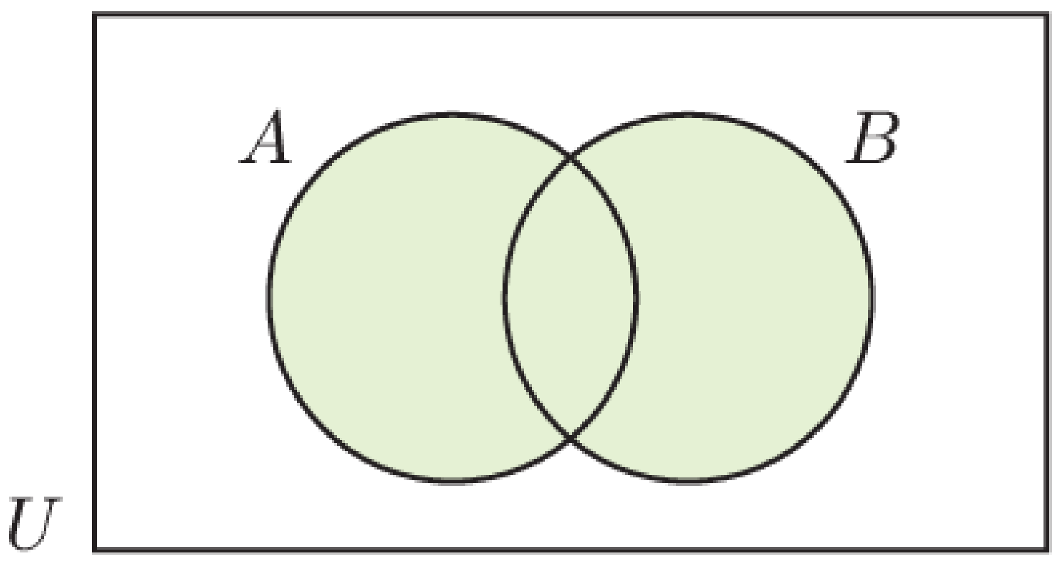


- The **intersection** $A \cap B$ consists of all elements common to both A and B.
  It is the region where the circles overlap.



$A \cap B$

- The **union** $A \cup B$ consists of all elements in A or B or both.
  It is the region which includes the two circles.

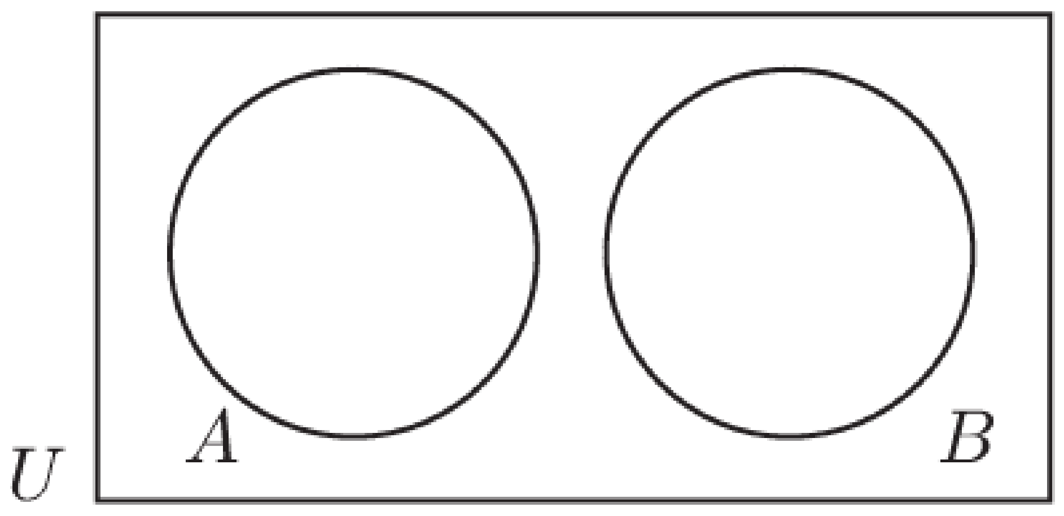


$A \cup B$

DISJOINT OR MUTUALLY EXCLUSIVE SETS

Disjoint sets do not have common elements.

They are represented by non-overlapping circles.



$A \cap B = \varnothing$

Example 5

Self Tutor

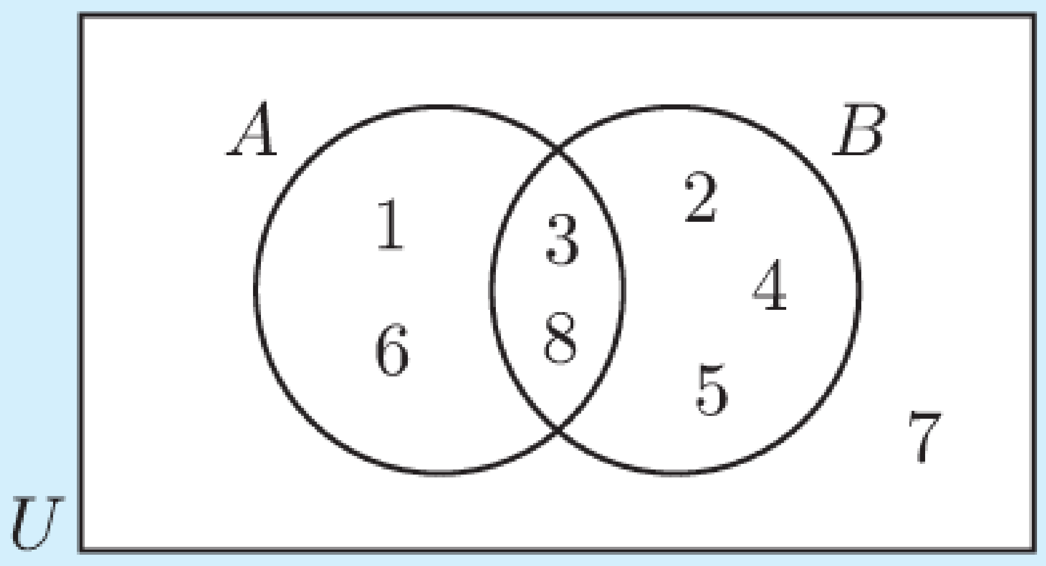
Suppose $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Illustrate on a Venn diagram the sets:

a $A = \{1, 3, 6, 8\}$ and $B = \{2, 3, 4, 5, 8\}$

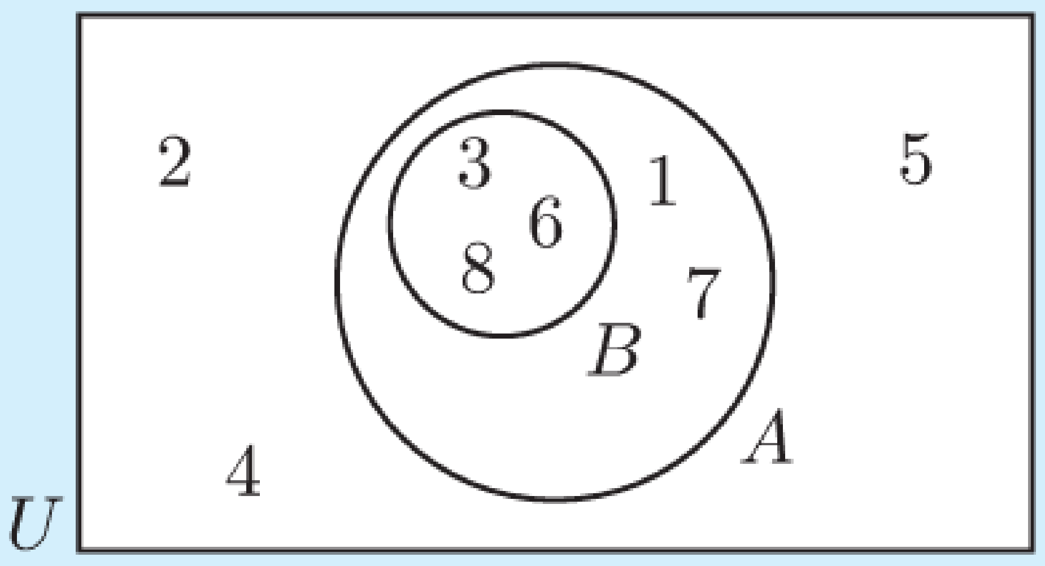
b $A = \{1, 3, 6, 7, 8\}$ and $B = \{3, 6, 8\}$

c $A = \{2, 4, 8\}$ and $B = \{1, 3, 5\}$.

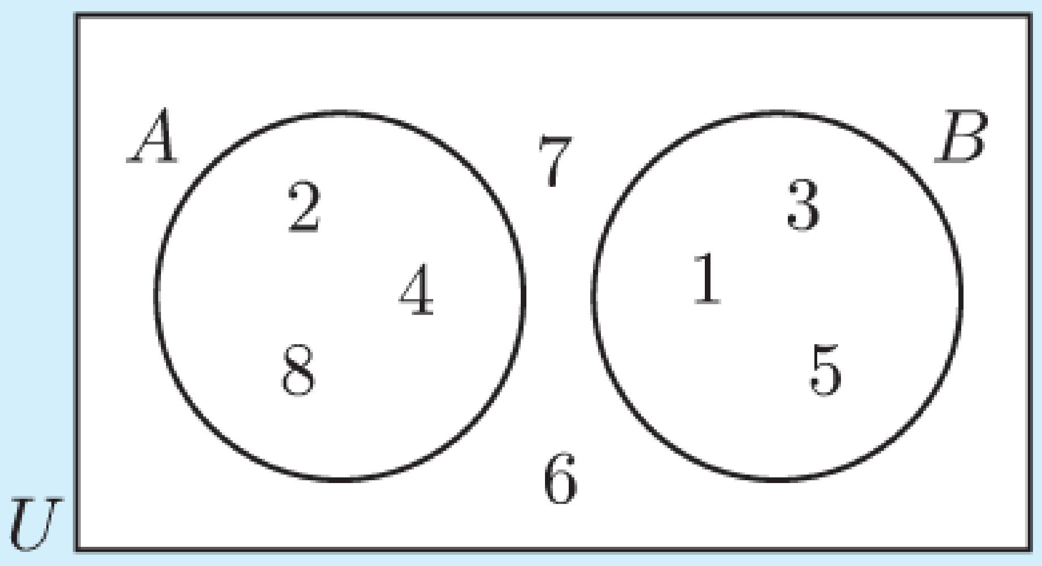
a $A \cap B = \{3, 8\}$



b $A \cap B = \{3, 6, 8\} = B$ and $B \neq A$, so $B \subset A$.



c $A \cap B = \varnothing$



EXERCISE 2F

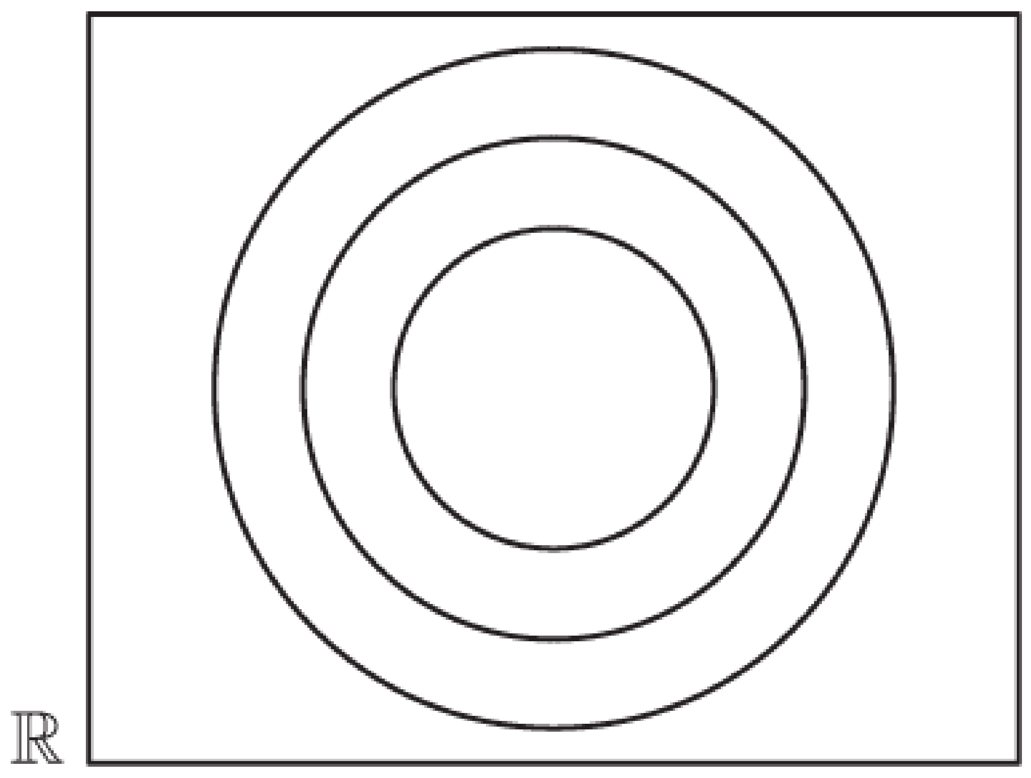
1 Represent sets A and B on a Venn diagram, given:

- **a** $U = \{2, 3, 4, 5, 6, 7\}$, $A = \{2, 4, 6\}$, and $B = \{5, 7\}$
- **b** $U = \{2, 3, 5, 7, 11, 13\}$, $A = \{2, 3, 7\}$, and $B = \{3, 5, 11\}$
- **c** $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{2, 4, 5, 6\}$, and $B = \{1, 4, 6, 7\}$
- **d** $U = \{1, 3, 4, 5, 7\}$, $A = \{3, 4, 5, 7\}$, and $B = \{3, 5\}$

2 Suppose $U = \{x \in \mathbb{Z} \mid 1 \leqslant x \leqslant 10\}$, $A = \{\text{odd numbers} < 10\}$, and $B = \{\text{primes} < 10\}$.

- **a** List sets A and B.
- **b** Find $A \cap B$ and $A \cup B$.
- **c** Represent the sets A and B on a Venn diagram.

3 Suppose $U = \mathbb{R}$.
Copy the Venn diagram and label the sets $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$.
Shade the region representing $\mathbb{Q}'$.
Place these numbers on the Venn diagram:
$7,\ \frac{1}{5},\ 2.\overline{8},\ -\pi,\ 0,\ \sqrt{7},\ -2,\ -0.35$.

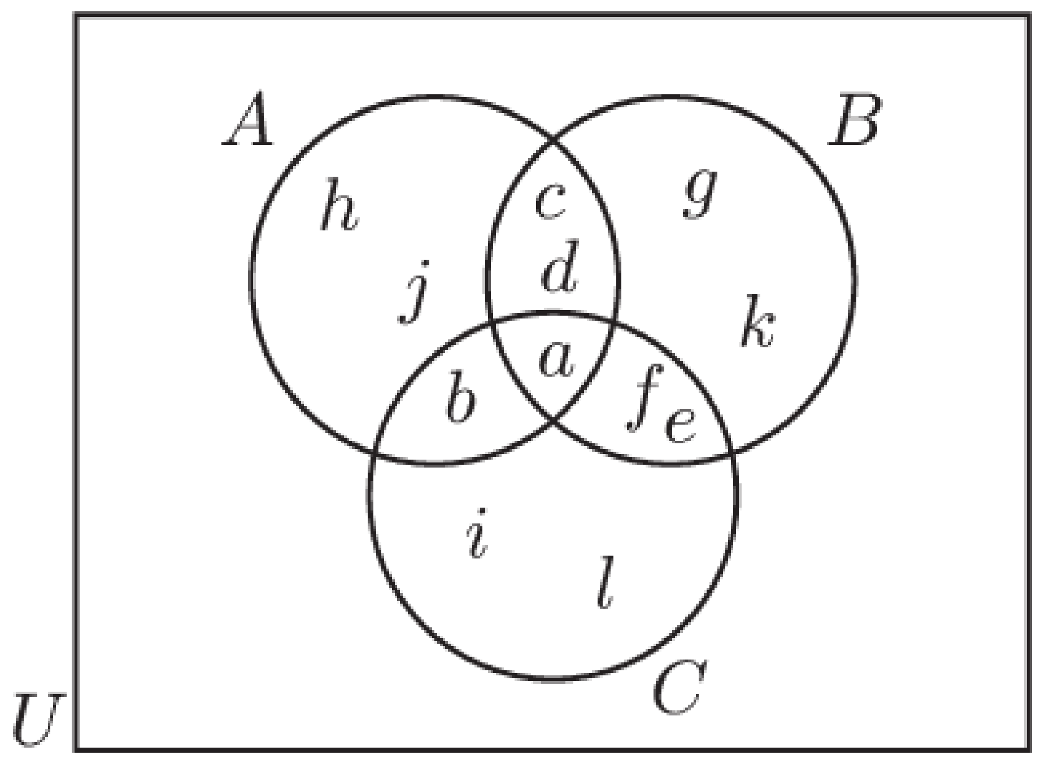


4 Display on a Venn diagram:

a $U = \{\text{parallelograms}\}$, $R = \{\text{rectangles}\}$, $S = \{\text{squares}\}$

b $U = \{\text{polygons}\}$, $Q = \{\text{quadrilaterals}\}$, $T = \{\text{triangles}\}$.

5 This Venn diagram consists of three overlapping circles A, B, and C.



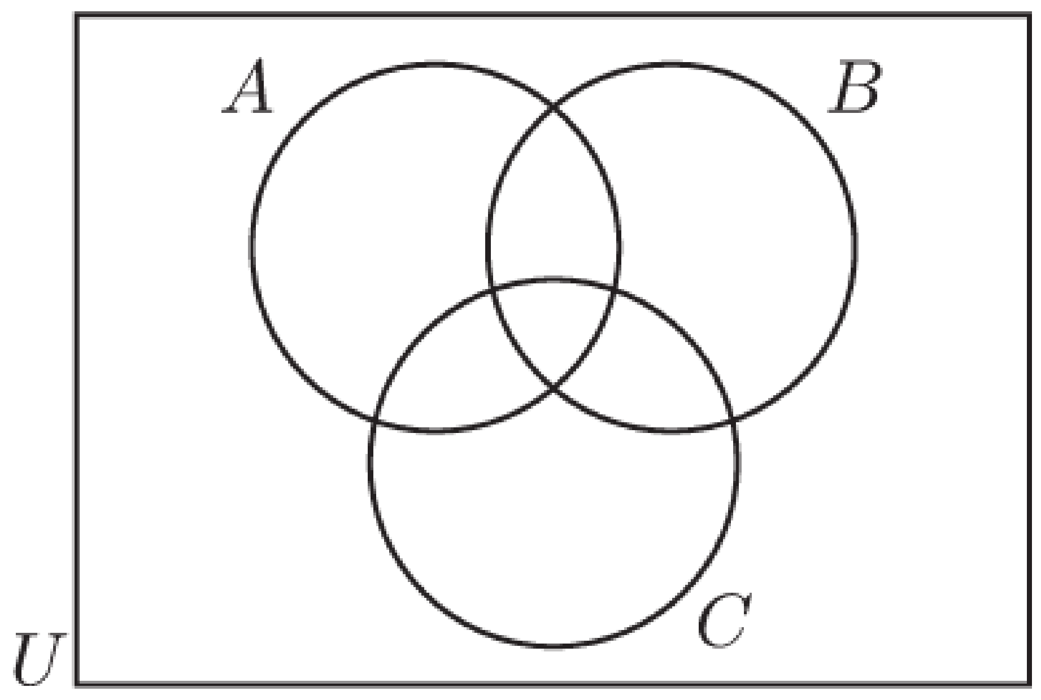
a List the letters in set:

i A
ii B
iii C
iv $A \cap B$
v $A \cup B$
vi $B \cap C$
vii $A \cap B \cap C$
viii $A \cup B \cup C$

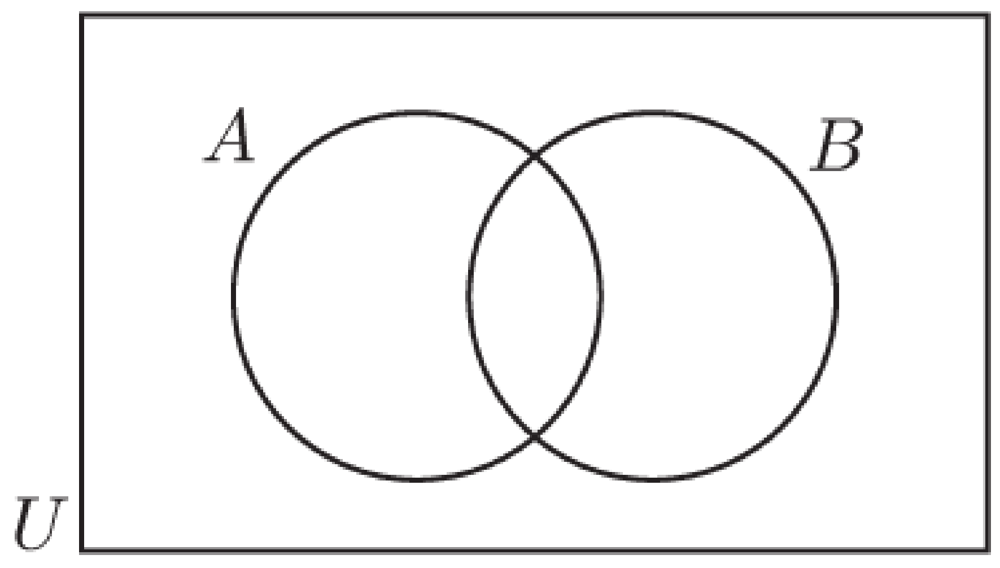
b Show that

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C).$$

6 Suppose $U = \{x \in \mathbb{Z}^+ \mid 40 \leqslant x \leqslant 60\}$,
$A = \{\text{multiples of } 2\}$, $B = \{\text{multiples of } 3\}$, and
$C = \{\text{multiples of } 5\}$.
Display the elements of these sets using a Venn diagram.



7

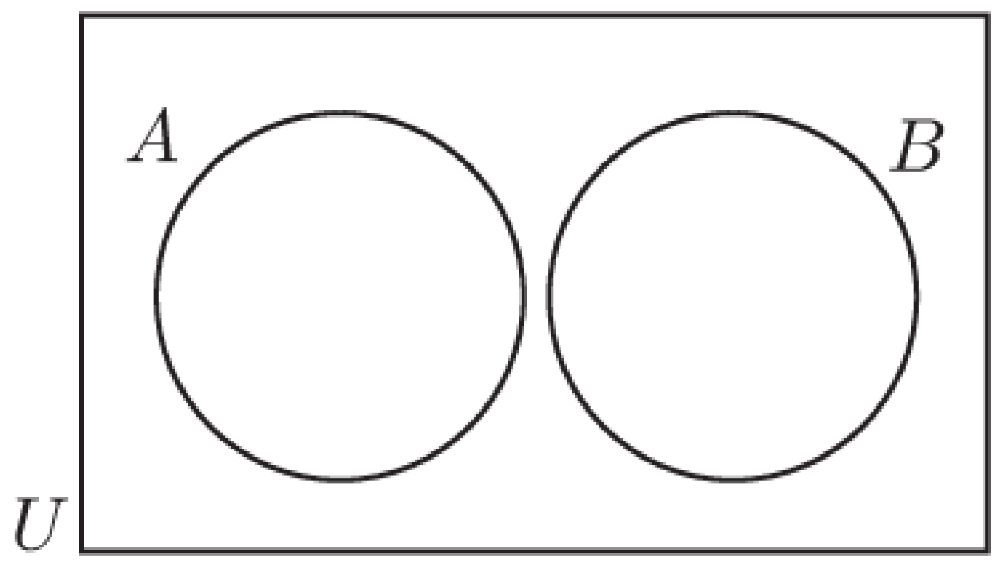


On separate Venn diagrams, shade regions for:

a A
b A'
c $A \cap B$
d $A \cap B'$
e $A' \cup B$
f $A \cup B'$
g $(A \cap B)'$
h $(A \cup B)'$

PRINTABLE VENN DIAGRAMS (OVERLAPPING)

8

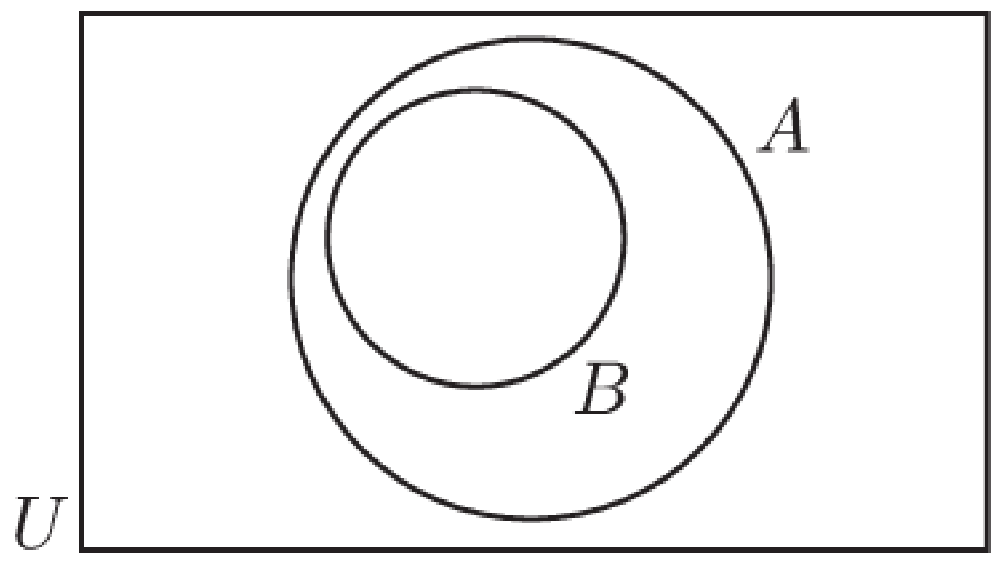


Suppose A and B are two disjoint sets. Shade on separate Venn diagrams:

a A
b B
c A'
d B'
e $A \cap B$
f $A \cup B$
g $A' \cap B$
h $A \cup B'$
i $(A \cap B)'$

PRINTABLE VENN DIAGRAMS (DISJOINT)

9

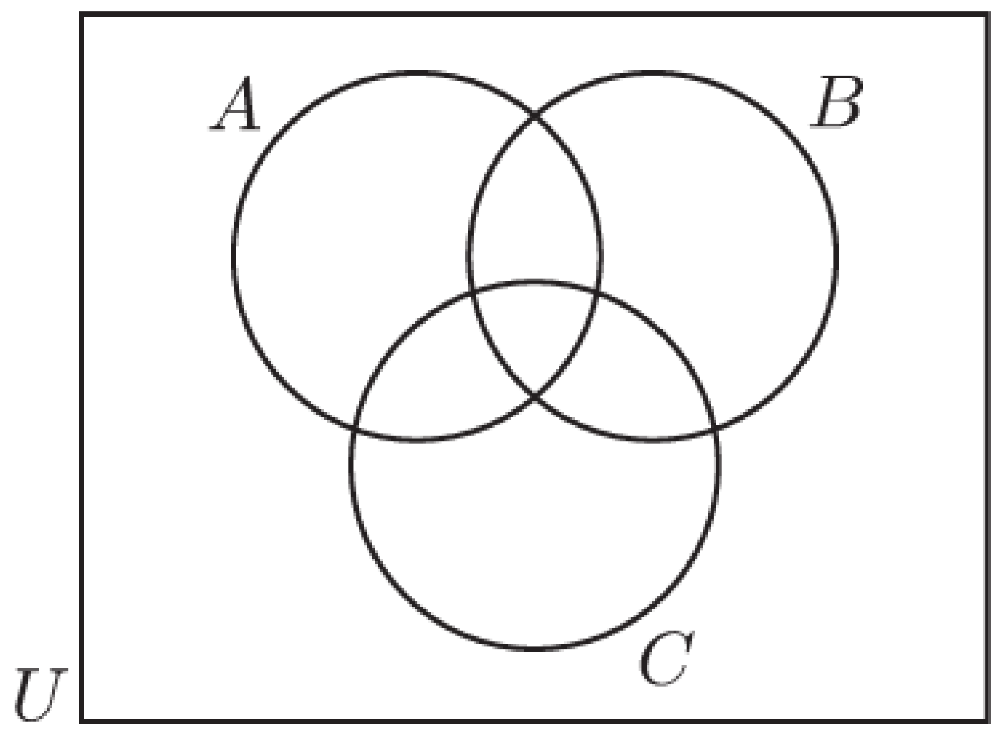


Suppose $B \subseteq A$. Shade on separate Venn diagrams:

a A
b B
c A'
d B'
e $A \cap B$
f $A \cup B$
g $A' \cap B$
h $A \cup B'$
i $(A \cap B)'$

PRINTABLE VENN DIAGRAMS (SUBSET)

10



This Venn diagram consists of three intersecting sets. Shade on separate Venn diagrams:

a A
b B'
c $B \cap C$
d $A \cup B$
e $A \cap B \cap C$
f $A \cup B \cup C$
g $(A \cap B \cap C)'$
h $(A \cup B) \cap C$
i $(B \cap C) \cup A$
j $A' \cap (B \cup C)$

PRINTABLE VENN DIAGRAMS (3 SETS)

ACTIVITY **VENN DIAGRAMS**

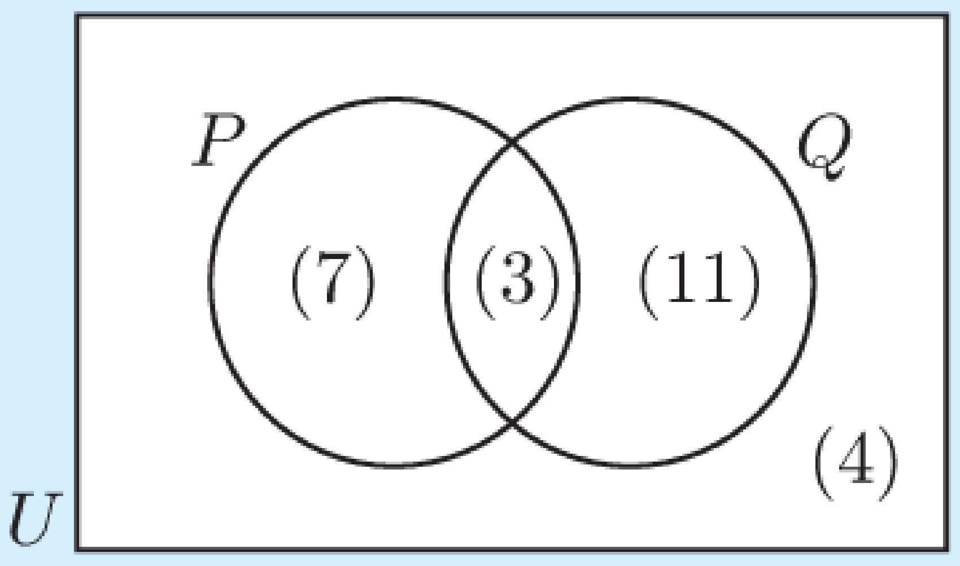
Click on the icon to practise shading regions representing various subsets. You can practise with both two and three intersecting sets.

VENN DIAGRAMS

G VENN DIAGRAM REGIONS

There are many situations where we are only interested in the **number of elements** that are in each region of a Venn diagram. We do not need to show all the elements on the diagram, so instead we write the number of elements in each region in brackets.

Example 6 **Self Tutor**



Use the Venn diagram to find the number of elements in:

a P

b Q'

c $P \cup Q$

d P, but not Q

e Q, but not P

f neither P nor Q.

a $n(P) = 7 + 3 = 10$

b $n(Q') = 7 + 4 = 11$

c $n(P \cup Q) = 7 + 3 + 11 = 21$

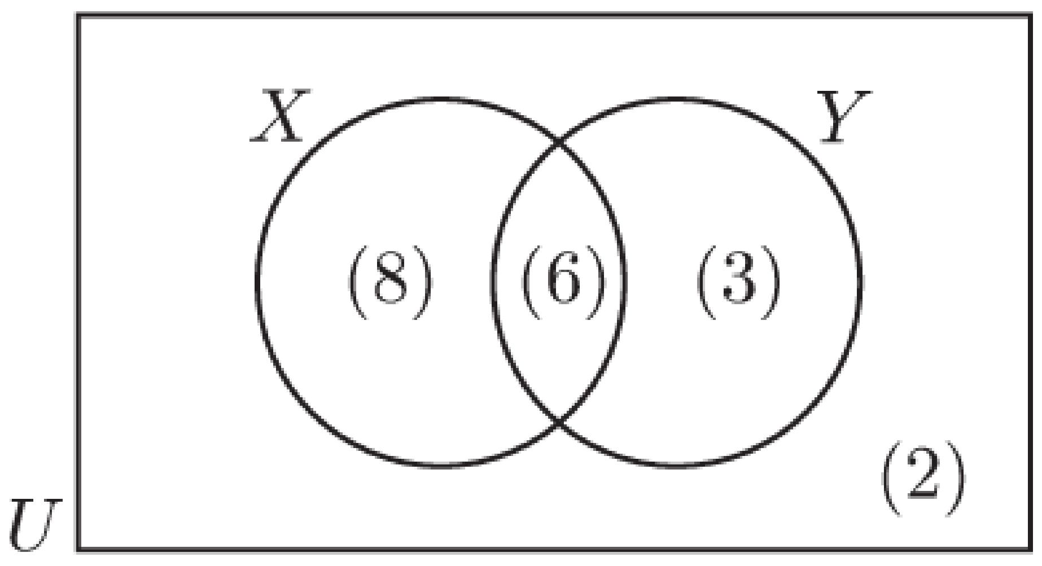
d $n(P, \text{ but not } Q) = 7$

e $n(Q, \text{ but not } P) = 11$

f $n(\text{neither } P \text{ nor } Q) = 4$

EXERCISE 2G

1



Find the number of elements in:

a X'

b $X \cap Y$

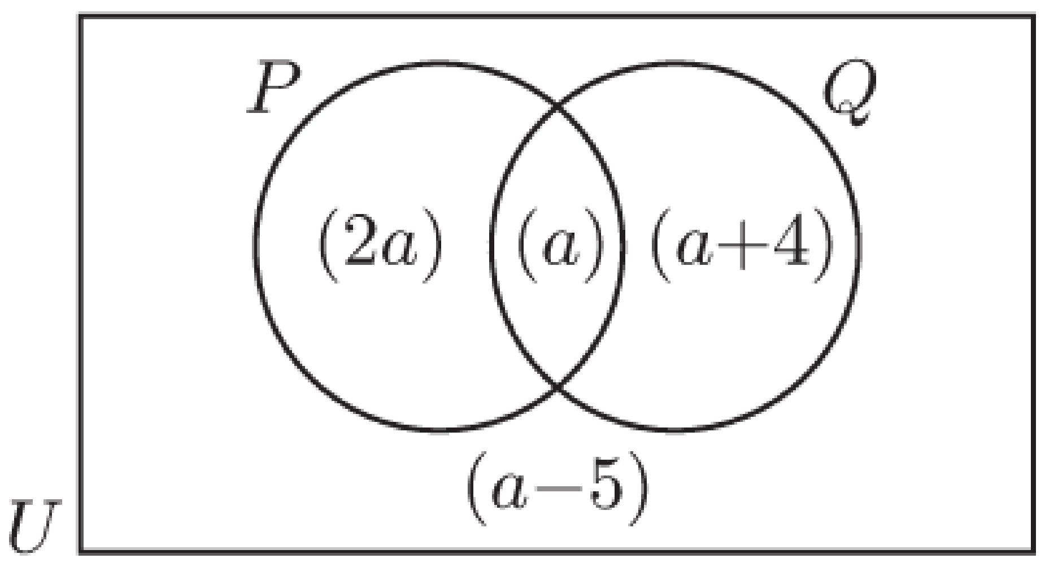
c $X \cup Y$

d X, but not Y

e Y, but not X

f neither X nor Y.

2



a Use the Venn diagram to find:

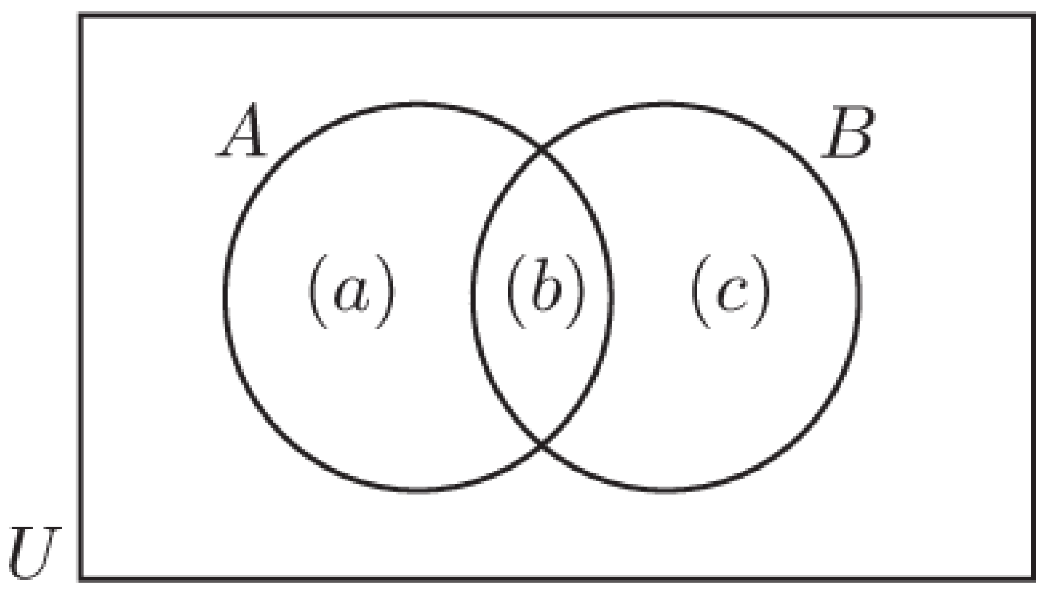
- **i** $n(P \cap Q)$
- **ii** $n(P)$
- **iii** $n(Q)$
- **iv** $n(P \cup Q)$
- **v** $n(Q')$
- **vi** $n(U)$

b Find the value of a if:

- **i** $n(U) = 29$
- **ii** $n(U) = 31$

Comment on your results.

3



Use the Venn diagram to show that:

a $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

b $n(A \cap B') = n(A) - n(A \cap B)$

4 Suppose A and B are disjoint sets. Use a Venn diagram to show that $n(A \cup B) = n(A) + n(B)$.

Example 7 **Self Tutor**

Given $n(U) = 30$, $n(A) = 14$, $n(B) = 17$, and $n(A \cap B) = 6$, find:

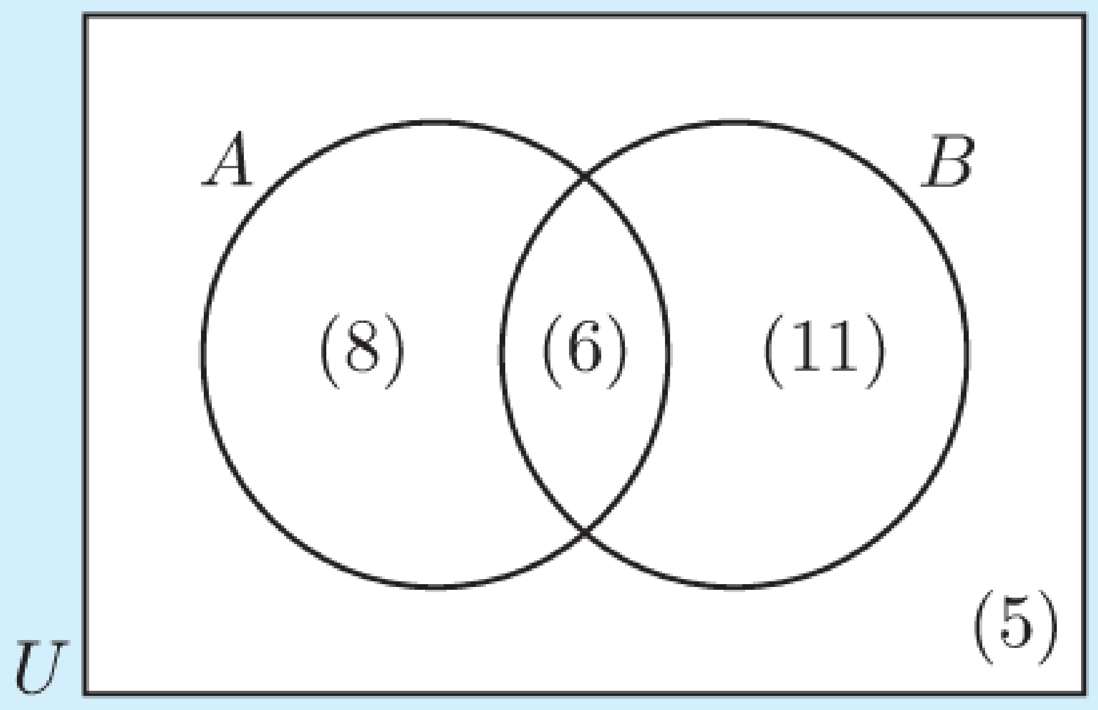
a $n(A \cup B)$ **b** $n(A$, but not $B)$

We are given $n(A \cap B) = 6$

$\therefore$ $n(A \cap B') = 14 - 6 = 8$

and $n(A' \cap B) = 17 - 6 = 11$

$\therefore$ $n(A' \cap B') = 30 - 6 - 8 - 11 = 5$



a $n(A \cup B) = 8 + 6 + 11 = 25$

b $n(A \cap B') = 8$

5 Given $n(U) = 26$, $n(A) = 11$, $n(B) = 12$, and $n(A \cap B) = 8$, find:

a $n(A \cup B)$ **b** $n(B$, but not $A)$

6 Given $n(U) = 32$, $n(M) = 13$, $n(M \cap N) = 5$, and $n(M \cup N) = 26$, find:

a $n(N)$ **b** $n((M \cup N)')$

7 Given $n(U) = 50$, $n(S) = 30$, $n(R) = 25$, and $n(R \cup S) = 48$, find:

a $n(R \cap S)$ **b** $n(S$, but not $R)$

H PROBLEM SOLVING WITH VENN DIAGRAMS

When we solve problems with Venn diagrams, we are generally only interested in the number of individuals in each region, rather than where a particular individual is placed.

Example 8 **Self Tutor**

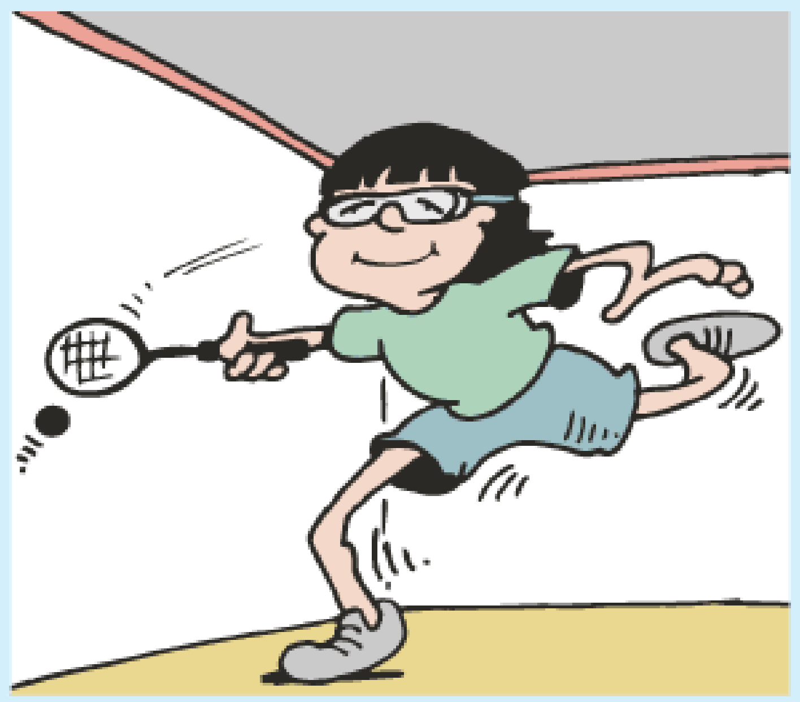
A squash club has 27 members. 19 have black hair, 14 have brown eyes, and 11 have both black hair and brown eyes.

a Place this information on a Venn diagram.

b Hence find the number of members with:

  i black hair or brown eyes

  ii black hair, but not brown eyes.



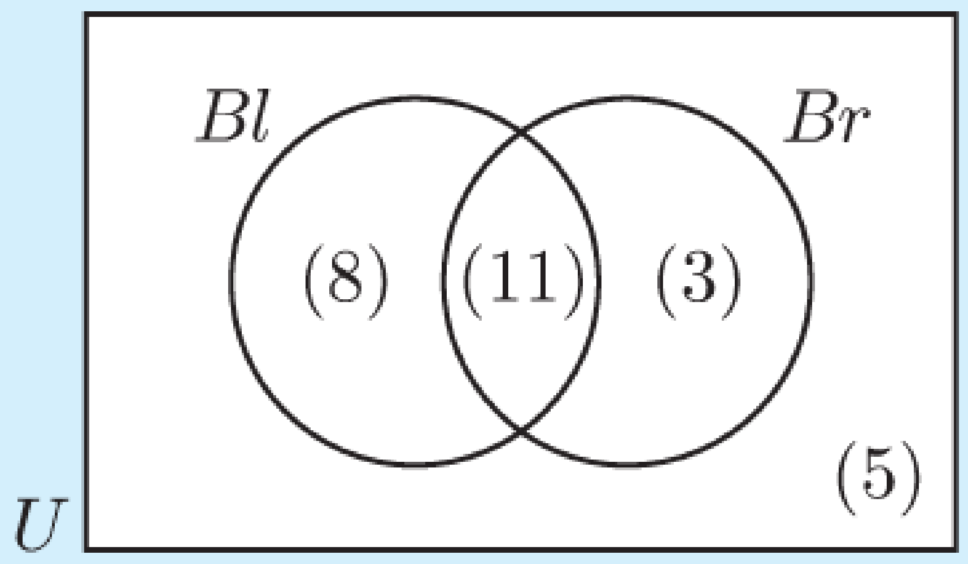
a Let Bl represent those with black hair and Br represent those with brown eyes.

$n(Bl \cap Br) = 11$

$\therefore$ $n(Bl \cap Br') = 19 - 11 = 8$

and $n(Bl' \cap Br) = 14 - 11 = 3$

$\therefore$ $n(Bl' \cap Br') = 27 - 11 - 8 - 3 = 5$



b **i** $n(Bl \cup Br) = 8 + 11 + 3 = 22$

22 members have black hair or brown eyes.

ii $n(Bl \cap Br') = 8$

8 members have black hair, but not brown eyes.

EXERCISE 2H

1 Pelé has 14 cavies as pets. Five have long hair and eight are brown. Two are both brown and have long hair.

- **a** Place this information on a Venn diagram.
- **b** Hence find the number of cavies that:
  - **i** do not have long hair
  - **ii** have long hair and are not brown
  - **iii** are neither long-haired nor brown.

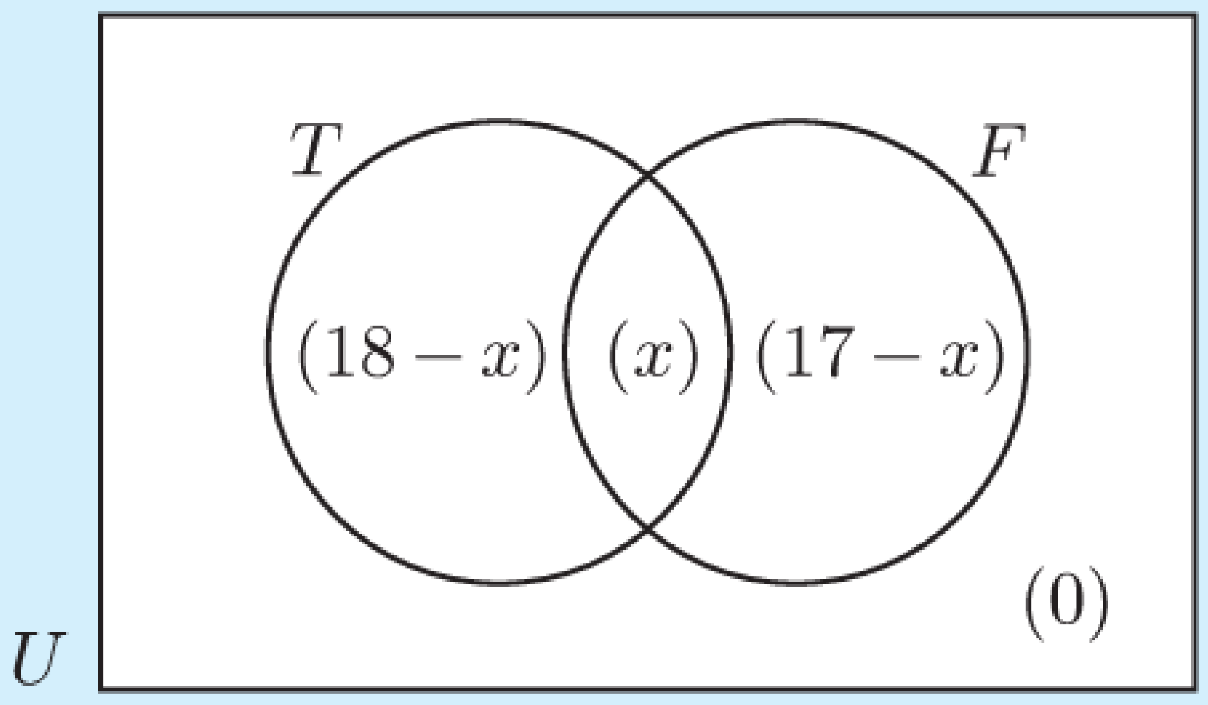
2 During a 2 week period, Murielle took her umbrella with her on 8 days. It rained on 9 days, and Murielle took her umbrella on five of the days when it rained.

- **a** Display this information on a Venn diagram.
- **b** Hence find the number of days that:
  - **i** Murielle did not take her umbrella and it rained
  - **ii** Murielle did not take her umbrella and it did not rain.

Example 9 **Self Tutor**

A platform diving squad of 25 has 18 members who dive from 10 m, and 17 who dive from 4 m. How many dive from both platforms?

Let T represent those who dive from 10 m and
F represent those who dive from 4 m.



Let $n(T \cap F) = x$

$\therefore \ n(T \cap F') = 18 - x$ and $n(T' \cap F) = 17 - x$

$n(T' \cap F') = 0$ since every diver in the squad must dive from at least one platform.

But $n(U) = 25$, so $(18 - x) + x + (17 - x) = 25$

$\therefore \ 35 - x = 25$

$\therefore \ x = 10$

10 members dive from both platforms.

3 In a factory, 56 people work on the assembly line. 47 work day shifts and 29 work night shifts. How many work both day shifts and night shifts?

4 There are 38 stalls in a marketplace. 21 stalls sell food, 14 stalls sell craft, and 7 stalls sell neither food nor craft. Find the number of stalls which sell:

- **a** both food and craft
- **b** food or craft but not both.

5 From the selection of 86 movies on a plane, Sandra has seen 13 and Robert has seen 14. 69 of the movies have been seen by neither. Find the number of movies seen by:

- **a** both Sandra and Robert
- **b** Robert but not Sandra.

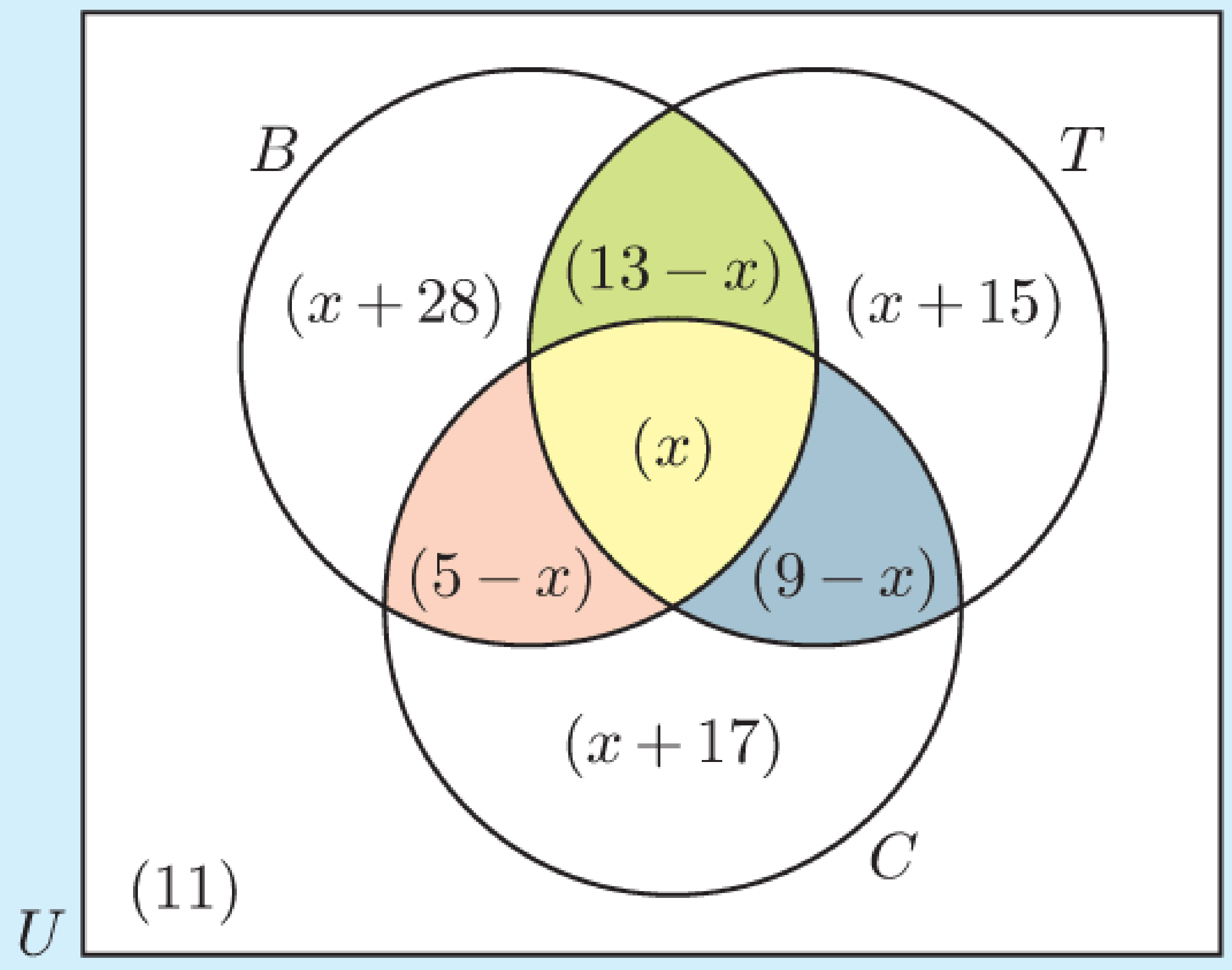
Example 10 **Self Tutor**

The students at a school in Delhi were asked what modes of transportation they had used to travel to school in the past week. 46% had caught a bus, 37% had caught a train, and 31% had been driven in a car. 13% had caught a bus and a train, 5% had caught a bus and been driven in a car, and 9% had caught a train and been driven in a car. 11% of students used none of these modes of transport.

Find the percentage of students who used all three modes of transport.

Let B, T, and C represent the students taking a bus, train, and car respectively.

$$\begin{aligned} \text{Let} \quad n(B \cap T \cap C) &= x \\ \therefore \quad n(B \cap T \cap C') &= 13 - x, \\ n(B \cap T' \cap C) &= 5 - x, \\ \text{and} \quad n(B' \cap T \cap C) &= 9 - x. \end{aligned}$$



$$\begin{aligned} \text{Now} \quad n(B) = 46, \quad \text{so} \quad n(B \cap T' \cap C') &= 46 - x - (13 - x) - (5 - x) = x + 28 \\ n(T) = 37, \quad \text{so} \quad n(B' \cap T \cap C') &= 37 - x - (13 - x) - (9 - x) = x + 15 \\ \text{and} \quad n(C) = 31, \quad \text{so} \quad n(B' \cap T' \cap C) &= 31 - x - (5 - x) - (9 - x) = x + 17 \end{aligned}$$

$$\begin{aligned} \therefore \quad (x + 28) + (13 - x) + x + (5 - x) + (x + 15) + (9 - x) + (x + 17) + 11 &= 100 \\ \therefore \quad x + 98 &= 100 \\ \therefore \quad x &= 2 \end{aligned}$$

2% of students used all three modes of transport.

6 In a year group of 63 students, 22 study Biology, 26 study Chemistry, and 25 study Physics. 18 study both Physics and Chemistry, 4 study both Biology and Chemistry, and 3 study both Physics and Biology. 1 student studies all three subjects.

- **a** Display this information on a Venn diagram.
- **b** How many students study:
  - **i** Biology only
  - **ii** Physics or Chemistry
  - **iii** none of Biology, Physics, or Chemistry
  - **iv** Physics but not Chemistry?

7 36 students participated in the mid-year adventure trip. 19 students went paragliding, 21 went abseiling, and 16 went white water rafting. 7 went abseiling and rafting, 8 went paragliding and rafting, and 11 went paragliding and abseiling. 5 students did all three activities. Find the number of students who:

- **a** went paragliding or abseiling
- **b** only went white water rafting
- **c** did not participate in any of the activities mentioned
- **d** did exactly two of the activities mentioned.

8 There are 29 students in the woodwind section of the school orchestra. 11 students can play the flute, 15 can play the clarinet, and 12 can play the saxophone. 4 can play the flute and the saxophone, 4 can play the flute and the clarinet, and 6 can play the clarinet and the saxophone. 3 students can play none of the three instruments.

- **a** Display the information on a Venn diagram.
- **b** Hence find the number of students in the woodwind section who can play:
  - **i** all of the instruments mentioned
  - **ii** only the saxophone
  - **iii** the saxophone and the clarinet, but not the flute
  - **iv** exactly one of the clarinet, saxophone, or flute.

9 In a particular region, most farms have livestock and crops. A survey of 21 farms showed that 15 grow crops, 9 have cattle, and 11 have sheep. 4 have sheep and cattle, 7 have cattle and crops, and 8 have sheep and crops. 2 have neither animals nor crops.

- **a** Display the information on a Venn diagram.
- **b** Hence find the number of farms with:
  - **i** only crops
  - **ii** only animals
  - **iii** exactly one type of animal, and crops.

10 Answer the **Opening Problem** parts **a** to **c** on page **34**. Discuss part **d** with your class.

REVIEW SET 2A

1 Let A be the set of letters in the word VENN and B be the set of letters in the word DIAGRAM.

- **a** List the elements of A and B.
- **b** Find $n(A)$ and $n(B)$.
- **c** Find $A \cap B$ and comment on your answer.
- **d** Decide whether the statement is true or false:
  - **i** $\text{V} \notin A$
  - **ii** $\text{G} \in B$
  - **iii** $n(A \cup B) = n(A) + n(B)$

2 If $U = \{\text{multiples of 6 less than 70}\}$ and $A = \{6, 6^2, 66\}$, find A'.

3 Decide whether each statement is true or false:

- **a** $\mathbb{N} \subset \mathbb{Q}$
- **b** $0 \in \mathbb{Z}^+$
- **c** $0 \in \mathbb{Q}$
- **d** $\mathbb{R} \subseteq \mathbb{Q}$

4 Represent on a number line:

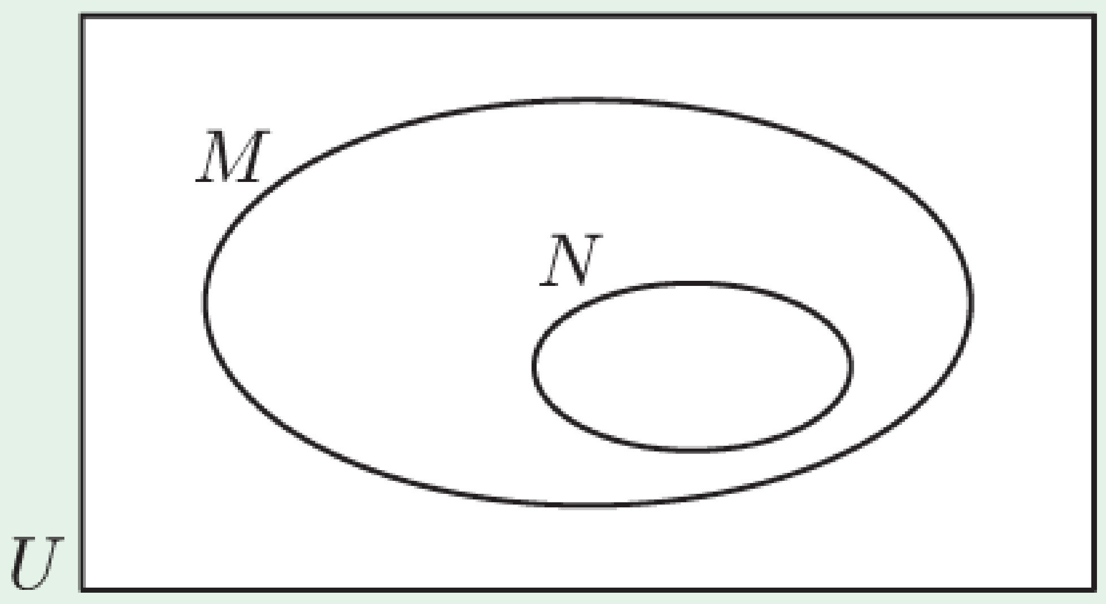
- **a** $\{x \in \mathbb{N} \mid x \leqslant 6\}$
- **b** $\{x \in \mathbb{R} \mid -3 \leqslant x < 2\}$
- **c** $x \in [0, 4] \ \cup \ x \in [10, \infty[$

5 Let $U = \{x \in \mathbb{Z}^+ \mid x \leqslant 30\}$, $P = \{\text{factors of 24}\}$, and $Q = \{\text{factors of 30}\}$.

- **a** List the elements of:
  - **i** P
  - **ii** Q
  - **iii** $P \cap Q$
  - **iv** $P \cup Q$
- **b** Show that $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$.

6 On separate Venn diagrams like the one alongside, shade:

- **a** N'
- **b** $M \cap N$
- **c** $M \cap N'$

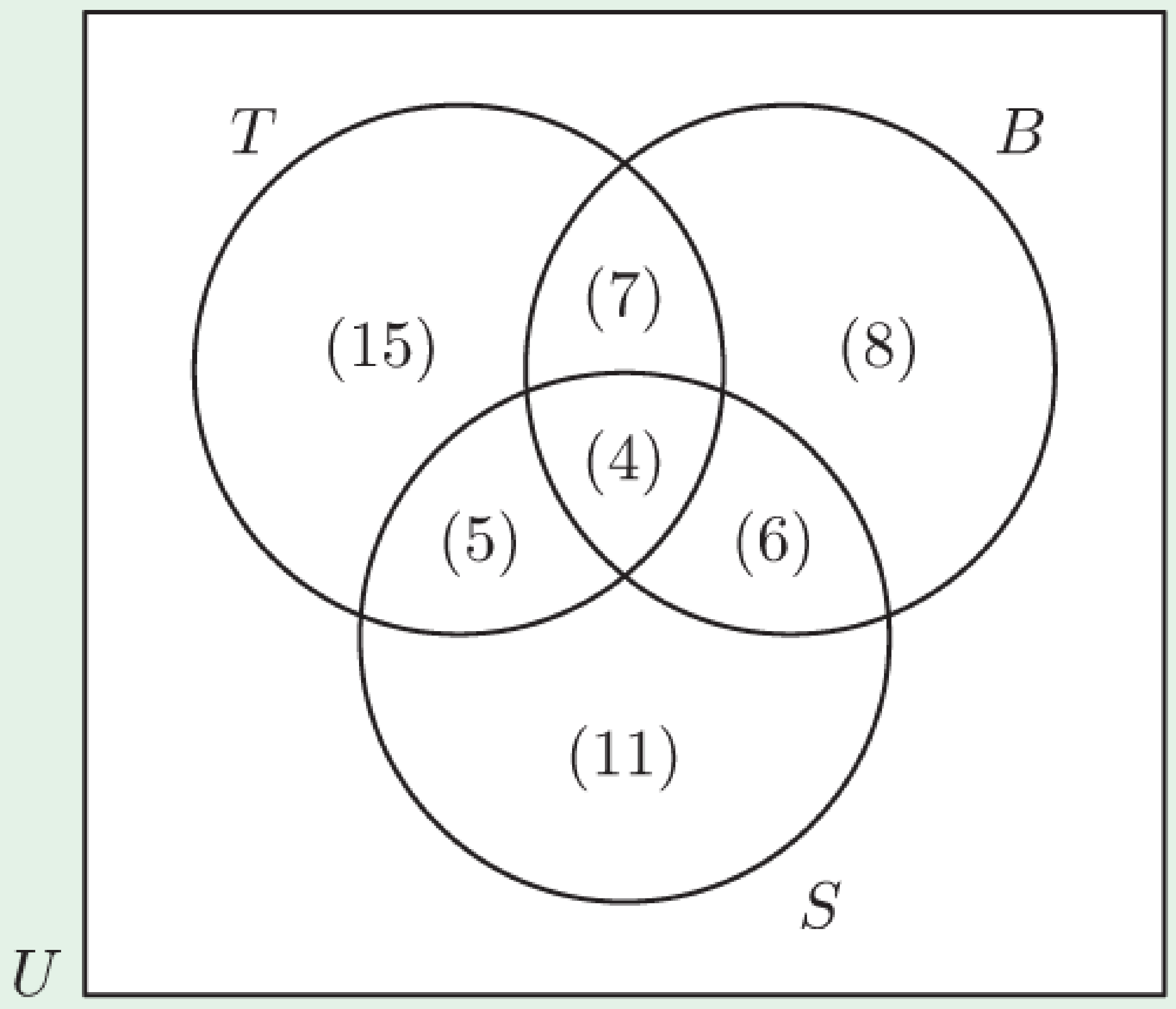


7 Suppose $P \subseteq Q$. Show that P and Q' are disjoint.

8 Consider the sets $U = \{x \in \mathbb{Z}^+ \mid x \leqslant 10\}$, $P = \{\text{odd numbers less than } 10\}$, and $Q = \{\text{even numbers less than } 11\}$.

- **a** List the sets P and Q.
- **b** What can be said about sets P and Q?
- **c** Illustrate sets P and Q on a Venn diagram.

9 The racquet sports offered at a local club are tennis (T), badminton (B), and squash (S). The Venn diagram shows the number of members involved in these activities. All of the members play at least one racquet sport.



- **a** Write down the number of members in the club.
- **b** Write down the number of members who:
  - **i** only play badminton
  - **ii** do not play tennis
  - **iii** play both tennis and squash, but not badminton.
- **c** Copy the diagram above, and shade the region that represents $S \cap (T \cup B)$.
- **d** Write down the number of members in $S \cap (T \cup B)'$.

10 A school has 564 students. During Term 1, 229 of them were absent for at least one day due to sickness, and 111 students missed some school because of family holidays. 296 students attended every day of Term 1.

- **a** Display this information on a Venn diagram.
- **b** Find how many students:
  - **i** missed school for both illness and holidays
  - **ii** were away for holidays but not sickness
  - **iii** were absent during Term 1 for any reason.

11 The main courses at a restaurant all contain rice or onion. Of the 23 choices, 17 contain onion and 14 contain rice. How many dishes contain both rice and onion?

12 38 students were asked what life skills they had. 15 could swim, 12 could drive, and 23 could cook. 9 could cook and swim, 5 could swim and drive, and 6 could drive and cook. There was 1 student who could do all three. Find the number of students who:

- **a** could only cook
- **b** could not do any of these things
- **c** had exactly two of these life skills.

REVIEW SET 2B

1 List the subsets of $\{1, 3, 5\}$.

2 S and T are disjoint sets. $n(S) = s$ and $n(T) = t$. Find: **a** $S \cap T$ **b** $n(S \cup T)$

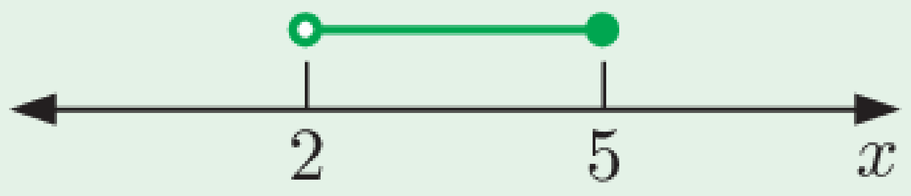
3 Write in interval notation:

- **a** the real numbers between 5 and 12
- **b** the integers between -4 and 7, including -4
- **c** the natural numbers greater than 45.

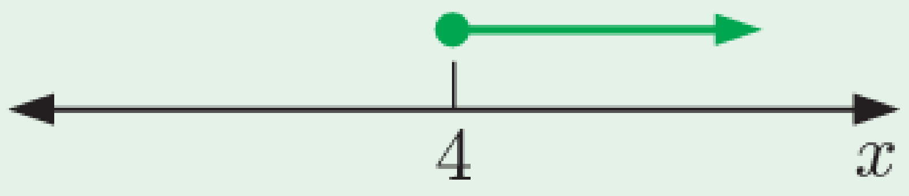
State whether each set is finite or infinite.

4 Write in bracket notation:

a



b



c

5 Let $S = \{x \in \mathbb{Z} \mid 2 < x \leqslant 7\}$.

a List the elements of S. **b** Display S on a number line. **c** Find $n(S)$.

6 Determine whether $A \subseteq B$:

a $A = \{2, 4, 6, 8\}$ and $B = \{x \in \mathbb{Z} \mid 0 < x < 10\}$

b $A = \varnothing$ and $B = \{x \mid 2 < x < 3\}$

c $A = \{x \in \mathbb{Q} \mid 2 < x \leqslant 4\}$ and $B = \{x \in \mathbb{R} \mid 0 \leqslant x < 4\}$

d $A = \{x \mid x < 3\}$ and $B = \{x \mid x \leqslant 4\}$

7 Find the complement of X given that:

a $U = \{$the 7 colours of the rainbow$\}$ and $X = \{$red, indigo, violet$\}$

b $U = \{x \in \mathbb{Z} \mid -5 \leqslant x \leqslant 5\}$ and $X = \{-4, -1, 3, 4\}$

c $U = \{x \in \mathbb{Q}\}$ and $X = \{x \in \mathbb{Q} \mid x < -8\}$

8 Consider the sets $U = \{x \in \mathbb{Z}^+ \mid x \leqslant 40\}$, $A = \{$factors of $40\}$, and $B = \{$factors of $20\}$.

a List the sets A and B. **b** What can be said about sets A and B?

c Illustrate sets A and B on a Venn diagram.

9 Consider $P = \{x \in \mathbb{Z} \mid 3 \leqslant x < 10\}$, $Q = \{2, 9, 15\}$, and $R = \{$multiples of 3 less than $12\}$.

a List the elements of P. **b** Write down $n(P)$.

c State whether P is finite or infinite.

d Explain why: **i** $Q \not\subset P$ **ii** $R \subset P$

e List the elements of: **i** $P \cap Q$ **ii** $R \cap Q$ **iii** $R \cup Q$

10 In a car club, 13 members drive a manual car and 15 members have a car with a sunroof. 5 have manual cars with a sunroof, and 4 have neither.

a Display this information on a Venn diagram.

b How many members:

i are in the club **ii** drive a manual car without a sunroof

iii do not drive a manual car?

11 All of the 30 students on a camp left something at home. 11 forgot to bring their towel. 23 forgot their hat. How many had neither a hat nor a towel?

12 At a conference, the 58 delegates speak many different languages. 28 speak Arabic, 27 speak Chinese, and 39 speak English. 12 speak Arabic and Chinese, 16 speak both Chinese and English, and 17 speak Arabic and English. 2 speak all three languages. How many delegates speak:

a Chinese only

b none of these languages

c neither Arabic nor Chinese?

Chapter 3

Surds and exponents

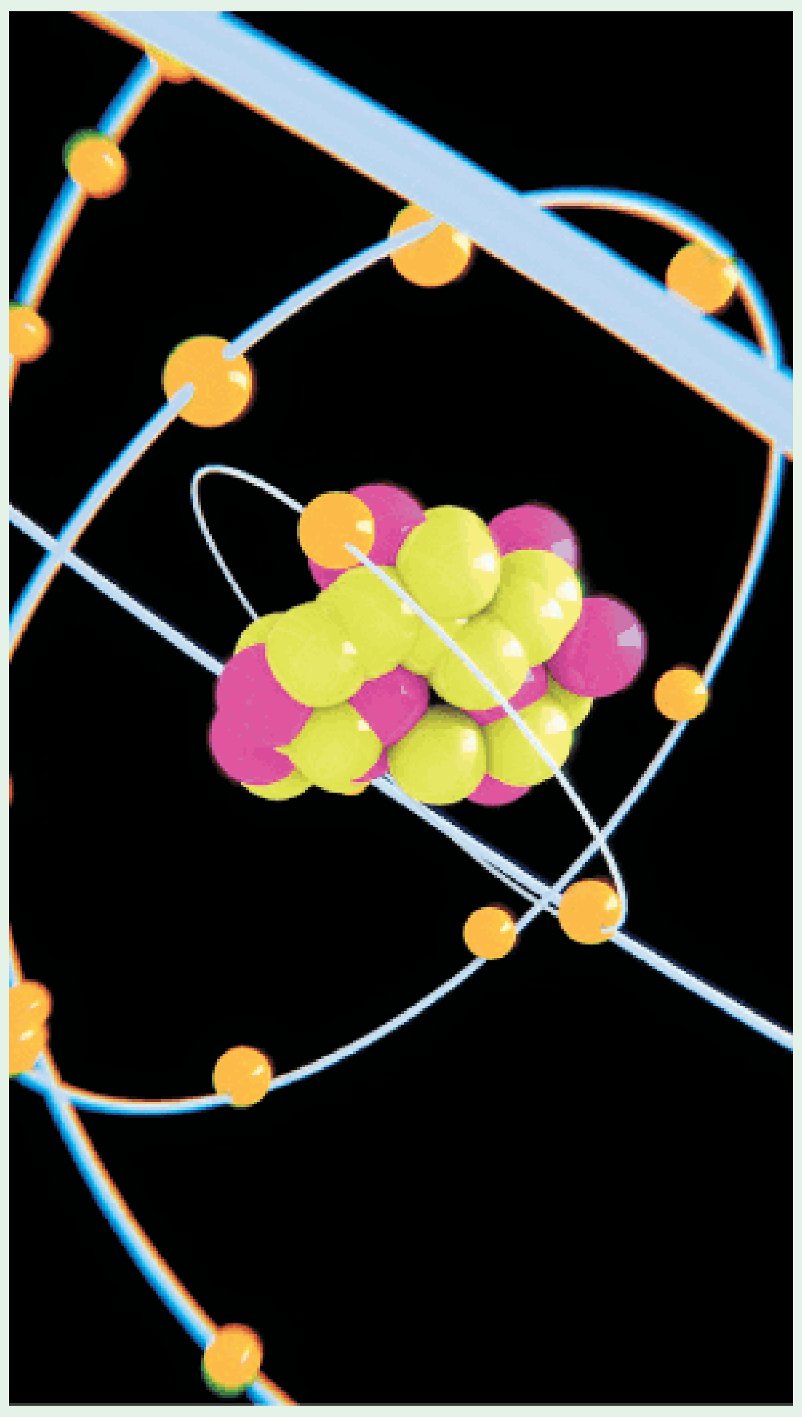
Contents:

OPENING PROBLEM

Sir Joseph John Thomson (1856 - 1940) won the Nobel Prize in Physics in 1906 for his experiments in the conduction of electricity through gases. He is also credited with the discovery of the first subatomic particle.

The table below shows the mass and electric charge for the three subatomic particles which, for many decades, were thought to be the smallest parts of an atom.

Particle	*Mass* (kg)	*Charge* (coulombs)
electron	$9.109\,383\,56 \times 10^{-31}$	$-1.602\,176\,620\,8 \times 10^{-19}$
proton	$1.672\,621\,898 \times 10^{-27}$	$+1.602\,176\,620\,8 \times 10^{-19}$
neutron	$1.674\,927\,471 \times 10^{-27}$	0



Things to think about:

- **a** Most of the numbers in the table are written in a form involving a power of 10. Why would we choose to write numbers in this form?
- **b** How many times more massive is:
  - **i** a neutron than an electron
  - **ii** a proton than an electron
  - **iii** a neutron than a proton?
- **c** An atom of silver has 47 protons, has no charge, and has mass $1.791\,193\,4 \times 10^{-25}$ kg. Find how many electrons and neutrons it has.
- **d** A large bolt of lightning transfers 350 coulombs of charge. How many electrons does it transfer?

In this Chapter we consider **surds** and **radicals**, and see how they relate to **exponents**. We will review the **laws of exponents** and apply them in algebra and to very large and very small numbers.

A SURDS AND OTHER RADICALS

A **radical** is any number which is written with the **radical sign** $\sqrt{\ }$.

A **surd** is a real, irrational radical.

For example:

- $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, and $\sqrt{6}$ are surds.
- $\sqrt{4}$ is a radical but not a surd, since $\sqrt{4} = 2$.
- $\sqrt{\frac{1}{4}}$ is a radical but not a surd, since $\sqrt{\frac{1}{4}} = \frac{1}{2}$.

By definition, $\sqrt{a}$ is the non-negative number such that $\sqrt{a} \times \sqrt{a} = a$.

Notice that:

- $\sqrt{a}$ is only a real number if $a \geqslant 0$.
- For any $a \geqslant 0$, $\sqrt{a} \geqslant 0$.

INVESTIGATION PROPERTIES OF RADICALS

What to do:

1 **a** Discuss each step of this argument with your class to make sure you are convinced it is valid:

$$\begin{aligned}(\sqrt{2\times 3})^2 &= 2\times 3 && \{\text{definition of square root}\}\\ &= (\sqrt{2}\times\sqrt{2})\times(\sqrt{3}\times\sqrt{3}) && \{\text{definition of square root}\}\\ &= (\sqrt{2}\times\sqrt{3})\times(\sqrt{2}\times\sqrt{3}) && \{\text{changing order of multiplication}\}\\ &= (\sqrt{2}\times\sqrt{3})^2 && \{\text{definition of perfect square}\}\end{aligned}$$

$\therefore$ since square roots are non-negative, $\sqrt{2\times 3}=\sqrt{2}\times\sqrt{3}$.

b Use the same argument to write a direct proof that $\sqrt{a\times b}=\sqrt{a}\times\sqrt{b}$ for any $a \geqslant 0$, $b \geqslant 0$.

2 **a** Discuss each step of this argument with your class to make sure you are convinced it is valid:

$$\begin{aligned}\left(\sqrt{\frac{2}{3}}\right)^2 &= \frac{2}{3} && \{\text{definition of square root}\}\\ &= \frac{\sqrt{2}\times\sqrt{2}}{\sqrt{3}\times\sqrt{3}} && \{\text{definition of square root}\}\\ &= \frac{\sqrt{2}}{\sqrt{3}}\times\frac{\sqrt{2}}{\sqrt{3}} && \{\text{multiplication of fractions}\}\\ &= \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\end{aligned}$$

$\therefore$ since square roots are non-negative, $\sqrt{\frac{2}{3}}=\frac{\sqrt{2}}{\sqrt{3}}$.

b Use the same argument to write a direct proof that $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$ for any $a \geqslant 0$, $b > 0$.

In the **Investigation** you should have proven that:

- $\sqrt{ab}=\sqrt{a}\times\sqrt{b}$ for $a \geqslant 0$ and $b \geqslant 0$.
- $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$ for $a \geqslant 0$ and $b > 0$.

Example 1 **Self Tutor**

Write as a single surd: **a** $\sqrt{2}\times\sqrt{3}$ **b** $\dfrac{\sqrt{18}}{\sqrt{6}}$

a
$$\begin{aligned}&\sqrt{2}\times\sqrt{3}\\ &=\sqrt{2\times 3}\\ &=\sqrt{6}\end{aligned}$$

b
$$\begin{aligned}&\frac{\sqrt{18}}{\sqrt{6}}\\ &=\sqrt{\tfrac{18}{6}}\\ &=\sqrt{3}\end{aligned}$$

SIMPLEST FORM

A radical is in **simplest form** when the number under the radical sign is the smallest possible integer.

Example 2 Self Tutor

Write $\sqrt{72}$ in simplest form.

$$\begin{aligned} &\sqrt{72} \\ &= \sqrt{36 \times 2} \qquad \{36 \text{ is the largest perfect square factor of } 72\} \\ &= \sqrt{36} \times \sqrt{2} \\ &= 6\sqrt{2} \end{aligned}$$

EXERCISE 3A

1 Write as a single surd or rational number:

a $\sqrt{11} \times \sqrt{11}$ **b** $\sqrt{3} \times \sqrt{5}$ **c** $\sqrt{5} \times \sqrt{6}$ **d** $3\sqrt{7} \times 2\sqrt{7}$

e $(3\sqrt{5})^2$ **f** $3\sqrt{2} \times \sqrt{5}$ **g** $-2\sqrt{3} \times 3\sqrt{5}$ **h** $2\sqrt{6} \times \sqrt{12}$

2 Write as a single surd or rational number:

a $\dfrac{\sqrt{12}}{\sqrt{2}}$ **b** $\dfrac{\sqrt{18}}{\sqrt{3}}$ **c** $\dfrac{\sqrt{20}}{\sqrt{5}}$ **d** $\dfrac{\sqrt{3}}{\sqrt{12}}$

e $\dfrac{\sqrt{6}}{\sqrt{18}}$ **f** $\dfrac{\sqrt{6} \times \sqrt{10}}{\sqrt{12}}$ **g** $\dfrac{\sqrt{3}}{\sqrt{6} \times \sqrt{8}}$ **h** $\dfrac{\sqrt{5}}{2\sqrt{10} \times \sqrt{2}}$

3 Write in simplest form:

a $\sqrt{12}$ **b** $\sqrt{20}$ **c** $\sqrt{27}$ **d** $\sqrt{54}$

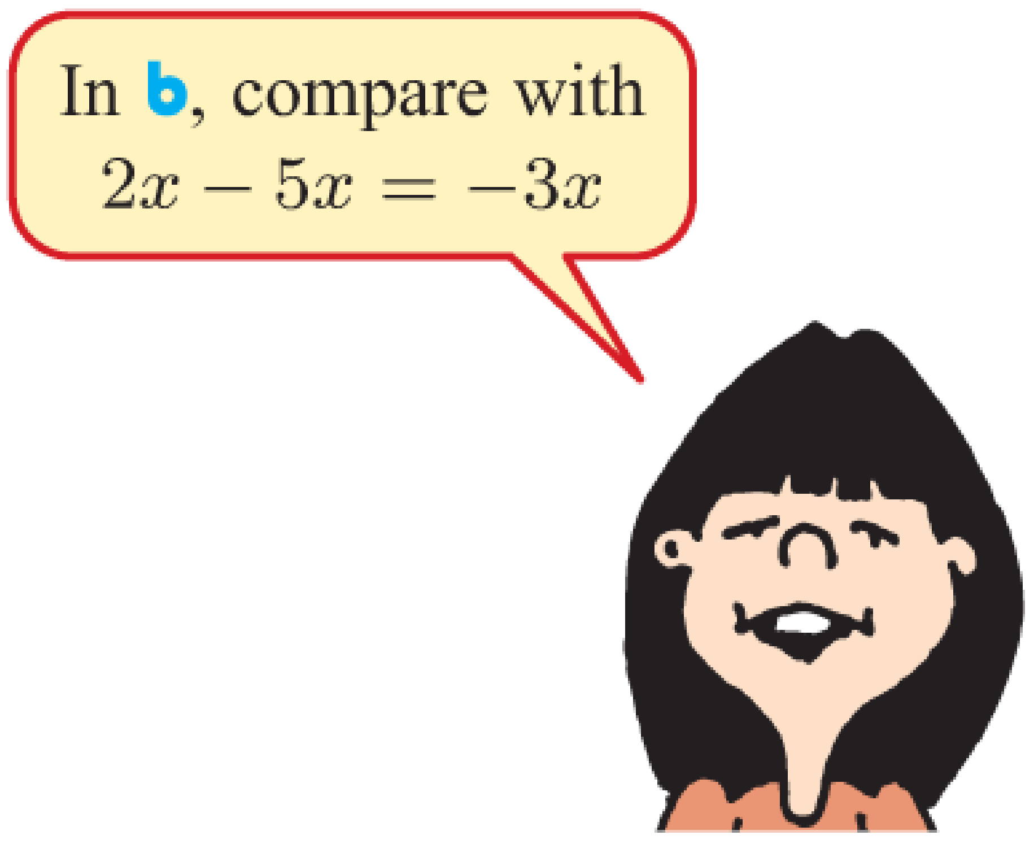
e $\sqrt{50}$ **f** $\sqrt{80}$ **g** $\sqrt{96}$ **h** $\sqrt{108}$

Example 3 Self Tutor

Simplify:

a $3\sqrt{3} + 5\sqrt{3}$ **b** $2\sqrt{2} - 5\sqrt{2}$

a $\begin{aligned} &3\sqrt{3} + 5\sqrt{3} \\ &= 8\sqrt{3} \end{aligned}$ **b** $\begin{aligned} &2\sqrt{2} - 5\sqrt{2} \\ &= -3\sqrt{2} \end{aligned}$



4 Simplify:

a $2\sqrt{2} + 3\sqrt{2}$ **b** $2\sqrt{2} - 3\sqrt{2}$ **c** $5\sqrt{5} - 3\sqrt{5}$ **d** $5\sqrt{5} + 3\sqrt{5}$

e $3\sqrt{5} - 5\sqrt{5}$ **f** $7\sqrt{3} + 2\sqrt{3}$ **g** $9\sqrt{6} - 12\sqrt{6}$ **h** $\sqrt{2} + \sqrt{2} + \sqrt{2}$

Example 4 **Self Tutor**

Simplify: $2\sqrt{75} - 5\sqrt{27}$

$$\begin{aligned} &2\sqrt{75} - 5\sqrt{27} \\ &= 2\sqrt{25 \times 3} - 5\sqrt{9 \times 3} \\ &= 2 \times 5 \times \sqrt{3} - 5 \times 3 \times \sqrt{3} \\ &= 10\sqrt{3} - 15\sqrt{3} \\ &= -5\sqrt{3} \end{aligned}$$

5 Simplify:

a $4\sqrt{3} - \sqrt{12}$
b $3\sqrt{2} + \sqrt{50}$
c $3\sqrt{6} + \sqrt{24}$
d $2\sqrt{27} + 2\sqrt{12}$
e $\sqrt{75} - \sqrt{12}$
f $\sqrt{2} + \sqrt{8} - \sqrt{32}$

Example 5 **Self Tutor**

Expand and simplify:

a $\sqrt{5}(6 - \sqrt{5})$
b $(6 + \sqrt{3})(1 + 2\sqrt{3})$

a

$$\begin{aligned} &\sqrt{5}(6 - \sqrt{5}) \\ &= \sqrt{5} \times 6 + \sqrt{5} \times (-\sqrt{5}) \\ &= 6\sqrt{5} - 5 \end{aligned}$$

b

$$\begin{aligned} &(6 + \sqrt{3})(1 + 2\sqrt{3}) \\ &= 6 + 6(2\sqrt{3}) + \sqrt{3}(1) + \sqrt{3}(2\sqrt{3}) \\ &= 6 + 12\sqrt{3} + \sqrt{3} + 6 \\ &= 12 + 13\sqrt{3} \end{aligned}$$

6 Simplify:

a $\sqrt{2}(3 - \sqrt{2})$
b $2\sqrt{6}(\sqrt{6} - 7)$
c $-\sqrt{8}(\sqrt{8} - 5)$
d $-3\sqrt{2}(4 - 6\sqrt{2})$
e $(5 + \sqrt{2})(4 + \sqrt{2})$
f $(9 - \sqrt{7})(4 + 2\sqrt{7})$
g $(\sqrt{3} + 1)(2 - 3\sqrt{3})$
h $(\sqrt{8} - 6)(2\sqrt{8} - 3)$
i $(2\sqrt{5} - 7)(1 - 4\sqrt{5})$

Example 6 **Self Tutor**

Simplify:

a $(5 - \sqrt{2})^2$
b $(7 + 2\sqrt{5})(7 - 2\sqrt{5})$

a

$$\begin{aligned} &(5 - \sqrt{2})^2 \\ &= 5^2 + 2(5)(-\sqrt{2}) + (\sqrt{2})^2 \\ &= 25 - 10\sqrt{2} + 2 \\ &= 27 - 10\sqrt{2} \end{aligned}$$

b

$$\begin{aligned} &(7 + 2\sqrt{5})(7 - 2\sqrt{5}) \\ &= 7^2 - (2\sqrt{5})^2 \\ &= 49 - (4 \times 5) \\ &= 29 \end{aligned}$$

7 Simplify:

a $(3 + \sqrt{2})^2$
b $(6 - \sqrt{3})^2$
c $(3\sqrt{5} + 1)^2$
d $(7 - 2\sqrt{10})^2$
e $(3 + \sqrt{7})(3 - \sqrt{7})$
f $(\sqrt{2} + 5)(\sqrt{2} - 5)$
g $(2\sqrt{2} + 1)(2\sqrt{2} - 1)$
h $(9\sqrt{3} - 5)(9\sqrt{3} + 5)$
i $(\sqrt{3} + 1)^3$

B DIVISION BY SURDS

Numbers like $\frac{6}{\sqrt{2}}$ and $\frac{9}{5+\sqrt{2}}$ involve division by a surd.

It is customary to "simplify" these numbers by rewriting them without the surd in the denominator.

For any fraction of the form $\frac{b}{\sqrt{a}}$, we can **rationalise the denominator** by multiplying by $\frac{\sqrt{a}}{\sqrt{a}}$.

Since $\frac{\sqrt{a}}{\sqrt{a}} = 1$, this does not change the value of the number.

Example 7 Self Tutor

Write with an integer denominator:

a $\frac{6}{\sqrt{5}}$

b $\frac{35}{\sqrt{7}}$

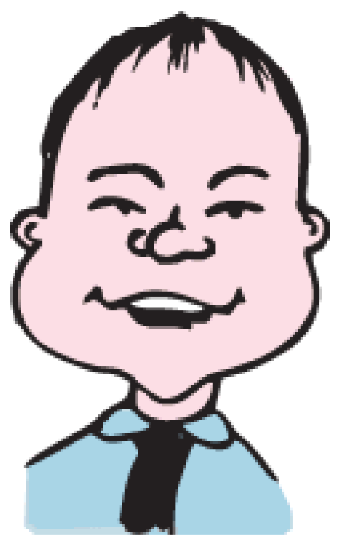
a

$$\frac{6}{\sqrt{5}}$$
$$= \frac{6}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$
$$= \frac{6\sqrt{5}}{5}$$

b

$$\frac{35}{\sqrt{7}}$$
$$= \frac{35}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$$
$$= \frac{\overset{5}{\cancel{35}}\sqrt{7}}{\underset{1}{\cancel{7}}}$$
$$= 5\sqrt{7}$$

Multiplying the original number by $\frac{\sqrt{5}}{\sqrt{5}}$ or $\frac{\sqrt{7}}{\sqrt{7}}$ does not change its value.



For any fraction of the form $\frac{c}{a+\sqrt{b}}$, we can remove the surd from the denominator by multiplying by $\frac{a-\sqrt{b}}{a-\sqrt{b}}$.

Expressions such as $a+\sqrt{b}$ and $a-\sqrt{b}$ are known as **radical conjugates**. They are identical except for the sign in front of the radical.

Example 8 Self Tutor

Write $\frac{5}{3-\sqrt{2}}$ with an integer denominator.

$$\frac{5}{3-\sqrt{2}} = \left(\frac{5}{3-\sqrt{2}}\right)\left(\frac{3+\sqrt{2}}{3+\sqrt{2}}\right)$$
$$= \frac{5(3+\sqrt{2})}{3^2-(\sqrt{2})^2} \quad \{\text{using the difference between two squares}\}$$
$$= \frac{15+5\sqrt{2}}{7}$$

The radical conjugate of $3-\sqrt{2}$ is $3+\sqrt{2}$.

EXERCISE 3B

1 Write with an integer denominator:

a $\dfrac{1}{\sqrt{3}}$ **b** $\dfrac{11}{\sqrt{3}}$ **c** $\dfrac{\sqrt{2}}{3\sqrt{3}}$ **d** $\dfrac{12}{\sqrt{2}}$

e $\dfrac{\sqrt{3}}{\sqrt{2}}$ **f** $\dfrac{1}{4\sqrt{2}}$ **g** $\dfrac{15}{\sqrt{5}}$ **h** $\dfrac{-3}{\sqrt{5}}$

i $\dfrac{1}{3\sqrt{5}}$ **j** $\dfrac{21}{\sqrt{7}}$ **k** $\dfrac{2}{\sqrt{11}}$ **l** $\dfrac{1}{(\sqrt{3})^3}$

2 Write with an integer denominator:

a $\dfrac{1}{3+\sqrt{2}}$ **b** $\dfrac{2}{3-\sqrt{2}}$ **c** $\dfrac{10}{\sqrt{6}-1}$ **d** $\dfrac{\sqrt{3}}{\sqrt{7}+2}$

e $\dfrac{1+\sqrt{2}}{1-\sqrt{2}}$ **f** $\dfrac{-2\sqrt{2}}{1-\sqrt{2}}$ **g** $\dfrac{1+\sqrt{5}}{2-\sqrt{5}}$ **h** $\dfrac{\sqrt{10}-7}{\sqrt{10}+4}$

i $\dfrac{3+\sqrt{5}}{4+\sqrt{5}}$ **j** $\dfrac{6-\sqrt{2}}{5-\sqrt{2}}$ **k** $\dfrac{\sqrt{7}+5}{\sqrt{7}-2}$ **l** $\dfrac{\sqrt{11}-3}{4-\sqrt{11}}$

3 Write in the form $a + b\sqrt{2}$:

a $\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$ **b** $\dfrac{5-\sqrt{2}}{6-\sqrt{2}}$ **c** $\dfrac{1}{(\sqrt{2}+1)^2}$ **d** $\dfrac{1}{(3-\sqrt{2})^2}$

e $\dfrac{1}{3+2\sqrt{2}}$ **f** $\dfrac{1}{2\sqrt{2}-7}$ **g** $\dfrac{\sqrt{2}+1}{(5+\sqrt{2})^2}$ **h** $\dfrac{1}{(3-\sqrt{2})^3}$

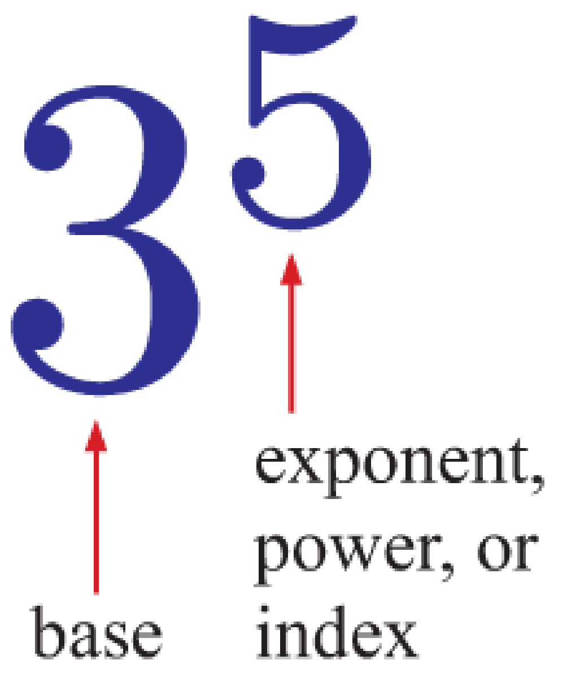
C EXPONENTS

Rather than writing $3 \times 3 \times 3 \times 3 \times 3$, we can write this product as 3^5.

If n is a positive integer, then a^n is the product of n factors of a.

$$\boldsymbol{a^n} = \underbrace{\boldsymbol{a} \times \boldsymbol{a} \times \boldsymbol{a} \times \boldsymbol{a} \times \; .... \; \times \boldsymbol{a}}_{n \text{ factors}}$$

We say that a is the **base**, and n is the **exponent**, **power**, or **index**.



NEGATIVE BASES

$(-1)^1 = -1$
$(-1)^2 = (-1) \times (-1) = 1$
$(-1)^3 = (-1) \times (-1) \times (-1) = -1$
$(-1)^4 = (-1) \times (-1) \times (-1) \times (-1) = 1$

$(-2)^1 = -2$
$(-2)^2 = (-2) \times (-2) = 4$
$(-2)^3 = (-2) \times (-2) \times (-2) = -8$
$(-2)^4 = (-2) \times (-2) \times (-2) \times (-2) = 16$

From the patterns above we can see that:

A **negative** base raised to an **odd** power is **negative**.
A **negative** base raised to an **even** power is **positive**.

EXERCISE 3C

1 List the first six powers of:

a 2 **b** 3 **c** 4

2 Simplify, then use a calculator to check your answer:

a $(-1)^5$ **b** $(-1)^6$ **c** $(-1)^{14}$
d $(-1)^{19}$ **e** $(-1)^8$ **f** -1^8
g $-(-1)^8$ **h** $(-2)^5$ **i** -2^5
j $-(-2)^6$ **k** $(-5)^4$ **l** $-(-5)^4$

GRAPHICS CALCULATOR INSTRUCTIONS

3 Use your calculator to find the value of the following, recording the entire display:

a 4^7 **b** 7^4 **c** -5^5 **d** $(-5)^5$ **e** 8^6
f $(-8)^6$ **g** -8^6 **h** 2.13^9 **i** -2.13^9 **j** $(-2.13)^9$

4 Use your calculator to evaluate the following. Comment on your results.

a 9^{-1} and $\frac{1}{9^1}$ **b** 6^{-2} and $\frac{1}{6^2}$
c 3^{-4} and $\frac{1}{3^4}$ **d** 17^0 and $(0.366)^0$

5 Consider $3^1, 3^2, 3^3, 3^4, 3^5, \ldots$. Look for a pattern and hence find the last digit of 3^{101}.

6 What is the last digit of 7^{217}?

7 **Nicomachus** was born in Roman Syria (now Jerash, Jordan) around 100 AD. He wrote in Greek and was a Pythagorean. He discovered an interesting number pattern involving cubes and the sums of odd numbers.
Find the series of odd numbers with sum equal to:

$$\begin{aligned} 1 &= 1^3 \\ 3+5 = 8 &= 2^3 \\ 7+9+11 = 27 &= 3^3 \\ &\vdots \end{aligned}$$

a 5^3 **b** 7^3 **c** 12^3

D LAWS OF EXPONENTS

When n is a positive integer, the notation a^n means a multiplied together n times.

From this definition, $a^m a^n = a^{m+n}$ and $(a^m)^n = a^{m \times n}$.

We observe that to transform a^n to a^{n+1}, we need to multiply by a.

So, to transform a^n to a^{n-1} we need to divide by a.

$\therefore$ we define $a^0 = \dfrac{a^1}{a} = 1$ to be consistent with the existing rules.

Dividing further by a, we find that $a^{-1} = \dfrac{1}{a}$, and more generally $a^{-n} = \dfrac{1}{a^n}$.

Using arguments like this, we arrive at the **laws of exponents** for $m, n \in \mathbb{Z}$:

$\boldsymbol{a^m \times a^n = a^{m+n}}$	To **multiply** numbers with the **same base**, keep the base and **add** the exponents.
$\boldsymbol{\dfrac{a^m}{a^n} = a^{m-n}, \quad a \neq 0}$	To **divide** numbers with the same base, keep the base and **subtract** the exponents.
$\boldsymbol{(a^m)^n = a^{m \times n}}$	When **raising** a **power** to a **power**, keep the base and **multiply** the exponents.
$\boldsymbol{(ab)^n = a^n b^n}$	The power of a product is the product of the powers.
$\boldsymbol{\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, \quad b \neq 0}$	The power of a quotient is the quotient of the powers.
$\boldsymbol{a^0 = 1, \quad a \neq 0}$	Any non-zero number raised to the power zero is 1.
$\boldsymbol{a^{-n} = \dfrac{1}{a^n}}$ and $\boldsymbol{\dfrac{1}{a^{-n}} = a^n}$ and in particular $\boldsymbol{a^{-1} = \dfrac{1}{a}, \quad a \neq 0}$.	

Example 9 **Self Tutor**

Simplify:

a $7^4 \times 7^5$ **b** $\dfrac{5^6}{5^3}$ **c** $(x^3)^k$

a $7^4 \times 7^5$
$= 7^{4+5}$
$= 7^9$

b $\dfrac{5^6}{5^3}$
$= 5^{6-3}$
$= 5^3$

c $(x^3)^k$
$= x^{3 \times k}$
$= x^{3k}$

EXERCISE 3D

1 Simplify:

a $k^4 \times k^2$
b $c^8 \times c^m$
c $r^2 \times r^5 \times r^4$
d $\dfrac{7^8}{7^3}$
e $\dfrac{m^{10}}{m^4}$
f $\dfrac{x^{3a}}{x^2}$
g $(7^6)^d$
h $(m^3)^t$
i $(11^x)^{2y}$
j $\dfrac{7^6}{7^n}$
k $(x^{2s})^3$
l $3^2 \times 3^7 \times 3^4$
m $(j^4)^{3x}$
n $\dfrac{z^7}{z^{4t}}$
o $(13^c)^{5d}$
p $w^{7p} \div w$
q $k^{5t} \div k^3$
r $\dfrac{(x^m)^3}{x^n}$

Example 10 **Self Tutor**

Simplify using the laws of exponents:

a $4x^3 \times 2x^6$ **b** $\dfrac{15t^7}{3t^5}$ **c** $\dfrac{k^2 \times k^6}{(k^3)^2}$

a
$$\begin{aligned} &4x^3 \times 2x^6 \\ &= 4 \times 2 \times x^3 \times x^6 \\ &= 8 \times x^{3+6} \\ &= 8x^9 \end{aligned}$$

b
$$\begin{aligned} &\frac{15t^7}{3t^5} \\ &= \tfrac{15}{3} \times t^{7-5} \\ &= 5t^2 \end{aligned}$$

c
$$\begin{aligned} &\frac{k^2 \times k^6}{(k^3)^2} \\ &= \frac{k^{2+6}}{k^{3\times 2}} \\ &= \frac{k^8}{k^6} \\ &= k^2 \end{aligned}$$

2 Simplify using the laws of exponents:

a $\dfrac{4b^5}{b^2}$ **b** $2w^4 \times 3w$ **c** $\dfrac{12p^4}{3p^2}$ **d** $5c^7 \times 6c^4$

e $\dfrac{d^2 \times d^7}{d^5}$ **f** $\dfrac{18a^2b^3}{6ab}$ **g** $\dfrac{24m^2n^4}{6m^2n}$ **h** $\dfrac{t^5 \times t^8}{(t^2)^3}$

i $5s^2t \times 4t^3$ **j** $\dfrac{(k^4)^5}{k^3 \times k^6}$ **k** $\dfrac{12x^2y^5}{8xy^2}$ **l** $\dfrac{(b^3)^4 \times b^5}{b^2 \times b^6}$

Example 11 **Self Tutor**

Write as a power of 2:

a 16 **b** $\frac{1}{16}$ **c** 1 **d** 4×2^n **e** $\dfrac{2^m}{8}$

a
$$\begin{aligned} &16 \\ &= 2 \times 2 \times 2 \times 2 \\ &= 2^4 \end{aligned}$$

b
$$\begin{aligned} &\tfrac{1}{16} \\ &= \frac{1}{2^4} \\ &= 2^{-4} \end{aligned}$$

c
$$\begin{aligned} &1 \\ &= 2^0 \end{aligned}$$

d
$$\begin{aligned} &4 \times 2^n \\ &= 2^2 \times 2^n \\ &= 2^{2+n} \end{aligned}$$

e
$$\begin{aligned} &\frac{2^m}{8} \\ &= \frac{2^m}{2^3} \\ &= 2^{m-3} \end{aligned}$$

3 Write as a power of 2:

a 4 **b** $\frac{1}{4}$ **c** 8 **d** $\frac{1}{8}$ **e** 32 **f** $\frac{1}{32}$

g 2 **h** $\frac{1}{2}$ **i** 64 **j** $\frac{1}{64}$ **k** 128 **l** $\frac{1}{128}$

4 Write as a power of 3:

a 9 **b** $\frac{1}{9}$ **c** 27 **d** $\frac{1}{27}$ **e** 3 **f** $\frac{1}{3}$

g 81 **h** $\frac{1}{81}$ **i** 1 **j** 243 **k** $\frac{1}{243}$

5 Write as a single power of 2:

a 2×2^a **b** 4×2^b **c** 8×2^t **d** $(2^{x+1})^2$ **e** $(2^{1-n})^{-1}$

f $\dfrac{2^c}{4}$ **g** $\dfrac{2^m}{2^{-m}}$ **h** $\dfrac{4}{2^{1-n}}$ **i** $\dfrac{2^{x+1}}{2^x}$ **j** $\dfrac{4^x}{2^{1-x}}$

6 Write as a single power of 3:

a 9×3^p **b** 27^a **c** 3×9^n **d** 27×3^d **e** 9×27^t

f $\frac{3^y}{3}$ **g** $\frac{3}{3^y}$ **h** $\frac{9}{27^t}$ **i** $\frac{9^a}{3^{1-a}}$ **j** $\frac{9^{n+1}}{3^{2n-1}}$

Example 12 **Self Tutor**

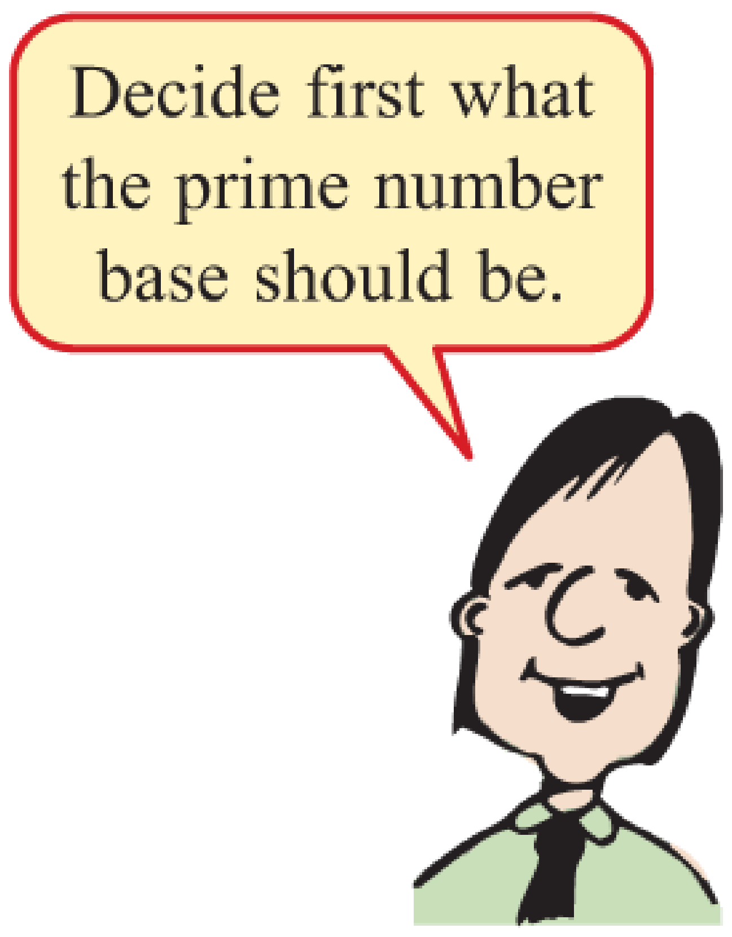
Express in exponent form with a prime number base:

a 9^4 **b** $\frac{3^x}{9^y}$ **c** 25^x

a 9^4
$= (3^2)^4$
$= 3^{2\times 4}$
$= 3^8$

b $\frac{3^x}{9^y}$
$= \frac{3^x}{(3^2)^y}$
$= \frac{3^x}{3^{2y}}$
$= 3^{x-2y}$

c 25^x
$= (5^2)^x$
$= 5^{2x}$



7 Express in exponent form with a prime number base:

a 32 **b** 25^3 **c** 16^p **d** $5^a \times 25$

e $4^n \times 8^n$ **f** $\frac{8^m}{16^n}$ **g** $\frac{25^p}{5^4}$ **h** 9^{t+2}

i 32^{2-r} **j** $\frac{81}{3^{y+1}}$ **k** $\frac{16^k}{4^k}$ **l** $\frac{5^{a+1} \times 125}{25^{2a}}$

Example 13 **Self Tutor**

Simplify:

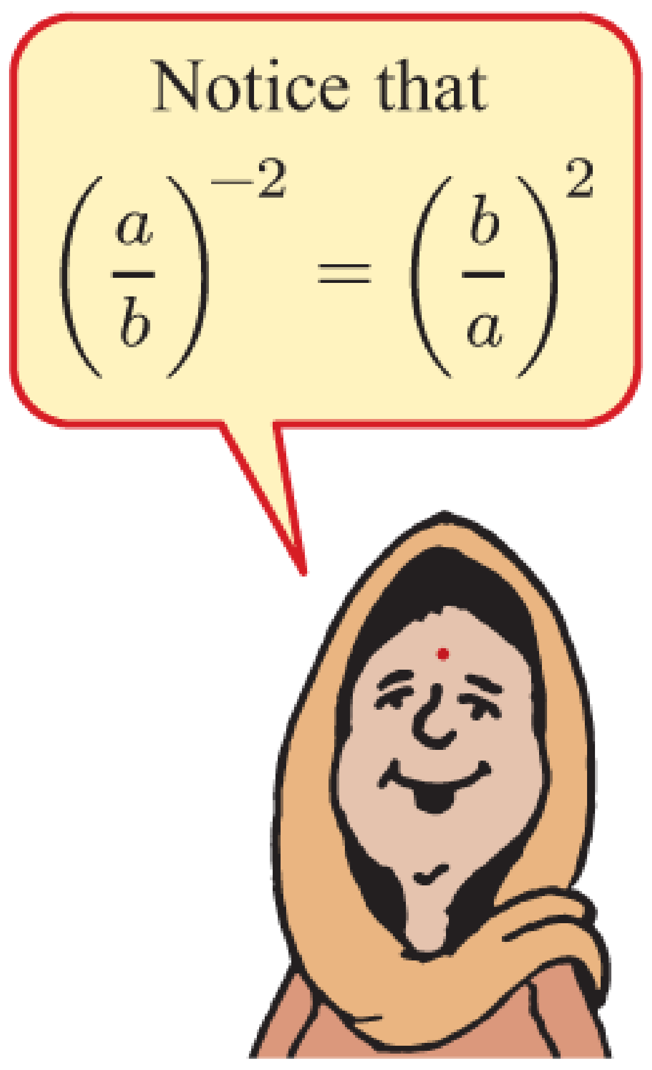
a 7^0 **b** 3^{-2} **c** $3^0 - 3^{-1}$ **d** $\left(\frac{5}{3}\right)^{-2}$

a 7^0
$= 1$

b 3^{-2}
$= \frac{1}{3^2}$
$= \frac{1}{9}$

c $3^0 - 3^{-1}$
$= 1 - \frac{1}{3}$
$= \frac{2}{3}$

d $\left(\frac{5}{3}\right)^{-2}$
$= \left(\frac{3}{5}\right)^2$
$= \frac{9}{25}$



8 Simplify:

a 4^0 **b** 3^{-1} **c** 7^{-2} **d** x^{-3}

e $5^0 + 5^{-1}$ **f** $\left(\frac{5}{3}\right)^0$ **g** $\left(\frac{7}{4}\right)^{-1}$ **h** $\left(\frac{1}{6}\right)^{-1}$

i $\left(\frac{4}{3}\right)^{-2}$ **j** $2^1 + 2^{-1}$ **k** $\left(1\frac{2}{3}\right)^{-3}$ **l** $5^2 + 5^1 + 5^{-1}$

9 Write using powers of 2, 3, and/or 5:

a $\frac{1}{9}$ **b** $\frac{1}{16}$ **c** $\frac{1}{125}$ **d** $\frac{3}{5}$

e $\frac{4}{27}$ **f** $\frac{2^c}{8 \times 9}$ **g** $\frac{9^k}{10}$ **h** $\frac{6^p}{75}$

Example 14 **Self Tutor**

Write in simplest form, without brackets:

a $(-3a^2)^4$ **b** $\left(-\dfrac{2a^2}{b}\right)^3$

a $$\begin{aligned}&(-3a^2)^4\\&=(-3)^4\times(a^2)^4\\&=81\times a^{2\times 4}\\&=81a^8\end{aligned}$$

b $$\begin{aligned}&\left(-\dfrac{2a^2}{b}\right)^3\\&=\dfrac{(-2)^3\times(a^2)^3}{b^3}\\&=\dfrac{-8a^6}{b^3}\end{aligned}$$

These have the form $(ab)^n = a^n b^n$ or $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

10 Write without brackets:

a $(2a)^2$ **b** $(3n)^2$ **c** $(5m)^3$ **d** $(mn)^3$

e $\left(\dfrac{a}{2}\right)^3$ **f** $\left(\dfrac{3}{m}\right)^2$ **g** $\left(\dfrac{p}{q}\right)^4$ **h** $\left(\dfrac{t}{5}\right)^2$

11 Write in simplest form, without brackets:

a $(2ab)^2$ **b** $(-2a)^2$ **c** $(6b^2)^2$ **d** $(-2a)^3$

e $(-3m^2n^2)^3$ **f** $(-2ab^4)^4$ **g** $\left(\dfrac{2a}{b}\right)^0$ **h** $\left(\dfrac{m}{3n}\right)^4$

i $\left(\dfrac{xy}{2}\right)^3$ **j** $\left(\dfrac{-2a^2}{b^2}\right)^3$ **k** $\left(\dfrac{-4a^3}{b}\right)^2$ **l** $\left(\dfrac{-3p^2}{q^3}\right)^2$

12 Expand the brackets and write in simplest form:

a $x^2(x^3+x)$ **b** $x^2(x^2-2x+3)$ **c** $x(x^2+1)(x^2-1)$

d $(x^3-x^2)(x^2+2)$ **e** $(x^3-x)^2$ **f** $x^2(x-2+x^{-1})$

g $x^{-1}(x^3+x^2-x)$ **h** $(x^2+x^{-1})^2$ **i** $(x^2+x^{-1})(x^2-x^{-1})$

Example 15 **Self Tutor**

Write without negative exponents:

a $(2c^3)^{-4}$ **b** $\dfrac{a^{-3}b^2}{c^{-1}}$

a $$\begin{aligned}(2c^3)^{-4}&=\dfrac{1}{(2c^3)^4}\\&=\dfrac{1}{2^4c^{3\times 4}}\\&=\dfrac{1}{16c^{12}}\end{aligned}$$

b $a^{-3}=\dfrac{1}{a^3}$ and $\dfrac{1}{c^{-1}}=c^1$

$$\therefore\ \dfrac{a^{-3}b^2}{c^{-1}}=\dfrac{b^2c}{a^3}$$

13 Write without negative exponents:

a ab^{-2} **b** $(ab)^{-2}$ **c** $(2ab^{-1})^2$ **d** $(5m^2)^{-2}$

e $(3a^{-2}b)^2$ **f** $(3xy^4)^{-3}$ **g** $\frac{a^2b^{-1}}{c^2}$ **h** $\frac{a^2b^{-1}}{c^{-2}}$

i $\frac{1}{a^{-3}}$ **j** $\frac{a^{-2}}{b^{-3}}$ **k** $\frac{2a^{-1}}{d^2}$ **l** $\frac{12a}{m^{-3}}$

Example 16 **Self Tutor**

Write $\frac{1}{2^{1-n}}$ without a fraction.

$$\frac{1}{2^{1-n}} = 2^{-(1-n)}$$
$$= 2^{-1+n}$$
$$= 2^{n-1}$$

14 Write without a fraction:

a $\frac{1}{a^n}$ **b** $\frac{5}{a^m}$ **c** $\frac{1}{b^{-n}}$ **d** $\frac{1}{2^{n-3}}$

e $\frac{1}{3^{2-n}}$ **f** $\frac{3}{a^{4-m}}$ **g** $\frac{a^n}{b^{-m}}$ **h** $\frac{a^{-n}}{a^{2+n}}$

15 Write without fractions:

a $\frac{1}{x^2}$ **b** $\frac{2}{x}$ **c** $x+\frac{1}{x}$ **d** $x^2-\frac{2}{x^3}$

e $\frac{1}{x}+\frac{3}{x^2}$ **f** $\frac{4}{x}-\frac{5}{x^3}$ **g** $7x-\frac{4}{x}+\frac{5}{x^2}$ **h** $\frac{3}{x}-\frac{2}{x^2}+\frac{5}{x^4}$

Example 17 **Self Tutor**

Write without fractions:

a $\frac{x^2+3x+2}{x}$ **b** $\frac{2x^5+x^2+3x}{x^{-2}}$

a
$$\frac{x^2+3x+2}{x}$$
$$= \frac{x^2}{x}+\frac{3x}{x}+\frac{2}{x}$$
$$= x+3+2x^{-1}$$

b
$$\frac{2x^5+x^2+3x}{x^{-2}}$$
$$= \frac{2x^5}{x^{-2}}+\frac{x^2}{x^{-2}}+\frac{3x}{x^{-2}}$$
$$= 2x^{5-(-2)}+x^{2-(-2)}+3x^{1-(-2)}$$
$$= 2x^7+x^4+3x^3$$

16 Write without fractions:

a $\frac{x+3}{x}$ **b** $\frac{3-2x}{x}$ **c** $\frac{5-x}{x^2}$ **d** $\frac{x+2}{x^3}$

e $\frac{x^2+5}{x}$ **f** $\frac{x^2+x-2}{x}$ **g** $\frac{2x^2-3x+4}{x}$ **h** $\frac{x^3-3x+5}{x^2}$

i $\frac{5-x-x^2}{x}$ **j** $\frac{8+5x-2x^3}{x}$ **k** $\frac{16-3x+x^3}{x^2}$ **l** $\frac{5x^4-3x^2+x+6}{x^2}$

17 Write without fractions:

a $\dfrac{4+2x}{x^{-1}}$ **b** $\dfrac{5-4x}{x^{-2}}$ **c** $\dfrac{6+3x}{x^{-3}}$ **d** $\dfrac{x^2+3}{x^{-1}}$

e $\dfrac{x^2+x-4}{x^{-2}}$ **f** $\dfrac{x^3-3x+6}{x^{-3}}$ **g** $\dfrac{x^3-6x+10}{x^{-2}}$ **h** $\dfrac{x^2+4+x^{-1}}{x^{-3}}$

E SCIENTIFIC NOTATION

Many people doing scientific work deal with very large or very small numbers. To avoid having to write and count lots of zeros, they write these numbers using **scientific notation** or **standard form**.

> **Scientific notation** or **standard form** involves writing any given number as a number between 1 inclusive and 10, multiplied by a power of 10.
>
> The result has the form $a \times 10^k$ where $1 \leqslant a < 10$ and $k \in \mathbb{Z}$.

Example 18 **Self Tutor**

Express in scientific notation:

a 37 600 **b** 0.000 008 6

a $37\,600 = 3.76 \times 10\,000$
$= 3.76 \times 10^4$
{shift decimal point 4 places to the left and $\times$ 10 000}

b $0.000\,008\,6 = 8.6 \div 10^6$
$= 8.6 \times 10^{-6}$
{shift decimal point 6 places to the right and $\div$ 1 000 000}

A number such as 4.62 is already between 1 and 10. We write it in scientific notation as 4.62×10^0 since $10^0 = 1$.

EXERCISE 3E

1 Which of the following numbers are *not* written in scientific notation?

A 3.7×10^4 **B** 4.2×10^{-7} **C** 0.3×10^5 **D** 21×10^{11}

2 Write in scientific notation:

a 259
b 259 000
c 2 590 000 000
d 2.59
e 0.259
f 0.000 259
g 40.7
h 4070
i 0.0407
j 407 000
k 407 000 000
l 0.000 040 7

3 Write each number in scientific notation:

a The mass of the R.M.S Titanic was approximately 47 450 000 kg.
b The ball bearing in a pen nib has diameter 0.003 m.
c There are about 2 599 000 different 5-card poker hands which can be dealt.
d The wavelength of blue light is about 0.000 000 47 m.

Example 19 **Self Tutor**

Write as an ordinary number:

a 3.2×10^2 **b** 5.76×10^{-5}

a 3.2×10^2
$= 3.20 \times 100$
$= 320$

b 5.76×10^{-5}
$= 000005.76 \div 10^5$
$= 0.000\,057\,6$

4 Write as an ordinary number:

a 4×10^3 **b** 3.8×10^5 **c** 8.6×10^1 **d** 4.33×10^7
e 4×10^{-3} **f** 3.8×10^{-5} **g** 8.6×10^{-1} **h** 4.33×10^{-7}

5 Write as a decimal number:

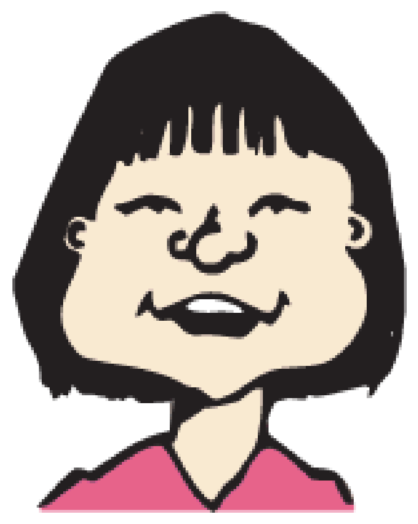
a It is estimated that the population of the world in 2020 will be 7.4×10^9 people.
b The mass of a Ryukyu mouse is 1.12×10^{-2} kg.
c The bacterium *bordetella pertussis* is about 5×10^{-7} m long.
d The Eiffel tower in Paris weighs approximately 7.3×10^6 kg.

6 Write these calculator displays in scientific notation and as decimal numbers:

a 4.5E07 **b** 3.8E-04 **c** 2.1E05 **d** 4.0E-03
e 6.1E03 **f** 1.6E-06 **g** 3.9E04 **h** 6.7E-02

GRAPHICS CALCULATOR INSTRUCTIONS

In an exam it is **not** acceptable to write your answer as a calculator display.

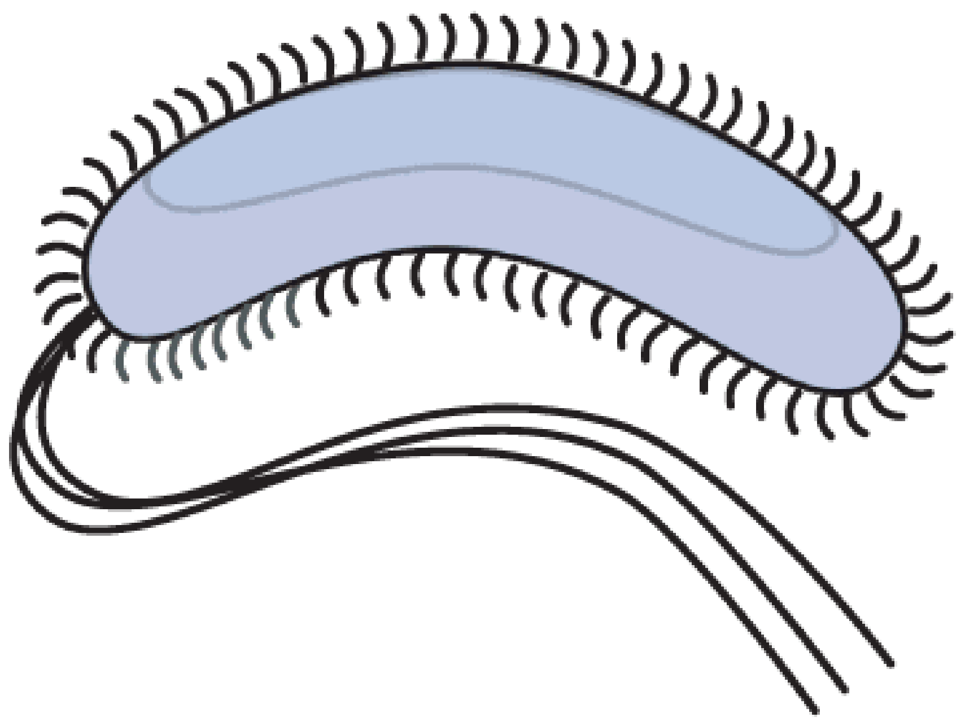


7 Use your calculator to evaluate the following, giving your answer in scientific notation:

a $680\,000 \times 73\,000\,000$ **b** $0.0006 \div 15\,000$ **c** $(0.0007)^3$
d $(3.42 \times 10^5) \times (4.8 \times 10^4)$ **e** $(6.42 \times 10^{-2})^2$ **f** $\dfrac{3.16 \times 10^{-10}}{6 \times 10^7}$
g $(9.8 \times 10^{-4}) \div (7.2 \times 10^{-6})$ **h** $\dfrac{1}{3.8 \times 10^5}$ **i** $(1.2 \times 10^3)^3$

8 Last year a peanut farmer produced 6×10^4 kg of peanuts. If the average weight of the peanuts was 8×10^{-4} kg, how many peanuts did the farm produce? Give your answer in scientific notation.

9 A bacterial cell is 4.6×10^{-7} m long. Behind it are flagella 2.15×10^{-6} m long which allow the bacterium to move. Find the total length of the bacterium.



10 The closest distance between Earth and Venus in their orbits is about 3.8×10^9 m.
The closest distance between Venus and Mercury in their orbits is about 7.7×10^9 m.
A spacecraft is sent from Earth to Venus, waits for the opportune time, then travels on to Mercury.

- **a** Find the minimum distance the spacecraft must travel.
- **b** Discuss the assumptions you have made in your answer.

11 In a vacuum, light travels at approximately 2.9979×10^8 m s^{-1}.

- **a** How far will light travel in:
  - **i** 1 minute
  - **ii** 1 day?
- **b** Assuming a year ≈ 365.25 days, calculate one **light-year**, the distance light will travel in a vacuum in a year.
- **c** Apart from our own sun, the closest star to Earth is Proxima Centauri, which is an average distance of 4.22 light-years from Earth. Write this length in metres.
- **d** The table alongside compares the sizes of some galaxies.
  - **i** Write the diameter of M87 in metres.
  - **ii** If Hercules A was mapped on a scale diagram 26 cm wide, what would be the scale factor of the diagram?
  - **iii** If a spaceship was able to travel at 100 000 km h^{-1}, how long would it take to cross the Milky Way?

Galaxy	*Diameter* (light-years)
Milky Way	100 000
M87	980 000
Hercules A	1 500 000

12 Answer the **Opening Problem** on page **54**.

DISCUSSION

The product of the first n integers can be written using **factorial notation**:

$$n! = 1 \times 2 \times 3 \times .... \times (n-1) \times n$$

When a Mahjong set is arranged ready for play, the total number of orders in which the tiles can be selected is $\frac{144!}{(4!)^{34}} \approx 6.563 \times 10^{202}$.

Can your calculator evaluate numbers this large? If not, how can we evaluate expressions like $\frac{144!}{(4!)^{34}}$?

REVIEW SET 3A

1 Simplify:

a $7\sqrt{5}-3\sqrt{5}$ **b** $2\sqrt{6}-\sqrt{54}$ **c** $5\sqrt{3}(4-\sqrt{3})$

d $(1+\sqrt{2})(2+\sqrt{2})$ **e** $(6-5\sqrt{2})^2$ **f** $(3+\sqrt{5})(3-\sqrt{5})$

2 Simplify: **a** $-(-1)^{10}$ **b** $-(-3)^3$ **c** 3^0-3^{-1}

3 Simplify: **a** $x^4\times x^2$ **b** $\left(2^{-1}\right)^7$ **c** $\left(ab^3\right)^6$

4 Write without negative exponents:

a 3^{-3} **b** $x^{-1}y$ **c** $\left(\frac{a}{b}\right)^{-1}$

5 Express in exponent form with a prime number base:

a 27 **b** 9^t **c** $\frac{4}{2^{m-1}}$

6 Simplify using the laws of exponents:

a $\frac{15xy^2}{3y^4}$ **b** $\frac{j^6}{j^5\times j^8}$ **c** $\frac{36g^3h^5}{12h^2}$

7 Express in simplest form, without brackets:

a $\left(\frac{t}{4s}\right)^3$ **b** $\left(\frac{m^2}{5n}\right)^0$ **c** $\left(5p^3q\right)^2$

8 Write without fractions:

a $\frac{x^2+8}{x}$ **b** $\frac{4+x+x^3}{x^{-2}}$ **c** $\frac{k^{-x}}{k^{x+6}}$

9 Simplify using the laws of exponents:

a $a^4b^5\times a^2b^2$ **b** $6xy^5\div 9x^2y^5$ **c** $\frac{5(x^2y)^2}{(5x^2)^2}$

10 Write as a decimal number:

a 4.6×10^{11} **b** 1.9×10^0 **c** 3.2×10^{-3}

11 Write in scientific notation:

a The diameter of the Earth is approximately 12.74 million metres.

b An extremely fine needle tip has width 0.000 12 m.

12 Sheets of paper are 3.2×10^{-4} m thick. How many sheets are required to make a pile of paper 10 cm high?

13 At a particular moment, Earth is 4.3×10^9 km from Neptune and 1.5×10^9 km from Saturn. How much further away is Neptune than Saturn?

REVIEW SET 3B

1 Simplify:

a $4\sqrt{11}-5\sqrt{11}$ **b** $\sqrt{32}-3\sqrt{2}$

c $(7+2\sqrt{3})(5-3\sqrt{3})$ **d** $(6+2\sqrt{2})(6-2\sqrt{2})$

2 Write with an integer denominator:

a $\frac{2}{\sqrt{3}}$ **b** $\frac{\sqrt{7}}{\sqrt{5}}$ **c** $\frac{3}{\sqrt{3}+2}$ **d** $\frac{1}{4+\sqrt{7}}$

3 Simplify:

a $\frac{m^9}{m^5}$ **b** y^0 **c** $\left(\frac{7z}{w}\right)^{-2}$

4 Simplify:

a $\frac{k^x}{k^2}$ **b** $11^r \times 11^{-4}$ **c** 9×3^b

5 Write without fractions:

a $\frac{a}{b^2}$ **b** $\frac{jk^4}{l^a}$ **c** $\frac{1}{x^3} - \frac{2}{x^5}$

6 Express in exponent form with a prime number base:

a $\frac{1}{16}$ **b** $3^k \times 81$ **c** $\frac{125^a}{5^b}$

7 Expand the brackets and write in simplest form:

a $x^{-1}(x^2 - 5x + 6)$ **b** $(x^3 + x^{-1})^2$ **c** $(x^2 - 3)(x + x^{-1})$

8 Express in simplest form, without brackets:

a $\left(\frac{2a^6}{8b^2}\right)^3$ **b** $\left(5d \times d^{-5}\right)^2$ **c** $\frac{16z^2 \times z^5}{(2z)^3}$

9 Write without brackets or negative exponents:

a $x^{-2} \times x^{-3}$ **b** $2(ab)^{-2}$ **c** $2ab^{-2}$

10 Write as a single power of 3:

a $\frac{27}{9^a}$ **b** $81^{1-x} \times 9^{1-2x}$

11 Express as a decimal number:

a Jupiter has a radius of 1.43×10^5 km.

b The ebola virus is about 8.2×10^{-8} m wide.

12 Sound travels along a telephone cable at 1.91×10^8 m s^{-1}. Find how long it takes Tetsuo's voice to travel from his office phone in Tokyo to:

a his wife's phone, 3740 m away

b his brother in Beijing, 2.1×10^6 m away.

13 Gold is special not just for its looks, but also its properties. Just 3 g of gold can be beaten out to form 1 m^2 of gold leaf approximately 1.8×10^{-7} m thick. By comparison, a US dime has a thickness of approximately 1.35×10^{-3} m. How many sheets of gold leaf would you need to create a stack the same height as the dime?

Chapter 4

Equations

Contents:

OPENING PROBLEM

Consider the equation $2x(x+4) = 8(x+k)$ where k is a constant.

Things to think about:

- **a** Suppose $k = 9$. Can you solve the equation:
  - **i** using algebra
  - **ii** by graphing each side of the equation on the same set of axes?
- **b** Can you show that for any value of k, the equation can be written in the form $x^2 = 4k$?
- **c** For what values of k does the equation have:
  - **i** two real solutions
  - **ii** one real solution
  - **iii** no real solutions?

In this Chapter we consider several types of equations and methods for finding their **solutions** or **roots**.

These methods are also used to find the **zeros** of an algebraic expression, which are the values of the variable for which the expression is equal to zero.

For example:
- The **roots** of $x^2 + x - 2 = 0$ are $x = 1$ and $x = -2$.
- The **zeros** of $x^2 + x - 2$ are 1 and -2.

We pay particular attention to quadratic equations, considering several methods for their solution. These are useful because we see many examples of quadratic equations in problem solving, and because the analysis of these equations teaches us principles we can apply elsewhere. We also solve equations using technology, giving us a tool for solving much harder problems.

A POWER EQUATIONS

A **power equation** is an equation which can be written in the form $x^n = k$, $n \neq 0$.

For example, $x^3 = 5$ and $2x^4 = 20$ are power equations.

You should have seen in previous years that:

$$\text{If } x^2 = k \text{ then } \begin{cases} x = \pm\sqrt{k} & \text{if } k > 0 \\ x = 0 & \text{if } k = 0 \\ \text{there are no real solutions if } k < 0. \end{cases}$$

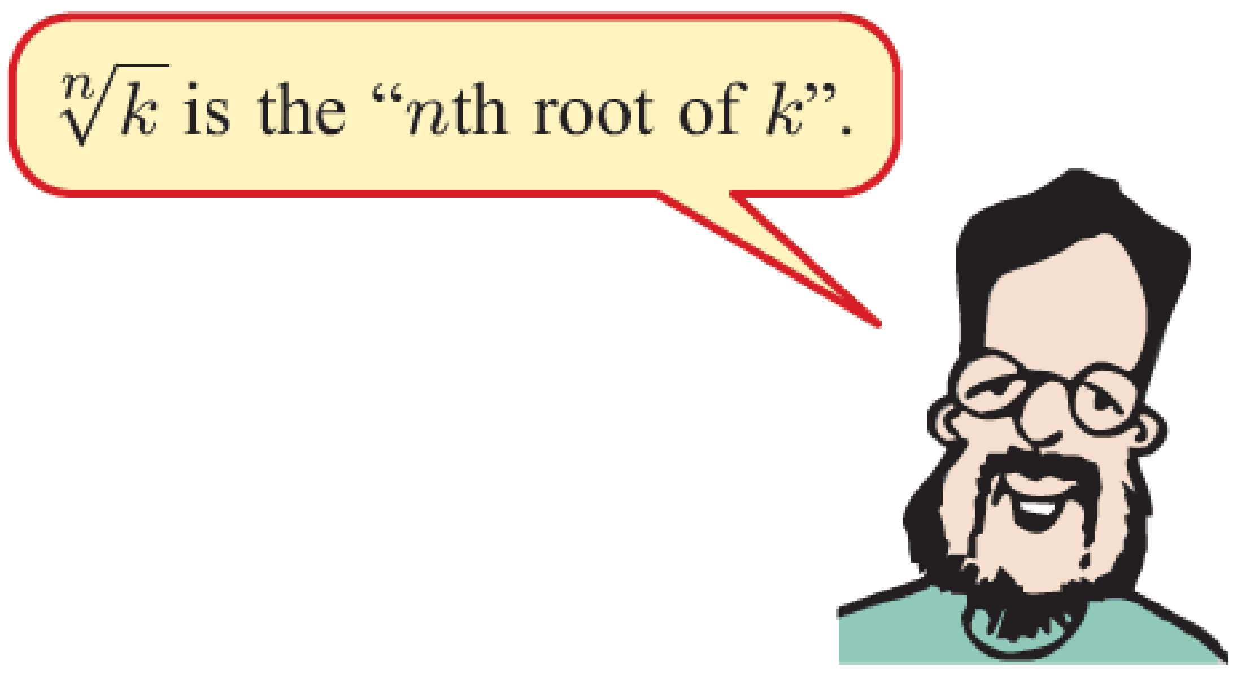
Notice that $x^2 = k$ has two real solutions if $k > 0$, one real solution if $k = 0$, and no real solutions if $k < 0$.

This is, in fact, true for all power equations with an *even power*.

If a power equation has an *odd power*, there will always be one real solution.

For $n > 0$, we write our solutions in terms of the nth root of k.

- If $x^n = k$ where n is **even**, then
$$\begin{cases} x = \pm\sqrt[n]{k} & \text{if } k > 0 \\ x = 0 & \text{if } k = 0 \\ \text{there are no real solutions if } k < 0. \end{cases}$$
- If $x^n = k$ where n is **odd**, then $x = \sqrt[n]{k}$.



Example 1

Self Tutor

Solve for x:

a $x^3 = -27$

b $x^4 + 5 = 15$

c $5 - 2x^2 = 11$

a
$$x^3 = -27$$
$$\therefore\ x = \sqrt[3]{-27}$$
$$\therefore\ x = -3$$

b
$$x^4 + 5 = 15$$
$$\therefore\ x^4 = 10$$
$$\therefore\ x = \pm\sqrt[4]{10}$$

c
$$5 - 2x^2 = 11$$
$$\therefore\ -2x^2 = 6$$
$$\therefore\ x^2 = -3$$

which has no real solutions as x^2 cannot be negative.

The same principle can be applied to equations written in the form $(x + a)^n = k$.

Example 2

Self Tutor

Solve for x:

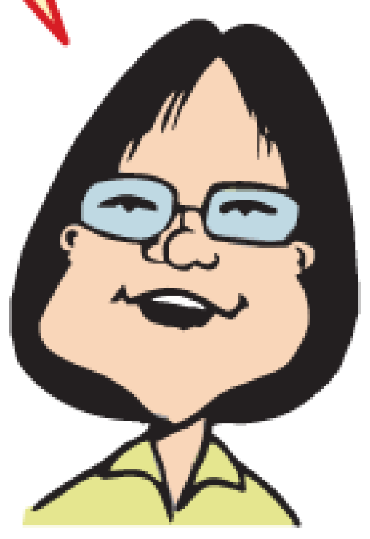
a $(x + 3)^2 = 36$

b $(x - 2)^5 = 11$

a
$$(x + 3)^2 = 36$$
$$\therefore\ x + 3 = \pm\sqrt{36}$$
$$\therefore\ x + 3 = \pm 6$$
$$\therefore\ x = -3 \pm 6$$
$$\therefore\ x = 3 \text{ or } -9$$

b
$$(x - 2)^5 = 11$$
$$\therefore\ x - 2 = \sqrt[5]{11}$$
$$\therefore\ x = 2 + \sqrt[5]{11}$$

For equations of the form $(x + a)^n = k$ we do not need to expand the brackets.



EXERCISE 4A

1 Solve for x:

a $3x^2 = 48$

b $5x^2 = 35$

c $2x^2 = -10$

d $6x^2 = 0$

e $4x^2 - 5 = 15$

f $7 - 3x^2 = 19$

2 Solve for x:

a $x^3 = 27$

b $x^4 = 16$

c $x^6 = -10$

d $x^5 = -13$

e $x^3 + 8 = 0$

f $2x^3 = 14$

g $5x^4 = 30$

h $x^3 = \frac{8}{27}$

i $x^4 = \frac{1}{16}$

j $3x^5 = 1$

k $4x^3 + 5 = -19$

l $2x^4 - 55 = 107$

3 Solve for x:

a $(x - 3)^2 = 16$

b $(x + 4)^2 = -25$

c $(x + 4)^2 = 13$

d $(x - 7)^2 = 0$

e $(2x - 3)^2 = 25$

f $\frac{1}{2}(3x + 1)^2 = 7$

g $(x - \sqrt{2})^2 = 2$

h $(2x - \sqrt{3})^2 = 2$

i $(2x + 1)^2 = 7$

4 Solve for x:

a $(x - 1)^3 = 17$

b $(x + 3)^5 = -1$

c $(x - 2)^4 = 20$

d $(x + 5)^4 = -16$

e $2(x + 4)^5 = -24$

f $(3x - 1)^6 = 1$

g $(2x - 3)^4 = 15$

h $\frac{1}{4}(1 - x)^5 = 8$

i $14 = 3 - \frac{1}{2}(4 - x)^3$

5 Find the zeros of:

a $3x^3 - 24$

b $(x - 1)^4 - 11$

c $(2x + 3)^5 + 1$

Example 3 **Self Tutor**

Solve for x: **a** $x^{-5} = -\frac{1}{32}$ **b** $x^{-4} = 256$

a $x^{-5} = -\frac{1}{32}$

$\therefore\ x^5 = -32$

$\therefore\ x = \sqrt[5]{-32}$

$\therefore\ x = -2$

b $x^{-4} = 256$

$\therefore\ x^4 = \frac{1}{256}$

$\therefore\ x = \pm\sqrt[4]{\frac{1}{256}}$

$\therefore\ x = \pm\frac{1}{4}$

For $n < 0$, remove the negative exponent using the appropriate law.

6 Solve for x:

a $x^{-1} = \frac{1}{6}$ **b** $x^{-2} = \frac{1}{9}$ **c** $x^{-3} = -\frac{1}{27}$ **d** $x^{-2} = 49$

e $x^{-4} = \frac{1}{16}$ **f** $x^{-3} = -64$ **g** $(x+1)^{-2} = -4$ **h** $(2x-5)^{-3} = \frac{1}{5}$

B EQUATIONS IN FACTORED FORM

An expression is **factored** if it is written as a product of factors.

For example, $(x^2+1)(x-1)$ is factored but $(x+1)(x-1)-3$ is not.

An equation is written in **factored form** if one side is fully factorised and the other side is zero.

For example, $ab = 0$, $x(x+1) = 0$, $(x+2)(x-3) = 0$ are all written in factored form.

If an equation is written in factored form, we can apply the **null factor law**:

> If the product of two (or more) numbers is zero then at least one of them must be zero.
>
> So, if $ab = 0$ then $a = 0$ or $b = 0$.

Example 4 **Self Tutor**

Solve for x using the null factor law:

a $3x(x-5) = 0$ **b** $(x-4)(3x+7) = 0$

a $3x(x-5) = 0$

$\therefore\ 3x = 0$ or $x - 5 = 0$

$\therefore\ x = 0$ or 5

b $(x-4)(3x+7) = 0$

$\therefore\ x - 4 = 0$ or $3x + 7 = 0$

$\therefore\ x = 4$ or $3x = -7$

$\therefore\ x = 4$ or $-\frac{7}{3}$

EXERCISE 4B

1 Solve using the null factor law:

a $4x = 0$ **b** $ab = 0$ **c** $2xy = 0$ **d** $3abc = 0$

2 Solve for x using the null factor law:

a $x(x-5) = 0$ **b** $2x(x+3) = 0$ **c** $(x+1)(x-3) = 0$

d $3x(7-x) = 0$ **e** $4(x+6)(2x-3) = 0$ **f** $(2x+1)(2x-1) = 0$

3 Solve for x using the null factor law:

a $x^2(x+5)=0$

b $4(5-x)^2=0$

c $-3(3x-1)^2=0$

d $-6(x-5)(3x+2)^2=0$

e $x(x+1)(x-2)=0$

f $3(x+2)(x+4)(2x-1)=0$

4 Solve, if possible, using the null factor law:

a $\frac{a}{b}=0$

b $\frac{3xy}{z}=0$

c $\frac{2}{xy}=0$

d $-\frac{x}{2y}=0$

C QUADRATIC EQUATIONS

A **quadratic equation** is an equation which can be written in the form $ax^2+bx+c=0$ where a, b, and c are constants, $a\neq 0$.

We have seen that a linear equation such as $2x+3=11$ will usually have *one* solution. In contrast, a quadratic equation may have *two*, *one*, or *zero* solutions.

Here are some quadratic equations which show the truth of this statement:

Equation	$ax^2+bx+c=0$ *form*	*Solutions*	*Number of solutions*
$(x+2)(x-2)=0$	$x^2+0x-4=0$	$x=2$ or $x=-2$	**two**
$(x-2)^2=0$	$x^2-4x+4=0$	$x=2$	**one**
$x^2=-4$	$x^2+0x+4=0$	none as x^2 is always $\geqslant 0$	**zero**

To solve quadratic equations we have the following methods to choose from:

- rewrite the quadratic into **factored form** then use the **null factor law**
- rewrite the quadratic into **completed square form** then solve $(x-h)^2=k$
- use the **quadratic formula**
- use **technology**.

SOLVING BY FACTORISATION

Step 1: If necessary, rearrange the equation so one side is zero.
Step 2: Fully factorise the other side.
Step 3: Use the null factor law: If $ab=0$ then $a=0$ or $b=0$.
Step 4: Solve the resulting linear equations.

Example 5 **Self Tutor**

Solve for x: **a** $3x^2+5x=0$ **b** $x^2=5x+6$

a
$$3x^2+5x=0$$
$$\therefore\ x(3x+5)=0$$
$$\therefore\ x=0 \text{ or } 3x+5=0$$
$$\therefore\ x=0 \text{ or } x=-\tfrac{5}{3}$$

b
$$x^2=5x+6$$
$$\therefore\ x^2-5x-6=0$$
$$\therefore\ (x-6)(x+1)=0$$
$$\therefore\ x=6 \text{ or } -1$$

Example 6 **Self Tutor**

Solve for x:

a $4x^2 + 1 = 4x$ **b** $6x^2 = 11x + 10$

a

$$\begin{aligned} 4x^2 + 1 &= 4x \\ \therefore\ 4x^2 - 4x + 1 &= 0 \\ \therefore\ (2x-1)^2 &= 0 \\ \therefore\ x &= \tfrac{1}{2} \end{aligned}$$

b

$$\begin{aligned} 6x^2 &= 11x + 10 \\ \therefore\ 6x^2 - 11x - 10 &= 0 \\ \therefore\ (2x-5)(3x+2) &= 0 \\ \therefore\ x &= \tfrac{5}{2} \text{ or } -\tfrac{2}{3} \end{aligned}$$

Caution:

- Do not be tempted to divide both sides by an expression involving x.
  If you do this then you may lose one of the solutions.
  For example, consider $x^2 = 5x$.

Correct solution

$$\begin{aligned} x^2 &= 5x \\ \therefore\ x^2 - 5x &= 0 \\ \therefore\ x(x-5) &= 0 \\ \therefore\ x &= 0 \text{ or } 5 \end{aligned}$$

Incorrect solution

$$\begin{aligned} x^2 &= 5x \\ \therefore\ \frac{x^{\cancel{2}}}{\cancel{x}} &= \frac{5\cancel{x}}{\cancel{x}} \\ \therefore\ x &= 5 \end{aligned}$$

We cannot divide by 0.
In dividing both sides by x, we assume $x \neq 0$.
For this reason, the solution $x = 0$ is lost.

- Be careful when taking square roots of both sides of an equation.
  If you do this then you may lose one of the solutions.
  For example, consider $(2x-7)^2 = (x+1)^2$.

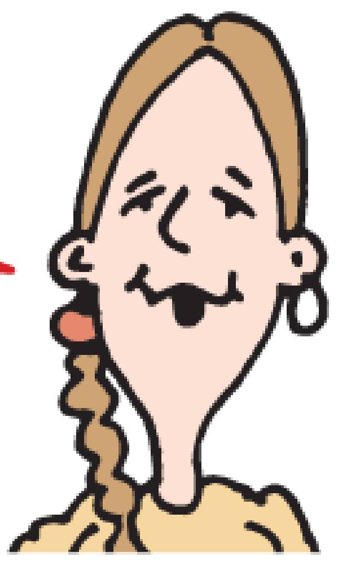
Correct solution

$$\begin{aligned} (2x-7)^2 &= (x+1)^2 \\ \therefore\ (2x-7)^2 - (x+1)^2 &= 0 \\ \therefore\ (2x-7+x+1)(2x-7-x-1) &= 0 \\ \therefore\ (3x-6)(x-8) &= 0 \\ \therefore\ x &= 2 \text{ or } 8 \end{aligned}$$

Incorrect solution

$$\begin{aligned} (2x-7)^2 &= (x+1)^2 \\ \therefore\ 2x-7 &= x+1 \\ \therefore\ x &= 8 \end{aligned}$$

If $a^2 = b^2$ then $a = \pm b$.
If we take the square root of both sides, we consider only the case $a = b$. The solution from $a = -b$ is lost.



EXERCISE 4C.1

1 Solve for x:

a $4x^2 + 7x = 0$ **b** $6x^2 + 2x = 0$ **c** $3x^2 - 7x = 0$
d $2x^2 - 11x = 0$ **e** $3x^2 = 8x$ **f** $9x = 6x^2$

2 Solve for x:

a $x^2 - 5x + 6 = 0$ **b** $x^2 - 2x + 1 = 0$ **c** $x^2 + 2x - 8 = 0$
d $x^2 + 7x + 12 = 0$ **e** $x^2 = 2x + 8$ **f** $x^2 + 21 = 10x$
g $9 + x^2 = 6x$ **h** $x^2 + x = 12$ **i** $x^2 + 8x = 33$
j $3x^2 + 9x = 12$ **k** $4x = 70 - 2x^2$ **l** $50 - 5x^2 = -15x$

3 Solve for x:

a $9x^2 - 12x + 4 = 0$
b $2x^2 - 13x - 7 = 0$
c $3x^2 = 16x + 12$
d $3x^2 + 5x = 2$
e $2x^2 + 3 = 5x$
f $3x^2 + 8x + 4 = 0$
g $3x^2 = 10x + 8$
h $4x^2 + 4x = 3$
i $4x^2 = 11x + 3$
j $12x^2 = 11x + 15$
k $7x^2 + 6x = 1$
l $15x^2 + 2x = 56$

Example 7 **Self Tutor**

Solve for x: $3x + \frac{2}{x} = -7$

$$\begin{aligned} 3x + \frac{2}{x} &= -7 \\ \therefore\ 3x^2 + 2 &= -7x && \{\text{multiplying both sides by } x\} \\ \therefore\ 3x^2 + 7x + 2 &= 0 && \{\text{making the RHS} = 0\} \\ \therefore\ (x+2)(3x+1) &= 0 && \{\text{factorising}\} \\ \therefore\ x &= -2 \text{ or } -\tfrac{1}{3} \end{aligned}$$

4 Solve for x:

a $(x+1)^2 = 2x^2 - 5x + 11$
b $(x+2)(1-x) = -4$
c $5 - 4x^2 = 3(2x+1) + 2$
d $x + \frac{2}{x} = 3$
e $2x - \frac{1}{x} = -1$
f $\frac{x+3}{1-x} = -\frac{9}{x}$
g $(x+3)(2-x) = 4$
h $(x-4)(x+2) = 16$
i $(x-5)(x+3) = 20$
j $(4x-5)(4x-3) = 143$

LEARNING ALGEBRA

SOLVING BY "COMPLETING THE SQUARE"

As you would be aware by now, not all quadratics factorise easily.

For example, $x^2 + 4x + 1$ cannot be factorised by easily identifying factors. In particular, we cannot write $x^2 + 4x + 1$ in the form $(x-a)(x-b)$ where a and b are rational numbers.

An alternative method is to rewrite the equation in the form $(x-h)^2 = k$. We refer to this process as "completing the square".

Start with the quadratic equation in the form $ax^2 + bx + c = 0$.

Step 1: If $a \neq 1$, divide both sides by a.

Step 2: Rearrange the equation so that only the constant is on the RHS.

Step 3: Add to both sides $\left(\frac{\text{coefficient of } x}{2}\right)^2$.

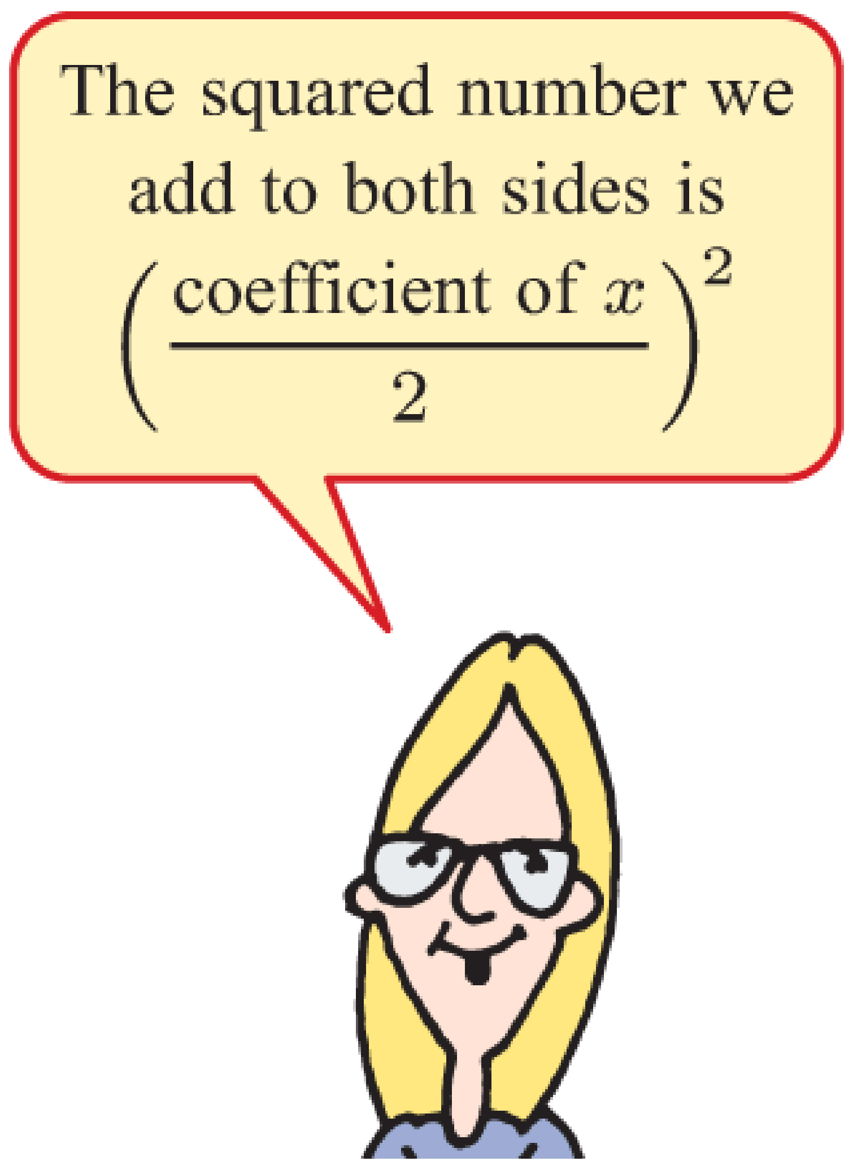
Step 4: Factorise the LHS.

Step 5: Use the rule: If $X^2 = k$ then $X = \pm\sqrt{k}$.

Example 8 **Self Tutor**

Solve exactly for x: $x^2 + 4x + 1 = 0$

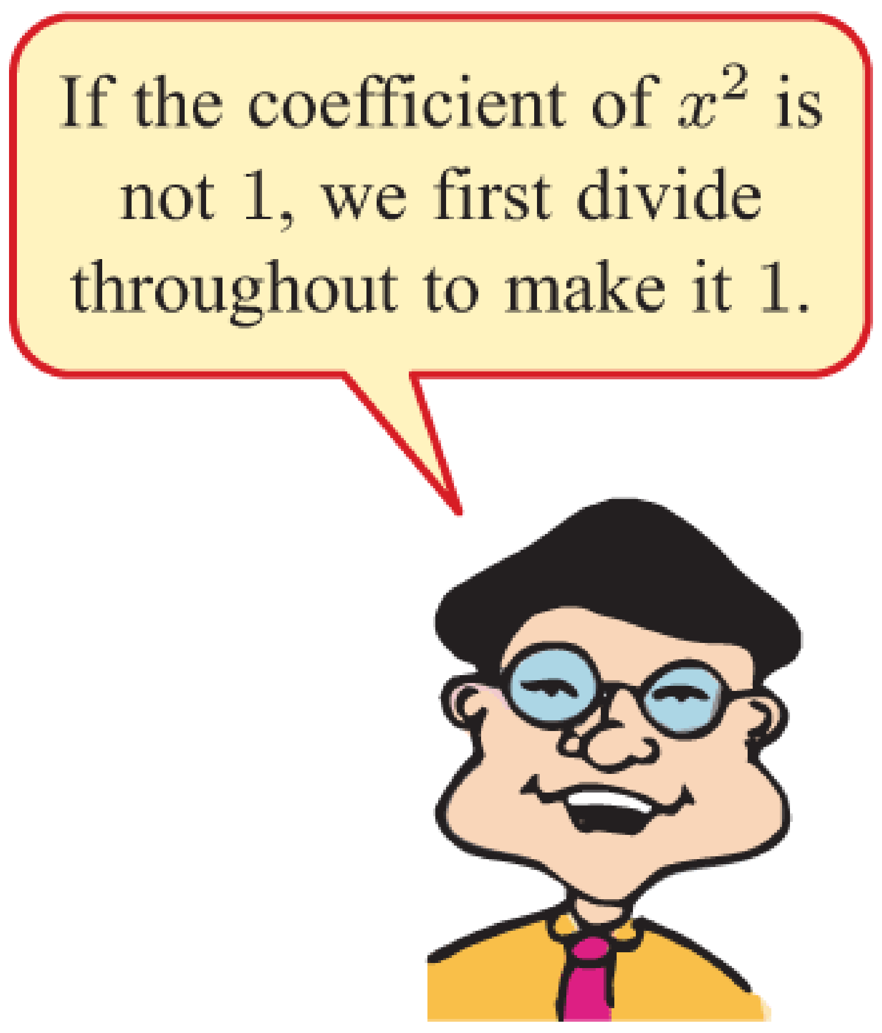
$$\begin{aligned}
x^2 + 4x + 1 &= 0 \\
\therefore\ x^2 + 4x &= -1 && \{\text{writing the constant on the RHS}\} \\
\therefore\ x^2 + 4x + 2^2 &= -1 + 2^2 && \{\text{completing the square}\} \\
\therefore\ (x+2)^2 &= 3 && \{\text{factorising the LHS}\} \\
\therefore\ x + 2 &= \pm\sqrt{3} \\
\therefore\ x &= -2 \pm \sqrt{3}
\end{aligned}$$



Example 9 **Self Tutor**

Solve exactly for x: $-3x^2 + 12x + 5 = 0$

$$\begin{aligned}
-3x^2 + 12x + 5 &= 0 \\
\therefore\ x^2 - 4x - \tfrac{5}{3} &= 0 && \{\text{dividing both sides by } -3\} \\
\therefore\ x^2 - 4x &= \tfrac{5}{3} && \{\text{writing the constant on the RHS}\} \\
\therefore\ x^2 - 4x + (-2)^2 &= \tfrac{5}{3} + (-2)^2 && \{\text{completing the square}\} \\
\therefore\ (x-2)^2 &= \tfrac{17}{3} && \{\text{factorising the LHS}\} \\
\therefore\ x - 2 &= \pm\sqrt{\tfrac{17}{3}} \\
\therefore\ x &= 2 \pm \sqrt{\tfrac{17}{3}}
\end{aligned}$$



EXERCISE 4C.2

1 Solve exactly by completing the square:

a $x^2 - 4x + 1 = 0$ **b** $x^2 + 6x + 2 = 0$ **c** $x^2 - 14x + 46 = 0$

d $x^2 = 4x + 3$ **e** $x^2 + 6x + 7 = 0$ **f** $x^2 = 2x + 6$

g $x^2 + 6x = 2$ **h** $x^2 + 10 = 8x$ **i** $x^2 + 6x = -11$

2 Solve exactly by completing the square:

a $2x^2 + 4x + 1 = 0$ **b** $2x^2 - 10x + 3 = 0$ **c** $3x^2 + 12x + 5 = 0$

d $3x^2 = 6x + 4$ **e** $5x^2 - 15x + 2 = 0$ **f** $4x^2 + 4x = 5$

3 Solve for x:

a $3x - \dfrac{2}{x} = 4$ **b** $1 - \dfrac{1}{x} = -5x$ **c** $3 + \dfrac{1}{x^2} = -\dfrac{5}{x}$

4 Suppose $ax^2 + bx + c = 0$ where a, b, and c are constants, $a \neq 0$.
Solve for x by completing the square.

LEARNING ALGEBRA

THE QUADRATIC FORMULA

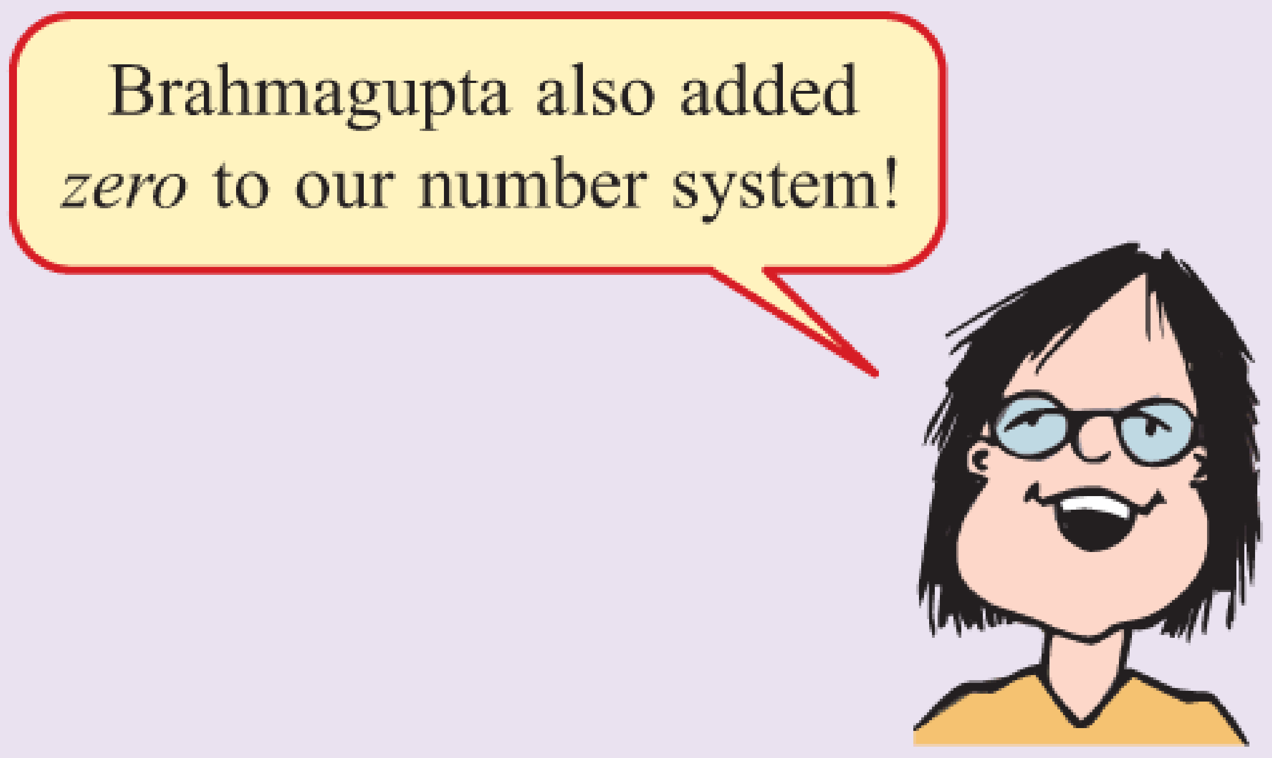
HISTORICAL NOTE THE QUADRATIC FORMULA

Thousands of years ago, people knew how to calculate the area of a rectangular shape given its side lengths. When they wanted to find the side lengths necessary to give a certain area, however, they ended up with a quadratic equation which they needed to solve.

The first known solution of a quadratic equation is written on the Berlin Papyrus from the Middle Kingdom (2160 - 1700 BC) in Egypt. By 400 BC, the Babylonians were using the method of "completing the square".

Pythagoras and **Euclid** both used geometric methods to explore the problem. Pythagoras noted that the square root was not always an integer, but he refused to accept that irrational solutions existed. Euclid also discovered that the square root was not always rational, but concluded that irrational numbers *did* exist.

A major jump forward was made in India around 700 AD, when Hindu mathematician **Brahmagupta** devised a general (but incomplete) solution for the quadratic equation $ax^2 + bx = c$ which was equivalent to $x = \dfrac{\sqrt{4ac + b^2} - b}{2a}$. Taking into account the sign of c, this is one of the two solutions we know today.

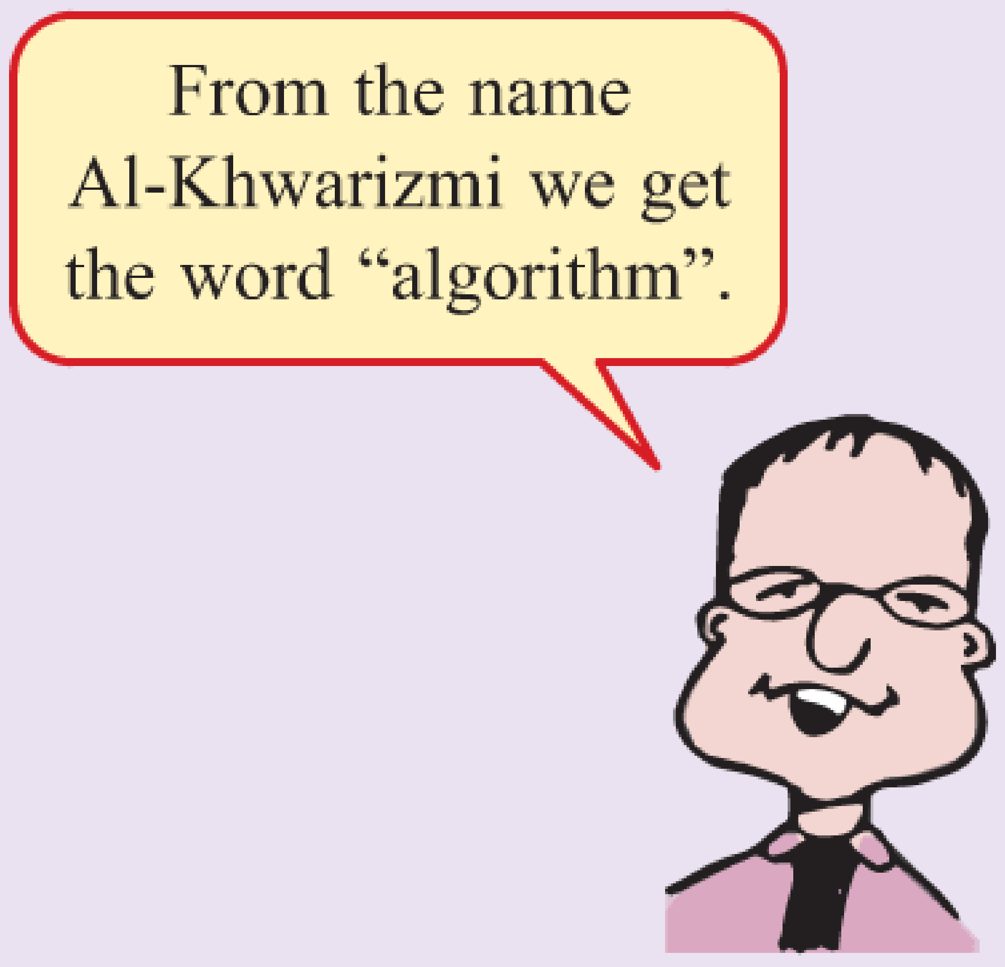


The final, complete solution as we know it today first came around 1100 AD, by another Hindu mathematician called **Bhaskhara**. He was the first to recognise that any positive number has two square roots, which could be negative or irrational. In fact, the quadratic formula is known in some countries today as "Bhaskhara's Formula".

While the Indians had knowledge of the quadratic formula even at this early stage, it took somewhat longer for the quadratic formula to arrive in Europe.

Around 820 AD, the Islamic mathematician **Muhammad bin Musa Al-Khwarizmi**, who was familiar with the work of Brahmagupta, recognised that for a quadratic equation to have real solutions, the value $b^2 - 4ac$ could not be negative.

Al-Khwarizmi's work was brought to Europe by the Jewish mathematician and astronomer **Abraham bar Hiyya** (also known as Savasorda) who lived in Barcelona around 1100 AD.



By 1545, **Girolamo Cardano** had blended the algebra of Al-Khwarizmi with Euclidean geometry. His work allowed for the existence of roots which are not real, as well as negative and irrational roots.

At the end of the 16th Century the mathematical notation and symbolism was introduced by **François Viète** in France.

In 1637, when **René Descartes** published *La Géométrie*, the quadratic formula adopted the form we see today.

In many cases, factorising a quadratic or completing the square can be long or difficult. We can instead use the **quadratic formula**:

$$\text{If}\quad \boldsymbol{ax^2+bx+c=0,\quad a\neq 0,}\quad \text{then}\quad \boldsymbol{x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}}.$$

Proof:

$$\begin{aligned}
&\text{If}\quad ax^2+bx+c=0,\quad a\neq 0\\
&\text{then}\quad x^2+\frac{b}{a}x+\frac{c}{a}=0 && \{\text{dividing each term by } a, \text{ as } a\neq 0\}\\
&\therefore\ x^2+\frac{b}{a}x = -\frac{c}{a}\\
&\therefore\ x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2 = -\frac{c}{a}+\left(\frac{b}{2a}\right)^2 && \{\text{completing the square}\}\\
&\therefore\ \left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2} && \{\text{factorising}\}\\
&\therefore\ x+\frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}\\
&\therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

Example 10 **Self Tutor**

Solve for x:

a $x^2-2x-6=0$

b $2x^2+3x-6=0$

a $x^2-2x-6=0$ has
$a=1,\ b=-2,\ c=-6$

$$\therefore\ x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-6)}}{2(1)}$$

$$\therefore\ x=\frac{2\pm\sqrt{28}}{2}$$

$$\therefore\ x=\frac{2\pm2\sqrt{7}}{2}=1\pm\sqrt{7}$$

b $2x^2+3x-6=0$ has
$a=2,\ b=3,\ c=-6$

$$\therefore\ x=\frac{-3\pm\sqrt{3^2-4(2)(-6)}}{2(2)}$$

$$\therefore\ x=\frac{-3\pm\sqrt{57}}{4}$$

EXERCISE 4C.3

1 Use the quadratic formula to solve exactly for x:

a $x^2-4x-3=0$
b $x^2+6x+7=0$
c $x^2+1=4x$
d $x^2+4x=1$
e $x^2-4x+2=0$
f $2x^2-2x-3=0$
g $3x^2-5x-1=0$
h $-x^2+4x+6=0$
i $-2x^2+7x-2=0$

2 Rearrange the following equations so they are written in the form $ax^2+bx+c=0$, then use the quadratic formula to solve exactly for x.

a $(x+2)(x-1)=2-3x$
b $(2x+1)^2=3-x$
c $(x-2)^2=1+x$
d $(3x+1)^2=-2x$
e $(x+3)(2x+1)=9$
f $(2x+3)(2x-3)=x$
g $\dfrac{x-1}{2-x}=2x+1$
h $x-\dfrac{1}{x}=1$
i $2x-\dfrac{1}{x}=3$

THE DISCRIMINANT OF A QUADRATIC

We can determine how many real solutions a quadratic equation has, without actually solving the equation.

In the quadratic formula, the quantity $b^2 - 4ac$ under the square root sign is called the **discriminant**.

The symbol **delta** Δ is used to represent the discriminant, so $\boxed{\Delta = b^2 - 4ac.}$

The quadratic formula becomes $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ where Δ replaces $b^2 - 4ac$.

- If $\Delta > 0$, $\sqrt{\Delta}$ is a positive real number, so there are **two distinct real roots**
  $$x = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{\Delta}}{2a}.$$
- If $\Delta = 0$, $x = \dfrac{-b}{2a}$ is the **only solution**, which we call a **repeated root**.
- If $\Delta < 0$, $\sqrt{\Delta}$ is not a real number and so there are **no real roots**.
- If a, b, and c are rational and Δ is a **square** then the equation has two rational roots which can be found by factorisation.

Example 11 **Self Tutor**

Use the discriminant to determine the nature of the roots of:

a $2x^2 - 2x + 3 = 0$ **b** $3x^2 - 4x - 2 = 0$

a $\Delta = b^2 - 4ac$
$= (-2)^2 - 4(2)(3)$
$= -20$

Since $\Delta < 0$, there are no real roots.

b $\Delta = b^2 - 4ac$
$= (-4)^2 - 4(3)(-2)$
$= 40$

Since $\Delta > 0$, but 40 is not a square, there are 2 distinct irrational roots.

EXERCISE 4C.4

1 Consider the quadratic equation $x^2 - 7x + 9 = 0$.
   - **a** Find the discriminant.
   - **b** Hence state the nature of the roots of the equation.
   - **c** Check your answer to **b** by solving the equation.

2 Consider the quadratic equation $4x^2 - 4x + 1 = 0$.
   - **a** Find the discriminant.
   - **b** Hence state the nature of the roots of the equation.
   - **c** Check your answer to **b** by solving the equation.

3
   - **a** Without using the discriminant, explain why the equation $x^2 + 5 = 0$ has no real roots.
   - **b** Check that $\Delta < 0$ for this equation.

4 Using the discriminant only, state the nature of the solutions of:
   - **a** $x^2 + 7x - 3 = 0$
   - **b** $x^2 - 3x + 2 = 0$
   - **c** $3x^2 + 2x - 1 = 0$
   - **d** $5x^2 + 4x - 3 = 0$
   - **e** $x^2 + x + 5 = 0$
   - **f** $16x^2 - 8x + 1 = 0$

5 Using the discriminant only, determine which of the following quadratic equations have rational roots which can be found by factorisation.

a $6x^2 - 5x - 6 = 0$ **b** $2x^2 - 7x - 5 = 0$ **c** $3x^2 + 4x + 1 = 0$

d $6x^2 - 47x - 8 = 0$ **e** $4x^2 - 3x + 2 = 0$ **f** $8x^2 + 2x - 3 = 0$

Example 12 **Self Tutor**

Consider $x^2 - 2x + m = 0$. Find the discriminant Δ, and hence find the values of m for which the equation has:

a a repeated root **b** two distinct real roots **c** no real roots.

$x^2 - 2x + m = 0$ has $a = 1$, $b = -2$, and $c = m$

$$\begin{aligned} \therefore \quad \Delta &= b^2 - 4ac \\ &= (-2)^2 - 4(1)(m) \\ &= 4 - 4m \end{aligned}$$

a For a repeated root

$$\begin{aligned} \Delta &= 0 \\ \therefore \quad 4 - 4m &= 0 \\ \therefore \quad 4 &= 4m \\ \therefore \quad m &= 1 \end{aligned}$$

b For two distinct real roots

$$\begin{aligned} \Delta &> 0 \\ \therefore \quad 4 - 4m &> 0 \\ \therefore \quad -4m &> -4 \\ \therefore \quad m &< 1 \end{aligned}$$

c For no real roots

$$\begin{aligned} \Delta &< 0 \\ \therefore \quad 4 - 4m &< 0 \\ \therefore \quad -4m &< -4 \\ \therefore \quad m &> 1 \end{aligned}$$

6 For each of the following quadratic equations, find the discriminant Δ in simplest form. Hence find the values of m for which the equation has:

i a repeated root **ii** two distinct real roots **iii** no real roots.

a $x^2 + 4x + m = 0$ **b** $mx^2 + 3x + 2 = 0, \quad m \neq 0$ **c** $mx^2 - 3x + 1 = 0, \quad m \neq 0$

7 The quadratic equation $4x^2 + kx + (3 - k) = 0$ has a repeated root.
Find the possible values of k, and the repeated root in each case.

THE SUM AND PRODUCT OF THE ROOTS

If $ax^2 + bx + c = 0$ has roots α and β, then $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$.

Proof:

Let α and β be the roots of $ax^2 + bx + c = 0$.

$$\begin{aligned} \therefore \quad ax^2 + bx + c &= a(x - \alpha)(x - \beta) \\ &= a(x^2 - [\alpha + \beta]x + \alpha\beta) \end{aligned}$$

$$\therefore \quad x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - [\alpha + \beta]x + \alpha\beta$$

Equating coefficients, $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$.

Example 13 **Self Tutor**

Find the sum and product of the roots of $25x^2 - 20x + 1 = 0$.
Check your answer by solving the quadratic.

If α and β are the roots then $\alpha+\beta=-\dfrac{b}{a}=\dfrac{20}{25}=\dfrac{4}{5}$

and $\alpha\beta=\dfrac{c}{a}=\dfrac{1}{25}$

Check: $25x^2-20x+1=0$ has roots

$$\frac{20\pm\sqrt{400-4(25)(1)}}{50}=\frac{20\pm\sqrt{300}}{50}=\frac{20\pm10\sqrt{3}}{50}=\frac{2\pm\sqrt{3}}{5}$$

These have sum $=\dfrac{2+\sqrt{3}}{5}+\dfrac{2-\sqrt{3}}{5}=\dfrac{4}{5}$ ✓

and product $=\left(\dfrac{2+\sqrt{3}}{5}\right)\left(\dfrac{2-\sqrt{3}}{5}\right)=\dfrac{4-3}{25}=\dfrac{1}{25}$ ✓

EXERCISE 4C.5

1 For each of the following quadratic equations:

i Find the sum and product of the roots.

ii Check your answer by solving the quadratic.

a $x^2+4x-21=0$

b $x^2-5x+5=0$

c $4x^2-12x+5=0$

d $3x^2-4x-2=0$

2 For the equation $kx^2-(1+k)x+(3k+2)=0$, the sum of the roots is twice their product. Find k and the two roots.

3 The quadratic equation $ax^2-6x+a-2=0$, $a\neq0$, has one root which is double the other.

a Let the roots be α and 2α. Hence find two equations involving α.

b Find a and the two roots of the quadratic equation.

4 The quadratic equation $kx^2+(k-8)x+(1-k)=0$, $k\neq0$, has one root which is two more than the other. Find k and the two roots.

Example 14

Self Tutor

The roots of the equation $4x^2+5x-1=0$ are α and β.
Find a quadratic equation with roots 3α and 3β.

If α and β are the roots of $4x^2+5x-1=0$, then $\alpha+\beta=-\frac{5}{4}$ and $\alpha\beta=-\frac{1}{4}$.

For the quadratic equation with roots 3α and 3β,

the sum of the roots $=3\alpha+3\beta$
$=3(\alpha+\beta)$
$=3(-\frac{5}{4})$
$=-\frac{15}{4}$

and the product of the roots $=(3\alpha)(3\beta)$
$=9\alpha\beta$
$=9(-\frac{1}{4})$
$=-\frac{9}{4}$

So, we have $-\dfrac{b}{a}=-\dfrac{15}{4}$ and $\dfrac{c}{a}=-\dfrac{9}{4}$.

The simplest solution is $a=4$, $b=15$, $c=-9$

$\therefore$ the quadratic equation is $4x^2+15x-9=0$.

All quadratics of the form $k(4x^2+15x-9)=0$, $k\in\mathbb{R}$, $k\neq0$, have roots 3α and 3β.

5 Find a quadratic equation with roots:

a 3 and -5

b $2+\sqrt{3}$ and $2-\sqrt{3}$.

6 The roots of the equation $3x^2+2x-4=0$ are α and β.
Find a quadratic equation with roots:

a $-\alpha$ and $-\beta$

b 2α and 2β.

7 The roots of the equation $-2x^2+5x+1=0$ are α and β.

a Find a quadratic equation with roots $\alpha+2$ and $\beta+2$.

b Find *all* quadratic equations with roots $\frac{\alpha}{2}$ and $\frac{\beta}{2}$.

8 The roots of the equation $x^2-4x-11=0$ are α and β.
Find *all* quadratic equations with roots $\alpha\beta$ and $\alpha+\beta$.

9 The roots of $2x^2+5x-9=0$ are α and β.
Find *all* quadratic equations with roots α^2 and β^2.

10 The roots of the equation $x^2-6x+7=0$ are α and β.
Find a quadratic equation with roots $\alpha+\frac{1}{\beta}$ and $\beta+\frac{1}{\alpha}$.

11 The roots of $2x^2-3x-5=0$ are p and q.
Find *all* quadratic equations with roots p^2+q and q^2+p.

D SOLVING POLYNOMIAL EQUATIONS USING TECHNOLOGY

We have already seen that:

- a **linear** equation can be written in the form $ax+b=0$, $a\neq 0$
- a **quadratic** equation can be written in the form $ax^2+bx+c=0$, $a\neq 0$.

These are the first two members of a family called **polynomial equations**.

The next two members of the family are:

- a **cubic** equation which can be written in the form $ax^3+bx^2+cx+d=0$, $a\neq 0$
- a **quartic** equation which can be written in the form $ax^4+bx^3+cx^2+dx+e=0$, $a\neq 0$.

The highest power of x in a polynomial equation is called its **degree**.

If a polynomial equation has degree n then it may have up to n real solutions.

You can use your graphics calculator to solve polynomial equations. You may need to first rearrange them into polynomial form.

Some calculator models will show exact solutions as well as numerical approximations.

GRAPHICS CALCULATOR INSTRUCTIONS

Example 15

Self Tutor

Use technology to solve:

a $2x^2 + 4x = 7$

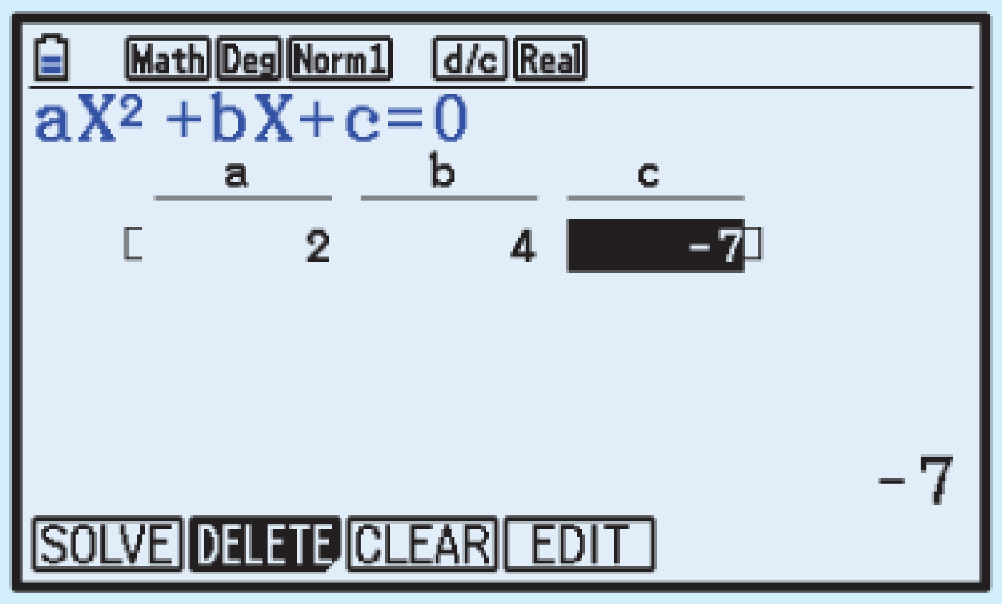
b $x^3 + 4x^2 = 6 - x$

a $2x^2 + 4x = 7$

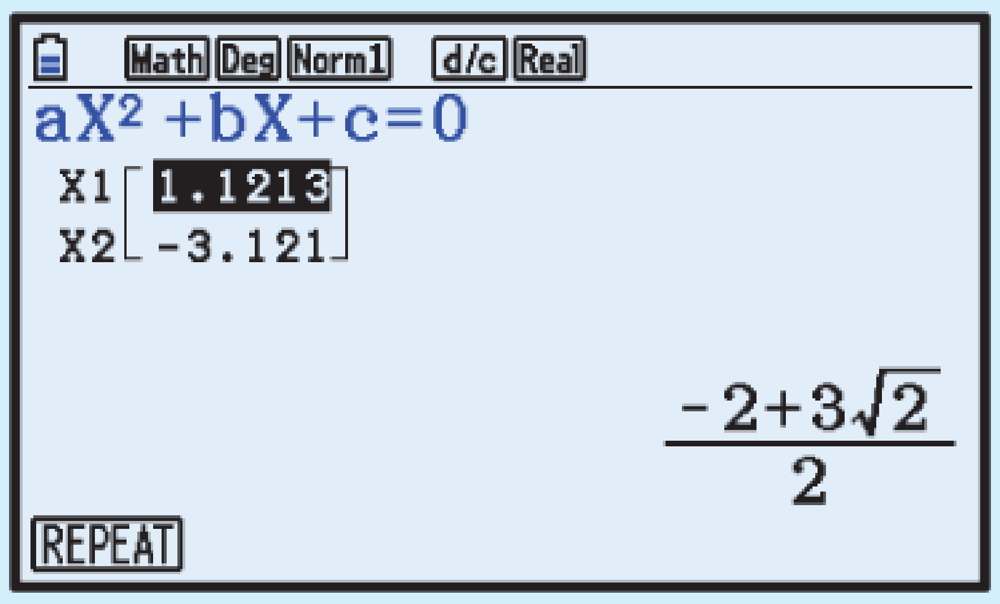
$\therefore\ 2x^2 + 4x - 7 = 0$

Using technology,

$x \approx 1.12$ or -3.12



```
aX²+bX+c=0
   a     b     c
[  2     4    -7]
                 -7
```



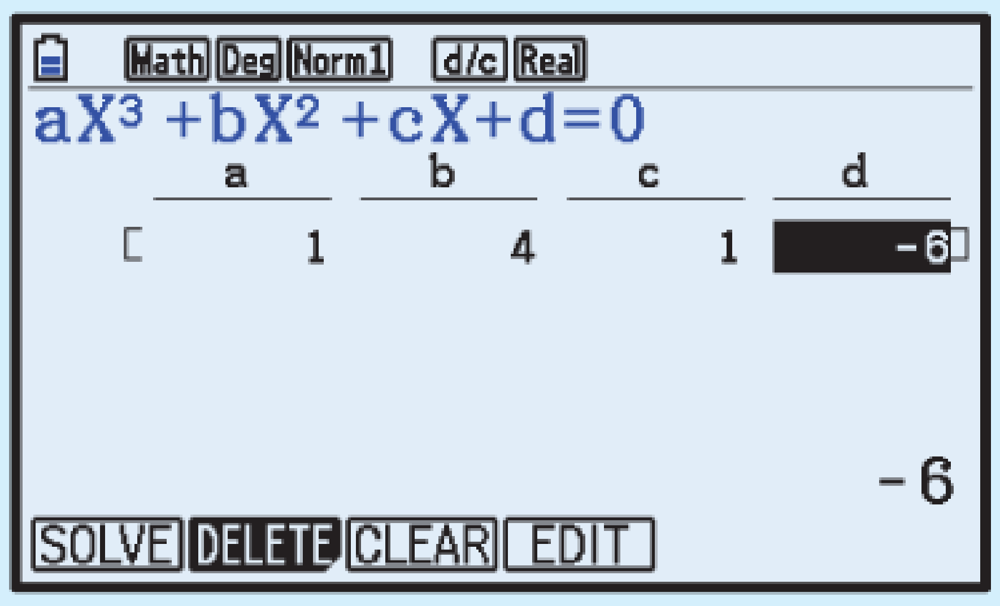
```
aX²+bX+c=0
X1 [ 1.1213]
X2 [-3.121 ]
             -2+3√2
             ------
               2
```

b $x^3 + 4x^2 = 6 - x$

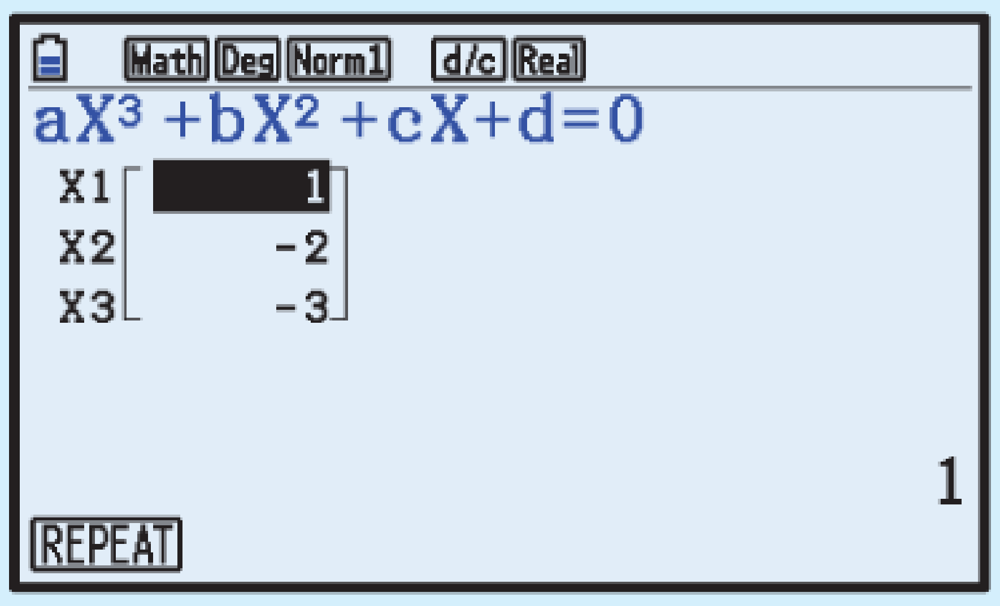
$\therefore\ x^3 + 4x^2 + x - 6 = 0$

Using technology,

$x = -3, -2,$ or 1



```
aX³+bX²+cX+d=0
   a     b     c     d
[  1     4     1    -6]
                       -6
```



```
aX³+bX²+cX+d=0
X1 [  1]
X2 [ -2]
X3 [ -3]
                    1
```

EXERCISE 4D

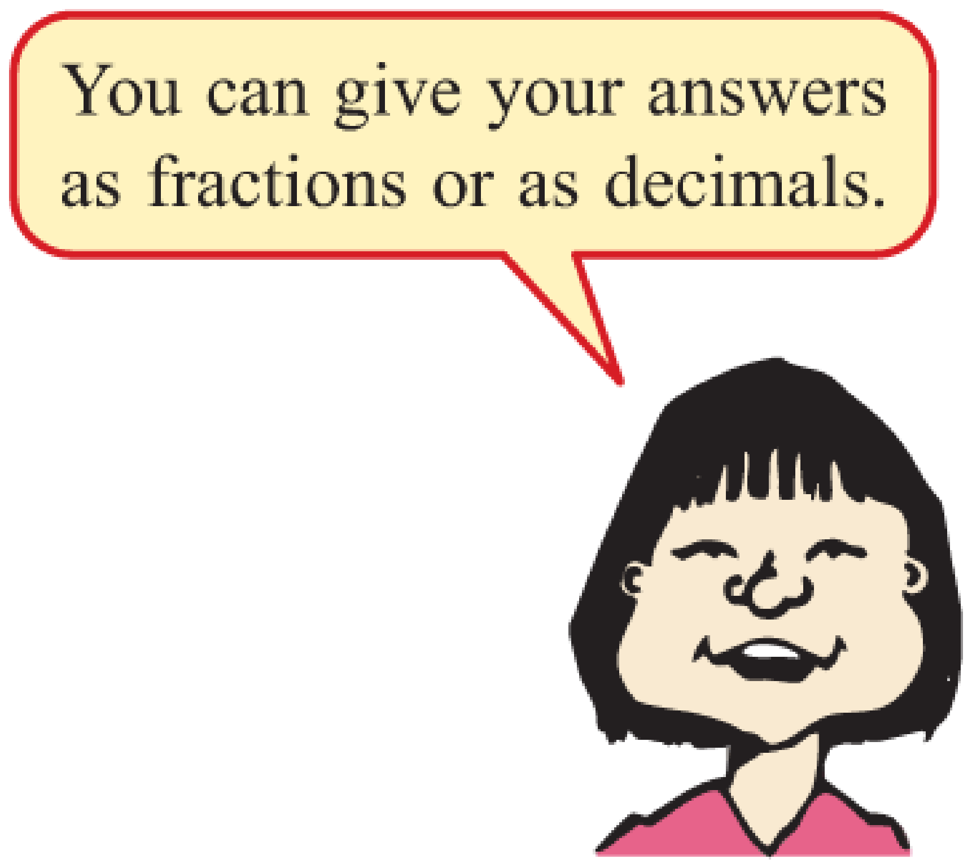
1 Use technology to solve:

a $x^2 + 9x + 14 = 0$ **b** $x^2 - 8x + 16 = 0$

c $4x^2 + x - 8 = 0$ **d** $4x^2 + 4x = 15$

e $-3x^2 + 12x = 10$ **f** $6 = 2x - 5x^2$

g $7x - 2 = 4x^2$ **h** $3.8x + 2.1x^2 = 52.6$



2 Solve for x:

a $x(x+5) + 2(x+6) = 0$ **b** $(x-1)(x+9) = 5x$

c $3x(x+2) - 5(x-3) = 18$ **d** $2x(x-6) = x - 25$

3 Use technology to solve:

a $x^3 - 9x = 0$ **b** $x^3 - 2x^2 + 4 = 0$

c $x^3 - x^2 - 14x + 24 = 0$ **d** $-x^3 + 2 = 2x - x^2$

e $2x^3 + x^2 = 3x - 1$ **f** $2x^3 + 8 = 5x^2 + 18x$

4 Use technology to solve:

a $x^4 - x^3 + 2 = 0$ **b** $x^4 + 2x^3 - 3x^2 + x - 4 = 0$

c $x^4 - 2x^2 + 1 = 0$ **d** $x^4 - x^3 + 3x^2 - x + 6 = 0$

e $x^4 - x = 5x^2 + 3$ **f** $x^4 + 2x^3 + 3 = x(9x + 14)$

5 Use technology to solve:

a $x(x^2 - 1) = 2x$ **b** $(x-2)(x+1) = x^3$

6 Consider the equation $\dfrac{x}{x-3} - \dfrac{2}{x^2} = 2$.

a Write the equation in polynomial form.

b Hence use technology to solve the equation.

E SOLVING OTHER EQUATIONS USING TECHNOLOGY

In **Chapter 1** we solved simultaneous linear equations by graphing the equations on a set of axes, and finding the point of intersection. We can use this graphical method to solve more complicated equations.

- To solve an equation graphically, we graph each side of the equation on the same set of axes. The solutions to the equation are the x-coordinates of the points where the graphs meet.
- If one side of the equation is zero, we graph the other side of the equation. The solutions to the equation are the x-intercepts of the graph.

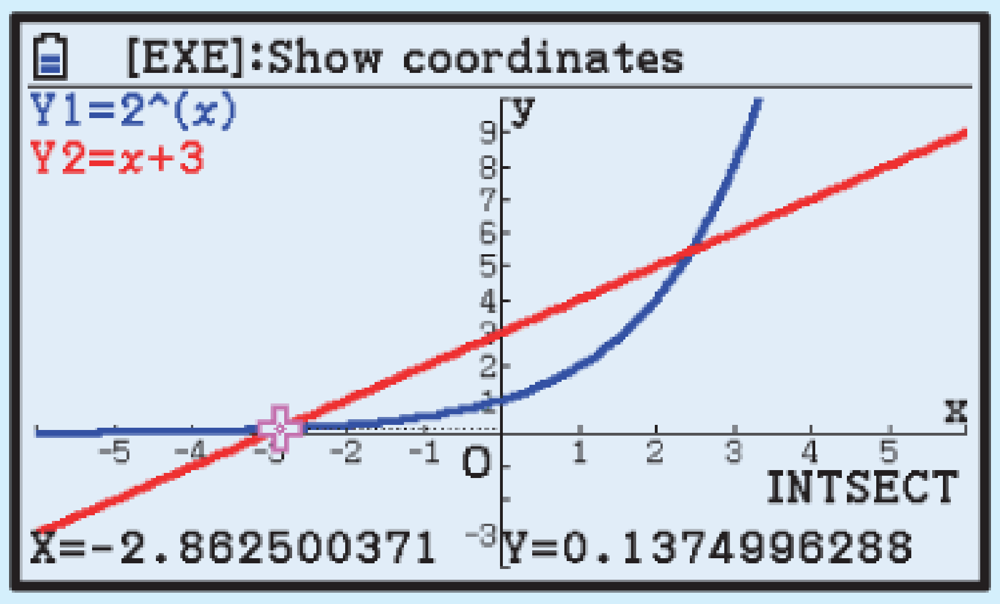
Example 16 **Self Tutor**

Use technology to solve:

a $2^x = x + 3$

b $x^2 - \frac{5}{x} = 0$

a We graph $y = 2^x$ and $y = x + 3$ on the same set of axes.



[EXE]:Show coordinates

Y1=2^(x)

Y2=x+3

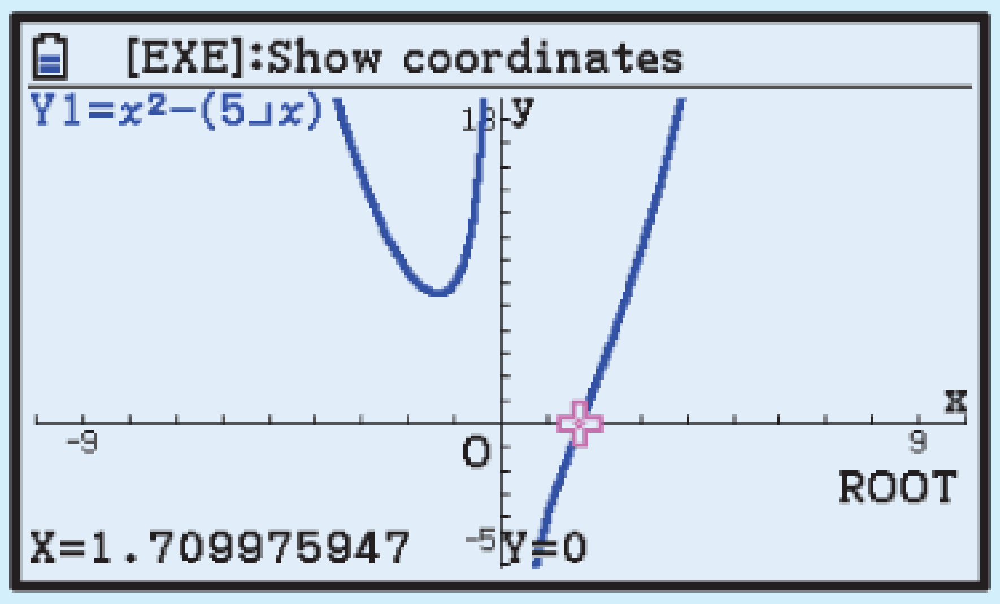
INTSECT

X=-2.862500371 Y=0.1374996288

The graphs intersect at $(-2.86,\ 0.137)$ and $(2.44,\ 5.44)$.

$\therefore$ the solutions are $x \approx -2.86$ or 2.44.

b We graph $y = x^2 - \frac{5}{x}$.



[EXE]:Show coordinates

Y1=x²−(5⌟x)

ROOT

X=1.709975947 Y=0

The x-intercept is ≈ 1.71.

$\therefore$ the solution is $x \approx 1.71$.

EXERCISE 4E

1 Solve the equation $x^2 - 5 = x + 1$ using: **a** algebra **b** technology.

2 Use technology to solve:

a $x = 5 - \sqrt{x}$ **b** $\frac{2}{x} = 3x + 4$ **c** $3^x = 15$

d $5 \times 2^{x-1} = 90$ **e** $x^2 + 5 = 4x + \sqrt{x}$ **f** $(x+1)(x-3) = \sqrt{x+3}$

3 Use technology to solve:

a $x^2 + \frac{4}{x} = 0$ **b** $2^x - x^4 = 0$ **c** $x + \sqrt[3]{x} + 4 = 0$

d $2x^2 - \sqrt{x+1} = 0$ **e** $x^4 + 2^{-x} - 30 = 0$ **f** $\frac{4}{x^2+5} - \frac{1}{x^2+1} = 0$

4 **a** Use technology to solve:

i $2x^2 - 12x + 11 = 1$ **ii** $2x^2 - 12x + 11 = -7$ **iii** $2x^2 - 12x + 11 = -10$

b For which values of k does the equation $2x^2 - 12x + 11 = k$ have:

i two real solutions **ii** one real solution **iii** no real solutions?

DISCUSSION

Many polynomial equations need significant rearrangement before they are in a form suitable for the polynomial solver on a calculator. For example, consider $(x-4)(x^2-1)=(2-x^2)(x+1)$.

- What are the advantages and disadvantages of:
    - rearranging the equation using algebra and hence using the polynomial solver
    - graphing both sides of the equation on the same set of axes then finding the x-coordinates of any points of intersection?
- How reliable is technology? For example, can you use your calculator to solve:
    - $x^x=-1$
    - $(x^2-2)^2(70x-99)=0$?

REVIEW SET 4A

1 Solve for x:

a $2x^2=38$ **b** $(x-2)^2=25$ **c** $3(x-\sqrt{2})^2=6$

2 Solve for x:

a $x^4=-9$ **b** $x^3=\frac{1}{27}$ **c** $(x-1)^5=2$

3 Solve for x using the null factor law:

a $x(x+2)=0$ **b** $-(x+3)(2x-7)=0$ **c** $(x+5)(x+1)(x-6)=0$

4 Solve for x:

a $3x^2-5x=0$ **b** $x^2-4x-5=0$ **c** $x^2+6x+9=0$

d $4x-3=x^2$ **e** $3x^2=2-5x$ **f** $2x^2-108=6x$

5 Solve for x:

a $(x+3)^2=5x+29$ **b** $x(x-4)-(x-6)=0$ **c** $(1-2x)(4-x)=39$

6 Use the quadratic formula to solve:

a $x^2-7x+2=0$ **b** $-x^2+2x-4=0$ **c** $-3x^2-x+3=0$

7 Consider the quadratic equation $6x^2-x-2=0$.

a Find the discriminant and hence state the nature of the roots of the equation.

b Check your answer to **a** by solving the equation.

8 Using the discriminant only, state the nature of the solutions of $2x^2-5x+4=0$.

9 Find the values of m for which $2x^2-3x+m=0$ has:

a a repeated root **b** two distinct real roots **c** no real roots.

10 Find *all* quadratic equations with roots $\frac{1}{2}$ and -3.

11 The roots of $2x^2-3x=4$ are α and β. Find the simplest quadratic equation which has roots $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$.

12 One of the roots of $kx^2+(1-3k)x+(k-6)=0$ is the negative reciprocal of the other root. Find k and the two roots.

13 Solve for x:

a $2x^3 - 3x^2 - 9x + 10 = 0$
b $3x^3 = x(7x - 2)$
c $x^3 + 60 = 23x + 2x^2$
d $x^2(x^2 - 3) = 64 - 6x^3 - 14x$

14 Use technology to solve:

a $10 \times 2^{x-1} = 35$
b $\sqrt{x} = \frac{4}{x} - 1$
c $x^3 - \sqrt[3]{x} + 5 = 0$

REVIEW SET 4B

1 Solve for x:

a $-7x^2 = 0$
b $-4x^3 = \frac{125}{2}$
c $(x - \sqrt{3})^2 = 16$

2 Solve for x:

a $x^4 = \frac{81}{16}$
b $x^5 = -18$
c $(x - 1)^{-2} = 4$

3 Solve, if possible, using the null factor law:

a $\dfrac{p}{q} = 0$
b $\dfrac{2xz}{y} = 0$
c $-\dfrac{5}{ab} = 0$

4 Solve for x:

a $2x^2 - 5x = 0$
b $3x^2 - 12x = 0$
c $x^2 - 7x + 6 = 0$
d $x^2 + 4 = -4x$
e $x^2 - 12 = 4x$
f $3x^2 - x - 10 = 0$

5 Write in the form $ax^2 + bx + c = 0$, and hence solve for x:

a $x = \dfrac{9}{x}$
b $\dfrac{1}{x} = \dfrac{3}{x^2} - 2$
c $3x - 1 = \dfrac{2}{x}$

6 Solve using the quadratic formula:

a $x^2 + 5x + 3 = 0$
b $3x^2 + 11x - 2 = 0$

7 Solve exactly by completing the square:

a $x^2 - 6x + 4 = 0$
b $2x^2 + 8x = 1$
c $4x^2 - 5x = 6$

8 Using the discriminant only, state the nature of the solutions of:

a $x^2 - 8x + 16 = 0$
b $2x^2 - x - 5 = 0$
c $3x^2 + 5x + 3 = 0$

9 Answer the **Opening Problem** on page **72**.

10 Use the quadratic formula to explain why the sum of the solutions to the equation $ax^2 + bx + c = 0$, $a \neq 0$, is always $-\dfrac{b}{a}$.

11 One of the roots of $2x^2 + kx + 12 = 0$ is three times the other. Find the possible values of k, and the two roots in each case.

12 $4x^2 - 3x - 3 = 0$ has roots p and q. Find *all* quadratic equations with roots p^3 and q^3.

13 Solve for x:

a $x^3 - 15x = 2x^2$
b $x^3 - 4x^2 - 6x + 7 = 0$
c $4x^4 - 4x^3 - 11x^2 - 8x + 6 = 0$

14 Use technology to solve:

a $2^x = 7$
b $x^3 = 9 - 2\sqrt{x}$
c $\dfrac{x^2}{5} - \sqrt{x + 3} = 0$

Chapter 5

Sequences and series

Contents:

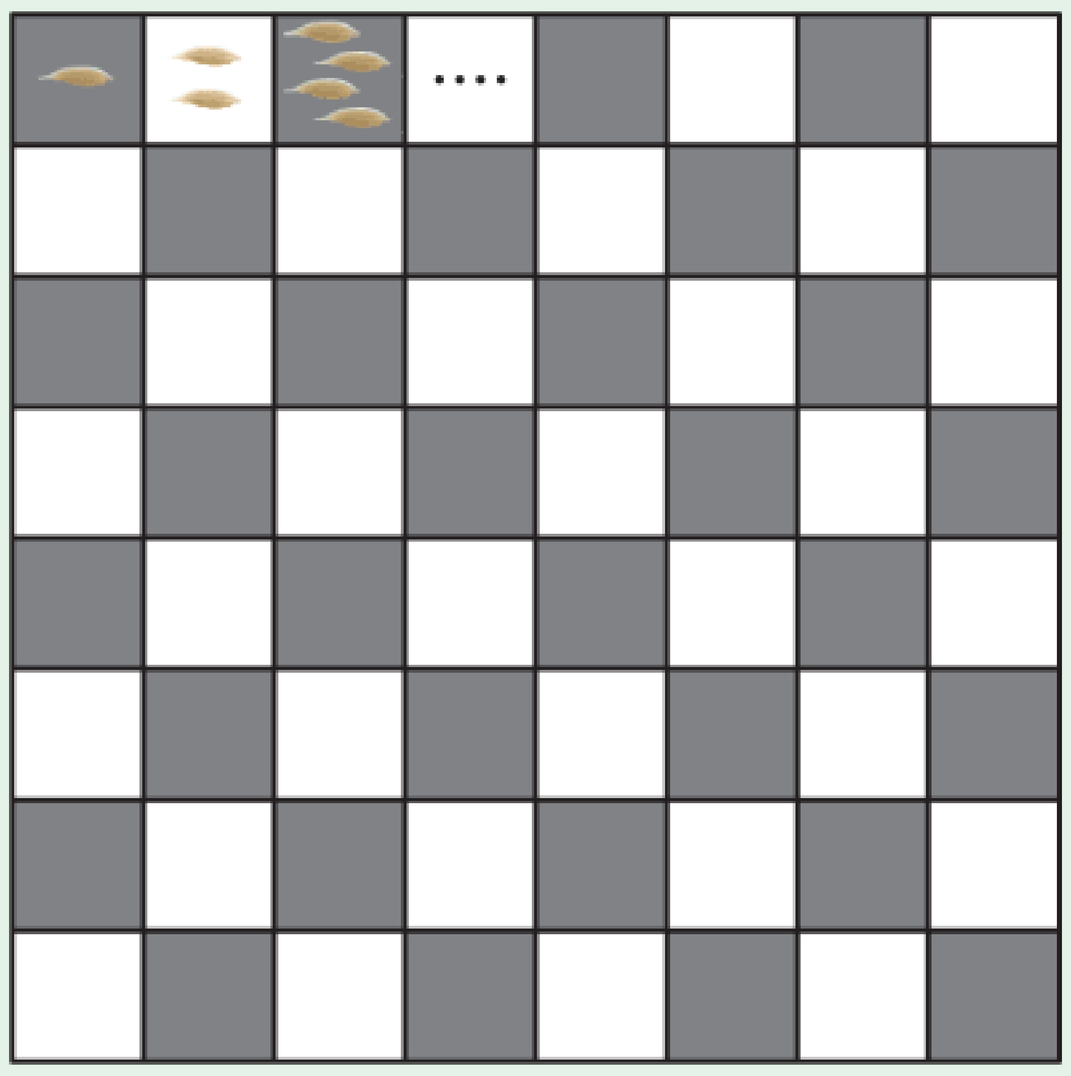
OPENING PROBLEM THE LEGEND OF SISSA IBN DAHIR

Around 1260 AD, the Kurdish historian Ibn Khallikān recorded the following story about Sissa ibn Dahir and a chess game against the Indian King Shihram.

> King Shihram was a tyrant king, and his subject Sissa ibn Dahir wanted to teach him how important all of his people were. He invented the game of chess for the king, and the king was greatly impressed. He insisted on Sissa ibn Dahir naming his reward, and the wise man asked for one grain of wheat for the first square, two grains of wheat for the second square, four grains of wheat for the third square, and so on, doubling the wheat on each successive square on the board.
>
> The king laughed at first and agreed, for there was so little grain on the first few squares. By halfway he was surprised at the amount of grain being paid, and soon he realised his great error: that he owed more grain than there was in the world.

Things to think about:

- **a** How can we describe the number of grains of wheat for each square?
- **b** What expression gives the number of grains of wheat for the nth square?
- **c** Find the total number of grains of wheat that the king owed.



To help understand problems like the **Opening Problem**, we need to study **sequences** and their sums which are called **series**.

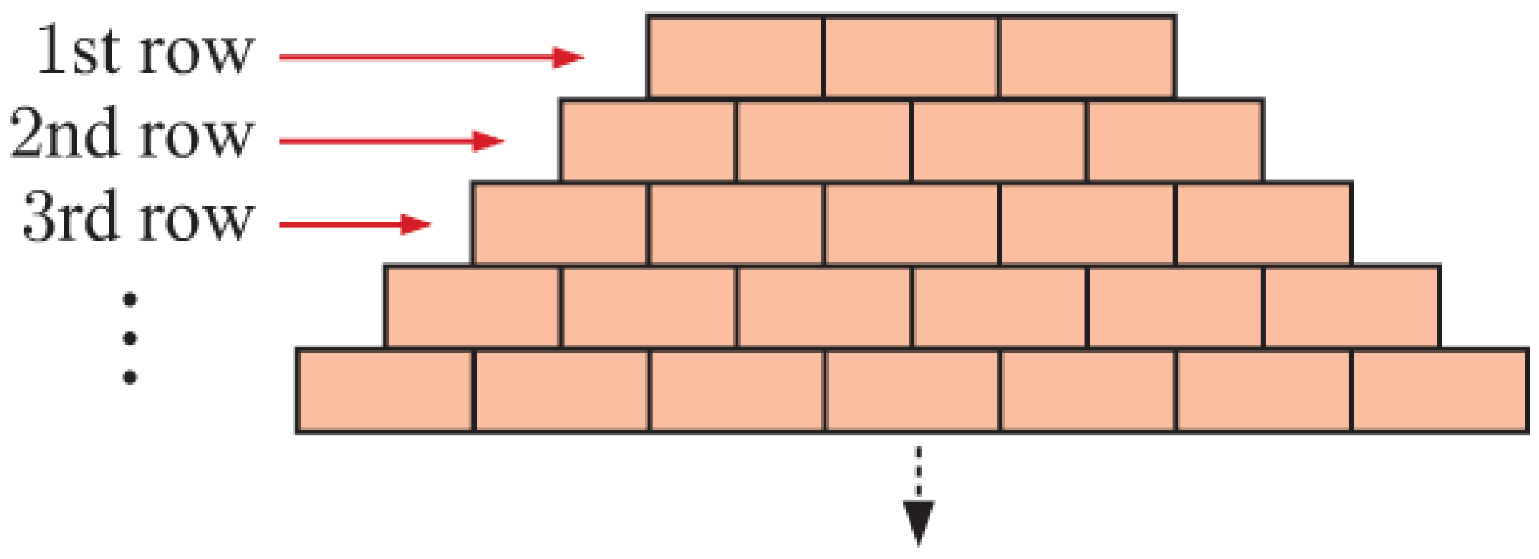
A NUMBER SEQUENCES

In mathematics it is important that we can:

- **recognise** a pattern in a set of numbers
- **describe** the pattern in words
- **continue** the pattern.

> A **number sequence** is an ordered list of numbers defined by a rule.
>
> The numbers in a sequence are called the **terms** of the sequence.

Consider the illustrated tower of bricks:



The first row has 3 bricks.
The second row has 4 bricks.
The third row has 5 bricks.
The fourth row has 6 bricks.

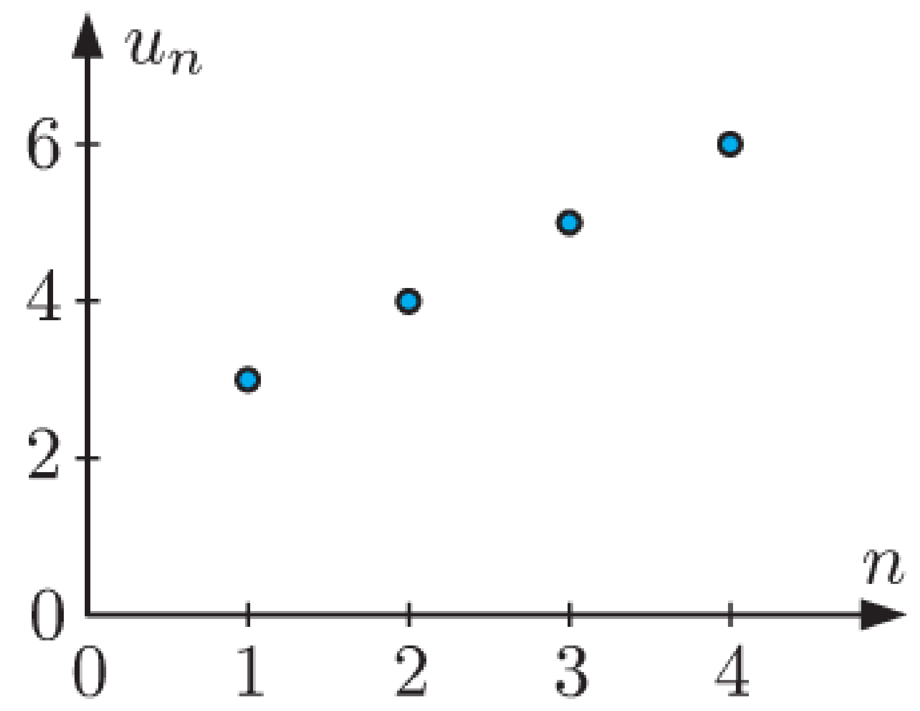
If we let u_n represent the number of bricks in the nth row, then $u_1 = 3$, $u_2 = 4$, $u_3 = 5$, and $u_4 = 6$.

The pattern could be continued forever, generating a **sequence** of numbers.

There are many ways to describe the sequence, including:

- **listing the terms**: 3, 4, 5, 6, 7, ...
- using **words**: "The sequence starts at 3, and increases by 1 each time".
- using the **explicit formula** $u_n = n + 2$ which gives the ***n*th term** or **general term** of the sequence in terms of n.
  We can use this formula to find, for example, the 20th term of the sequence, which is $u_{20} = 20 + 2 = 22$.
- a **graph** where each term of a sequence is represented by a dot. The dots *must not* be joined because n must be an integer.

The string of dots indicates that the pattern continues forever.



Example 1 **Self Tutor**

Describe the sequence: 14, 17, 20, 23, ... and write down the next two terms.

The sequence starts at 14, and each term is 3 more than the previous term.

The next two terms are 26 and 29.

THE GENERAL TERM OF A SEQUENCE

The **general term** or ***n*th term** of a sequence is represented by a symbol with a subscript, for example u_n, T_n, t_n, or A_n.

$\{u_n\}$ represents the sequence that can be generated by using u_n as the nth term.

Unless stated otherwise, we assume the first term of the sequence is u_1, and that the sequence is defined for $n \in \mathbb{Z}^+$. Sometimes we might choose for a sequence to start with u_0, particularly if n represents the number of time periods after the start of an experiment or investment.

You can use technology to help generate sequences from a formula.

GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 5A

1 Write down the first four terms of the sequence if you start with:

a 4 and add 9 each time

b 45 and subtract 6 each time

c 2 and multiply by 3 each time

d 96 and divide by 2 each time.

2 The sequence of prime numbers is 2, 3, 5, 7, 11, 13, 17, 19, ... Write down the value of:

a u_2

b u_5

c u_{10}.

3 Consider the sequence $4, 7, 10, 13, 16, ...$

a Describe the sequence in words.
b Write down the values of u_1 and u_4.
c Assuming the pattern continues, find the value of u_8.

4 A sequence is defined by the explicit formula $u_n = 2n + 5$.

a Write down the first four terms of the sequence.
b Display these terms on a graph.

5 A sequence is defined by the explicit formula $u_n = 3n - 2$. Find:

a u_1 **b** u_5 **c** u_{27}.

6 Consider the number sequence $-9, -6, -1, 6, 15, ...$

a Which of these is the correct explicit formula for this sequence?

A $u_n = n - 10$ **B** $u_n = n^2 - 10$ **C** $u_n = n^3 - 10$

b Use the correct formula to find the 20th term of the sequence.

7 Write a description of the sequence and find the next 2 terms:

a $8, 16, 24, 32, ...$ **b** $2, 5, 8, 11, ...$ **c** $36, 31, 26, 21, ...$
d $96, 89, 82, 75, ...$ **e** $1, 4, 16, 64, ...$ **f** $2, 6, 18, 54, ...$
g $480, 240, 120, 60, ...$ **h** $243, 81, 27, 9, ...$ **i** $50\,000, 10\,000, 2000, 400, ...$

8 Describe the sequence and write down the next 3 terms:

a $1, 4, 9, 16, ...$ **b** $1, 8, 27, 64, ...$ **c** $2, 6, 12, 20, ...$

9 Find the next two terms of the sequence:

a $95, 91, 87, 83, ...$ **b** $5, 20, 80, 320, ...$ **c** $1, 16, 81, 256, ...$
d $2, 3, 5, 7, 11, ...$ **e** $2, 4, 7, 11, ...$ **f** $9, 8, 10, 7, 11, ...$

10 Evaluate the first *five* terms of the sequence:

a $\{2n\}$ **b** $\{2n - 3\}$ **c** $\{2n + 11\}$ **d** $\{4n - 3\}$
e $\{2^n\}$ **f** $\{6 \times (\frac{1}{2})^n\}$ **g** $\{(-2)^n\}$ **h** $\{15 - (-2)^n\}$

ACTIVITY 1 RECURRENCE FORMULAE

A **recurrence formula** describes the nth term of a sequence using a formula which involves the preceding terms.

Click on the icon to obtain this Activity.

RECURRENCE FORMULAE

B ARITHMETIC SEQUENCES

An **arithmetic sequence** is a sequence in which each term differs from the previous one by the same fixed number. We call this number the **common difference** d.

A sequence is arithmetic $\Leftrightarrow$ $\boldsymbol{u_{n+1} - u_n = d}$ for all $n \in \mathbb{Z}^+$.

An arithmetic sequence can also be referred to as an **arithmetic progression**.

For example:

- the tower of bricks in the previous Section forms an arithmetic sequence with common difference 1
- 2, 5, 8, 11, 14, ... is arithmetic with common difference 3 since
  $5 - 2 = 3$
  $8 - 5 = 3$
  $11 - 8 = 3$, and so on.
- 30, 25, 20, 15, 10, ... is arithmetic with common difference -5 since
  $25 - 30 = -5$
  $20 - 25 = -5$
  $15 - 20 = -5$, and so on.

The name "arithmetic" is given because the middle term of any three consecutive terms is the **arithmetic mean** of the terms on either side.

If the terms are a, b, c, then $b - a = c - b$ {equating common differences}

$\therefore\ 2b = a + c$

$\therefore\ b = \dfrac{a+c}{2}$ which is the arithmetic mean.

THE GENERAL TERM FORMULA

If we know that a sequence is arithmetic, we can use a formula to find the value of any term of the sequence.

Suppose the first term of an arithmetic sequence is u_1 and the common difference is d.

Then $u_2 = u_1 + d$, $u_3 = u_1 + 2d$, $u_4 = u_1 + 3d$, and so on.

Hence $$\underset{\text{term number}}{u_n} = u_1 + \underbrace{(n-1)}_{\text{the coefficient of } d \text{ is one less than the term number}} d$$

For an **arithmetic sequence** with **first term** $\boldsymbol{u_1}$ and **common difference** $\boldsymbol{d}$, the **general term** or **$\boldsymbol{n}$th term** is $\boldsymbol{u_n = u_1 + (n-1)d}$.

Example 2 **Self Tutor**

Consider the sequence 2, 9, 16, 23, 30, ...

a Show that the sequence is arithmetic.

b Find a formula for the general term u_n.

c Find the 100th term of the sequence.

d Is **i** 828 **ii** 2341 a term of the sequence?

a $9 - 2 = 7$, $16 - 9 = 7$, $23 - 16 = 7$, $30 - 23 = 7$
The difference between successive terms is constant.
$\therefore$ the sequence is arithmetic with $u_1 = 2$ and $d = 7$.

b
$u_n = u_1 + (n-1)d$
$\therefore\ u_n = 2 + 7(n-1)$
$\therefore\ u_n = 7n - 5$

c
$u_{100} = 7(100) - 5$
$= 695$

d **i**
Let $u_n = 828$
$\therefore\ 7n - 5 = 828$
$\therefore\ 7n = 833$
$\therefore\ n = 119$
$\therefore$ 828 is a term of the sequence, and in fact is the 119th term.

ii
Let $u_n = 2341$
$\therefore\ 7n - 5 = 2341$
$\therefore\ 7n = 2346$
$\therefore\ n = 335\frac{1}{7}$
But n must be an integer, so 2341 is not a member of the sequence.

EXERCISE 5B.1

1 For each of these arithmetic sequences:

- **i** State u_1 and d.
- **ii** Find the formula for the general term u_n.
- **iii** Find the 15th term of the sequence.

a 19, 25, 31, 37, ...
b 101, 97, 93, 89, ...
c $8, 9\frac{1}{2}, 11, 12\frac{1}{2}, ...$
d 31, 36, 41, 46, ...
e $5, -3, -11, -19, ...$
f $a, a+d, a+2d, a+3d, ...$

2 Consider the sequence 6, 17, 28, 39, 50, ...

a Show that the sequence is arithmetic.
b Find the formula for its general term.
c Find the 50th term.
d Is 325 a member?
e Is 761 a member?

3 Consider the sequence 87, 83, 79, 75, 71, ...

a Show that the sequence is arithmetic.
b Find the formula for its general term.
c Find the 40th term.
d Which term of the sequence is -297?

4 A sequence is defined by $u_n = 3n - 2$.

a By finding $u_{n+1} - u_n$, prove that the sequence is arithmetic.
b Find u_1 and d.
c Find the 57th term.
d What is the largest term of the sequence that is smaller than 450? Which term is this?

5 A sequence is defined by $u_n = \dfrac{71 - 7n}{2}$.

a Prove that the sequence is arithmetic.
b Find u_1 and d.
c Find u_{75}.
d For what values of n are the terms of the sequence less than -200?

6 An arithmetic sequence starts 23, 36, 49, 62, ... Find the first term of the sequence to exceed 100 000.

7 A sequence is defined by the formula $u_1 = -12$, $u_{n+1} = u_n + 7$, $n \geqslant 1$.

a Prove that the sequence is arithmetic.
b Find the 200th term of the sequence.
c Is 1000 a member of the sequence?

Example 3 **Self Tutor**

Find k given that $3k+1$, k, and -3 are consecutive terms of an arithmetic sequence.

Since the terms are consecutive, $k - (3k+1) = -3 - k$ {equating differences}

$$\begin{aligned} \therefore\ k - 3k - 1 &= -3 - k \\ \therefore\ -2k - 1 &= -3 - k \\ \therefore\ -1 + 3 &= -k + 2k \\ \therefore\ k &= 2 \end{aligned}$$

8 Find k given the consecutive arithmetic terms:

a $32, k, 3$
b $k, 7, 10$
c $k,\ 2k-1,\ 13$
d $k,\ 2k+1,\ 8-k$
e $2k+7,\ 3k+5,\ 5k-4$
f $2k+18,\ -2-k,\ 2k+2$
g $k,\ k^2,\ k^2+6$
h $5,\ k,\ k^2-8$

9 Suppose $10k+1$, $2k$, and $4k^2-5$ are consecutive terms of an arithmetic sequence.

- **a** Find the possible values of k.
- **b** For each value of k, find the common difference of the sequence.

Example 4 **Self Tutor**

Find the general term u_n for an arithmetic sequence with $u_3 = 8$ and $u_8 = -17$.

$u_3 = 8 \quad \therefore \; u_1 + 2d = 8 \quad .... \; (1) \qquad \{\text{using } u_n = u_1 + (n-1)d\}$

$u_8 = -17 \quad \therefore \; u_1 + 7d = -17 \quad .... \; (2)$

We now solve (1) and (2) simultaneously:

$$\begin{aligned} -u_1 - 2d &= -8 && \{\text{multiplying both sides of (1) by } -1\} \\ u_1 + 7d &= -17 && \\ \hline \therefore \; 5d &= -25 && \{\text{adding the equations}\} \\ \therefore \; d &= -5 && \end{aligned}$$

So, in (1):

$$\begin{aligned} u_1 + 2(-5) &= 8 \\ \therefore \; u_1 - 10 &= 8 \\ \therefore \; u_1 &= 18 \\ \text{Now} \quad u_n &= u_1 + (n-1)d \\ \therefore \; u_n &= 18 - 5(n-1) \\ \therefore \; u_n &= 18 - 5n + 5 \\ \therefore \; u_n &= 23 - 5n \end{aligned}$$

Check:

$$\begin{aligned} u_3 &= 23 - 5(3) \\ &= 23 - 15 \\ &= 8 \; ✓ \\ u_8 &= 23 - 5(8) \\ &= 23 - 40 \\ &= -17 \; ✓ \end{aligned}$$

10 Find the general term u_n for an arithmetic sequence with:

- **a** $u_7 = 41$ and $u_{13} = 77$
- **b** $u_5 = -2$ and $u_{12} = -12\frac{1}{2}$
- **c** seventh term 1 and fifteenth term -39
- **d** eleventh and eighth terms being -16 and $-11\frac{1}{2}$ respectively.

11 Suppose a sequence $u_1, u_2, u_3, u_4, u_5, u_6, ...$ is arithmetic.

- **a** Show that $u_1 + u_2, \; u_3 + u_4, \; u_5 + u_6, \; ....$ is also arithmetic.
- **b** Given that $u_2 = 5$ and $u_{10} = 61$, find the 30th term of the sequence in **a**.

Example 5 **Self Tutor**

Insert four numbers between 3 and 12 so that all six numbers are in arithmetic sequence.

Suppose the common difference is d.

$\therefore$ the numbers are $3, \; 3+d, \; 3+2d, \; 3+3d, \; 3+4d,$ and 12

$$\begin{aligned} \therefore \; 3 + 5d &= 12 \\ \therefore \; 5d &= 9 \\ \therefore \; d &= \tfrac{9}{5} = 1.8 \end{aligned}$$

So, the sequence is 3, 4.8, 6.6, 8.4, 10.2, 12.

12 Insert three numbers between 5 and 10 so that all five numbers are in arithmetic sequence.

13 Insert six numbers between -1 and 32 so that all eight numbers are in arithmetic sequence.

14
- **a** Insert three numbers between 50 and 44 so that all five numbers are in arithmetic sequence.
- **b** Assuming the sequence continues, find the first negative term of the sequence.

15 $\frac{1}{k}$, k, k^2+1 are respectively the 3rd, 4th, and 6th terms of an arithmetic sequence.

Given that $k \in \mathbb{Q}$, find: **a** k **b** the general term u_n.

16 The three numbers x, y, and z are such that $x > y > z > 0$.

Show that if $\frac{1}{x}$, $\frac{1}{y}$, and $\frac{1}{z}$ are consecutive terms of an arithmetic sequence, then $x-z$, y, and $x-y+z$ could be the lengths of the sides of a right angled triangle.

17 It is known that there are infinitely many prime numbers. Is it possible to construct an infinite arithmetic sequence such that all of the terms are distinct primes? Explain your answer.

Example 6 **Self Tutor**

Ryan is a cartoonist. His comic strip has just been bought by a newspaper, so he sends them the 28 comic strips he has drawn so far. Each week after the first he sends 3 more comic strips to the newspaper.

- **a** Find the total number of comic strips sent after 1, 2, 3, and 4 weeks.
- **b** Show that the total number of comic strips sent after n weeks forms an arithmetic sequence.
- **c** Find the number of comic strips sent after 15 weeks.
- **d** When does Ryan send his 120th comic strip?

a *Week 1*: 28 comic strips
Week 2: $28+3=31$ comic strips
Week 3: $31+3=34$ comic strips
Week 4: $34+3=37$ comic strips

b Every week, Ryan sends 3 comic strips, so the difference between successive weeks is always 3. We have an arithmetic sequence with $u_1 = 28$ and common difference $d = 3$.

c
$$\begin{aligned} u_n &= u_1 + (n-1)d \\ &= 28 + (n-1)\times 3 \\ &= 25 + 3n \end{aligned} \qquad \begin{aligned} \therefore\ u_{15} &= 25 + 3\times 15 \\ &= 70 \end{aligned}$$

After 15 weeks Ryan has sent 70 comic strips.

d We want to find n such that
$$\begin{aligned} u_n &= 120 \\ \therefore\ 25+3n &= 120 \\ \therefore\ 3n &= 95 \\ \therefore\ n &= 31\tfrac{2}{3} \end{aligned}$$

Ryan sends the 120th comic strip in the 32nd week.

18 A luxury car manufacturer sets up a factory for a new model vehicle. In the first month only 5 cars are made. After this, 13 cars are made every month.

- **a** List the total number of cars that have been made by the end of each of the first six months.
- **b** Explain why the total number of cars made after n months forms an arithmetic sequence.
- **c** How many cars are made in the first year?
- **d** How long is it until the 250th car is made?

19 At the start of the dry season, Yafiah's 3000 L water tank is full. She uses 183 L of water each week to water her garden.

- **a** Find the amount of water left in the tank after 1, 2, 3, and 4 weeks.
- **b** Explain why the amount of water left in the tank after n weeks forms an arithmetic sequence.
- **c** When will Yafiah's tank run out of water?

APPROXIMATIONS USING ARITHMETIC SEQUENCES

In the real-world questions we have just seen, exactly the same number of items is added or subtracted in each time period. We can therefore use an arithmetic sequence to model the number of items exactly.

Most real-world scenarios will not be this exact. Instead, random variation may give us a sequence where the difference between terms is *similar*, but not the same. In these cases we can use an arithmetic sequence as an *approximation*.

For example, the table below shows the total mass of people in a lift as they walk in:

n (people)	1	2	3	4	5	6
Mass (kg)	86.2	147.5	210.1	298.4	385.0	459.8

No two people will have exactly the same mass, so the total mass of people will not form an exact arithmetic sequence. However, since the *average* mass of the people is $\frac{459.8}{6} \approx 76.6$ kg, a reasonable model for the total mass would be the arithmetic sequence $u_n = 76.6n$.

EXERCISE 5B.2

1 Halina is measuring the mass of oranges on a scale. When there are 8 oranges, the total mass is 1.126 kg.

- **a** Find the average mass of the oranges on the scale.
- **b** Hence write an arithmetic sequence for u_n, the approximate total mass when n oranges have been placed on the scale.

2 Nadir has placed an empty egg carton on a set of scales. Its mass is 32 g. When the carton is filled with 12 eggs, the total mass of eggs and carton is 743 g.

- **a** Find the average mass of the eggs in the carton.
- **b** Hence write an arithmetic sequence for u_n, the approximate total mass when n eggs have been added to the carton.
- **c** For what values of n is your model valid?

3 A farmer has 580 square bales of hay in his shed with total mass 9850 kg.
The farm has 8 yards of animals. Each day, the farmer feeds out 2 bales of hay to each yard.

- **a** Write an arithmetic sequence for the number of bales of hay remaining after n days.
- **b** Write an arithmetic sequence which *approximates* the mass of hay remaining after n days.

4 Valéria joins a social networking website. After 1 week she has 34 online friends, and after 9 weeks she has 80 online friends.

- **a** Find the average number of online friends Valéria has made each week from week 1 to week 9.
- **b** Assuming that her total number of online friends after n weeks forms an arithmetic sequence, find a model which approximates the number of online friends after n weeks.

- **c** Do you think it is a problem that the common difference in your model is not an integer? Explain your answer.
- **d** Use your model to predict how many online friends Valéria will have after 20 weeks.

5 A wedding reception venue advertises all-inclusive venue hire and catering costs of €6950 for 50 guests or €11 950 for 100 guests.

Assume that the cost of venue hire and catering for n guests forms an arithmetic sequence.

- **a** Write a formula for the general term u_n of the sequence.
- **b** Explain the significance of:
  - **i** the common difference
  - **ii** the constant term.
- **c** Estimate the cost of venue hire and catering for a reception with 85 guests.

C GEOMETRIC SEQUENCES

A **geometric sequence** is a sequence in which each term can be obtained from the previous one by multiplying by the same non-zero number. We call this number the **common ratio** r.

A sequence is geometric $\Leftrightarrow$ $\boldsymbol{\dfrac{u_{n+1}}{u_n} = r}$ for all $n \in \mathbb{Z}^+$.

A geometric sequence can also be referred to as a **geometric progression**.

For example:

- 2, 10, 50, 250, ... is a geometric sequence as each term can be obtained by multiplying the previous term by 5.

  The common ratio is 5 since $\frac{10}{2} = \frac{50}{10} = \frac{250}{50} = 5$.
- 2, -10, 50, -250, ... is a geometric sequence with common ratio -5.

The name "geometric" is given because the middle term of any three consecutive terms is the **geometric mean** of the terms on either side.

If the terms are a, b, c then $\dfrac{b}{a} = \dfrac{c}{b}$ {common ratio}

$\therefore$ $b^2 = ac$

$\therefore$ $b = \pm\sqrt{ac}$ where $\sqrt{ac}$ is the geometric mean.

THE GENERAL TERM FORMULA

Suppose the first term of a geometric sequence is u_1 and the common ratio is r.

Then $u_2 = u_1\,r$, $u_3 = u_1\,r^2$, $u_4 = u_1\,r^3$, and so on.

Hence $u_n = u_1\,r^{n-1}$

term number

The power of r is one less than the term number.

For a **geometric sequence** with **first term** $\boldsymbol{u_1}$ and **common ratio** $\boldsymbol{r}$, the **general term** or $\boldsymbol{n}$**th term** is $\boldsymbol{u_n = u_1 r^{n-1}}$.

Example 7

Self Tutor

Consider the sequence $8, 4, 2, 1, \frac{1}{2}, ...$

a Show that the sequence is geometric.

b Find the general term u_n.

c Hence find the 12th term as a fraction.

a $\frac{4}{8} = \frac{2}{4} = \frac{1}{2} = \frac{\frac{1}{2}}{1}$, so consecutive terms have the common ratio $\frac{1}{2}$.

$\therefore$ the sequence is geometric with $u_1 = 8$ and $r = \frac{1}{2}$.

b $u_n = u_1\, r^{n-1}$

$$\therefore\ u_n = 8\left(\tfrac{1}{2}\right)^{n-1} \quad \textit{or} \quad \begin{aligned} u_n &= 2^3 \times 2^{-(n-1)} \\ &= 2^3 \times 2^{1-n} \\ &= 2^{4-n} \end{aligned}$$

c $$\begin{aligned} u_{12} &= 8 \times \left(\tfrac{1}{2}\right)^{11} \\ &= \tfrac{1}{256} \end{aligned}$$

EXERCISE 5C

1 For each of these geometric sequences:

- **i** State u_1 and r.
- **ii** Find the formula for the general term u_n.
- **iii** Find the 9th term of the sequence.

a $3, 6, 12, 24, ...$

b $2, 10, 50, ...$

c $512, 256, 128, ...$

d $1, 3, 9, 27, ...$

e $12, 18, 27, ...$

f $\frac{1}{16}, -\frac{1}{8}, \frac{1}{4}, -\frac{1}{2}, ...$

2 **a** Show that the sequence $5, 10, 20, 40, ...$ is geometric.

b Find u_n, and hence find the 15th term.

3 **a** Show that the sequence $12, -6, 3, -\frac{3}{2}, ...$ is geometric.

b Find u_n, and hence write the 13th term as a rational number.

4 **a** Show that the sequence $8, -6, 4.5, -3.375, ...$ is geometric.

b Hence find the 10th term as a decimal.

5 **a** Show that the sequence $8, 4\sqrt{2}, 4, 2\sqrt{2}, ...$ is geometric.

b Hence show that the general term of the sequence is $u_n = 2^{\frac{7}{2}-\frac{1}{2}n}$.

Example 8

Self Tutor

$k-1$, $2k$, and $21-k$ are consecutive terms of a geometric sequence. Find k.

Since the terms are geometric, $\dfrac{2k}{k-1} = \dfrac{21-k}{2k}$ {equating the common ratio r}

$$\begin{aligned} \therefore\ 4k^2 &= (21-k)(k-1) \\ \therefore\ 4k^2 &= 21k - 21 - k^2 + k \\ \therefore\ 5k^2 - 22k + 21 &= 0 \\ \therefore\ (5k-7)(k-3) &= 0 \\ \therefore\ k &= \tfrac{7}{5} \text{ or } 3 \end{aligned}$$

Check: If $k = \frac{7}{5}$ the terms are: $\frac{2}{5}, \frac{14}{5}, \frac{98}{5}$. ✓ $\{r = 7\}$

If $k = 3$ the terms are: $2, 6, 18$. ✓ $\{r = 3\}$

6 Find k given that the following are consecutive terms of a geometric sequence:

a $k,\ 3k,\ 54$ **b** $1000,\ 4k,\ k$ **c** $7,\ k,\ 28$

d $18,\ k,\ \frac{2}{9}$ **e** $k,\ 12,\ \frac{k}{9}$ **f** $k,\ 20,\ \frac{25}{4}k$

g $k,\ 3k,\ 20-k$ **h** $k,\ k+8,\ 9k$

7 The first three terms of a geometric sequence are $k-1$, 6, and $3k$.

a Find the possible values of k.

b For each value of k, find the next term in the sequence.

Example 9

Self Tutor

A geometric sequence has $u_2 = -6$ and $u_5 = 162$. Find its general term.

$$u_2 = u_1 r = -6 \quad .... (1)$$

$$\text{and} \quad u_5 = u_1 r^4 = 162 \quad .... (2)$$

$$\text{Now} \quad \frac{u_1 r^4}{u_1 r} = \frac{162}{-6} \qquad \{(2) \div (1)\}$$

$$\therefore \quad r^3 = -27$$

$$\therefore \quad r = -3$$

$$\text{Using (1),} \quad u_1(-3) = -6$$

$$\therefore \quad u_1 = 2$$

$$\text{Thus} \quad u_n = 2 \times (-3)^{n-1}$$

8 Find the general term u_n of the geometric sequence which has:

a $u_4 = 24$ and $u_7 = 192$ **b** $u_3 = 8$ and $u_6 = -1$

c $u_7 = 24$ and $u_{15} = 384$ **d** $u_3 = 5$ and $u_7 = \frac{5}{4}$

9 A geometric sequence has $u_3 = 80$ and $u_6 = 270$.

a Find the first term and common ratio of the sequence.

b Find the 10th term of the sequence.

Example 10

Self Tutor

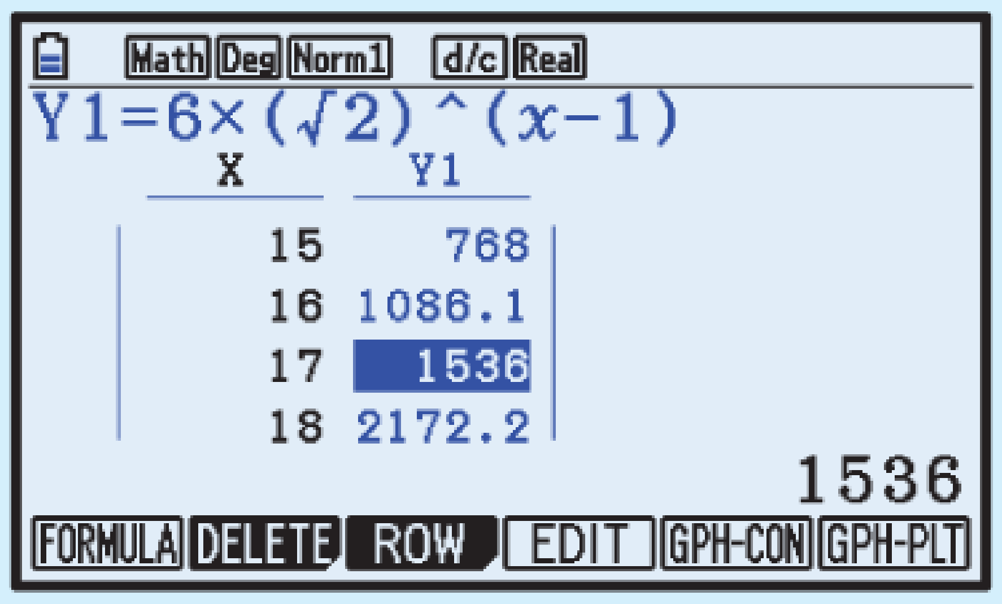
Find the first term of the sequence $6,\ 6\sqrt{2},\ 12,\ 12\sqrt{2},\ ....$ which exceeds 1400.

The sequence is geometric with $u_1 = 6$ and $r = \sqrt{2}$

$$\therefore \quad u_n = 6 \times (\sqrt{2})^{n-1}$$

We need to find n such that $u_n > 1400$.

Using a graphics calculator with $Y_1 = 6 \times (\sqrt{2})$^$(X-1)$, we view a table of values:



```
Math Deg Norm1 d/c Real
Y1=6×(√2)^(x−1)
      X     Y1
     15    768
     16 1086.1
     17   1536
     18 2172.2
                1536
FORMULA DELETE ROW EDIT GPH-CON GPH-PLT
```

GRAPHICS CALCULATOR INSTRUCTIONS

The first term to exceed 1400 is $u_{17} = 1536$.

10 **a** Find the first term of the sequence $2, 6, 18, 54, ...$ which exceeds 10 000.

b Find the first term of the sequence $4, 4\sqrt{3}, 12, 12\sqrt{3}, ...$ which exceeds 4800.

c Find the first term of the sequence $12, 6, 3, 1.5, ...$ which is less than 0.0001.

d Find the first term of the sequence $5, -\frac{15}{2}, \frac{45}{4}, -\frac{135}{8}, ...$ which is less than -100.

11 What is the maximum number of distinct prime numbers that can occur in a geometric sequence? Give an example of such a sequence.

12 A geometric sequence with common ratio r and an arithmetic sequence with common difference d have the same first two terms. The third terms of the geometric and arithmetic sequences are in the ratio $2 : 1$.

a Find the possible values of r.

b For each value of r, find the ratio of the fourth terms of the sequences.

D GROWTH AND DECAY

We now turn our attention to applications of geometric sequences, the first of which being growth and decay problems. Typically, geometric sequences are observed when a quantity increases or decreases by a fixed percentage of its size each time period.

For most real-world situations, we have an **initial condition** corresponding to time zero. This may be an initial investment or an initial population. We therefore allow a "zeroeth" term u_0 to begin the sequence.

Typically, geometric sequences are useful for short term population growth models, or longer term models of radioactive decay.

For situations where there is **growth** or **decay** in a geometric sequence:

$$\boldsymbol{u_n = u_0 \times r^n}$$

where u_0 = initial amount
r = growth multiplier for each time period
n = number of time periods
u_n = amount after n time periods.

Example 11 **Self Tutor**

The initial population of rabbits on a farm was 50. The population increased by 7% each week.

a How many rabbits were present after:

i 15 weeks **ii** 30 weeks?

b How long will it take for the population to reach 500?

There is a fixed percentage increase each week, so the population forms a geometric sequence.

$u_0 = 50$ and $r = 1.07$

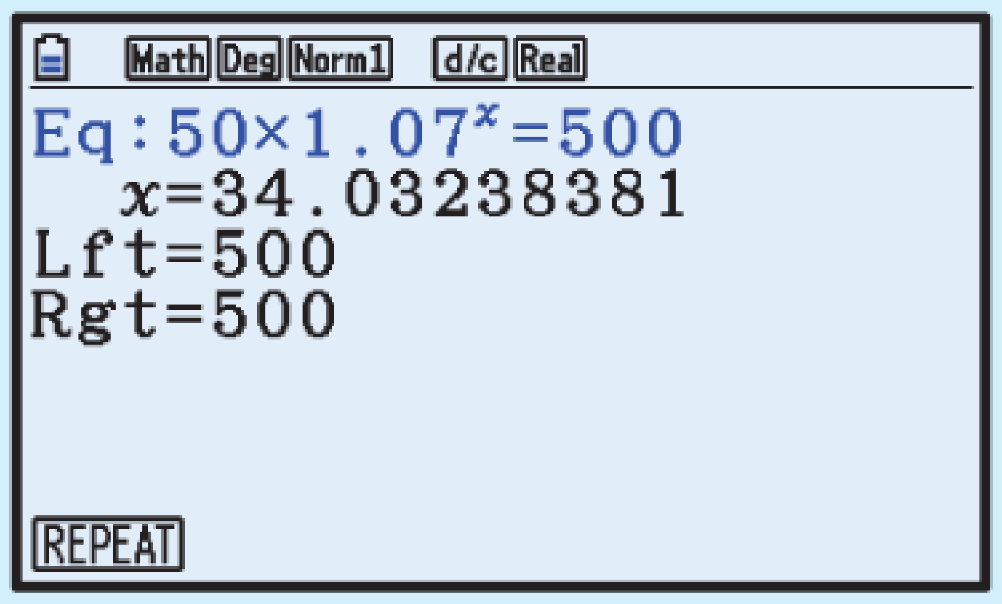
$\therefore$ the population after n weeks is $u_n = 50 \times 1.07^n$.

a **i** $u_{15} = 50 \times (1.07)^{15} \approx 137.95$
There were 138 rabbits.

ii $u_{30} = 50 \times (1.07)^{30} \approx 380.61$
There were 381 rabbits.

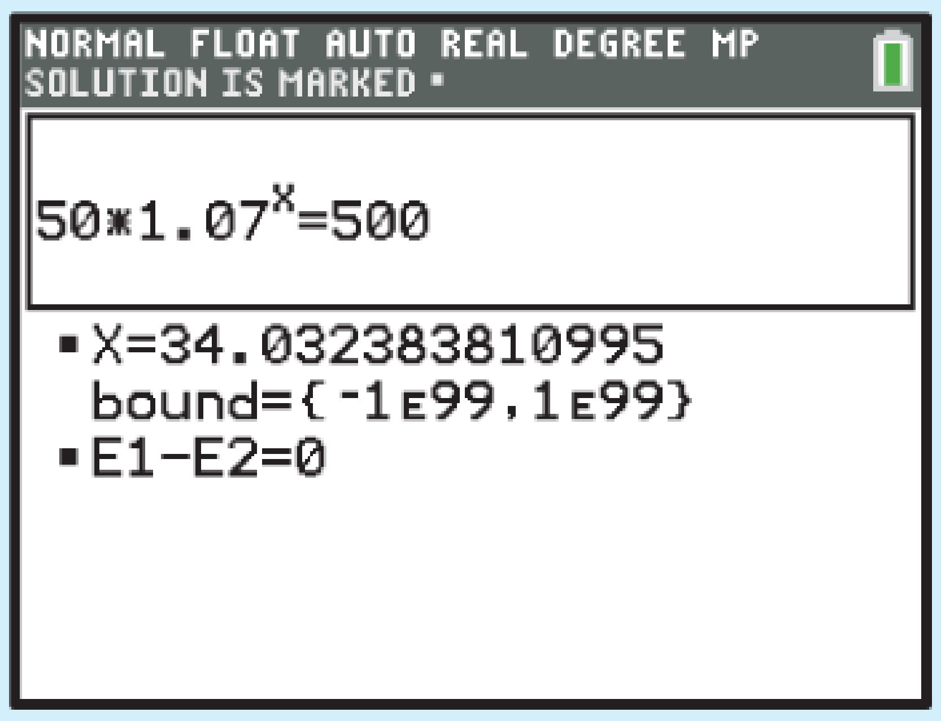
b We need to find when $50 \times (1.07)^n = 500$.

Casio fx-CG50



```
Eq:50×1.07^x=500
  x=34.03238381
Lft=500
Rgt=500
```

TI-84 Plus CE

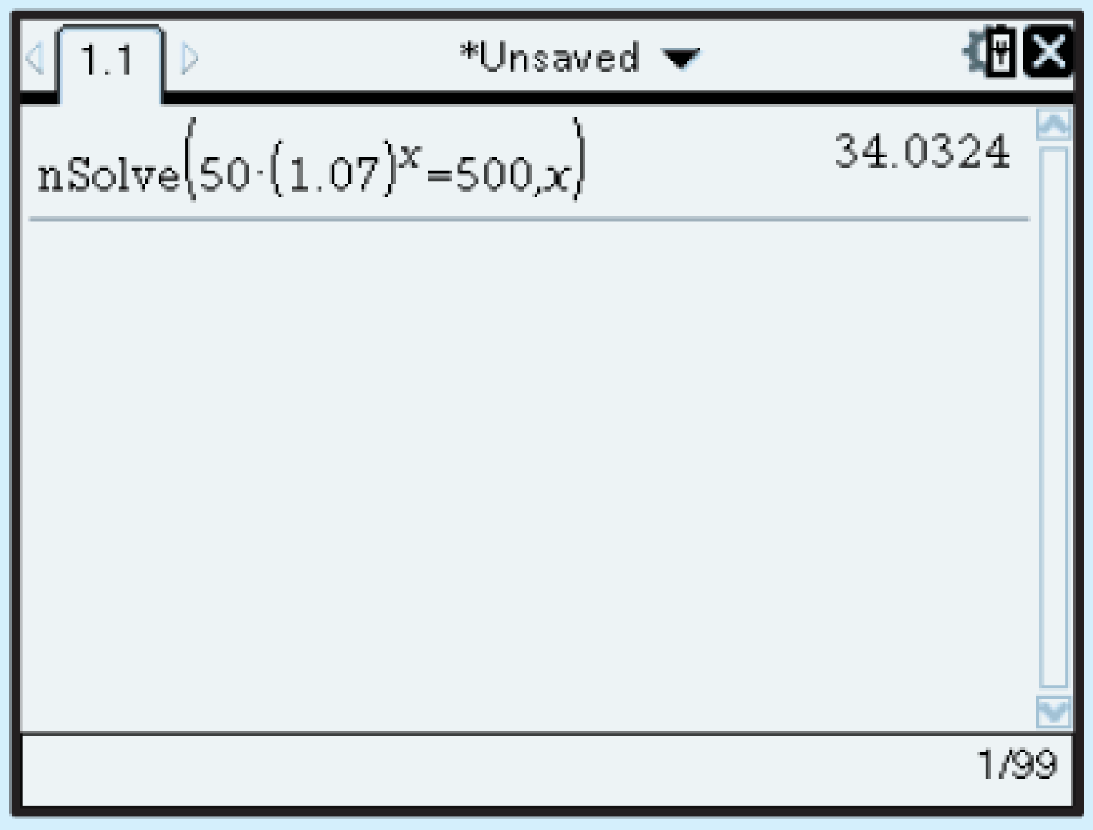


```
NORMAL FLOAT AUTO REAL DEGREE MP
SOLUTION IS MARKED ▪

50*1.07^X=500

▪X=34.032383810995
 bound={-1E99,1E99}
▪E1-E2=0
```

TI-nspire



```
nSolve(50·(1.07)^x=500,x)          34.0324
```

So, it will take approximately 34.0 weeks.

EXERCISE 5D

1 A nest of ants initially contains 500 individuals. The population is increasing by 12% each week.

- **a** How many ants will there be after:
  - **i** 10 weeks
  - **ii** 20 weeks?
- **b** How many weeks will it take for the ant population to reach 2000?

2 The animal *Eraticus* is endangered. Since 2005 there has only been one colony remaining, and in 2005 the population of that colony was 555. The population has been steadily decreasing by 4.5% per year.

- **a** Estimate the population in the year 2020.
- **b** In what year do we expect the population to have declined to 50?

3 A herd of 32 deer is to be left unchecked in a new sanctuary. It is estimated that the size of the herd will increase each year by 18%.

- **a** Estimate the size of the herd after:
  - **i** 5 years
  - **ii** 10 years.
- **b** How long will it take for the herd size to reach 5000?

4 An endangered species of marsupial has a population of 178. However, with a successful breeding program it is expected to increase by 32% each year.

- **a** Find the expected population size after:
  - **i** 10 years
  - **ii** 25 years.
- **b** Estimate how long it will take for the population to reach 10 000.

5 Each year, a physicist measures the remaining radioactivity in a sample. She finds that it reduces by 18% each year. If there was 1.52 g of radioactive material left after 4 years:

- **a** Find the initial quantity of radioactive material.
- **b** How many *more* years will it take for the amount of radioactive material to reduce to 0.2 g?

6 Each year, Maria's salary is increased by 2.3%. She has been working for her company for 10 years, and she currently earns €49 852 per annum.

- **a** What was Maria's salary when she joined the company?
- **b** If she stays with the company for another 4 years, what will her salary be?

E FINANCIAL MATHEMATICS

At some stage in life, most people need to either **invest** or **borrow** money. It is very important that potential investors and borrowers understand these procedures so they can make the right decisions according to their circumstances and their goals.

When money is lent, the person lending the money is known as the **lender**, and the person receiving the money is known as the **borrower**. The amount borrowed is called the **principal**.

The lender usually charges a fee called **interest** to the borrower. This fee represents the cost of using the other person's money. The borrower must repay the principal borrowed as well as the interest charged for using that money.

The rate at which interest is charged is usually expressed as a percentage of the principal. This percentage is known as the **interest rate**, and it is an important factor when deciding where to invest your money and where to borrow money from.

The total amount of interest charged on a loan depends on the principal, the time the money is borrowed for, and the interest rate.

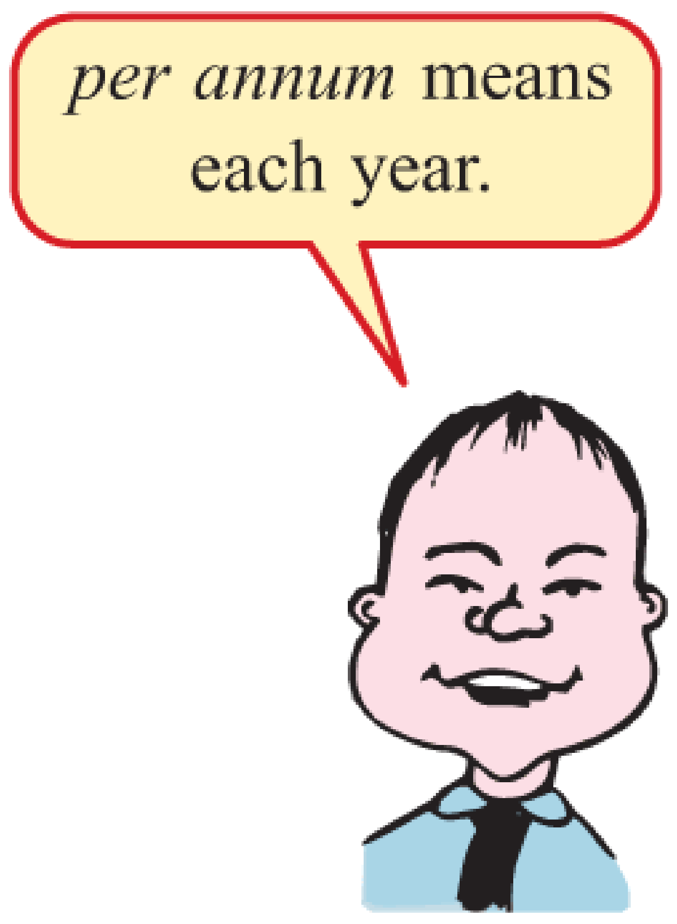
COMPOUND INTEREST

When money is deposited in a bank, it will usually earn **compound interest**.

After a certain amount of time called the **period**, the bank pays interest, which is calculated as a percentage of the money already in the account.

It is called *compound* interest because the interest generated in one period will itself earn more interest in the next period.

For example, suppose you invest $1000 in the bank. The account pays an interest rate of 4% per annum (p.a.). The interest is added to your investment each year, so at the end of each year you will have $100\% + 4\% = 104\%$ of the value at its start. This corresponds to a *multiplier* of 1.04.



After one year your investment is worth $\$1000 \times 1.04 = \$1040$.

After two years it is worth

$$\begin{aligned} &\$1040 \times 1.04 \\ =\ &\$1000 \times 1.04 \times 1.04 \\ =\ &\$1000 \times (1.04)^2 \\ =\ &\$1081.60 \end{aligned}$$

After three years it is worth

$$\begin{aligned} &\$1081.60 \times 1.04 \\ =\ &\$1000 \times (1.04)^2 \times 1.04 \\ =\ &\$1000 \times (1.04)^3 \\ \approx\ &\$1124.86 \end{aligned}$$

Observe that:

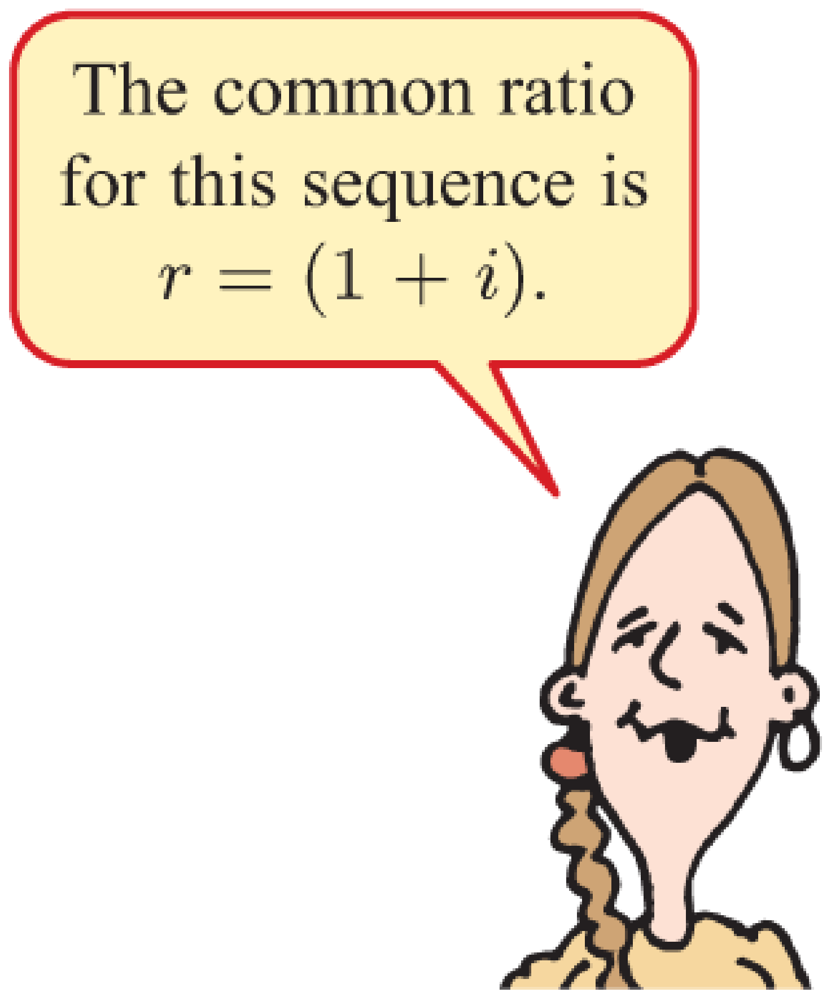
$$\begin{aligned} u_0 &= \$1000 && = \text{initial investment} \\ u_1 &= u_0 \times 1.04 && = \text{amount after 1 year} \\ u_2 &= u_0 \times (1.04)^2 && = \text{amount after 2 years} \\ u_3 &= u_0 \times (1.04)^3 && = \text{amount after 3 years} \\ &\ \vdots \\ u_n &= u_0 \times (1.04)^n && = \text{amount after } n \text{ years} \end{aligned}$$

So the amount in the account after each year forms a geometric sequence!

The value of a compound interest investment after n time periods is

$$u_n = u_0(1+i)^n$$

where u_0 is the initial investment
and i is the interest rate per compounding period.



Example 12 **Self Tutor**

\$5000 is invested for 4 years at 7% p.a. interest compounded annually.

a What will it amount to at the end of this period?

b How much interest has been earned?

a The interest is calculated annually, so $n = 4$ time periods.

$$\begin{aligned} u_4 &= u_0 \times (1+i)^4 \\ &= 5000 \times (1.07)^4 \qquad \{7\% = 0.07\} \\ &\approx 6553.98 \end{aligned}$$

The investment will amount to \$6553.98 .

b The interest earned $=$ \$6553.98 $-$ \$5000
$=$ \$1553.98

EXERCISE 5E.1

1 Lucy invested £7000 at 6% p.a. interest compounded annually. Find the value of this investment after 5 years.

2 €2000 is invested for 4 years at 2.8% p.a. interest compounded annually.

- **a** What will it amount to at the end of this period?
- **b** How much interest has been earned?

3 How much compound interest is earned by investing \$8000 at 2.9% p.a. over a 3 year period?

Example 13 **Self Tutor**

£5000 is invested for 4 years at 3% p.a. interest compounded quarterly.
Find the value of the investment at the end of this period.

There are $n = 4 \times 4 = 16$ time periods.

Each time period the investment increases by $i = \frac{3\%}{4} = 0.75\%$.

$$\therefore \text{ the amount after 4 years is } \begin{aligned}[t] u_{16} &= u_0 \times (1+i)^{16} \\ &= 5000 \times (1.0075)^{16} \qquad \{0.75\% = 0.0075\} \\ &\approx 5634.96 \end{aligned}$$

The investment will amount to £5634.96 .

4 \$20 000 is invested at 4.8% p.a. interest compounded quarterly. Find the value of this investment after:

- **a** 1 year
- **b** 3 years.

5 **a** What will an investment of €30 000 at 5.6% p.a. interest compounded annually amount to after 4 years?

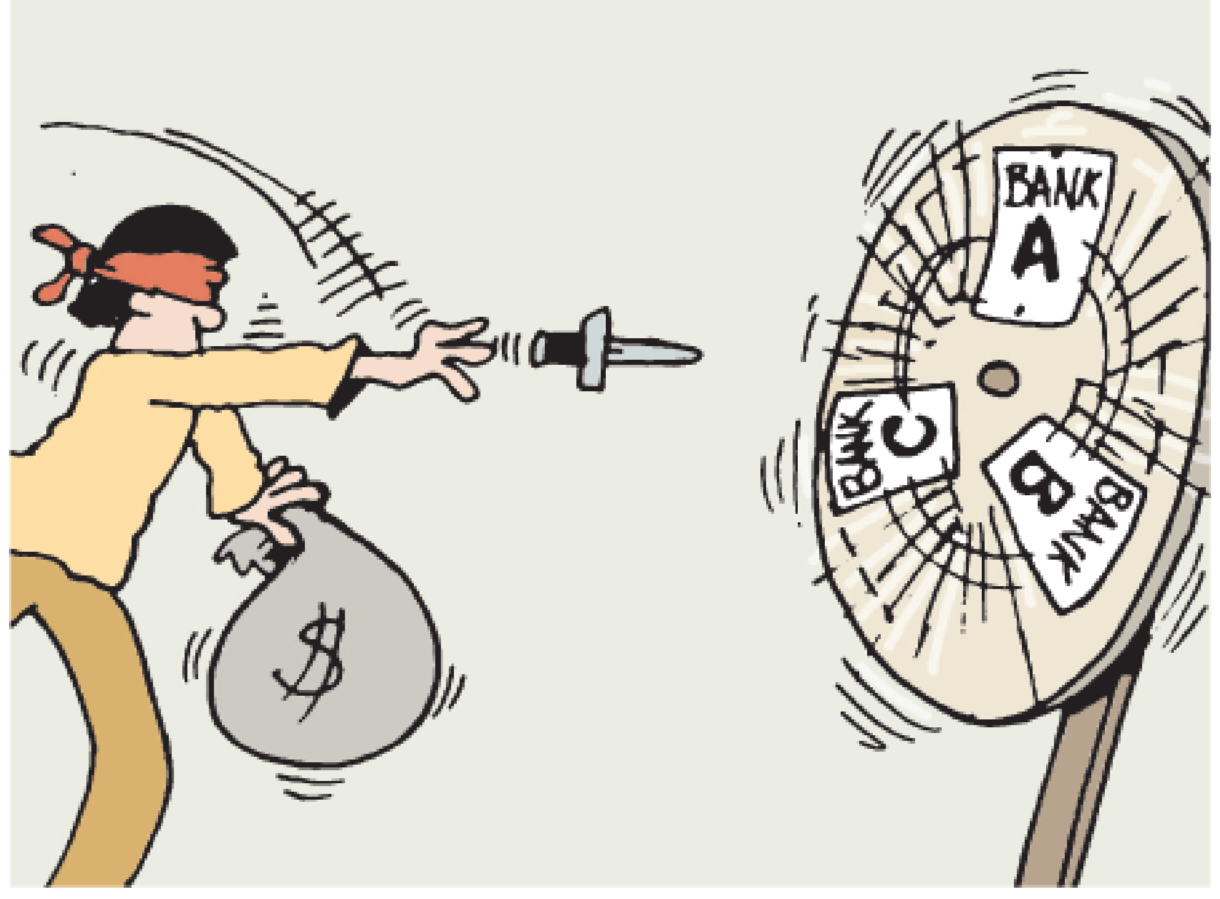
- **b** How much of this is interest?

6 How much interest is earned by investing \$80 000 at 4.4% p.a. for a 3 year period with interest compounded quarterly?

7 Jai recently inherited \$92 000. He decides to invest it for 10 years before he spends any of it. The three banks in his town offer the following terms:

Bank A: $5\frac{1}{2}\%$ p.a. compounded yearly.
Bank B: $5\frac{1}{4}\%$ p.a. compounded quarterly.
Bank C: 5% p.a. compounded monthly.

Which bank offers Jai the greatest interest on his inheritance?



Example 14

Self Tutor

How much does Ivana need to invest now, to get a maturing value of \$10 000 in 4 years' time, given interest at 8% p.a. compounded twice annually? Give your answer to the nearest dollar.

The initial investment u_0 is unknown.

There are $n = 4 \times 2 = 8$ time periods.

Each time period the investment increases by $i = \frac{8\%}{2} = 4\%$.

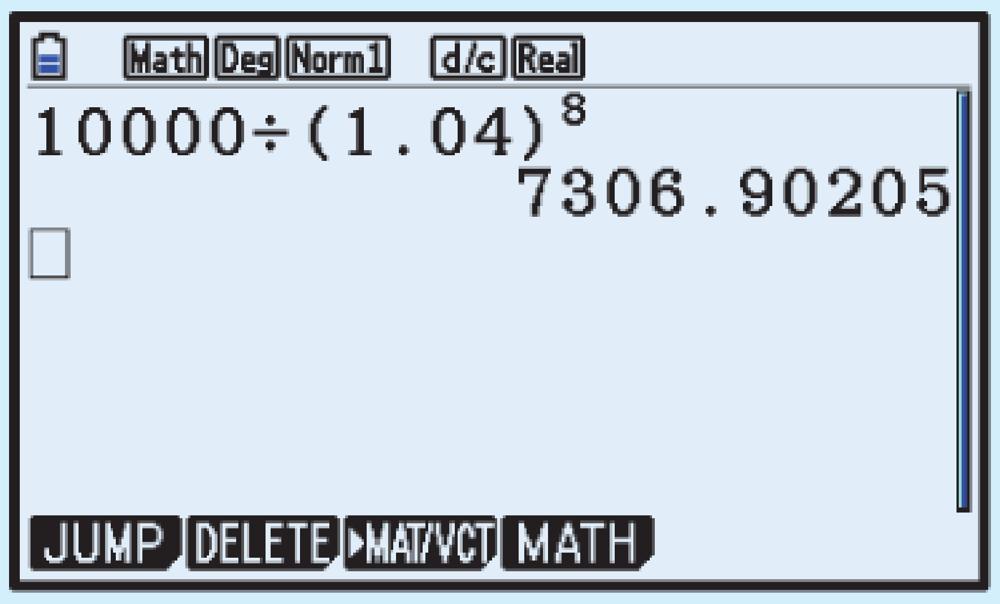
Now $u_8 = u_0 \times (1+i)^8$

$\therefore\ 10\,000 = u_0 \times (1.04)^8 \qquad \{4\% = 0.04\}$

$\therefore\ u_0 = \frac{10\,000}{(1.04)^8}$

$\therefore\ u_0 \approx 7306.90$

Ivana needs to invest \$7307 now.



```
10000÷(1.04)^8
                 7306.90205
```

8 How much does Habib need to invest now, to get a maturing value of £20 000 in 4 years' time, given the money can be invested at a fixed rate of 7.5% p.a. compounded annually? Round your answer up to the next pound.

9 What initial investment is required to produce a maturing amount of \$15 000 in 60 months' time given a guaranteed fixed interest rate of 5.5% p.a. compounded annually? Round your answer up to the next dollar.

10 How much should I invest now to yield \$25 000 in 3 years' time, if the money can be invested at a fixed rate of 4.2% p.a. compounded quarterly?

11 What initial investment will yield ¥4 000 000 in 8 years' time if your money can be invested at 3.6% p.a. compounded monthly?

INFLATION

Inflation is the general increase in the price of goods and services over time. Inflation reduces the **purchasing power** of your money, because a fixed amount of money will buy less goods and services as prices rise over time.

Inflation needs to be taken into account when setting investment goals. For example, suppose you see a painting that you would like to buy, and it costs \$5000 today. If it takes you 3 years to accumulate the \$5000, the price of the painting is likely to have increased in that time. So, you will need to accumulate more than \$5000 to purchase the painting.

To find how much you need to accumulate, you must **index** the value of the painting for inflation.

Example 15 **Self Tutor**

Georgia would like to purchase a painting that is currently worth \$5000. She makes monthly deposits into an investment account, so that she can purchase the painting in 3 years' time.

If inflation averages 2.5% per year, calculate the value of the painting indexed for inflation for 3 years.

To index the value of the painting for inflation, we increase it by 2.5% each year for 3 years.

$$\begin{aligned}\therefore\quad \text{indexed value} &= \$5000 \times (1.025)^3 \\ &= \$5384.45\end{aligned}$$

EXERCISE 5E.2

1 If inflation averages 3% per year, calculate the value of:

- **a** \$8000 indexed for inflation over 2 years
- **b** \$14 000 indexed for inflation over 5 years
- **c** \$22 500 indexed for inflation over 7 years.

2 Hoang currently requires \$1000 per week to maintain his lifestyle. Assuming inflation averages 2% per year, how much will Hoang require per week for him to maintain his current lifestyle in:

- **a** 10 years' time
- **b** 20 years' time
- **c** 30 years' time?

3 A holiday package is valued at \$15 000 today.
If inflation averages 2% per year, calculate the value of the holiday package indexed for inflation over 4 years.

THE REAL VALUE OF AN INVESTMENT

To understand how well an investment will perform, we can consider its final value in terms of today's purchasing power. We call this the **real value** of the investment.

We have seen that to index a value for inflation, we *multiply* its value by the inflation multiplier each year. So, to consider the final value of an investment in today's dollars, we *divide* its value by the inflation multiplier each year.

Example 16 **Self Tutor**

Gemma invested \$4000 in an account for 5 years at 4.8% p.a. interest compounded half-yearly. Inflation over the period averaged 3% per year.

a Calculate the value of the investment after 5 years.

b Find the real value of the investment by indexing it for inflation.

a There are $n = 5 \times 2 = 10$ time periods.

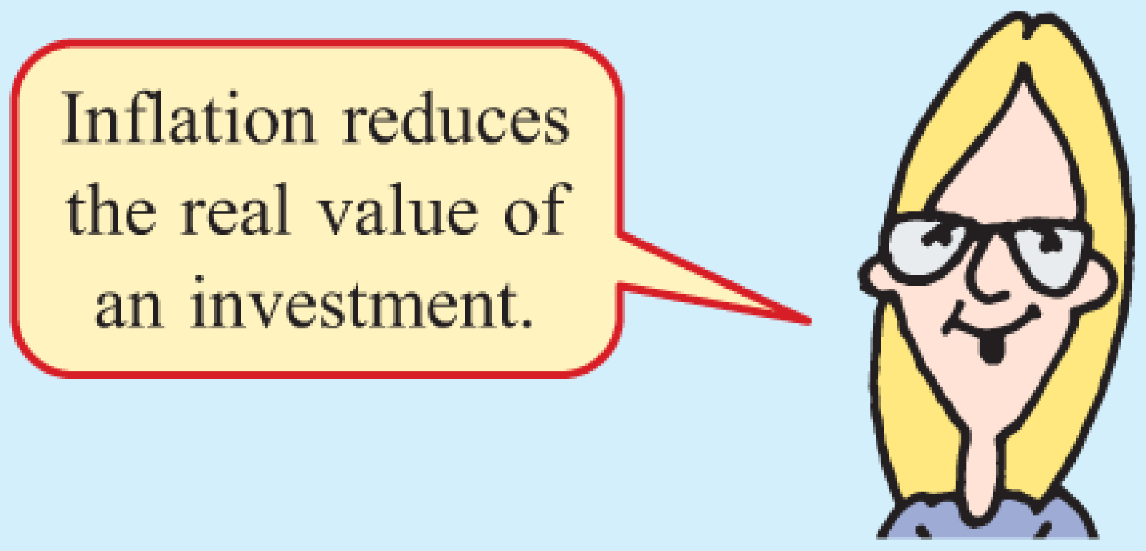
Each period, the investment increases by $i = \frac{4.8\%}{2} = 2.4\%$.

$$\therefore \text{ the amount after 5 years is } \quad u_{10} = u_0 \times (1+i)^{10}$$
$$= 4000 \times (1.024)^{10}$$
$$\approx 5070.60$$

The investment will amount to \$5070.60.

b $\text{real value} \times (1.03)^5 = \$5070.60$

$$\therefore \quad \text{real value} = \frac{\$5070.60}{(1.03)^5}$$
$$= \$4373.94$$



EXERCISE 5E.3

1 Ernie invested \$5000 in an account for 3 years at 3.6% p.a. interest compounded quarterly. Inflation over the period averaged 2% per year.

- **a** Calculate the value of the investment after 3 years.
- **b** Find the real value of the investment by indexing it for inflation.

2 Gino invested €20 000 in an account for 4 years at 4.2% p.a. interest compounded monthly. Inflation over the period averaged 3.4% per year.

- **a** Find the value of Gino's investment after 4 years.
- **b** Find the real value of the investment.

3 Brooke invested \$4000 in an account that pays 3% p.a. interest compounded half-yearly for 6 years.

- **a** Calculate the final value of the investment.
- **b** How much interest did Brooke earn?
- **c** Given that inflation averaged 3.2% per year over the investment period, find the real value of the investment.
- **d** Discuss the effectiveness of the investment once inflation has been considered.

4 Jerome places \$6000 in an investment account. The account pays 0.5% interest per month, and inflation is 0.1% per month.

- **a** Explain why the real interest rate is approximately 0.4% per month.
- **b** Hence find the real value of the investment after 2 years.

5 Suppose \$$u_0$ is invested in an account which pays i% interest per quarter, and that inflation is r% per quarter. Write a formula for the *real value* of the investment after y years.

DEPRECIATION

Assets such as computers, cars, and furniture lose value as time passes. This is due to wear and tear, technology becoming old, fashions changing, and other reasons.

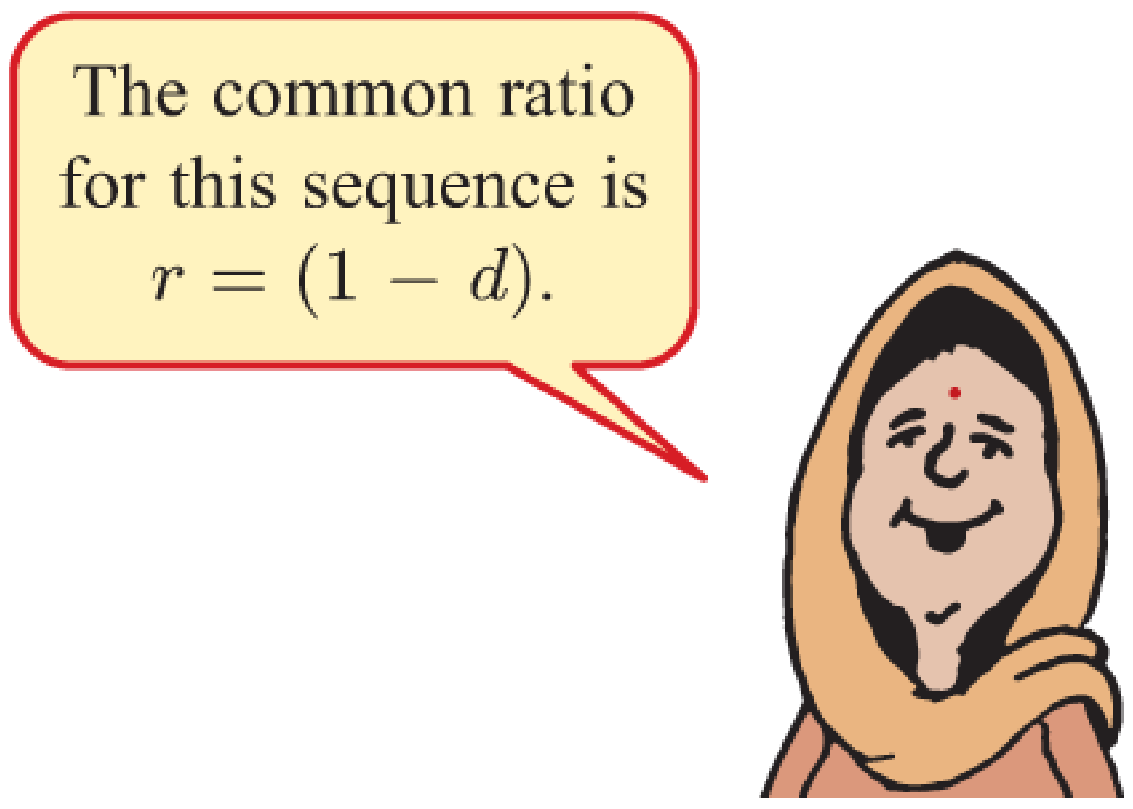
Depreciation is the loss in value of an item over time.

Mathematically, depreciation is similar to compound interest. In the case of compound interest, the investment increases by a fixed percentage each time period. For depreciation, the value *decreases* by a fixed percentage each time period.

The value of an object after n years is $u_n = u_0(1-d)^n$

where u_0 is the initial value of the object

and d is the rate of depreciation per annum.



Example 17 **Self Tutor**

An industrial dishwasher was purchased for £2400 and depreciated by 15% each year.

a Find its value after six years. **b** By how much did it depreciate?

a $u_6 = u_0 \times (1-d)^6$

$= 2400 \times (0.85)^6 \quad \{15\% = 0.15\}$

≈ 905.16

So, after 6 years the value is £905.16 .

b The depreciation $=$ £2400 $-$ £905.16

$=$ £1494.84

EXERCISE 5E.4

1 A lathe is purchased by a workshop for €2500. It depreciates by 20% each year. Find the value of the lathe after 3 years.

2 A tractor was purchased for €110 000, and depreciates at 25% p.a. for 5 years.

- **a** Find its value at the end of this period.
- **b** By how much has it depreciated?

3 Suppose I buy a laptop for ¥87 500 and keep it for 3 years. During this time it depreciates at an annual rate of 30%.

- **a** Find its value after 3 years.
- **b** By how much has the laptop depreciated?

4 A printing press costing $250 000 was sold 4 years later for $80 000. At what yearly rate did it depreciate in value?

USING TECHNOLOGY FOR FINANCIAL MODELS

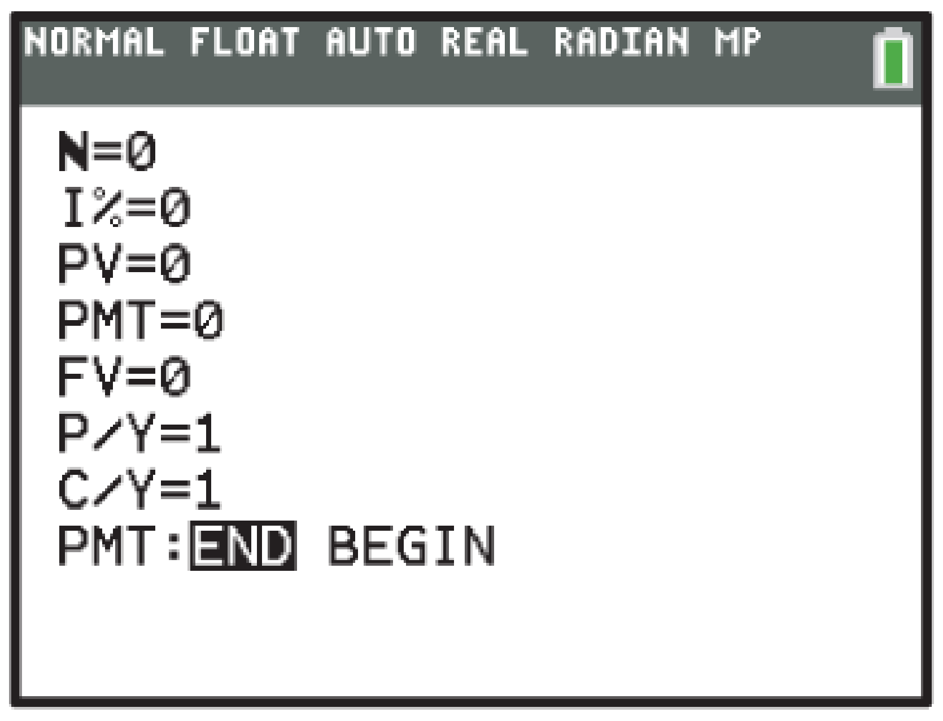
To solve more complicated problems involving financial models, we must use technology.

Most graphics calculators have an in-built **finance program** that can help with finance problems. This is called a **TVM Solver**, where **TVM** stands for **time value of money**.

The TVM Solver can be used to find any variable if all the other variables are given. For the **TI-84 Plus CE**, the abbreviations used are:

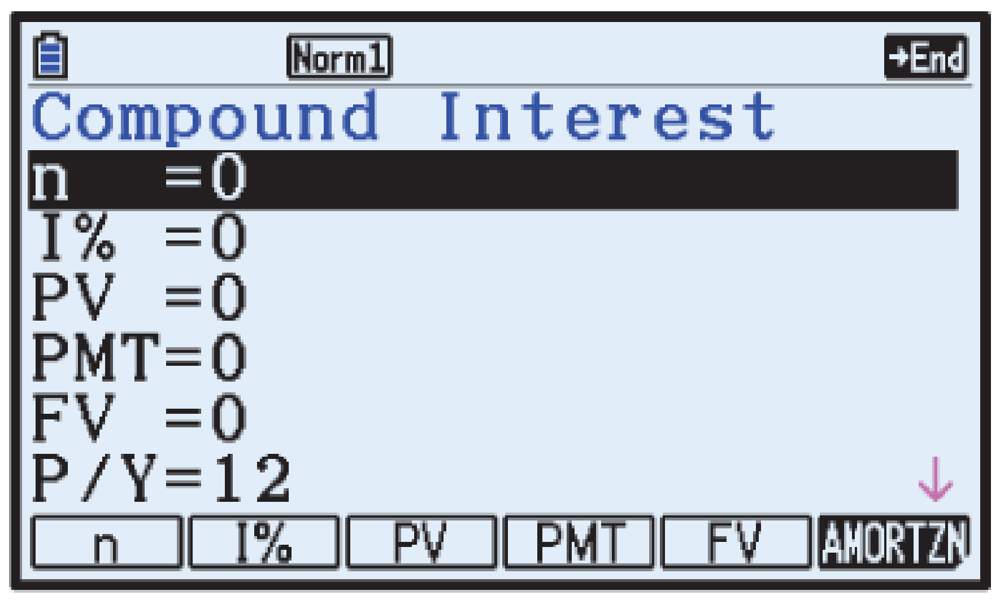
- N represents the **number of compounding periods**
- $I\%$ represents the **interest rate per year**
- PV represents the **present value** of the investment
- PMT represents the **payment each time period**
- FV represents the **future value** of the investment
- P/Y is the **number of payments per year**
- C/Y is the **number of compounding periods per year**
- PMT : END BEGIN lets you choose between payments at the end of a time period or payments at the beginning of a time period. Most interest payments are made at the end of the time periods.

TI-84 Plus CE



```
N=0
I%=0
PV=0
PMT=0
FV=0
P/Y=1
C/Y=1
PMT:END BEGIN
```

Casio fx-CG50



```
Compound Interest
n   =0
I%  =0
PV  =0
PMT=0
FV  =0
P/Y=12
```

Click on the icon to obtain instructions for using the finance program on your calculator.

GRAPHICS CALCULATOR INSTRUCTIONS

When calculating compound interest using electronic technology, notice that:

- The initial investment is entered as a negative value, because that money is moving from you to the bank. The future value is the money you receive at the end of the investment, so FV is positive.
- N represents the number of compounding periods, not the number of years.
- I is always the percentage interest rate *per annum*.

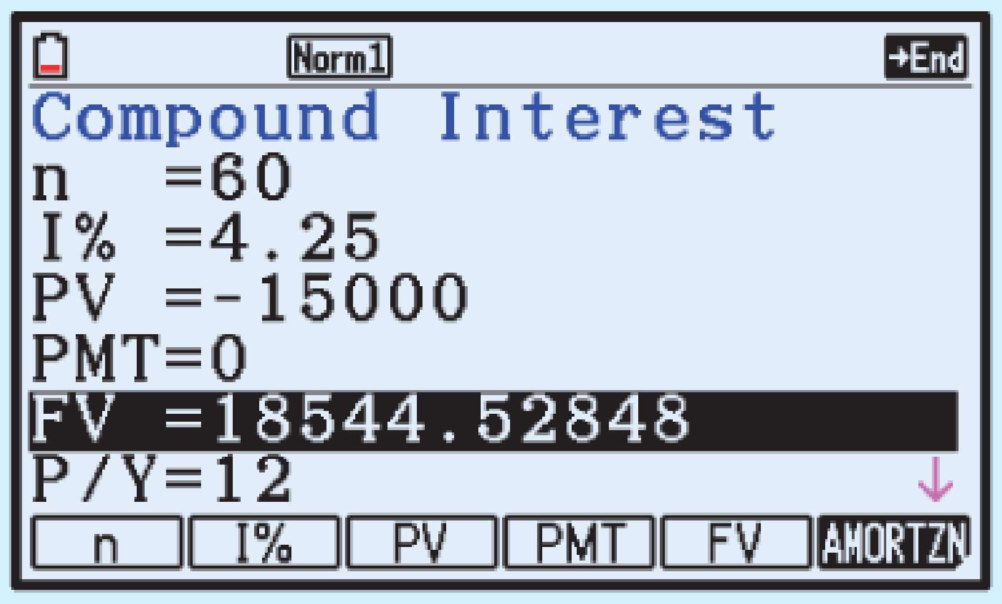
Example 18 **Self Tutor**

Sally invests \$15 000 in an account that pays 4.25% p.a. compounded monthly. How much is her investment worth after 5 years?

$N = 5 \times 12 = 60, \quad I\% = 4.25, \quad PV = -15\,000, \quad PMT = 0,$
$P/Y = 12, \quad C/Y = 12$

$\therefore \quad FV \approx 18\,544.53$

Sally's investment is worth \$18 544.53 after 5 years.



```
Compound Interest
n   =60
I%  =4.25
PV  =-15000
PMT=0
FV  =18544.52848
P/Y=12
```

EXERCISE 5E.5

1 Enrique invests 60 000 pesos at 3.7% p.a. compounded annually. Find the value of his investment after 6 years.

2 I deposit \$6000 in a bank account that pays 5% p.a. compounded monthly. How much will I have in my account after 2 years?

3 Kenneth sold his boat for \$8000, and deposited the money in a bank account paying 5.6% p.a. compounded quarterly. How much will Kenneth have in his account after:

a 3 years **b** 8 years?

4 Drew invested €5000 in an account paying 4.2% p.a. interest compounded monthly.

   a Find the amount in the account after 7 years.

   b Calculate the interest earned.

5 €4000 is invested in an account that pays 1.2% interest per quarter. At this time, inflation averages 0.5% per quarter.

   a Find the real rate of interest per year.

   b Hence find the real value of the investment after 5 years.

Example 19

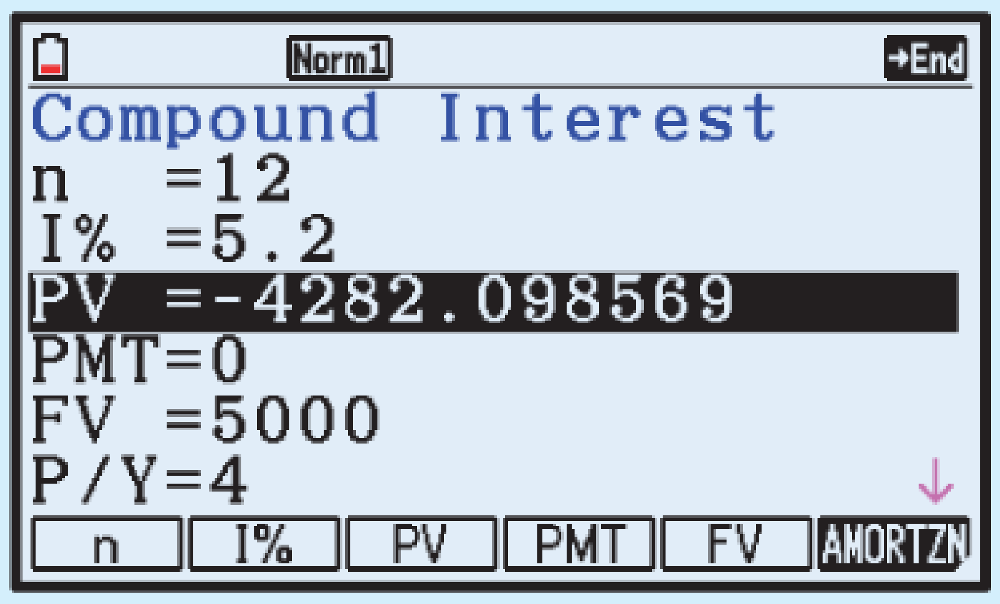
Self Tutor

Halena is investing money in a term deposit paying 5.2% p.a. compounded quarterly. How much does she need to deposit now, in order to collect \$5000 at the end of 3 years?

$N = 3 \times 4 = 12, \quad I\% = 5.2, \quad PMT = 0, \quad FV = 5000,$
$P/Y = 4, \quad C/Y = 4$

$\therefore \quad PV \approx -4282.10$

Thus, \$4282.10 needs to be deposited.



```
Compound Interest
n   =12
I%  =5.2
PV  =-4282.098569
PMT=0
FV  =5000
P/Y=4
```

6 How much would you need to invest now in order to accumulate \$2500 in 5 years' time, if the interest rate is 4.5% p.a. compounded monthly?

7 You have just won the lottery and decide to invest the money. Your accountant advises you to deposit your winnings in an account that pays 6.5% p.a. compounded annually. After four years your winnings have grown to \$102 917.31 . How much did you win in the lottery?

8 Donald bought a new stereo for \$458. If it depreciated in value by 25% p.a., find its value after 5 years.

To use the TVM Solver for depreciation, I will be negative.

Example 20

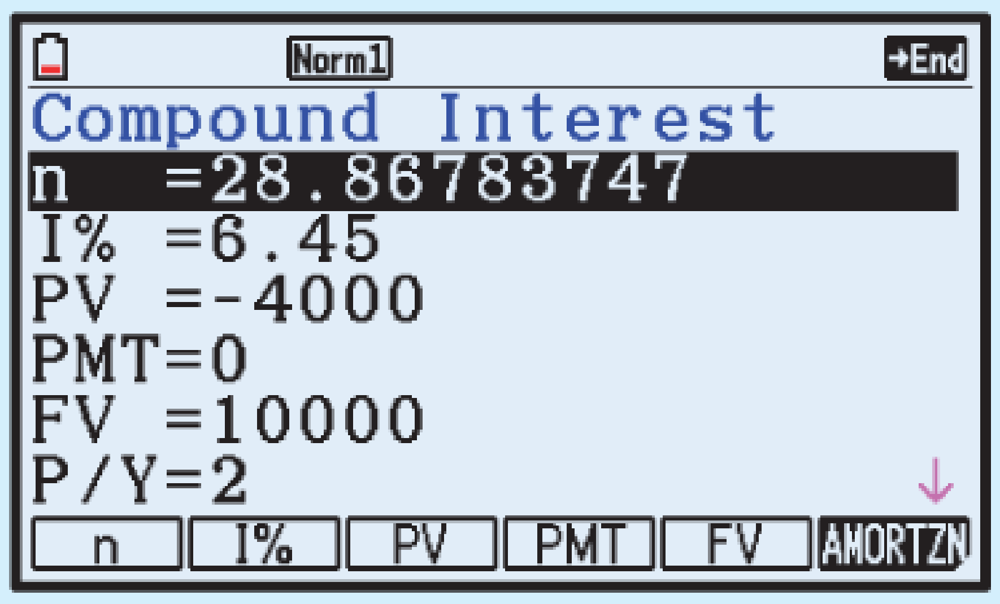
Self Tutor

For how long must Magnus invest \$4000 at 6.45% p.a. compounded half-yearly, for it to amount to \$10 000?

$I\% = 6.45, \quad PV = -4000, \quad PMT = 0, \quad FV = 10\,000,$
$P/Y = 2, \quad C/Y = 2$

$\therefore \quad N \approx 28.9$

So, 29 half-years are required, which is 14.5 years.



```
Compound Interest
n   =28.86783747
I%  =6.45
PV  =-4000
PMT=0
FV  =10000
P/Y=2
```

9 A couple inherited \$40 000 and deposited it in an account paying $4\frac{1}{2}\%$ p.a. compounded quarterly. They withdrew the money as soon as they had over \$45 000. How long did they keep the money in that account?

10 A business deposits €80 000 in an account that pays $5\frac{1}{4}\%$ p.a. compounded monthly. How long will it take before they double their money?

11 Farm vehicles are known to depreciate in value by 12% each year. If Susan buys a quadrunner for \$6800, how long will it take for the value to reduce to \$1000?

Example 21

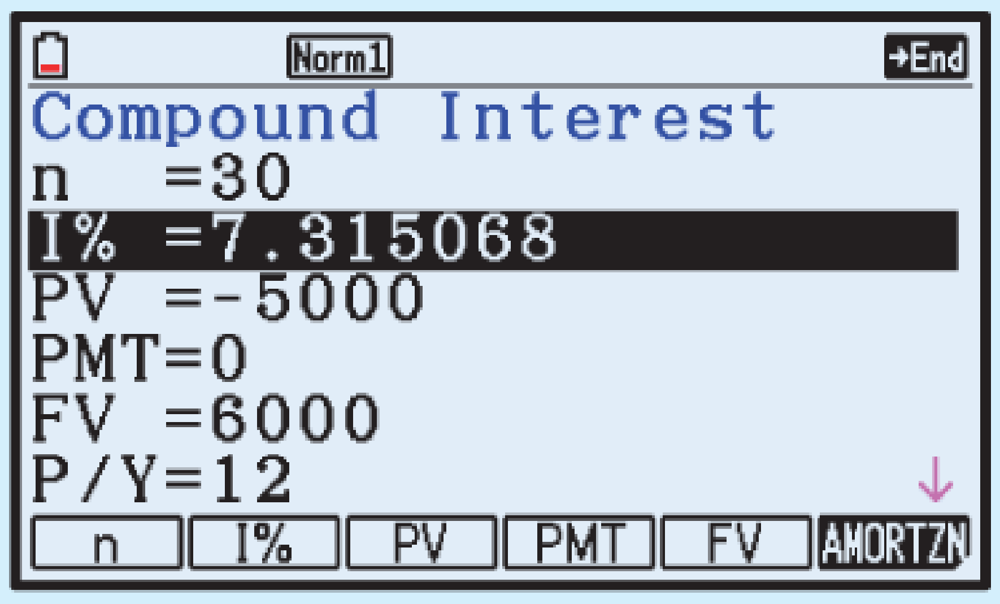
Self Tutor

Iman deposits \$5000 in an account that compounds interest monthly. 2.5 years later, the account has balance \$6000. What annual rate of interest has been paid?

$N = 2.5 \times 12 = 30, \quad PV = -5000, \quad PMT = 0, \quad FV = 6000, \quad P/Y = 12, \quad C/Y = 12$

$\therefore \quad I\% \approx 7.32$

The interest rate is 7.32% p.a.



```
Compound Interest
n   =30
I%  =7.315068
PV  =-5000
PMT=0
FV  =6000
P/Y=12
```

12 An investor has purchased rare medals for \$10 000 and hopes to sell them 3 years later for \$15 000. What must the annual percentage increase in the value of the medals be, in order for the investor's target to be reached?

13 I deposited €5000 into an account that compounds interest monthly. $3\frac{1}{2}$ years later the account has balance €6165. What annual rate of interest did the account pay?

14 A young couple invests their savings of \$9000 in an account where the interest is compounded quarterly. Three years later the account balance is \$10 493. What interest rate has been paid?

15 A new sports car devalues from £68 500 to £26 380 over 4 years. Find the annual rate of depreciation.

F SERIES

There are many situations where we are interested in finding the sum of the terms of a number sequence.

A **series** is the sum of the terms of a sequence.

For a **finite** sequence with n terms, the corresponding series is $u_1 + u_2 + u_3 + ... + u_n$.

The sum of this series is $S_n = u_1 + u_2 + u_3 + ... + u_n$ and this will always be a finite real number.

For an **infinite** sequence the corresponding series is $u_1 + u_2 + u_3 + ... + u_n + ...$

In many cases, the sum of an infinite series cannot be calculated. In some cases, however, it does **converge** to a finite number.

SIGMA NOTATION

$u_1 + u_2 + u_3 + u_4 + ... + u_n$ can be written more compactly using **sigma notation** or **summation notation**.

The symbol $\sum$ is called **sigma**. It is the equivalent of capital S in the Greek alphabet.

We write $u_1 + u_2 + u_3 + u_4 + ... + u_n$ as $\sum_{k=1}^{n} u_k$.

$\sum_{k=1}^{n} u_k$ reads "the sum of all numbers of the form u_k where $k = 1, 2, 3, ...,$ up to n".

Example 22 **Self Tutor**

Consider the sequence 1, 4, 9, 16, 25, ...

a Write down an expression for S_n.

b Find S_n for $n = 1, 2, 3, 4,$ and 5.

a $S_n = 1^2 + 2^2 + 3^2 + 4^2 + ... + n^2$ {all terms are squares}

$= \sum_{k=1}^{n} k^2$

b $S_1 = 1$

$S_2 = 1 + 4 = 5$

$S_3 = 1 + 4 + 9 = 14$

$S_4 = 1 + 4 + 9 + 16 = 30$

$S_5 = 1 + 4 + 9 + 16 + 25 = 55$

Example 23 **Self Tutor**

Expand and evaluate:

a $\sum_{k=1}^{7} (k+1)$

b $\sum_{k=1}^{5} \frac{1}{2^k}$

a $\sum_{k=1}^{7} (k+1) = 2 + 3 + 4 + 5 + 6 + 7 + 8$

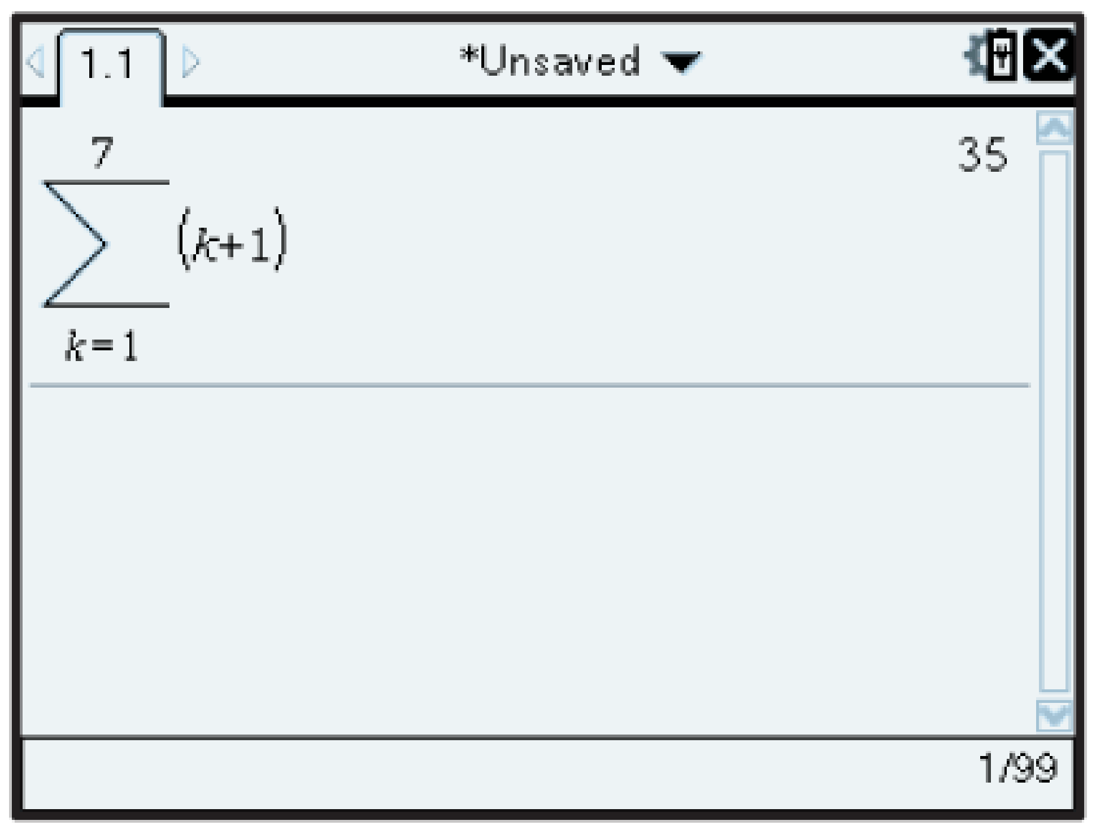
$= 35$

b $\sum_{k=1}^{5} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}$

$= \frac{31}{32}$

You can also use technology to evaluate the sum of a series in sigma notation. Click on the icon for instructions.

GRAPHICS CALCULATOR INSTRUCTIONS



$\sum_{k=1}^{7} (k+1)$ 35

PROPERTIES OF SIGMA NOTATION

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

If c is a constant, $\sum_{k=1}^{n} ca_k = c \sum_{k=1}^{n} a_k$ and $\sum_{k=1}^{n} c = cn$.

EXERCISE 5F

1 Consider the sequence of composite numbers $4, 6, 8, 9, 10, 12, 14, 15, 16, ...$
Find:

a S_3 **b** S_5 **c** S_{12}

2 Suppose a sequence has $S_4 = 13$ and $S_5 = 20$. Find the value of u_5.

3 For each of the following sequences:

i Write down an expression for S_n. **ii** Find S_5.

a $3, 11, 19, 27, ...$ **b** $42, 37, 32, 27, ...$ **c** $12, 6, 3, 1\frac{1}{2}, ...$

d $2, 3, 4\frac{1}{2}, 6\frac{3}{4}, ...$ **e** $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ...$ **f** $1, 8, 27, 64, ...$

4 Expand and evaluate:

a $\sum_{k=1}^{3} 4k$ **b** $\sum_{k=1}^{6} (k+1)$ **c** $\sum_{k=1}^{4} (3k-5)$

d $\sum_{k=1}^{5} (11-2k)$ **e** $\sum_{k=1}^{7} k(k+1)$ **f** $\sum_{k=1}^{5} 10 \times 2^{k-1}$

5 For $u_n = 3n - 1$, write $u_1 + u_2 + u_3 + ... + u_{20}$ using sigma notation and evaluate the sum.

6 Show that:

a $\sum_{k=1}^{n} c = cn$ **b** $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$ **c** $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

Example 24 **Self Tutor**

a Expand $\sum_{k=1}^{n} 2k$ then write the expansion again underneath but with the terms in the reverse order.

b Add the terms vertically, and hence write an expression for the sum S_n of the first n even integers.

a $\sum_{k=1}^{n} 2k = 2 + 4 + 6 + ... + (2n-2) + 2n$

or $2n + (2n-2) + (2n-4) + ... + 4 + 2$

b
$$\begin{aligned} 2\sum_{k=1}^{n} 2k &= (2n+2) + (2n+2) + (2n+2) + ... + (2n+2) + (2n+2) \\ &= n(2n+2) \\ &= 2n(n+1) \end{aligned}$$

$$\therefore \sum_{k=1}^{n} 2k = n(n+1)$$

7 **a** Expand $\sum_{k=1}^{n} k$ then write the expansion again underneath with the terms in the reverse order.

b Add the terms vertically, and hence write an expression for the sum S_n of the first n integers.

c Hence find a and b such that $\sum_{k=1}^{n} (ak+b) = 8n^2 + 11n$ for all positive integers n.

8 Write an expression for the sum of the first n positive odd integers.

9 **a** Show that $(x+1)^3 = x^3 + 3x^2 + 3x + 1$.

b Copy and complete:

$$\begin{aligned}(0+1)^3 &= 0^3 + 3(0)^2 + 3(0) + 1\\ (1+1)^3 &= 1^3 + 3(1)^2 + 3(1) + 1\\ (2+1)^3 &= 2^3 + ...\\ (3+1)^3 &= ....\\ &\vdots\\ (n+1)^3 &= ....\end{aligned}$$

c Add the terms vertically, and hence show that $(n+1)^3 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + (n+1)$.

d Given that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, show that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

10 Given that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ and $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, write $\sum_{k=1}^{n} (k+1)(k+2)$ in simplest form.

Check your answer in the case when $n = 10$.

G ARITHMETIC SERIES

An **arithmetic series** is the sum of the terms of an arithmetic sequence.

For example: 21, 23, 25, 27, ..., 49 is a finite arithmetic sequence.

21 + 23 + 25 + 27 + ... + 49 is the corresponding arithmetic series.

SUM OF A FINITE ARITHMETIC SERIES

Rather than adding all the terms individually, we can use a formula to find the sum of a finite arithmetic series.

If the first term is u_1, the final term is u_n, and the common difference is d, the terms are $u_1,\ u_1 + d,\ u_1 + 2d,\ ....,\ (u_n - 2d),\ (u_n - d),\ u_n$.

$$\therefore\ S_n = u_1 + (u_1 + d) + (u_1 + 2d) + ... + (u_n - 2d) + (u_n - d) + u_n$$

$$\text{But}\ \ S_n = u_n + (u_n - d) + (u_n - 2d) + ... + (u_1 + 2d) + (u_1 + d) + u_1 \qquad \{\text{reversing them}\}$$

Adding these two equations vertically, we get:

$$2S_n = \underbrace{(u_1+u_n) + (u_1+u_n) + (u_1+u_n) + ... + (u_1+u_n) + (u_1+u_n) + (u_1+u_n)}_{n \text{ of these}}$$

$$\therefore\ 2S_n = n(u_1 + u_n)$$

$$\therefore\ S_n = \frac{n}{2}(u_1 + u_n) \quad \text{where} \quad u_n = u_1 + (n-1)d$$

The sum of a finite arithmetic series with first term u_1, common difference d, and last term u_n, is

$$\boldsymbol{S_n = \frac{n}{2}(u_1 + u_n)} \qquad \textit{or} \qquad \boldsymbol{S_n = \frac{n}{2}(2u_1 + (n-1)d)}.$$

Example 25 **Self Tutor**

Find the sum of $4+7+10+13+....$ to 50 terms.

The series is arithmetic with $u_1=4$, $d=3$, and $n=50$.

$$\text{Now}\quad S_n=\frac{n}{2}(2u_1+(n-1)d)$$

$$\therefore\quad S_{50}=\tfrac{50}{2}(2\times4\;+\;49\times3)$$

$$=3875$$

Example 26 **Self Tutor**

Find the sum of $-6+1+8+15+....+141$.

The series is arithmetic with $u_1=-6$, $d=7$, and $u_n=141$.

First we need to find n.

$$\begin{aligned}\text{Now}\quad u_n&=141\\ \therefore\quad u_1+(n-1)d&=141\\ \therefore\quad -6+7(n-1)&=141\\ \therefore\quad 7(n-1)&=147\\ \therefore\quad n-1&=21\\ \therefore\quad n&=22\end{aligned}$$

$$\begin{aligned}\text{Using}\quad S_n&=\frac{n}{2}(u_1+u_n),\\ S_{22}&=\tfrac{22}{2}(-6+141)\\ &=11\times135\\ &=1485\end{aligned}$$

You can also use technology to evaluate series, although for some calculator models this is tedious.

GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 5G

1 Find the sum of:

- **a** $7+9+11+13+....$ to 10 terms
- **b** $3+7+11+15+....$ to 20 terms
- **c** $\frac{1}{2}+3+5\frac{1}{2}+8+....$ to 50 terms
- **d** $100+93+86+79+....$ to 40 terms
- **e** $(-31)+(-28)+(-25)+(-22)+....$ to 15 terms
- **f** $50+48\frac{1}{2}+47+45\frac{1}{2}+....$ to 80 terms.

2 Find the sum of:

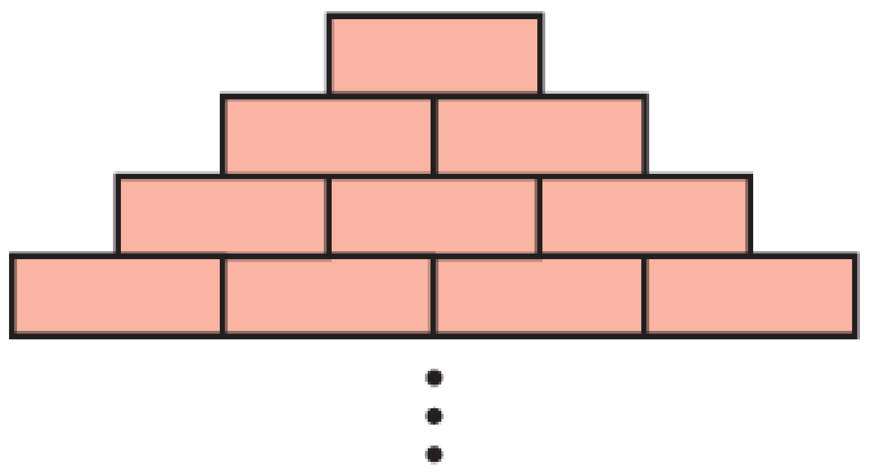
- **a** $5+8+11+14+....+101$
- **b** $37+33+29+25+....+9$
- **c** $50+49\frac{1}{2}+49+48\frac{1}{2}+....+(-20)$
- **d** $8+10\frac{1}{2}+13+15\frac{1}{2}+....+83$

3 Evaluate these arithmetic series:

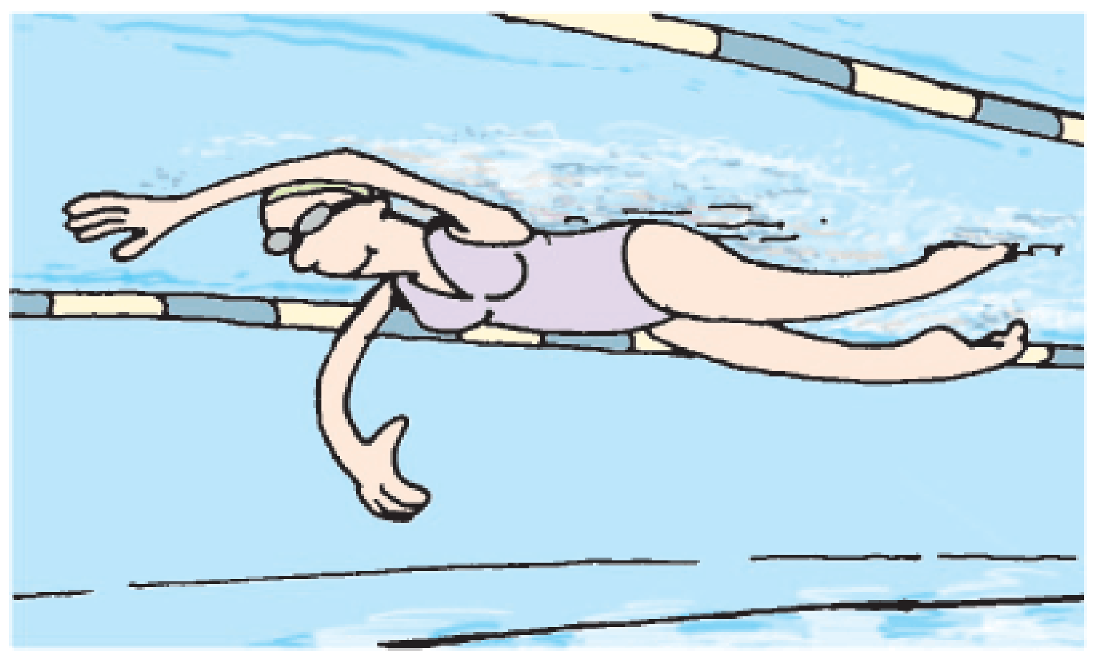
- **a** $\displaystyle\sum_{k=1}^{10}(2k+5)$
- **b** $\displaystyle\sum_{k=1}^{15}(k-50)$
- **c** $\displaystyle\sum_{k=1}^{20}\left(\frac{k+3}{2}\right)$

Check your answers using technology.

4 An arithmetic series has eleven terms. The first term is 6 and the last term is -27. Find the sum of the series.

5 

A bricklayer builds a triangular wall with layers of bricks as shown. If the bricklayer uses 171 bricks, how many layers did he build?



6 Vicki has 30 days to train for a swimming competition. She swims 20 laps on the first day, then each day after that she swims two more laps than the previous day. So, she swims 22 laps on the second day, 24 laps on the third day, and so on.

- **a** How many laps does Vicki swim on:
  - **i** the tenth day
  - **ii** the final day?
- **b** How many laps does Vicki swim in total?

7 A woman deposits \$100 into her son's savings account on his first birthday. She deposits \$125 on his second birthday, \$150 on his third birthday, and so on.

- **a** Calculate the amount of money she will deposit into her son's account on his 15th birthday.
- **b** Find the total amount she will have deposited over the 15 years.

8 A football stadium has 25 sections of seating. Each section has 44 rows of seats, with 22 seats in the first row, 23 in the second row, 24 in the third row, and so on. How many seats are there in:

- **a** row 44 of one section
- **b** each section
- **c** the whole stadium?

9 Find the sum of:

- **a** the first 50 multiples of 11
- **b** the multiples of 7 between 0 and 1000
- **c** the integers from 1 to 100 which are not divisible by 3
- **d** the three-digit numbers which start or end with a "4".

10 $k-1$, $2k+3$, and $27-k$ are the first three terms of an arithmetic sequence.

- **a** Find k.
- **b** Find the sum of the first 25 terms of the sequence.

11 The sixth term of an arithmetic sequence is 21, and the sum of the first seventeen terms is 0. Find the first two terms of the sequence.

12 An arithmetic series has $S_3 = 9$ and $S_6 = 90$. Find S_{10}.

Example 27 **Self Tutor**

An arithmetic sequence has first term 8 and common difference 2. The sum of the terms of the sequence is 170. Find the number of terms in the sequence.

Now $S_n = 170$, so

$$\begin{aligned} \frac{n}{2}(2u_1 + (n-1)d) &= 170 \\ \therefore\ \frac{n}{2}(16 + 2(n-1)) &= 170 \\ \therefore\ 8n + n(n-1) &= 170 \\ \therefore\ n^2 + 7n - 170 &= 0 \\ \therefore\ (n+17)(n-10) &= 0 \\ \therefore\ n &= 10 \quad \{\text{as } n > 0\} \end{aligned}$$

$\therefore$ there are 10 terms in the sequence.

13 An arithmetic sequence has first term 4 and common difference 6. The sum of the terms of the sequence is 200. Find the number of terms in the sequence.

14 An arithmetic sequence has $u_1 = 7$ and $S_2 = 17$.

- **a** Find the common difference of the sequence.
- **b** Find n such that $S_n = 242$.

15 Consider the arithmetic sequence $13, 21, 29, 37, ...$ How many terms are needed for the sum of the sequence terms to exceed 1000?

16 Use the arithmetic sequence formula to prove that the sum of the first n integers is $\dfrac{n(n+1)}{2}$.

17
- **a** Write a formula for the nth odd integer.
- **b** Hence prove that the sum of the first n odd integers is n^2.

18 Three consecutive terms of an arithmetic sequence have a sum of 12 and a product of -80. Find the terms.

19 The sum of the first 15 terms of an arithmetic sequence is 480. Find the 8th term of the sequence.

20 Five consecutive terms of an arithmetic sequence have a sum of 40. The product of the first, middle, and last terms is 224. Find the terms of the sequence.

21 The sum of the first n terms of an arithmetic sequence is $\dfrac{n(3n+11)}{2}$.

- **a** Find its first two terms.
- **b** Find the twentieth term of the sequence.

22 Find $3 - 5 + 7 - 9 + 11 - 13 + 15 - ....$ to 80 terms.

23 The sum of the first n terms of a sequence is $n^2 - \frac{9}{2}n$. Prove that the sequence is arithmetic.

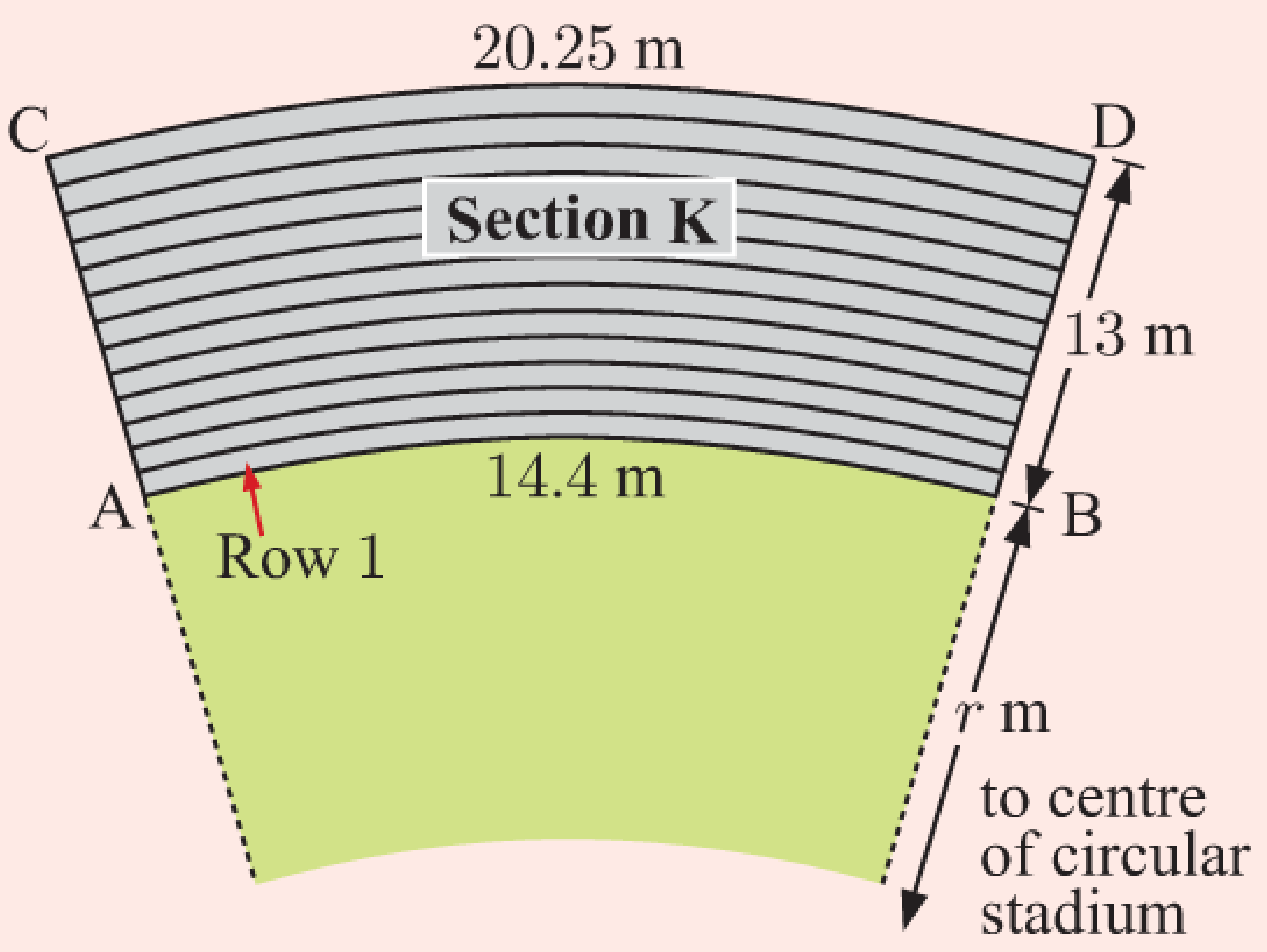
24 Let $u_n = 3 + 2n$.

- **a** For $n = 1, ..., 4,$ plot the points (n, u_n) on a graph, and draw rectangles with vertices (n, u_n), $(n+1, u_n)$, $(n, 0)$, and $(n+1, 0)$.
- **b** Explain how S_n relates to the areas of the rectangles.
- **c** Using your sketch, explain why:
  - **i** $u_{n+1} = u_n + 2$
  - **ii** $S_{n+1} = S_n + u_{n+1}$.

25 An arithmetic sequence has common difference d. The series sums S_2, S_5, and S_7 themselves form an arithmetic sequence. Find, in terms of d, the common difference for this sequence.

ACTIVITY 2 STADIUM SEATING

A circular stadium consists of sections as illustrated, with aisles in between. The diagram shows the 13 tiers of concrete steps for the final section, Section K. Seats are placed along every concrete step, with each seat 0.45 m wide. The arc AB at the front of the first row is 14.4 m long, while the arc CD at the back of the back row is 20.25 m long.



1. How wide is each concrete step?
2. What is the length of the arc of the back of Row 1, Row 2, Row 3, and so on?
3. How many seats are there in Row 1, Row 2, Row 3, ..., Row 13?
4. How many sections are there in the stadium?

5 What is the total seating capacity of the stadium?

6 What is the radius r of the "playing surface"?

THEORY OF KNOWLEDGE

The sequence of odd numbers 1, 3, 5, 7, ... is defined by $u_n = 2n - 1$, $n = 1, 2, 3, 4, ...$

By studying sums of the first few terms of the sequence, we might suspect that the sum of the first n odd numbers is n^2.

$$\begin{aligned} S_1 &= 1 = 1 = 1^2 \\ S_2 &= 1 + 3 = 4 = 2^2 \\ S_3 &= 1 + 3 + 5 = 9 = 3^2 \\ S_4 &= 1 + 3 + 5 + 7 = 16 = 4^2 \\ S_5 &= 1 + 3 + 5 + 7 + 9 = 25 = 5^2 \end{aligned}$$

But is this enough to *prove* that the statement is true for all positive integers n?

1 Can we prove that a statement is true in all cases by checking that it is true for some specific cases?

2 How do we know when we have proven a statement to be true?

In the case of the sum of the first n odd integers, you should have proven the result in the last Exercise using the known, proven formula for the sum of an arithmetic series.

However, in mathematics not all **conjectures** turn out to be true. For example, consider the sequence of numbers $u_n = n^2 + n + 41$.

We observe that:

$u_1 = 1^2 + 1 + 41 = 43$ which is prime
$u_2 = 2^2 + 2 + 41 = 47$ which is prime
$u_3 = 3^2 + 3 + 41 = 53$ which is prime
$u_4 = 4^2 + 4 + 41 = 61$ which is prime.

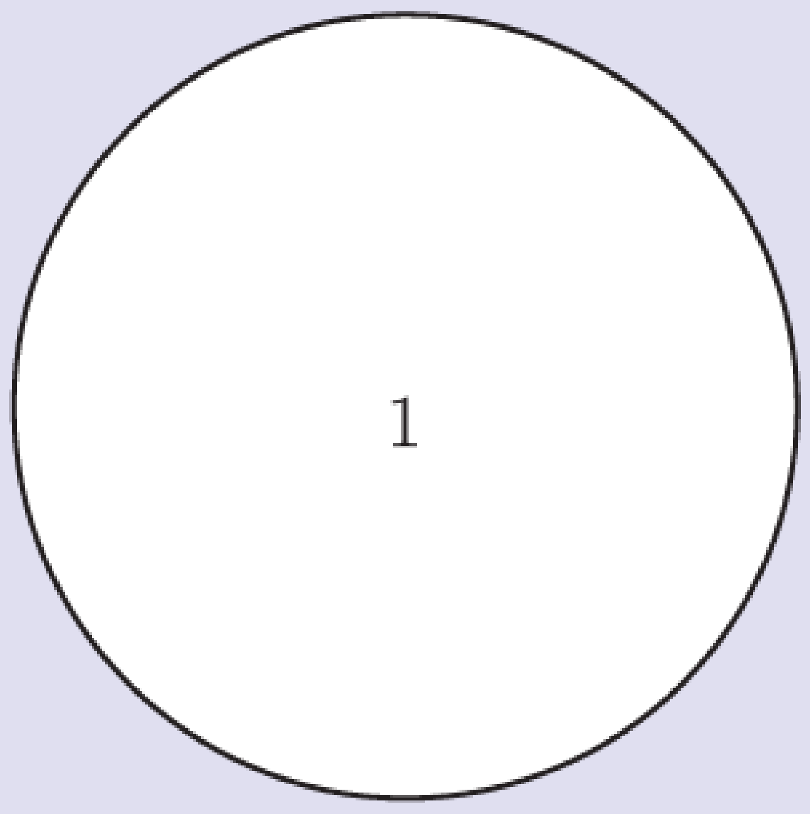
From this, we may *conjecture* that $n^2 + n + 41$ is prime for any positive integer n.

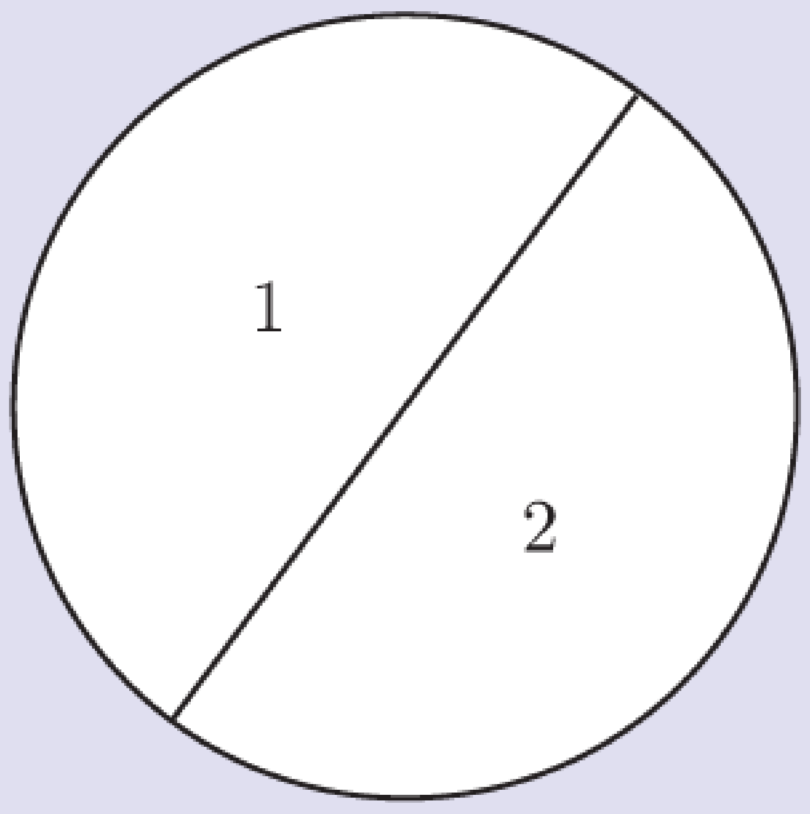
In fact, $n^2 + n + 41$ is prime for all positive integers n from 1 to 39.

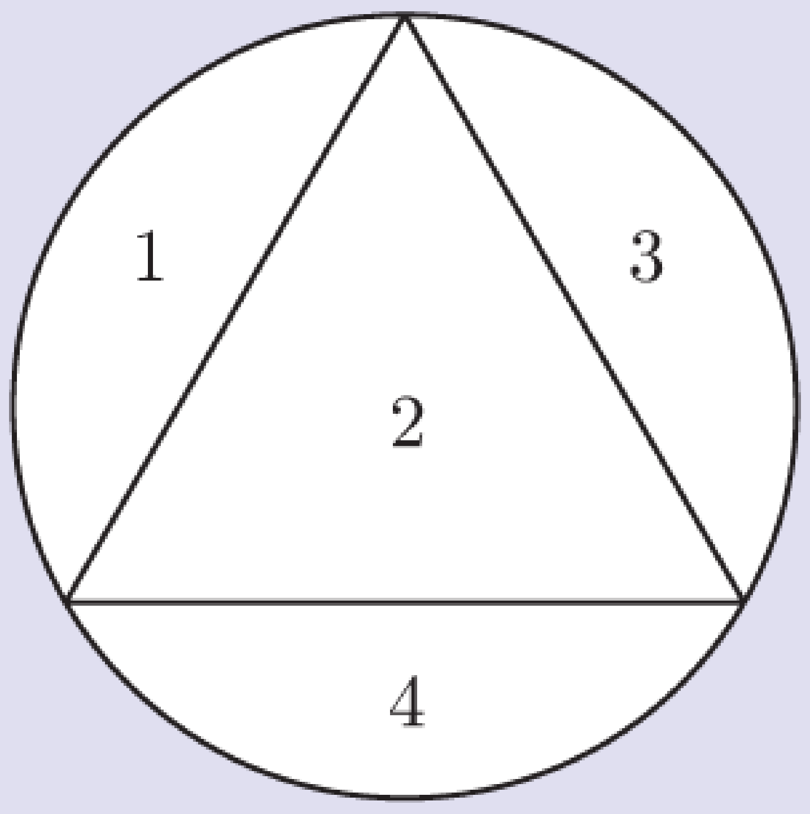
However, $u_{40} = 40^2 + 40 + 41 = 41^2$, so u_{40} is composite.

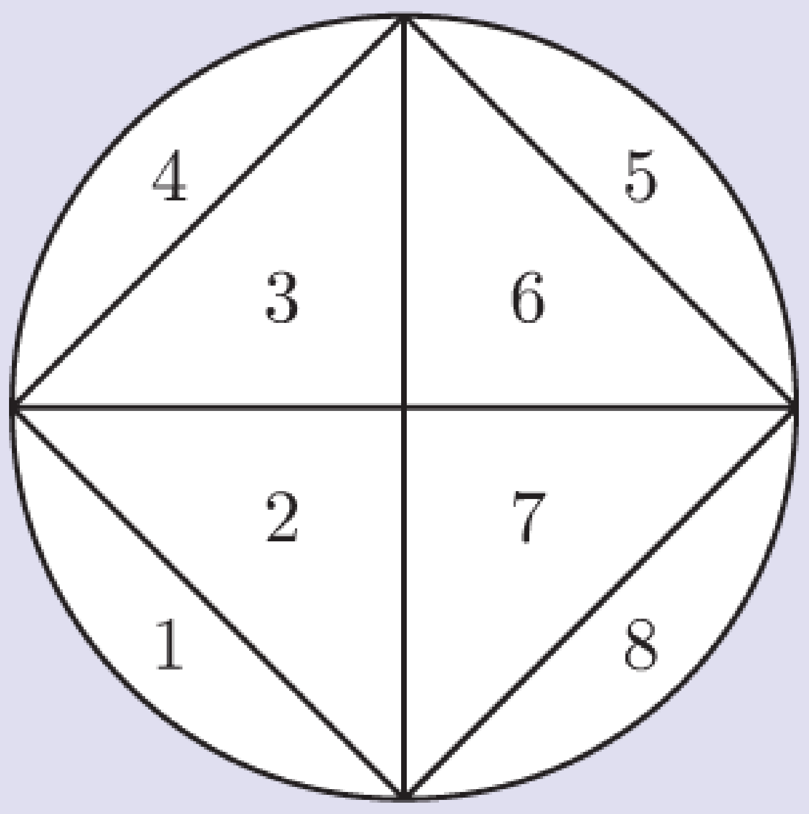
Suppose we place n points around a circle such that when we connect each point with every other point, no three lines intersect at the same point. We then count the number of regions that the circle is divided into.

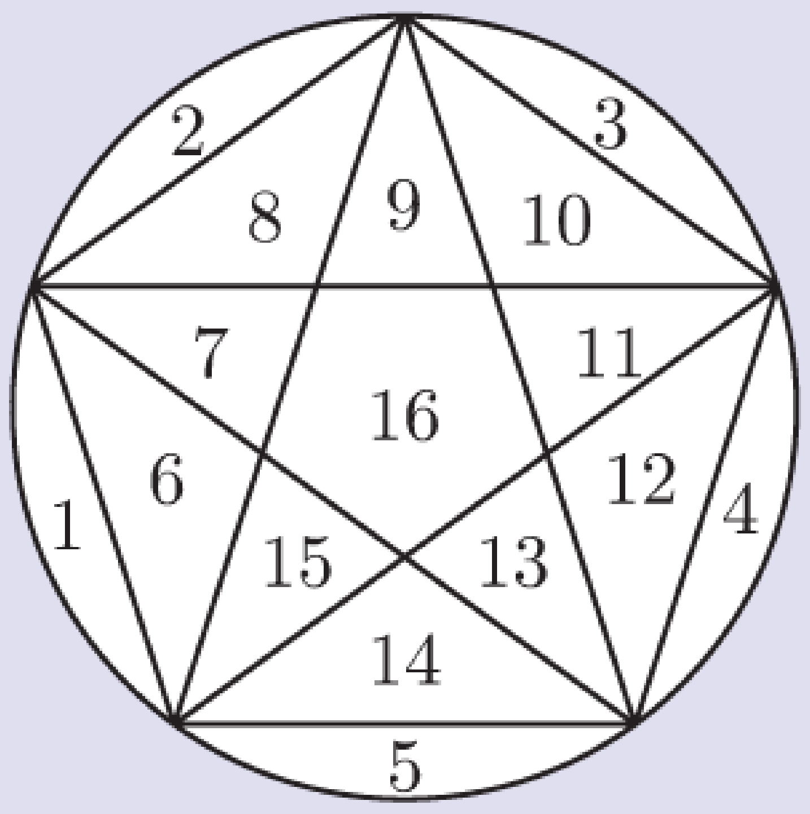
The first five cases are shown below:











From these cases we *conjecture* that for n points, the circle is divided into 2^{n-1} regions.

Draw the case $n = 6$ and see if the conjecture is true!

3 Is it reasonable for a mathematician to assume a conjecture is true until it has been formally proven?

H FINITE GEOMETRIC SERIES

A **geometric series** is the sum of the terms of a geometric sequence.

For example: 1, 2, 4, 8, 16, ..., 1024 is a finite geometric sequence.

$1 + 2 + 4 + 8 + 16 + ... + 1024$ is the corresponding finite geometric series.

If we are adding the first n terms of an infinite geometric sequence, we are then calculating a finite geometric series called the ***n*th partial sum** of the corresponding infinite series.

If we are adding all of the terms in an infinite geometric sequence, we have an **infinite geometric series**.

SUM OF A FINITE GEOMETRIC SERIES

If the first term is u_1 and the common ratio is r, then the terms are: $u_1, u_1r, u_1r^2, u_1r^3, ..., u_1r^{n-1}$.

So, $S_n = u_1 + u_1r + u_1r^2 + u_1r^3 + ... + u_1r^{n-2} + u_1r^{n-1}$

For a finite geometric series with $\boldsymbol{r \neq 1}$,

$$\boldsymbol{S_n = \frac{u_1(r^n - 1)}{r - 1}} \quad \textit{or} \quad \boldsymbol{S_n = \frac{u_1(1 - r^n)}{1 - r}}.$$

Proof:

$$\begin{aligned}
\text{If}\quad S_n &= u_1 + u_1r + u_1r^2 + u_1r^3 + ... + u_1r^{n-2} + u_1r^{n-1} \qquad (*)\\
\text{then}\quad rS_n &= (u_1r + u_1r^2 + u_1r^3 + u_1r^4 + ... + u_1r^{n-1}) + u_1r^n\\
\therefore\ rS_n &= (S_n - u_1) + u_1r^n \qquad \{\text{from } (*)\}\\
\therefore\ rS_n - S_n &= u_1r^n - u_1\\
\therefore\ S_n(r-1) &= u_1(r^n - 1)\\
\therefore\ S_n &= \frac{u_1(r^n-1)}{r-1} \quad \text{or} \quad \frac{u_1(1-r^n)}{1-r} \quad \text{provided } r \neq 1.
\end{aligned}$$

In the case $r = 1$ we have a sequence in which all terms are the same. The sequence is also arithmetic (with $d = 0$), and $S_n = u_1n$.

Example 28 **Self Tutor**

Find the sum of $2+6+18+54+....$ to 12 terms.

The series is geometric with $u_1=2$, $r=3$, and $n=12$.

$$S_n=\frac{u_1(r^n-1)}{r-1}$$

$$\therefore\ S_{12}=\frac{2(3^{12}-1)}{3-1}=531\,440$$

Example 29 **Self Tutor**

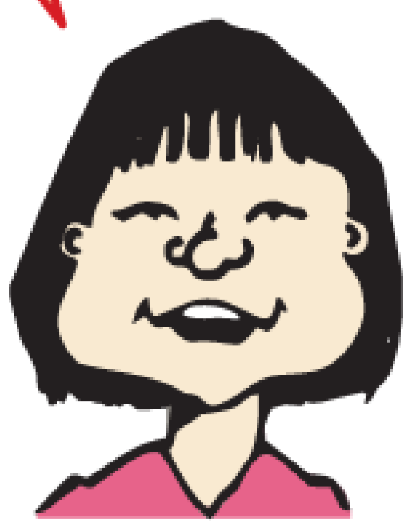
Find a formula for S_n, the sum of the first n terms of the series $9-3+1-\frac{1}{3}+....$

The series is geometric with $u_1=9$ and $r=-\frac{1}{3}$.

$$S_n=\frac{u_1(1-r^n)}{1-r}=\frac{9(1-(-\frac{1}{3})^n)}{\frac{4}{3}}$$

$$\therefore\ S_n=\tfrac{27}{4}(1-(-\tfrac{1}{3})^n)$$

This answer cannot be simplified as we do not know if n is odd or even.



EXERCISE 5H

1 Find the sum of the following series:

- **a** $2+6+18+54+....$ to 8 terms
- **b** $5+10+20+40+....$ to 10 terms
- **c** $12+6+3+1.5+....$ to 10 terms
- **d** $\sqrt{7}+7+7\sqrt{7}+49+....$ to 12 terms
- **e** $6-3+1\frac{1}{2}-\frac{3}{4}+....$ to 15 terms
- **f** $1-\frac{1}{\sqrt{2}}+\frac{1}{2}-\frac{1}{2\sqrt{2}}+....$ to 20 terms.

2 Find a formula for S_n, the sum of the first n terms of the series:

- **a** $\sqrt{3}+3+3\sqrt{3}+9+....$
- **b** $12+6+3+1\frac{1}{2}+....$
- **c** $0.9+0.09+0.009+0.0009+....$
- **d** $20-10+5-2\frac{1}{2}+....$

3 Evaluate these geometric series:

- **a** $\displaystyle\sum_{k=1}^{10} 3\times 2^{k-1}$
- **b** $\displaystyle\sum_{k=1}^{12} \left(\tfrac{1}{2}\right)^{k-2}$
- **c** $\displaystyle\sum_{k=1}^{25} 6\times(-2)^k$

4 At the end of each year, a salesperson is paid a bonus of \$2000 which is always deposited into the same account. It earns a fixed rate of interest of 6% p.a. with interest being paid annually. The total amount in the account at the end of each year will be:

$$A_1=2000$$
$$A_2=A_1\times 1.06+2000$$
$$A_3=A_2\times 1.06+2000 \quad \text{and so on.}$$

- **a** Show that $A_3=2000+2000\times 1.06+2000\times(1.06)^2$.
- **b** Show that $A_4=2000[1+1.06+(1.06)^2+(1.06)^3]$.
- **c** Find the total bank balance after 10 years, assuming there are no fees or withdrawals.

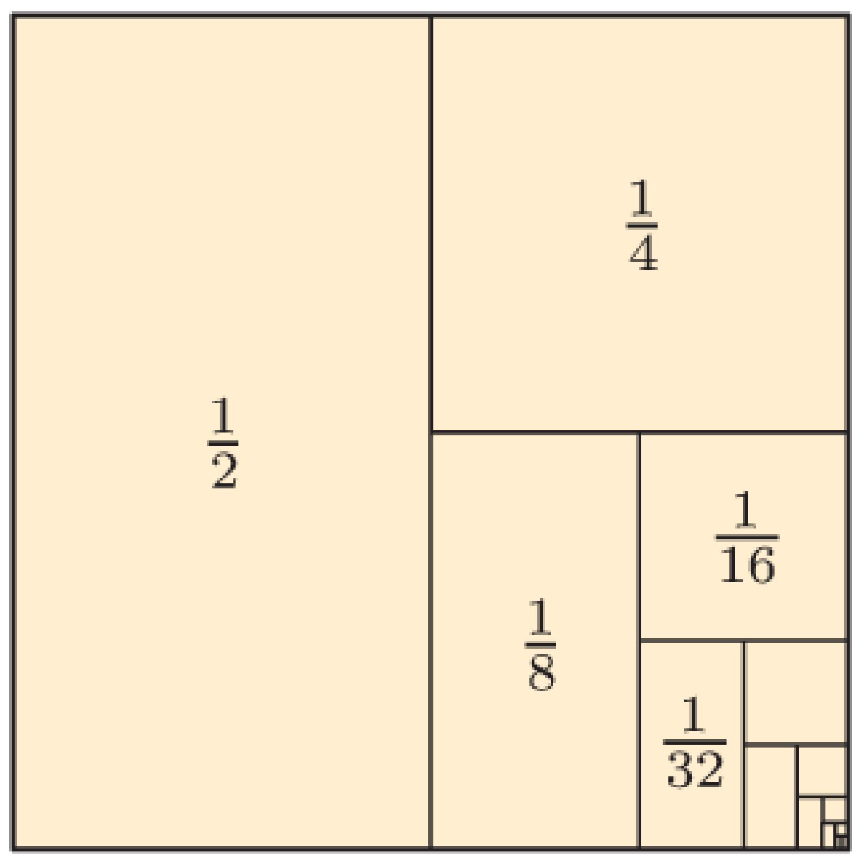
5 Answer the **Opening Problem** on page **90**.

6 Paula has started renting an apartment. She paid \$5000 rent in the first year, and the rent increased by 5% each year.

- **a** Find, to the nearest \$10, the rent paid by Paula in the 4th year.
- **b** Write an expression for the total rent paid by Paula during the first n years.
- **c** How much rent did Paula pay during the first 7 years? Give your answer to the nearest \$10.

7 Jim initially deposits £6000 in an account which earns 5% p.a. interest paid annually. At the end of each year, Jim invests another £1000 in the account. Find the value of the account after 8 years.

8 Consider $S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... + \frac{1}{2^n}$.



- **a** Find S_1, S_2, S_3, S_4, and S_5 in fractional form.
- **b** Hence guess the formula for S_n.
- **c** Find S_n using $S_n = \dfrac{u_1(1-r^n)}{1-r}$.
- **d** Comment on S_n as n gets very large.
- **e** Explain the relationship between the given diagram and **d**.

9 A geometric series has second term 6. The sum of its first three terms is -14. Find its fourth term.

10 An arithmetic and a geometric sequence both have first term 1, and their second terms are equal. The 14th term of the arithmetic sequence is three times the third term of the geometric sequence. Find the twentieth term of each sequence.

11 Suppose $u_1, u_2, ..., u_n$ is a geometric sequence with common ratio r. Show that

$$(u_1+u_2)^2 + (u_2+u_3)^2 + (u_3+u_4)^2 + ... + (u_{n-1}+u_n)^2 = \frac{2{u_1}^2(r^{2n-1}-1)}{r-1} - ({u_1}^2 + {u_n}^2).$$

Example 30 **Self Tutor**

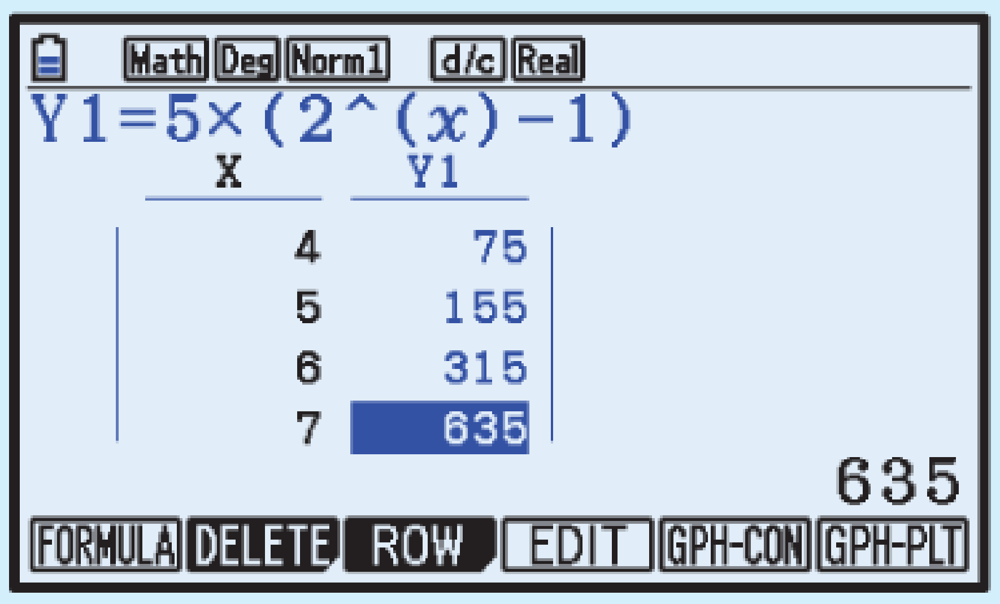
A geometric sequence has first term 5 and common ratio 2. The sum of the first n terms of the sequence is 635. Find n.

The sequence is geometric with $u_1 = 5$ and $r = 2$.

$$\therefore \quad S_n = \frac{u_1(r^n-1)}{r-1} = \frac{5(2^n-1)}{2-1} = 5(2^n-1)$$

To find n such that $S_n = 635$, we use a table of values with $Y_1 = 5 \times (2\hat{}\,X - 1)$:

Casio fx-CG50

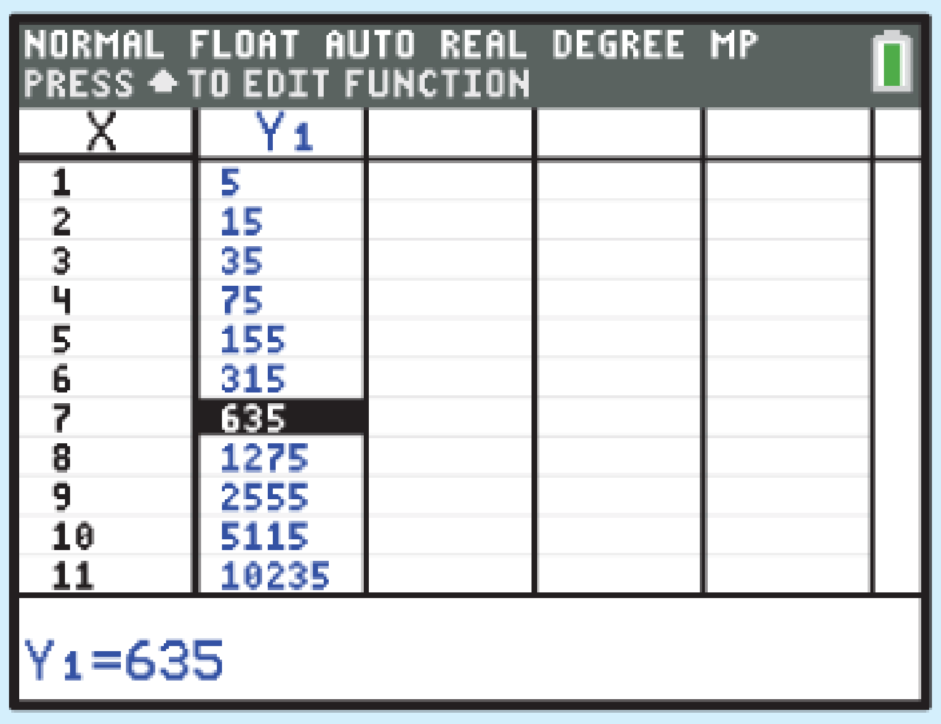


Y1=5×(2^(x)−1)

X	Y1
4	75
5	155
6	315
7	635

635

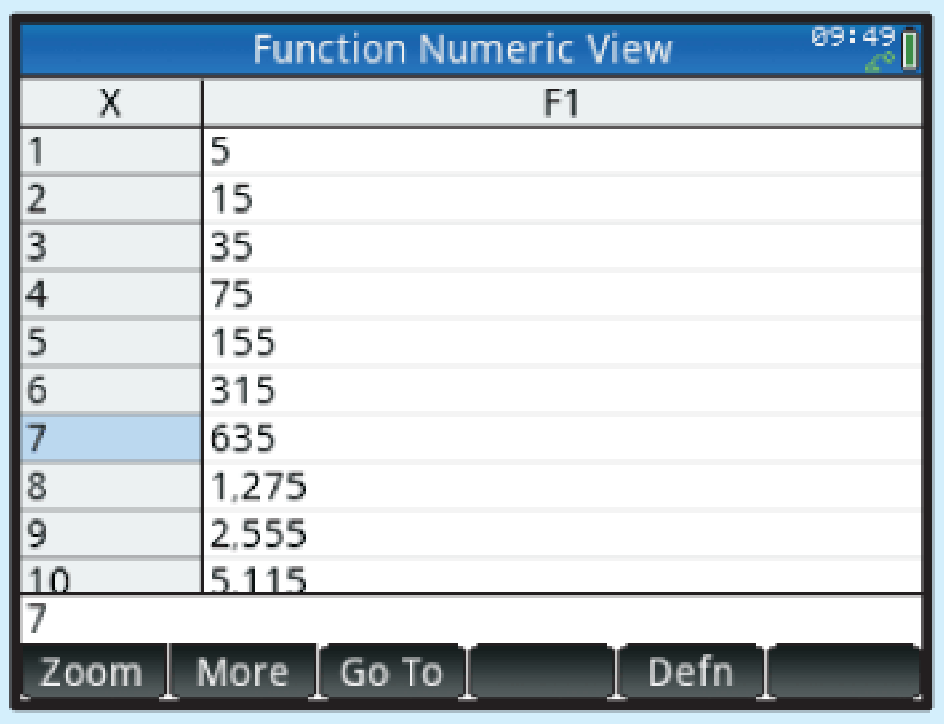
TI-84 Plus CE



X	Y₁
1	5
2	15
3	35
4	75
5	155
6	315
7	635
8	1275
9	2555
10	5115
11	10235

Y₁=635

HP Prime



X	F1
1	5
2	15
3	35
4	75
5	155
6	315
7	635
8	1,275
9	2,555
10	5,115

7

$S_7 = 635$, so $n = 7$.

12 A geometric sequence has first term 6 and common ratio 1.5. The sum of the first n terms of the sequence is 79.125. Find n.

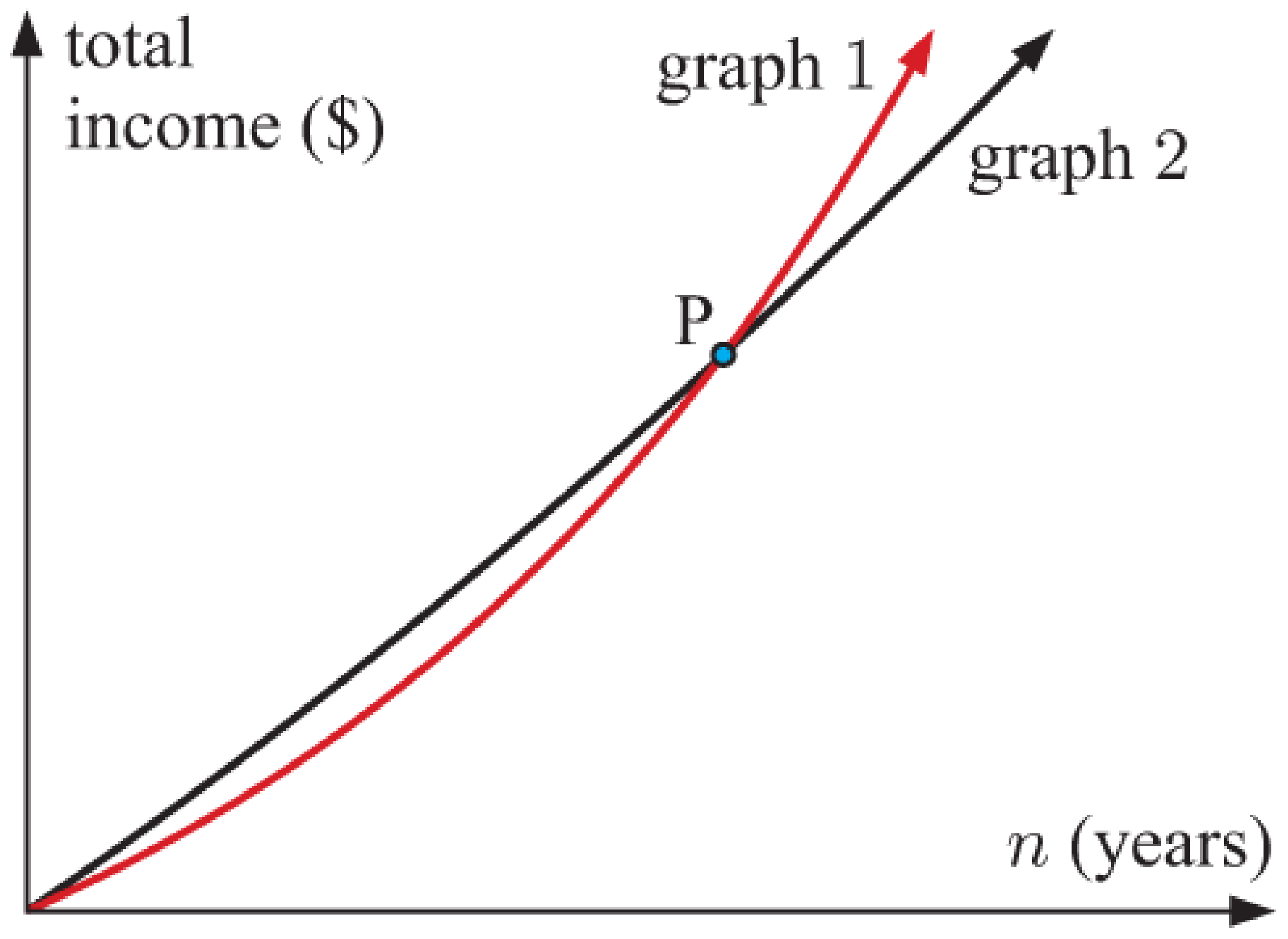
13 Find n given that $\sum_{k=1}^{n} 2 \times 3^{k-1} = 177\,146$.

14 Felicity is offered a new job, and is given two salary options to choose from:

Option A: $40 000 in the first year, and 5% extra each subsequent year.

Option B: $60 000 in the first year, and $1000 more each subsequent year.

- **a** If Felicity believed that she would work for 3 years in this new job, explain why *Option B* would be best for her.
- **b** Write down an expression for the amount of money earned in the nth year if she selects:
  - **i** *Option A*
  - **ii** *Option B*.
- **c** Find the minimum length of time Felicity would need to work before the amount of money earned per year from *Option A* exceeds that of *Option B*.
- **d** Felicity decides that the best way to compare the two options is to consider the *total* income accumulated after the first n years in each case. If T_A and T_B represent the total income earned over n years for *Options A* and *B* respectively, show that:
  - **i** $T_A = 800\,000(1.05^n - 1)$ dollars
  - **ii** $T_B = 500n^2 + 59\,500n$ dollars
- **e** The graph alongside shows T_A and T_B graphed against n.
  - **i** Which graph represents T_A and which graph represents T_B?
  - **ii** Use technology to find the coordinates of the point P, where T_A and T_B intersect.
  - **iii** Hence write down a time interval, in whole years, for which *Option B* provides the greater total income.



15 $8000 is borrowed over a 2-year period at a rate of 12% p.a. compounded quarterly. Quarterly repayments are made and the interest is adjusted each quarter, which means that at the end of each quarter, interest is charged on the previous balance and then the balance is reduced by the amount repaid.

There are $2 \times 4 = 8$ repayments and the interest per quarter is $\frac{12\%}{4} = 3\%$.

At the end of the first quarter, the amount owed is given by $A_1 = \$8000 \times 1.03 - R$, where R is the amount of each repayment.

At the end of the second quarter, the amount owed is given by:

$$\begin{aligned} A_2 &= A_1 \times 1.03 - R \\ &= (\$8000 \times 1.03 - R) \times 1.03 - R \\ &= \$8000 \times (1.03)^2 - 1.03R - R \end{aligned}$$

- **a** Write an expression for the amount owed at the end of the third quarter, A_3.
- **b** Write an expression for the amount owed at the end of the eighth quarter, A_8.
- **c** Given that $A_8 = 0$ for the loan to be fully repaid, deduce the value of R.
- **d** Now suppose the amount borrowed was $\$P$, r is the interest rate per repayment interval (as a decimal), and there are m repayments. Show that each repayment is

$$R = \frac{P(1+r)^m \times r}{(1+r)^m - 1} \text{ dollars.}$$

I INFINITE GEOMETRIC SERIES

To examine the sum of all the terms of an infinite geometric sequence, we need to consider $S_n = \frac{u_1(1-r^n)}{1-r}$ when n gets very large.

If $|r| > 1$, the series is said to be **divergent** and the sum becomes infinitely large.

For example, when $r = 2$, $1+2+4+8+16+....$ is infinitely large.

If $|r| < 1$, or in other words $-1 < r < 1$, then as n becomes very large, r^n approaches 0.

This means that S_n will get closer and closer to $\frac{u_1}{1-r}$.

$|r|$ is the *size* of r. If $|r| > 1$ then $r < -1$ or $r > 1$.

If $|r| < 1$, an infinite geometric series of the form $u_1 + u_1 r + u_1 r^2 + ... = \sum_{k=1}^{\infty} u_1 r^{k-1}$ will **converge** to the **limiting sum** $\boldsymbol{S = \frac{u_1}{1-r}}$.

Proof: If the first term is u_1 and the common ratio is r, the terms are $u_1, u_1 r, u_1 r^2, u_1 r^3, ...$

Suppose the sum of the corresponding infinite series is

$S = u_1 + u_1 r + u_1 r^2 + u_1 r^3 + ...$ $(*)$

$$\therefore\ rS = u_1 r + u_1 r^2 + u_1 r^3 + u_1 r^4 + ...$$

$$\therefore\ rS = S - u_1 \qquad \{\text{comparing with } (*)\}$$

$$\therefore\ S(r-1) = -u_1$$

$$\therefore\ S = \frac{u_1}{1-r} \qquad \{\text{provided } r \neq 1\}$$

This result can be used to find the value of recurring decimals.

Example 31 **Self Tutor**

Write $0.\overline{7}$ as a rational number.

$0.\overline{7} = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10\,000} + ...$ is an infinite geometric series with $u_1 = \frac{7}{10}$ and $r = \frac{1}{10}$.

$$\therefore\ S = \frac{u_1}{1-r} = \frac{\frac{7}{10}}{1-\frac{1}{10}} = \frac{7}{9}$$

$$\therefore\ 0.\overline{7} = \tfrac{7}{9}$$

EXERCISE 5I

1 **a** Explain why $0.\overline{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + ...$ is an infinite geometric series.

b Hence show that $0.\overline{3} = \frac{1}{3}$.

2 Write as a rational number: **a** $0.\overline{4}$ **b** $0.\overline{16}$ **c** $0.\overline{312}$

3 Use $S = \frac{u_1}{1-r}$ to check your answer to **Exercise 5H** question **8 d**.

4 Find the sum of each of the following infinite geometric series:

a $18 + 12 + 8 + \frac{16}{3} + ...$ **b** $18.9 - 6.3 + 2.1 - 0.7 + ...$

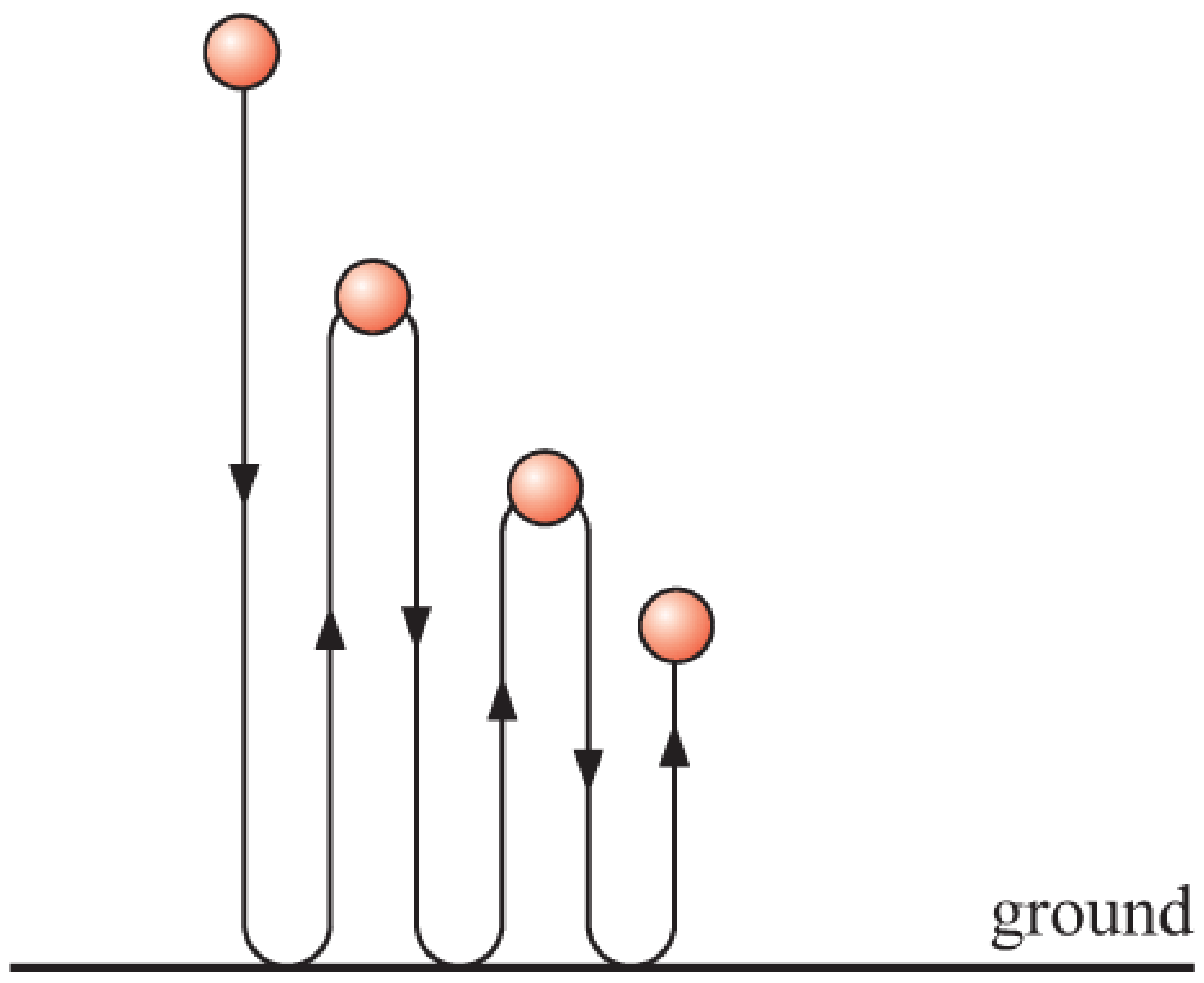
5 Find: **a** $\sum_{k=1}^{\infty} \frac{3}{4^k}$ **b** $\sum_{k=0}^{\infty} 6\left(-\frac{2}{5}\right)^k$

6 The sum of the first three terms of a convergent infinite geometric series is 19. The sum of the series is 27. Find the first term and the common ratio.

7 The second term of a convergent infinite geometric series is $\frac{8}{5}$. The sum of the series is 10. Show that there are two possible series, and find the first term and the common ratio in each case.

8 An infinite geometric series has $S = \frac{64}{3}$ and $S_3 = 21$. Find S_5.

9



When dropped, a ball takes 1 second to hit the ground. It then takes 90% of this time to rebound to its new height, and this continues until the ball comes to rest.

a Show that the total time of motion is given by $1 + 2(0.9) + 2(0.9)^2 + 2(0.9)^3 + ...$

b Find S_n for the series in **a**.

c How long does it take for the ball to come to rest?

10 When a ball is dropped, it rebounds 75% of its height after each bounce. If the ball travels a total distance of 490 cm, from what height was the ball dropped?

11 **a** Explain why $0.\overline{9} = 1$ exactly. **b** Show that if $u_n = \frac{9}{10^n}$, then $S_n = 1 - \frac{1}{10^n}$.

c On a graph, plot the points (n, u_n) and (n, S_n) for $n = 1, 2, ..., 10$. Connect each set of points with a smooth curve.

12 Find x if $\sum_{k=1}^{\infty}\left(\frac{3x}{2}\right)^{k-1} = 4$.

13 Suppose $u_1 + u_2 + u_3 + u_4 + ...$ is a convergent infinite geometric series, with $u_1, u_2, u_3, u_4, ... > 0$.

a Explain why $u_1 - u_2 + u_3 - u_4 + ...$ and $\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} + \sqrt{u_4} + ...$ must also be convergent infinite geometric series.

b Given $u_1 - u_2 + u_3 - u_4 + ... = \frac{81}{10}$ and $\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} + \sqrt{u_4} + ... = \frac{9}{2}$, find $u_1 + u_2 + u_3 + u_4 + ...$.

14 Show that $\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \frac{5}{64} + ... = 1$.

ACTIVITY 3

Click on the icon to run a card game for sequences and series.

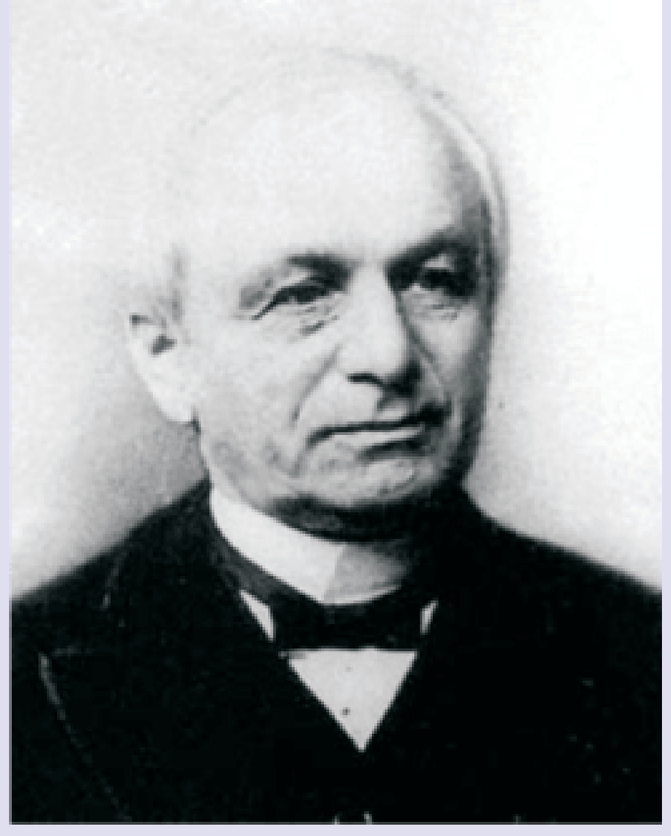
CARD GAME

THEORY OF KNOWLEDGE

The German mathematician Leopold Kronecker (1823 - 1891) made important contributions in number theory and algebra. Several things are named after him, including formulae, symbols, and a theorem.

Kronecker made several well-known quotes, including:

> *"God made integers; all else is the work of man."*
>
> *"A mathematical object does not exist unless it can be constructed from natural numbers in a finite number of steps."*



Leopold Kronecker

1 What do you understand by the term *infinity*?

2 If the entire world were made of grains of sand, could you count them? Would the number of grains of sand be infinite?

3 There are clearly an infinite number of positive integers, and an infinite number of positive even integers.

- **a** Construct an argument that:
  - **i** there are *more* positive integers than positive even integers
  - **ii** there is the *same number* of positive integers as positive even integers.
- **b** Can the traditional notions of "more than", "less than", and "equal to" be extended to infinity?

Consider an infinite geometric series with first term u_1 and common ratio r.

If $|\,r\,| < 1$, the series will converge to the sum $S = \dfrac{u_1}{1-r}$.

4 Can we explain through *intuition* how a sum of non-zero terms, which goes on and on for ever and ever, could actually be a finite number?

In the case $r = -1$, the terms are $u_1, -u_1, u_1, -u_1, ...$

If we take partial sums of the series, the answer is always u_1 or 0.

5 What is the sum of the infinite series when $r = -1$? Is it infinite? Is it defined?

Substituting $r = -1$ into the formula above gives $S = \dfrac{u_1}{2}$. Could this possibly be the answer?

ACTIVITY 4 VON KOCH'S SNOWFLAKE CURVE

The Haese Mathematics logo is the 4th member of a sequence of diagrams. The **limiting curve** which the sequence approaches is called **von Koch's snowflake curve**.

Click on the icon to obtain this Activity.

REVIEW SET 5A

1 Consider the number sequence 5, 9, 11, 12, 15, 19. Find:

a u_2 **b** u_6 **c** S_4.

2 Find k if $3k$, $k-2$, and $k+7$ are consecutive terms of an arithmetic sequence.

3 A sequence is defined by $u_n = 6\left(\frac{1}{2}\right)^{n-1}$.

a Prove that the sequence is geometric. **b** Find u_1 and r.

c Find the 16th term of the sequence to 3 significant figures.

4 Determine the general term of a geometric sequence given that its sixth term is $\frac{16}{3}$ and its tenth term is $\frac{256}{3}$.

5 Insert six numbers between 23 and 9 so that all eight numbers are in arithmetic sequence.

6 Theodore is squeezing the juice from some lemons. When he has squeezed 6 lemons, he has collected 274.3 mL of juice.

a Find the average amount of juice collected from each lemon.

b Hence write an arithmetic sequence u_n which approximates the amount of juice collected from squeezing n lemons.

c Predict the amount of juice collected from squeezing 13 lemons.

7 Find the sum of each of the following infinite geometric series:

a $18 - 12 + 8 - ....$ **b** $8 + 4\sqrt{2} + 4 + ...$

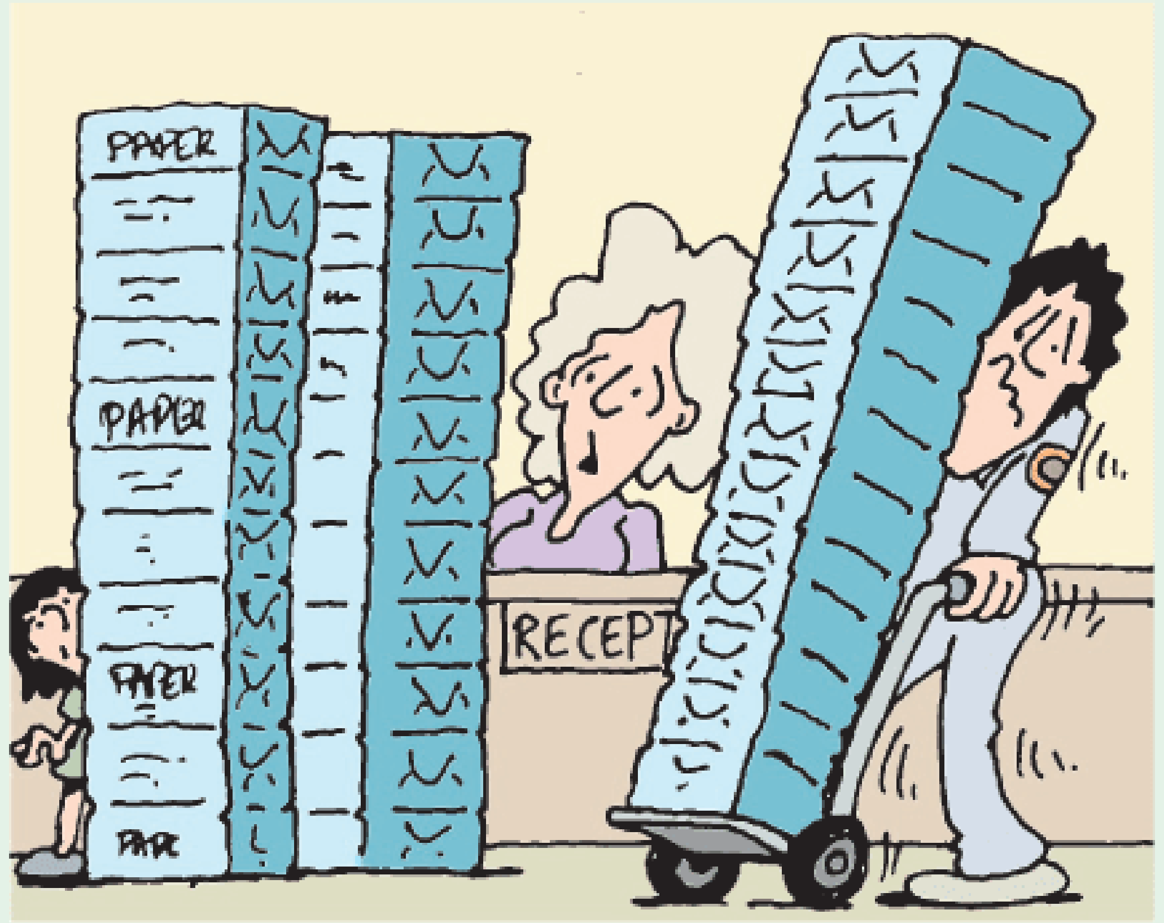
8 Find the sum of:

a $7 + 11 + 15 + 19 + ... + 99$ **b** $35 + 33\frac{1}{2} + 32 + 30\frac{1}{2} + ... + 20$

9 Each year, a school manages to use only 90% as much paper as the previous year. In the year 2010, they used 700 000 sheets of paper.

a Find how much paper the school used in the years 2011 and 2012.

b How much paper did the school use in total in the decade from 2008 to 2018?



10 Expand and hence evaluate: **a** $\sum_{k=1}^{7} k^2$ **b** $\sum_{k=1}^{4} \frac{k+3}{k+2}$

11 The sum of the first n terms of an infinite sequence is $\dfrac{3n^2+5n}{2}$ for all $n \in \mathbb{Z}^+$.

a Find the nth term. **b** Prove that the sequence is arithmetic.

12 £12 500 is invested in an account which pays 4.25% p.a. interest. Find the value of the investment after 5 years if the interest is compounded:

a half-yearly **b** monthly.

13 4 years ago, Chelsea invested some money in an account paying 6.5% p.a. interest compounded quarterly. There is currently \$6212.27 in the account. How much did Chelsea invest originally?

14 If inflation averages 2.5% per year, calculate the value of:

a €6000 indexed for inflation over 4 years

b €11 200 indexed for inflation over 7 years.

15 Georgina invested \$20 000 in an account which paid 6.2% p.a. interest, compounded monthly for 3 years. Inflation averaged 1.8% per year over this time.

a Calculate the future value of the investment.

b Find the real value of the investment.

16 Show that 28, 23, 18, 13, ... is an arithmetic sequence. Hence find u_n and the sum S_n of its first n terms in simplest form.

17 **a** Determine the number of terms in the sequence 128, 64, 32, 16, ..., $\frac{1}{512}$.

b Find the sum of these terms.

18 The sum of the first two terms of an infinite geometric series is 90. The third term is 24. Show that there are two possible series, and that both series converge.

19 After years of decline in his health, Tim has now realised that smoking is unhealthy. Until now, he has regularly smoked 120 cigarettes each week. To help him quit, he is determined to reduce this amount by 5 cigarettes every week from now on.

a Explain why the number of cigarettes Tim smokes each week will form an arithmetic sequence with $u_1 = 115$. State the common difference for the sequence.

b How many weeks will it take before Tim has smoked his last cigarette?

c Find the total number of cigarettes Tim will smoke before he successfully quits.

20 Consider the infinite geometric sequence $160,\ 80\sqrt{2},\ 80,\ 40\sqrt{2},\ ....$

a Write the 12th term of the sequence in the form $k\sqrt{2}$ where $k \in \mathbb{Q}$.

b Find, in the form $a + b\sqrt{2}$ where $a, b \in \mathbb{Z}$:

i S_{10} **ii** the sum S of the infinite series.

21 The first 3 terms of a geometric sequence have sum 39. If the middle term is increased by $66\frac{2}{3}\%$, the first three terms now form an arithmetic sequence. Find the smallest possible value of the first term.

22 The series sums S_1, S_2, and S_4 of an arithmetic sequence form a geometric sequence. Find the common ratio for this sequence.

23 x, y, and z are consecutive terms of a geometric sequence.
If $x + y + z = \frac{7}{3}$ and $x^2 + y^2 + z^2 = \frac{91}{9}$, find the values of x, y, and z.

24 A competition offers three options for the first prize, each of which pays the winner a monthly sum for 24 months.

Option 1: \$8000 per month.

Option 2: \$1000 in the first month, then each successive month pays \$600 more than the previous month.

Option 3: \$500 in the first month, then each successive month pays 20% more than the previous month.

- **a** Calculate the total prize value for *Option 1*.
- **b** For *Option 2*:
  - **i** Write down the amount won in each of the first three months.
  - **ii** Calculate the total amount won over the 24 month period.
- **c** For *Option 3*:
  - **i** Write down the amount won in each of the first three months.
  - **ii** Calculate the total amount won over the 24 month period.
- **d** Which option is worth the greatest amount of money overall?
- **e** The amount won in the first month under *Option 3* is to be altered so that the total prize over 24 months is \$250 000. Calculate the new initial amount, giving your answer to the nearest cent.

25 $2x$ and $x-2$ are the first two terms of a convergent geometric series. The sum of the series is $\frac{18}{7}$. Find x, clearly explaining why there is only one possible value.

26 a, b, and c are consecutive terms of an arithmetic sequence. Prove that the following are also consecutive terms of an arithmetic sequence:

a $b+c$, $c+a$, and $a+b$

b $\frac{1}{\sqrt{b}+\sqrt{c}}$, $\frac{1}{\sqrt{c}+\sqrt{a}}$, and $\frac{1}{\sqrt{a}+\sqrt{b}}$

REVIEW SET 5B

1 Evaluate the first five terms of the sequence:

a $\{(\frac{1}{3})^n\}$ **b** $\{12+5n\}$ **c** $\left\{\frac{4}{n+2}\right\}$

2 A sequence is defined by $u_n = 68 - 5n$.

- **a** Prove that the sequence is arithmetic.
- **b** Find u_1 and d.
- **c** Find the 37th term of the sequence.
- **d** State the first term of the sequence which is less than -200.

3
- **a** Find the general term of the arithmetic sequence with $u_7 = 31$ and $u_{15} = -17$.
- **b** Hence find the value of u_{34}.

4 Find the sum of the first 12 terms of:

a $3+9+15+21+....$ **b** $24+12+6+3+....$

5 Stacy runs a hot dog stand at a local fair. On the first day she served 25 customers and made £60 profit. On the second day she served 43 customers and made £135 profit.

- **a** Assuming that her profit from serving n customers forms an arithmetic sequence, find a model which approximates the profit from serving n customers.

b Explain the significance of the common difference and the constant term in your model.

c On the third day, Stacy served 36 customers. Use your model to estimate her profit.

6 Find the first term of the sequence 5, 10, 20, 40, ... which exceeds 10 000.

7 A ball bounces from a height of 3 metres and returns to 80% of its previous height on each bounce. Find the total distance travelled by the ball until it stops bouncing.

8 $7000 is invested at 6% p.a. compound interest. Find the value of the investment after 3 years if interest is compounded:

a annually **b** quarterly **c** monthly.

9 **a** Find k given that 4, k, and $k^2 - 1$ are consecutive terms of a geometric sequence.

b For each value of k, find the common ratio of the sequence.

10 Seve is training for a long distance walk. He walks for 10 km in the first week, then each week thereafter he walks 500 m further than the previous week. If he continues this pattern for a year, how far does Seve walk:

a in the last week **b** in total?

11 Find the sum of the infinite geometric series:

a $1.21 - 1.1 + 1 - ....$ **b** $\frac{14}{3} + \frac{4}{3} + \frac{8}{21} + ...$

12 Find the first term of the sequence $24, 8, \frac{8}{3}, \frac{8}{9}, ...$ which is less than 0.001 .

13 Vijay deposits 200 000 rupees in an account that compounds interest half-yearly. 6 years later, the account has balance 250 680 rupees. Use technology to calculate the annual rate of interest.

14 Frederik invests €5000 at 5.8% p.a. compounded monthly. Use technology to find how long it will take to amount to €12 000.

15 Richard sold his car for $7500. He invested the money in an account paying 3.7% p.a. interest compounded quarterly for 8 years. Inflation averaged 3.1% per year over this period.

a Find the future value of Richard's investment.

b Find the real value of Richard's investment.

16 A photocopier bought for $9800 will depreciate by 26% each year. Find its value after 5 years.

17 Evaluate:

a $\sum_{k=1}^{8}\left(\frac{31-3k}{2}\right)$ **b** $\sum_{k=1}^{15} 50(0.8)^{k-1}$ **c** $\sum_{k=7}^{\infty} 5\left(\frac{2}{5}\right)^{k-1}$

18 A geometric sequence has $u_6 = 24$ and $u_{11} = 768$.

a Determine the general term of the sequence. **b** Find the sum of the first 15 terms.

19 In 2004 there were 3000 iguanas on a Galapagos island. Since then, the population of iguanas on the island has increased by 5% each year.

a How many iguanas were on the island in 2007?

b In what year will the population first exceed 10 000?

20 **a** Under what conditions will the series $\sum_{k=1}^{\infty} 50(2x-1)^{k-1}$ converge? Explain your answer.

b Find $\sum_{k=1}^{\infty} 50(2x-1)^{k-1}$ if $x = 0.3$.

21 Suppose $u_1, u_2, u_3, \ldots.$ is an arithmetic sequence.

a Show that $2^{u_1}, 2^{u_2}, 2^{u_3}, \ldots.$ is a geometric sequence.

b Given that $u_3 = -4$ and $u_5 = -10$, find the sum of the infinite geometric series $2^{u_1} + 2^{u_2} + 2^{u_3} + \ldots.$

22 Michael is saving to buy a house and needs $400 000.

a Three years ago, he invested a sum of money in an account paying 6.5% p.a. interest compounded half-yearly. This investment has just matured at $100 000. How much did Michael invest three years ago?

b Michael decides to reinvest his $100 000 lump sum into an account for a period of n years at 6.0% p.a. interest compounded annually.

Copy and complete the table below showing the value V_n of Michael's investment after n years.

n (years)	0	1	2	3	4
V_n ($)	100 000	106 000	112 360		

c Write a formula for V_n in terms of n.

d Michael also decides to start an additional saving plan, whereby he deposits $6000 into a safe at the end of each year. Write down a formula for S_n, the amount of money in Michael's safe after n years.

e The total amount of money Michael has for his house after n years is given by $T_n = V_n + S_n$. Calculate the missing values in the table below.

n (years)	0	1	2	3	4
T_n ($)	100 000	112 000	124 360		

f After how many whole years will Michael have the $400 000 needed to buy his house?

23 The sum of an infinite geometric series is 49 and the second term of the series is 10. Find the possible values for the sum of the first three terms of the series.

24 Suppose n consecutive geometric terms are inserted between 1 and 2. Write the sum of these n terms, in terms of n.

25 The 3rd, 4th, and 8th terms of an arithmetic sequence are the first three terms of a geometric sequence, $r \neq 1$.

a Find the common ratio for the geometric sequence.

b Show that the 24th term of the arithmetic sequence is the 4th term of the geometric sequence.

26 Notice that $11 - 2 = 9 = 3^2$ and $1111 - 22 = 1089 = 33^2$.

Show that $\underbrace{(111111\ldots.1)}_{2n \text{ lots of } 1} - \underbrace{(22222\ldots.2)}_{n \text{ lots of } 2}$ is a perfect square.

Chapter 6

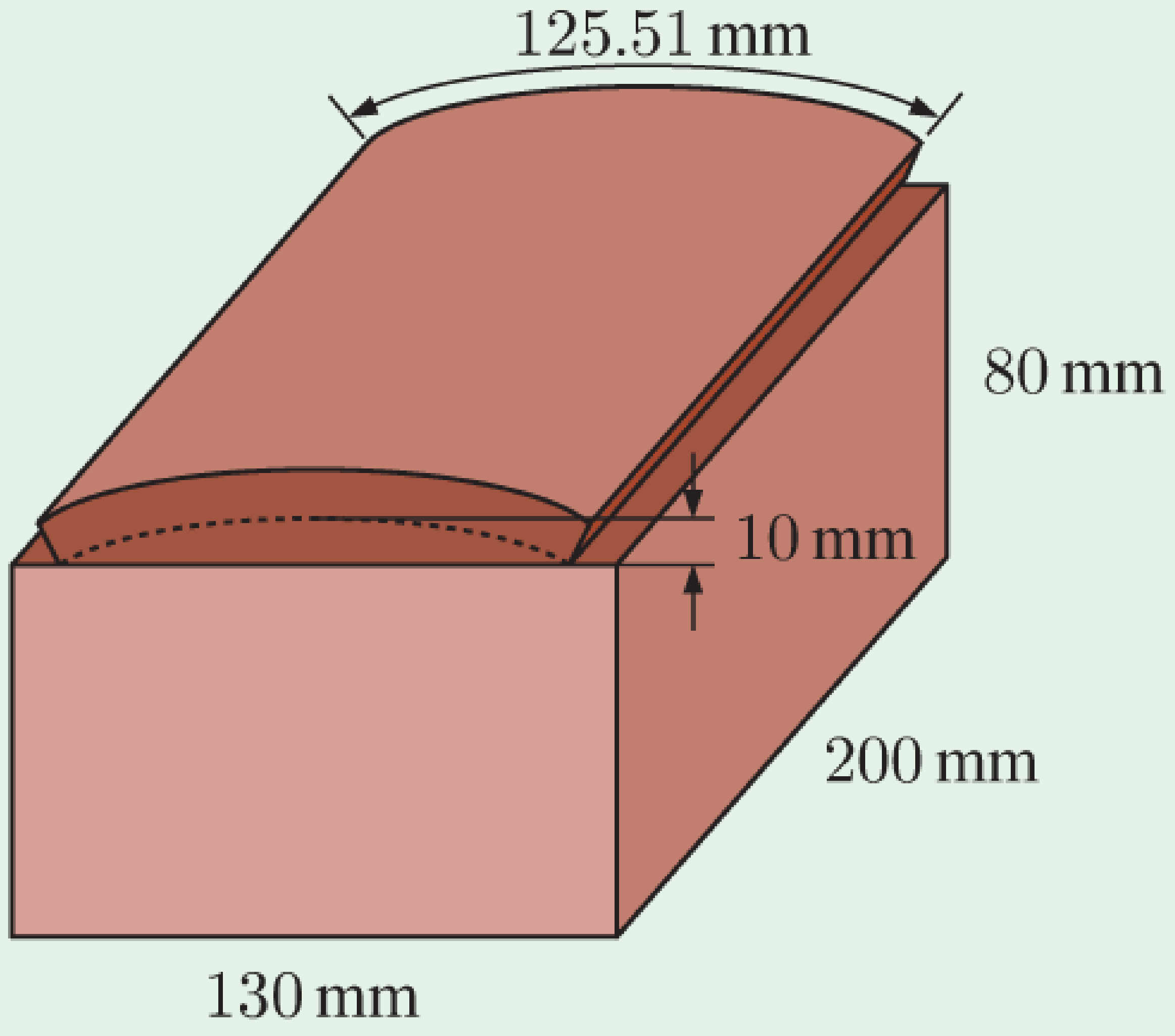
Measurement

Contents:

OPENING PROBLEM

A jewellery box is made of wood 5 mm thick.
When shut, its height is 95 mm.

The curved edge of the lid is an arc of a circle, and has length 125.51 mm.



Things to think about:

- **a** What is the *external* surface area of the container?
- **b** Why is it useful to specify the "external" surface area when talking about a container?
- **c** Can you find:
  - **i** the *volume* of jewellery the box can hold
  - **ii** the *capacity* of the box
  - **iii** the *volume* of wood used to make the box?

In previous years you should have studied measurement extensively. In this Chapter we revise measurements associated with parts of a circle, as well as the surface area and volume of 3-dimensional shapes.

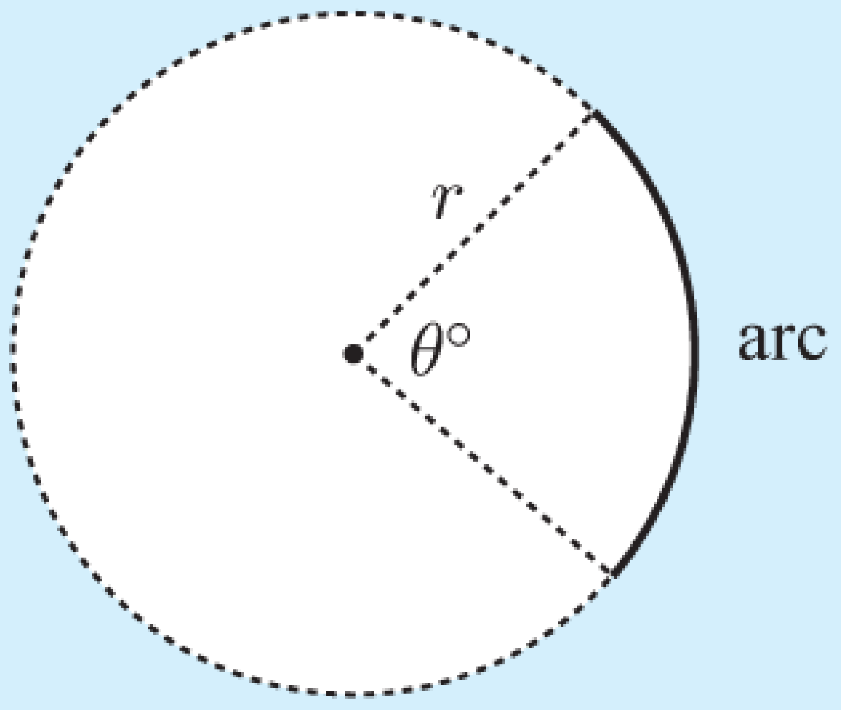
A CIRCLES, ARCS, AND SECTORS

For a **circle** with radius r:

- the **circumference** $C = 2\pi r$
- the **area** $A = \pi r^2$.

An **arc** is a part of a circle which joins any two different points. It can be measured using the angle $\theta°$ subtended by the points at the centre.

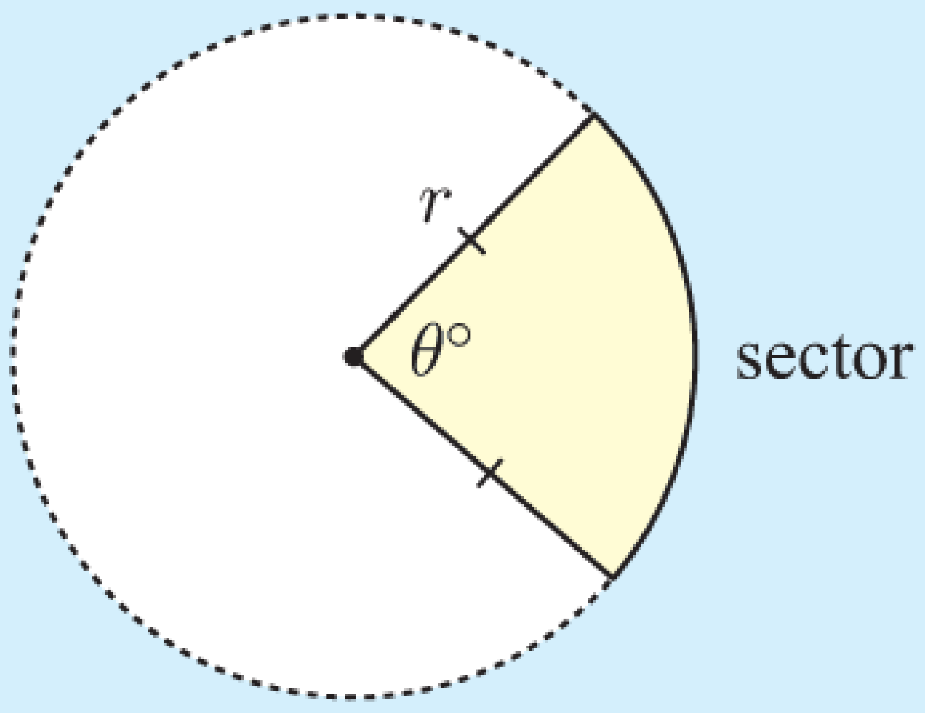
$$\text{Arc length} = \frac{\theta}{360} \times 2\pi r$$



A **sector** is the region between two radii of a circle and the arc between them.

$$\begin{aligned}\text{Perimeter} &= \text{two radii} + \text{arc length}\\ &= 2r + \frac{\theta}{360} \times 2\pi r\end{aligned}$$

$$\text{Area} = \frac{\theta}{360} \times \pi r^2$$



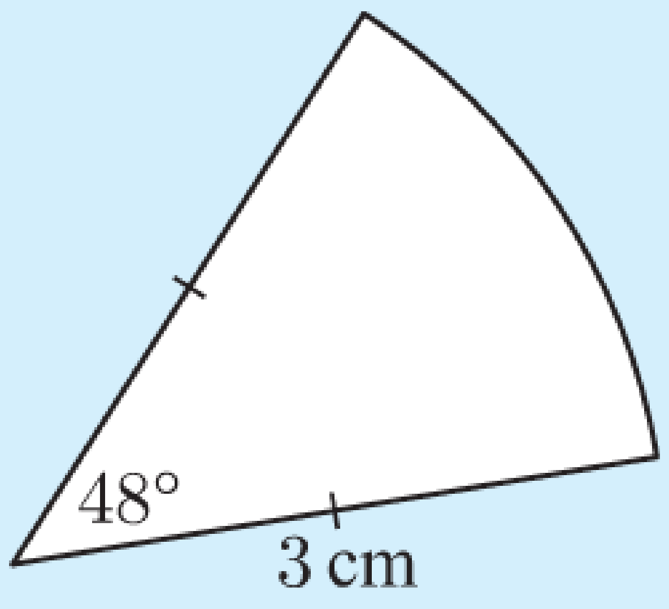
Example 1 **Self Tutor**

For the given figure, find to 3 significant figures:

a the length of the arc

b the perimeter of the sector

c the area of the sector.



a Arc length $= \dfrac{\theta}{360} \times 2\pi r$

$= \frac{48}{360} \times 2\pi \times 3$ cm

≈ 2.51 cm

b Perimeter $= 2r +$ arc length

$\approx 2 \times 3 + 2.51$ cm

≈ 8.51 cm

c Area $= \dfrac{\theta}{360} \times \pi r^2$

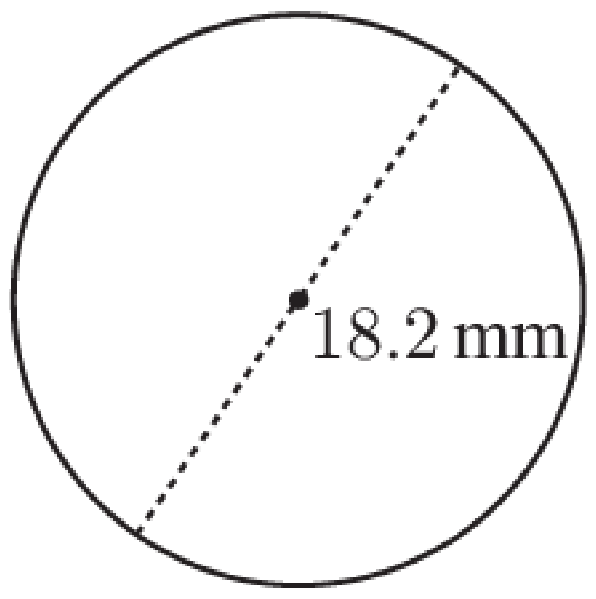
$= \frac{48}{360} \times \pi \times 3^2$ cm^2

≈ 3.77 cm^2

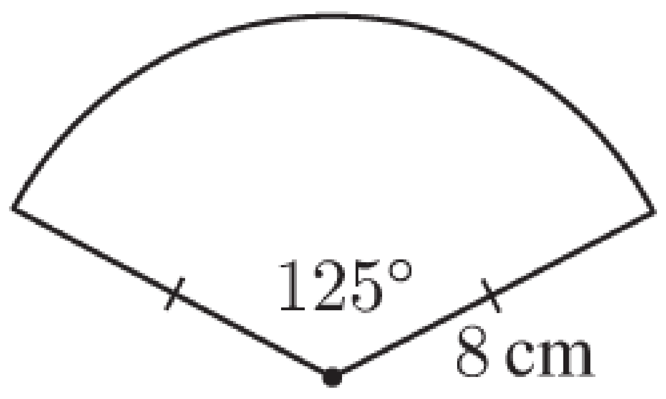
EXERCISE 6A

1 Find the perimeter of:

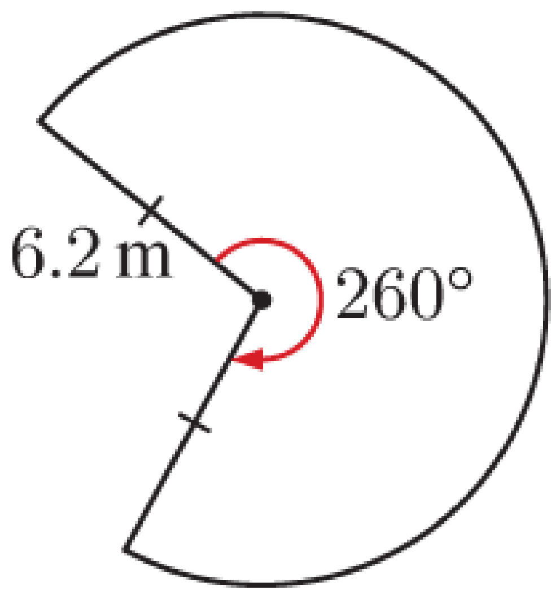
a



b



c



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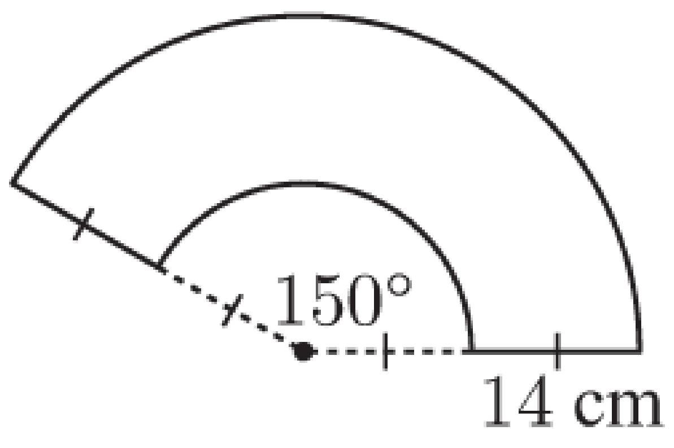
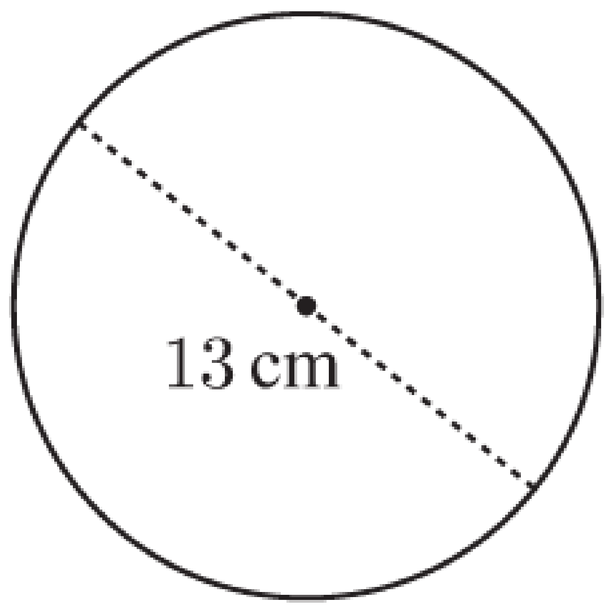


2 An arc of a circle makes a 36° angle at its centre. If the arc has length 26 cm, find the radius of the circle.

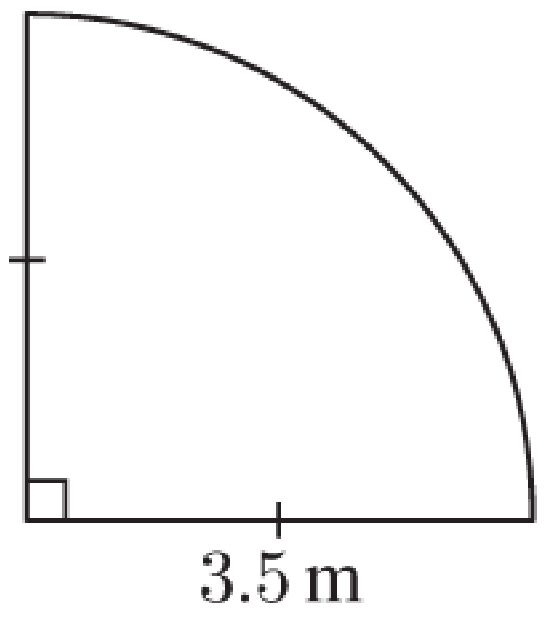
3 A sector of a circle makes a 127° angle at its centre. If the arc of the sector has length 36 mm, find the perimeter of the sector.

4 Find the area of:

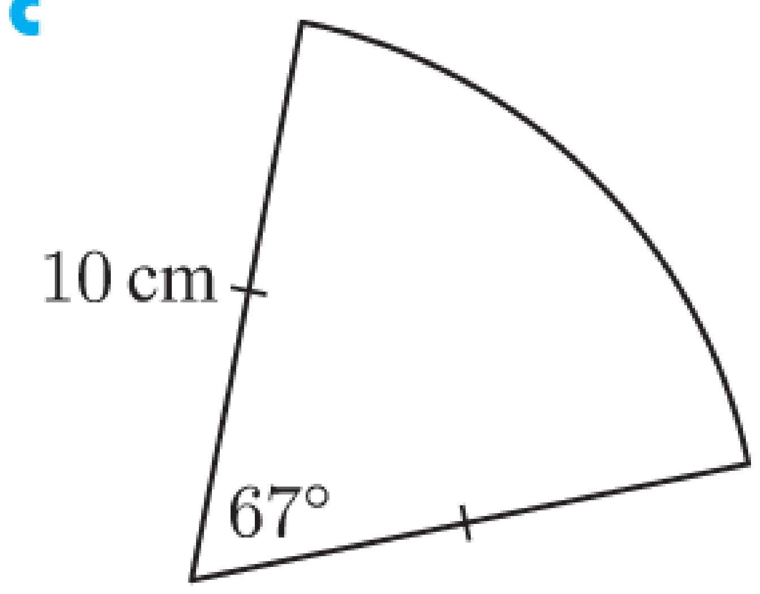
a



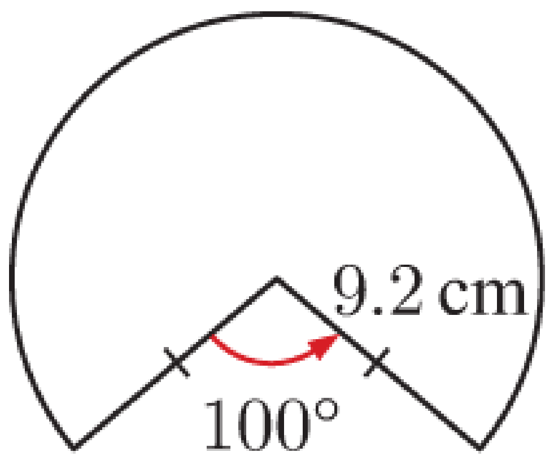
b



c



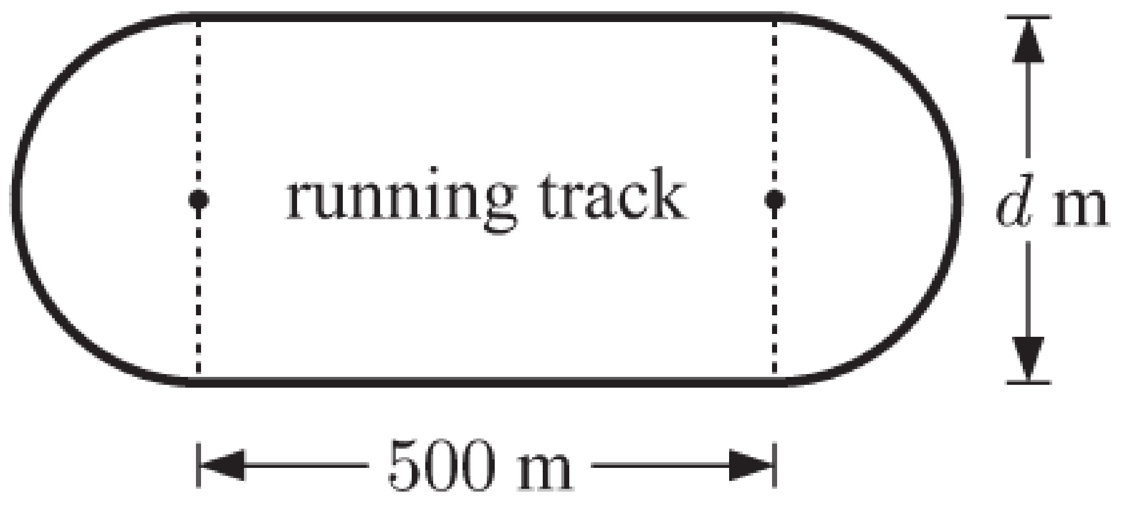
d



5 Find the radius of a sector with angle 67° and area 16.2 cm^2.

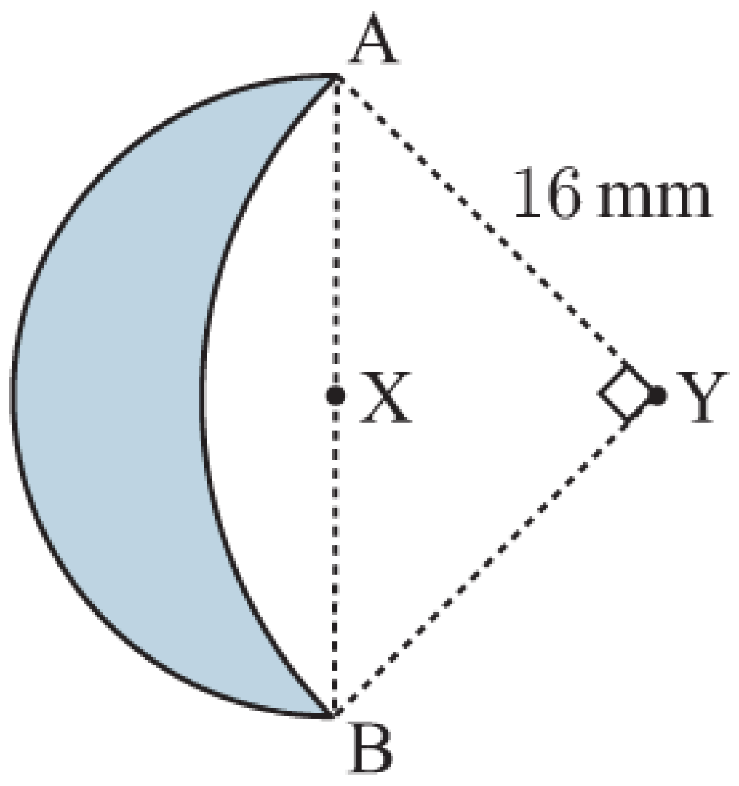
6 Find the perimeter of a sector with angle 136° and area 28.8 cm^2.

7 A running track consists of two straight segments joined by semi-circular ends, as shown. The total perimeter of the track is 1600 metres.



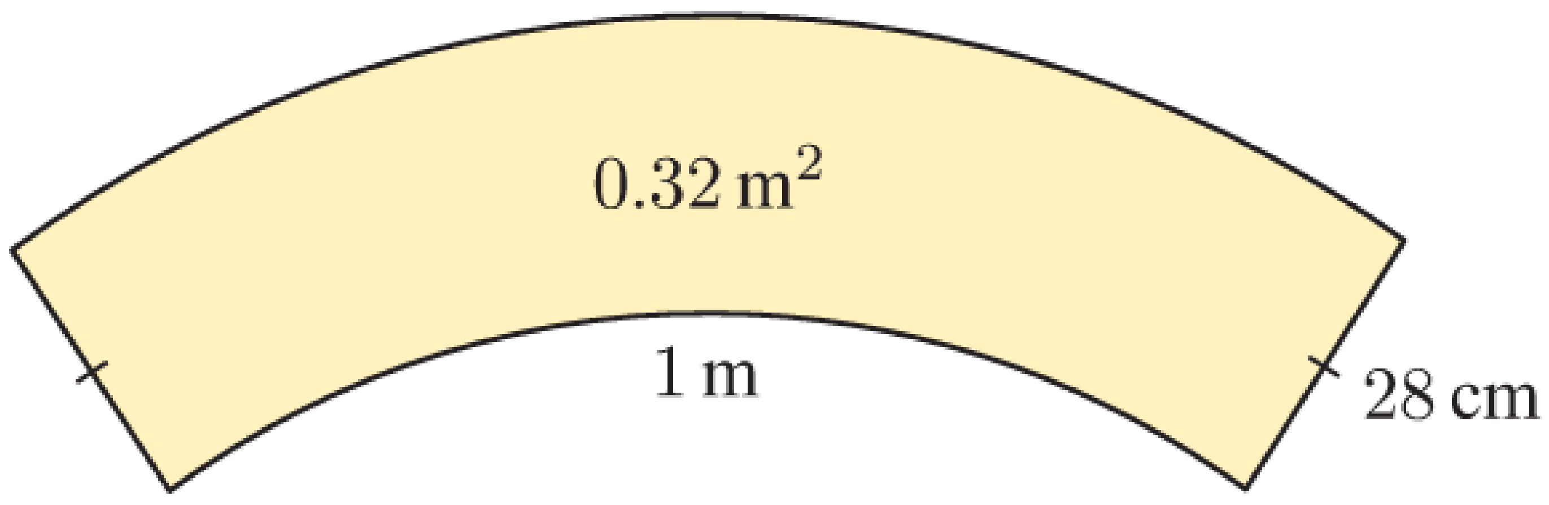
- **a** Determine the diameter of the semi-circular ends.
- **b** Jason takes 4 minutes and 25 seconds to complete a single lap of the track. Calculate Jason's average speed in m s^{-1}.

8 X and Y are the centres of the circles containing the two arcs AB shown.
Find:



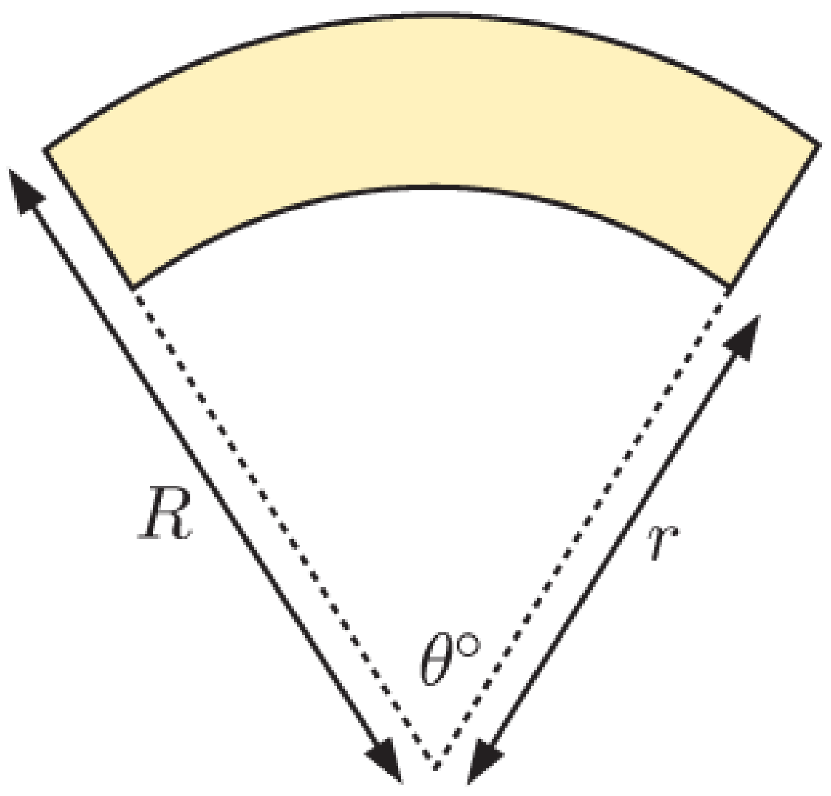
- **a** the length AX
- **b** the perimeter of the shaded crescent
- **c** the area of the crescent.

9



Belinda has made a lampshade with area $0.32\ \text{m}^2$. Its shorter arc has length 1 m, and its slant height is 28 cm.

Suppose the material is cut as the difference between two sectors with common angle $\theta°$, and radii r and R.



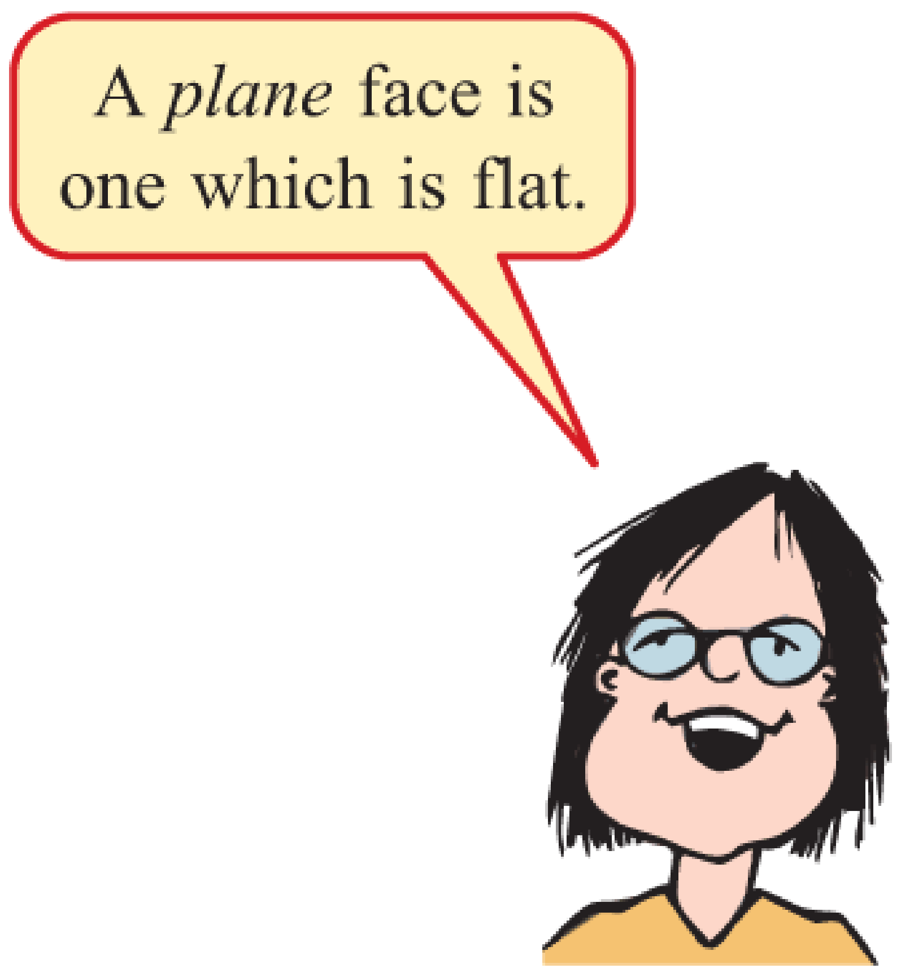
- **a** Show that the area of the lampshade is given by $A = \dfrac{0.28\theta}{360}\,\pi(2r + 0.28)\ \ \text{m}^2$.
- **b** Use the smaller sector to show that $\theta = \dfrac{180}{\pi r}$.
- **c** Find r and θ.
- **d** Hence find the length of the longer arc.

B SURFACE AREA

SOLIDS WITH PLANE FACES

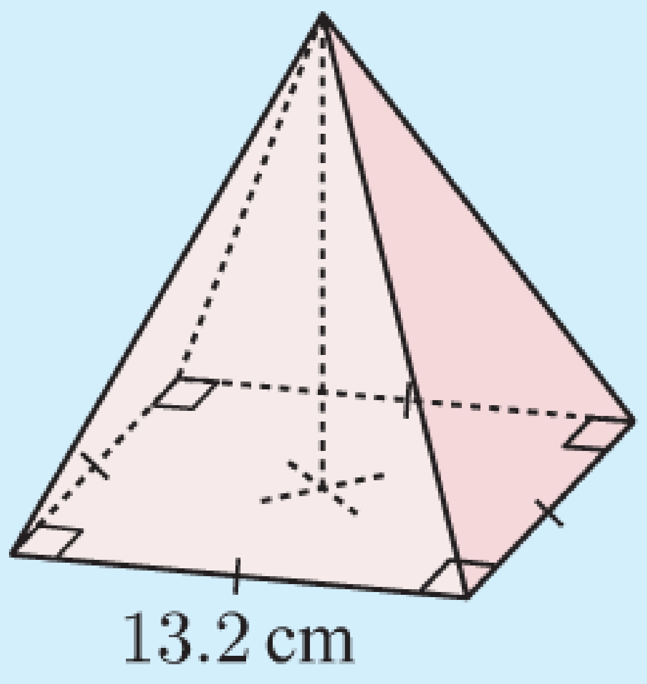
The **surface area** of a three-dimensional figure with plane faces is the sum of the areas of the faces.

The surface area is therefore the same as the area of the **net** required to make the figure.



Example 2 **Self Tutor**

The pyramid shown is 10.8 cm high.
Find its surface area.



The net of the pyramid includes one square with side length 13.2 cm, and four isosceles triangles with base 13.2 cm.

Let the height of the triangles be h cm.

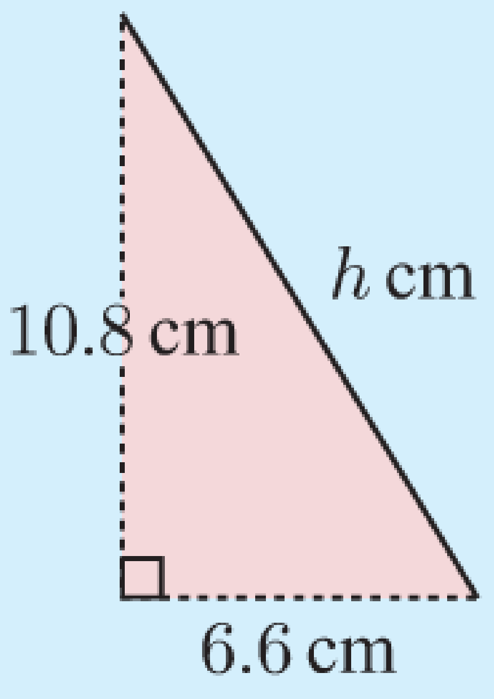
Now $h^2 = 10.8^2 + 6.6^2$ {Pythagoras}

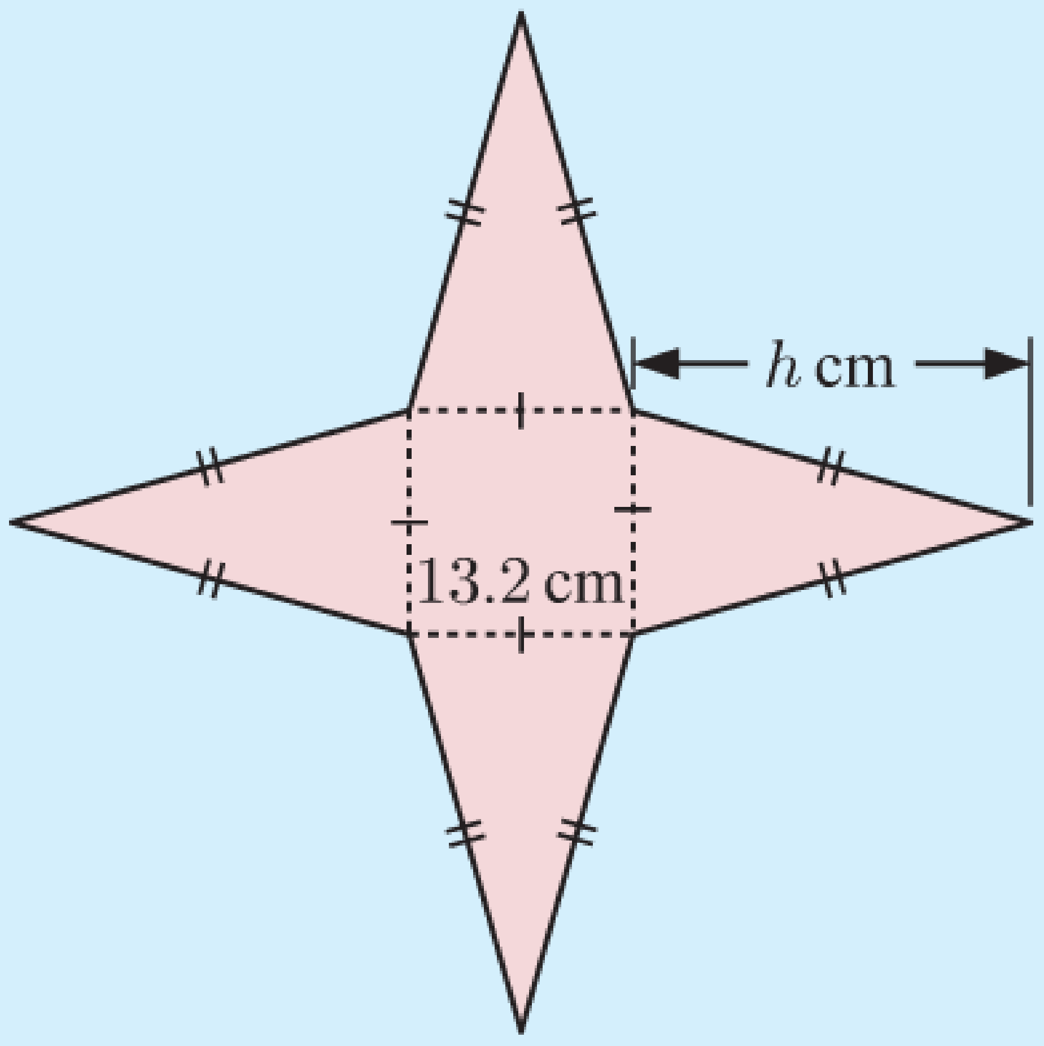
$\therefore$ $h = \sqrt{10.8^2 + 6.6^2} \approx 12.66$

$\therefore$ the surface area

$\approx 13.2^2 + 4 \times (\frac{1}{2} \times 13.2 \times 12.66)$ cm^2

≈ 508 cm^2

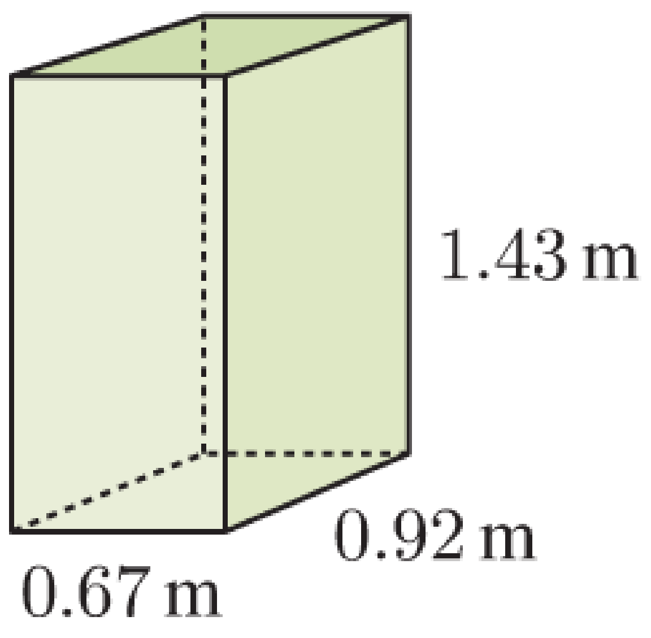




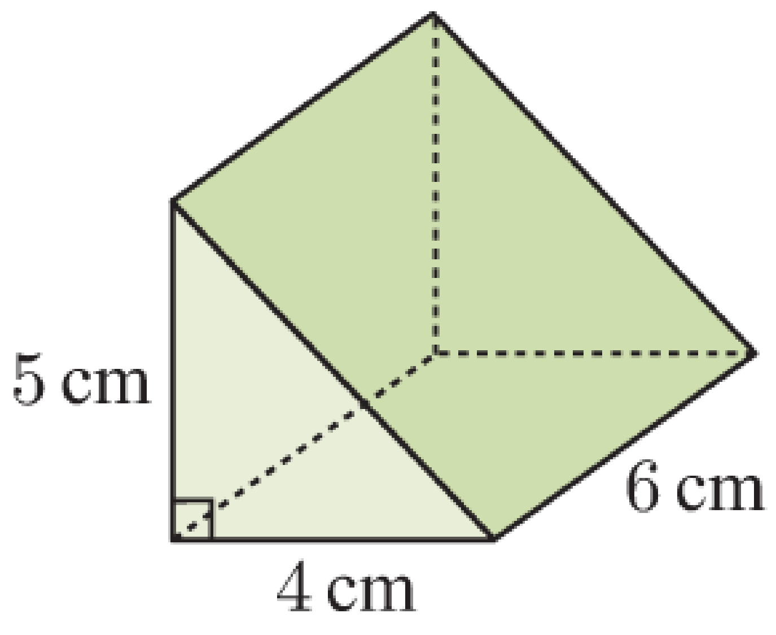
EXERCISE 6B.1

1 Find the surface area of each solid:

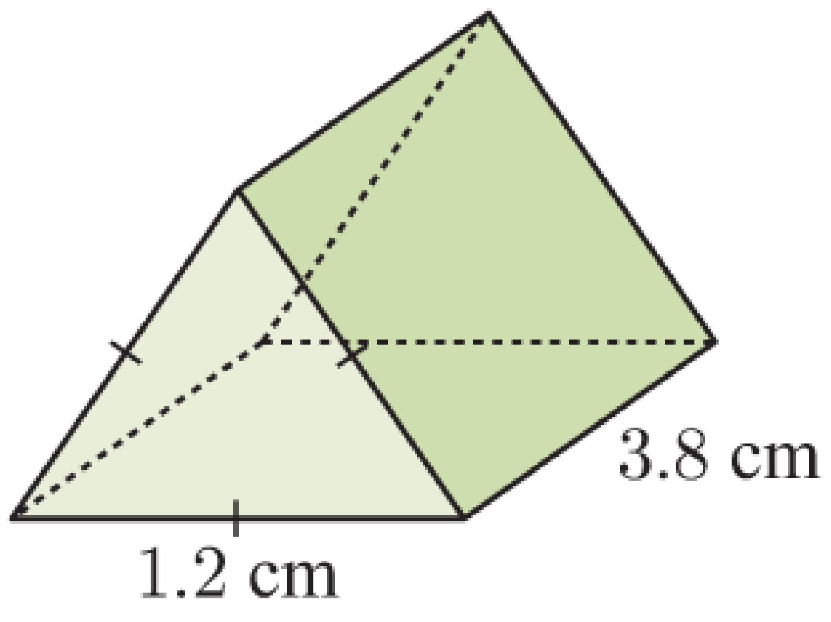
a



b

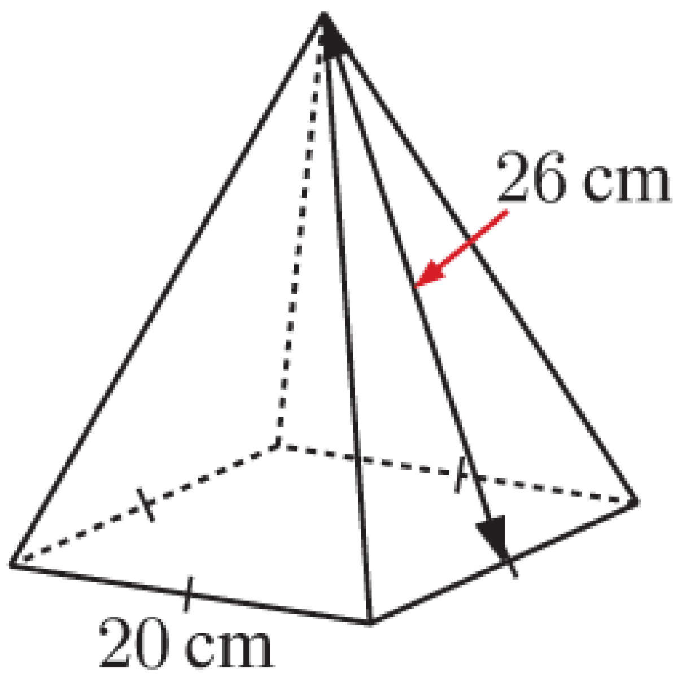


c



2 Find the surface area of each pyramid:

a



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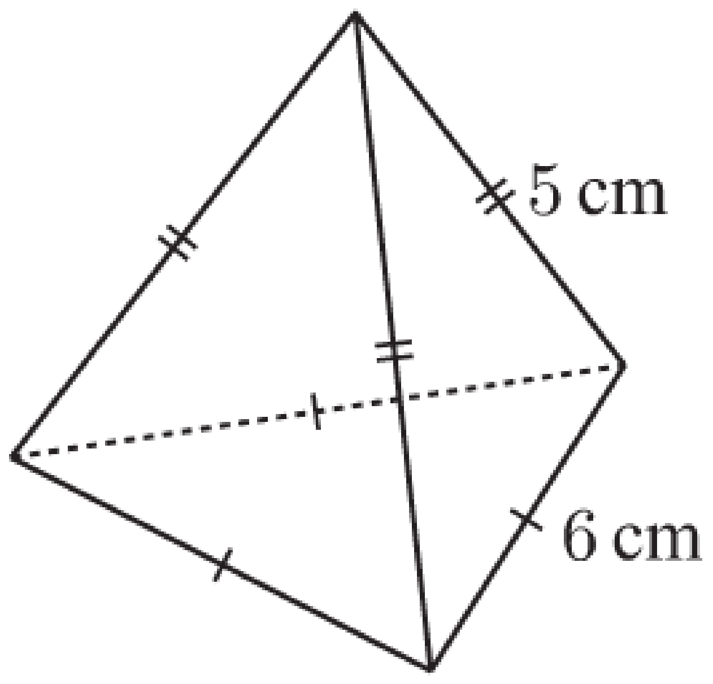
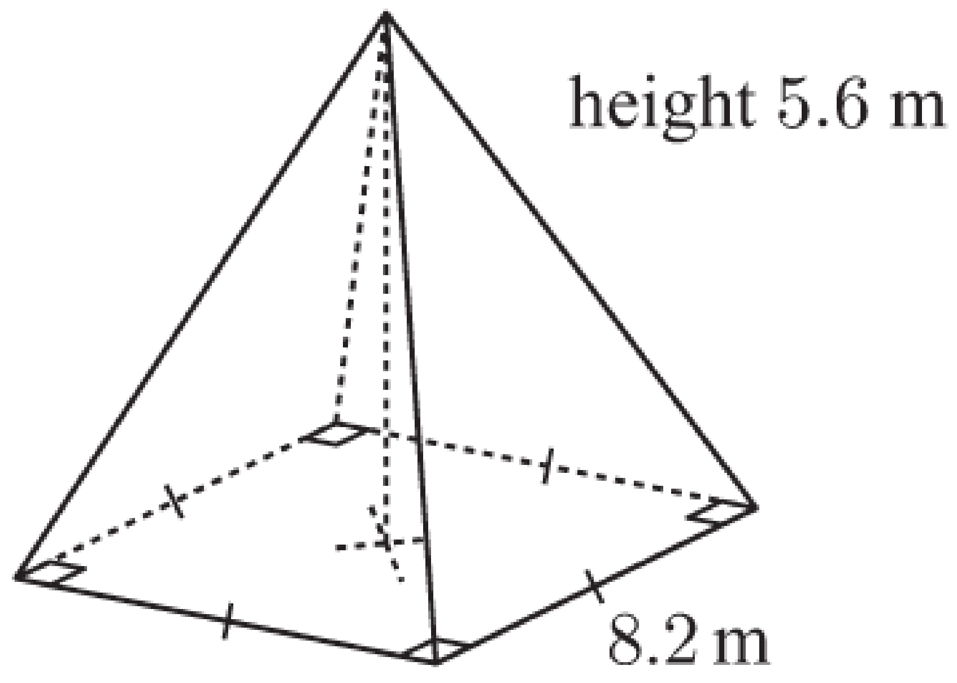
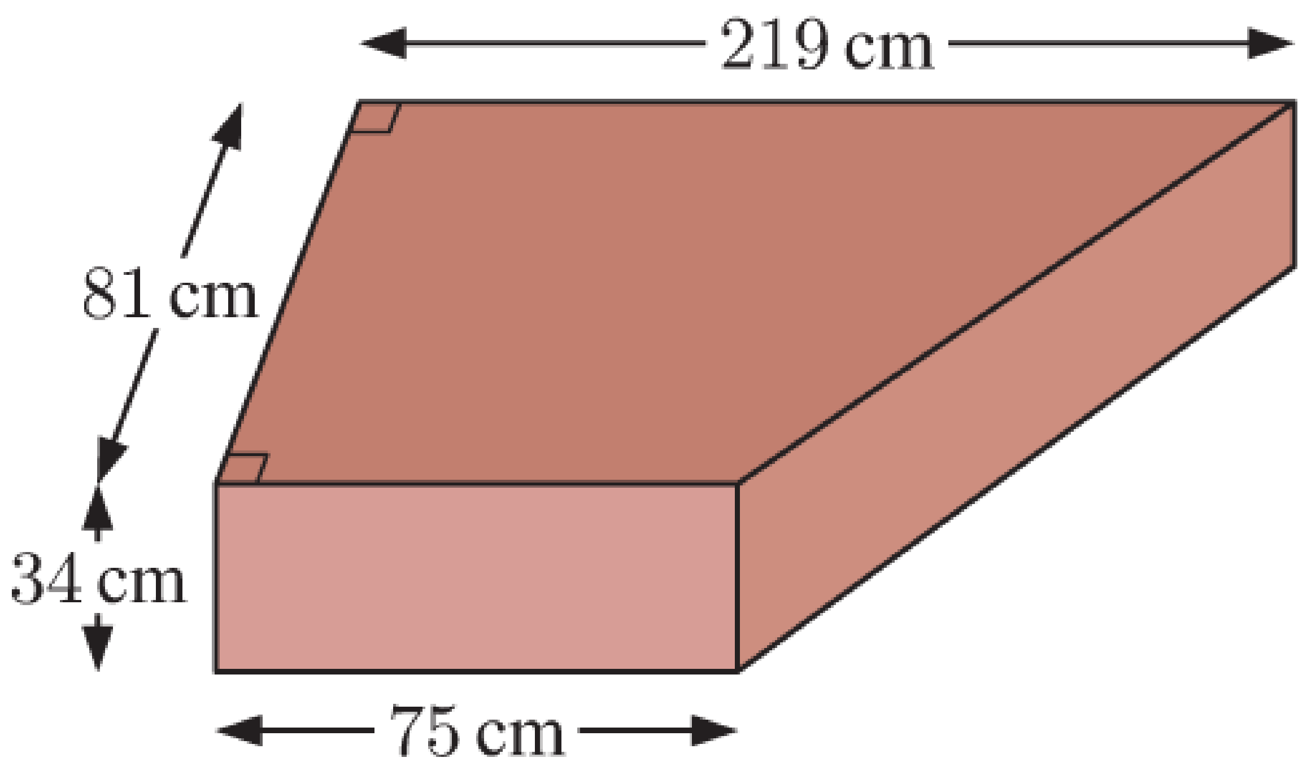


c



3



A harpsichord case has the dimensions shown.

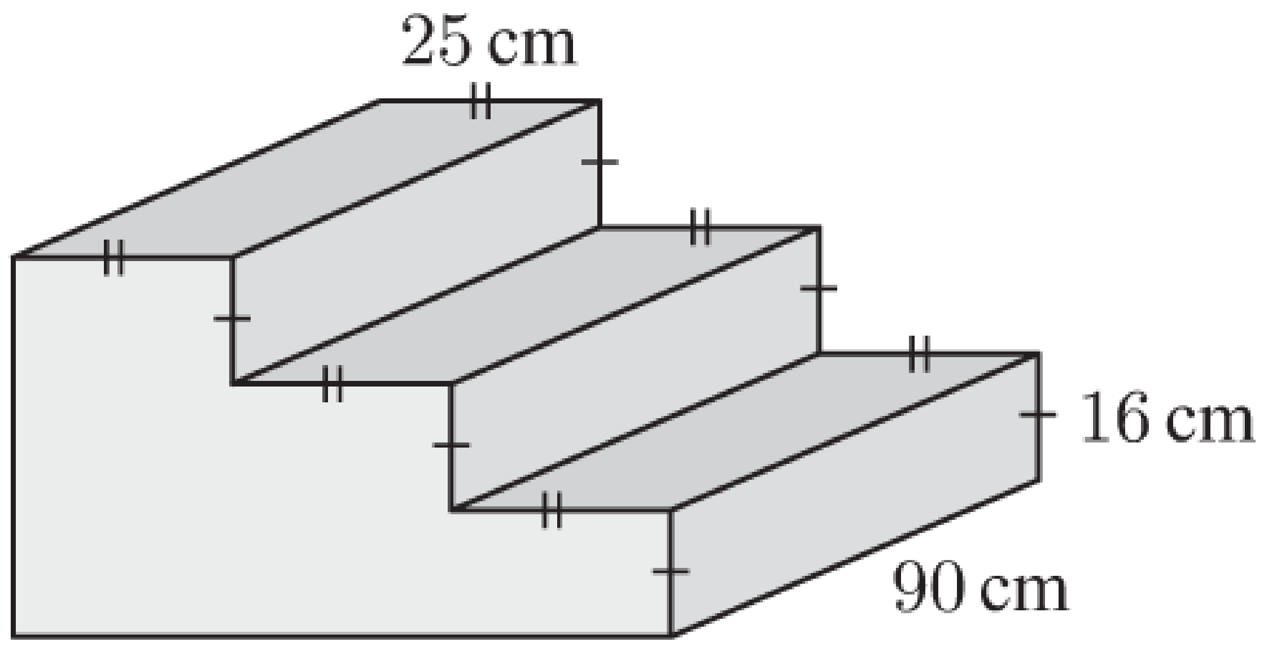
a Find the total area of the top and bottom surfaces.

b Find the area of each side of the case.

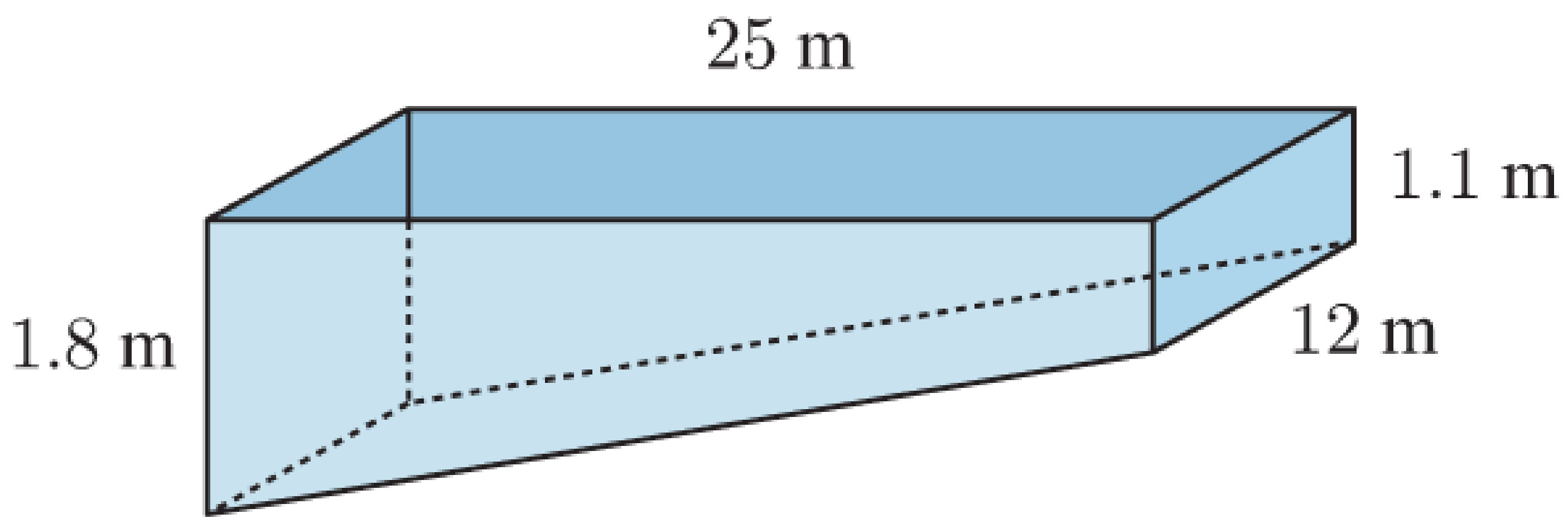
c If the timber costs €128 per square metre, find the value of the timber used to construct this case.

4 Find the surface area of:

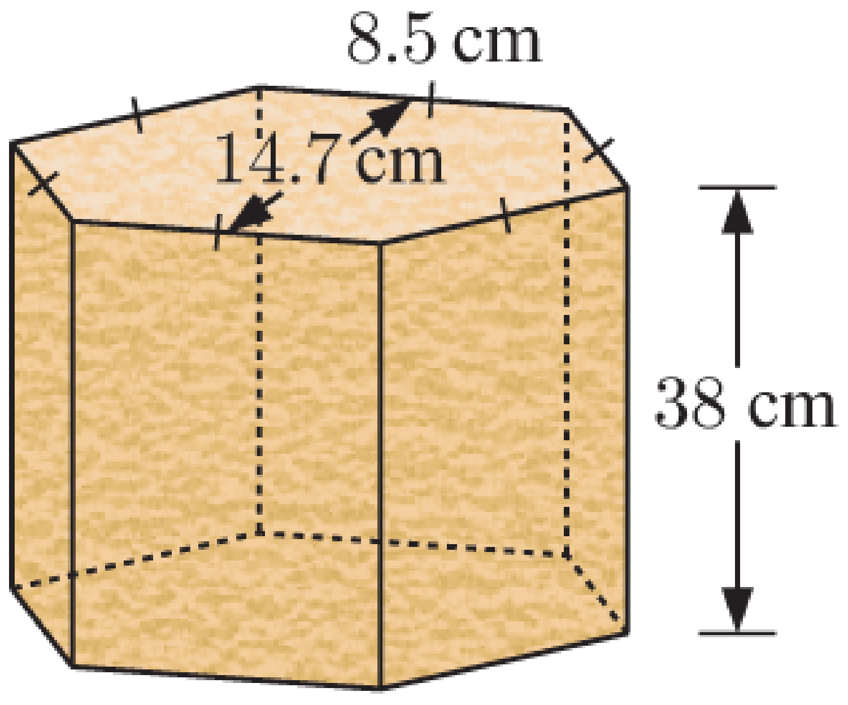
a this set of steps



b the sides and base of this swimming pool.



5 The **Taylor Prism** is a regular hexagonal prism made of clay with a historical record written on its sides. It was found by archaeologist **Colonel Taylor** in 1830.
If the ancient Assyrians had written on all the surfaces, what total surface area would the writing have covered?



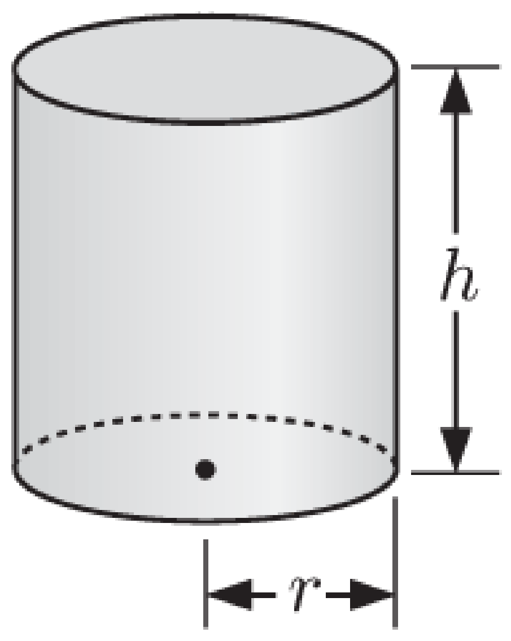
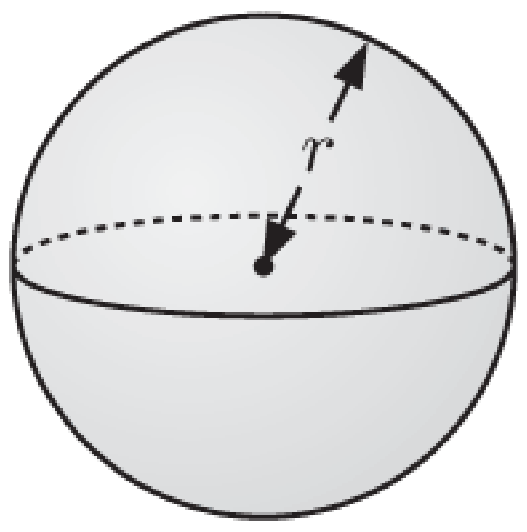
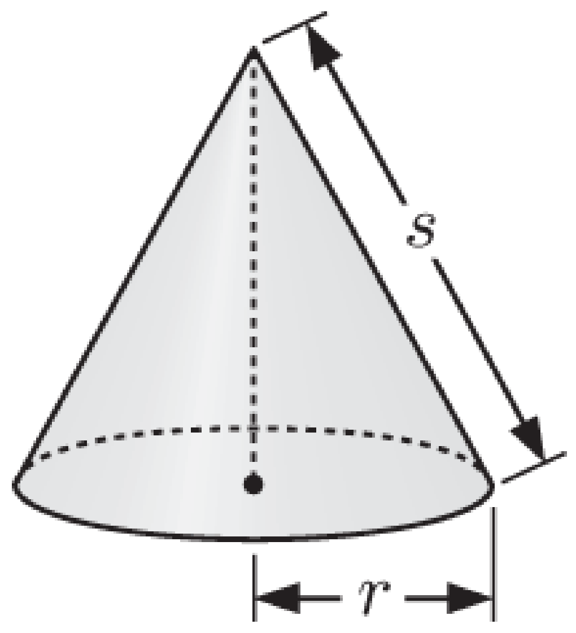
6 Write a formula for the surface area of:

a a rectangular prism with side lengths x cm, $(x+2)$ cm, and $2x$ cm

b a square-based pyramid for which every edge has length x cm.

SOLIDS WITH CURVED SURFACES

These objects have curved surfaces, but their surface areas can still be calculated using formulae.

Cylinder	**Sphere**	**Cone**
		
$A =$ curved surface $+$ 2 circular ends $= 2\pi rh + 2\pi r^2$	**$A = 4\pi r^2$**	**$A =$ curved surface $+$ circular base $= \pi rs + \pi r^2$**

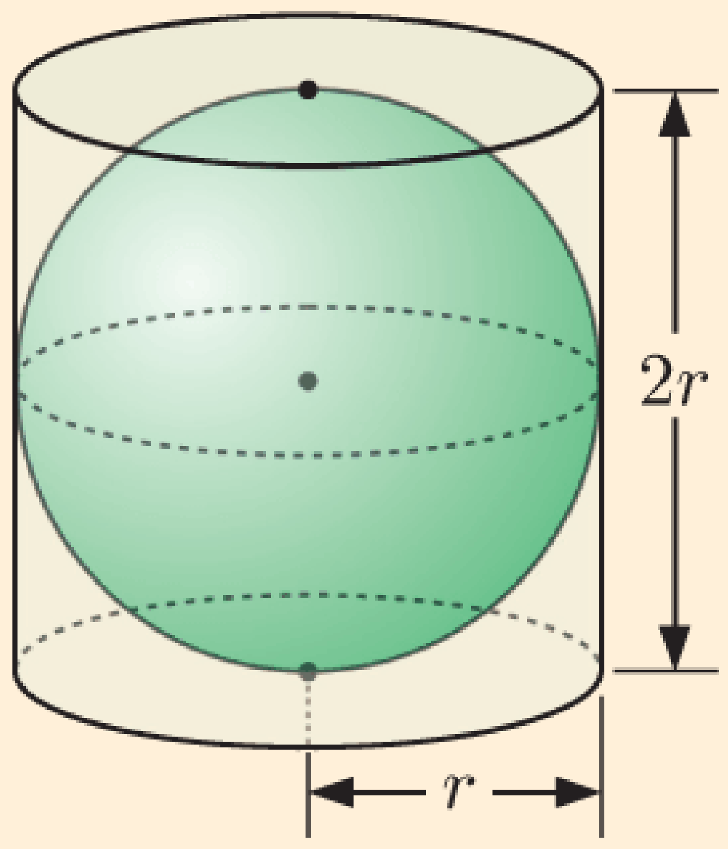
INVESTIGATION 1 ARCHIMEDES AND THE SPHERE

Archimedes of Syracuse (287 BC - 212 BC) was born on the island of Sicily. The son of a mathematician, he studied first in Syracuse and then Alexandria, Egypt, where he would have seen the works of **Euclid**. Returning to Syracuse, he was a notable inventor and problem solver for **King Heiro**. The **Archimedes screw** he invented is still used today as a primitive water pump. He also designed and constructed war machines for the defence of Syracuse, enabling the city to withstand the Roman siege for over two years. However, in 212 BC the Romans took Syracuse, and despite the orders of the Roman commander **Marcellus** to spare him, Archimedes was killed.

Upset at the death of his respected foe, Marcellus ensured that Archimedes was buried as he had requested: in recognition of his greatest mathematical achievement, the symbol of a sphere in a cylinder was engraved on his tombstone.

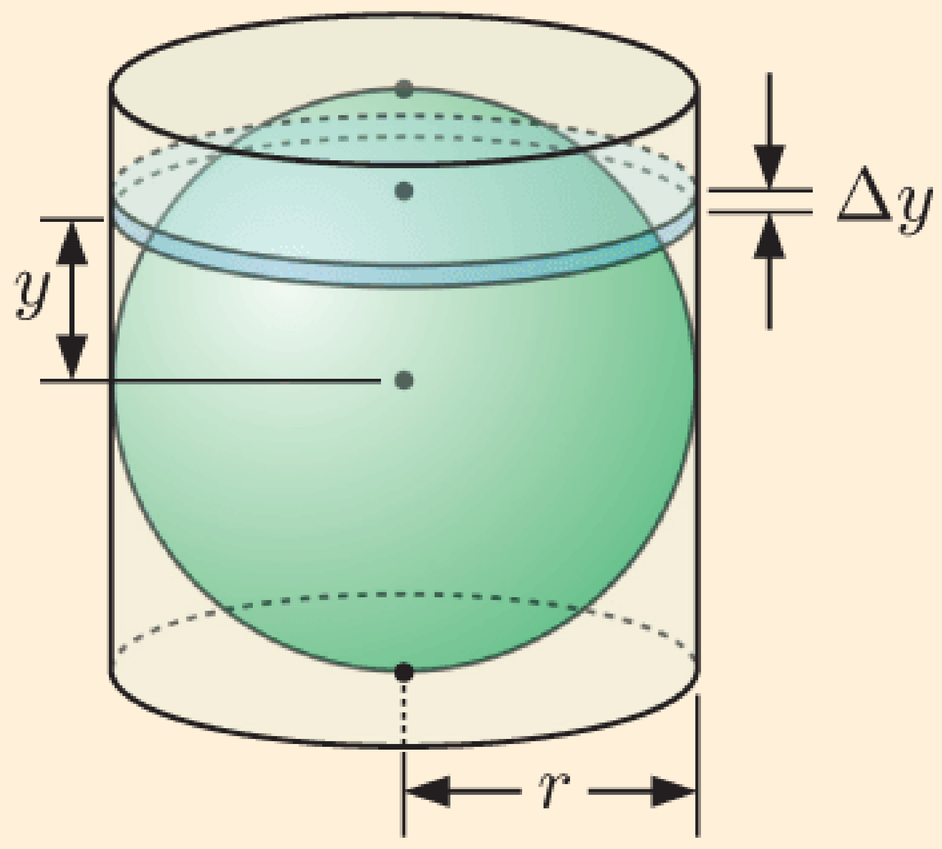
Archimedes was fascinated by the geometric properties of cylinders, cones, and spheres. In this Investigation we will follow Archimedes' proof of the formula for the surface area of a sphere. He then went on to prove the formula for the volume of a sphere and how it related to the volumes of a cone and cylinder with the same radius as the sphere and height twice that radius.

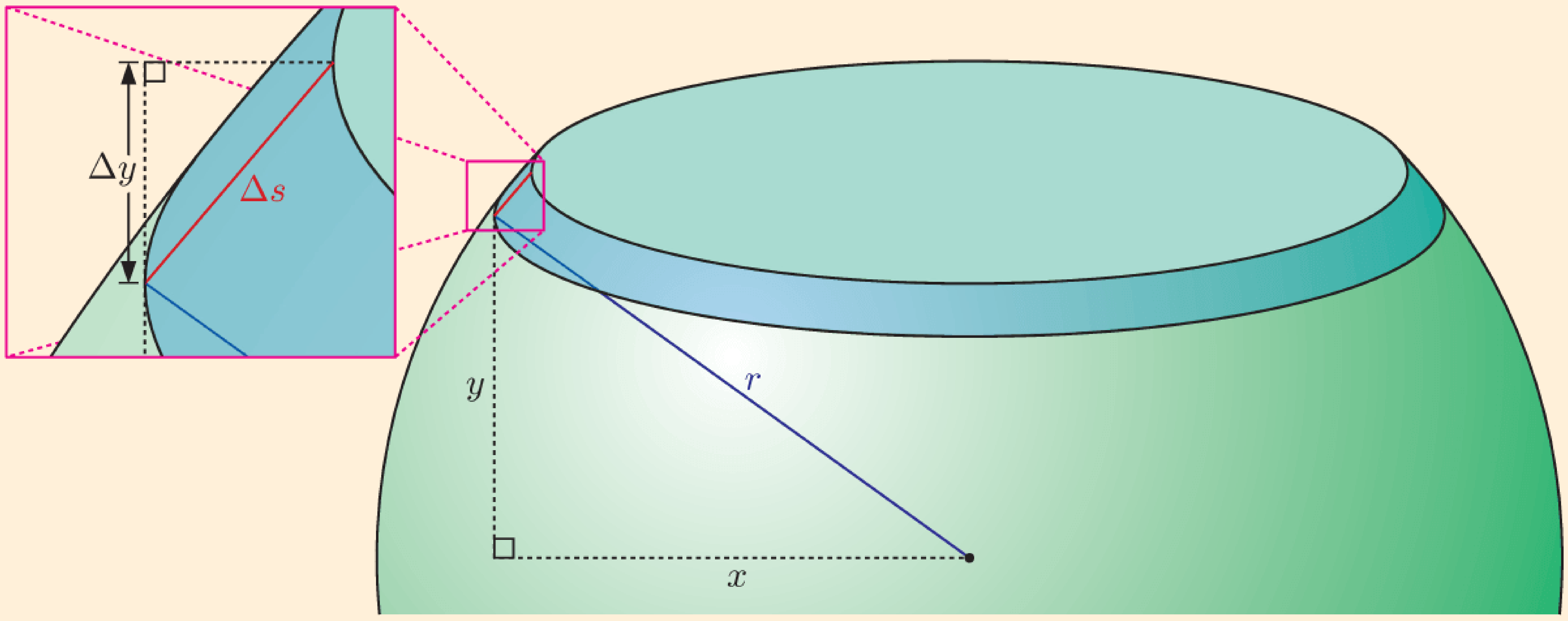
At the time of Archimedes, the relationship between the circumference of a circle and its radius was well known, so the surface area of the curved surface of a cylinder was also known. Archimedes supposed that a sphere of radius r was placed in a cylinder which only *just* contained it, so the cylinder had radius r and height $2r$.



What to do:

1. Find the area of the curved surface of the cylinder with radius r and height $2r$.
2. Suppose a thin slice of thickness Δy is taken at some distance y above or below the centre of the sphere. Let the radius of the cross-section of the sphere at that height be x.
   - **a** Considering the area of the curved surface of the cylinder, explain why the contribution from this slice is $2\pi r \Delta y$.
   - **b** Explain why $x^2 = r^2 - y^2$.



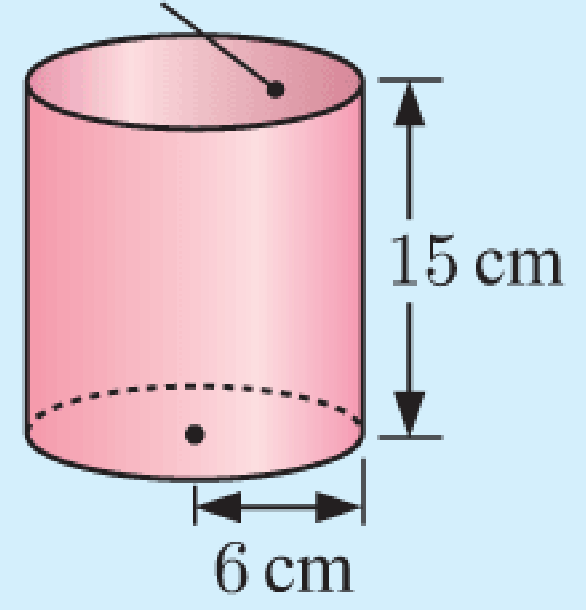


   - **c** Use similar triangles to show that $x\Delta s = r\Delta y$.
   - **d** Hence show that the contribution to the surface area of the sphere from this slice is also $2\pi r \Delta y$.
3. Hence explain why the surface area of a sphere is equal to the area of the curved surface of the cylinder which just contains it, and state the formula for this surface area.

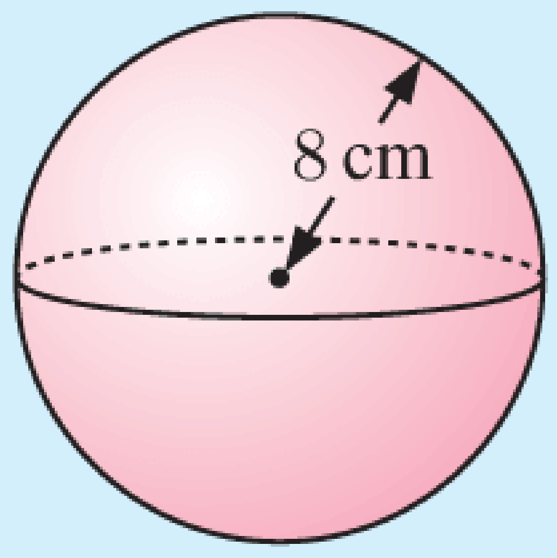
Example 3 **Self Tutor**

Find, to 1 decimal place, the outer surface area of:

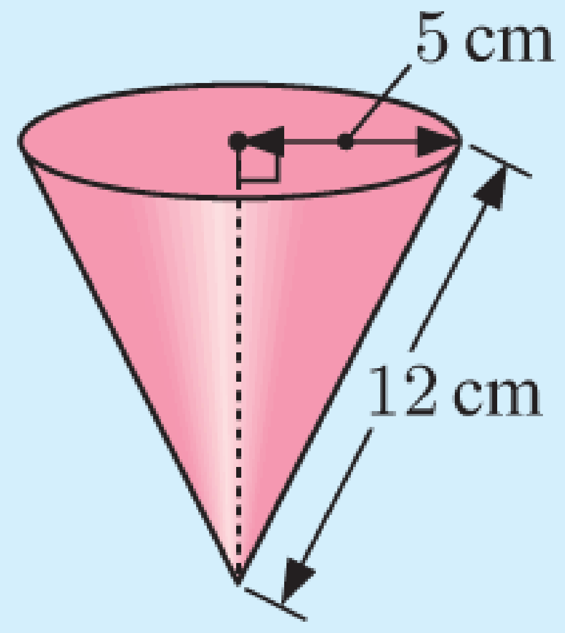
a



b



c



a The cylinder is hollow top and bottom, so we only have the curved surface.

$$A = 2\pi rh$$
$$= 2 \times \pi \times 6 \times 15 \text{ cm}^2$$
$$\approx 565.5 \text{ cm}^2$$

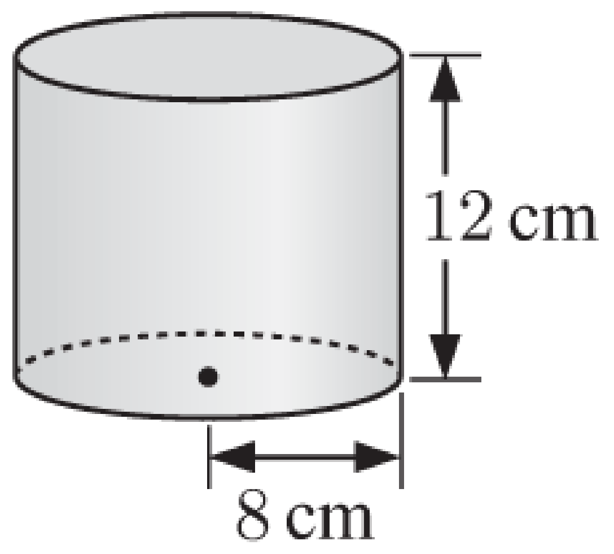
b
$$A = 4\pi r^2$$
$$= 4 \times \pi \times 8^2 \text{ cm}^2$$
$$\approx 804.2 \text{ cm}^2$$

c
$$A = \pi rs + \pi r^2$$
$$= \pi \times 5 \times 12 + \pi \times 5^2 \text{ cm}^2$$
$$\approx 267.0 \text{ cm}^2$$

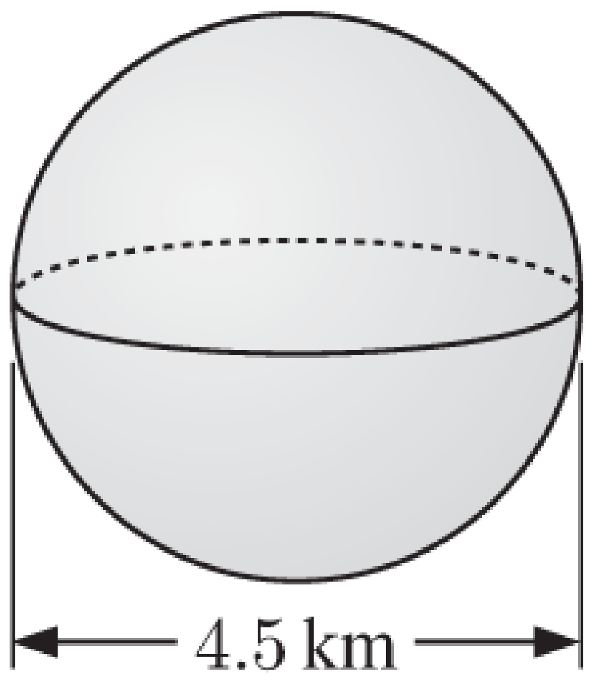
EXERCISE 6B.2

1 Find, to 1 decimal place, the outer surface area of:

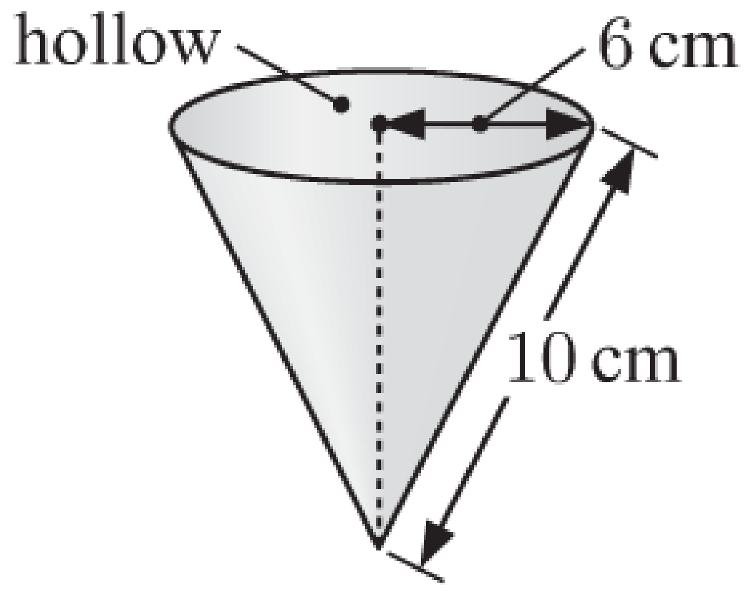
a



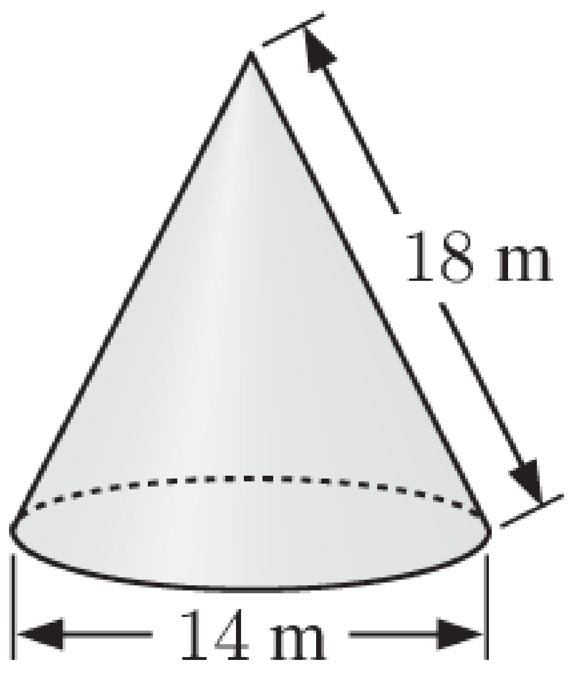
b



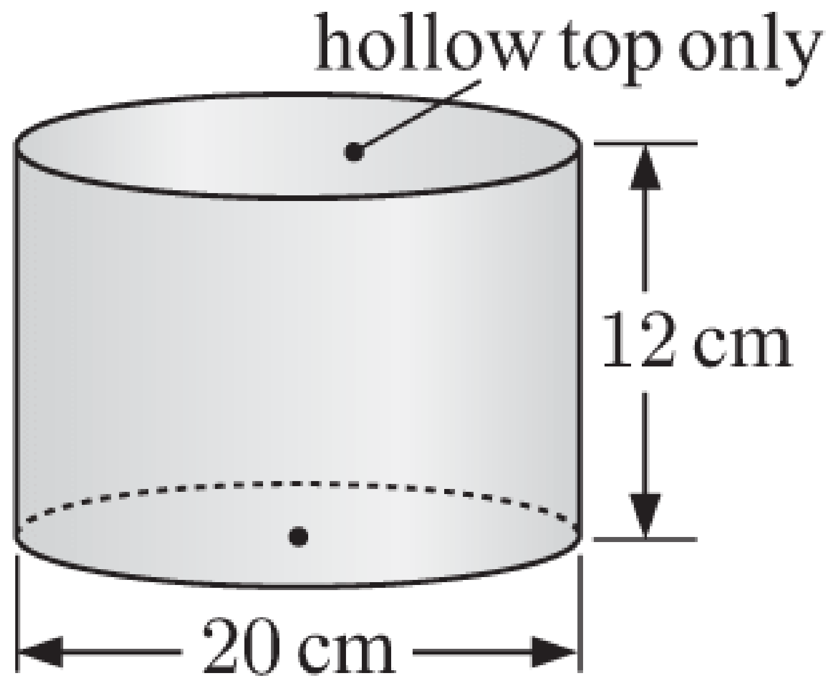
c



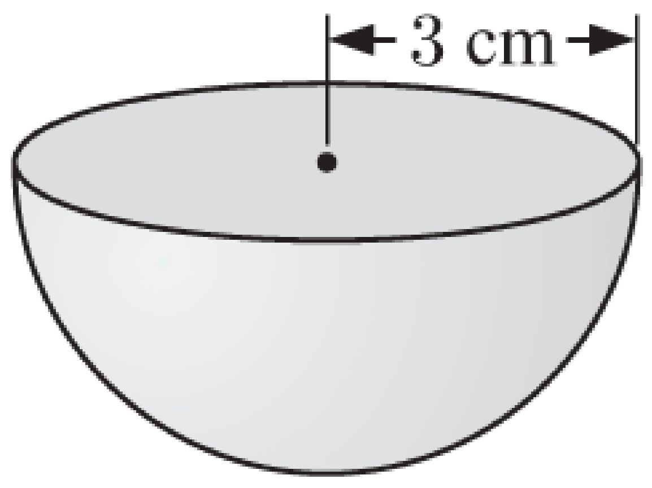
d



e



f



2 Find the surface area of:

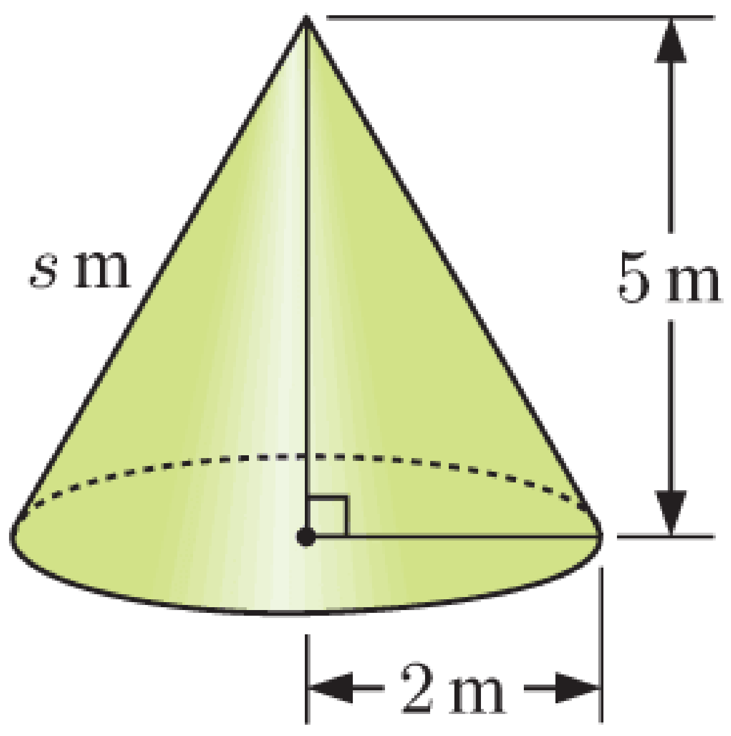
a a cylinder with height 36 cm and radius 8 cm

b a sphere with diameter 4.6 m

c a cone with radius 38 mm and slant height 86 mm

d a cone with radius 1.2 cm and height 1.6 cm.

3



A conical tent has base radius 2 m and height 5 m.

a Find the slant height s, to 2 decimal places.

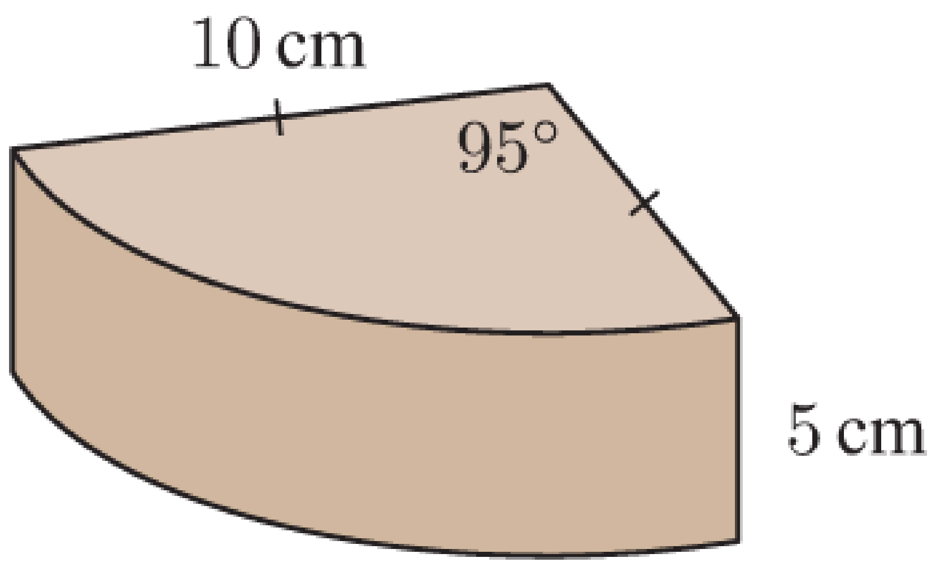
b Find the area of canvas necessary to make the tent, including the base.

c If canvas costs \$18 per m^2, find the cost of the canvas.

4 A cylindrical tank of base diameter 8 m and height 6 m requires a non-porous lining on its circular base and curved walls. The lining costs \$23.20 per m^2 for the base, and \$18.50 per m^2 for the sides.

- **a** Find the area of the base.
- **b** Find the cost of lining the base.
- **c** Find the area of the curved wall.
- **d** Find the cost of lining the curved wall.
- **e** Find the total cost of the lining, to the nearest \$10.

5 This slice of cake is to be covered with icing on all sides, excluding the bottom.
Find the surface area of the cake slice to be iced.



Example 4

Self Tutor

The length of a hollow pipe is three times its radius.

- **a** Write an expression for its outer surface area in terms of its radius r.
- **b** If the outer surface area is 301.6 m^2, find the radius of the pipe.

a Let the radius be r m, so the length is $3r$ m.

$$\begin{aligned}\text{Surface area} &= 2\pi r h\\ &= 2\pi r \times 3r\\ &= 6\pi r^2 \text{ m}^2\end{aligned}$$

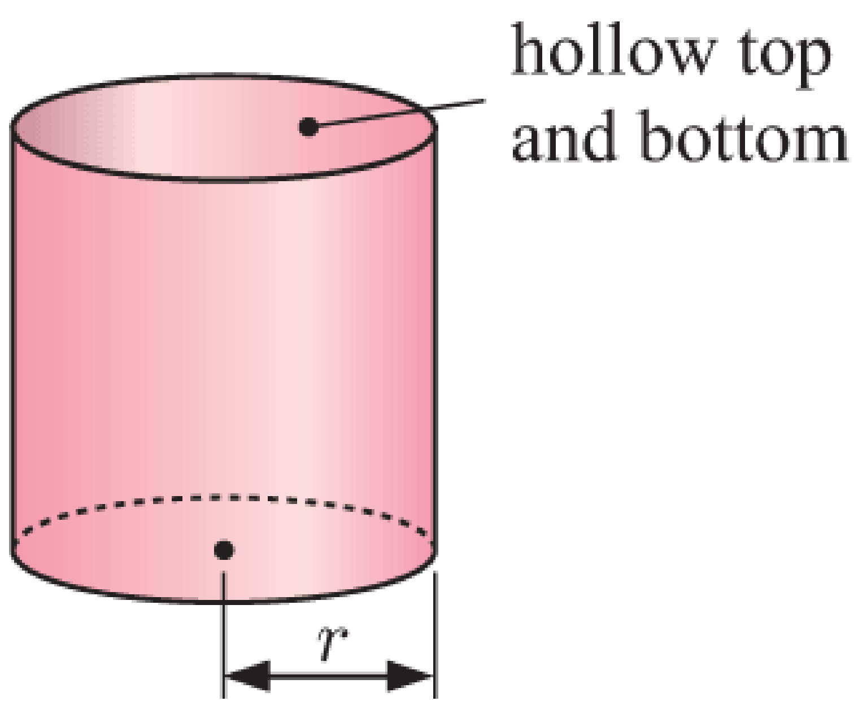
b The surface area is 301.6 m^2

$$\begin{aligned}\therefore\ 6\pi r^2 &= 301.6\\ \therefore\ r^2 &= \frac{301.6}{6\pi}\\ \therefore\ r &= \sqrt{\frac{301.6}{6\pi}} \quad \{\text{as } r > 0\}\\ \therefore\ r &\approx 4.00\end{aligned}$$

The radius of the pipe is 4 m.

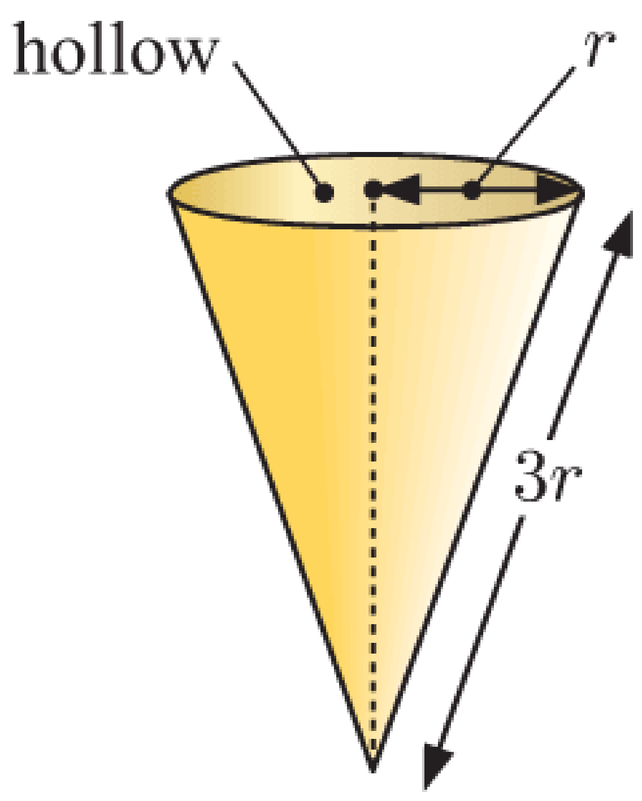
6 The height of a hollow cylinder is the same as its diameter.

- **a** Write an expression for the outer surface area of the cylinder in terms of its radius r.
- **b** Find the height of the cylinder if its surface area is 91.6 m^2.



7 The slant height of a hollow cone is three times its radius.

- **a** Write an expression for the outer surface area of the cone in terms of its radius r.
- **b** Given that the surface area is 21.2 cm^2, find the cone's:
  - **i** slant height
  - **ii** height.



8 Write a formula for the surface area of:

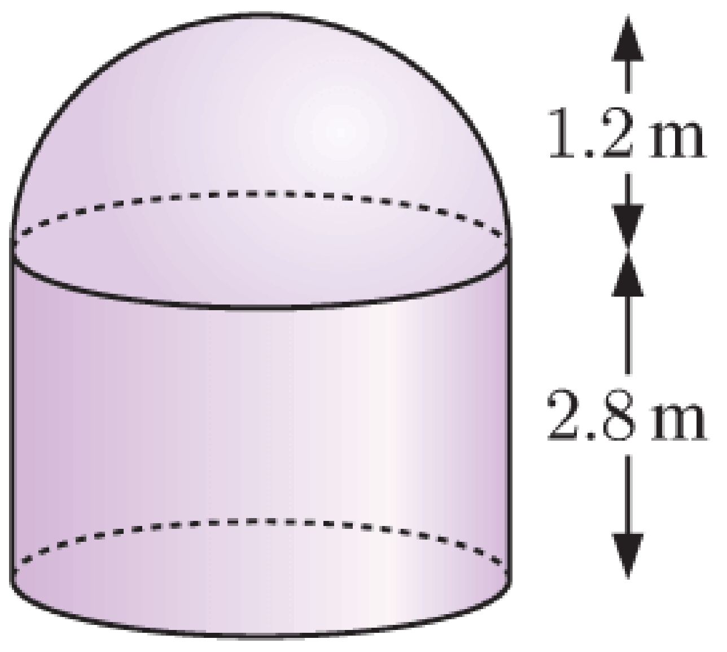
- **a** a cylinder with radius x cm and height $2x$ cm
- **b** a hemisphere with radius r cm
- **c** a cone with radius x cm and height $2x$ cm.

9 Find:

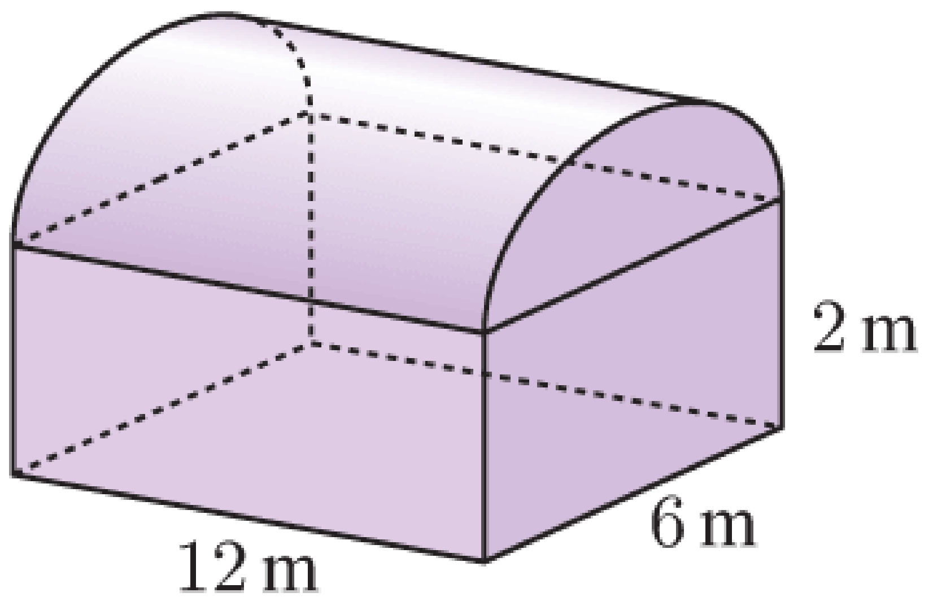
- **a** the radius of a sphere with surface area 64π cm^2
- **b** the height of a solid cylinder with radius 6.3 cm and surface area 1243 cm^2
- **c** the radius of a cone with slant height 143 mm and surface area 60 000 mm^2.

10 Find, correct to 1 decimal place, the surface area of each solid:

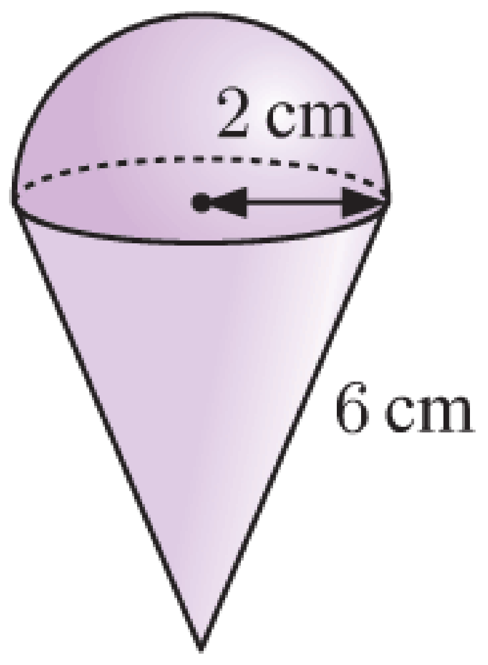
a



b

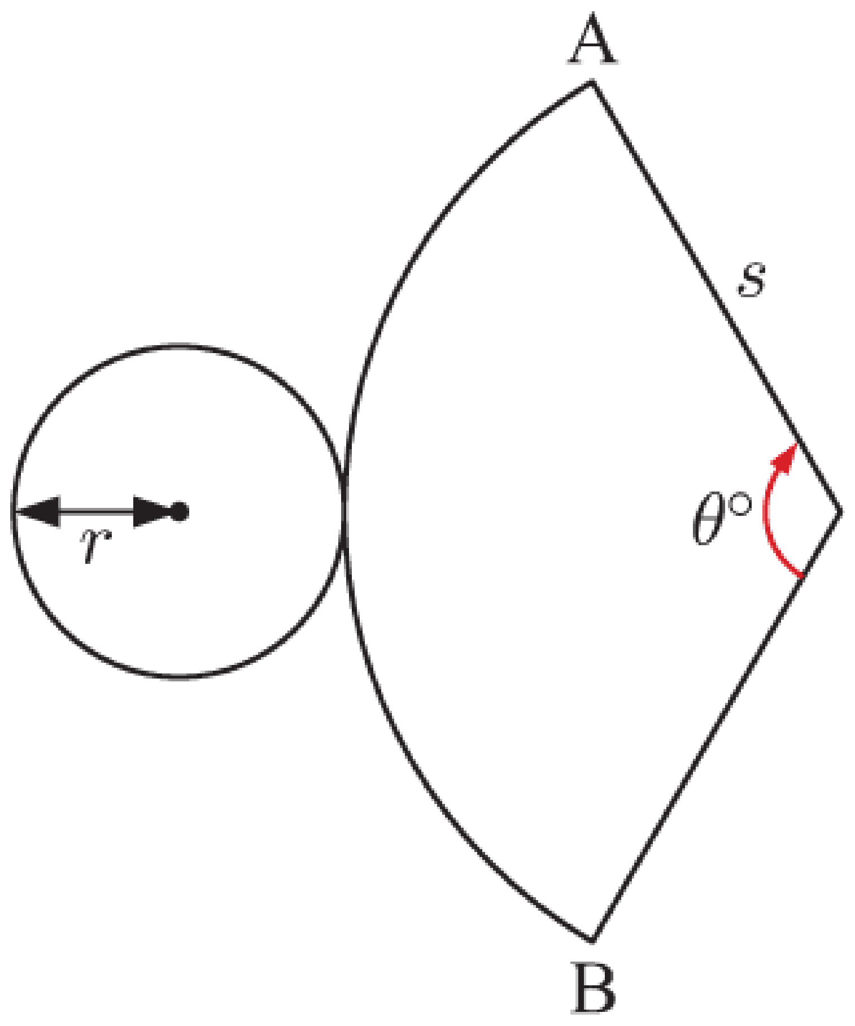


c



11 The planet Neptune is roughly spherical and has surface area $\approx 7.618 \times 10^9$ km^2. Estimate the radius of Neptune.

12



For the net of a cone alongside, notice that the length of arc AB must equal the circumference of the base circle.

- **a** Write the arc length AB in terms of s and θ.
- **b** Hence write θ in terms of r and s.
- **c** Show that the surface area of the cone is given by $A = \pi rs + \pi r^2$.

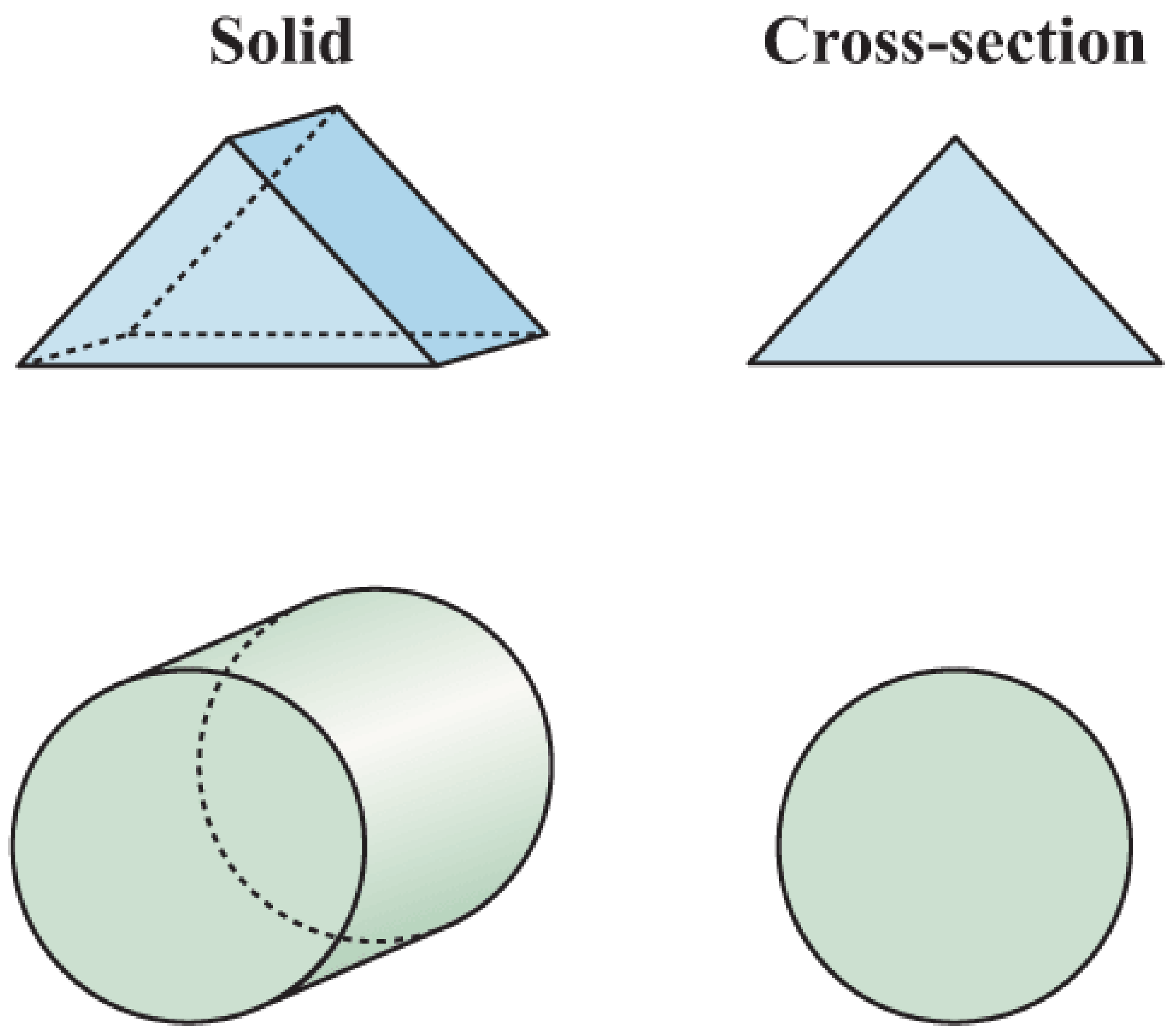
C VOLUME

The **volume** of a solid is the amount of space it occupies.

SOLIDS OF UNIFORM CROSS-SECTION

In the triangular prism alongside, any vertical slice parallel to the front triangular face will be the same size and shape as that face. Solids like this are called *solids of uniform cross-section*. The cross-section in this case is a triangle.

Another example is a cylinder which has a circular cross-section.

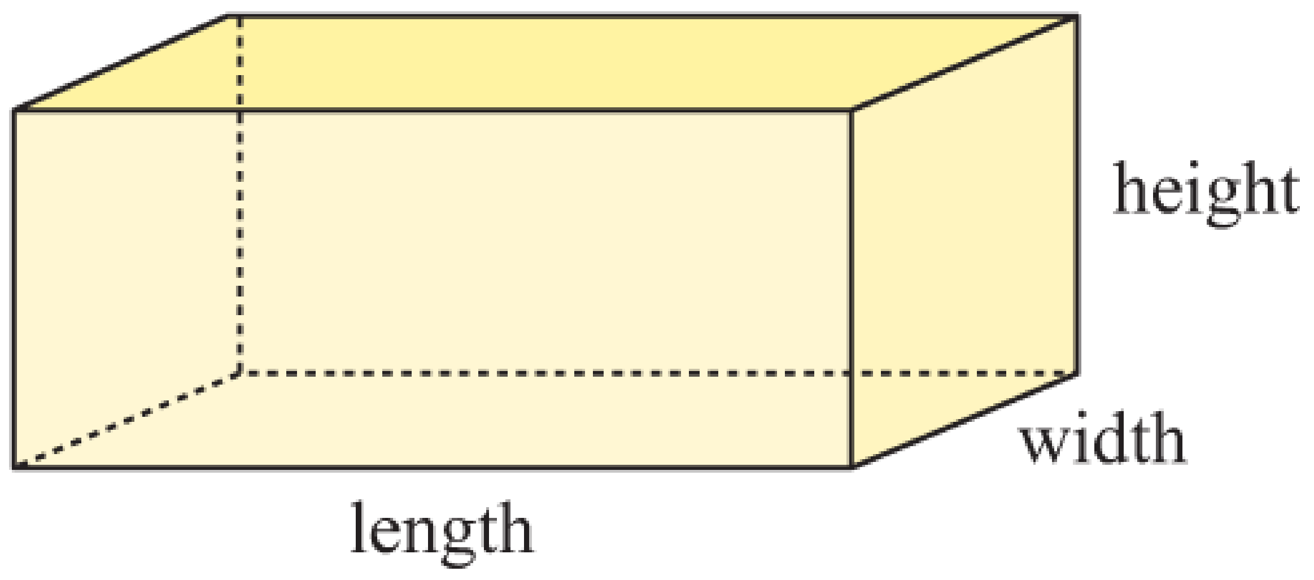


For any solid of uniform cross-section:

Volume = area of cross-section × length

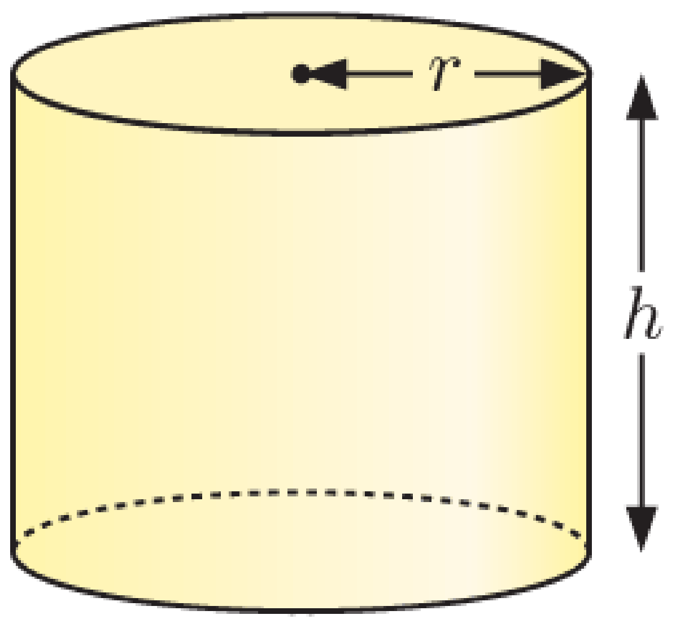
In particular, we can define formulae for the volume of:

- rectangular prisms



Volume = length × width × height

- cylinders



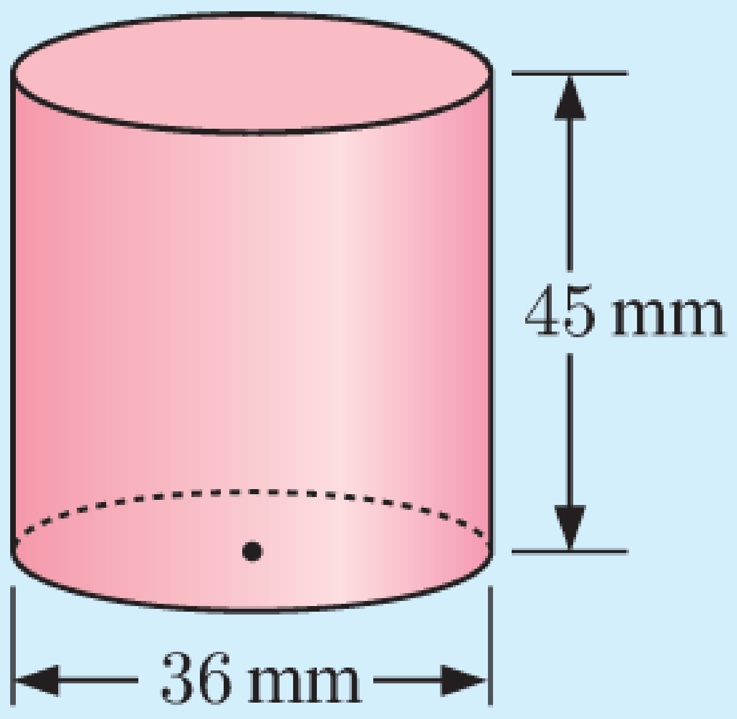
Volume = $\pi r^2 h$

Example 5

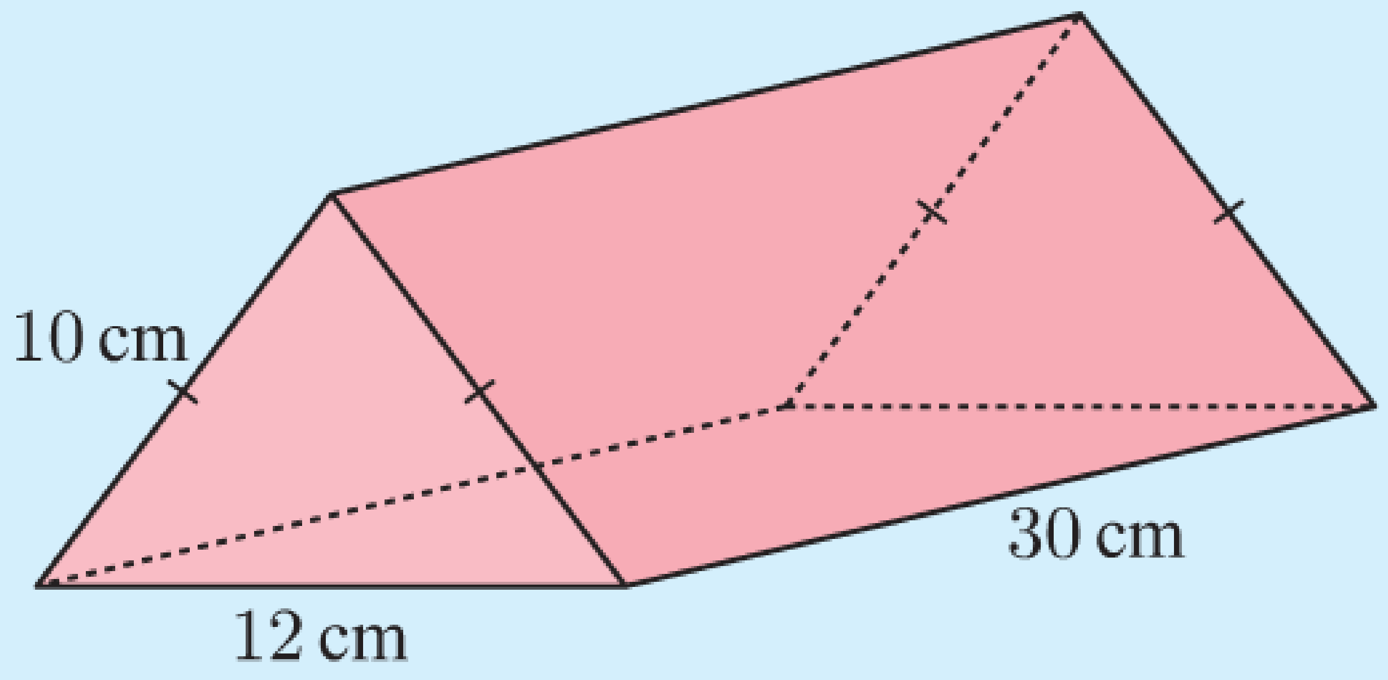
Self Tutor

Find the volume of:

a



b

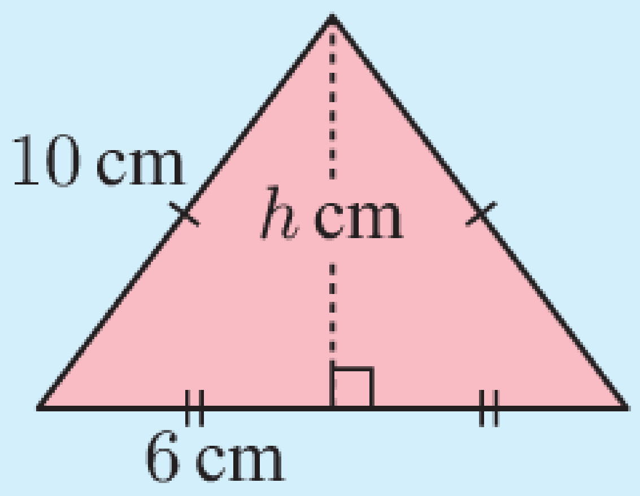


a $V = \pi r^2 h$

$= \pi \times 18^2 \times 45\ \text{mm}^3$

$\approx 45\,800\ \text{mm}^3$

b



Let the prism have height h cm.

$h^2 + 6^2 = 10^2$ {Pythagoras}

$\therefore\ h^2 + 36 = 100$

$\therefore\ h^2 = 64$

$\therefore\ h = 8$ {as $h > 0$}

Volume = area of cross-section × length

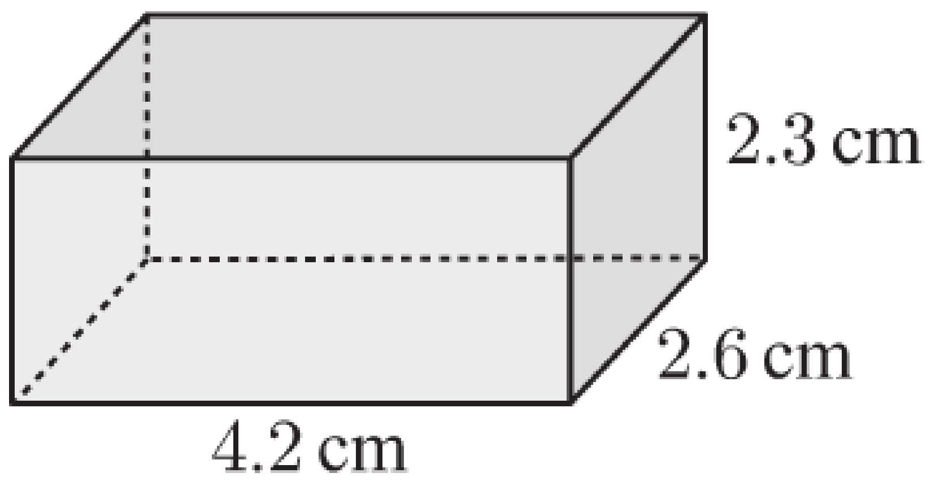
$= (\frac{1}{2} \times 12 \times 8) \times 30\ \text{cm}^3$

$= 1440\ \text{cm}^3$

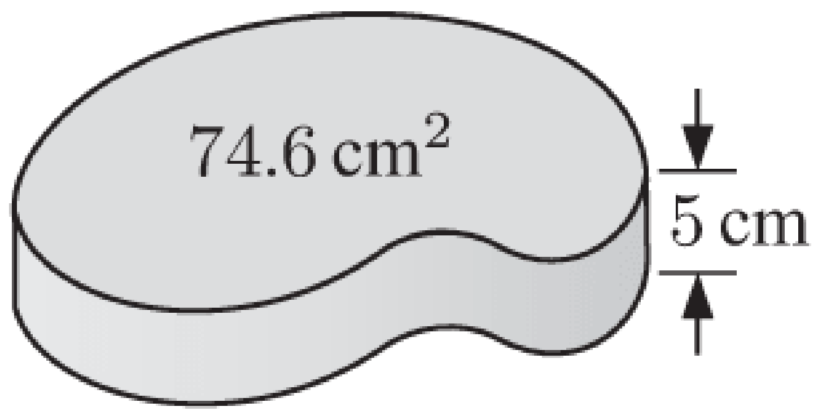
EXERCISE 6C.1

1 Find the volume of:

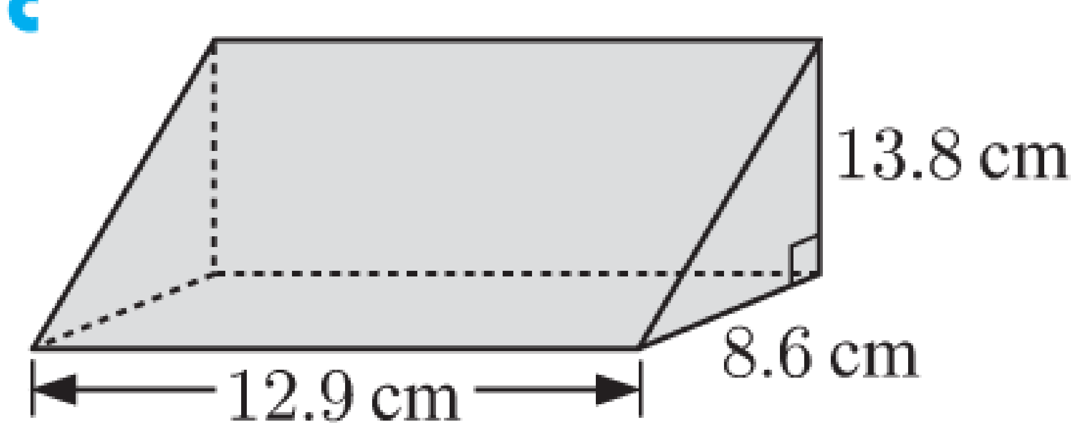
a



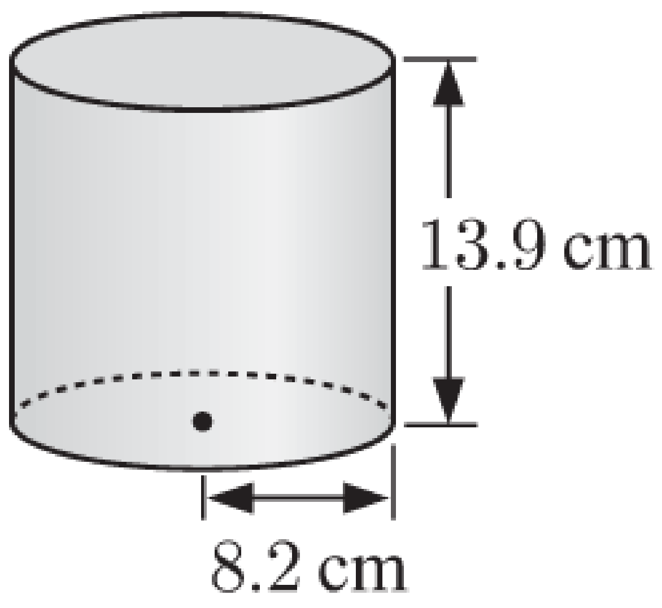
b



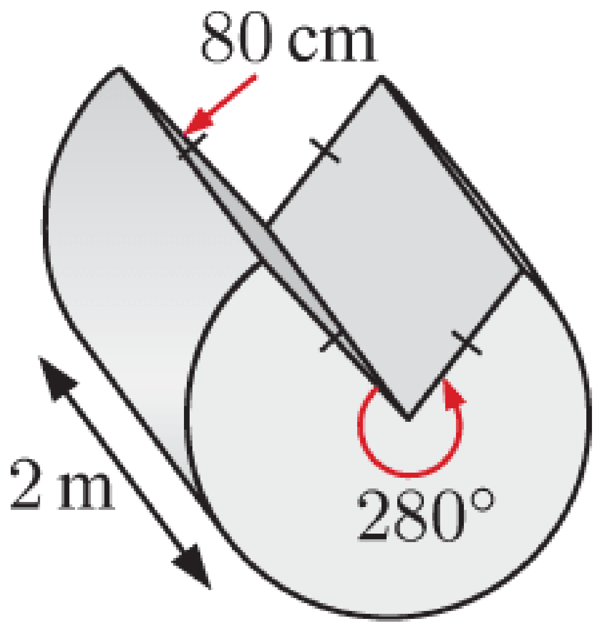
c



d



e



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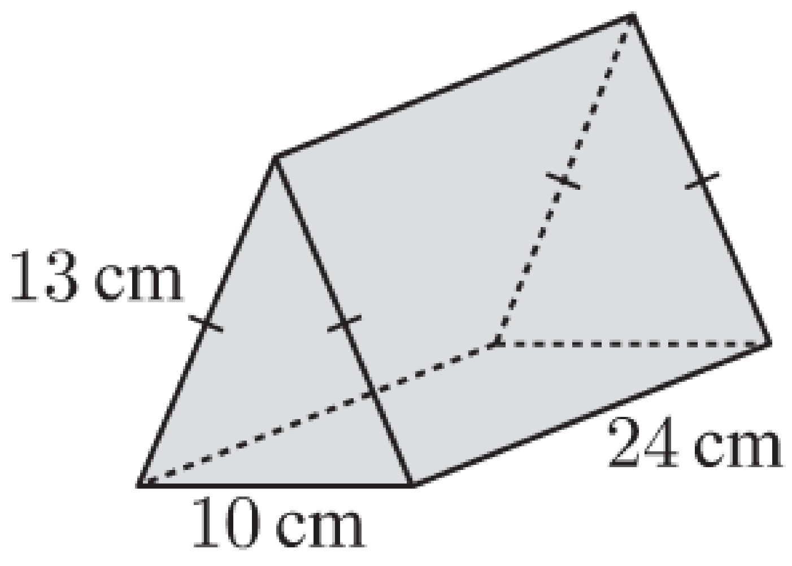
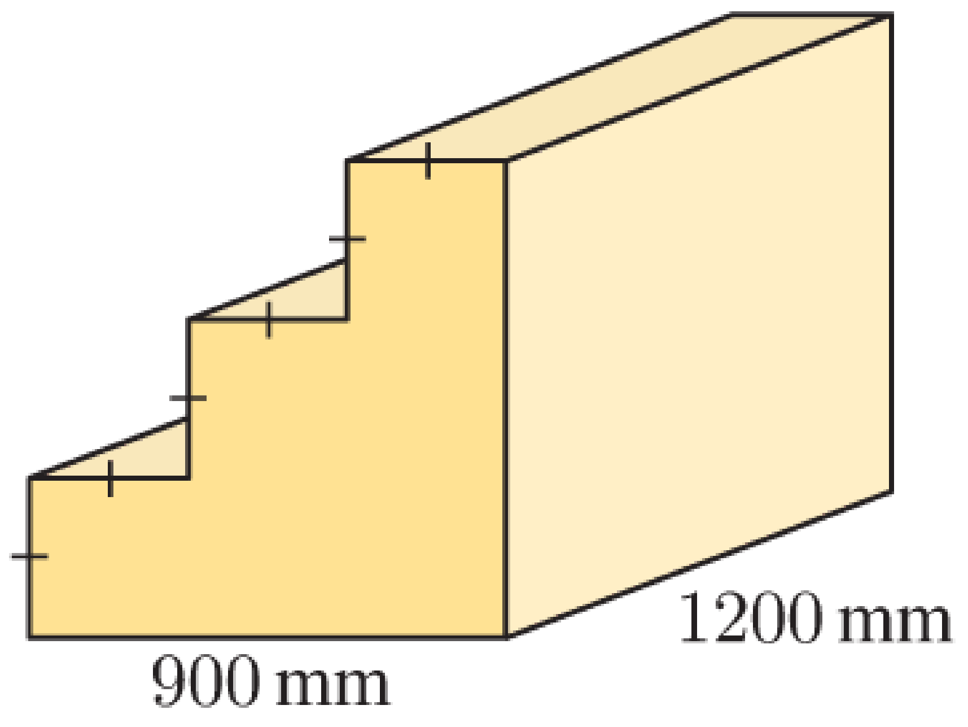


2 Find the volume of:

a



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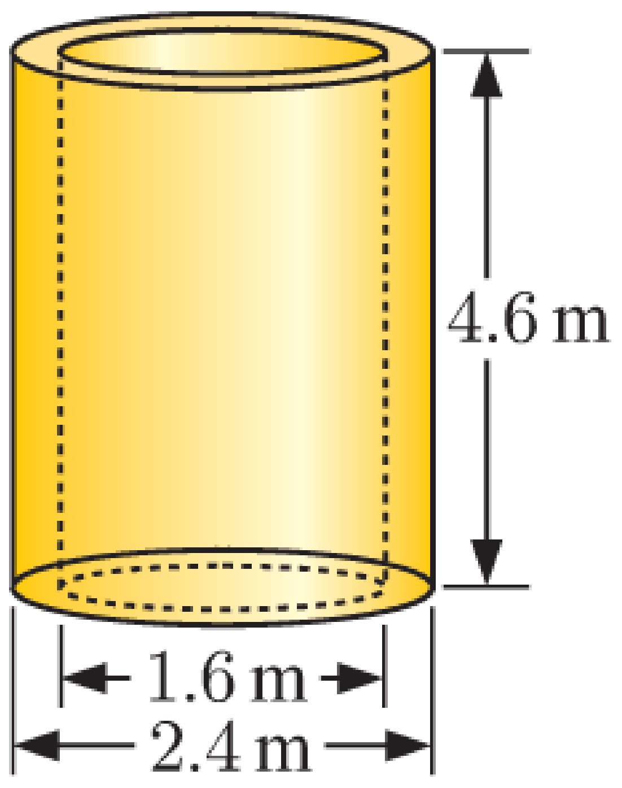
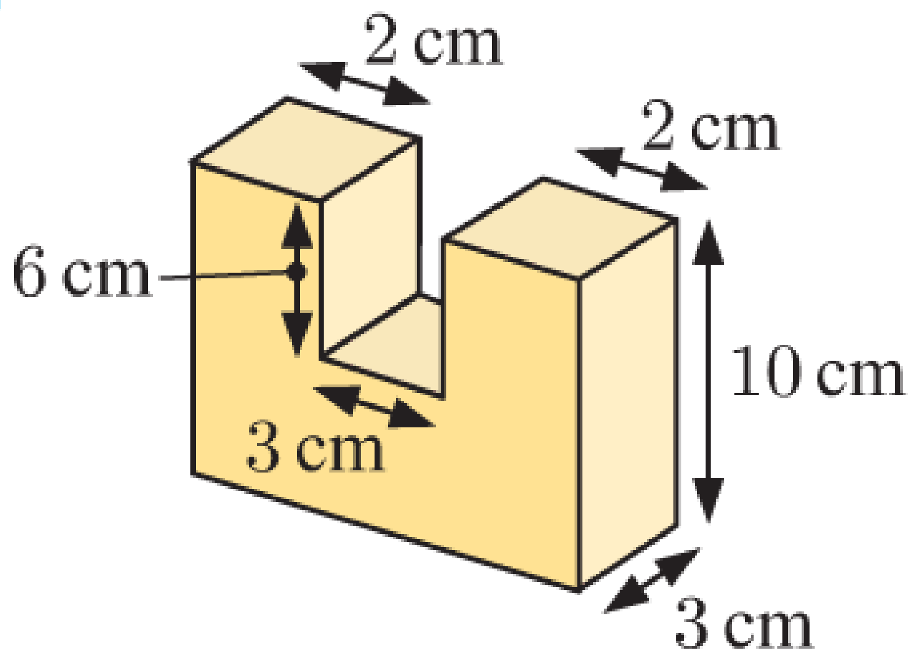
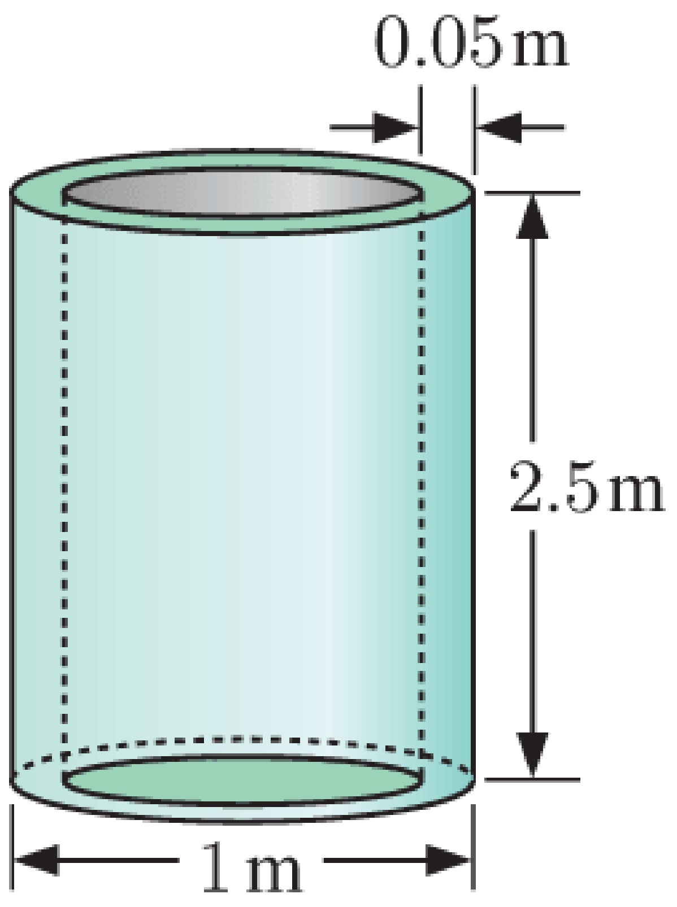


c



3 The Water Supply department uses huge concrete pipes to drain stormwater.

- **a** Find the external radius of a pipe.
- **b** Find the internal radius of a pipe.
- **c** Find the volume of concrete necessary to make one pipe.



4 A rectangular garage floor 9.2 m by 6.5 m is to be concreted to a depth of 120 mm.

- **a** What volume of concrete is required?
- **b** Concrete costs \$135 per m^3, and is only supplied in multiples of 0.2 m^3. How much will the concrete cost?

5 A concrete path 1 m wide and 10 cm deep is placed around a circular lighthouse of diameter 12 m.

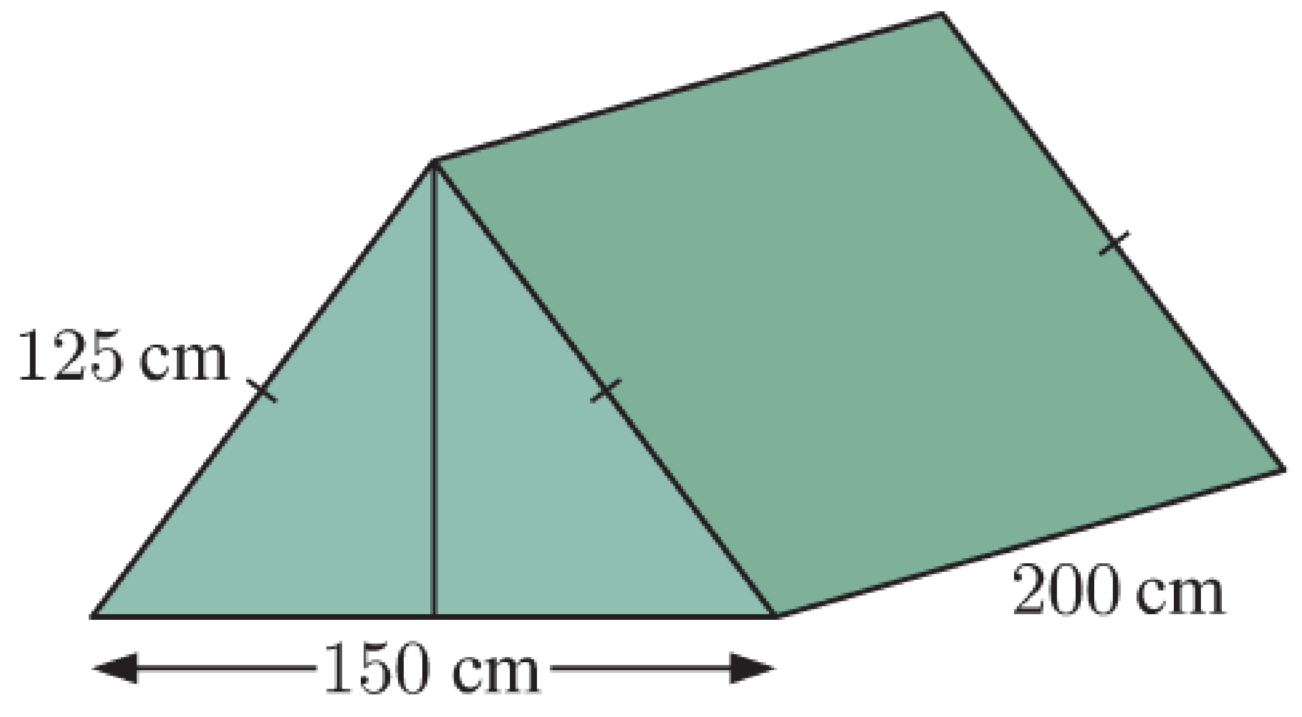
- **a** Draw an overhead view of the situation.
- **b** Find the surface area of the concrete.
- **c** Find the volume of concrete required for the path.

6 1000 km of black plastic cylindrical water piping with internal diameter 13 mm and walls of thickness 2 mm is required for a major irrigation project. The piping is made from bulk plastic which weighs 0.86 tonnes per cubic metre. How many tonnes of black plastic are required?

7 I am currently building a new rectangular garden which is 8.6 m by 2.4 m, and 15 cm deep. I have decided to purchase some soil from the local garden supplier, and will load it into my trailer which measures 2.2 m × 1.8 m × 60 cm. I will fill the trailer to within 20 cm from the top.

- **a** How many trailer loads of soil will I need?
- **b** Each load of soil costs \$87.30. What will the total cost of the soil be?
- **c** I decide to put bark on top of the soil in the garden. Each load covers 11 m^2 of garden bed.
  - **i** How many loads of bark will I need?
  - **ii** Each load of bark costs \$47.95. What is the total cost of the bark?
- **d** Calculate the total cost of establishing the garden.

8



A scout's tent is 150 cm wide and 200 cm long. It has the shape of an isosceles triangular prism as shown.

a Find the height of each vertical support post.

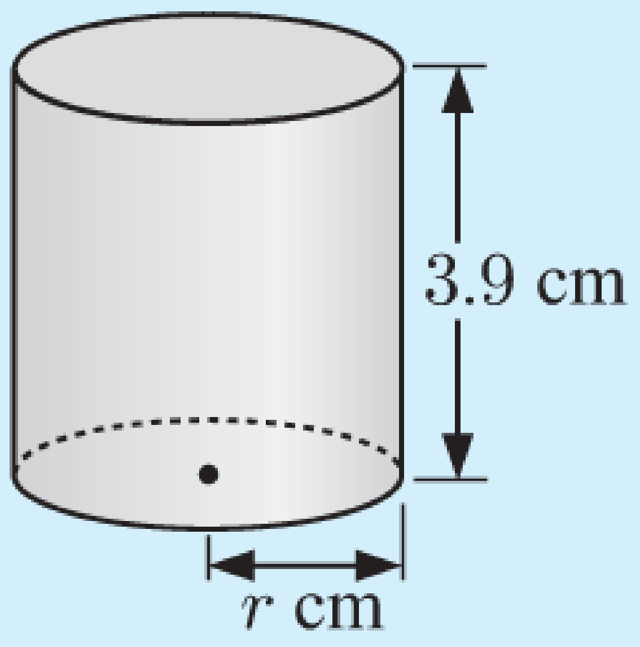
b Find the volume of the tent.

c Find the total area of the canvas in the tent, including the ends and floor.

Example 6

Self Tutor

Find, to 3 significant figures, the radius of a cylinder with height 3.9 cm and volume 54.03 cm^3.



$$V = 54.03\ \text{cm}^3$$

$$\therefore\ \pi \times r^2 \times 3.9 = 54.03 \qquad \{V = \text{area of cross-section} \times \text{length}\}$$

$$\therefore\ r^2 = \frac{54.03}{\pi \times 3.9} \qquad \{\text{dividing both sides by } \pi \times 3.9\}$$

$$\therefore\ r = \sqrt{\frac{54.03}{\pi \times 3.9}} \approx 2.10 \qquad \{\text{as } r > 0\}$$

The radius is approximately 2.10 cm.

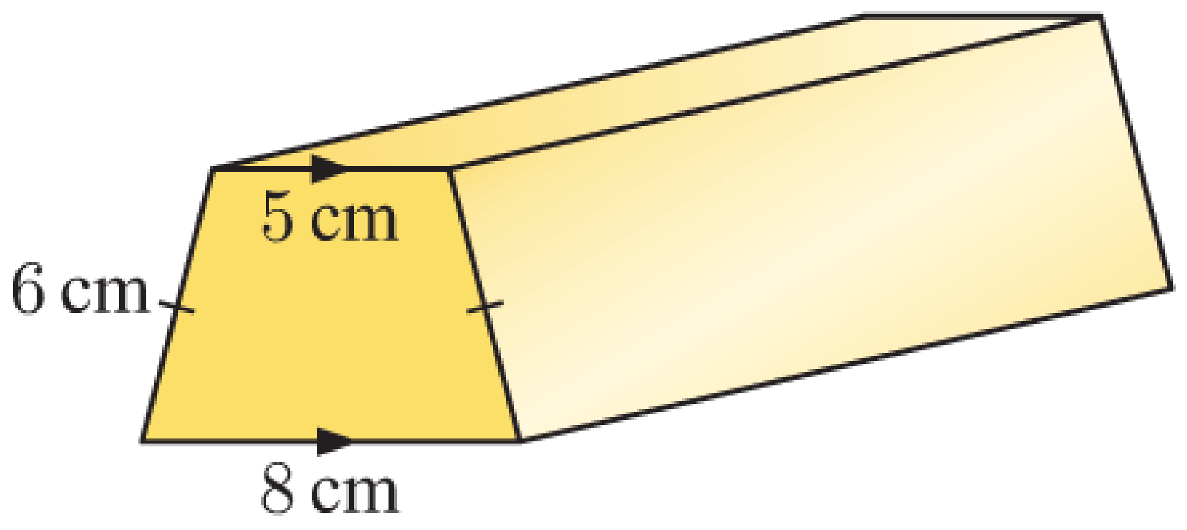
9 Find:

a the height of a rectangular prism with base 5 cm by 3 cm and volume 40 cm^3

b the side length of a cube of butter with volume 34.01 cm^3

c the radius of a steel cylinder with height 4.6 cm and volume 43.75 cm^3.

10



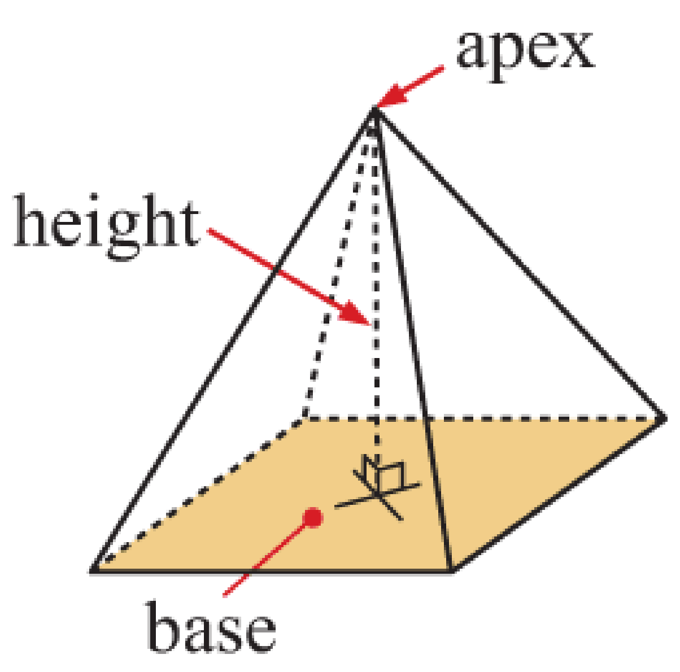
A gold bar has the trapezoidal cross-section shown. Its volume is 480 cm^3.

Find the length of the bar.

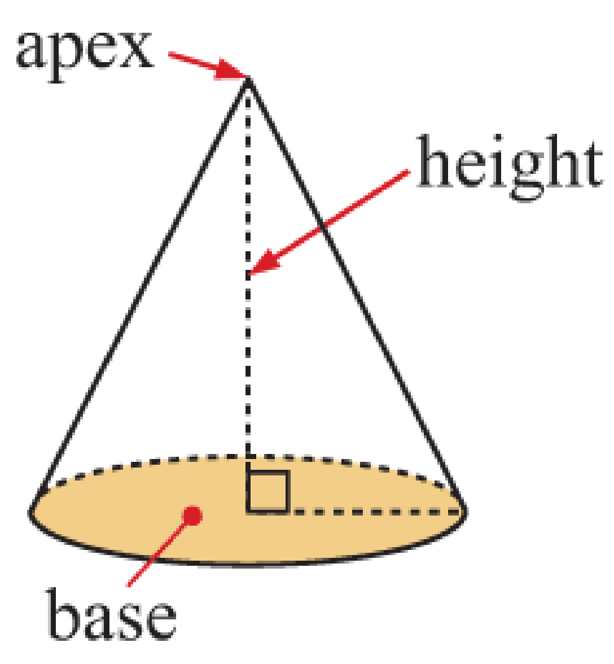
OTHER SOLIDS

We will now consider the volume of pyramids, cones, and spheres.

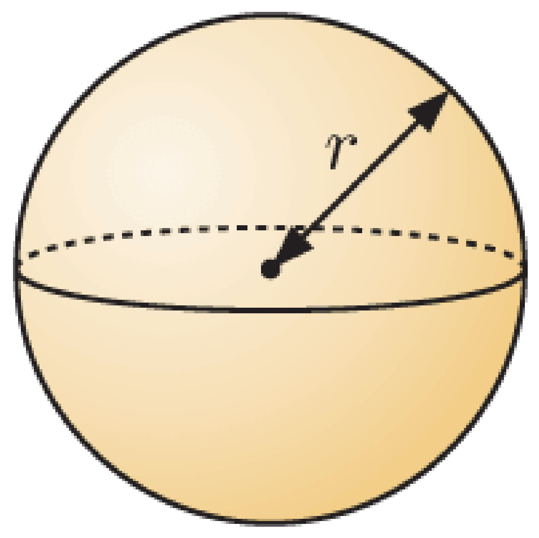
Pyramids and cones are called **tapered solids**. The cross-sections of tapered solids are not uniform. Rather, the cross-sections are a set of similar shapes which get smaller as we approach the apex.



pyramid



cone



sphere

INVESTIGATION 2 VOLUME FORMULAE

We have already seen formulae for the surface area and volume of many solids. We now seek to establish formulae for other solids including pyramids, cones, and spheres.

To achieve this, we make use of two mathematical series we proved in **Chapter 5**:

- the sum of the first n integers: $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- the sum of the first n perfect squares: $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

What to do:

1 Suppose the sum of the first n integers is divided by n^2.

a Evaluate $\frac{\sum_{k=1}^{n} k}{n^2}$ for:

i $n = 10$ **ii** $n = 100$ **iii** $n = 1000$ **iv** $n = 10\,000$.

b Predict the value of $\frac{\sum_{k=1}^{n} k}{n^2}$ as $n \to \infty$.

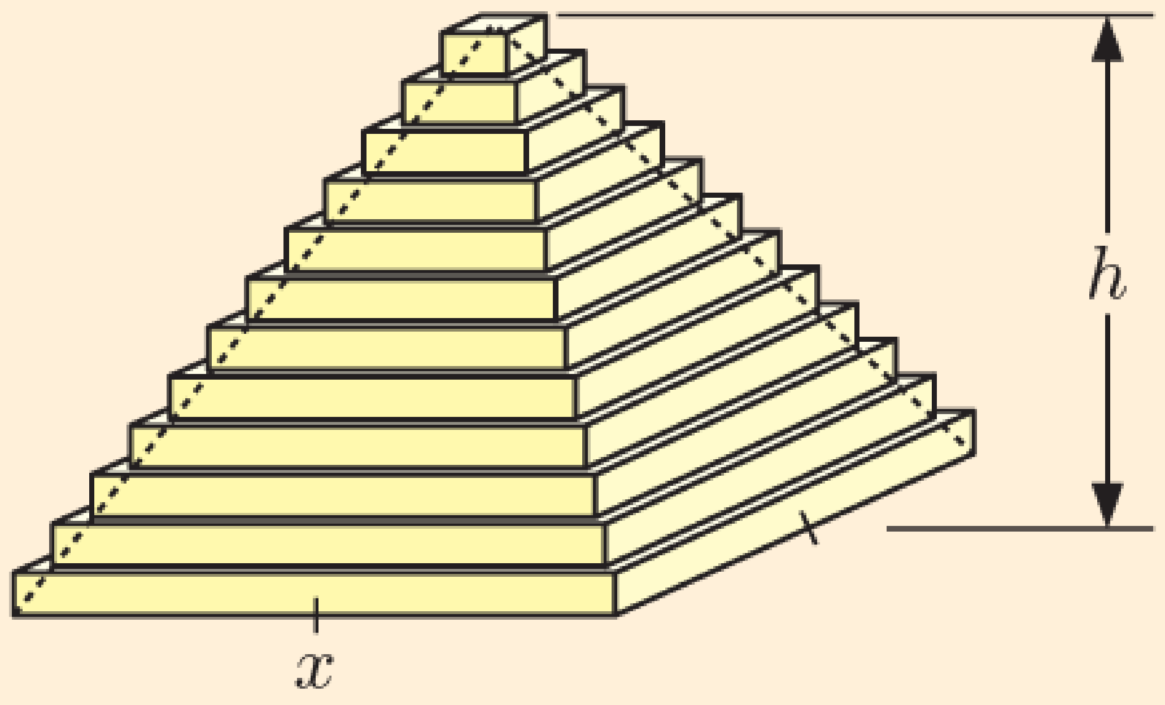
2 Suppose the sum of the first n perfect squares is divided by n^3.

a Evaluate $\frac{\sum_{k=1}^{n} k^2}{n^3}$ for:

i $n = 10$ **ii** $n = 100$ **iii** $n = 1000$ **iv** $n = 10\,000$.

b Predict the value of $\frac{\sum_{k=1}^{n} k^2}{n^3}$ as $n \to \infty$.

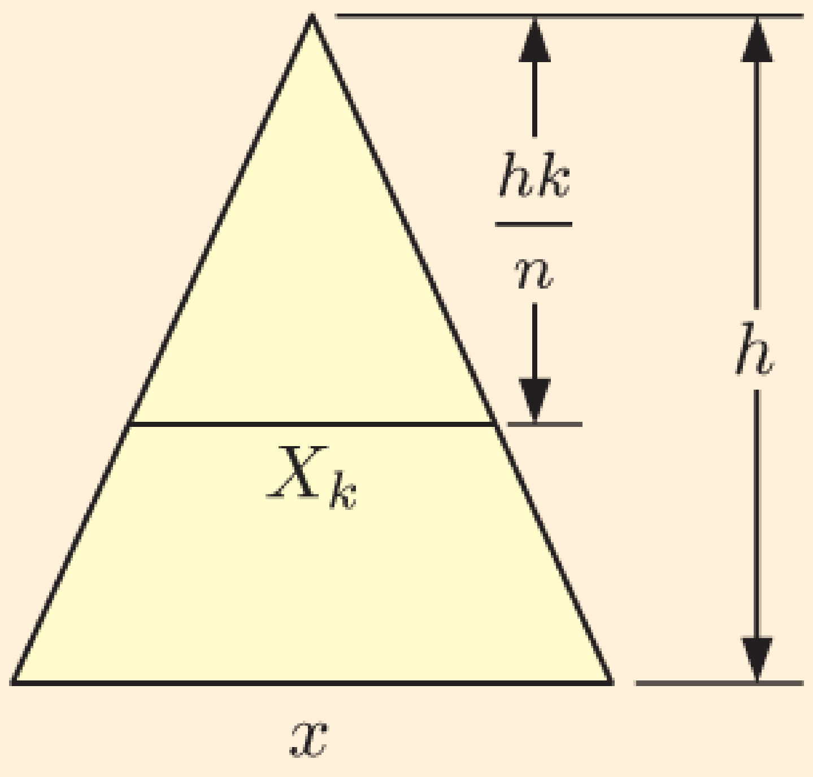
3 Consider a square-based pyramid with base side length x and height h. The pyramid can be approximated using a set of n rectangular prisms with equal thickness, and each with a square base, stacked on top of one another.



a Explain why the thickness of each prism is $\frac{h}{n}$.

b We suppose the base of each prism is the cross-section of the actual pyramid at the corresponding height. Let the kth prism have base $X_k \times X_k$. We start at the apex and move down, so the base of the nth prism will be the $x \times x$ base of the pyramid.

Use the diagram alongside to explain why $X_k = \frac{xk}{n}$.



c Explain why the volume of the pyramid can be approximated using the series

$$\sum_{k=1}^{n} \frac{h}{n}\left(\frac{xk}{n}\right)^2 = x^2 h \, \frac{\sum\limits_{k=1}^{n} k^2}{n^3}.$$

d Use **2** to explain why as the number of prisms we use in our approximation approaches infinity, the volume is given by $\frac{1}{3}x^2h = \frac{1}{3}$ × base area × height.

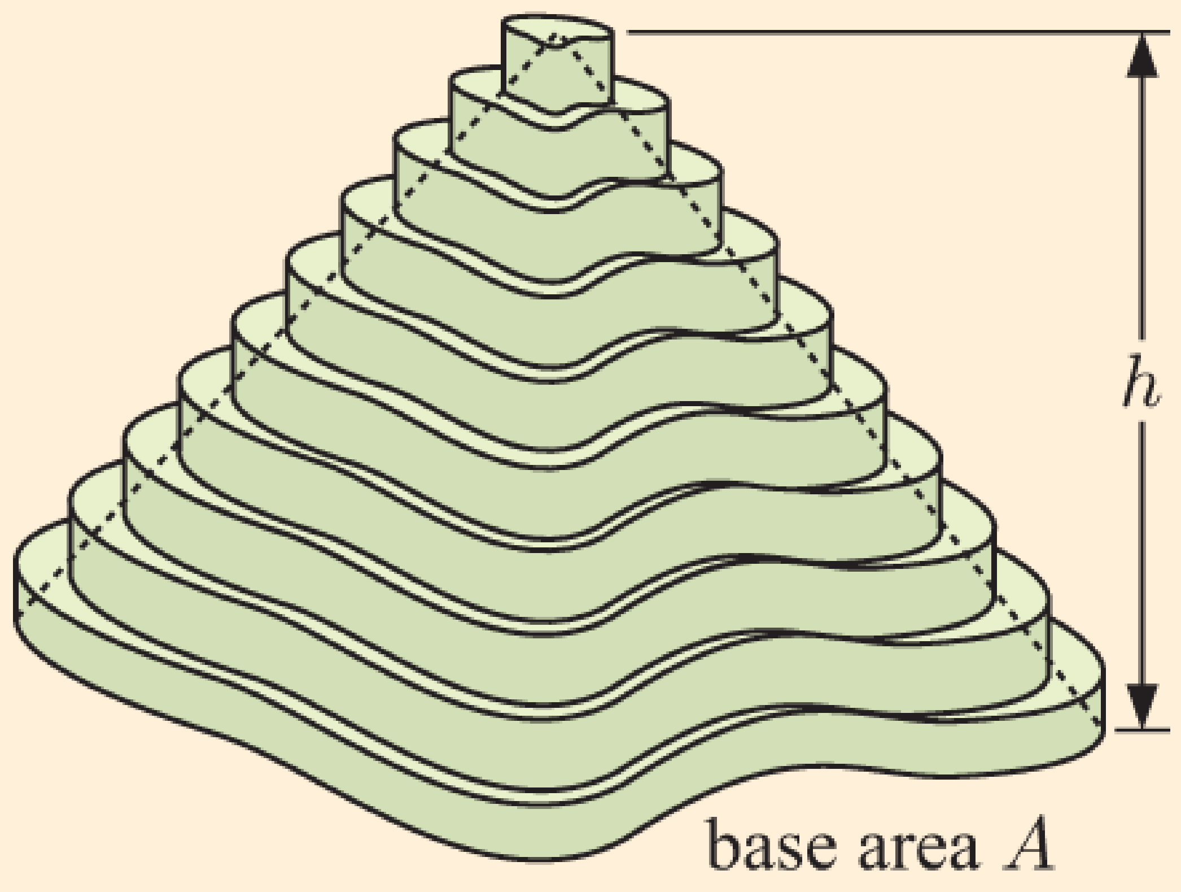
4 Approximate a cone with radius r and height h using a stack of n cylinders with equal thickness. Explain why:

a the kth cylinder has height $\dfrac{h}{n}$ and radius $R_k = \dfrac{rk}{n}$

b the volume of the cone can be approximated using the series $\displaystyle\sum_{k=1}^{n} \frac{h}{n}\pi\left(\frac{rk}{n}\right)^2 = \pi r^2 h \, \frac{\sum\limits_{k=1}^{n} k^2}{n^3}$

c the volume of the cone is given by $\frac{1}{3}\pi r^2 h = \frac{1}{3}$ × base area × height.

5 Suppose a tapered solid has height h and base area A. We can approximate the solid using n solids of uniform cross-section which have equal thickness, and whose bases are mathematically similar to the base of the tapered solid. Explain why:

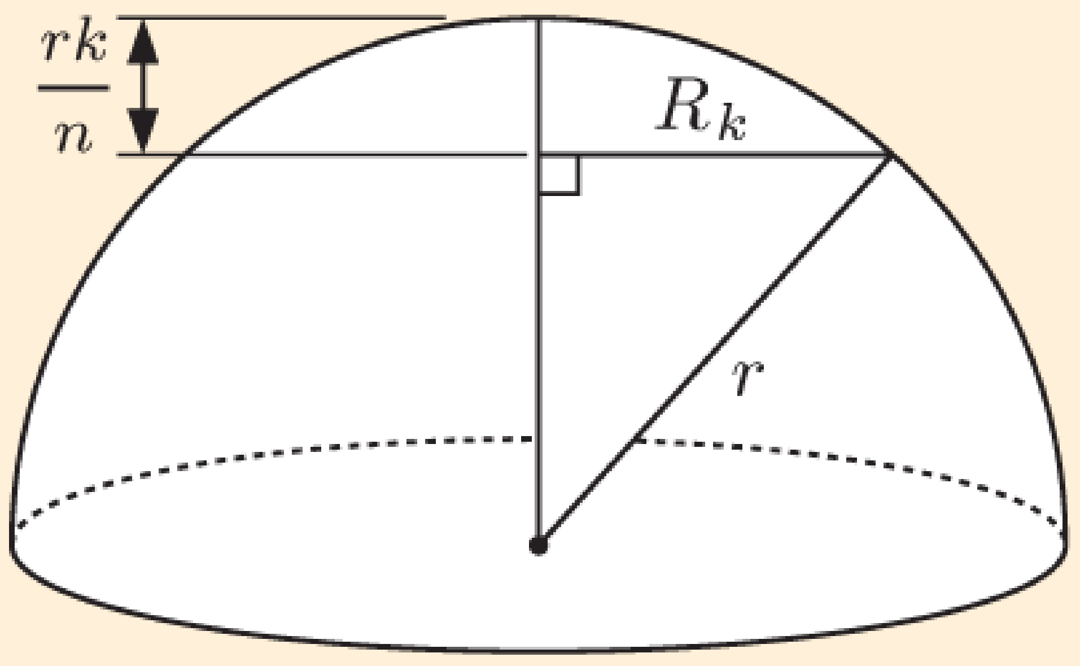


a the kth solid of uniform cross-section has height $\dfrac{h}{n}$ and area $A_k = A\left(\dfrac{k}{n}\right)^2$

b the volume of the tapered solid can be approximated using the series $\displaystyle\sum_{k=1}^{n} \frac{h}{n} A\left(\frac{k}{n}\right)^2$

c the volume of the tapered solid is given by $\frac{1}{3}$ × base area × height.

6 Approximate a hemisphere with radius r using a stack of n cylinders with equal thickness.



a Explain why the kth cylinder has height $\dfrac{r}{n}$.

b Let the radius of the kth cylinder be R_k.
Use the diagram alongside to explain why

$$\left(r - \frac{rk}{n}\right)^2 + R_k^2 = r^2.$$

Hence show that $R_k^2 = \dfrac{r^2}{n}\left(2k - \dfrac{k^2}{n}\right)$.

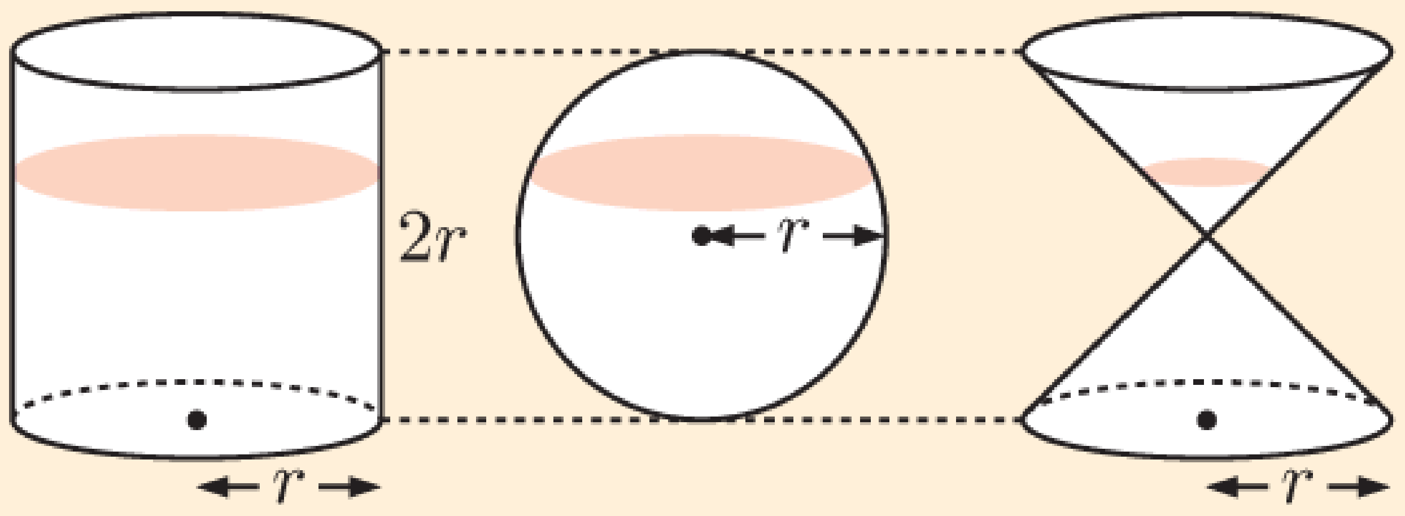
c Explain why the volume of the hemisphere can be approximated using the series

$$\sum_{k=1}^{n} \frac{r}{n}\pi\left(\frac{r^2}{n}\left(2k - \frac{k^2}{n}\right)\right) = \pi r^3\left(\frac{2\sum\limits_{k=1}^{n} k}{n^2} - \frac{\sum\limits_{k=1}^{n} k^2}{n^3}\right).$$

d Use **1** and **2** to explain why as the number of cylinders we use in our approximation approaches infinity, the volume is given by $\frac{2}{3}\pi r^3$. Hence find a formula for the volume of a sphere with radius r.

7 Archimedes used a different idea to find the formula for the volume of a sphere. The following is not exactly what he did, but is very much in the same spirit.

Consider the cylinder, sphere, and double cone alongside.

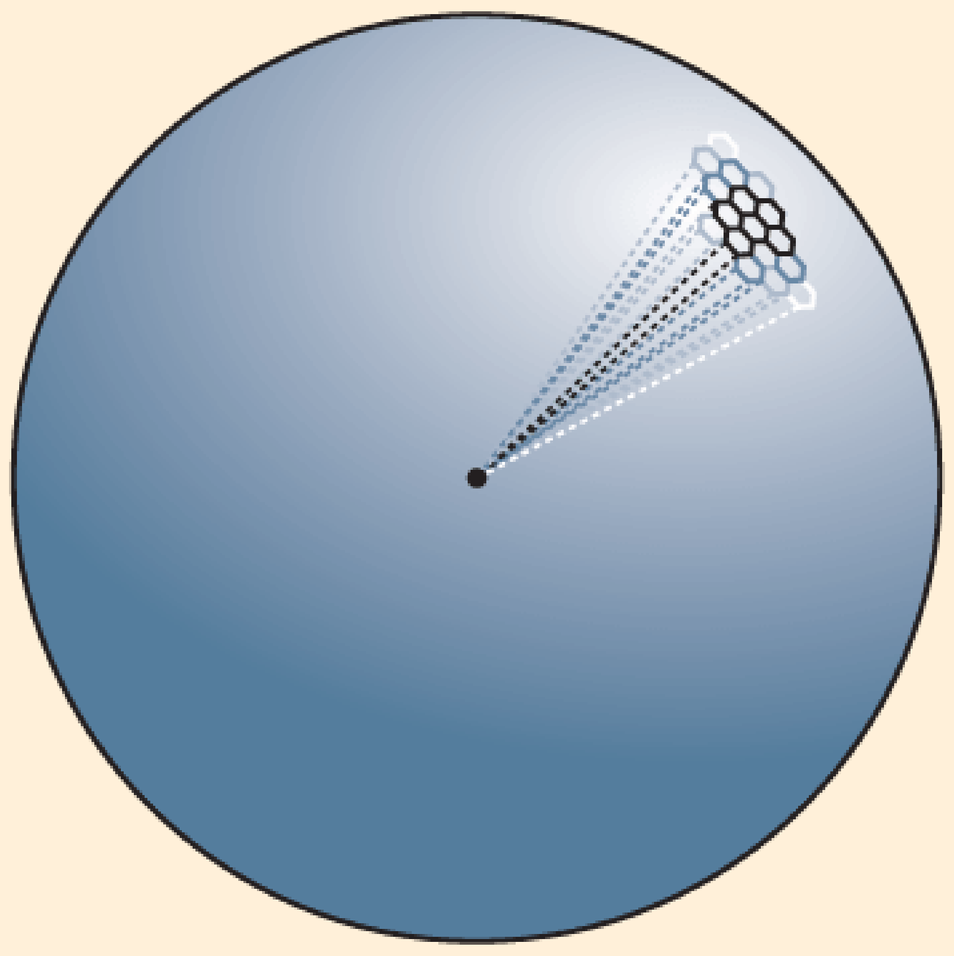


a Show that if *any* horizontal slice of the three solids is made, the sum of the areas from the sphere and the double cone equals the area from the cylinder.

b Explain why the sum of the volumes of the cylinder and double cone equals the volume of the cylinder.

c Use the known volume of a cylinder and cone formulae to find the volume of the sphere.

8 Now suppose we approximate a sphere with radius r using a large number n of tapered solids, each with height r and their apex at the centre of the sphere. Suppose the kth solid has base area A_k.



a Explain why $\sum\limits_{k=1}^{n} \frac{1}{3}A_k r = \frac{4}{3}\pi r^3$.

b Hence explain why the surface area of a sphere is given by $A = 4\pi r^2$.

From the **Investigation**, you should have established that:

- The volume of any tapered solid is given by $\boldsymbol{\textbf{Volume} = \frac{1}{3}(\textbf{area of base} \times \textbf{height})}$

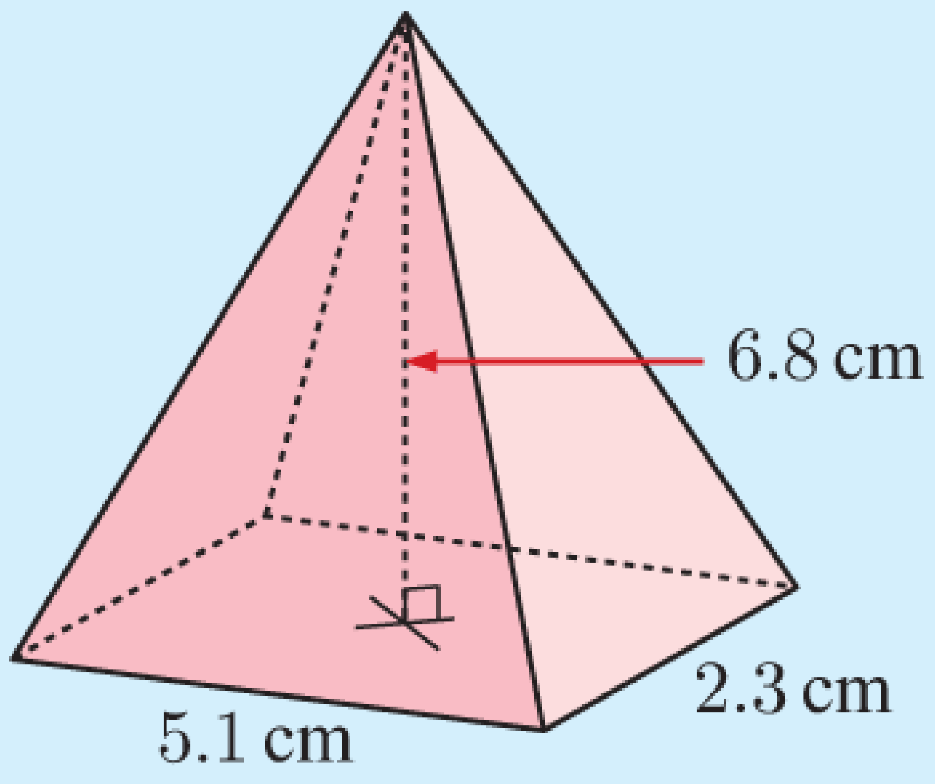
  It is a third of the volume of the solid of uniform cross-section with the same base and height.

- The volume of a sphere with radius r is given by $\boldsymbol{\textbf{Volume} = \frac{4}{3}\pi r^3}$

Example 7 **Self Tutor**

Find the volume of each solid:

a



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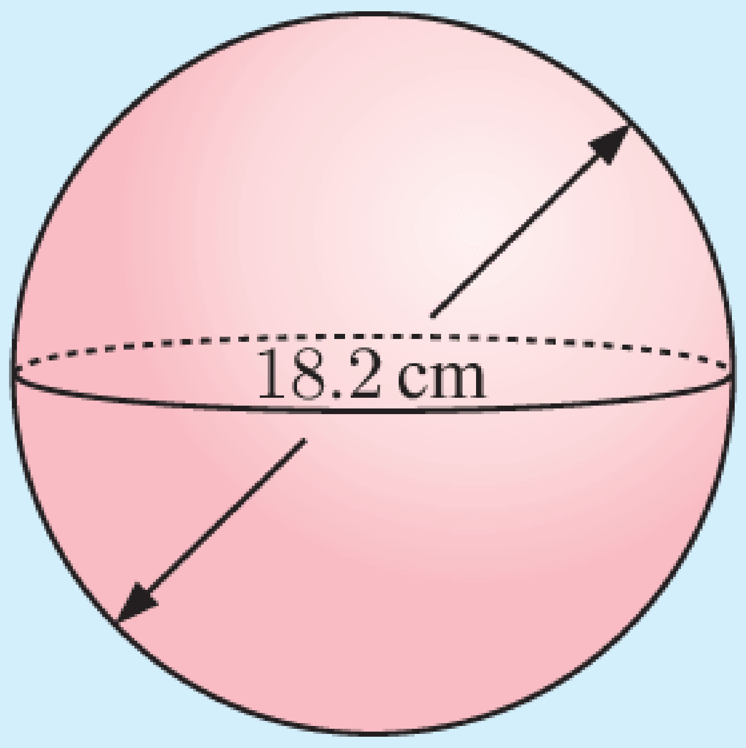


a $V = \frac{1}{3}(\text{area of base} \times \text{height})$

$= \frac{1}{3}(\text{length} \times \text{width} \times \text{height})$

$= \frac{1}{3}(5.1 \times 2.3 \times 6.8) \text{ cm}^3$

$\approx 26.6 \text{ cm}^3$

b $V = \frac{4}{3}\pi r^3$

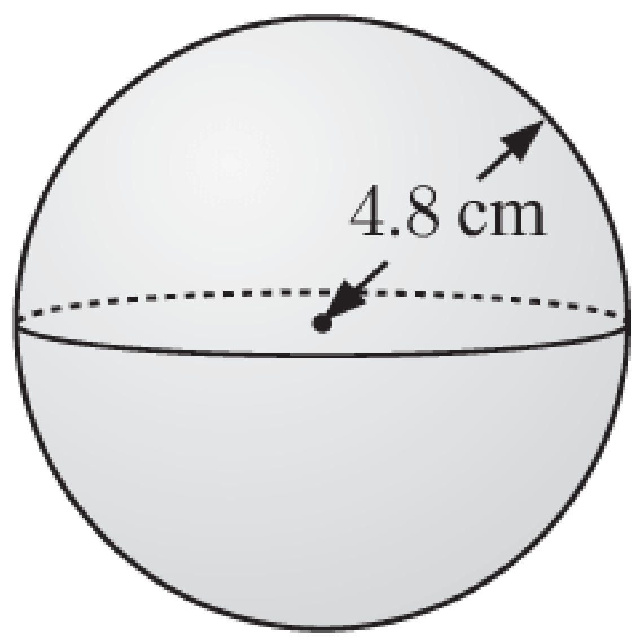
$= \frac{4}{3}\pi \left(\frac{18.2}{2}\right)^3 \text{ cm}^3$

$\approx 3160 \text{ cm}^3$

EXERCISE 6C.2

1 Find the volume of:

a



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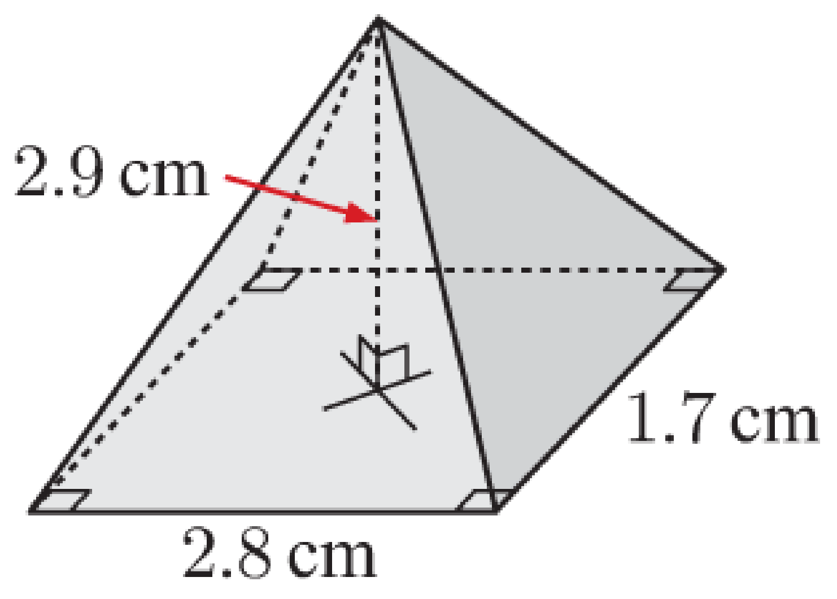

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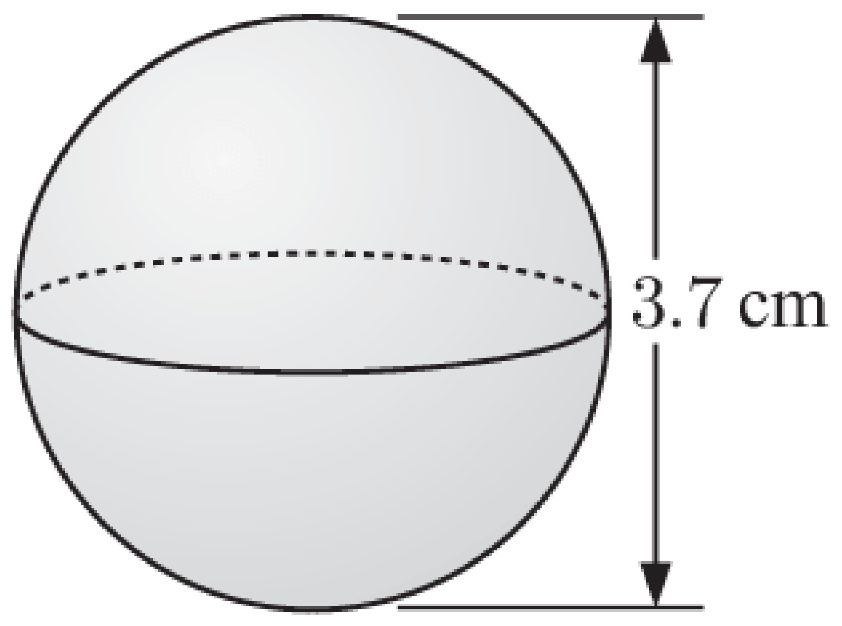

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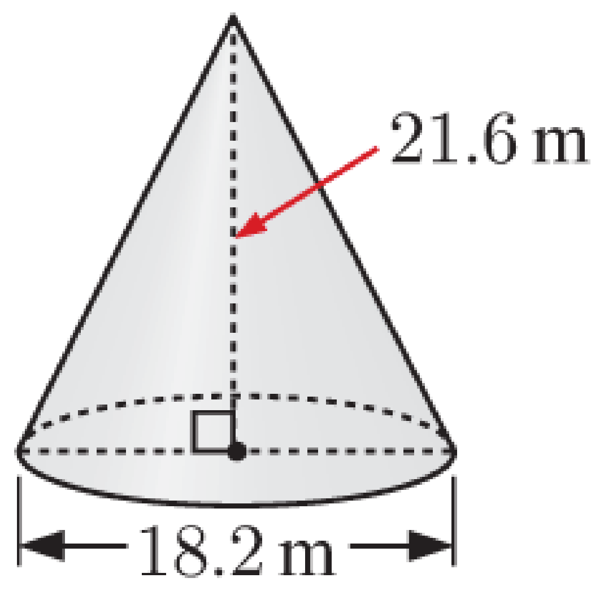

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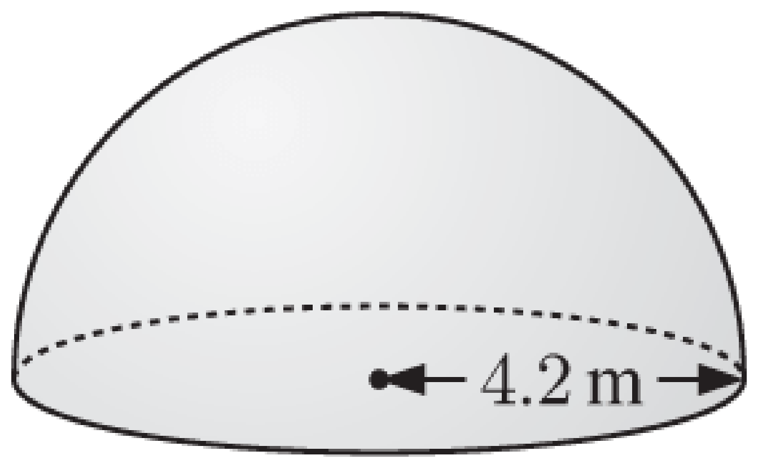

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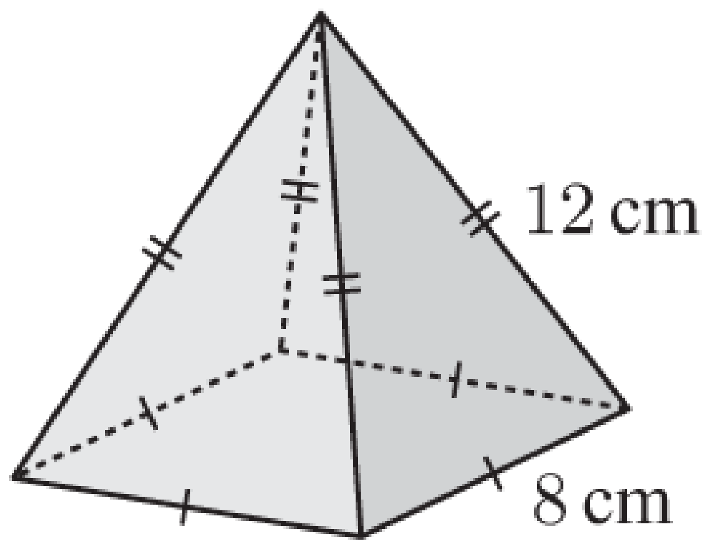
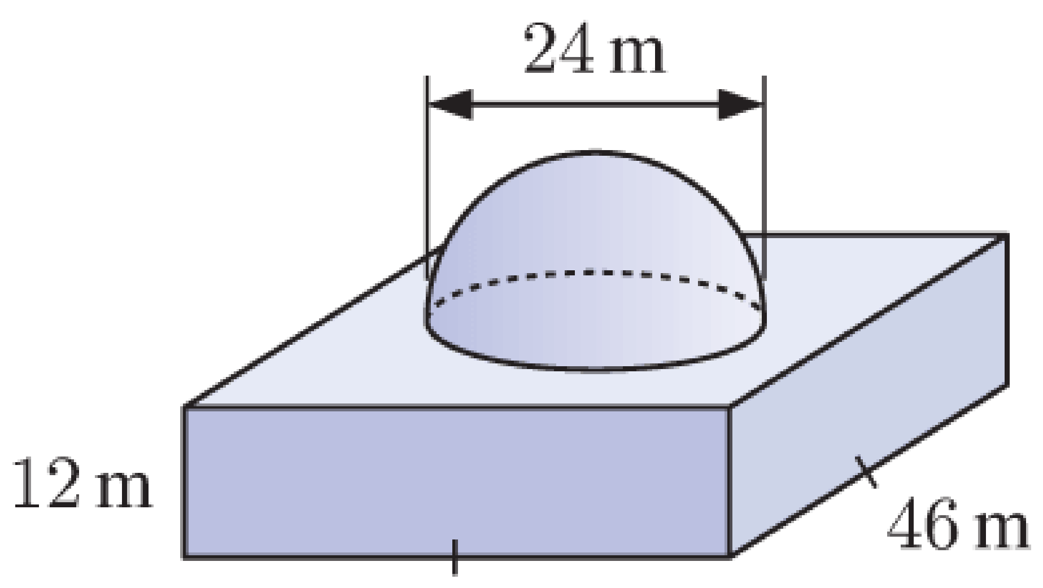


2 Find the volume of:

a



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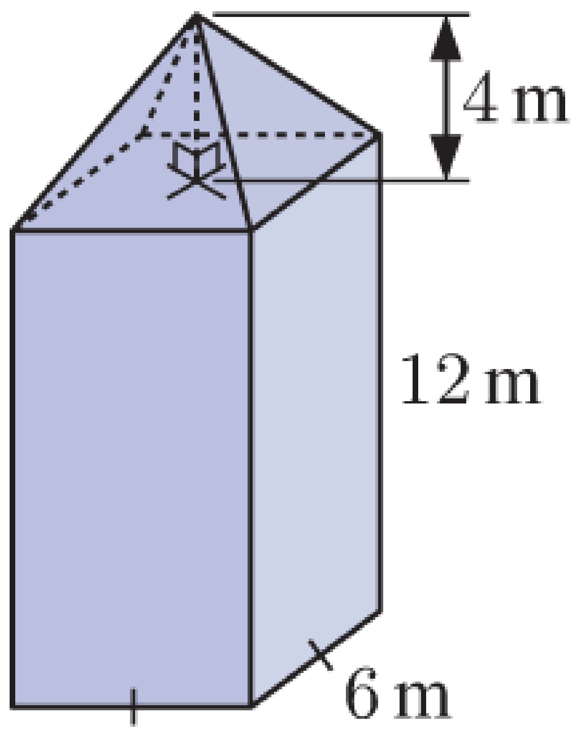

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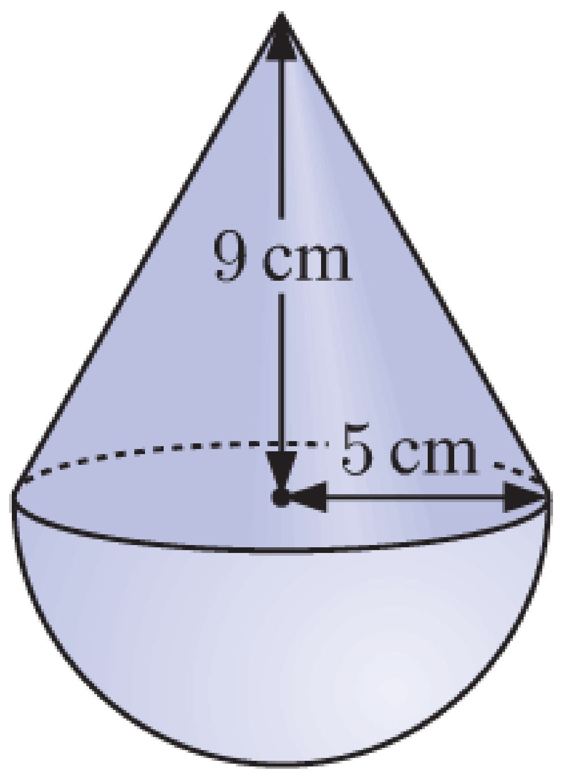
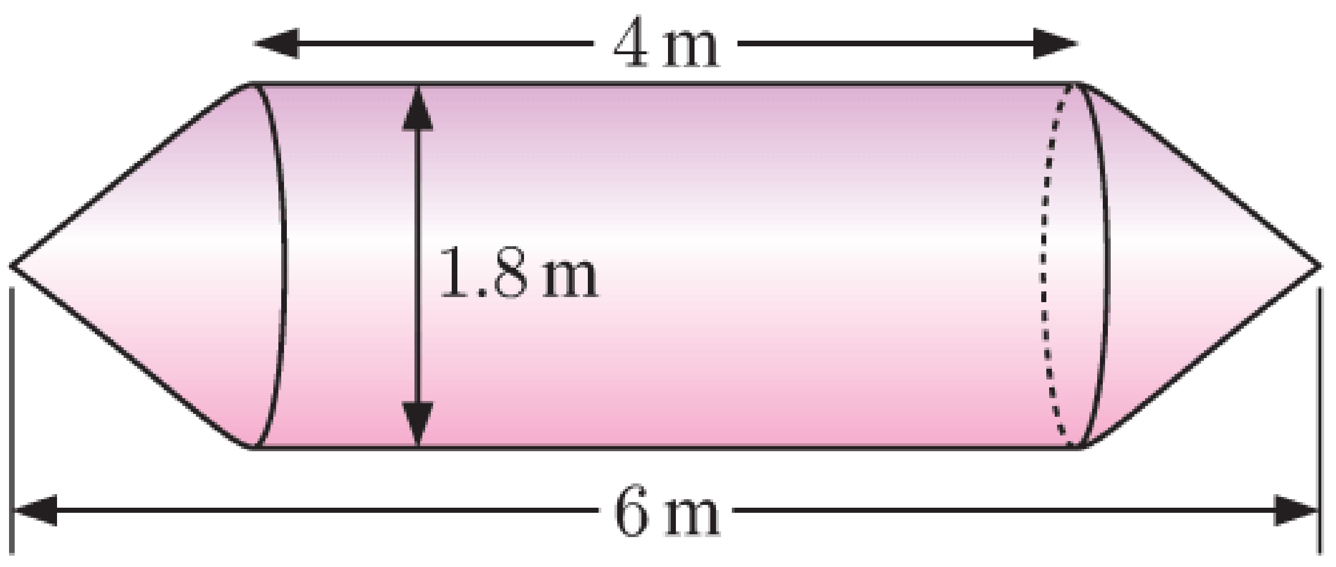


3 A ready mixed concrete tanker is to be constructed from steel as a cylinder with conical ends.



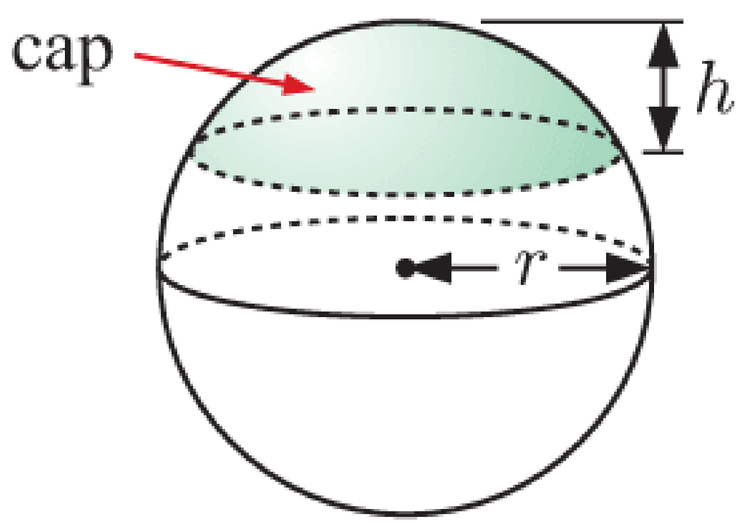
- **a** Calculate the total volume of concrete that can be held in the tanker.
- **b** How *long* would the tanker be if the ends were hemispheres instead of cones, but the cylindrical section remained the same?
- **c** How much more or less concrete would fit in the tanker if the ends were hemispheres instead of cones?
- **d** Show that the surface area of the tanker:
  - **i** with conical ends is about 30 m^2
  - **ii** with hemispherical ends is about 33 m^2.
- **e** Overall, which do you think is the better design for the tanker? Give reasons for your answer.

4 Find:

- **a** the height of a glass cone with base radius 12.3 cm and volume 706 cm^3
- **b** the radius of a spherical weather balloon with volume 73.62 m^3
- **c** the base radius of a cone with height 6.2 cm and volume 203.9 cm^3.

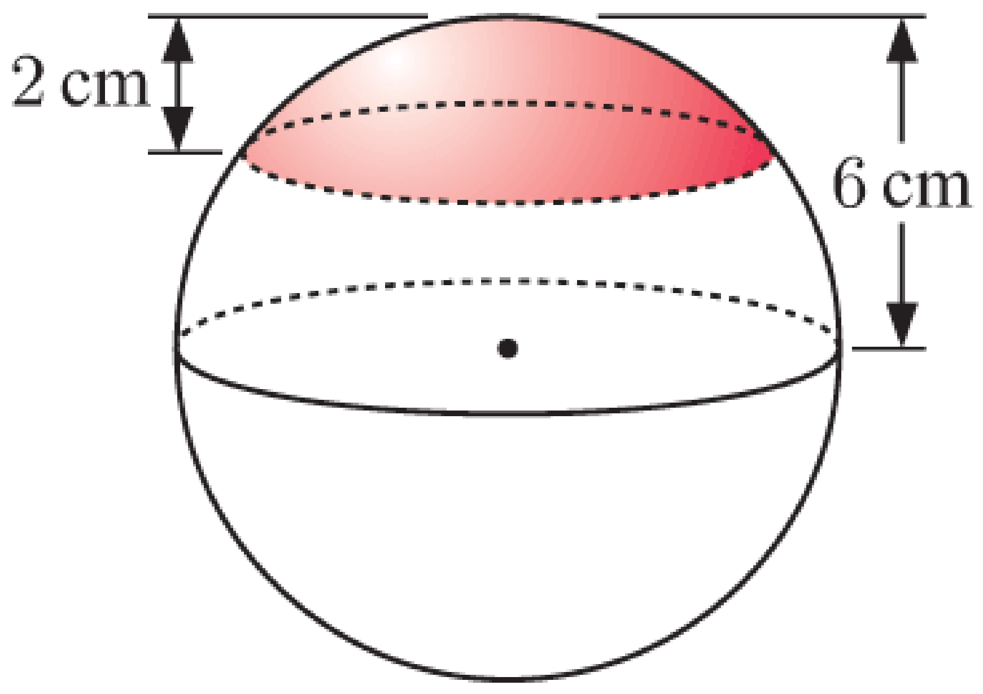
5 A cylinder of resin has height equal to its diameter. Some of it is used to form a cone with the same height and diameter as the cylinder. Show that the remainder is the exact amount needed to form a sphere with the same diameter.

6 For a sphere of radius r, the volume of the **cap** of height h is $V = \frac{\pi h^2}{3}(3r - h)$.

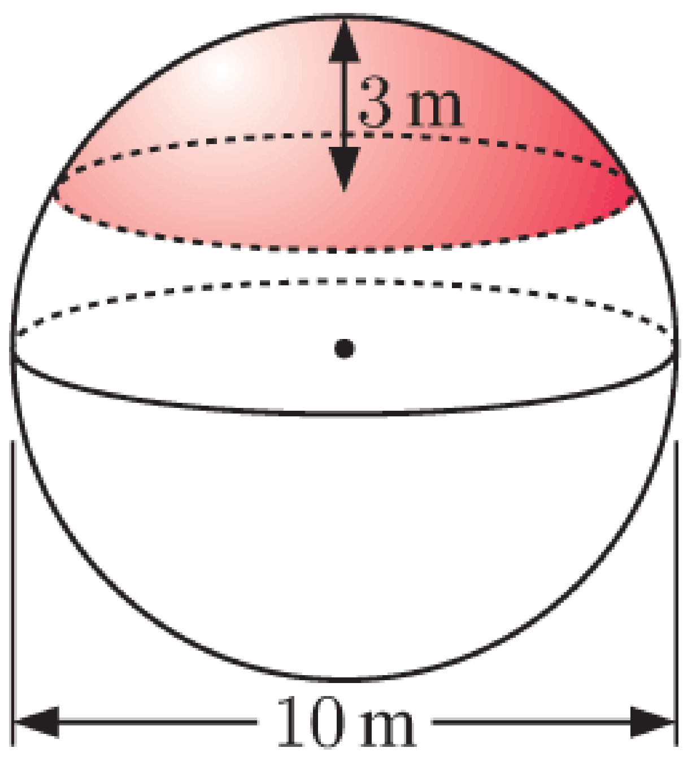


a Find the volumes of these caps:

i



ii



b Write an expression for the volume of the cap in the case that $h = r$. Compare this volume with the volume of the sphere. Explain your result.

ACTIVITY 1 DENSITY

The **density** of a substance is its mass per unit volume.

$$\textbf{density} = \frac{\textbf{mass}}{\textbf{volume}}$$

One **gram** is the mass of one cubic centimetre of pure water at 4°C. The density of pure water at 4°C is therefore $\frac{1 \text{ g}}{1 \text{ cm}^3} = 1 \text{ g cm}^{-3}$.

Some densities of common substances are shown in the table:

Substance	*Density* (g cm^{-3})
pine wood	0.41
paper	0.80
oil	0.92
water	1.00
steel	8.05
copper	8.96
lead	11.34

What to do:

1 Find the density of:

- **a** a metal rod with mass 10 g and volume 2 cm^3
- **b** a cube of chocolate with side length 2 cm and mass 10.6 g
- **c** a glass marble with radius 4.5 mm and mass 1.03 g.

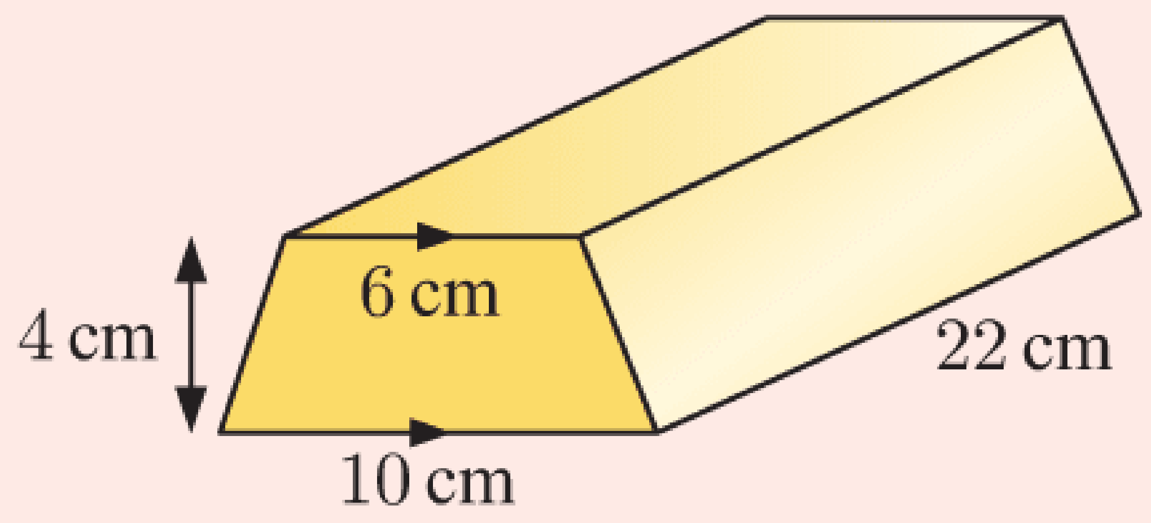
2 Rearrange the density formula to make:

- **a** mass the subject
- **b** volume the subject.

3 Find the volume of 80 g of salt with density 2.16 g cm^{-3}.

4 Find the mass of a copper wire with radius 1 mm and length 250 m.

5 The gold bar shown has mass 13.60 kg. Find the density of gold.



6 Jonathon has a steel ball bearing with radius 1.4 cm, and a lead sphere with radius 1.2 cm. Which sphere weighs more, and by what percentage?

7 Oil and water are *immiscible*, which means they do not mix. Does oil float on water, or water float on oil? Explain your answer.

8 **a** In general, as a substance is heated, it expands. What happens to the density of the substance?

b Water is unusual in that its solid state is less dense than its liquid state. How do we observe this in the world around us?

9 Determine the total mass of stone required to build a square-based pyramid with all edges of length 200 m. The density of the stone is 2.25 tonnes per m^3.

10 The planet Uranus is approximately spherical with radius 2.536×10^7 m and mass 8.681×10^{25} kg.

a Estimate the volume of Uranus. **b** Hence find its density.

PROJECT HOW BIG IS THE MOUNTAIN?

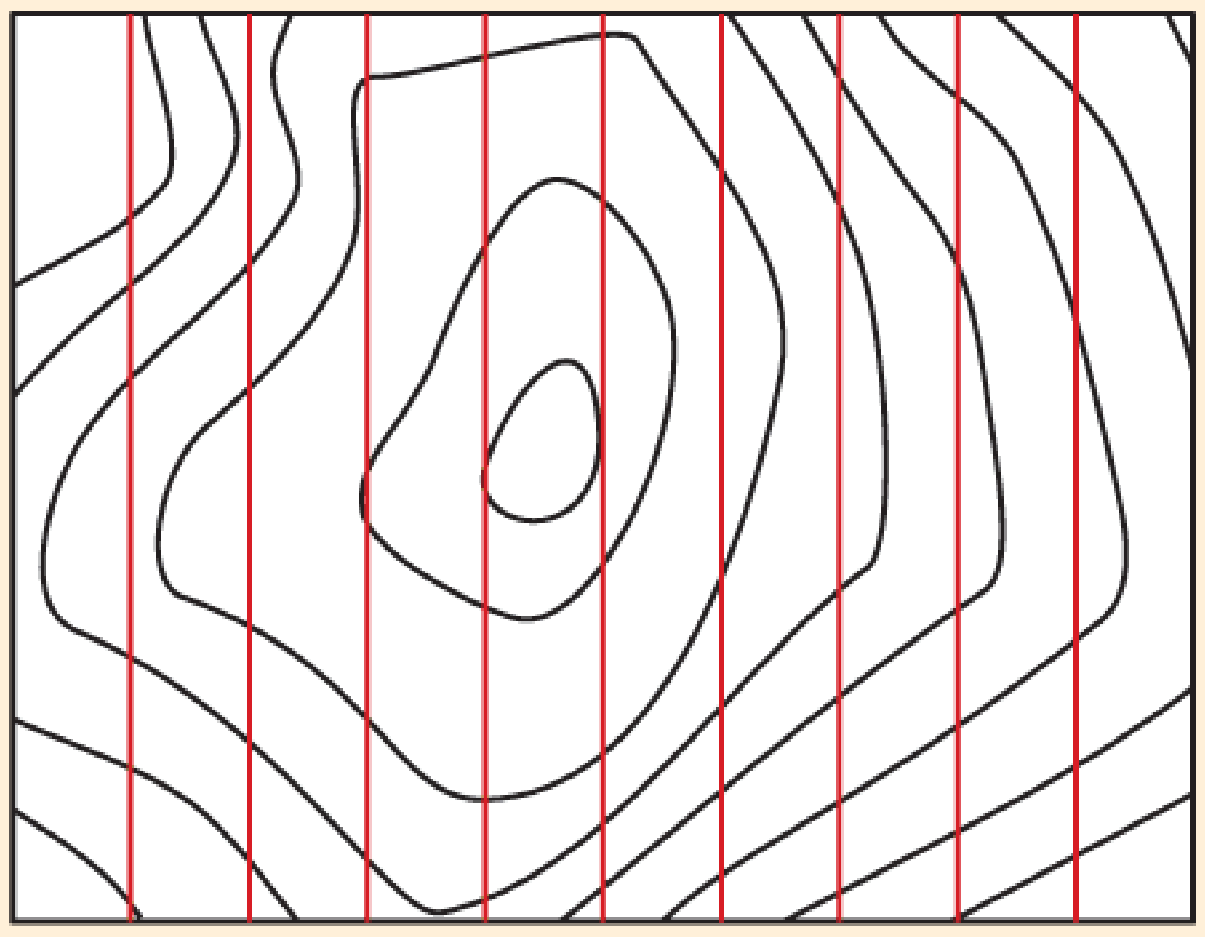
Choose an iconic mountain of the world. Your task is to estimate its volume.

To achieve this task, you will need:

- a topographic map of the mountain
- knowledge of Simpson's rule.

SIMPSON'S RULE

What to do:



1 **a** Use Simpson's rule to estimate the cross-sectional area of the mountain at each contour level.

b Hence estimate the volume of the mountain added at each change in altitude level.

c Use a solid of uniform cross-section to estimate the volume of the mountain from your lowest chosen contour down to sea level.

d Sum your results to estimate the total volume of the mountain.

e Discuss the assumptions you made in your calculations.

2 **a** Make regular slices across your contour map and use Simpson's rule to estimate the area of each slice.
   b Hence estimate the volume of the mountain for each interval between the slices.
   c Sum your results to give the total volume of the mountain.
   d Discuss the assumptions you made in your calculations.

3 **a** Overlay a fine grid on top of the topographical map. Use the contours to estimate the altitude at each vertex point of the grid. Hence estimate the average altitude of each grid square.
   b Hence estimate the volume of the mountain under each grid square.
   c Sum your results to give the total volume of the mountain.
   d Discuss the assumptions you have made in your calculations.

4 Compare the estimates you have obtained for the volume of the mountain.
   a What assumptions do you need to make in order to compare them fairly?
   b Which method do you think is the:
      i most elegant
      ii most accurate
      iii easiest to automate using software?

5 Can you suggest a more accurate method for estimating the volume of the mountain? Explain why you believe it is more accurate, and perform calculations.

6 Research the composition of your chosen mountain and use the information to estimate its mass.

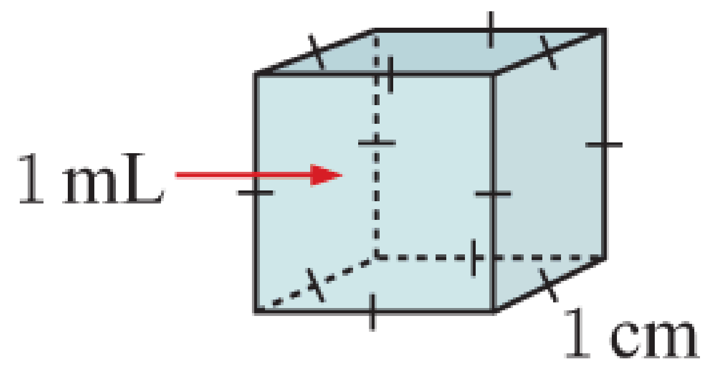
7 If you measured the volume of a mountain down to the base plane around it rather than to sea level, what is the "biggest" mountain on Earth?

D CAPACITY

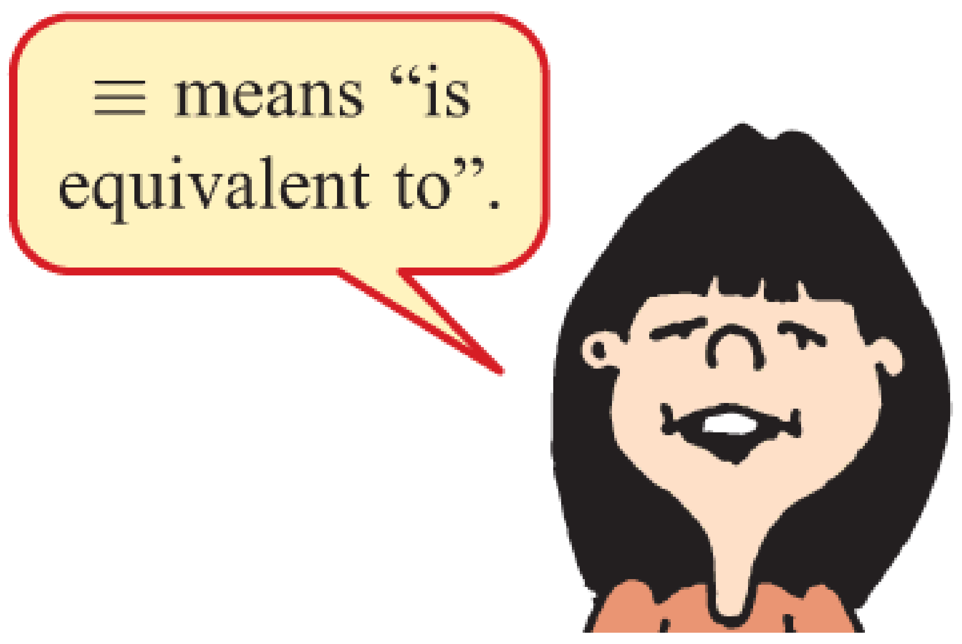
The **capacity** of a container is the quantity of fluid it is capable of holding.

Notice that the term "capacity" belongs to the container rather than the fluid itself. The capacity of the container tells us what *volume* of fluid fits inside it. The units of volume and capacity are therefore linked:

1 mL of water occupies 1 cm^3 of space.



Volume	*Capacity*
$1 \text{ cm}^3 \equiv$	1 mL
$1000 \text{ cm}^3 \equiv$	1 L
$1 \text{ m}^3 \equiv$	1 kL
$1 \text{ m}^3 \equiv$	1000 L

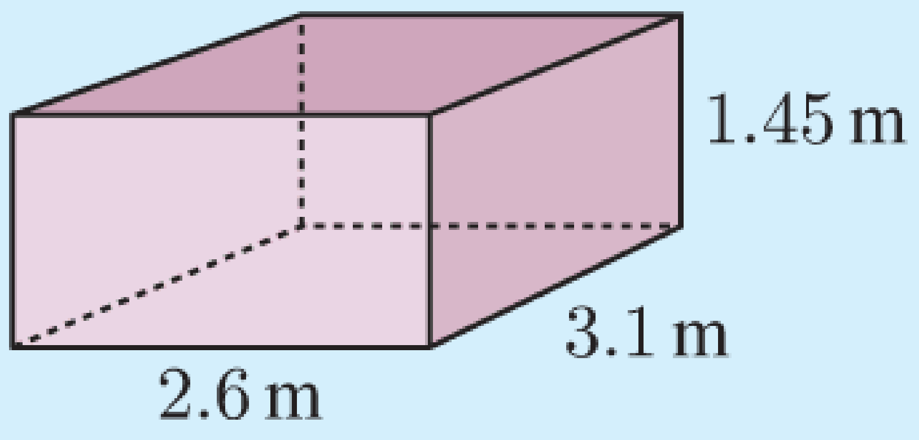


DISCUSSION

- In common language, are the terms *volume* and *capacity* used correctly?
- Which of the following statements are technically correct? Which of the statements are commonly accepted in language, even though they are not technically correct?
  - The jug has capacity 600 mL.
  - The jug can hold 600 mL of water.
  - The volume of the jug is 600 cm^3.
  - The jug can hold 600 cm^3 of water.
  - I am going to the supermarket to buy a 2 L bottle of milk.
  - I am going to the supermarket to buy 2 L of milk.

Example 8 **Self Tutor**

Find the capacity of a 2.6 m by 3.1 m by 1.45 m tank.



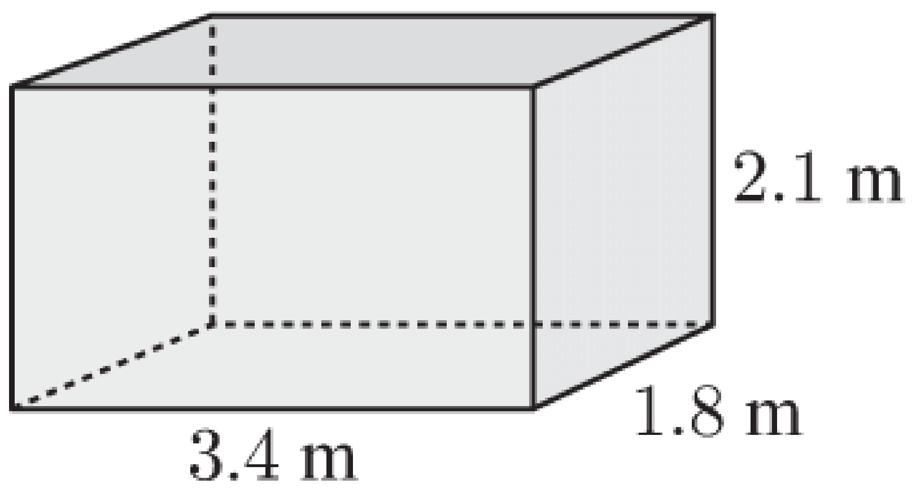
$$V = \text{length} \times \text{width} \times \text{height}$$
$$= 2.6 \times 3.1 \times 1.45 \text{ m}^3$$
$$= 11.687 \text{ m}^3$$

The tank's capacity is 11.687 kL.

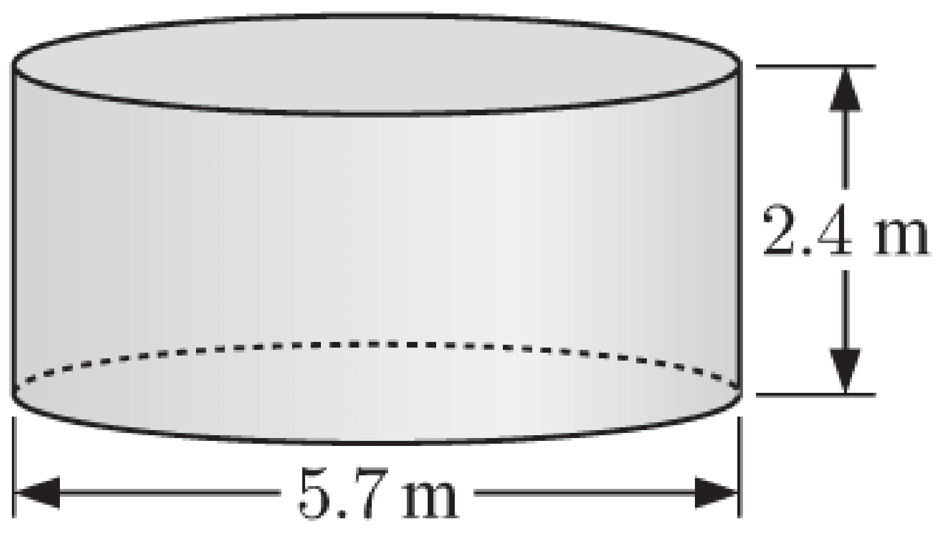
EXERCISE 6D

1 Find the capacity (in kL) of each tank:

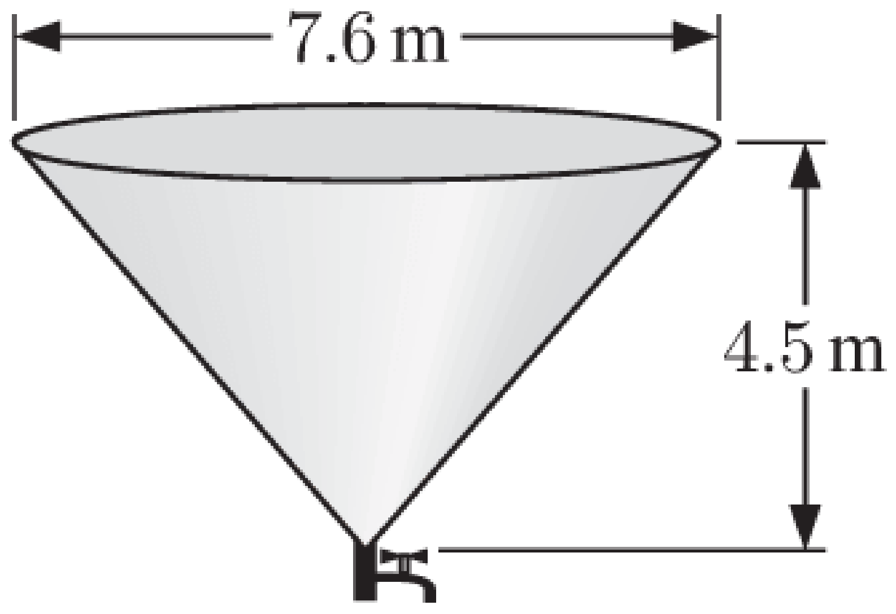
a



b

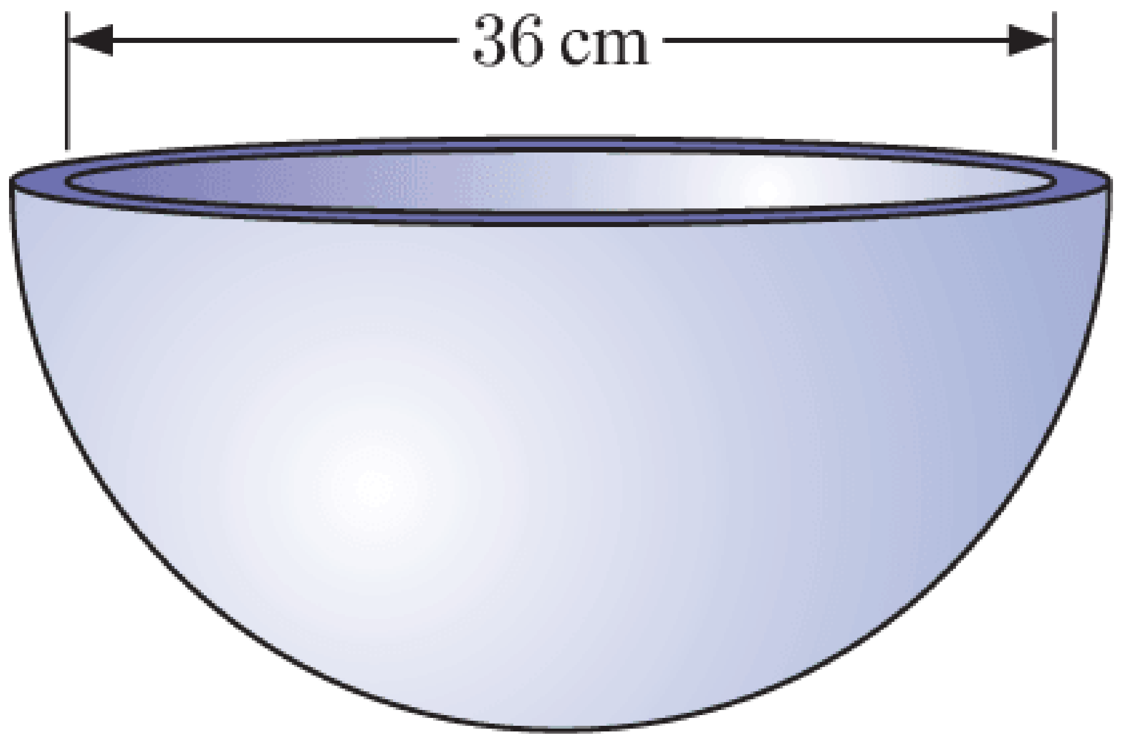


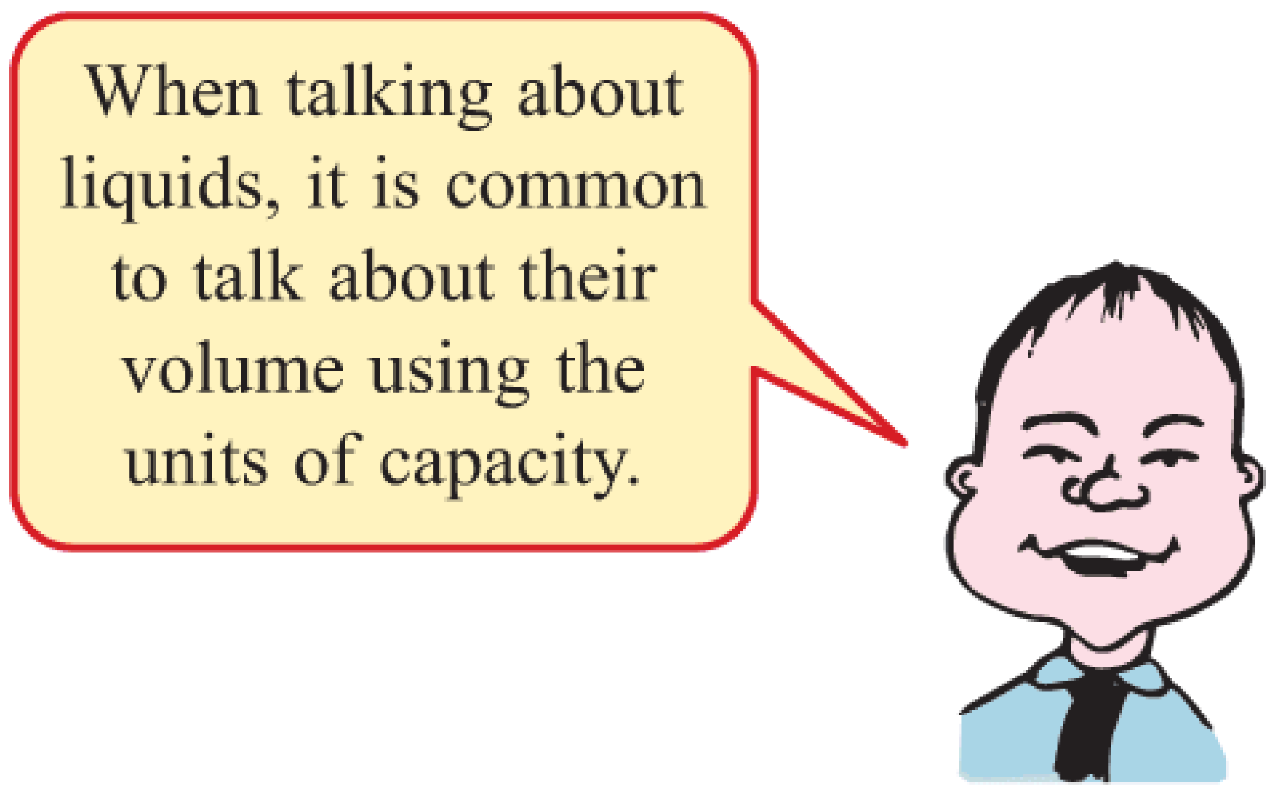
c



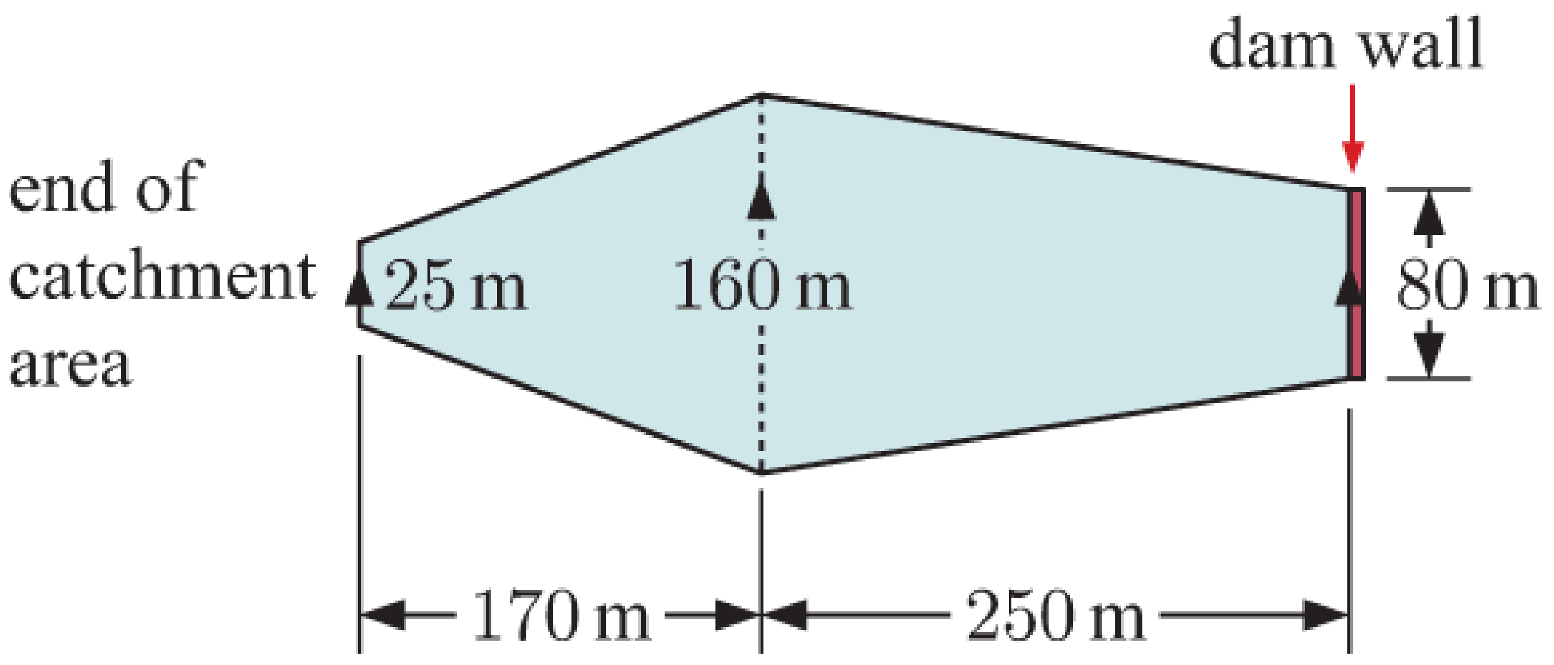
2 Find the volume of soup that will fit in this hemispherical pot. Give your answer in:

a cm^3 **b** L.



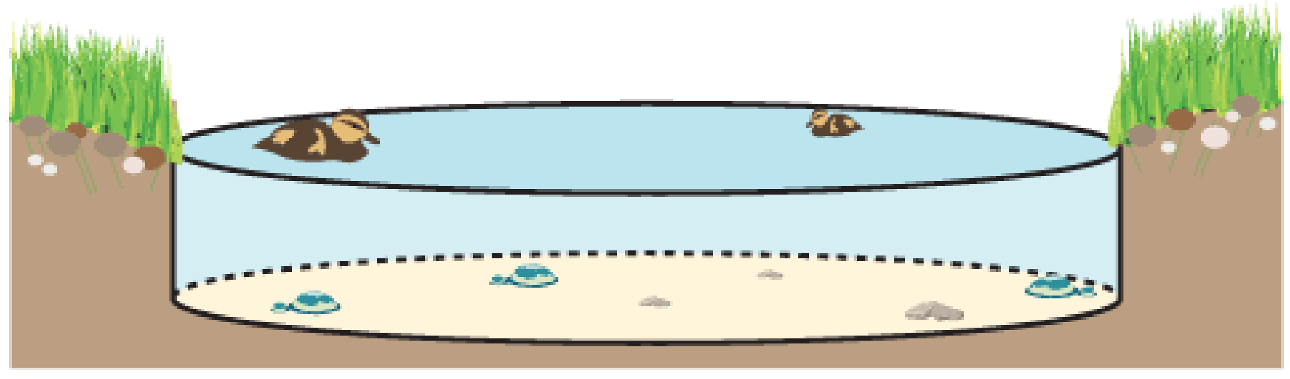


3 A dam wall is built at the narrow point of a river to create a small reservoir. When full, the reservoir has an average depth of 13 m, and has the shape shown in the diagram. Find the capacity of the reservoir.



4 Jam is packed into cylindrical tins which have radius 4.5 cm and height 15 cm. The mixing vat is also cylindrical with cross-sectional area 1.2 m^2 and height 4.1 m.

- **a** Find the capacity of each tin.
- **b** Find the capacity of the mixing vat.
- **c** How many tins of jam could be filled from one vat?
- **d** If the jam is sold at \$3.50 per tin, what is the value of one vat of jam?

5 

The circular pond in the park near my house has radius 2.4 m. It has just been filled with 10 kL of water. How deep is the pond?

Example 9 **Self Tutor**

17.3 mm of rain falls on a flat rectangular shed roof which has length 10 m and width 6.5 m. All of the water goes into a cylindrical tank with base diameter 4 m. By how many millimetres does the water level in the tank rise?

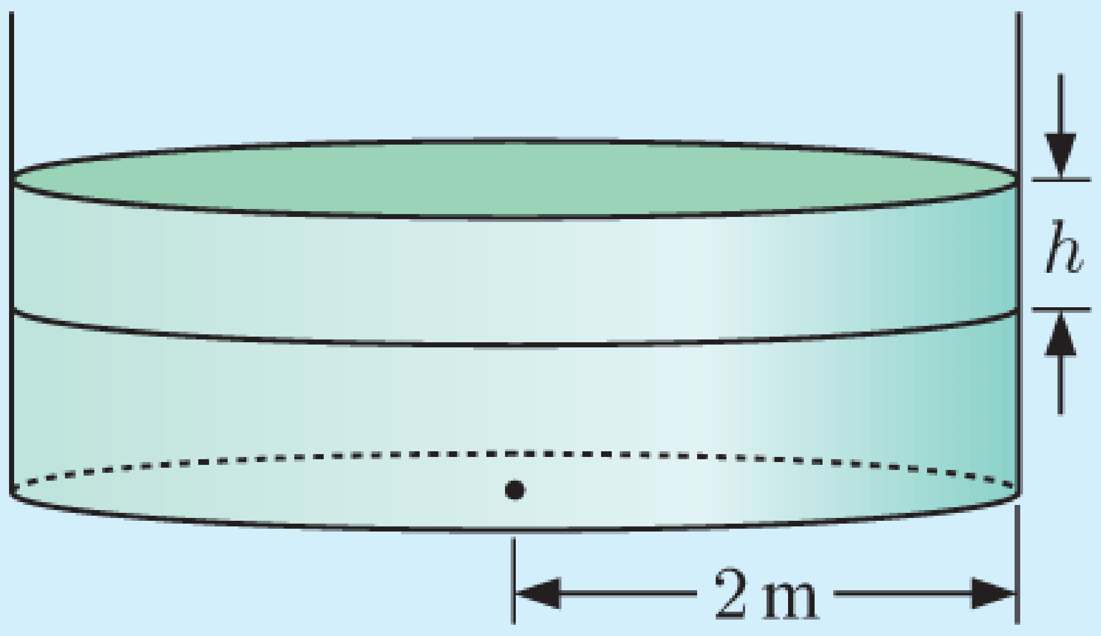
For the roof: The dimensions of the roof are in m, so we convert 17.3 mm to metres.

17.3 mm = (17.3 ÷ 1000) m = 0.0173 m

$$\begin{aligned}\text{The volume of water collected by the roof} &= \text{area of roof} \times \text{depth}\\ &= 10 \times 6.5 \times 0.0173 \text{ m}^3\\ &= 1.1245 \text{ m}^3\end{aligned}$$

For the tank: The volume added to the tank

$$\begin{aligned}&= \text{area of base} \times \text{height}\\ &= \pi \times 2^2 \times h \text{ m}^3\\ &= 4\pi \times h \text{ m}^3\end{aligned}$$



The volume added to the tank must equal the volume which falls on the roof, so

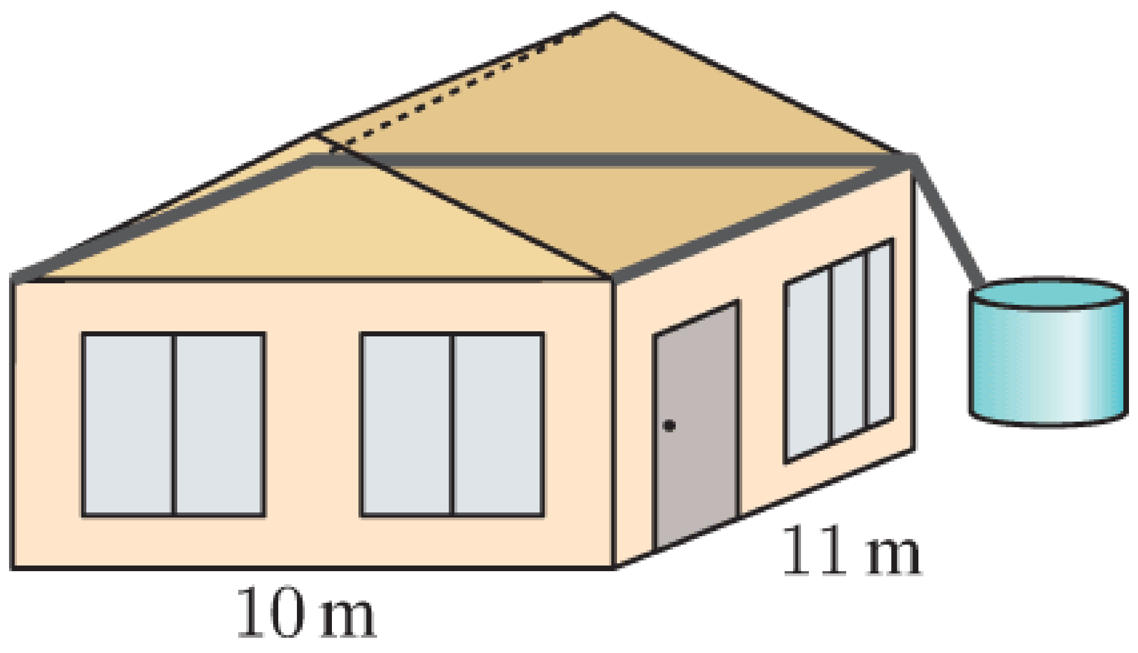
$4\pi \times h = 1.1245$

$\therefore \; h = \dfrac{1.1245}{4\pi}$ {dividing both sides by 4π}

$\therefore \; h \approx 0.0895$ m

$\therefore$ the water level rises by about 89.5 mm.

6 The base of a house has area 110 m^2. One night 12 mm of rain falls on the roof. All of the water goes into a tank which has base diameter 4 m.

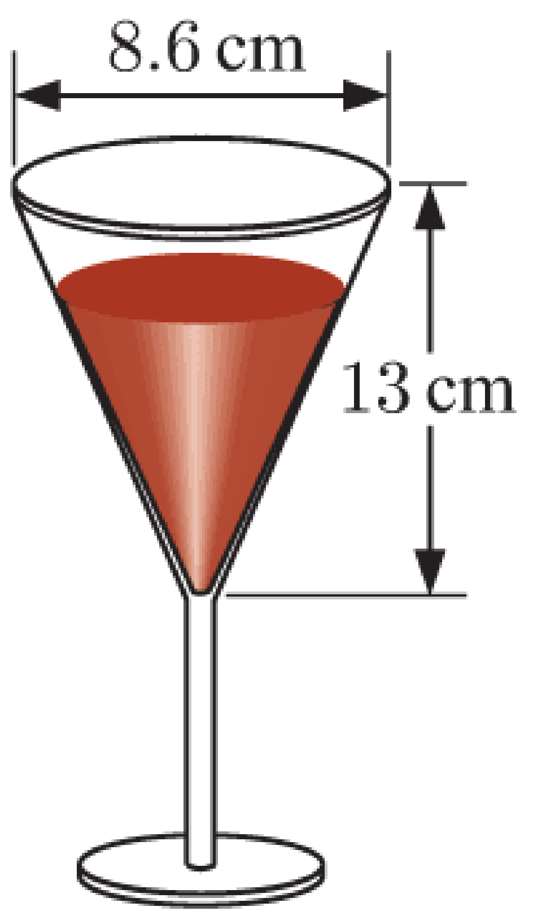


- **a** Find the volume of water which fell on the roof.
- **b** How many kL of water entered the tank?
- **c** By how much did the water level in the tank rise?

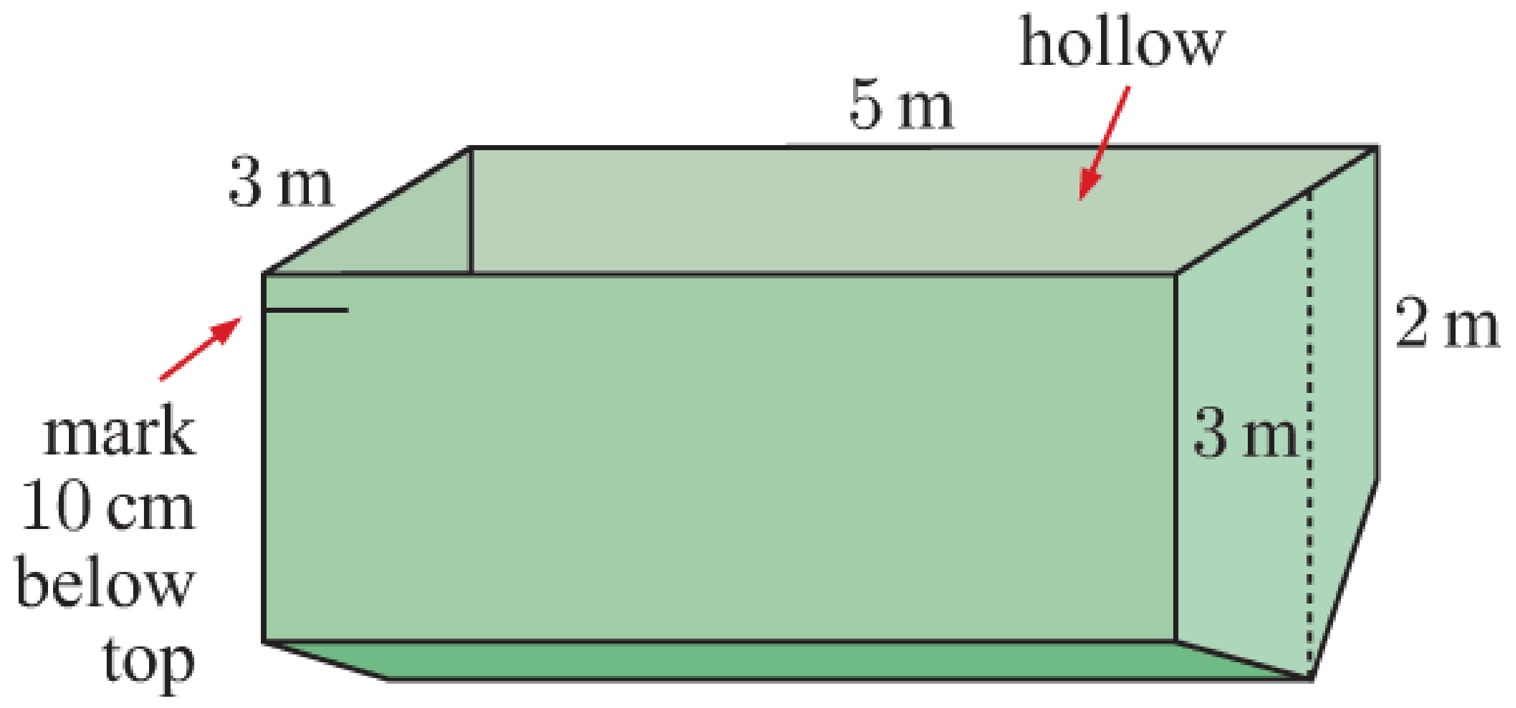
7 The design department of a fish canning company wants to change the size of their cylindrical tins. The original tin is 15 cm high and 7.2 cm in diameter. The new tin is to have approximately the same volume, but its diameter will be 10 cm. How high must it be, to the nearest mm?

8 A conical wine glass has the dimensions shown.

- **a** Find the capacity of the glass.
- **b** Suppose the glass is 75% full.
  - **i** How many mL of wine does it contain?
  - **ii** If the wine is poured into a cylindrical glass of the same diameter, how high will it rise?



9 A fleet of trucks have containers with the shape illustrated. Wheat is transported in these containers, and its level must not exceed a mark 10 cm below the top. How many truck loads of wheat are necessary to fill a cylindrical silo with internal diameter 8 m and height 25 m?



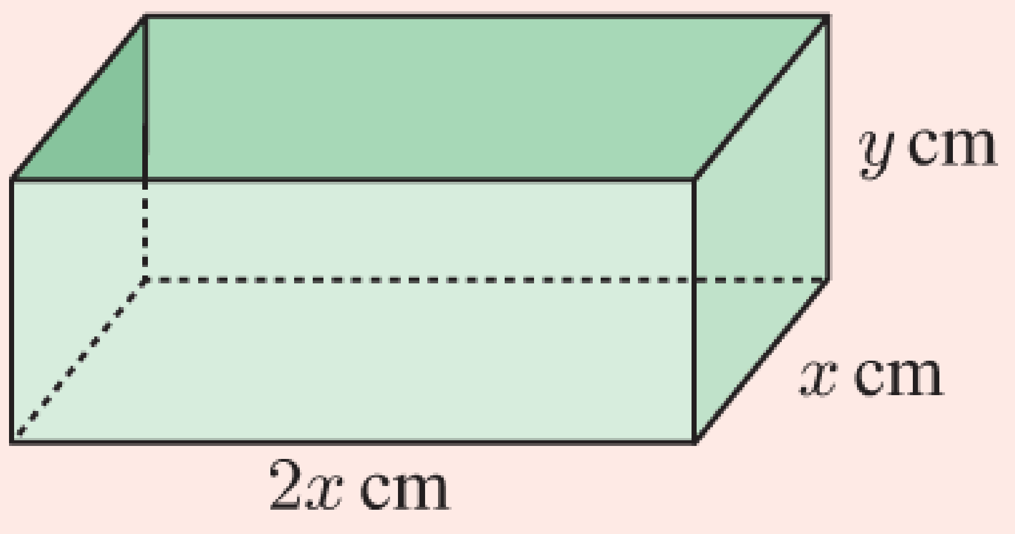
10 Answer the **Opening Problem** on page **132**.

ACTIVITY 2 MINIMISING MATERIAL

Your boss asks you to design a rectangular box-shaped container which is open at the top and contains exactly 1 litre of fluid. The base measurements must be in the ratio 2 : 1. She intends to manufacture millions of these containers, and wishes to keep manufacturing costs to a minimum. She therefore insists that the least amount of material is used.

What to do:

1 The base is to be in the ratio 2 : 1, so we let the dimensions be x cm and $2x$ cm. The height is also unknown, so we let it be y cm. As the values of x and y vary, the container changes size.



Explain why:

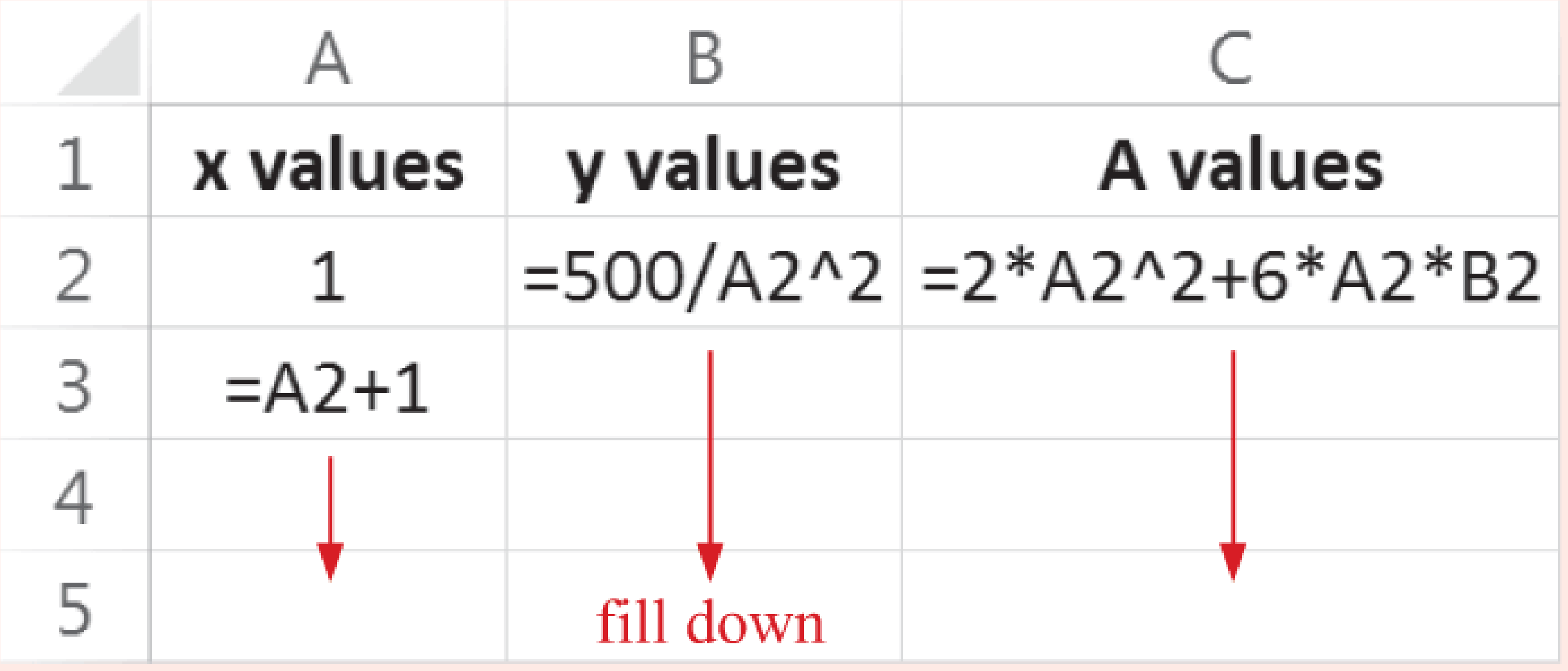
- **a** the volume $V = 2x^2y$
- **b** $2x^2y = 1000$
- **c** $y = \dfrac{500}{x^2}$

2 Show that the surface area is given by $A = 2x^2 + 6xy$.

3 Construct a spreadsheet which calculates the surface area for $x = 1, 2, 3, 4, ...$

SPREADSHEET

INSTRUCTIONAL VIDEO



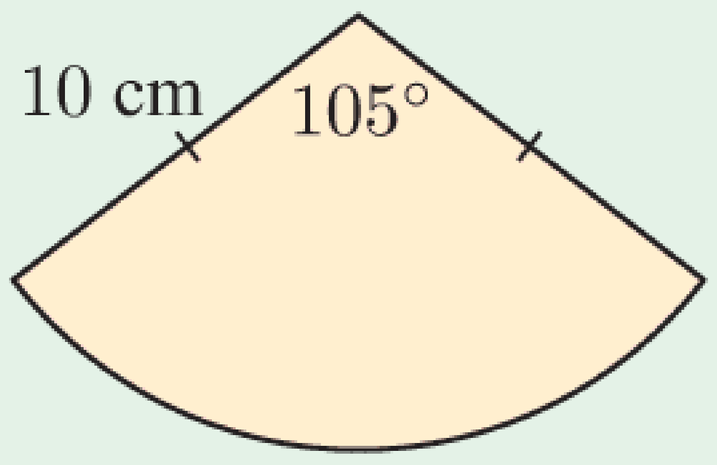
	A	B	C
1	x values	y values	A values
2	1	=500/A2^2	=2*A2^2+6*A2*B2
3	=A2+1		
4			
5		fill down	

4 Find the smallest value of A, and the value of x which produces it. Hence write down the dimensions of the box your boss desires.

REVIEW SET 6A

1 For the given sector, find to 3 significant figures:

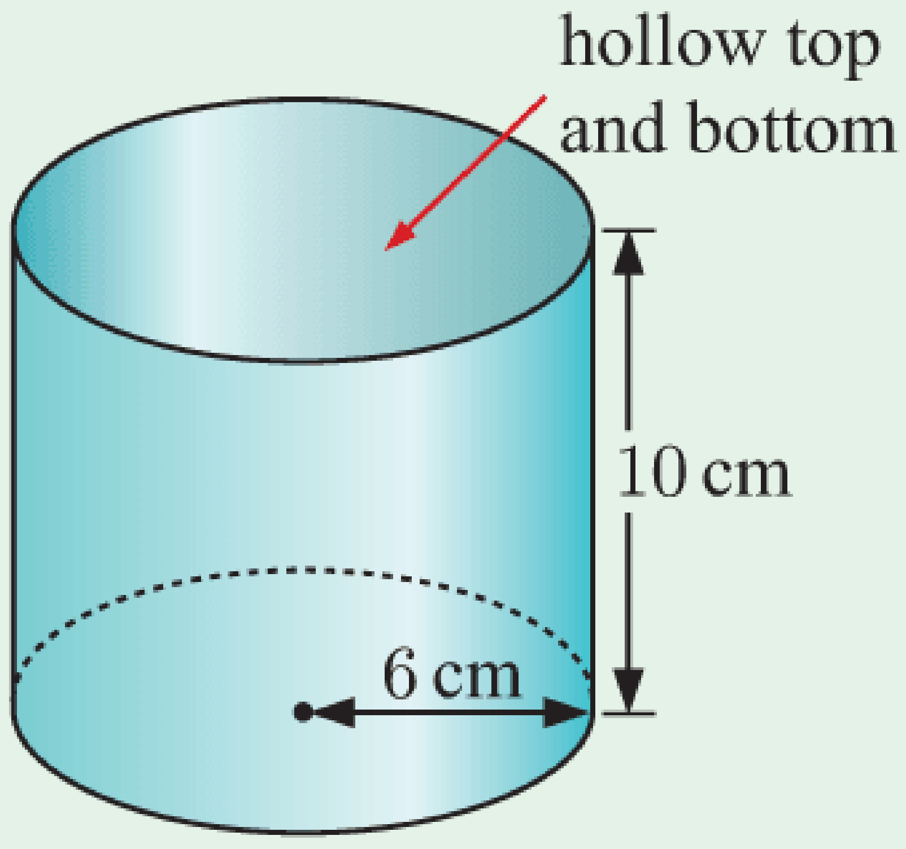
- **a** the length of the arc
- **b** the perimeter of the sector
- **c** the area of the sector.



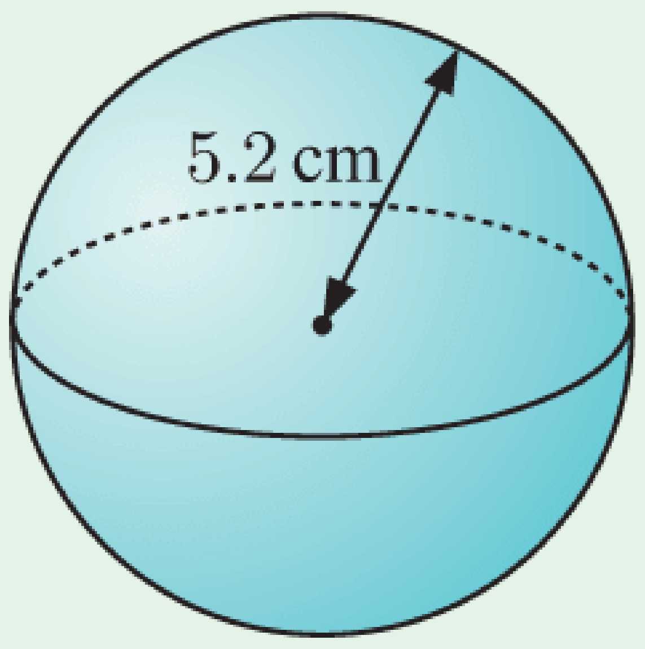
2 Find the radius of a sector with angle 80° and area 24π cm^2.

3 Find, to 1 decimal place, the outer surface area of:

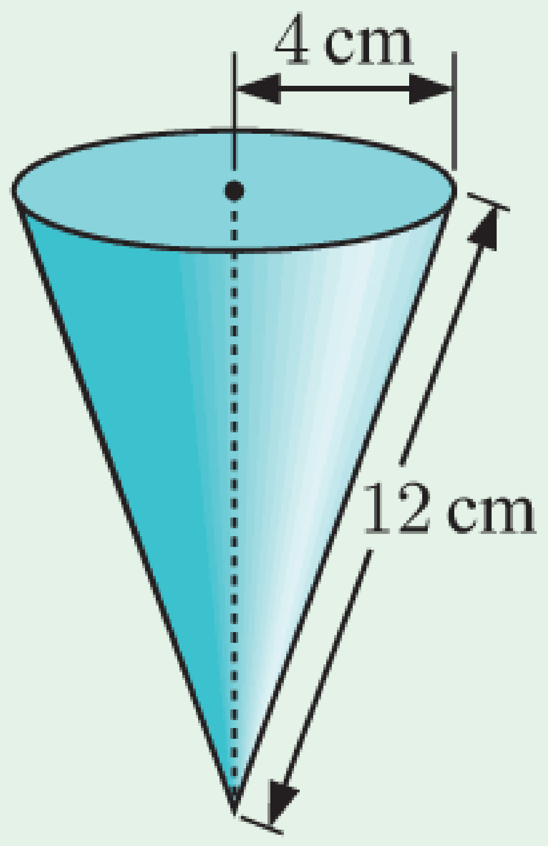
a



b

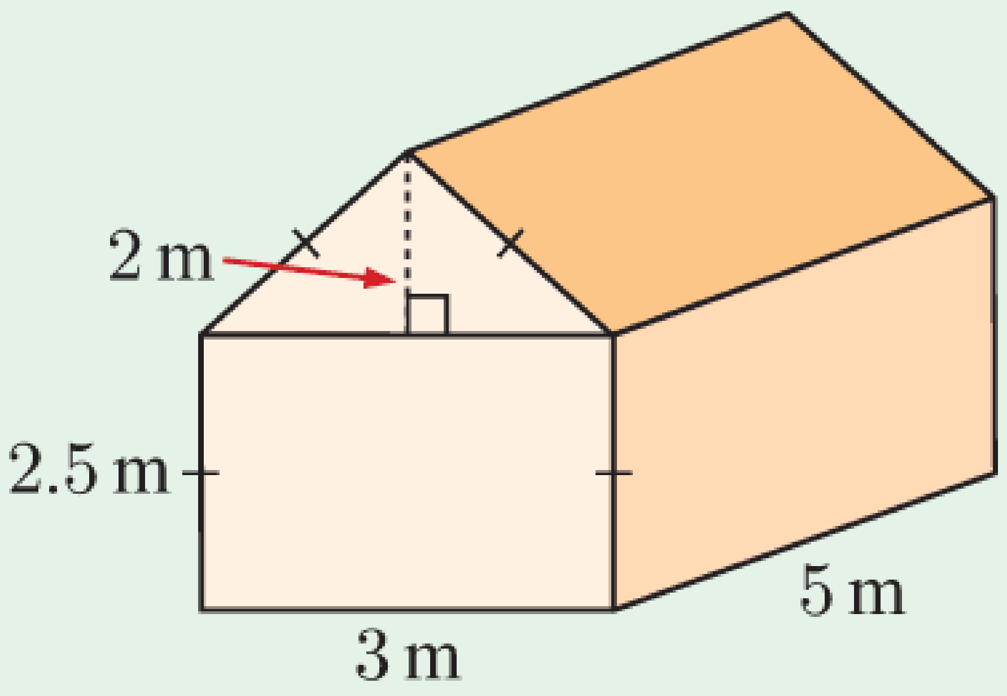


c



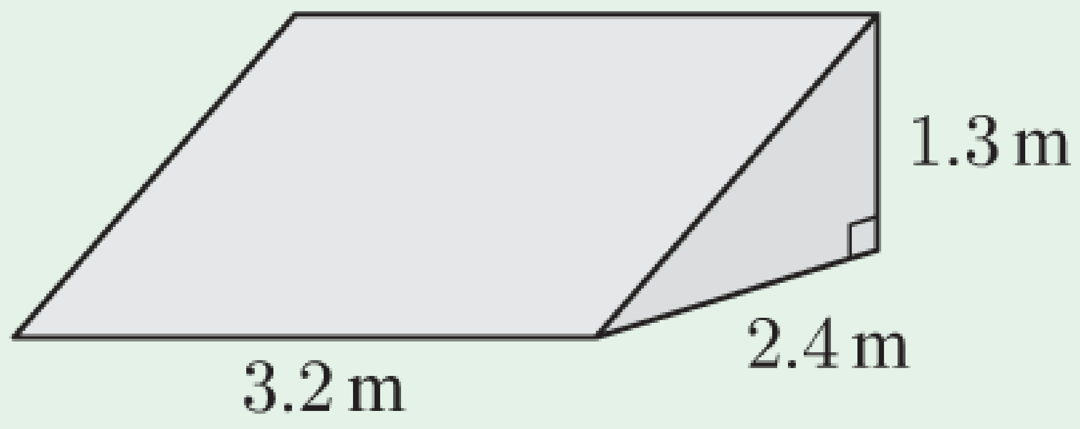
4 A tool shed with the dimensions illustrated is to be painted with two coats of zinc-aluminium. Each litre of zinc-aluminium covers 5 m^2 and costs \$8.25. It must be purchased in whole litres.

- **a** Find the area to be painted, including the roof.
- **b** Find the total cost of the zinc-aluminium.

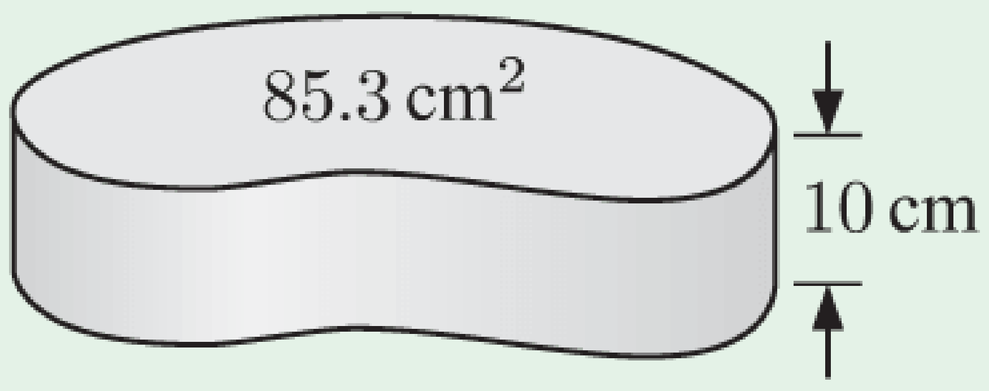


5 Calculate, to 3 significant figures, the volume of:

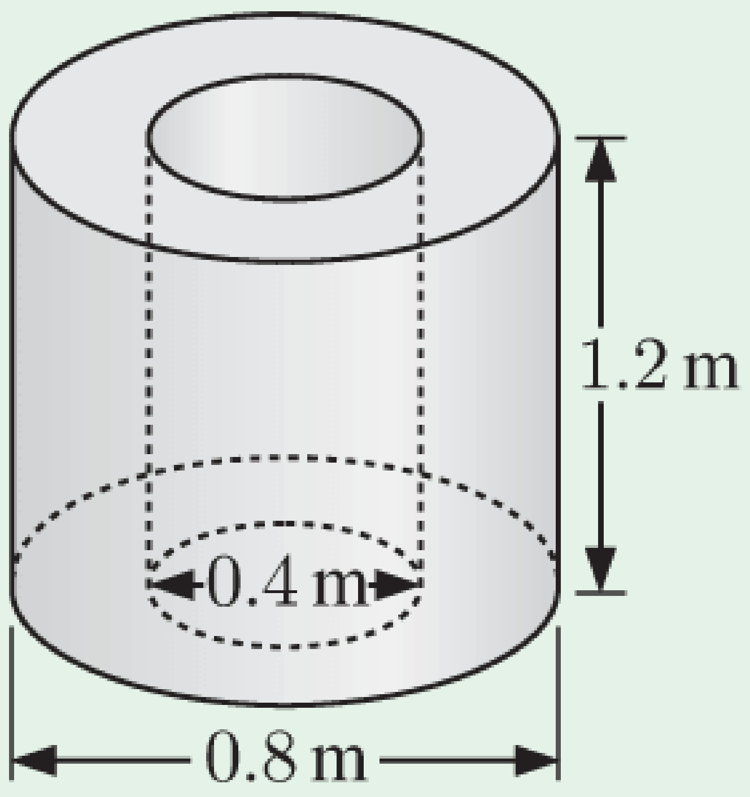
a



b



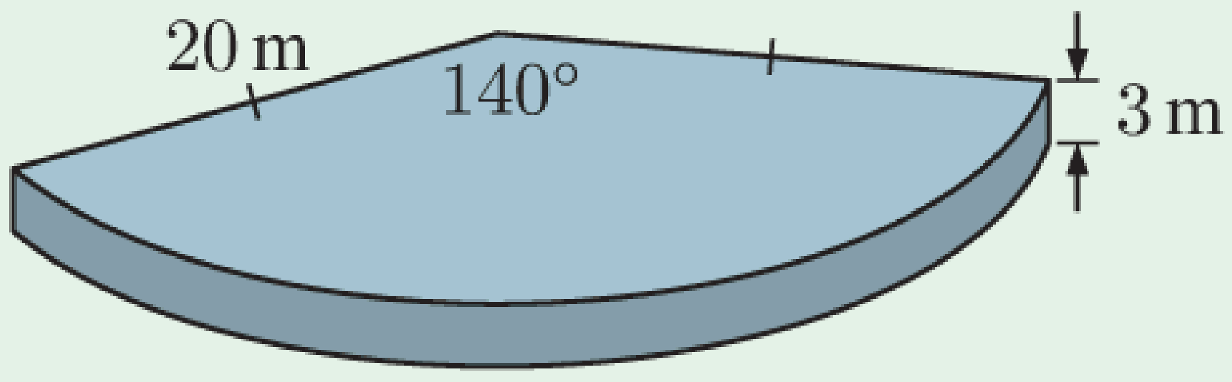
c



6 Tom has just had a load of sand delivered. The sand is piled in a cone with radius 1.6 m and height 1.2 m. Find the volume of the sand.

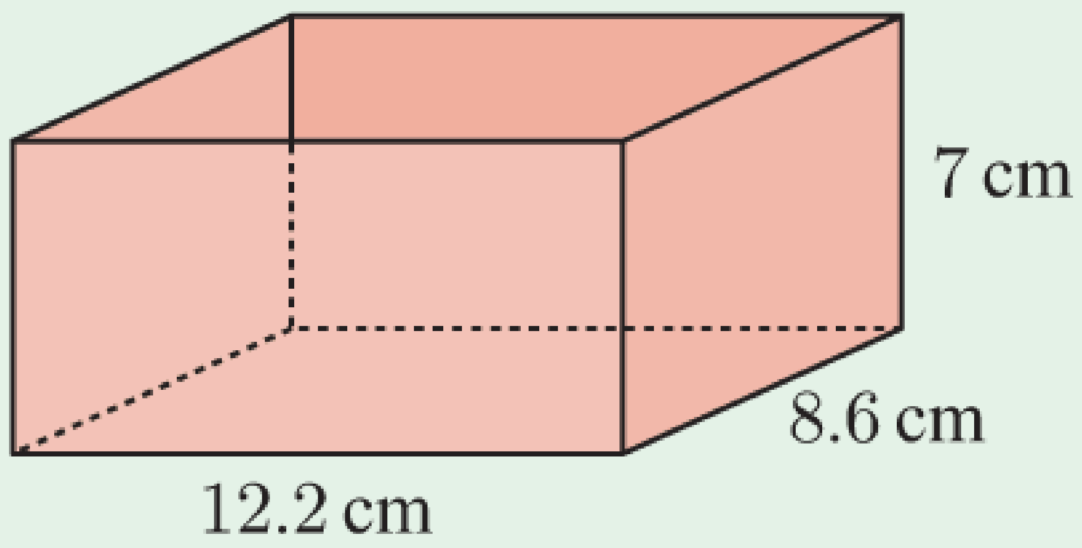
7 A plastic beach ball has radius 27 cm. Find its volume.

8 Find the volume of material required to construct this stage.

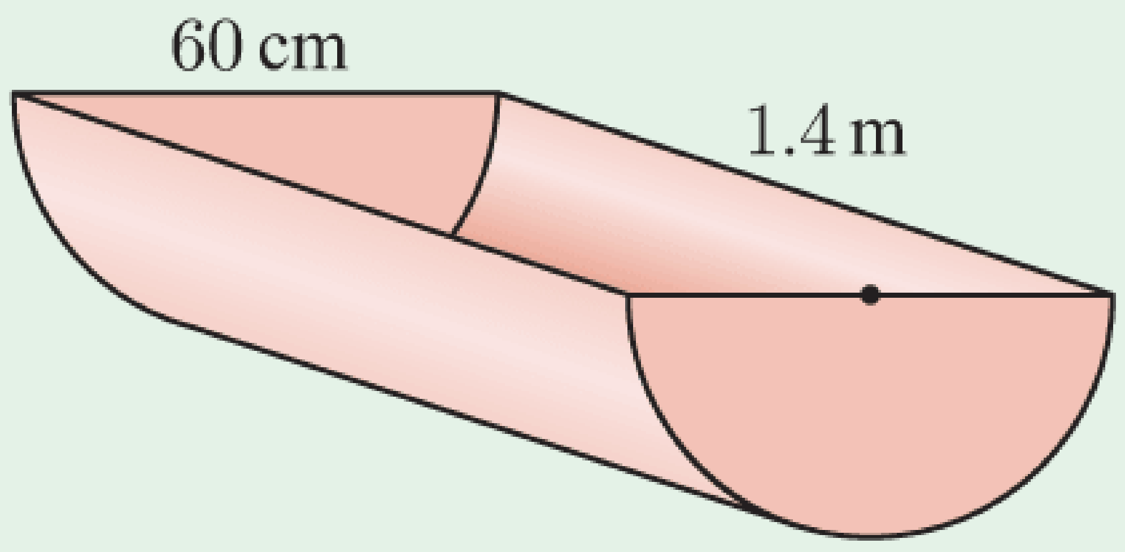


9 Find the capacity of:

a



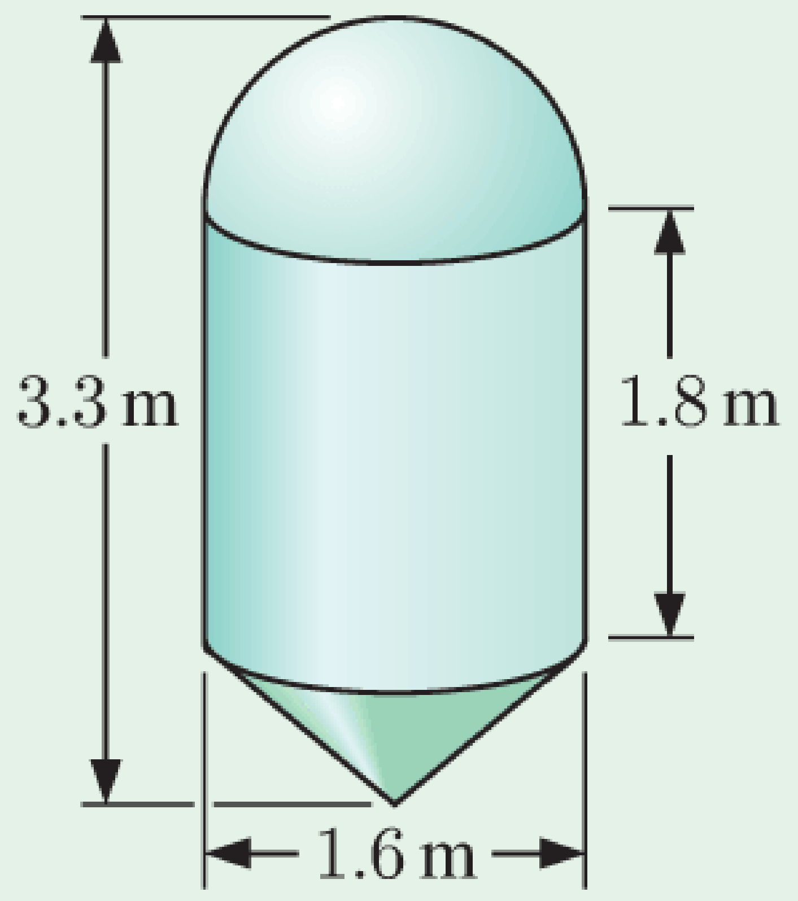
b



10 A rectangular shed has a roof of length 12 m and width 5.5 m. Rainfall from the roof runs into a cylindrical tank with base diameter 4.35 m. If 15.4 mm of rain falls, how many millimetres does the water level in the tank rise?

11 A feed silo is made out of sheet steel 3 mm thick using a hemisphere, a cylinder, and a cone.

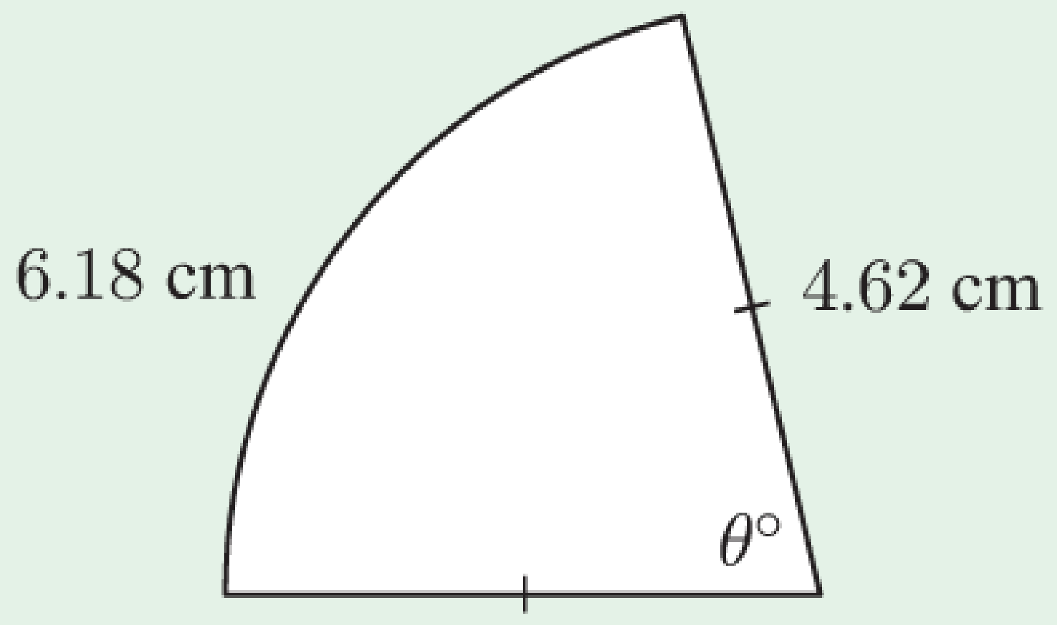
- **a** Explain why the height of the cone must be 70 cm.
- **b** Hence find the *slant height* of the conical section.
- **c** Calculate the total amount of steel used.
- **d** Show that the silo can hold about 5.2 cubic metres of grain.
- **e** Write the capacity of the silo in kL.



REVIEW SET 6B

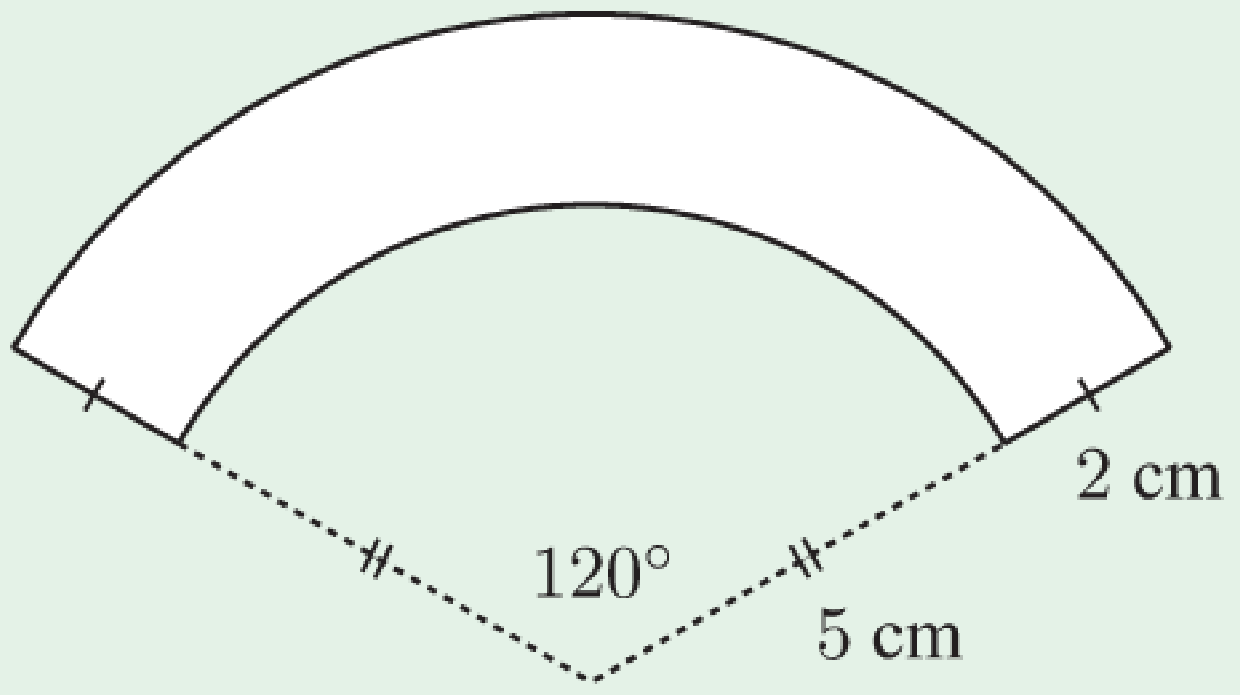
1 For the given sector, find to 3 significant figures:

- **a** the angle $\theta°$
- **b** the area of the sector.



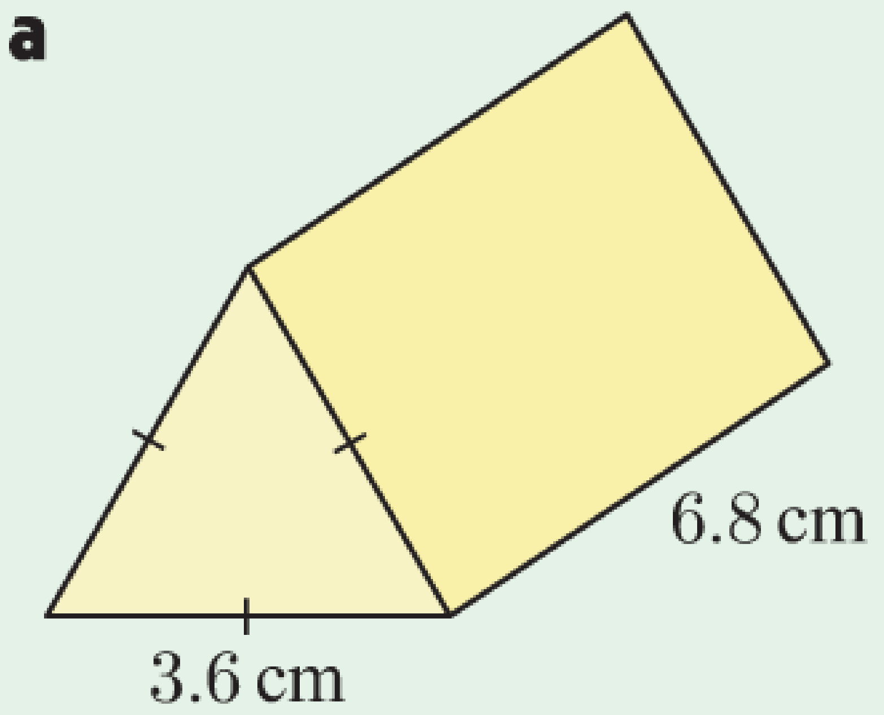
2 For the given figure, find the:

- **a** perimeter
- **b** area.

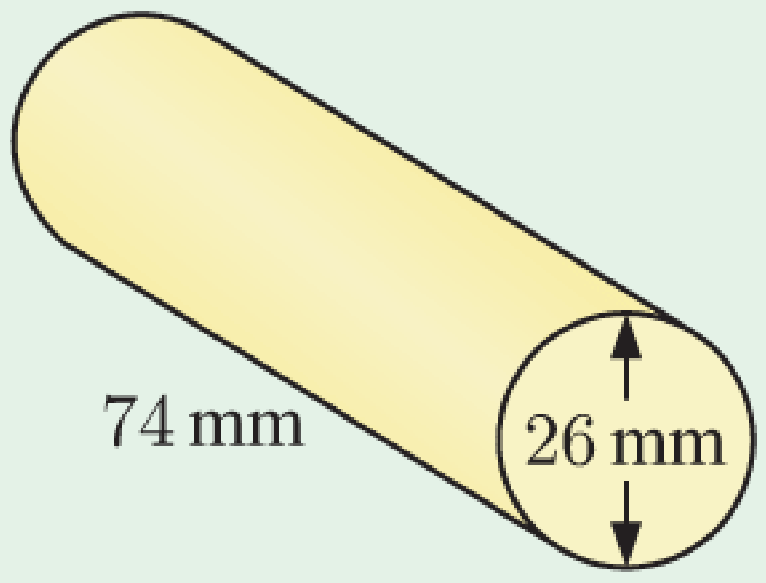


3 Find the surface area of:

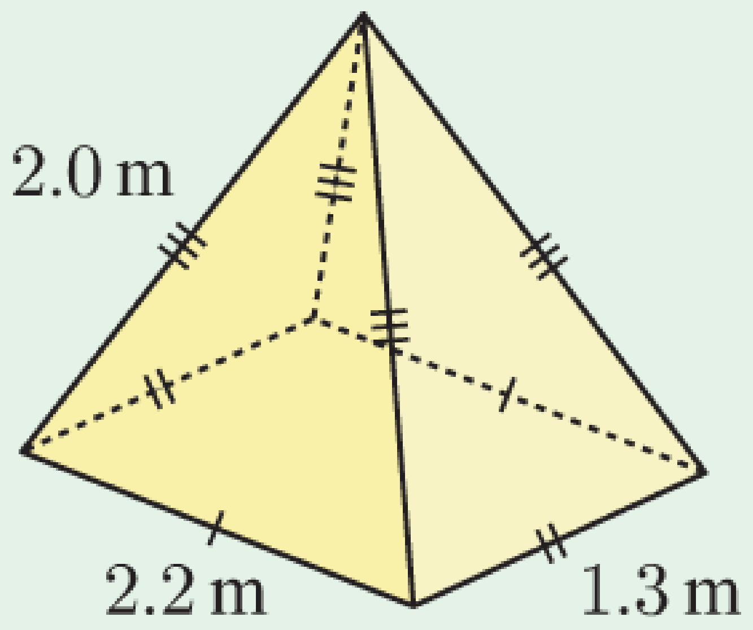
a



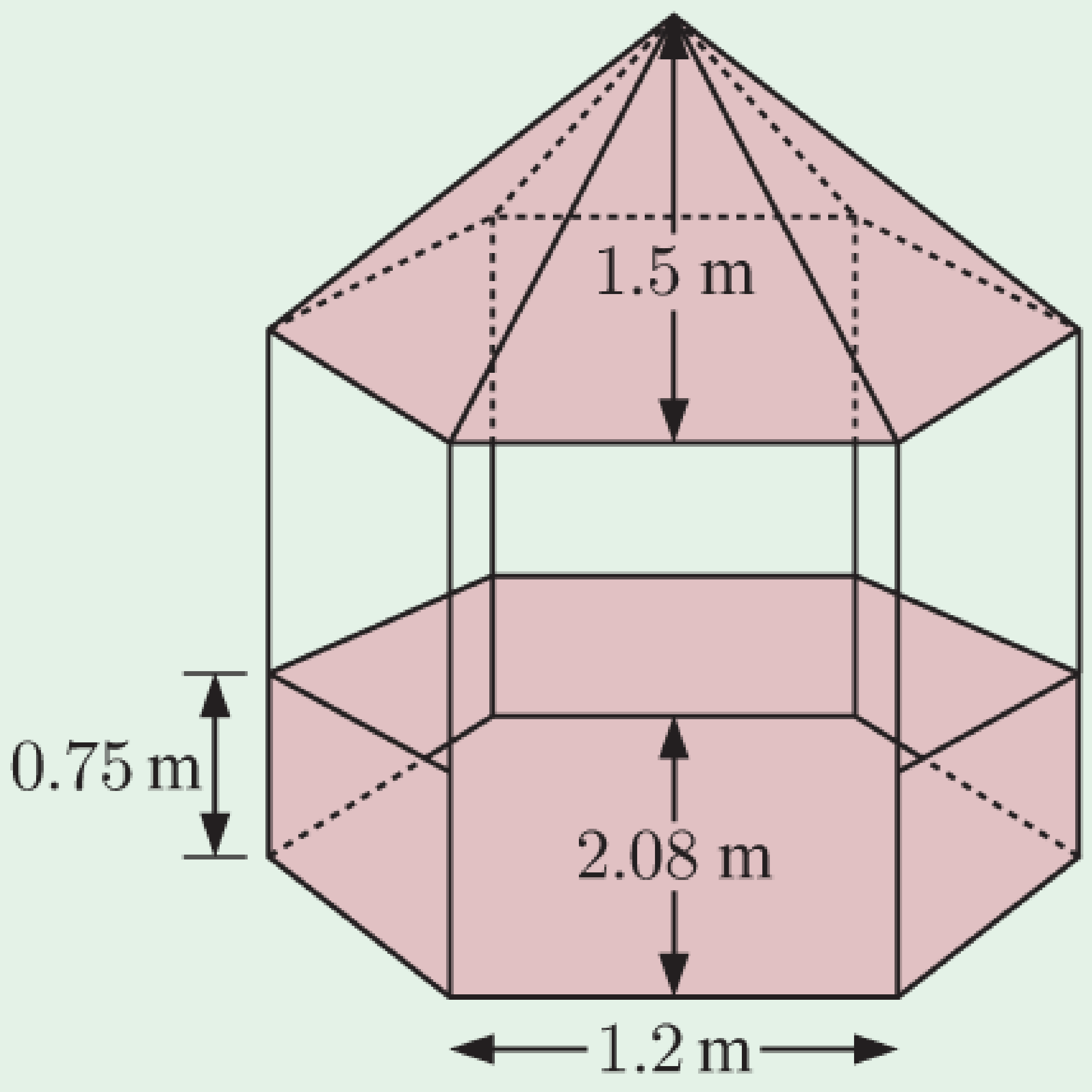
b



c



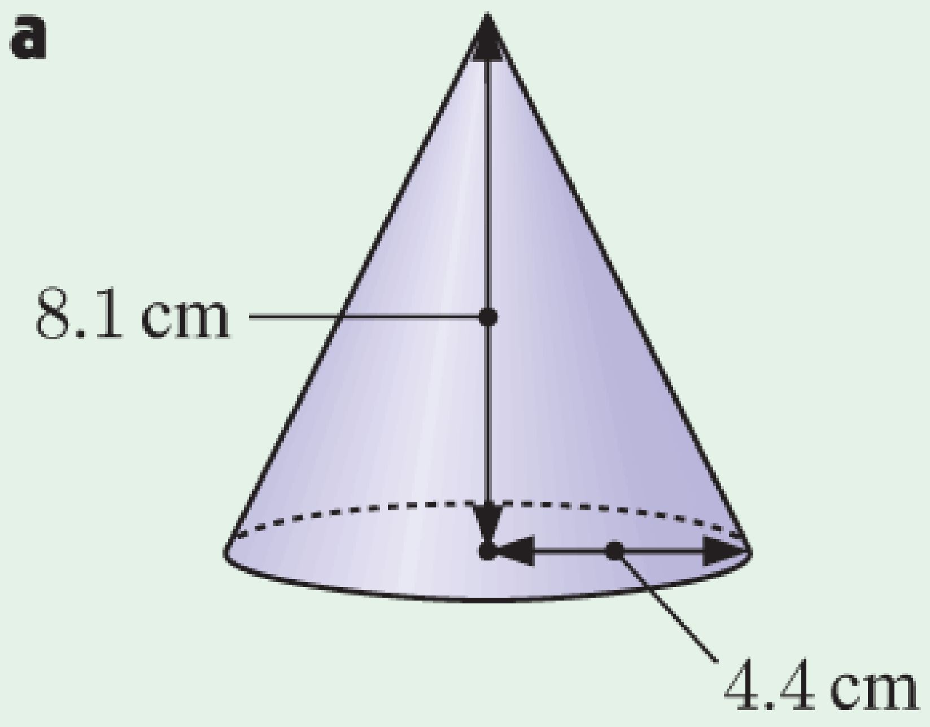
4 The hexagonal gazebo shown has wood panelling for the roof, floor, and part of five of the walls. Find the total surface area of wood panelling in the gazebo. Include the interior as well as the exterior.



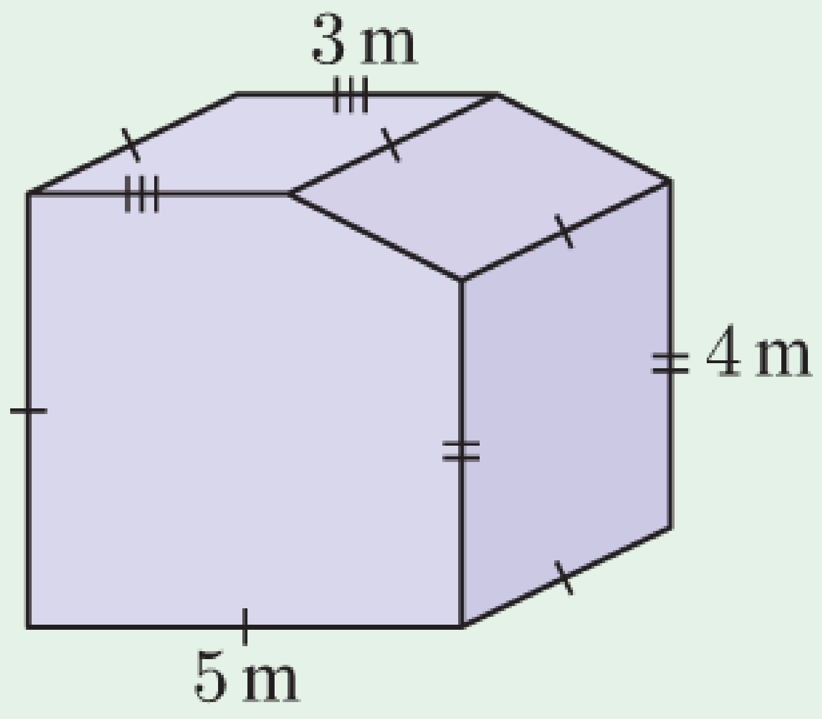
5 I am sending my sister some fragile objects inside a postal cylinder. The cylinder is 325 mm long and has diameter 40 mm. What area of bubble wrap do I need to line its inside walls?

6 Find the volume of:

a



b



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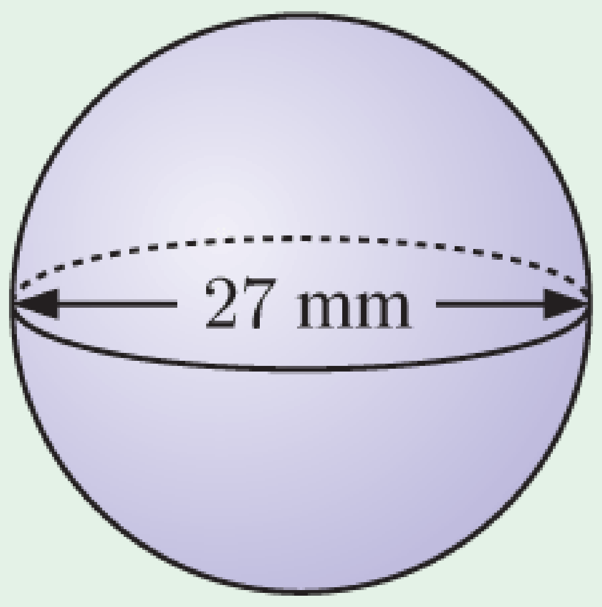
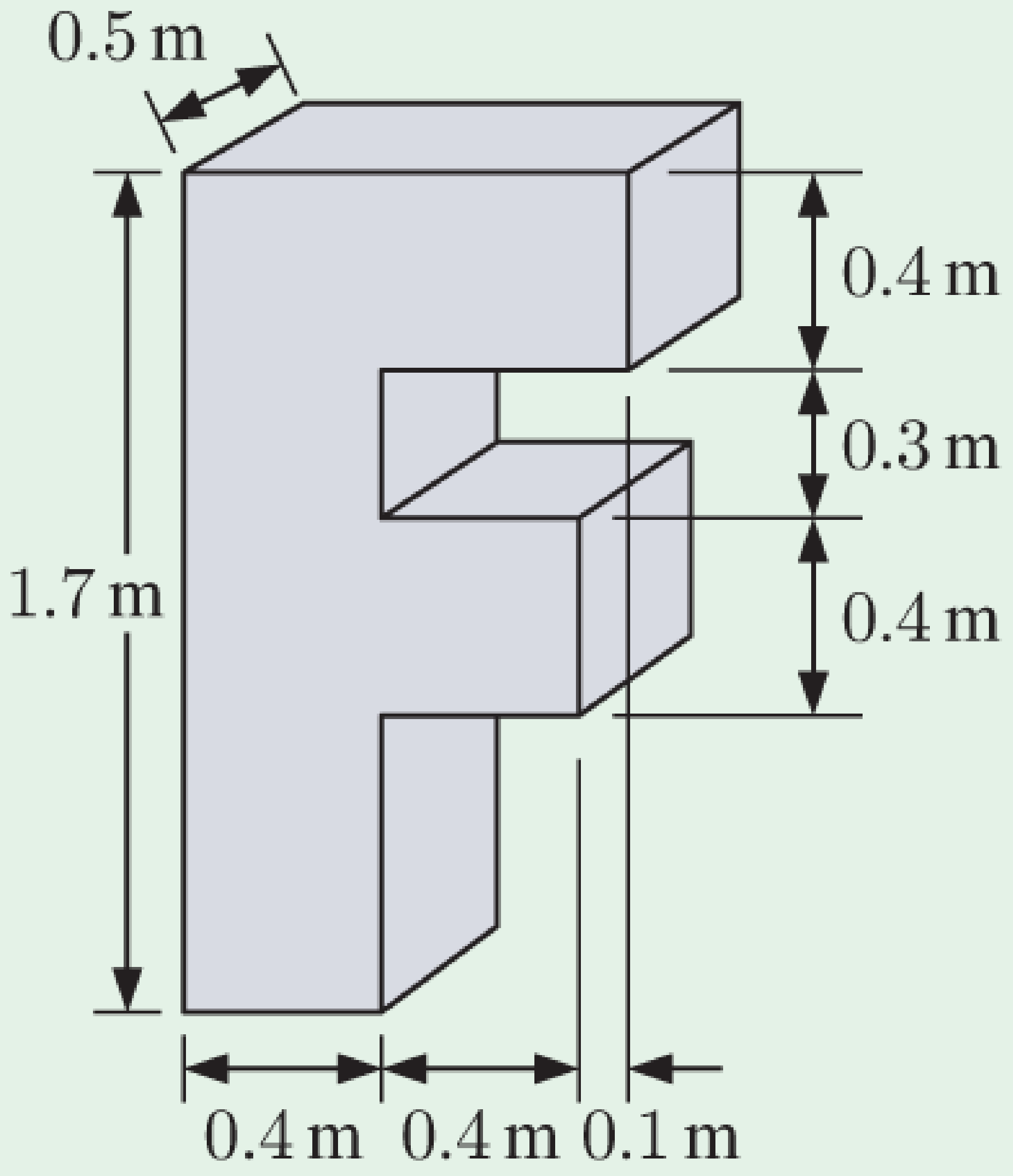


7 Frank wants to have a large F outside his shop for advertising. He designs one with the dimensions shown.

- **a** If the F is made from solid plastic, what volume of plastic is needed?
- **b** If the F is made from fibreglass as a hollow object, what surface area of fibreglass is needed?



8 A kitchen bench is a rectangular prism measuring 3845 mm by 1260 mm by 1190 mm. It contains a rectangular sink which is 550 mm wide, 750 mm long, and 195 mm deep. Find the storage capacity of the bench in litres.

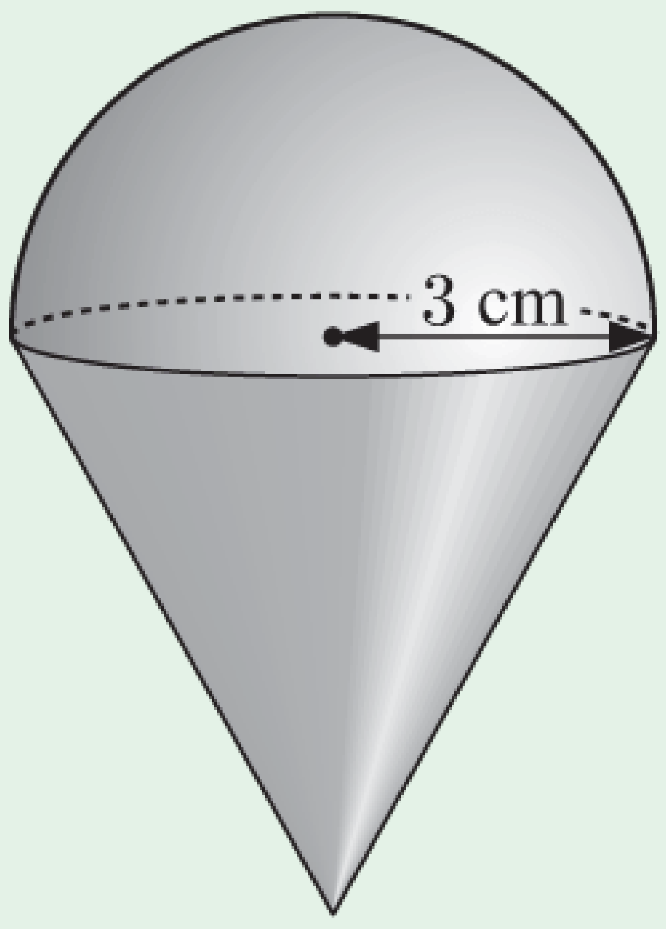
9 A cylindrical drum for storing industrial waste has capacity 10 kL. If the height of the drum is 3 m, find its radius.

10 The Sun is a nearly perfect sphere with radius $\approx 6.955 \times 10^8$ m.
Find, in scientific notation, the Sun's:

- **a** surface area
- **b** volume.

11 A solid metal spinning top is constructed by joining a hemispherical top to a cone-shaped base.
The radius of both the hemisphere and the base of the cone is 3 cm. The volume of the cone is half that of the hemisphere.
Calculate:

- **a** the volume of the hemispherical top
- **b** the height of the cone-shaped base
- **c** the outer surface area of the spinning top.



Chapter 7

Right angled triangle trigonometry

Contents:

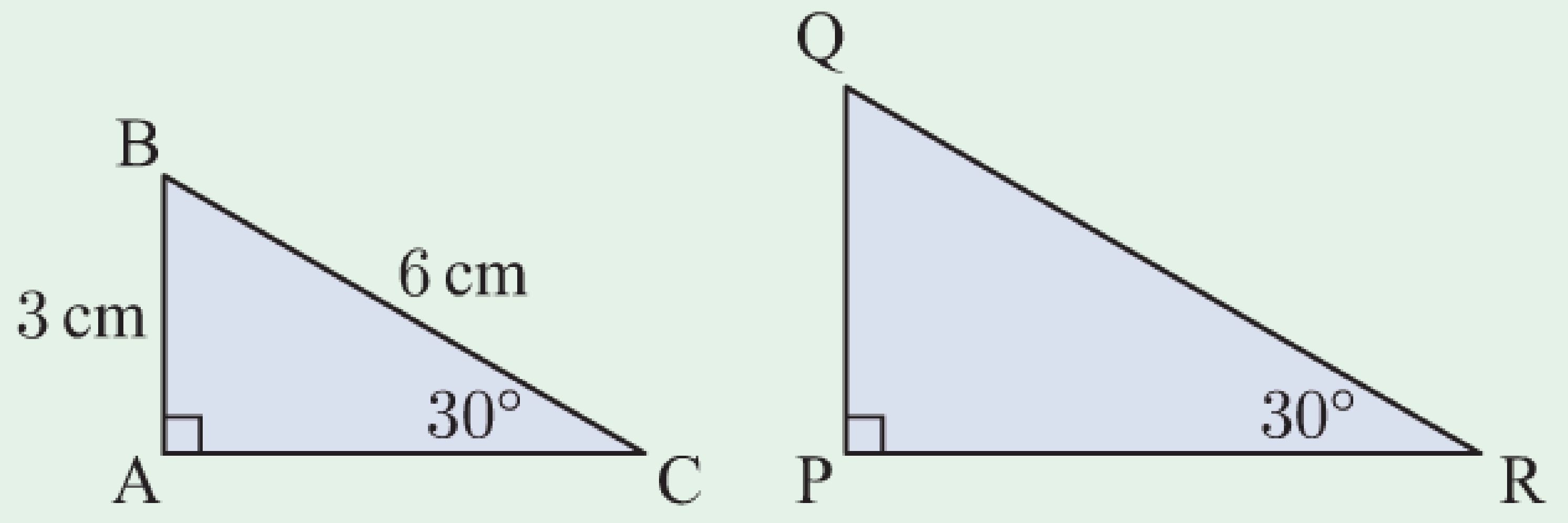
OPENING PROBLEM

Things to think about:

a Both of the right angled triangles alongside contain a 30° angle. From the information we are given, can we determine:

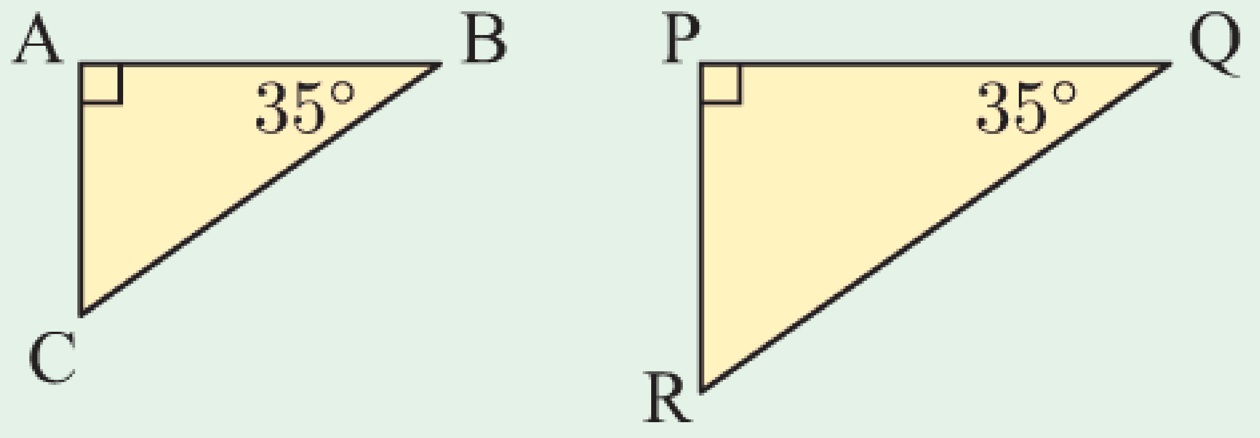
  i the lengths PQ and QR

  ii the *ratio* of lengths $\frac{PQ}{QR}$?



b Both of the right angled triangles alongside contain a 35° angle. Can you explain why:

  i $\frac{AB}{BC} = \frac{PQ}{QR}$



  ii any other right angled triangle containing a 35° angle will have corresponding sides in the same ratio?

Trigonometry is the study of the relationships between the side lengths and angles of triangles.

Trigonometry is extensively used in the real world, being essential for engineering, architecture, building, physics, astronomy, navigation, and many other industries.

THEORY OF KNOWLEDGE

The study of celestial objects such as the sun, moon, stars, and planets is called **astronomy**. It has been important to civilisations all over the world for thousands of years, not only because it allowed them to navigate at night, but because the celestial objects feature in so many of their myths and beliefs.

To create an accurate star map, astronomers measure the angles between objects in the sky. The oldest known star map was found in the Silk Road town of Dunhuang in 1907. It was made in the 7th century AD, presumably from the Imperial Observatory in either Chang'an (present day Xi'an) or Luoyang. A possible author of the map was the mathematician and astronomer **Li Chunfeng** (602 - 670). The map shows 1339 stars in 257 star groups recorded with great precision on 12 charts, each covering approximately 30 degree sections of the night sky.[1]

1 How much of what we *believe* comes from what we *observe*? Is it necessary to *understand* something, in order to *believe* it? How much of what we *study* is a quest to *understand* what we *observe*, and *prove* what we *believe*?

2 How much of what we want to know is a common desire of people and cultures all over the world?

3 How did ancient people calculate with such accuracy before computer technology?

[1] "The Dunhuang Chinese Sky: A comprehensive study of the oldest known star atlas", J-M Bonnet-Bidaud, F. Praderie, S. Whitfield, *J. Astronomical History and Heritage*, 12(1), 39-59 (2009).

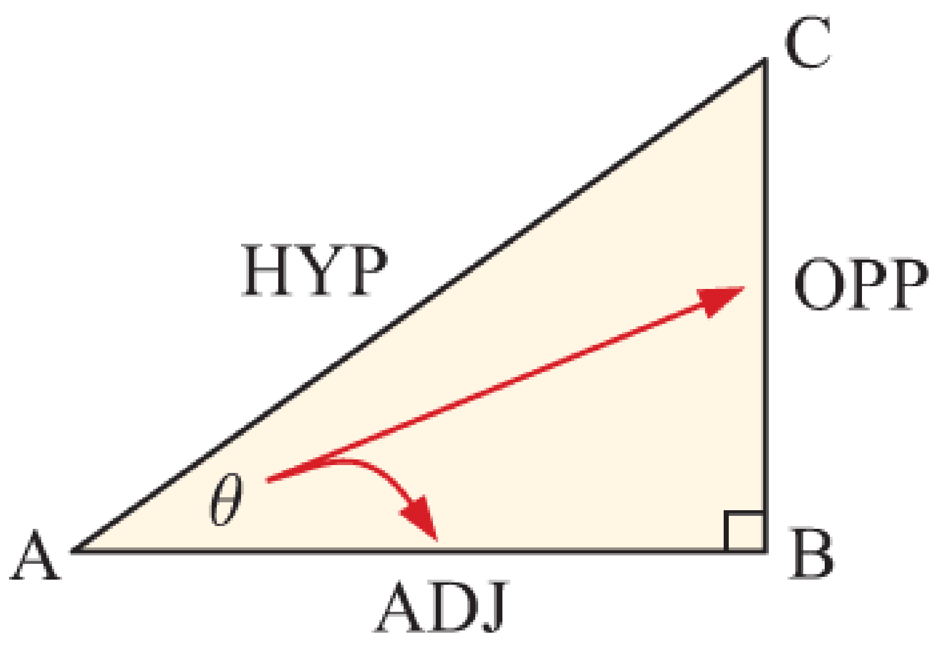
In this Chapter we will study the trigonometry of right angled triangles.

A TRIGONOMETRIC RATIOS

In previous years you should have seen that for any right angled triangle with a fixed angle θ, the side lengths are in the same ratio.

In the triangle alongside:

- The **hypotenuse (HYP)** is the side which is opposite the right angle. It is the longest side of the triangle.
- [BC] is the side **opposite (OPP)** angle θ.
- [AB] is the side **adjacent (ADJ)** to angle θ.



The **trigonometric ratios** for the angle θ are:

$$\sin\theta = \frac{\text{OPP}}{\text{HYP}} \qquad \cos\theta = \frac{\text{ADJ}}{\text{HYP}} \qquad \tan\theta = \frac{\text{OPP}}{\text{ADJ}}$$

They stand for **sine**, **cosine**, and **tangent**.

Notice that $\dfrac{\sin\theta}{\cos\theta} = \dfrac{\;\frac{\text{OPP}}{\text{HYP}}\;}{\;\frac{\text{ADJ}}{\text{HYP}}\;} = \dfrac{\text{OPP}}{\text{ADJ}} = \tan\theta$, so $\boxed{\tan\theta = \dfrac{\sin\theta}{\cos\theta}}$.

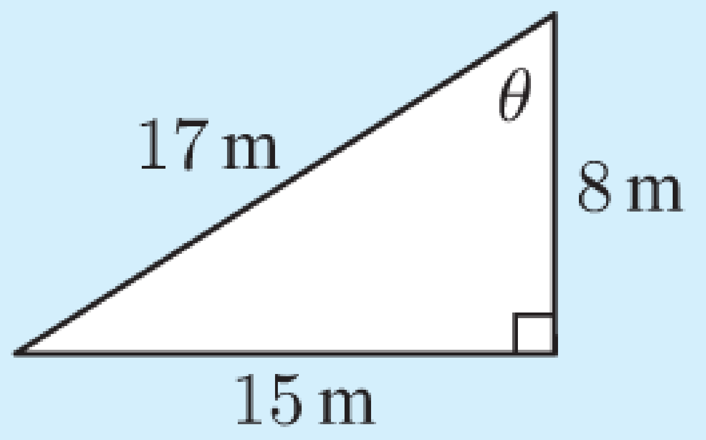
The trigonometric ratios for any angle can be found using a calculator.

GRAPHICS CALCULATOR INSTRUCTIONS

Example 1 **Self Tutor**

For the following triangle, find:

a $\sin\theta$ **b** $\cos\theta$ **c** $\tan\theta$



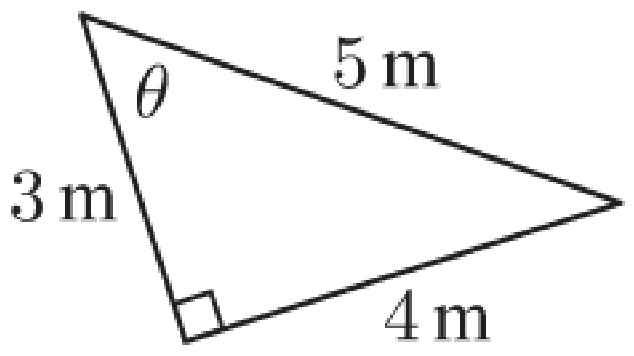
a $\sin\theta = \dfrac{\text{OPP}}{\text{HYP}} = \dfrac{15}{17}$ **b** $\cos\theta = \dfrac{\text{ADJ}}{\text{HYP}} = \dfrac{8}{17}$ **c** $\tan\theta = \dfrac{\text{OPP}}{\text{ADJ}} = \dfrac{15}{8}$

EXERCISE 7A

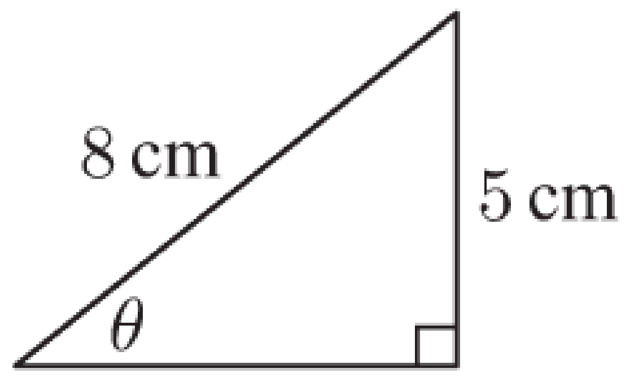
1 For each of the following triangles, find:

i $\sin\theta$ **ii** $\cos\theta$ **iii** $\tan\theta$

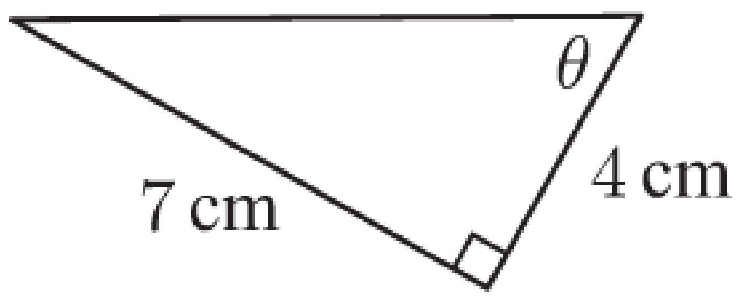
a



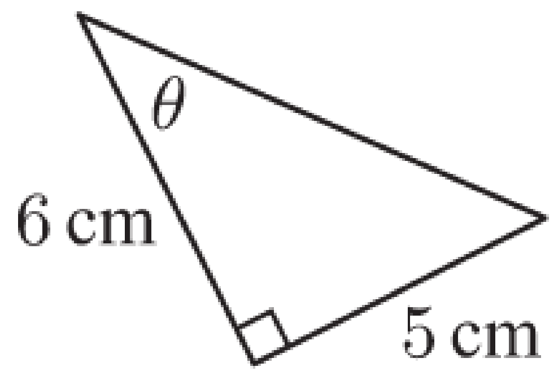
b



c



d



2 The right angled triangle alongside contains an angle of 56°.

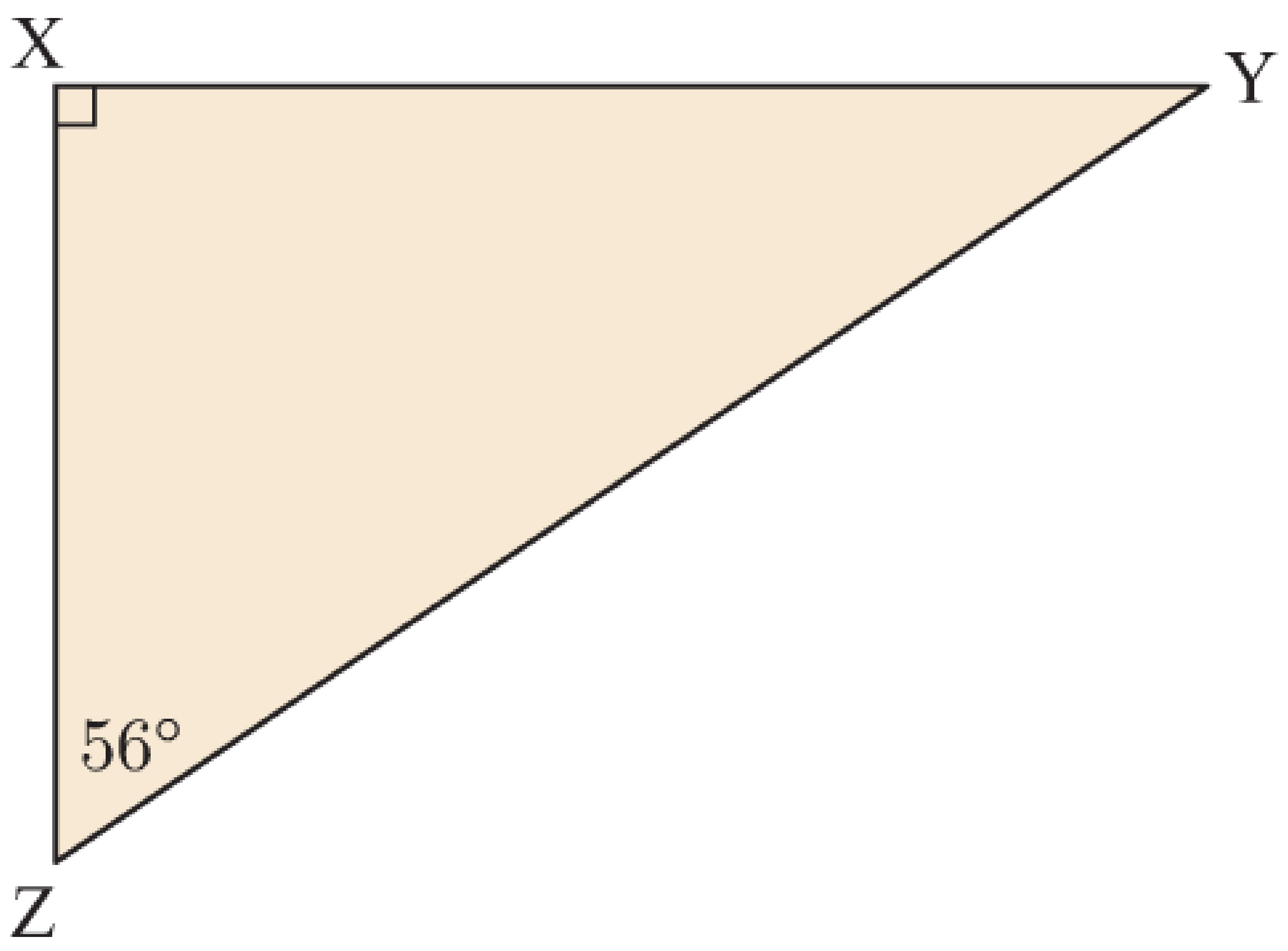
a Use a ruler to measure the length of each side, to the nearest millimetre.

b Hence estimate, to 2 decimal places:

i $\sin 56°$ **ii** $\cos 56°$

iii $\tan 56°$

c Check your answers using a calculator.



3 Consider the right angled isosceles triangle ABC alongside.

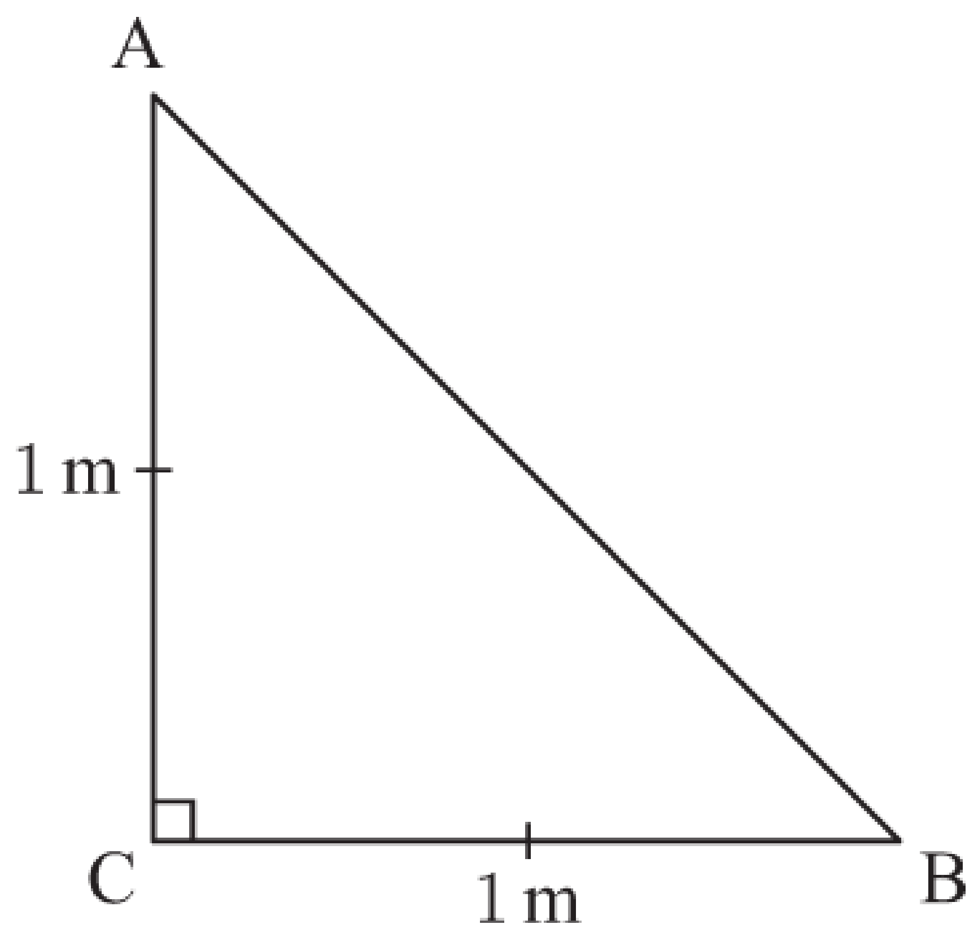
a Explain why $A\widehat{B}C = 45°$.

b Use Pythagoras' theorem to find AB.

c Hence find:

i $\sin 45°$ **ii** $\cos 45°$ **iii** $\tan 45°$.

d Check your answers using a calculator.



4 Explain why it is impossible for the sine or cosine of an angle to be greater than 1.

5 Consider the right angled triangle shown.

a Write expressions for:

i $\sin A$ **ii** $\cos A$ **iii** $\tan A$

iv $\sin B$ **v** $\cos B$ **vi** $\tan B$

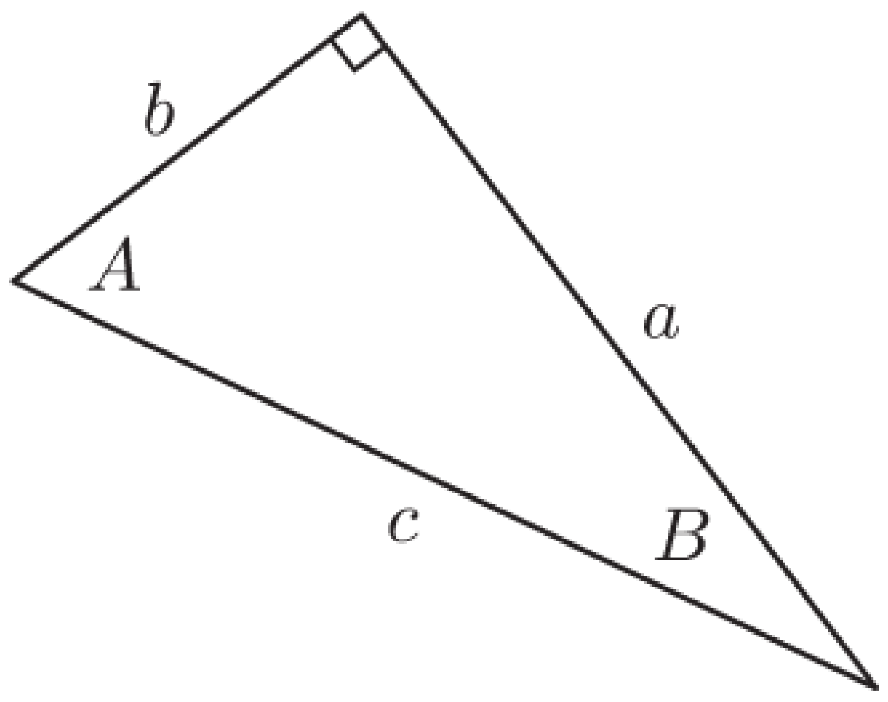
b State the relationship between A and B.

c Hence state the relationship between:

i $\sin\theta$ and $\cos(90° - \theta)$

ii $\cos\theta$ and $\sin(90° - \theta)$

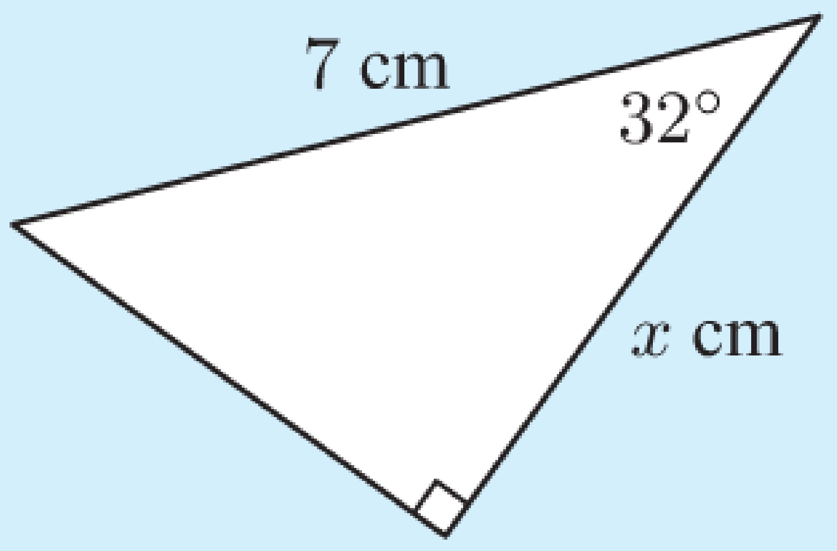
iii $\tan\theta$ and $\tan(90° - \theta)$



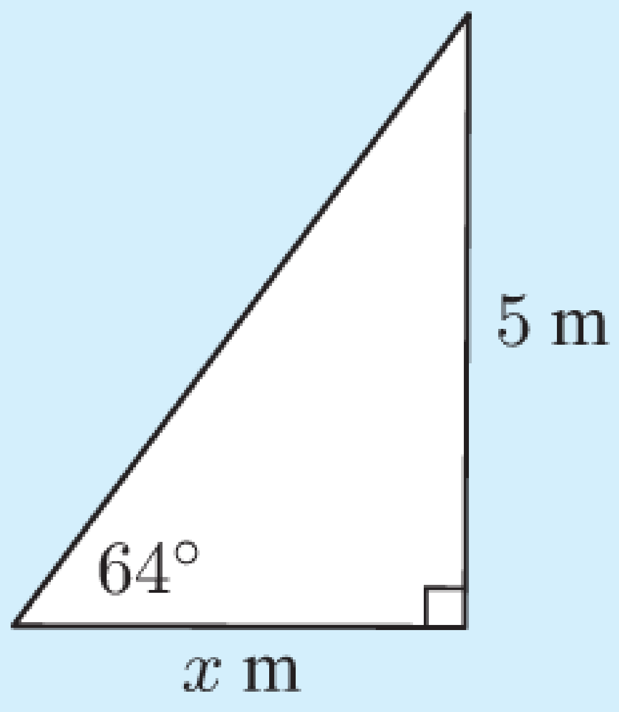
Example 2 **Self Tutor**

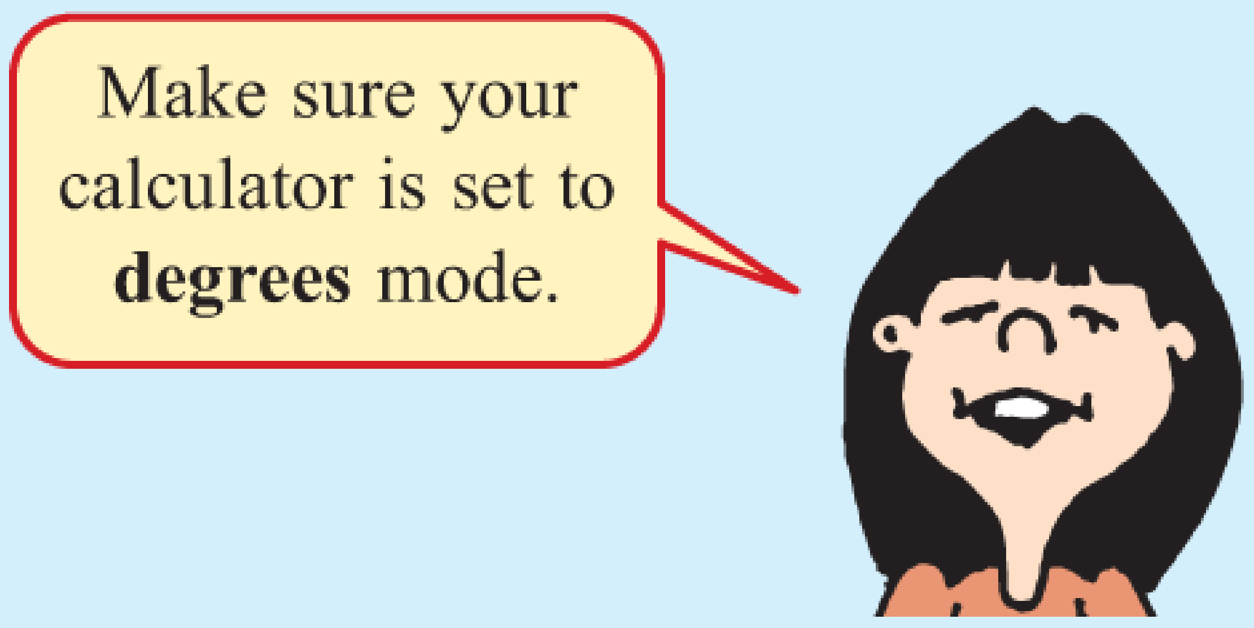
Find, correct to 3 significant figures, the unknown length in the following triangles:

a

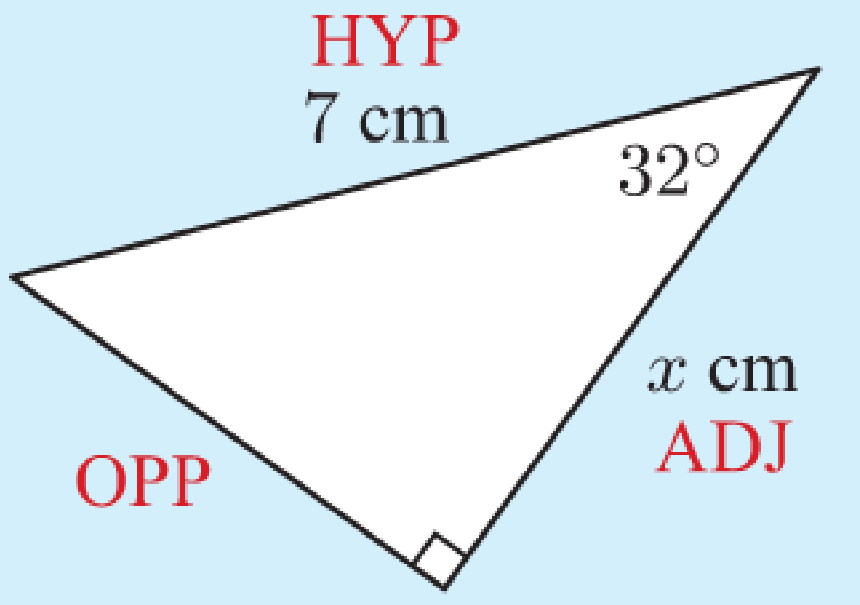


b





a



The relevant sides are ADJ and HYP, so we use the *cosine* ratio.

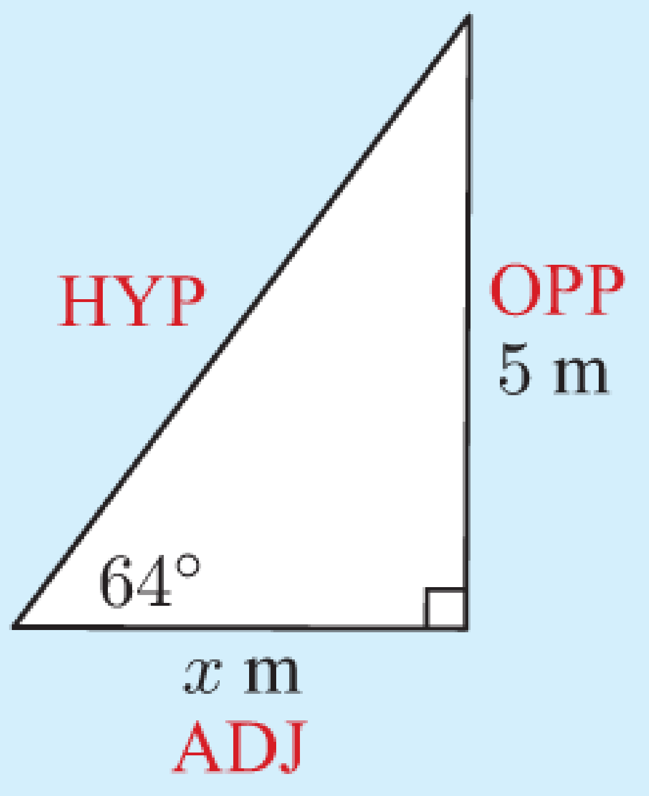
$$\cos 32^\circ = \frac{x}{7} \qquad \left\{\cos\theta = \frac{\text{ADJ}}{\text{HYP}}\right\}$$

$$\therefore\ 7 \times \cos 32^\circ = x \qquad \{\text{multiplying both sides by } 7\}$$

$$\therefore\ x \approx 5.94 \qquad \{\text{using technology}\}$$

So, the side is about 5.94 cm long.

b



The relevant sides are ADJ and OPP, so we use the *tangent* ratio.

$$\tan 64^\circ = \frac{5}{x} \qquad \left\{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\right\}$$

$$\therefore\ x \times \tan 64^\circ = 5 \qquad \{\text{multiplying both sides by } x\}$$

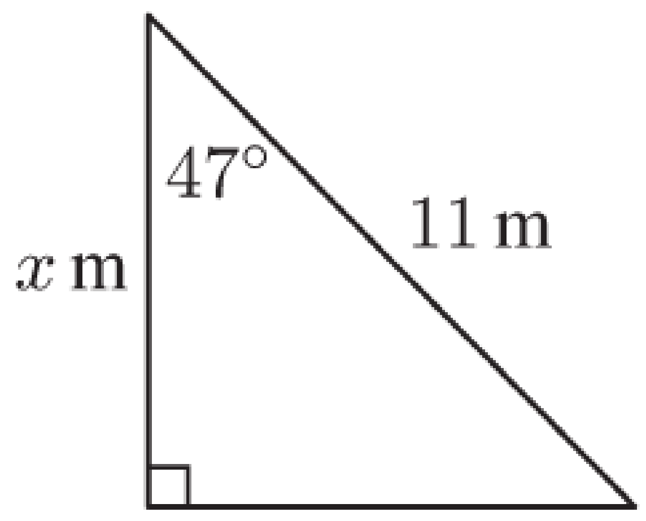
$$\therefore\ x = \frac{5}{\tan 64^\circ} \qquad \{\text{dividing both sides by } \tan 64^\circ\}$$

$$\therefore\ x \approx 2.44 \qquad \{\text{using technology}\}$$

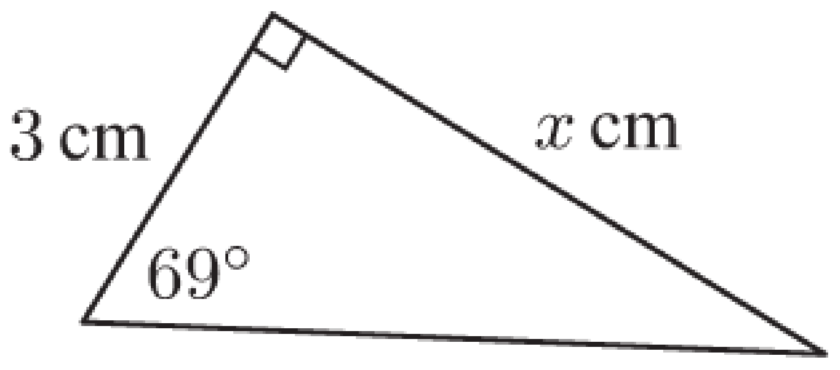
So, the side is about 2.44 m long.

6 Find, correct to 3 significant figures, the unknown length:

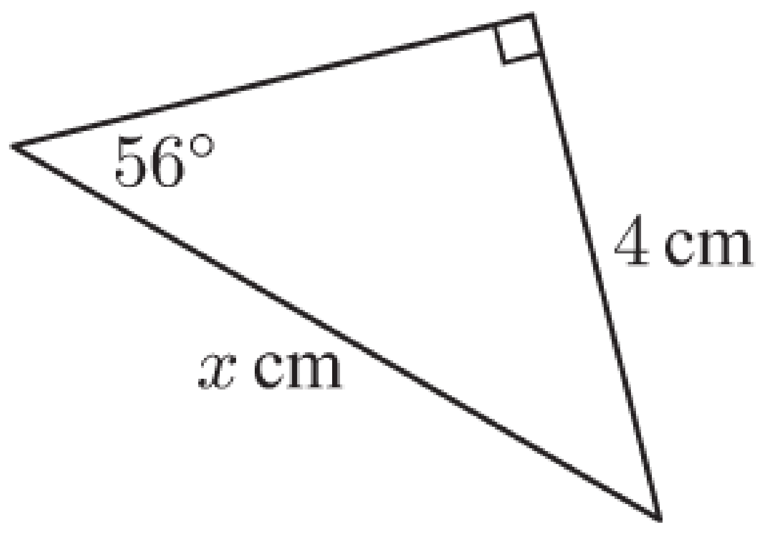
a



b



c



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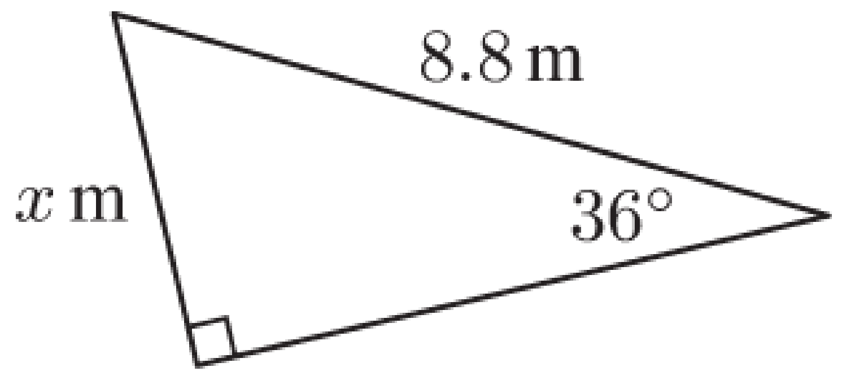

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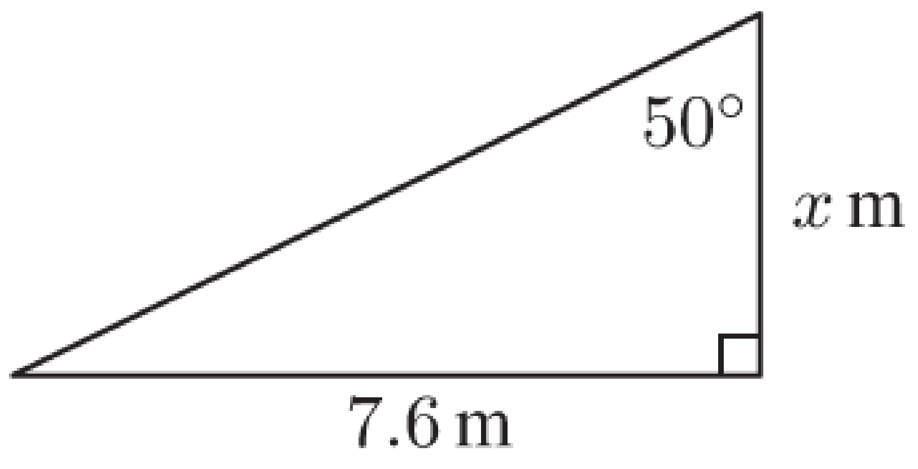

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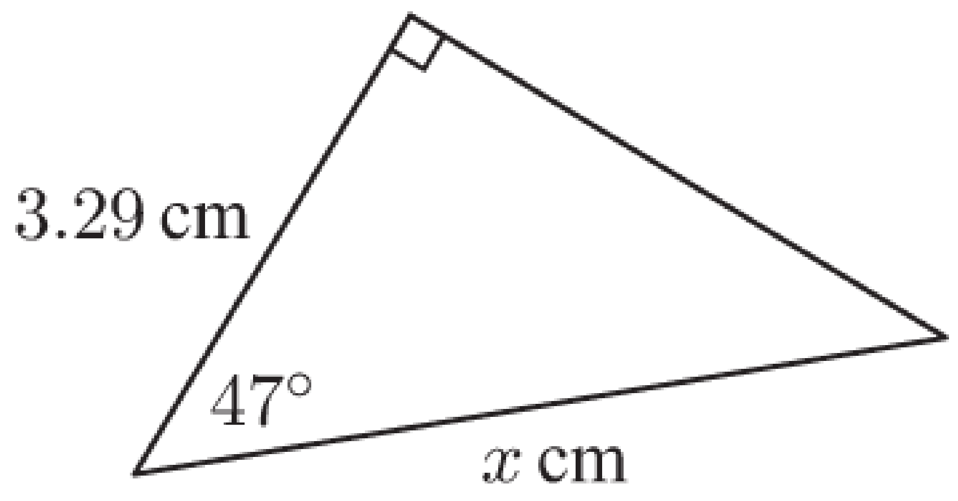


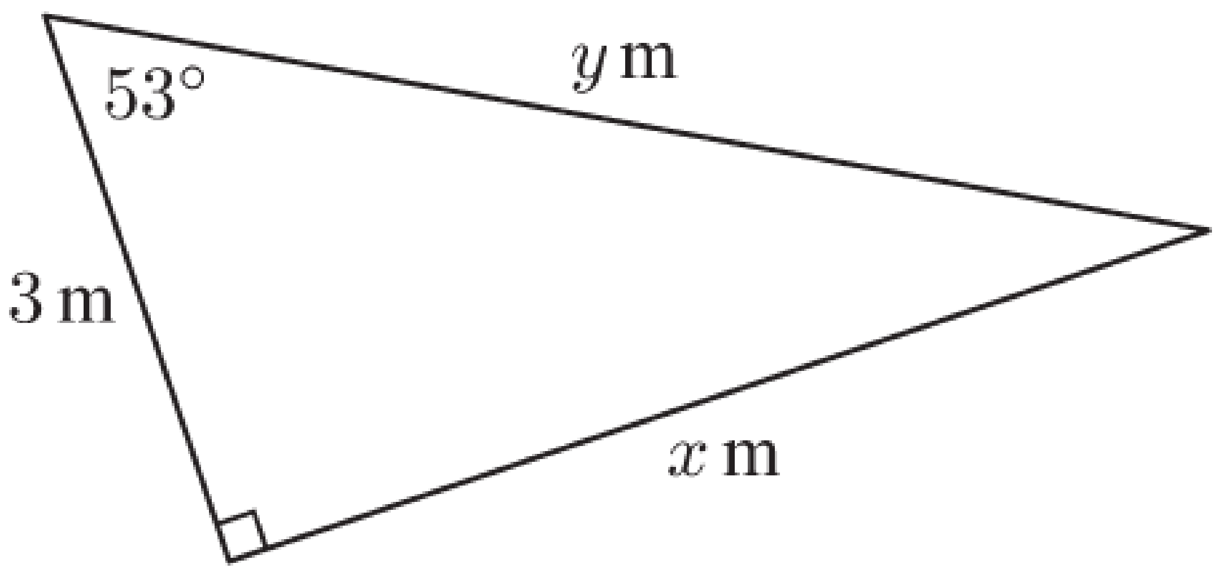
7 Consider the triangle alongside.

a Find x.

b Find y using:

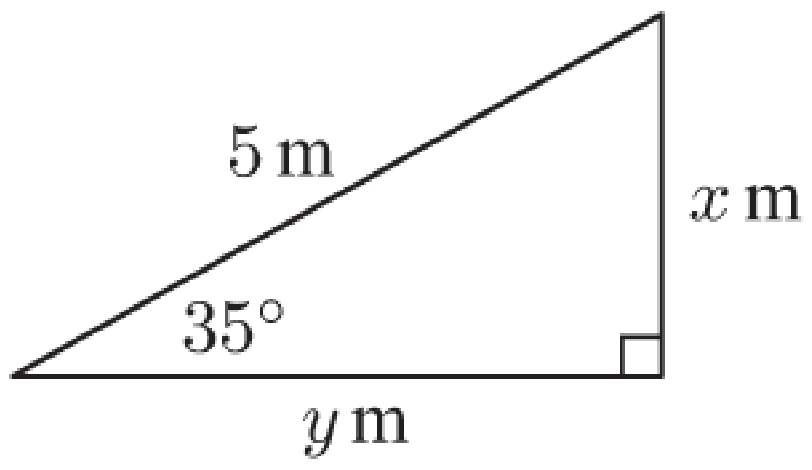
  i Pythagoras' theorem

  ii trigonometry.



8 Find, correct to 2 decimal places, *all* unknown sides:

a



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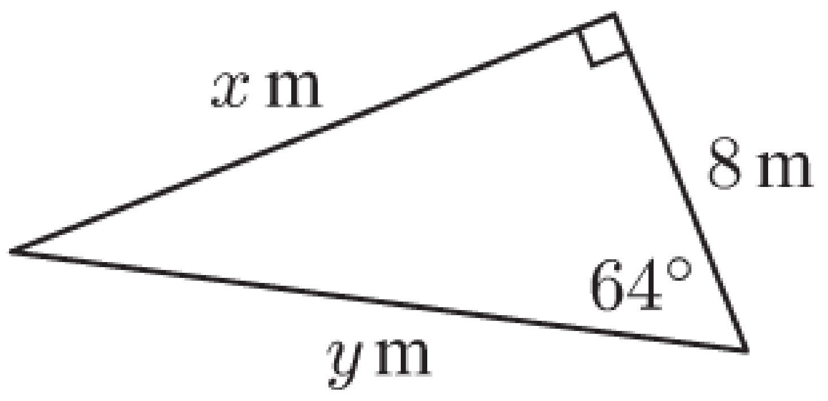

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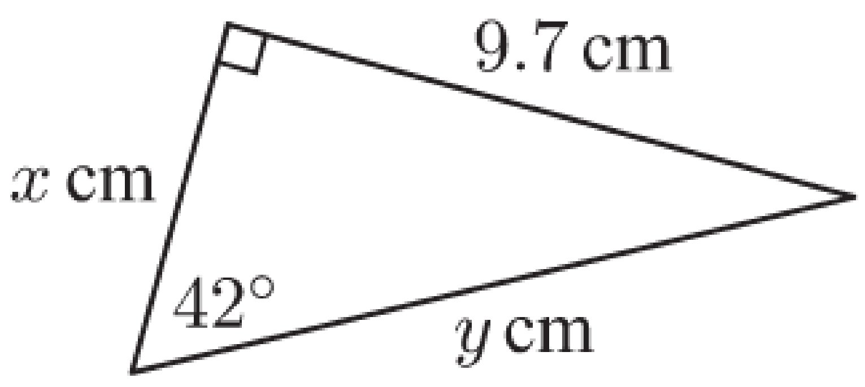
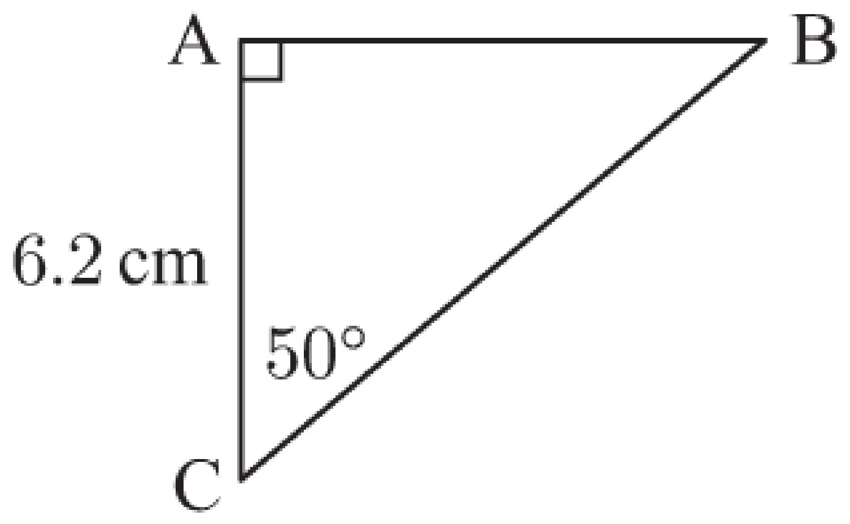


9 Find the perimeter and area of triangle ABC.

a



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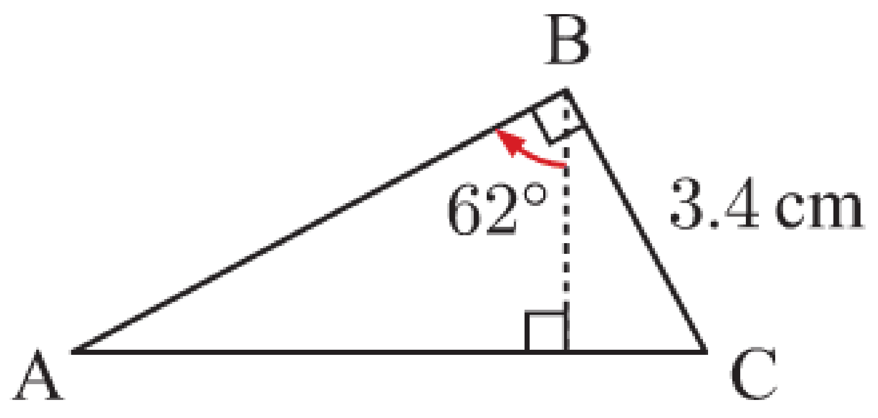
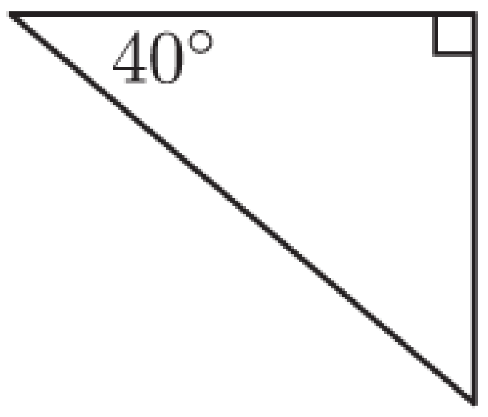


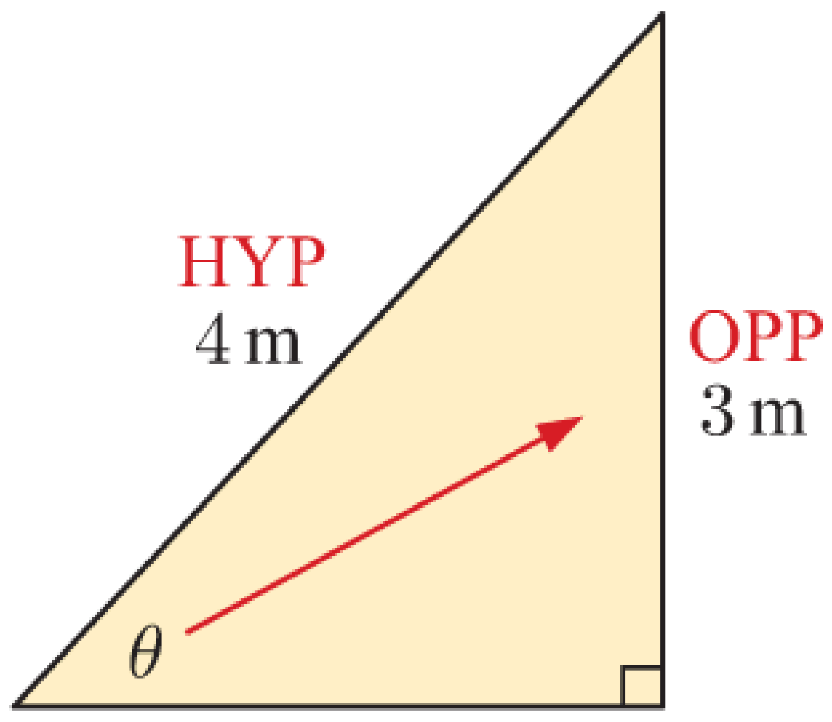
10 This triangle has area 20 cm^2. Find its perimeter.



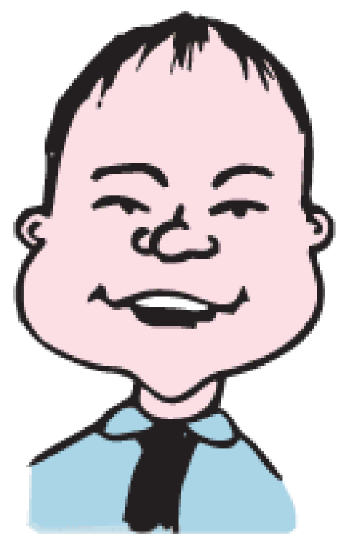
B INVERSE TRIGONOMETRIC RATIOS

In the triangle alongside, $\sin\theta = \dfrac{\text{OPP}}{\text{HYP}} = \dfrac{3}{4}$.

To find θ, we need to find the angle whose sine is $\frac{3}{4}$.

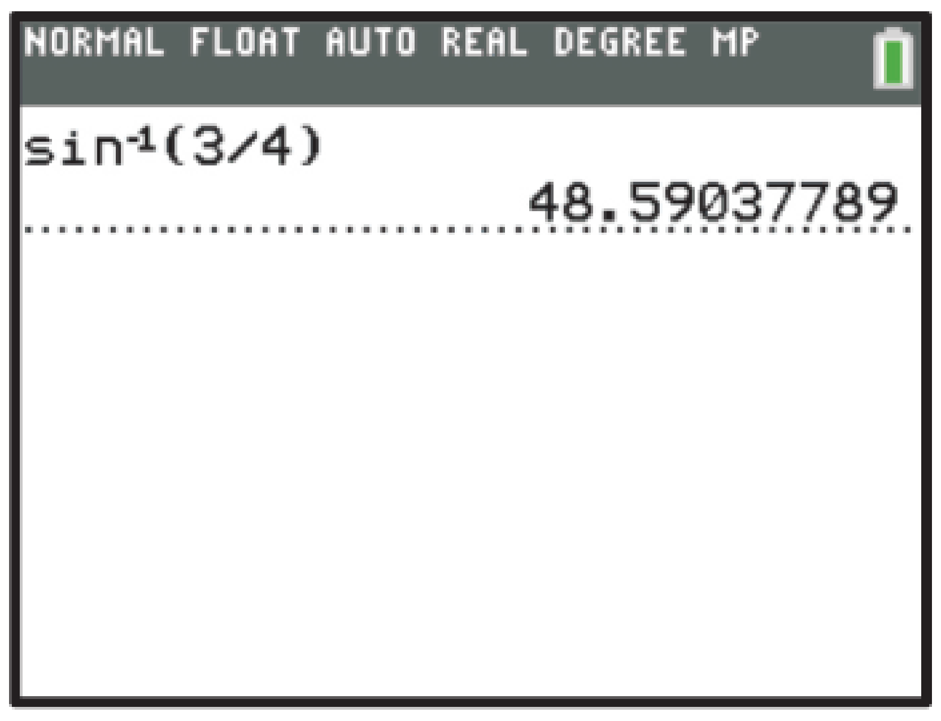


We say that θ is the **inverse sine** of $\frac{3}{4}$, and write $\theta = \sin^{-1}\left(\frac{3}{4}\right) \approx 48.6°$.



$\sin^{-1} x$ is the angle with a sine of x.

GRAPHICS CALCULATOR INSTRUCTIONS



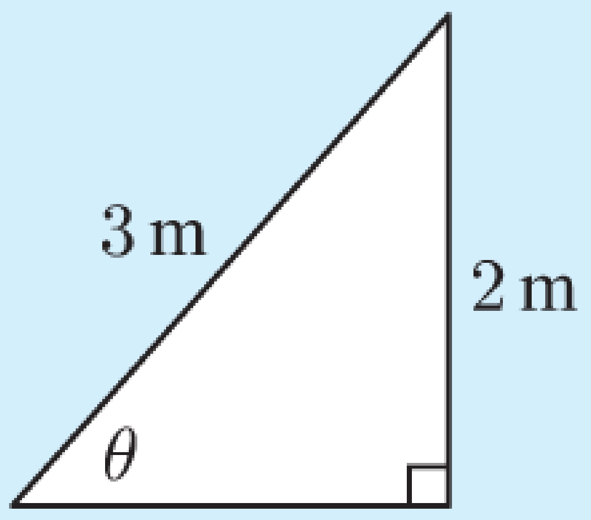
```
NORMAL FLOAT AUTO REAL DEGREE MP
sin⁻¹(3/4)
                    48.59037789
```

We define **inverse cosine** and **inverse tangent** in a similar way.

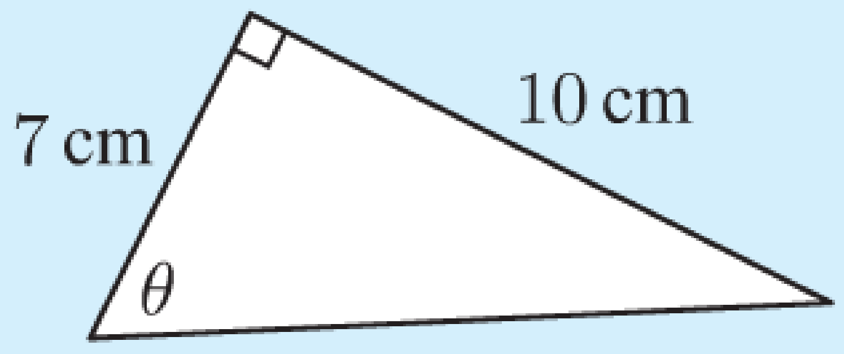
Example 3 Self Tutor

Find, correct to 3 significant figures, the measure of the angle marked θ:

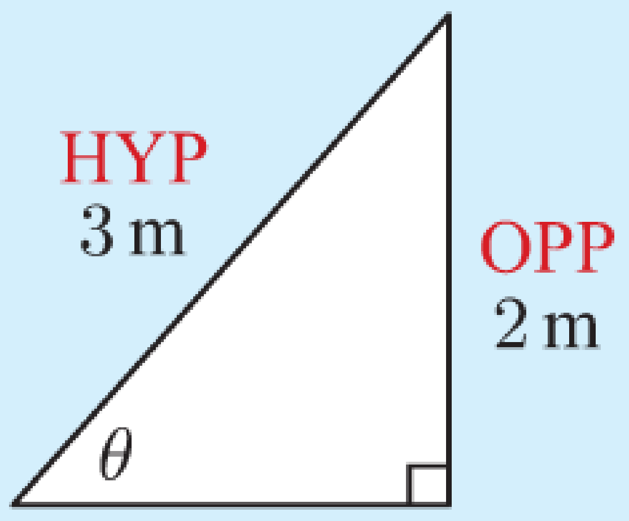
a



b



a

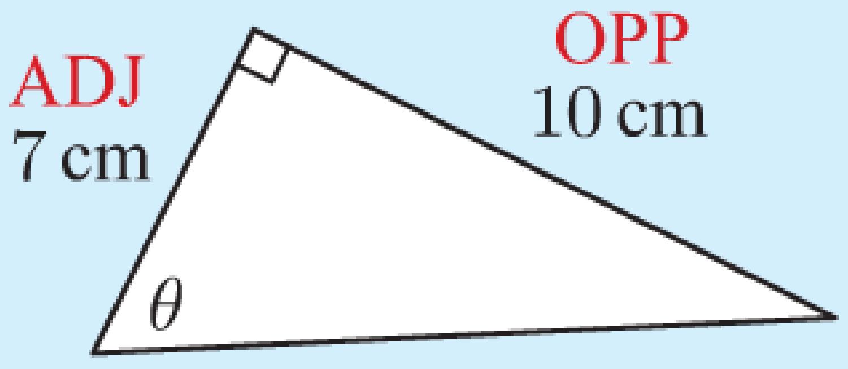


$\sin\theta = \frac{2}{3}$ $\{\sin\theta = \frac{\text{OPP}}{\text{HYP}}\}$

$\therefore \quad \theta = \sin^{-1}\left(\frac{2}{3}\right)$

$\therefore \quad \theta \approx 41.8^\circ$ {using technology}

b



$\tan\theta = \frac{10}{7}$ $\{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$

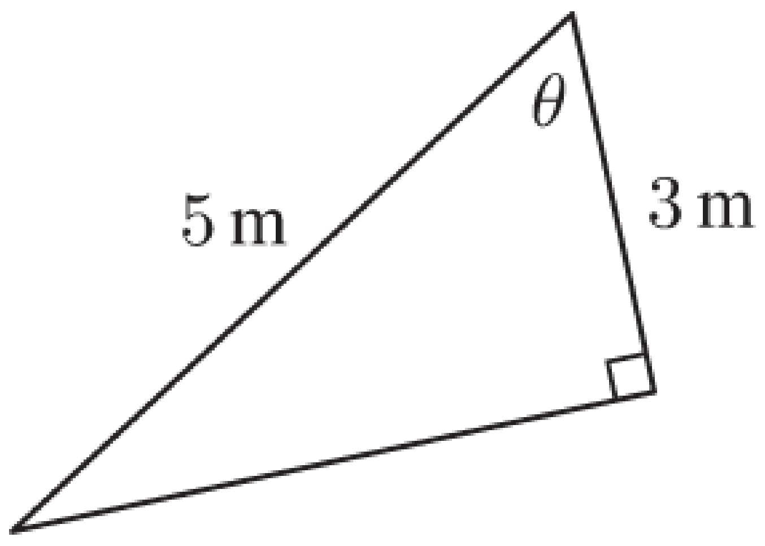
$\therefore \quad \theta = \tan^{-1}\left(\frac{10}{7}\right)$

$\therefore \quad \theta \approx 55.0^\circ$ {using technology}

EXERCISE 7B

1 Find, correct to 3 significant figures, the measure of the angle marked θ:

a



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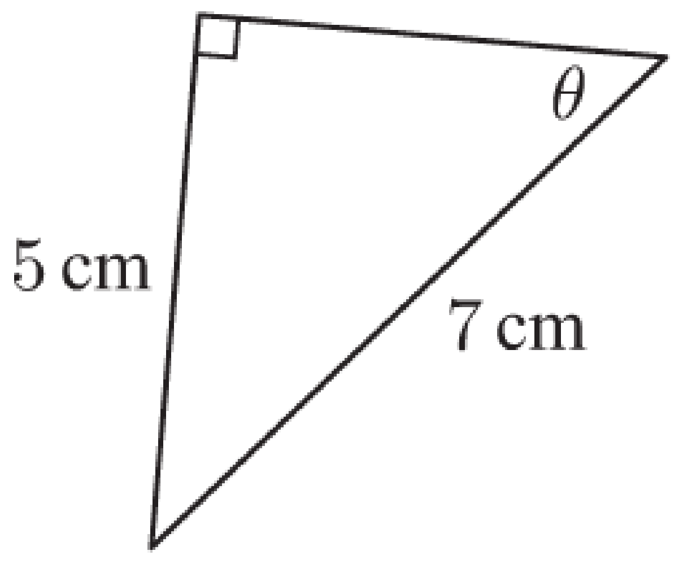

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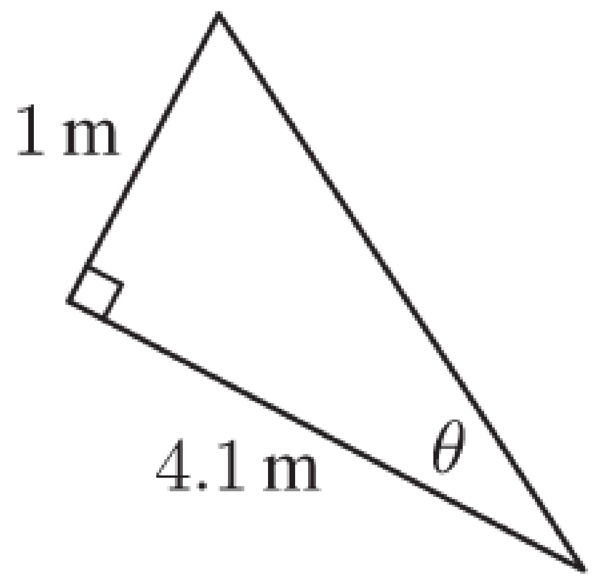
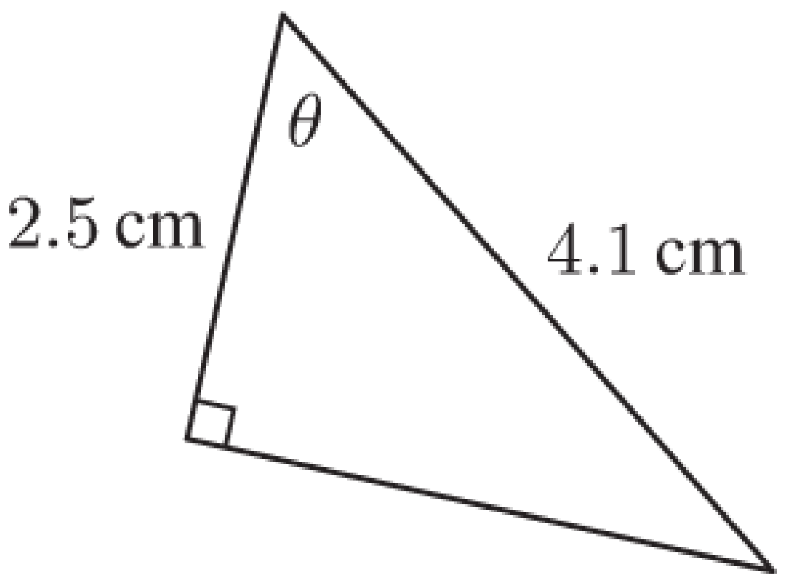
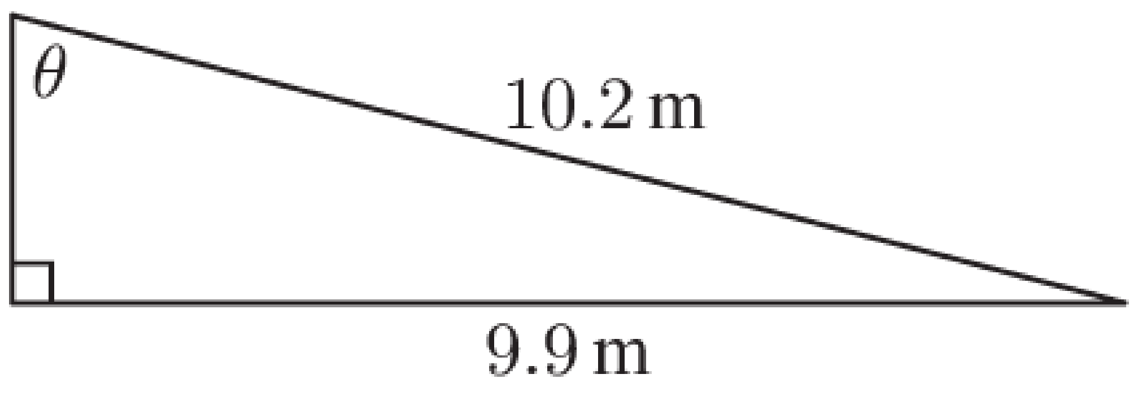


d



e



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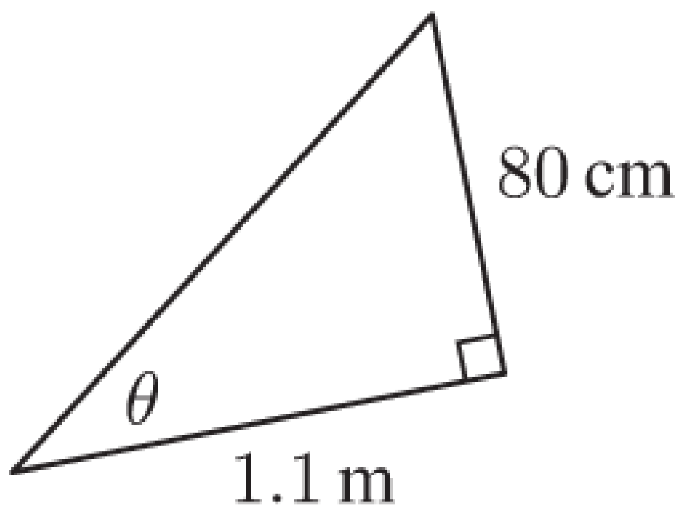


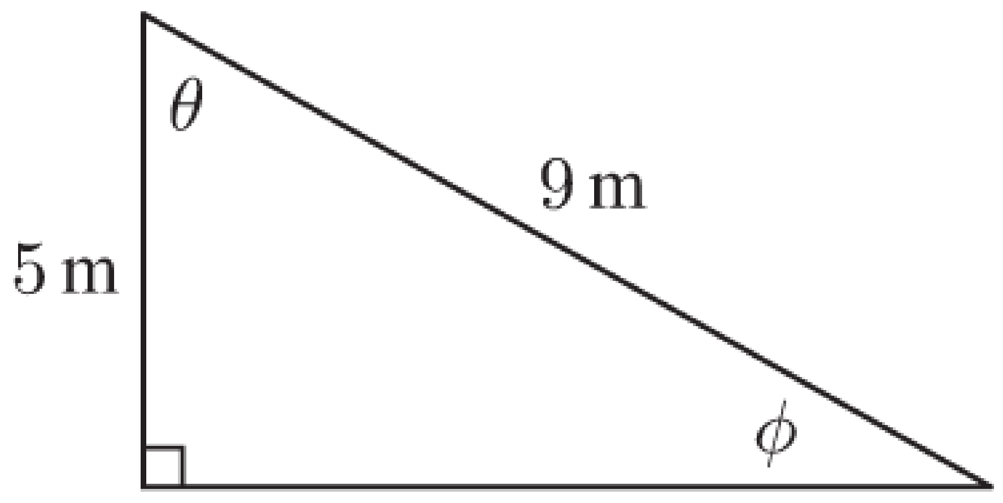
2 Consider the triangle alongside.

a Find θ, correct to 1 decimal place.

b Find ϕ using:

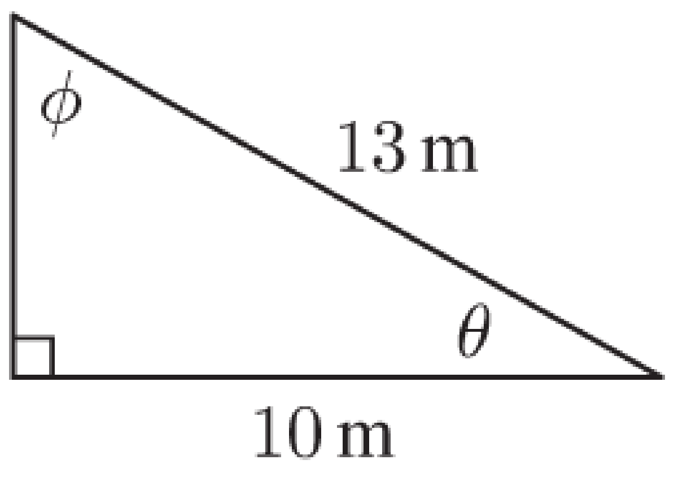
  i the angles in a triangle theorem

  ii trigonometry.

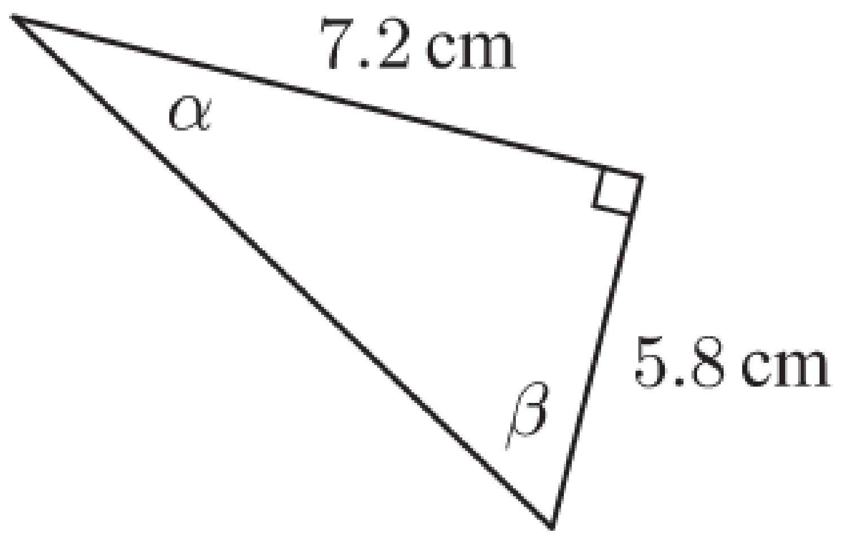


3 Find, correct to 1 decimal place, all unknown angles:

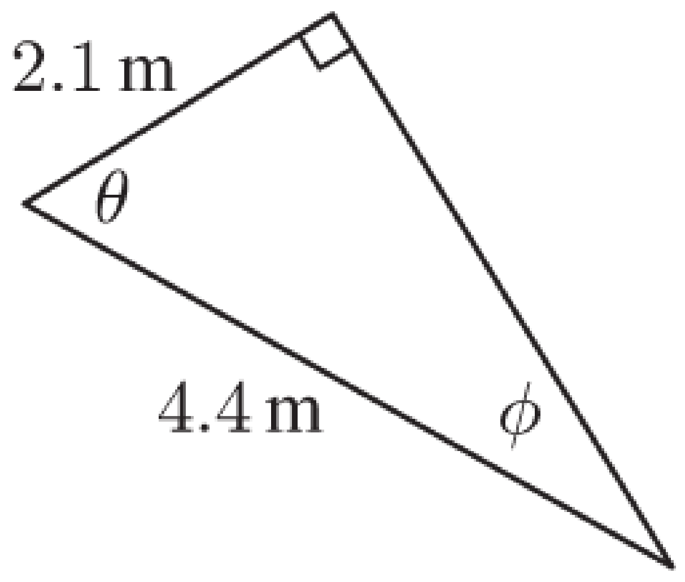
a



b

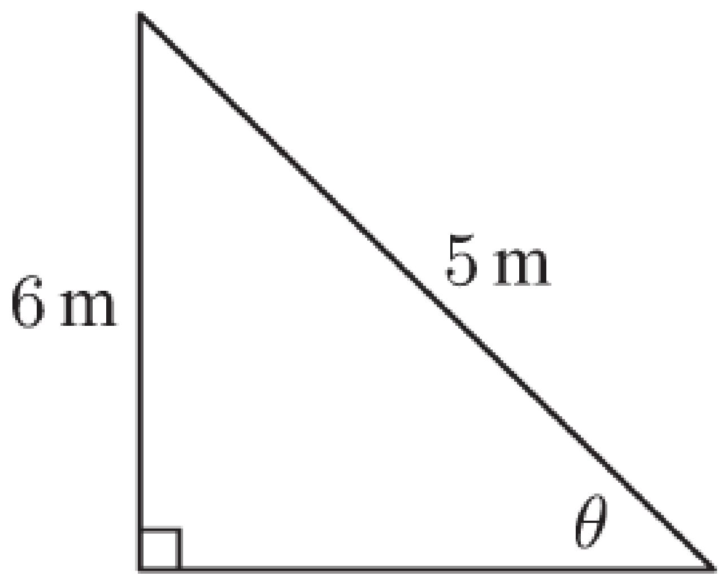


c



4 Try to find θ in the following diagrams. What conclusions can you draw?

a



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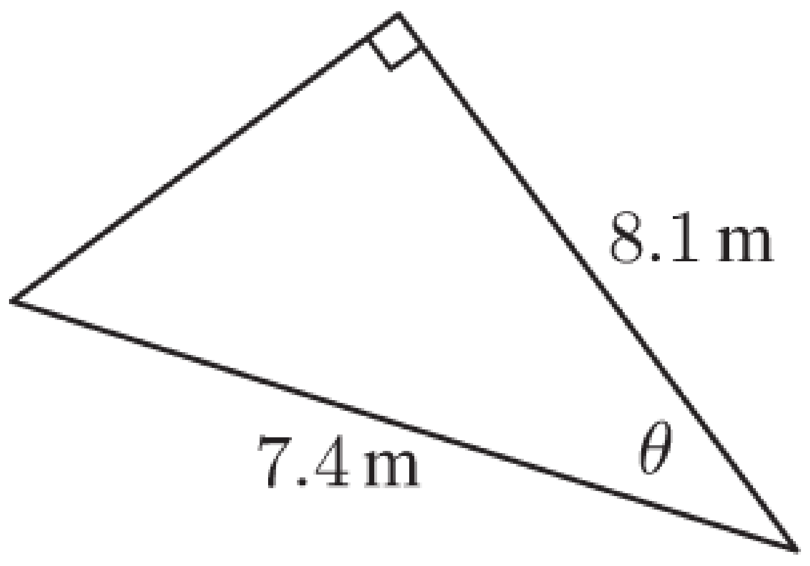

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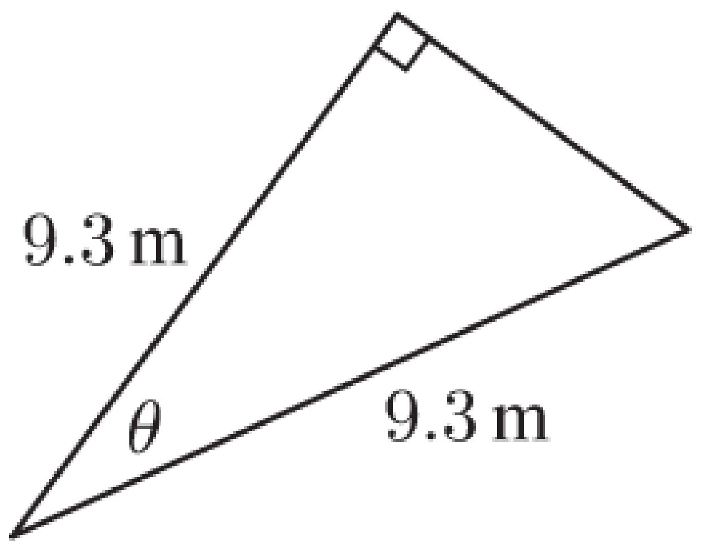
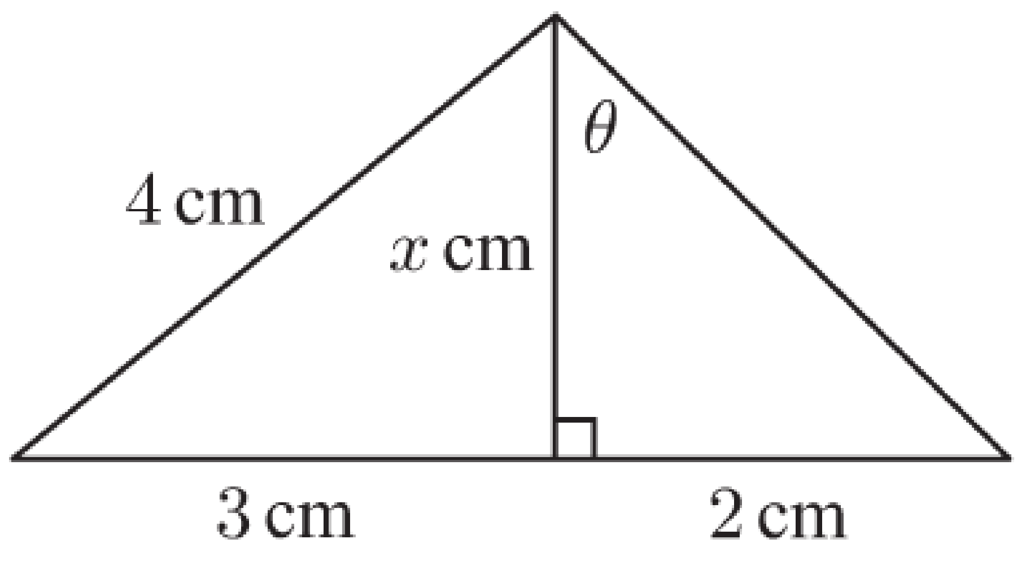


5 Find all unknown sides and angles in these figures:

a



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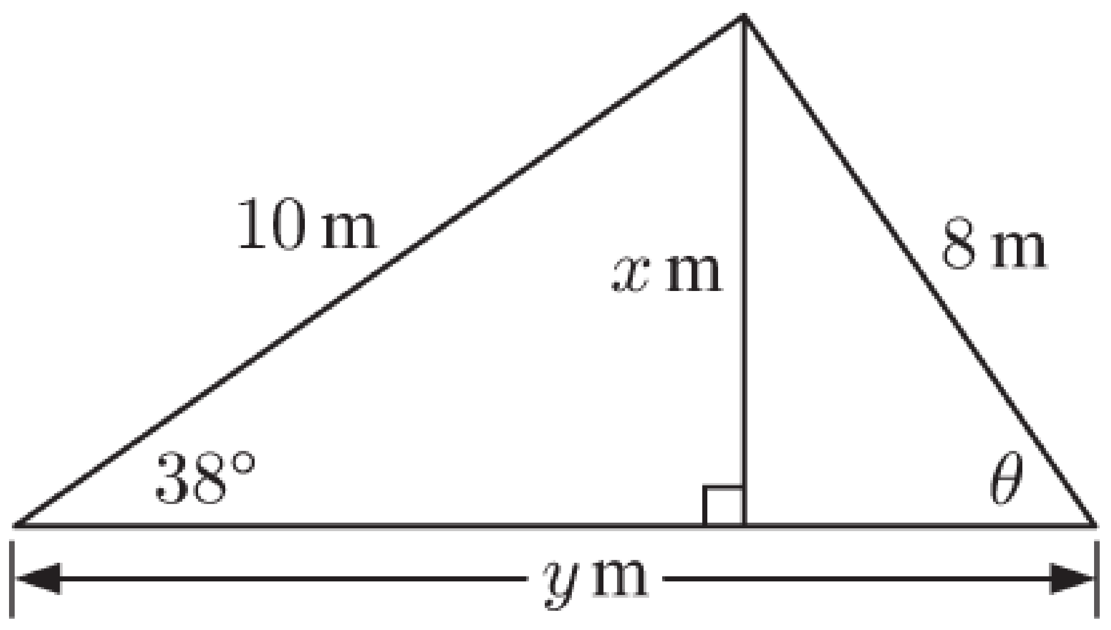
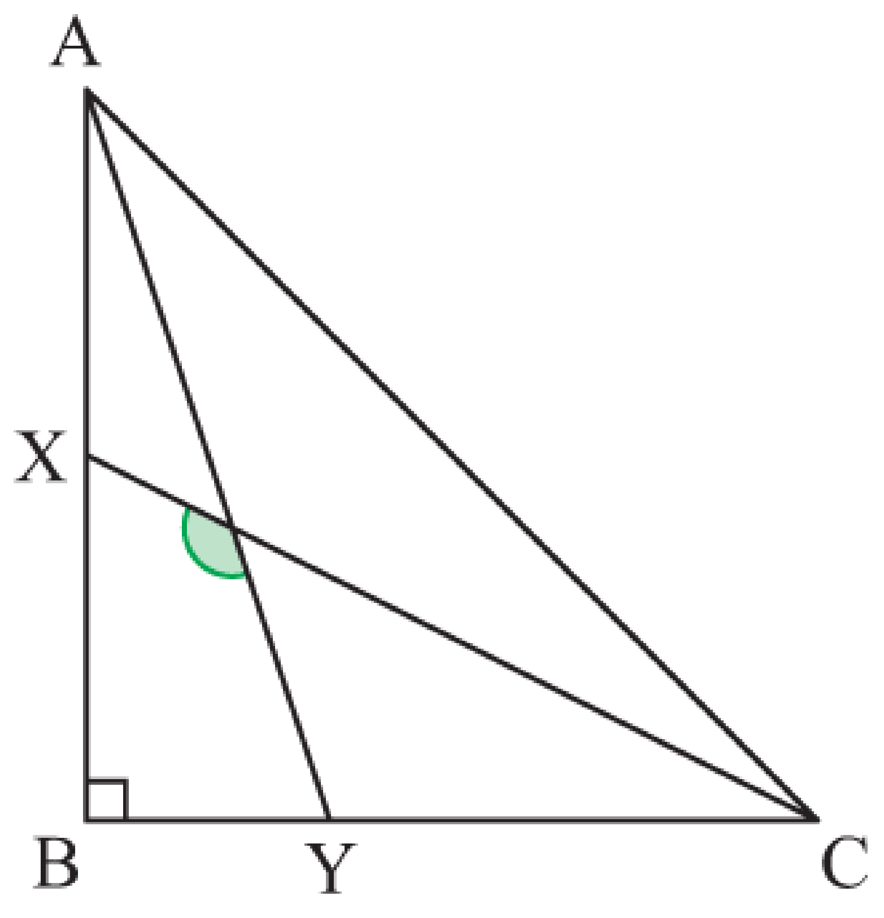
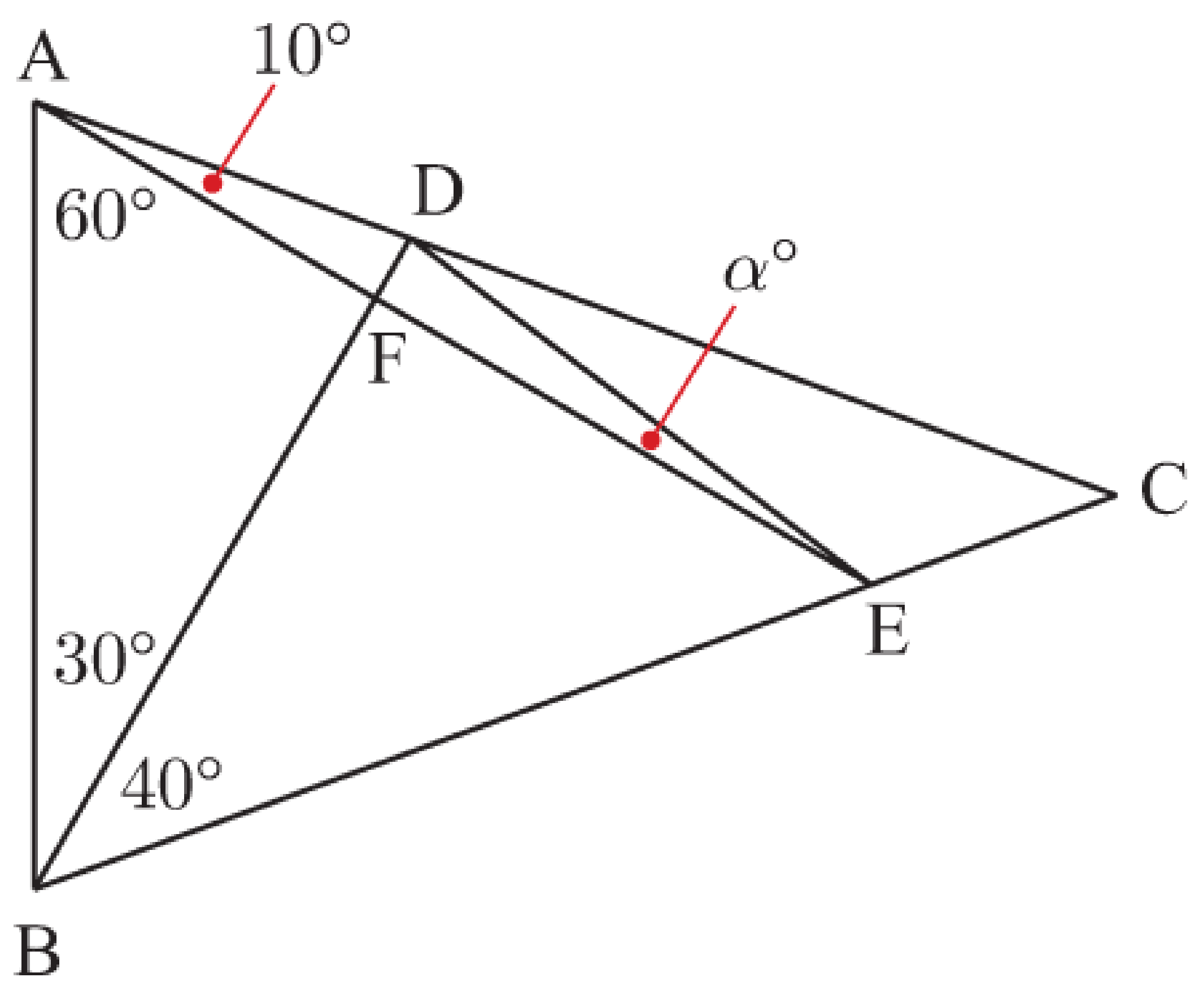


6 ABC is a right angled isosceles triangle. AX = XB and BY : YC = 1 : 2.
Find the measure of the shaded angle.



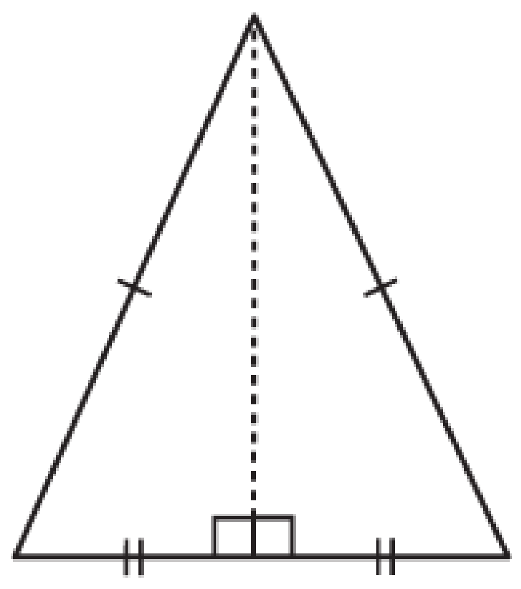
7 Find α:



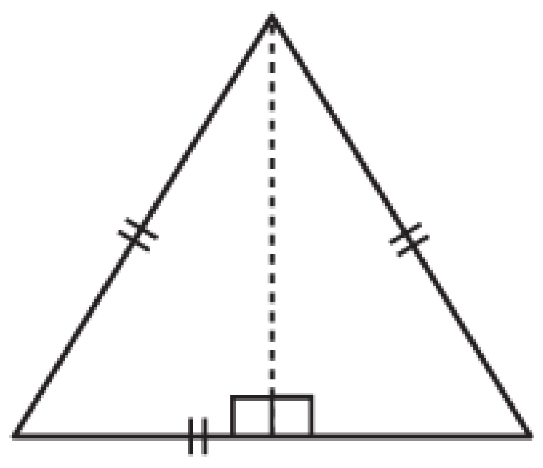
C RIGHT ANGLES IN GEOMETRIC FIGURES

Many geometric figures contain right angles which we can use to help solve problems:

- In an **isosceles triangle** and an **equilateral triangle**, the altitude bisects the base at right angles.

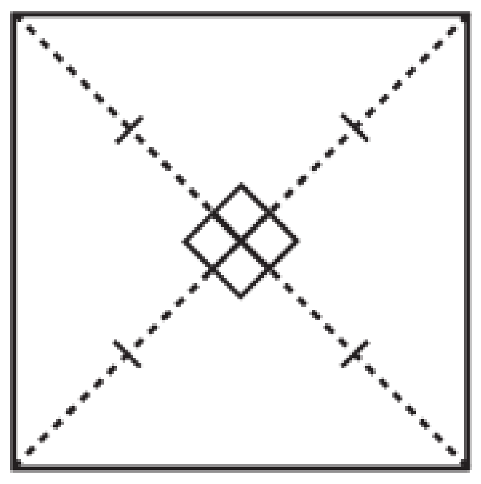


isosceles triangle

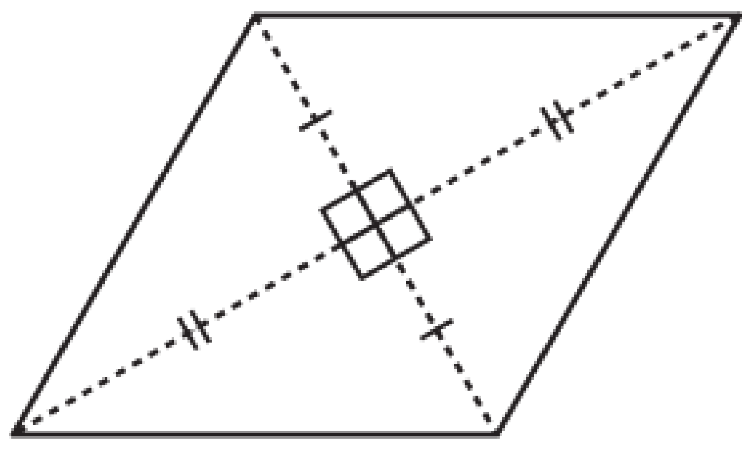


equilateral triangle

- In a **square** and a **rhombus**, the diagonals bisect each other at right angles.

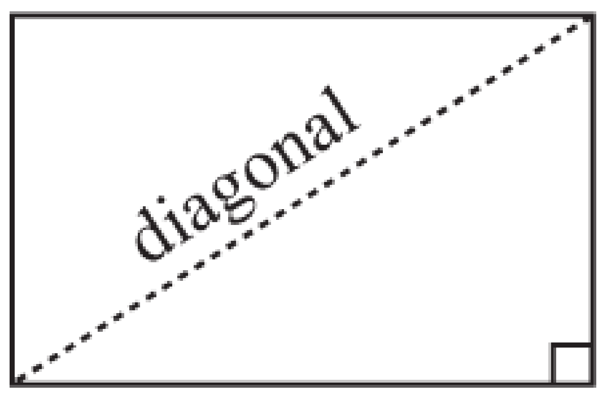


square

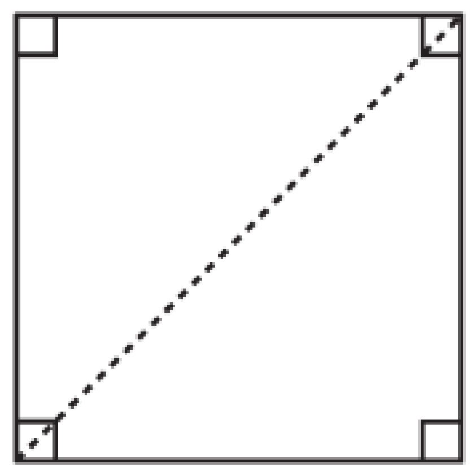


rhombus

- The corners of a **rectangle** and a **square** are right angles. We can construct a diagonal to form a right angled triangle.

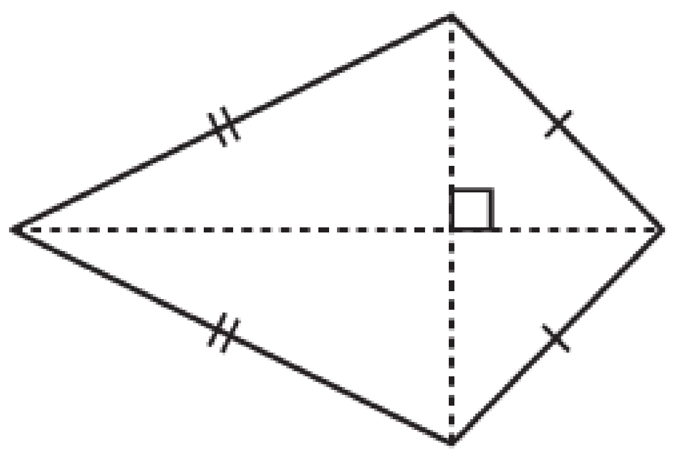


rectangle



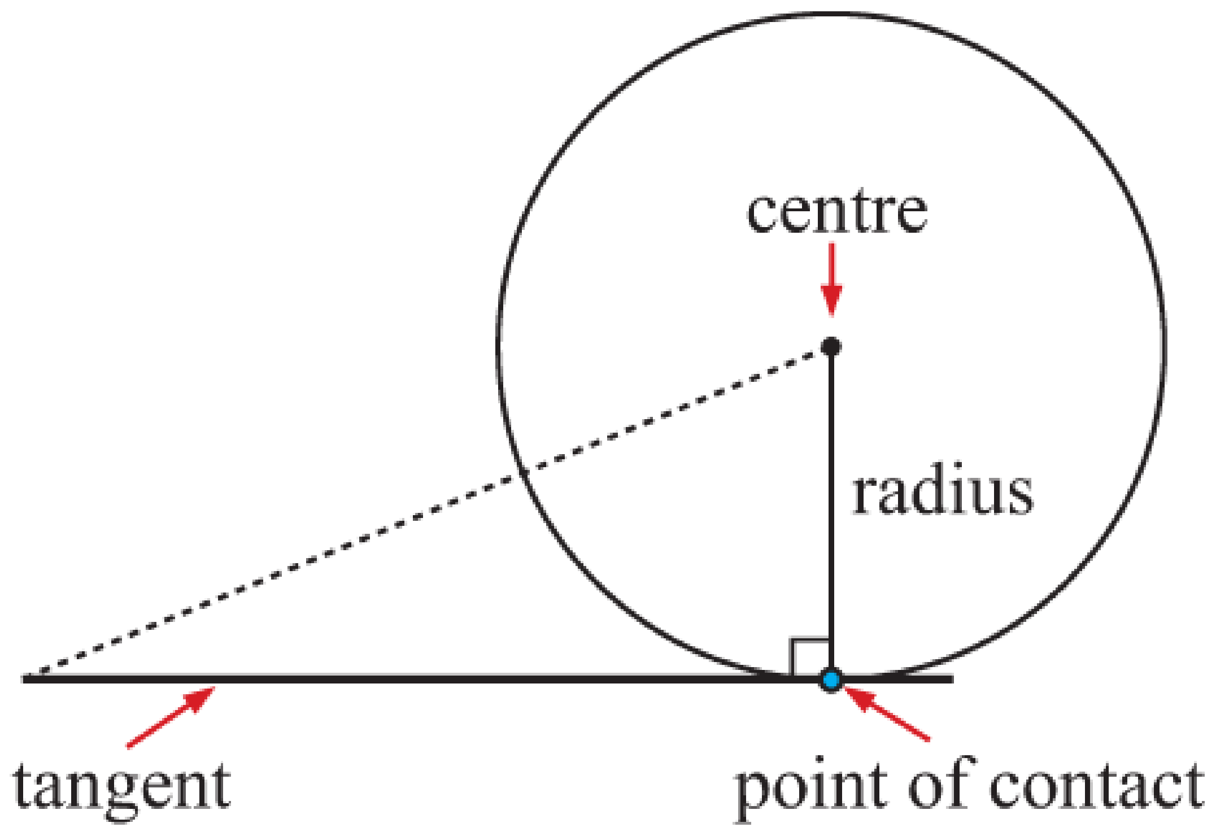
square

- In a **kite**, the diagonals intersect at right angles.

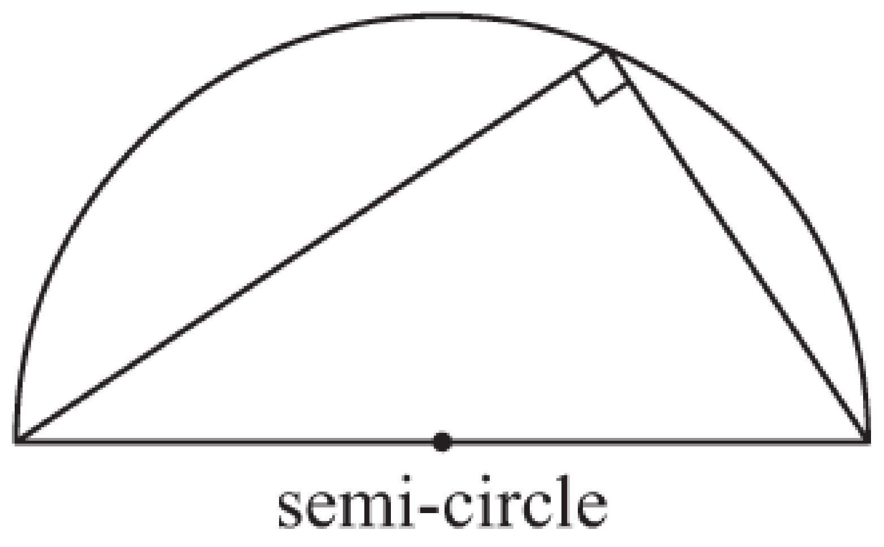


kite

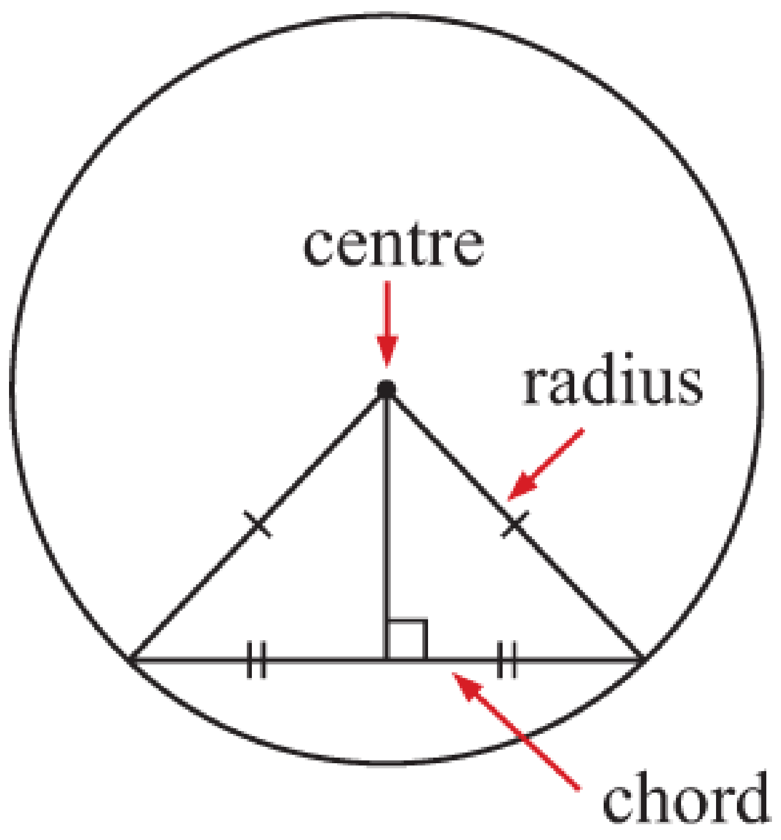
- A **tangent** to a circle and a radius at the point of contact meet at right angles.
  We can form a right angled triangle with another point on the tangent.



- The angle in a **semi-circle** is always a right angle.



- The line drawn from the centre of a circle at right angles to a **chord**, bisects the chord.

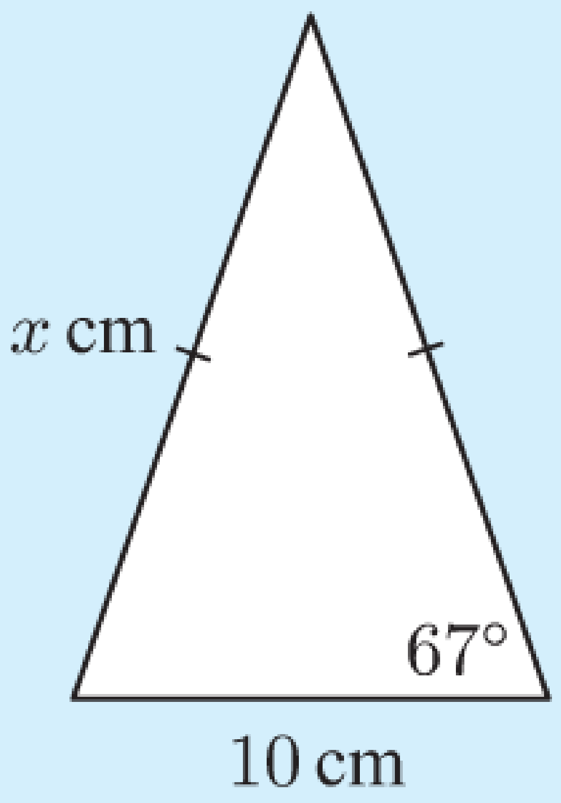


Example 4

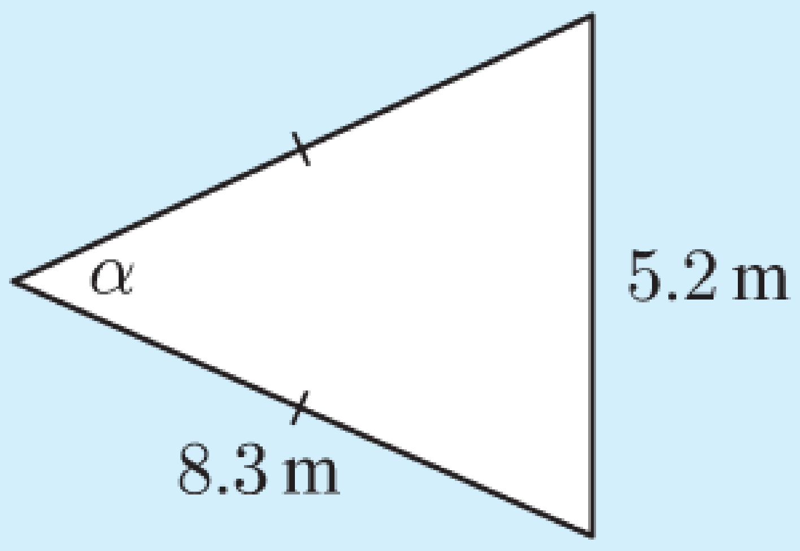
Self Tutor

Find the unknowns:

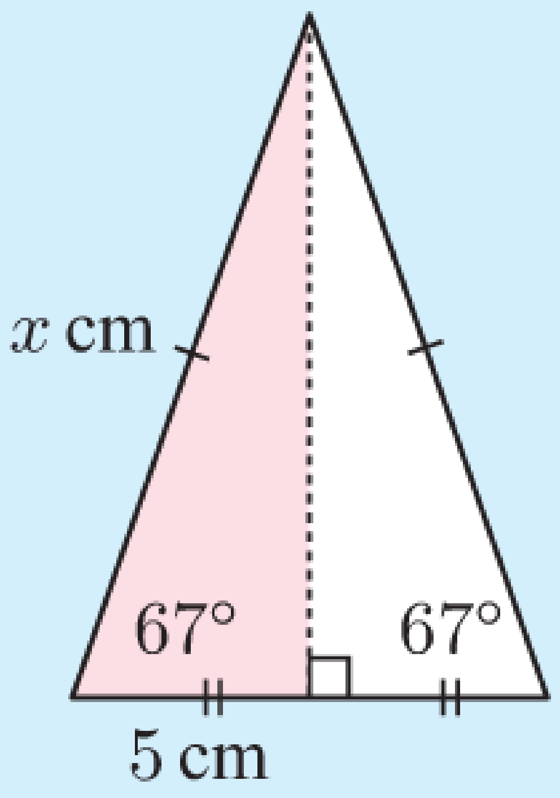
a



b



a

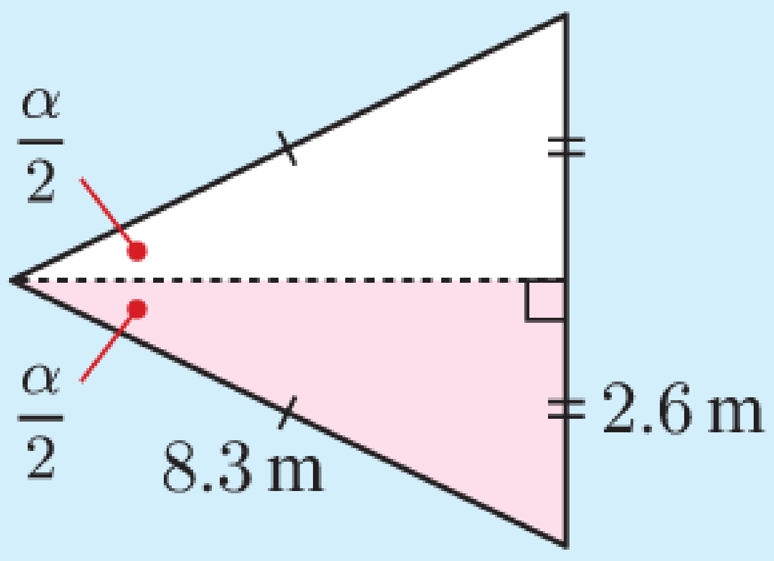


In the shaded right angled triangle,

$$\cos 67^\circ = \frac{5}{x}$$

$$\therefore \quad x = \frac{5}{\cos 67^\circ} \approx 12.8$$

b



In the shaded right angled triangle,

$$\sin \frac{\alpha}{2} = \frac{2.6}{8.3}$$

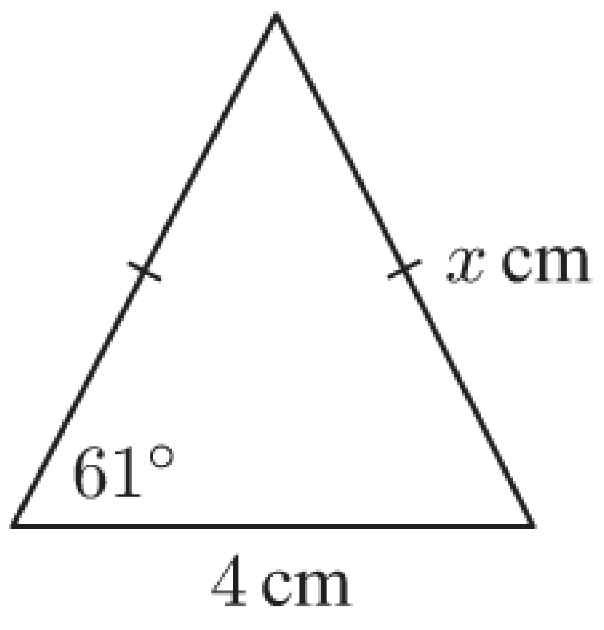
$$\therefore \quad \frac{\alpha}{2} = \sin^{-1}\left(\frac{2.6}{8.3}\right)$$

$$\therefore \quad \alpha = 2\sin^{-1}\left(\frac{2.6}{8.3}\right) \approx 36.5^\circ$$

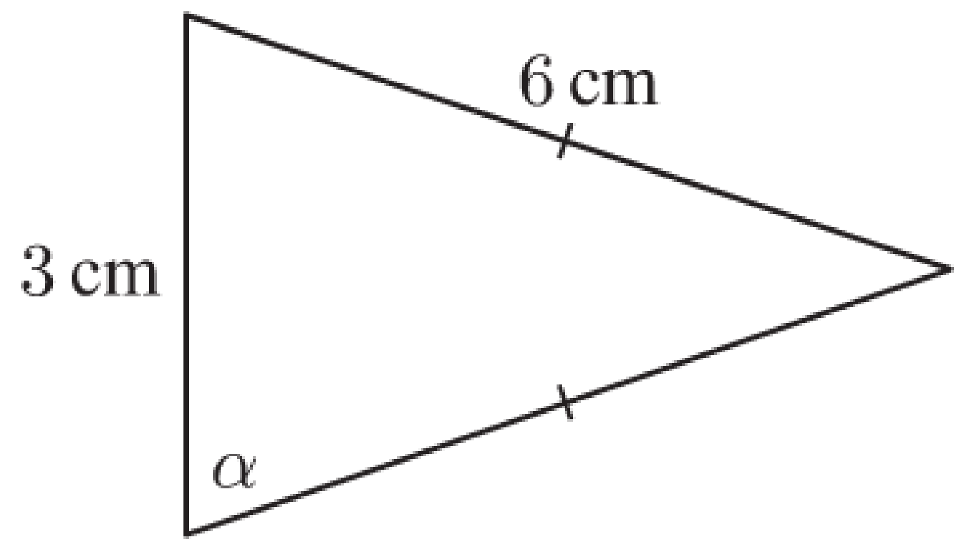
EXERCISE 7C

1 Find the unknown, correct to 3 significant figures:

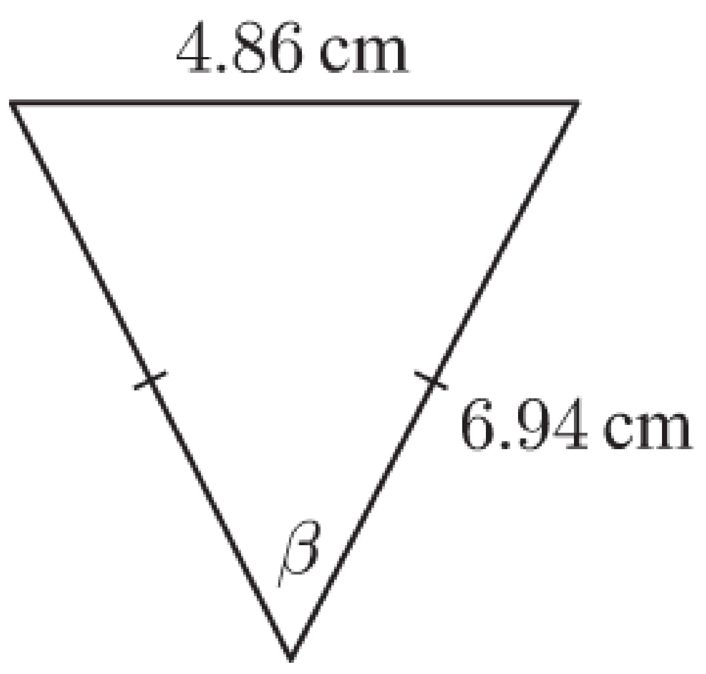
a



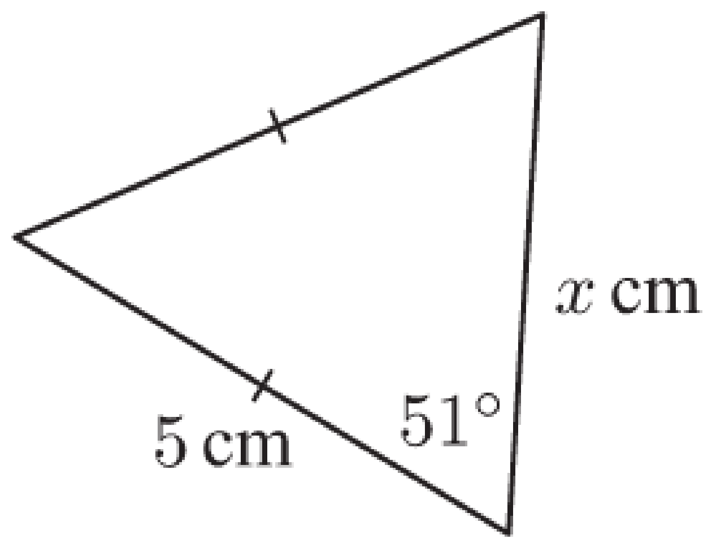
b



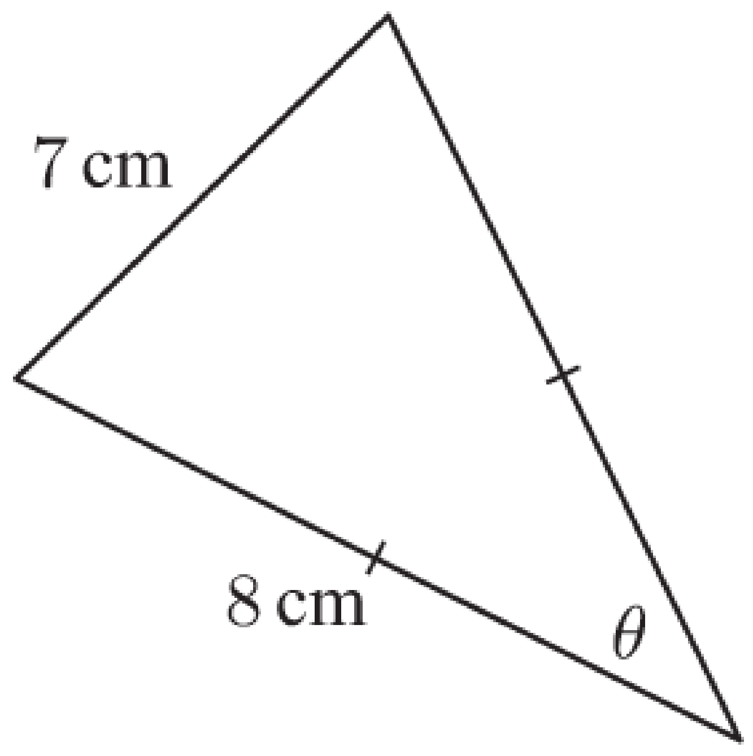
c



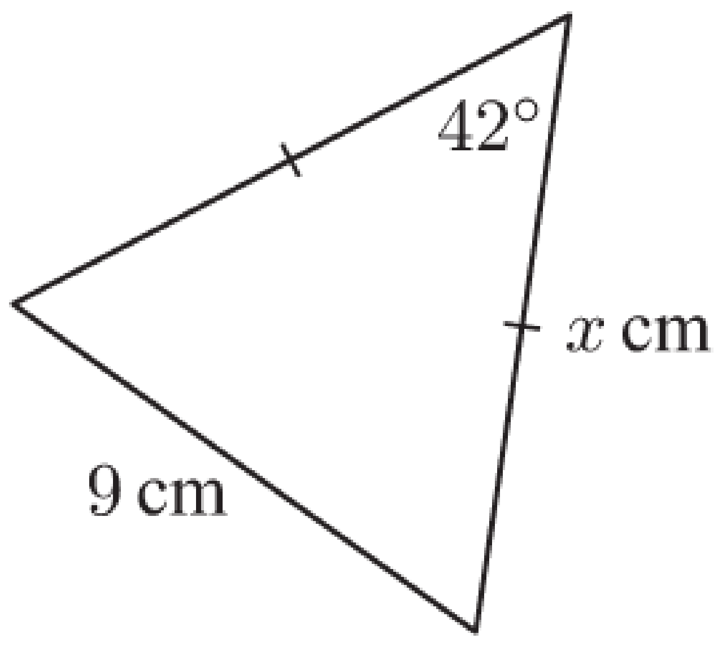
d



e



f

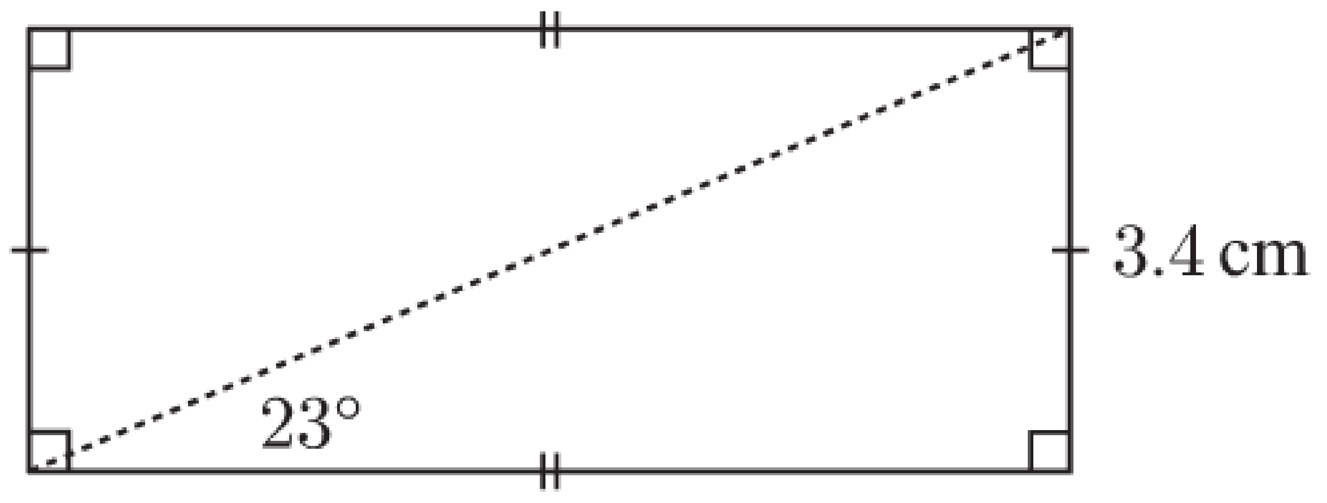


2 A rectangle is 9.2 m by 3.8 m. What angle does its diagonal make with its longer side?

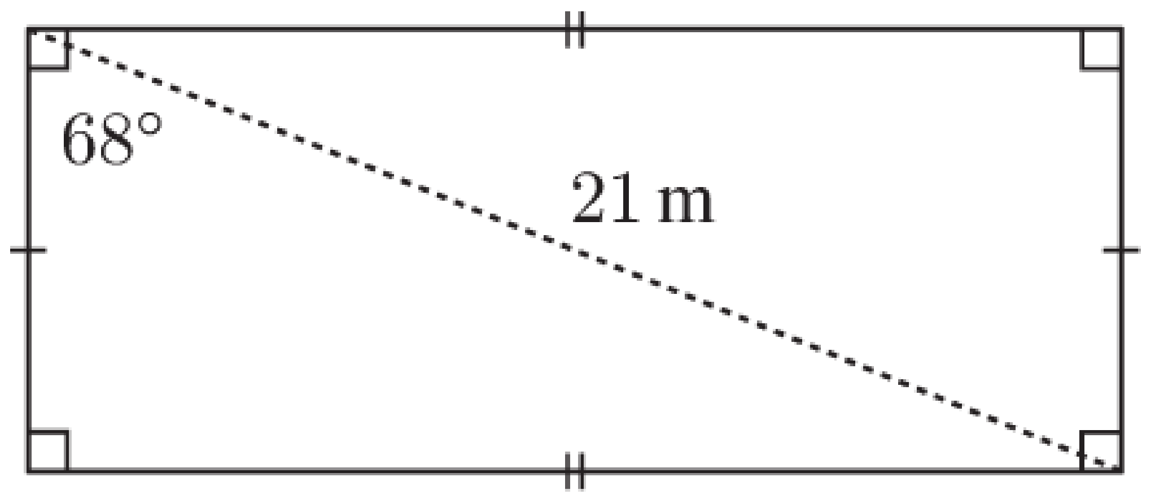
3 The diagonal and the longer side of a rectangle make an angle of 43.2°. If the longer side is 12.6 cm, find the length of the shorter side.

4 Find the area of the rectangle:

a



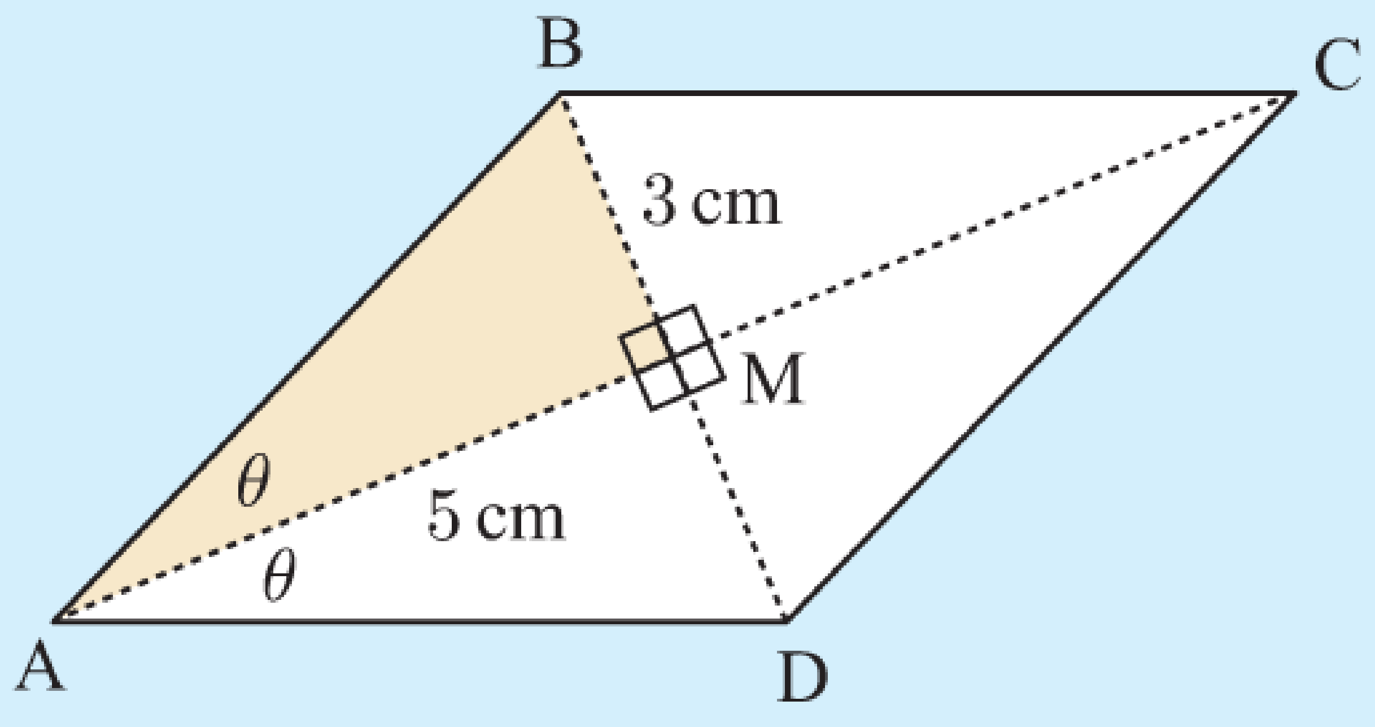
b



Example 5

Self Tutor

A rhombus has diagonals of length 10 cm and 6 cm. Find the smaller angle of the rhombus.



The diagonals bisect each other at right angles, so AM = 5 cm and BM = 3 cm.

In $\triangle$ABM, θ will be the smallest angle as it is opposite the shortest side.

$$\tan\theta = \tfrac{3}{5}$$
$$\therefore\ \theta = \tan^{-1}\left(\tfrac{3}{5}\right)$$
$$\therefore\ \theta \approx 30.964°$$

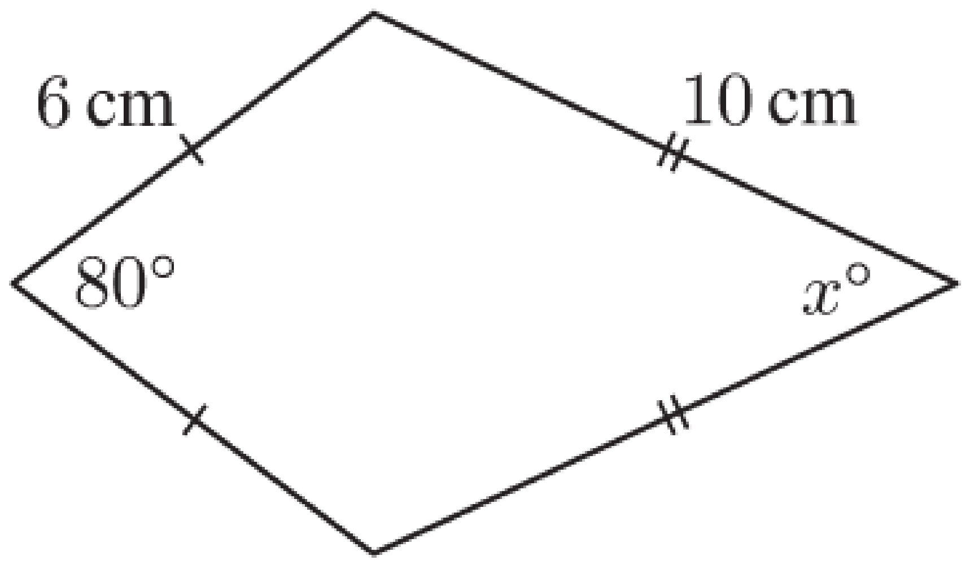
The required angle is 2θ as the diagonals bisect the angles at each vertex.
So, the angle is about 61.9°.

5 A rhombus has diagonals of length 12 cm and 7 cm. Find the larger angle of the rhombus.

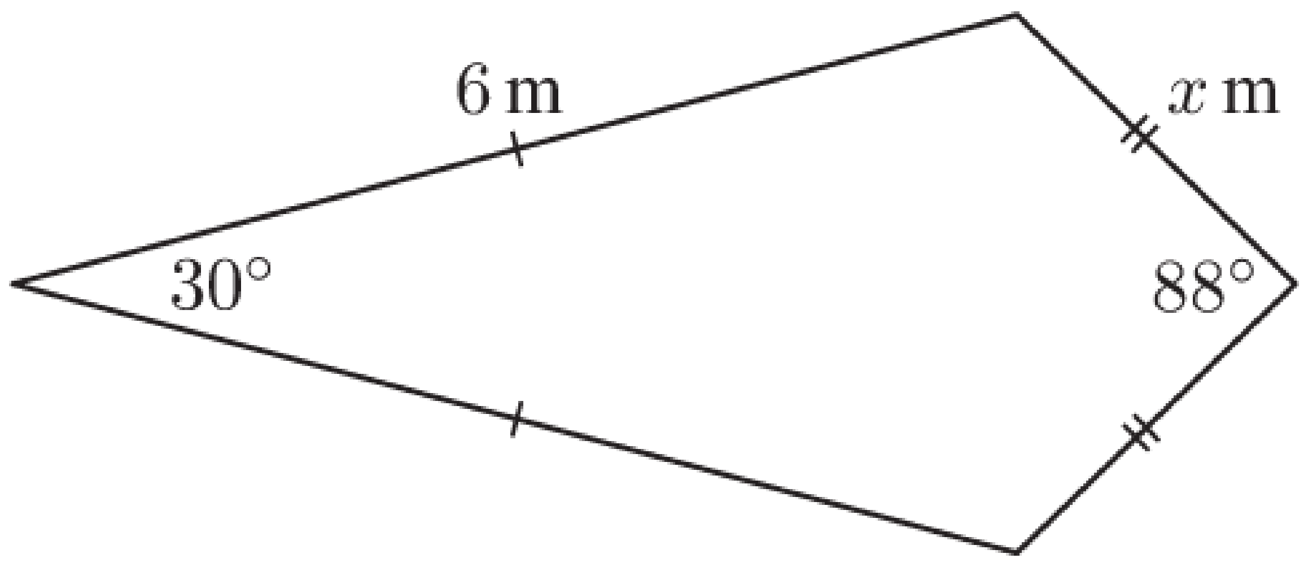
6 The smaller angle of a rhombus measures 21.8° and the shorter diagonal has length 13.8 cm. Find the lengths of the sides of the rhombus.

7 Find the value of x:

a

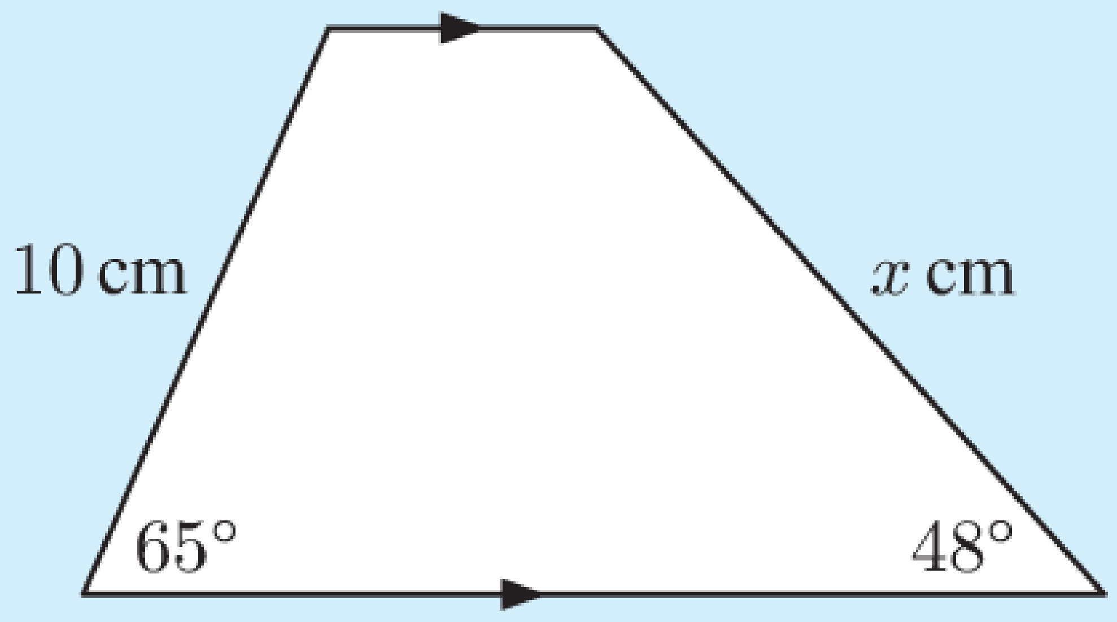


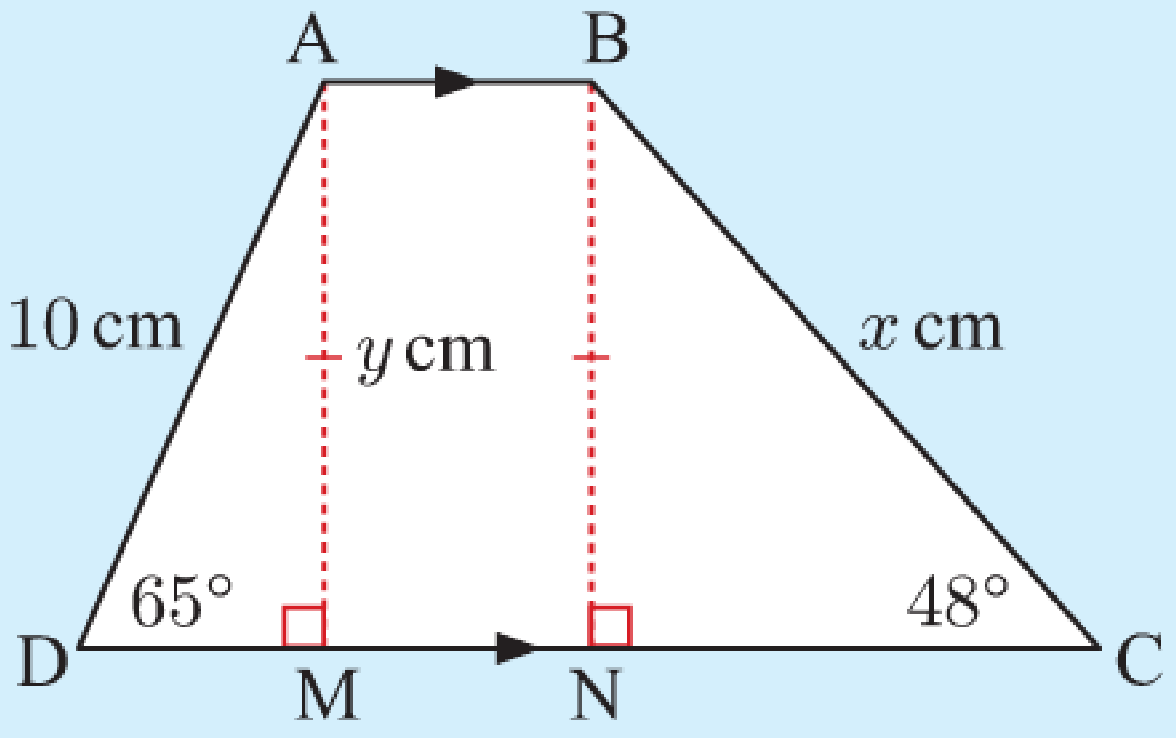
b



Example 6 **Self Tutor**

Find x:





We draw perpendiculars [AM] and [BN] to [DC], creating right angled triangles and the rectangle ABNM.

In $\triangle$ADM, $\sin 65° = \frac{y}{10}$

$\therefore\ y = 10\sin 65°$

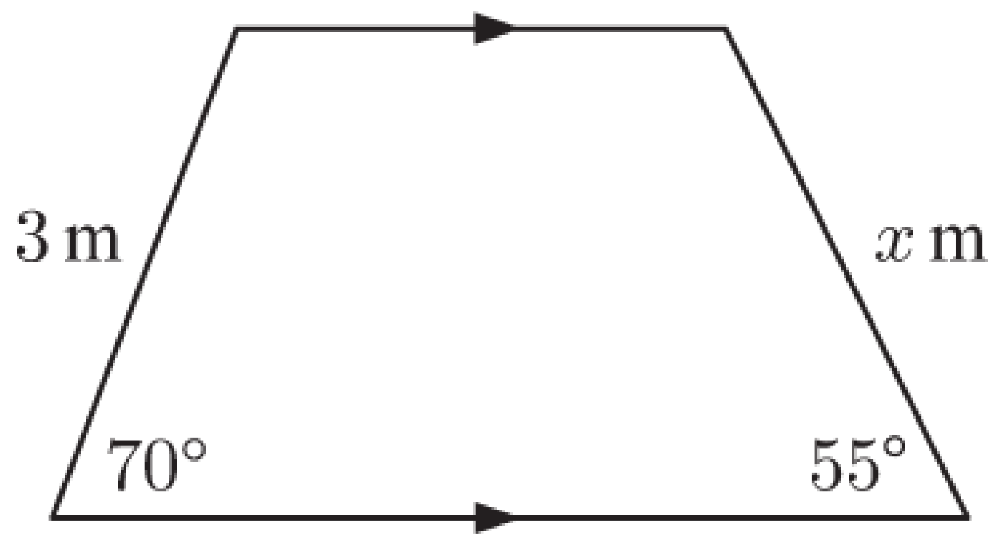
In $\triangle$BCN, $\sin 48° = \frac{y}{x}$

$= \frac{10\sin 65°}{x}$

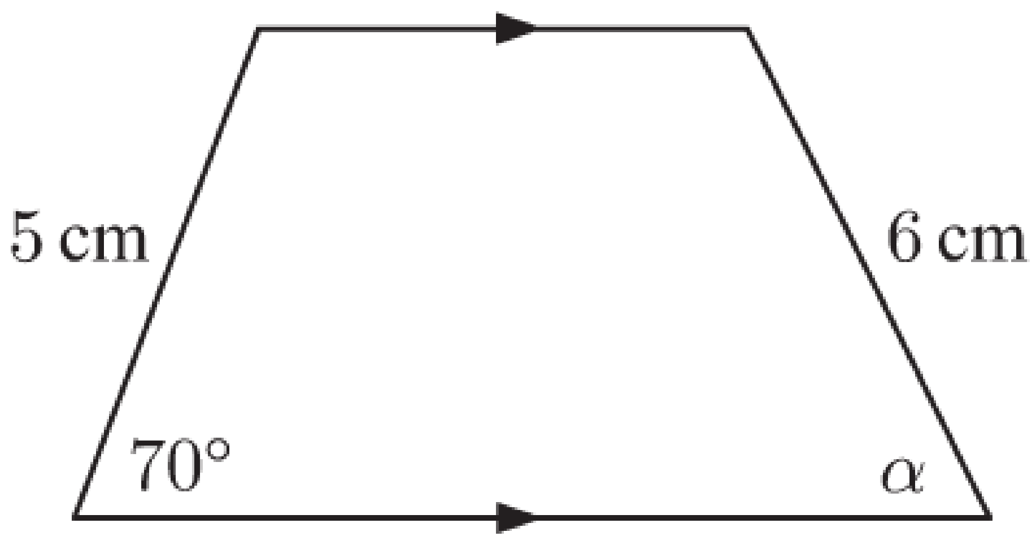
$\therefore\ x = \frac{10\sin 65°}{\sin 48°} \approx 12.2$

8 Find the unknown value:

a

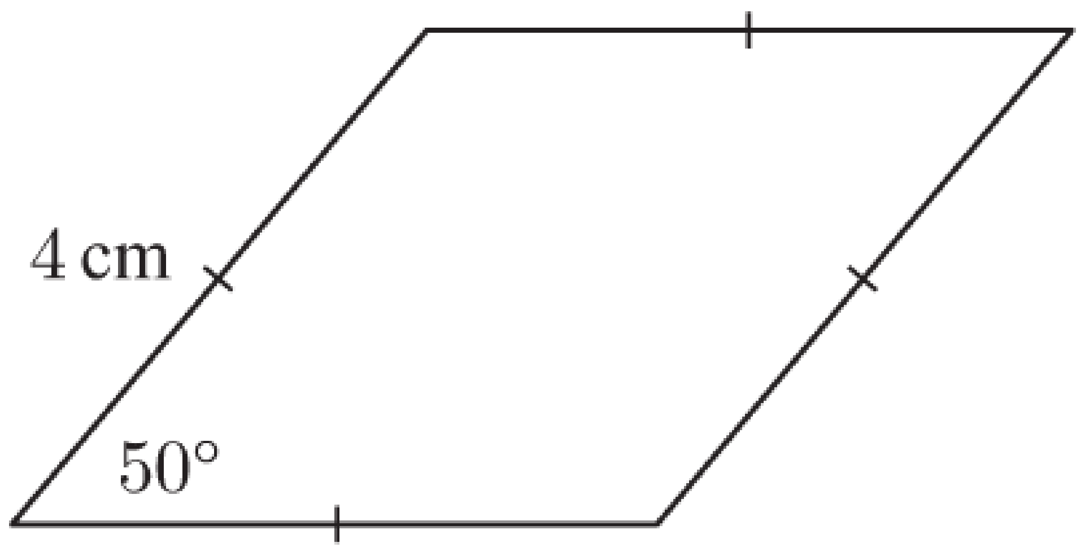


b



9 Find the area of the figure:

a



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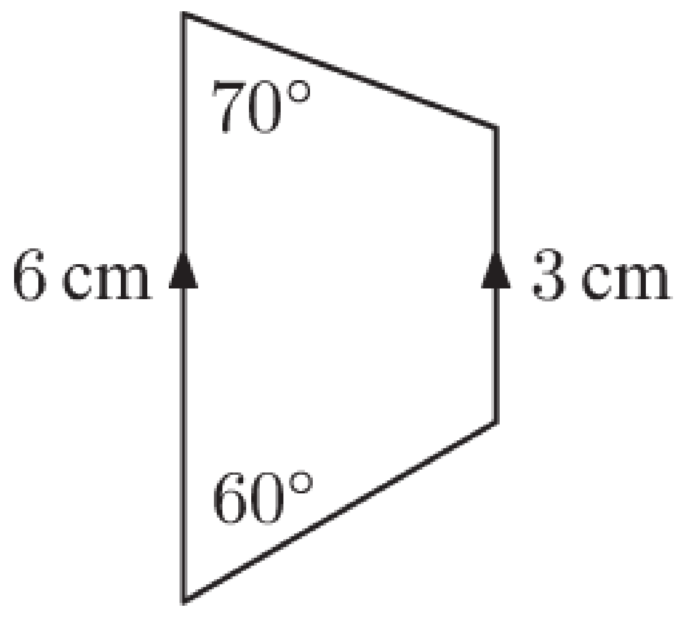


10 A rhombus has sides of length 8 metres, and the longer diagonal has length 13 metres.

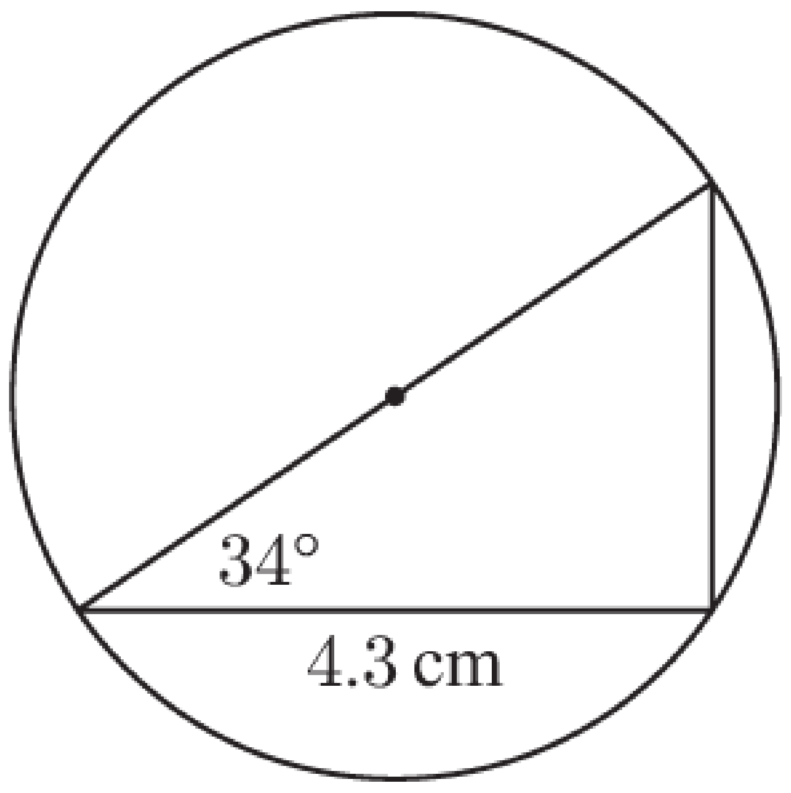
a Draw a diagram and label it with the given information.

b Find the length of the shorter diagonal of the rhombus.

c Find the measure of the smaller angle in the rhombus.

11 Find the radius of the circle:

a



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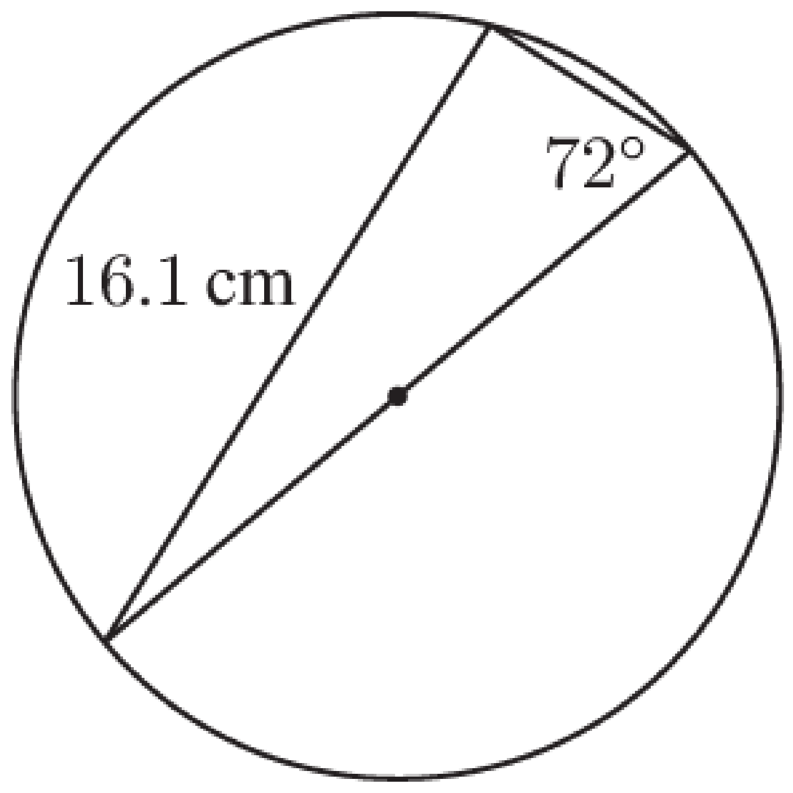
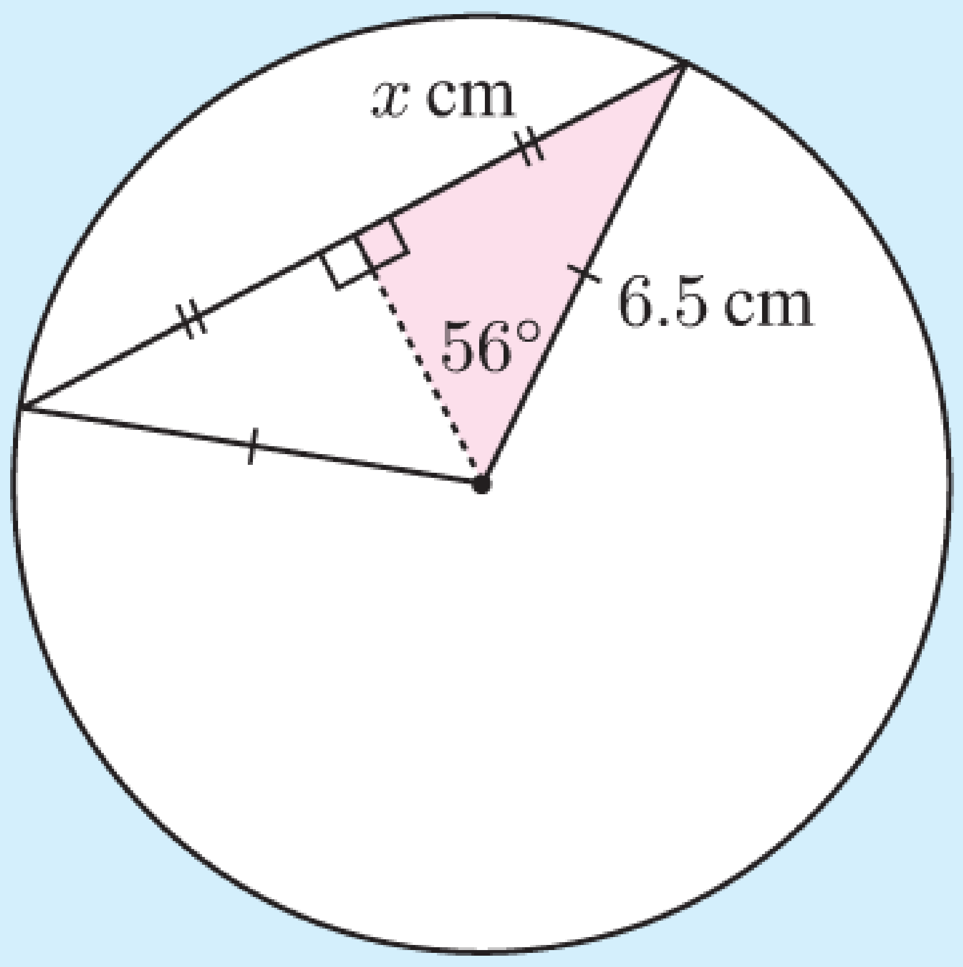


Example 7 **Self Tutor**

A circle has radius 6.5 cm. A chord of the circle subtends an angle of 112° at its centre. Find the length of the chord.



We complete an isosceles triangle and draw the line from the apex to the base.

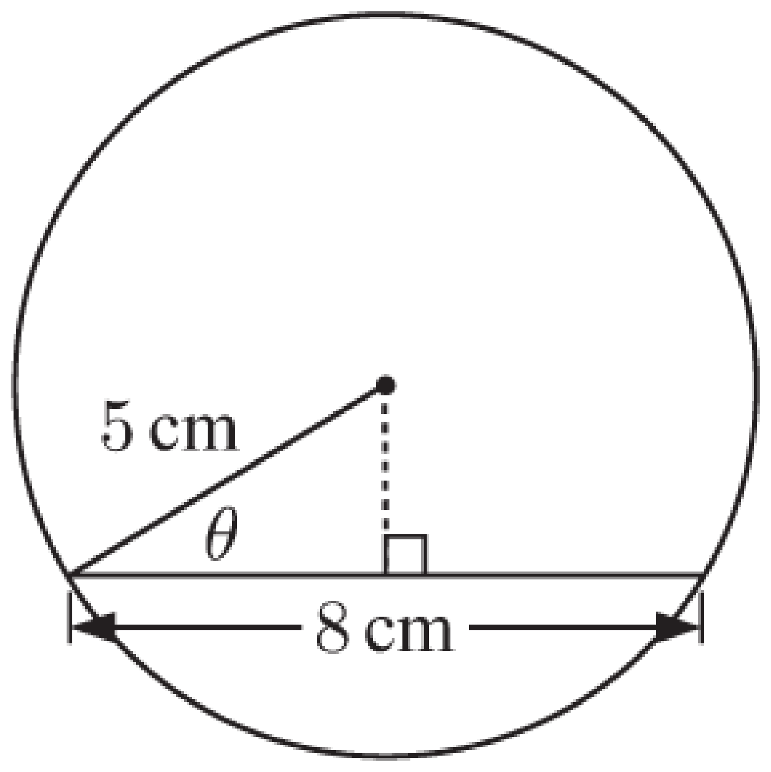
For the shaded triangle, $\sin 56^\circ = \dfrac{x}{6.5}$

$$\therefore \quad 6.5 \times \sin 56^\circ = x$$
$$\therefore \quad x \approx 5.389$$
$$\therefore \quad 2x \approx 10.78$$

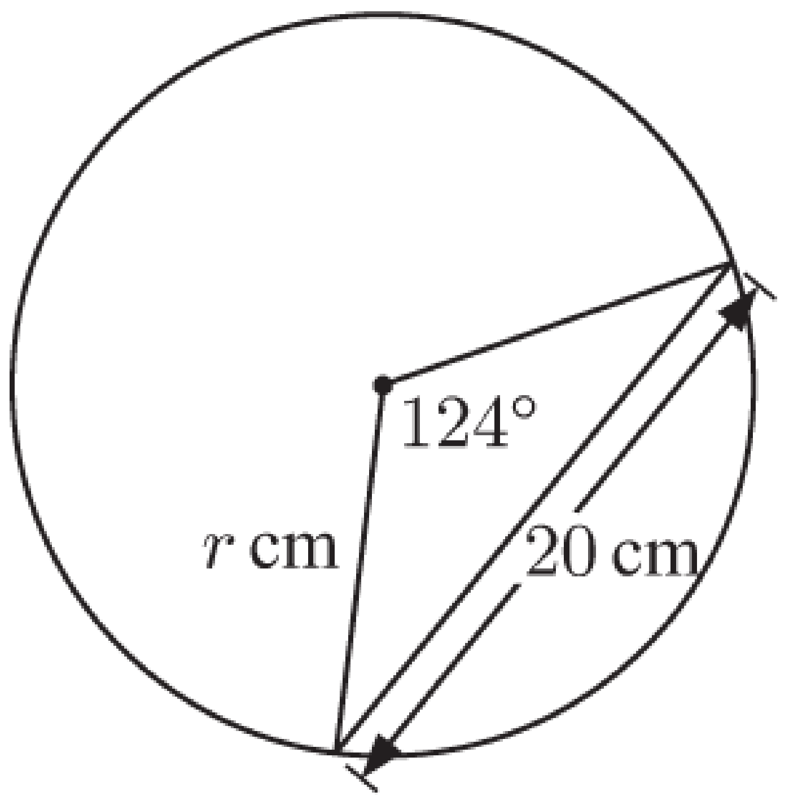
$\therefore$ the chord is about 10.8 cm long.

12 Find the value of the unknown:

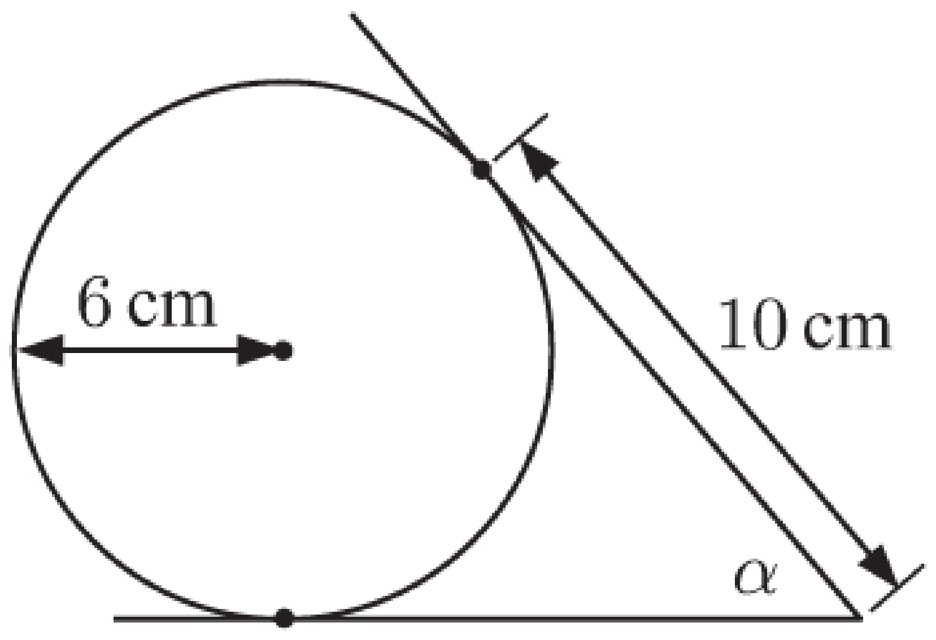
a



b



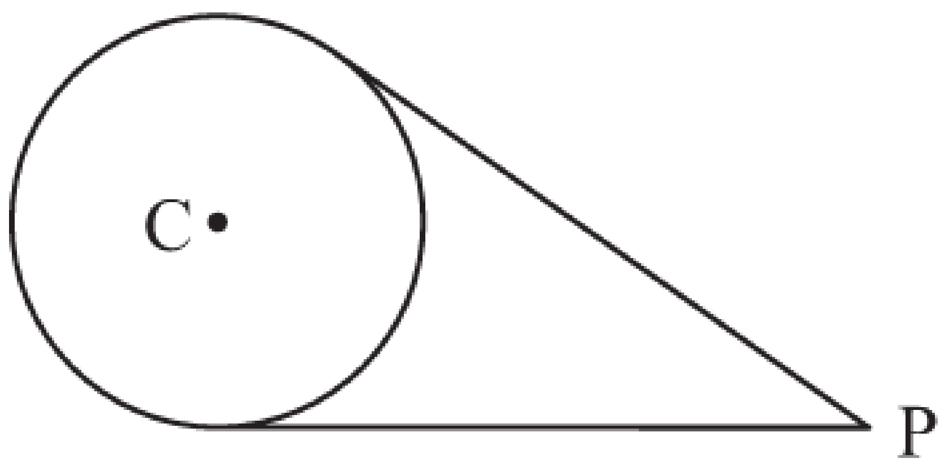
c



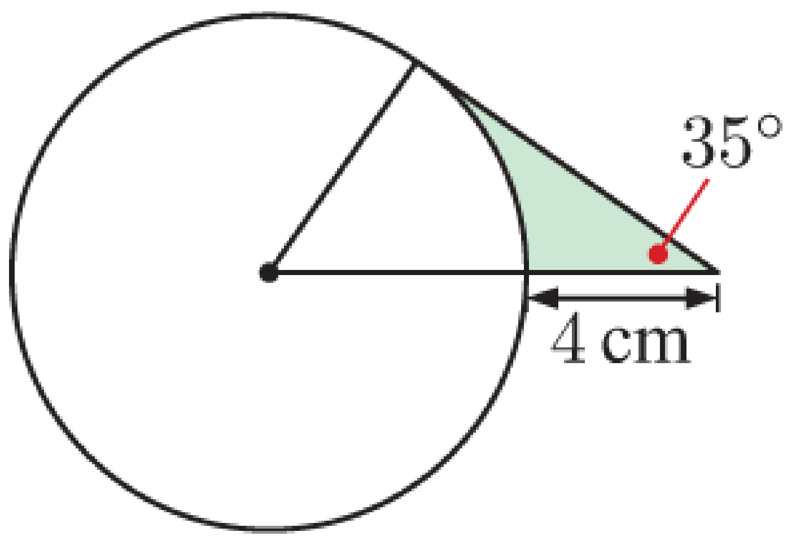
13 A circle has diameter 11.4 cm. A chord of the circle subtends an angle of 89° at its centre. Find the length of the chord.

14 A chord of a circle is 13.2 cm long and the circle's radius is 9.4 cm. Find the angle subtended by the chord at the centre of the circle.

15 PC = 10 cm, and the circle has radius 4 cm. Find the angle between the tangents.



16 Find the shaded area.



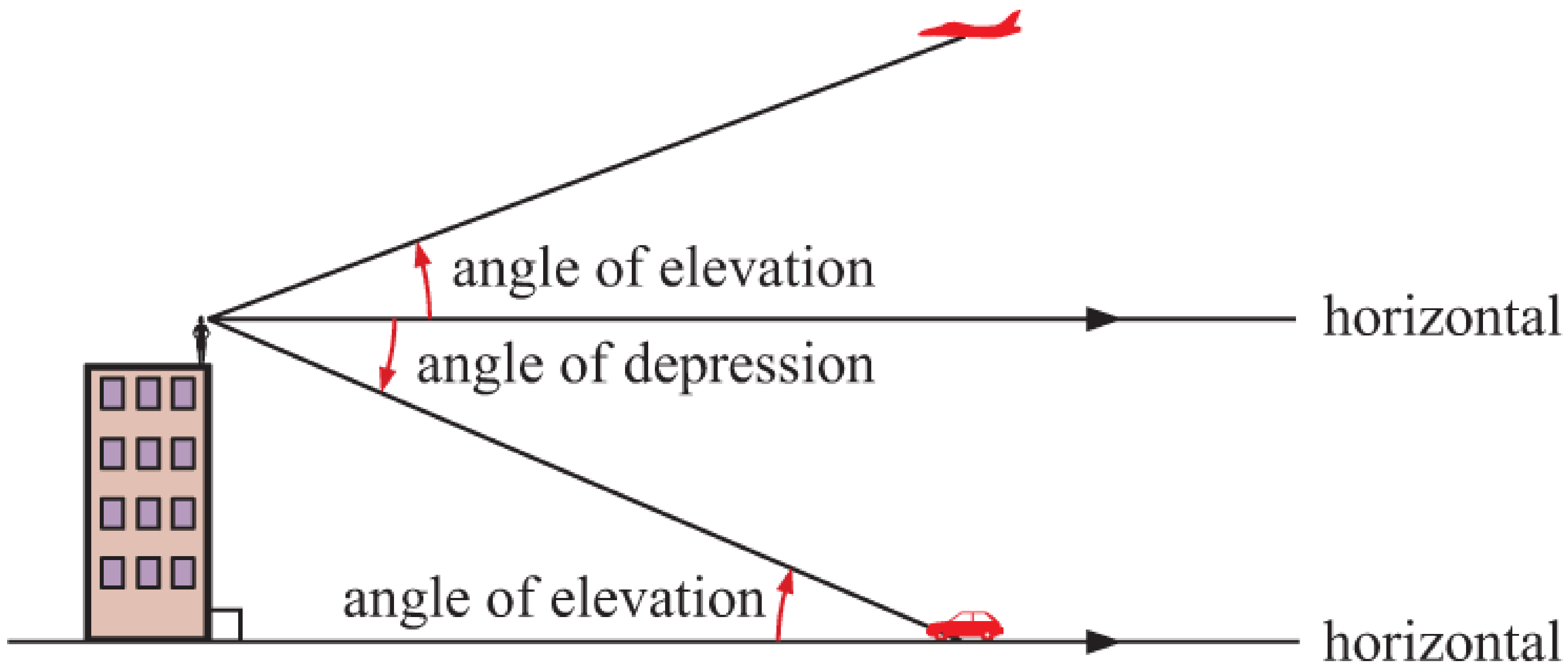
D PROBLEM SOLVING WITH TRIGONOMETRY

In this Section we consider practical applications of trigonometry. It allows us to find heights and distances which are very difficult or even impossible to measure directly.

ANGLES OF ELEVATION AND DEPRESSION

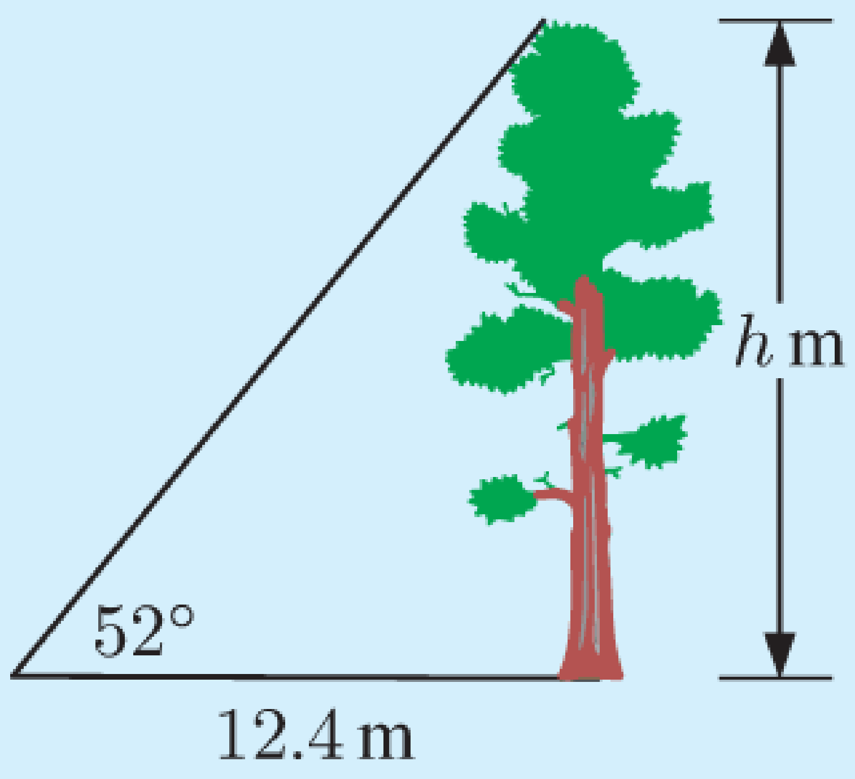
The angle between the horizontal and your line of sight to an object is called:

- the **angle of elevation** if you are looking upwards
- the **angle of depression** if you are looking downwards.



Example 8 **Self Tutor**

A tree casts a shadow of 12.4 m when the angle of elevation to the sun is 52°. Find the height of the tree.



Let the tree's height be h m.

For the 52° angle, OPP $= h$ m, ADJ $= 12.4$ m

$$\therefore \quad \tan 52^\circ = \frac{h}{12.4}$$

$$\therefore \quad 12.4 \times \tan 52^\circ = h$$

$$\therefore \quad h \approx 15.9$$

$\therefore$ the tree is about 15.9 m high.

EXERCISE 7D

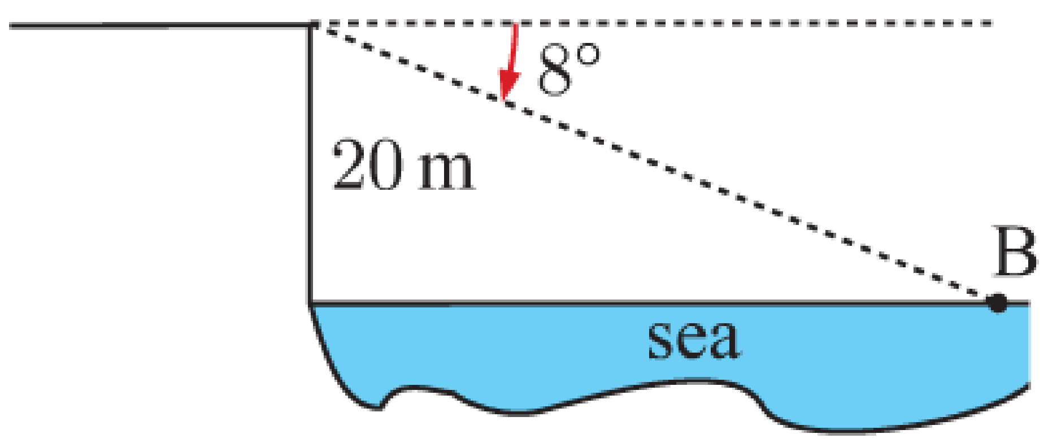
1 A flagpole casts a shadow of 9.32 m when the angle of elevation to the sun is 63°. Find the height of the flagpole.

2 A steep hill is inclined at 18° to the horizontal. It runs down to the beach so its base is at sea level.
  a If I walk 150 m up the hill, what is my height above sea level?
  b If I climb to a point 80 m above sea level, how far have I walked?

3 A train must climb a constant gradient of 5.5 m for every 200 m of track. Find the angle of incline.

4 **a** Find the angle of elevation to the top of a 56 m high building from point A which is 113 m from its base.
  b What is the angle of depression from the top of the building to A?

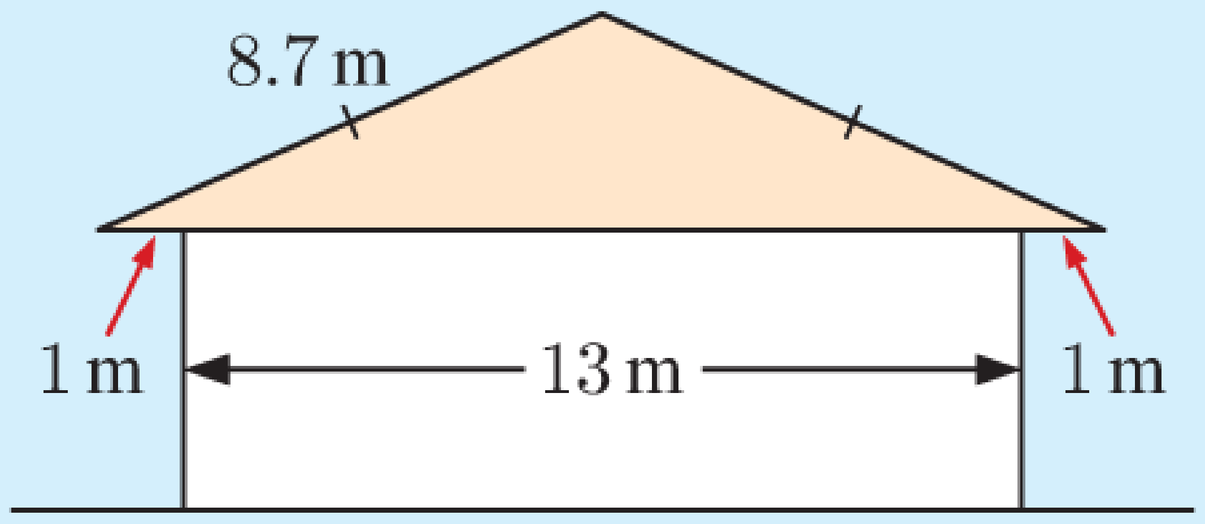
5 The angle of depression from the top of a 20 m high vertical cliff to a boat B is 8°.
How far is the boat from the base of the cliff?

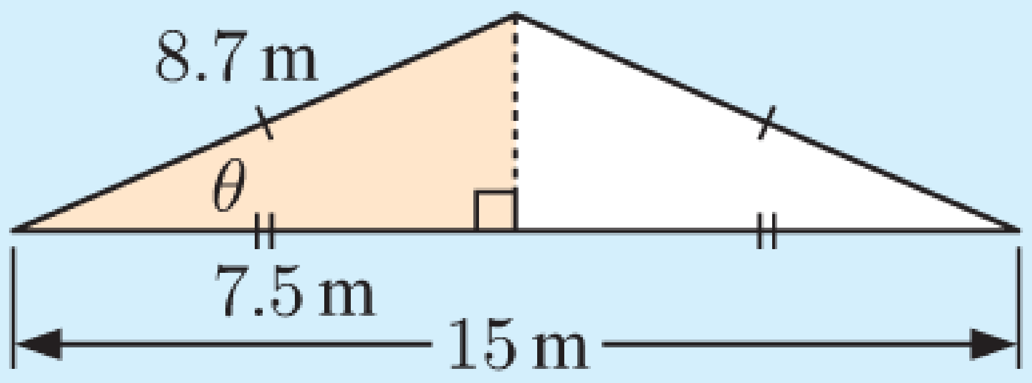


Example 9

Self Tutor

A builder has designed the roof structure illustrated. Find the pitch of this roof.





By constructing an altitude of the isosceles triangle, we form two right angled triangles.

$$\cos\theta = \frac{7.5}{8.7} \qquad \left\{\cos\theta = \frac{\text{ADJ}}{\text{HYP}}\right\}$$

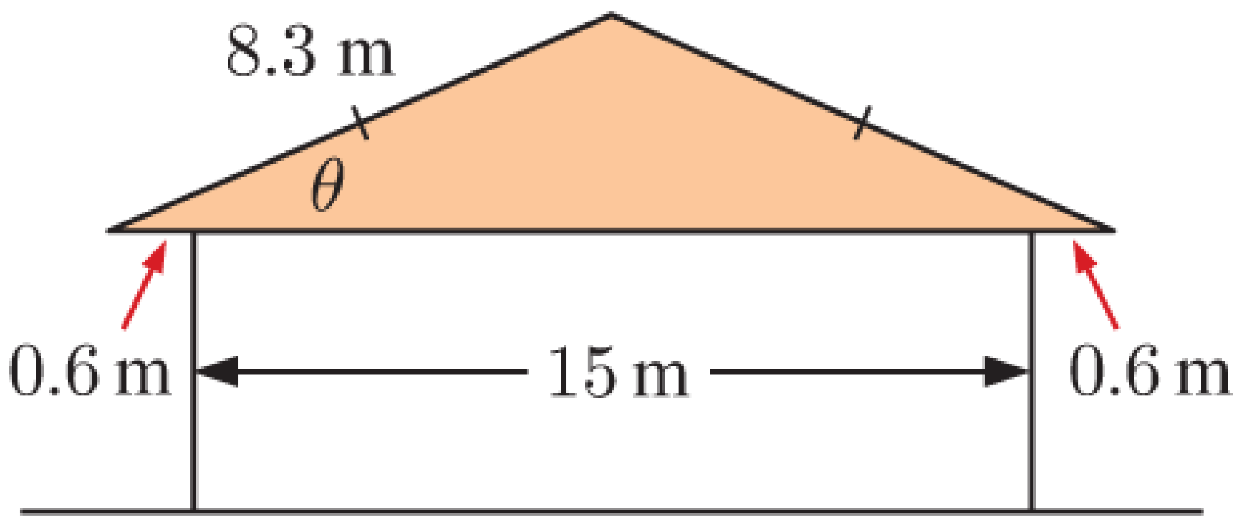
$$\therefore \quad \theta = \cos^{-1}\left(\frac{7.5}{8.7}\right)$$

$$\therefore \quad \theta \approx 30.5^\circ$$

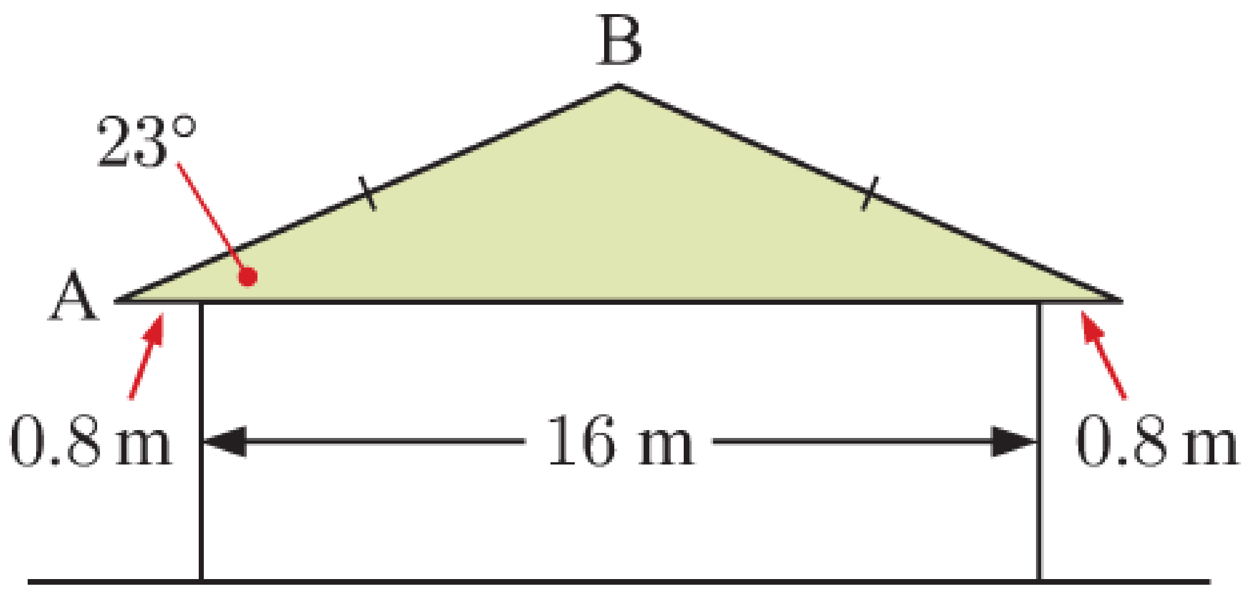
The pitch of the roof is approximately 30.5°.

The *pitch* of a roof is the angle that the roof makes with the horizontal.

6 Find θ, the pitch of the roof.

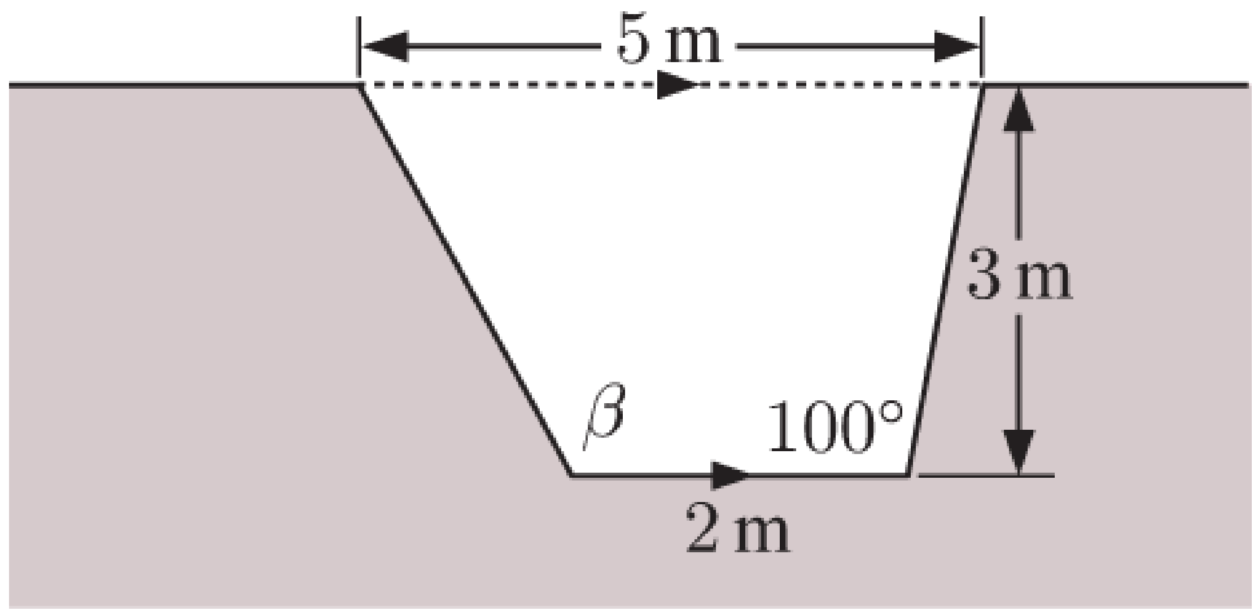


7 The pitch of the given roof is 23°. Find the length of the timber beam [AB].

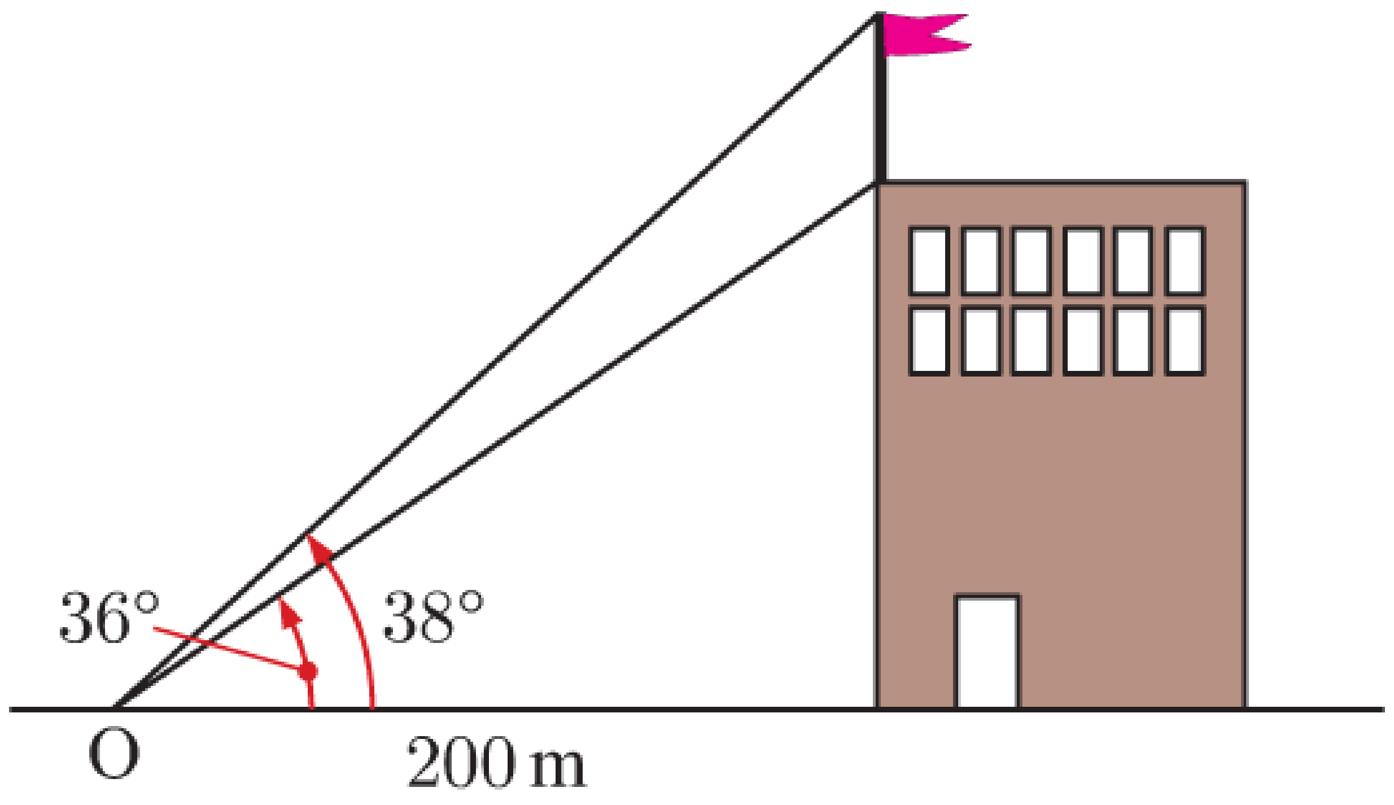


8 A rectangular field is 20 metres longer than it is wide. When Patrick walks from one corner to the opposite corner, he makes an angle of 55° with the shorter side of the field. Find the length of this shorter side.

9 A stormwater drain has the shape illustrated. Determine the angle β where the left hand side meets with the bottom of the drain.



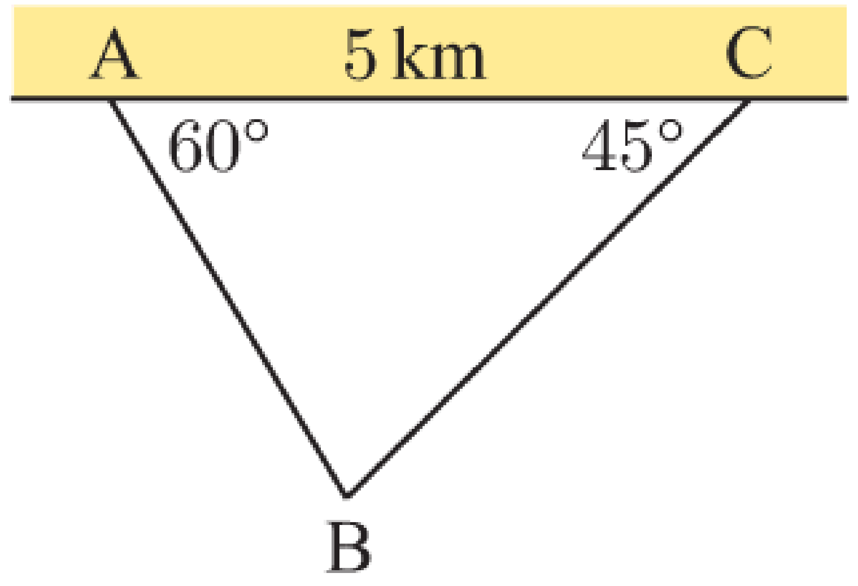
10 From an observer O who is 200 m from a building, the angles of elevation to the bottom and top of a flagpole are 36° and 38° respectively. Find the height of the flagpole.



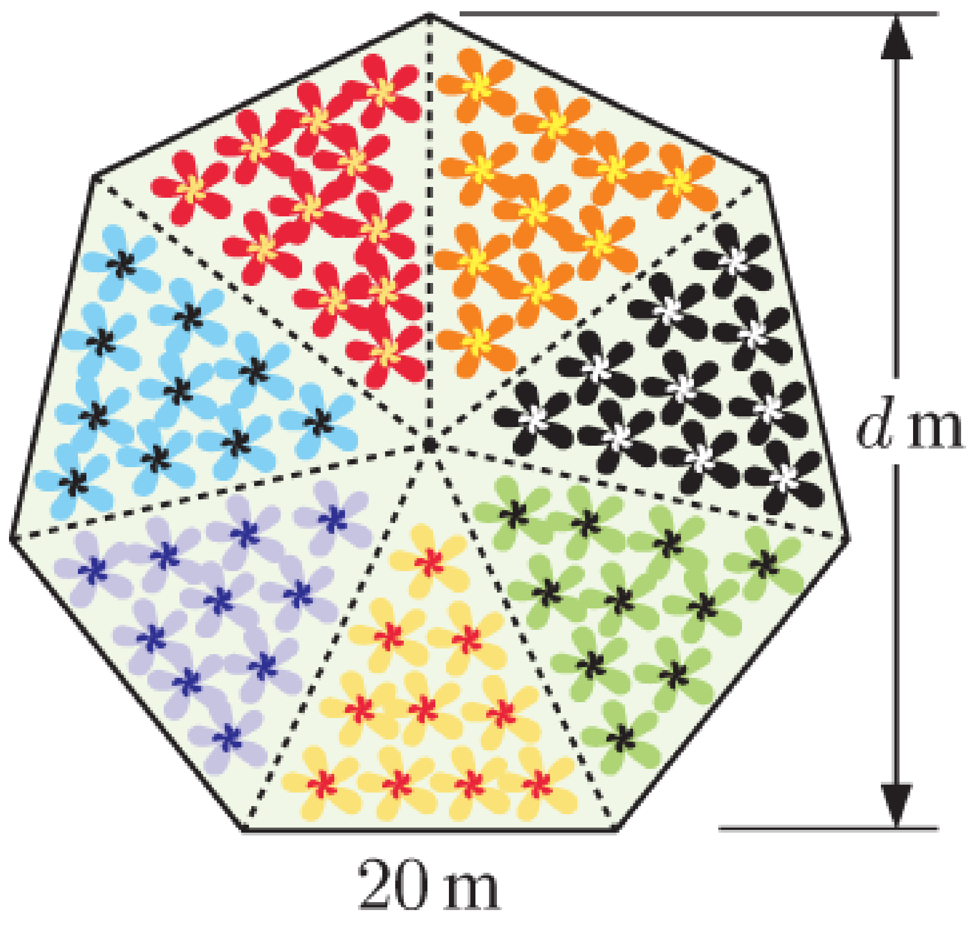
11 The angle of depression from the top of a 15 m high cliff to a boat at sea is 2.7°. How much closer to the cliff must the boat move for the angle of depression to become 4°?

12 A helicopter flies horizontally at 100 km h^{-1}. An observer notices that it takes 20 seconds for the helicopter to fly from directly overhead to being at an angle of elevation of 60°. Find the height of the helicopter above the ground.

13 [AC] is a straight shore line 5 km long. B is a boat out at sea. Find the shortest distance from the boat to the shore.



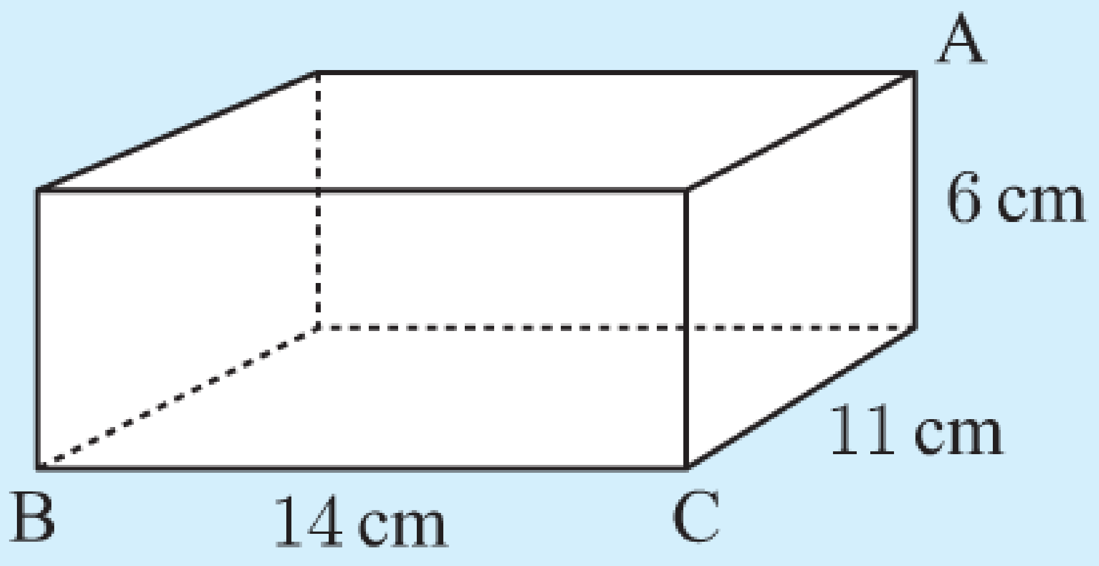
14 A new feature at the botanical gardens will be a regular heptagonal flower bed with sides of length 20 m.
Find the width of land d m required for the flower bed.



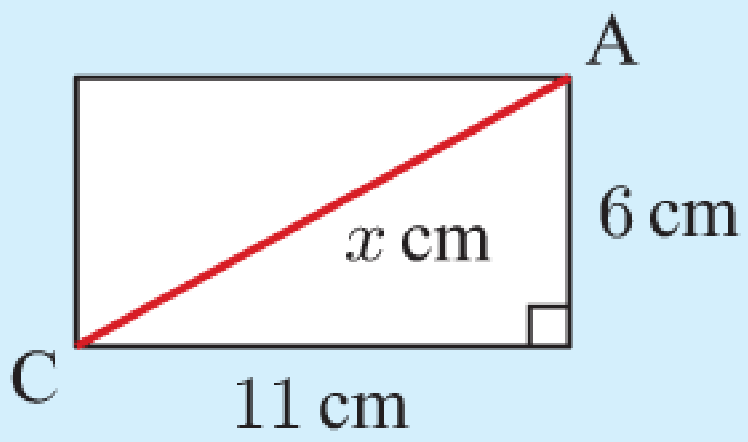
Example 10

Self Tutor

A rectangular prism has the dimensions shown.
Find the measure of $\widehat{ABC}$.



Consider the end of the prism containing points A and C.
Let $AC = x$ cm.

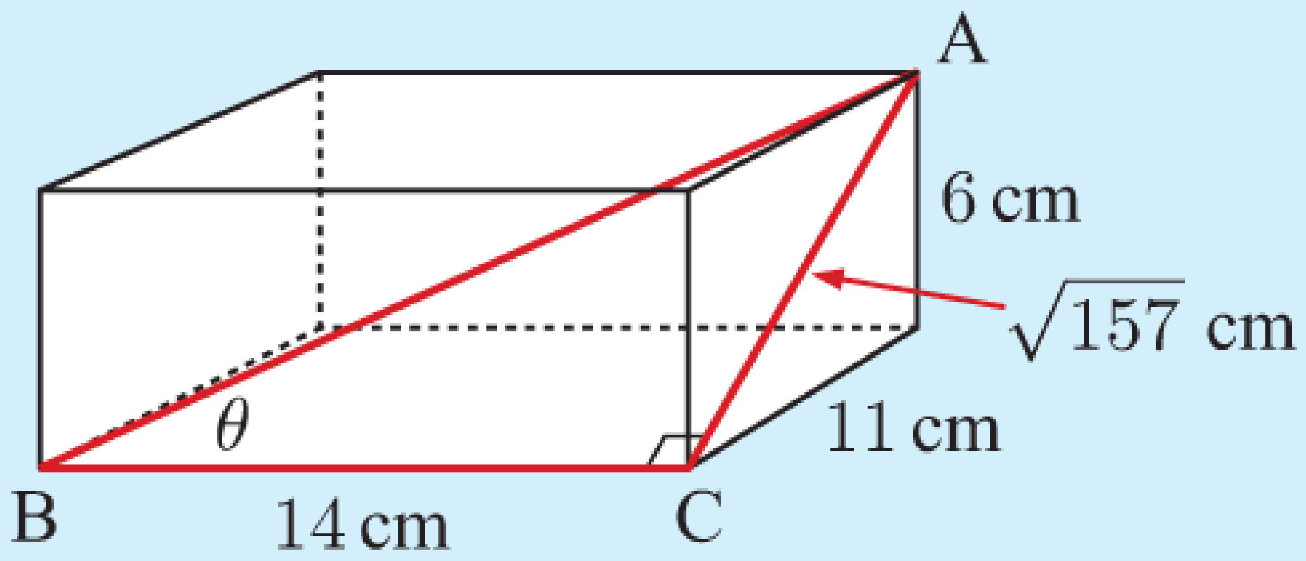


Using Pythagoras, $x^2 = 6^2 + 11^2$

$\therefore \; x^2 = 157$

$\therefore \; x = \sqrt{157}$ {as $x > 0$}

$\triangle$ABC is right angled at C.



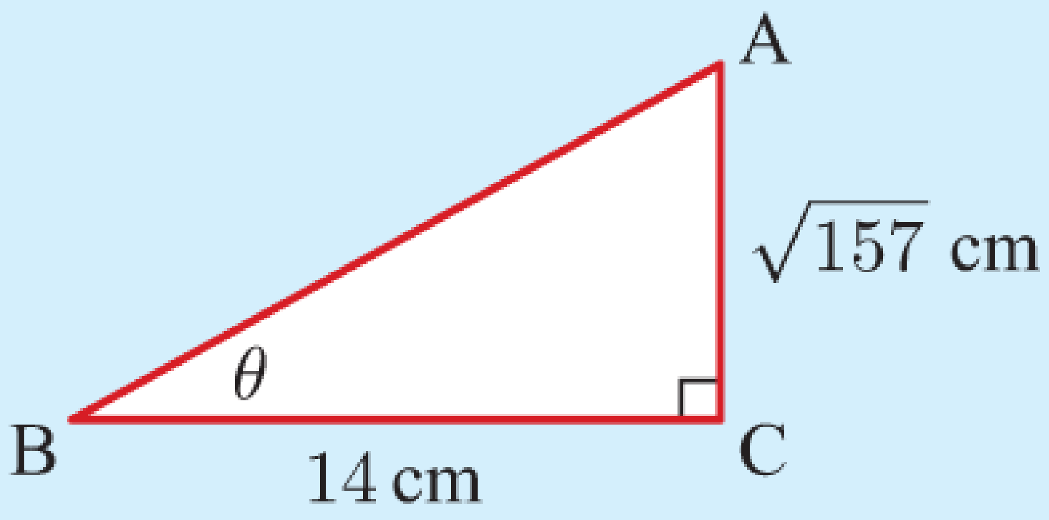
Let $\widehat{ABC} = \theta$

$\therefore \; \tan\theta = \frac{\sqrt{157}}{14}$ $\left\{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\right\}$

$\therefore \; \theta = \tan^{-1}\left(\frac{\sqrt{157}}{14}\right)$

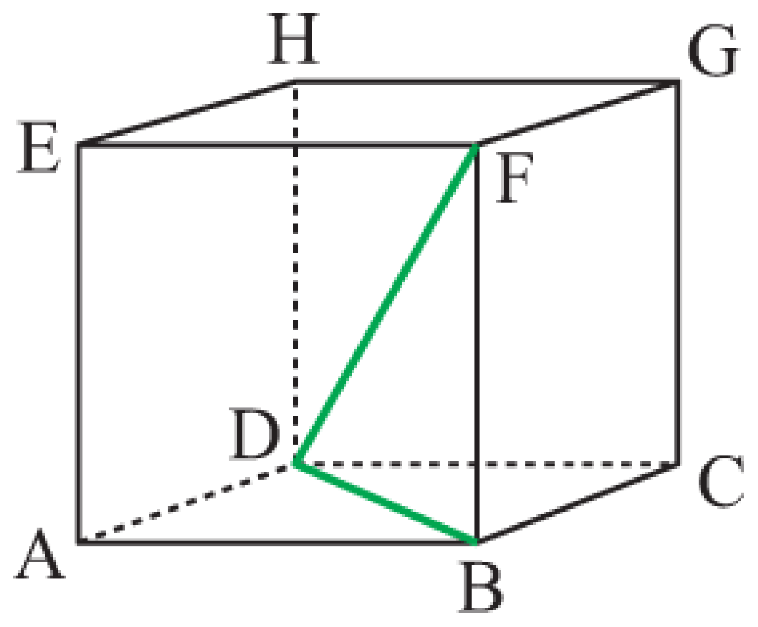
$\therefore \; \theta \approx 41.8°$

So, $\widehat{ABC} \approx 41.8°$.

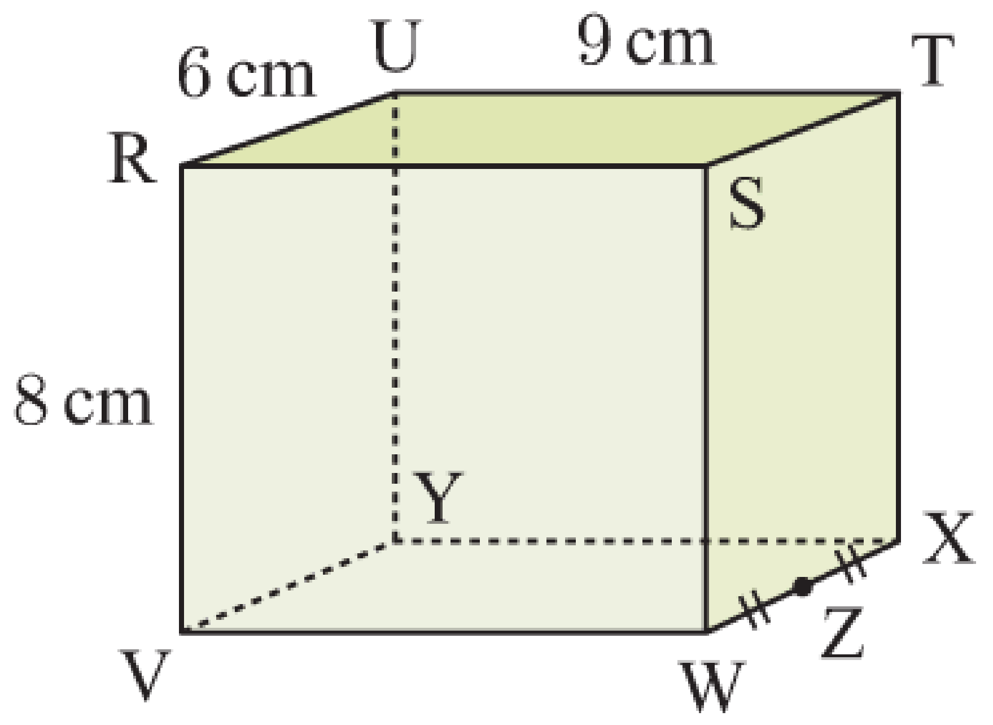


15 The cube shown has sides of length 13 cm. Find:

a BD **b** $F\widehat{D}B$.



16



In the rectangular prism shown, Z is the midpoint of [WX]. Find:

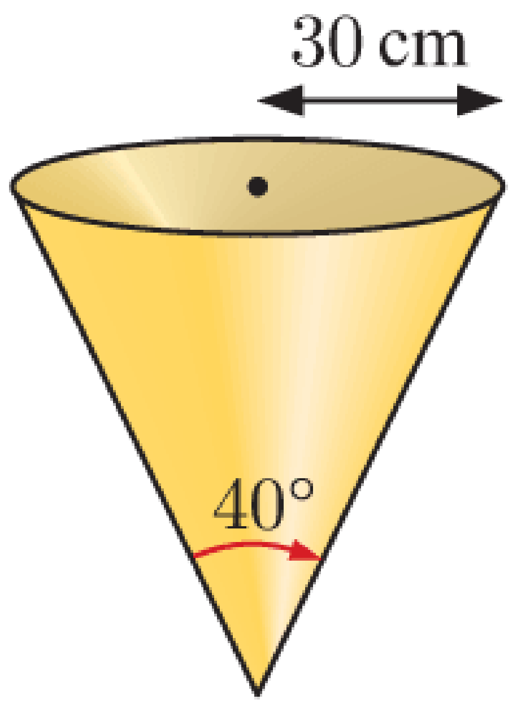
a VX **b** $R\widehat{X}V$

c YZ **d** $Y\widehat{Z}U$.

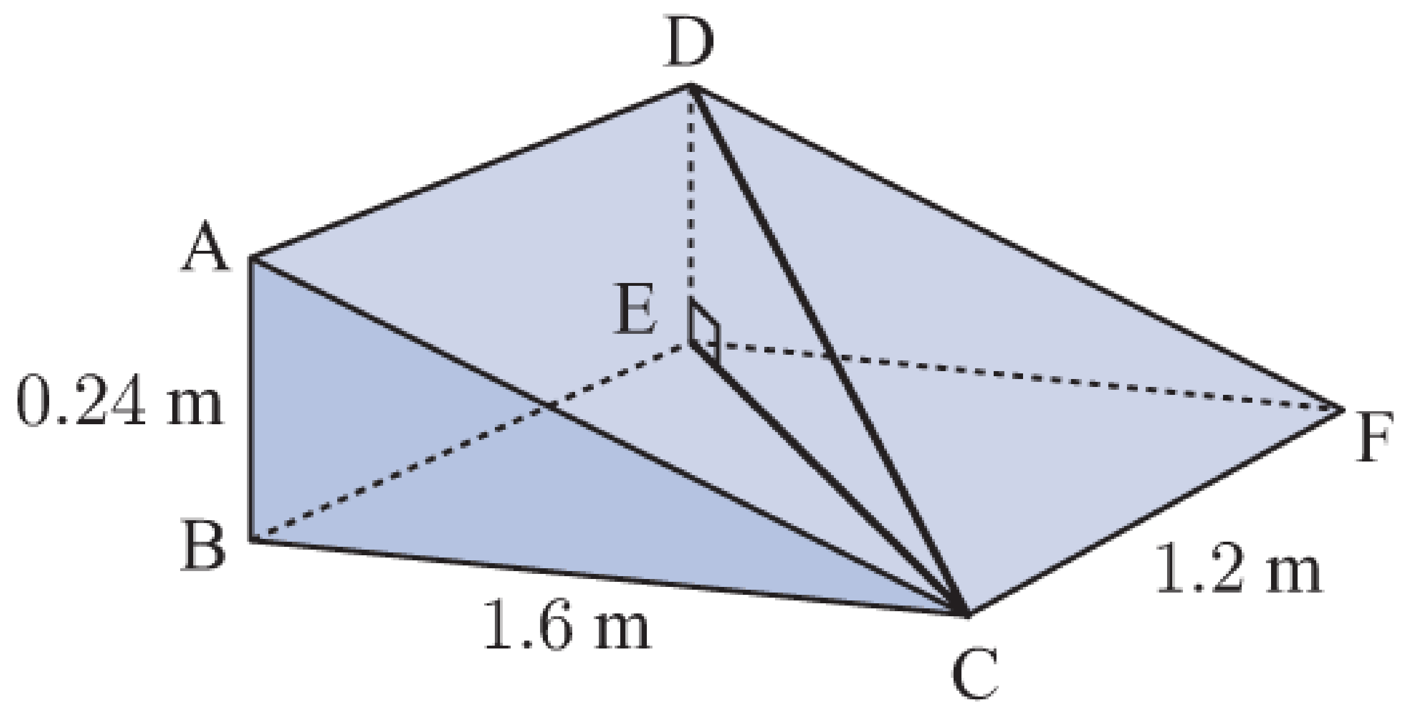
17 An open cone has a vertical angle measuring 40° and a base radius of 30 cm. Find:

a the height of the cone

b the capacity of the cone in litres.



18



A ramp is built as the triangular prism shown.

a Find the length:

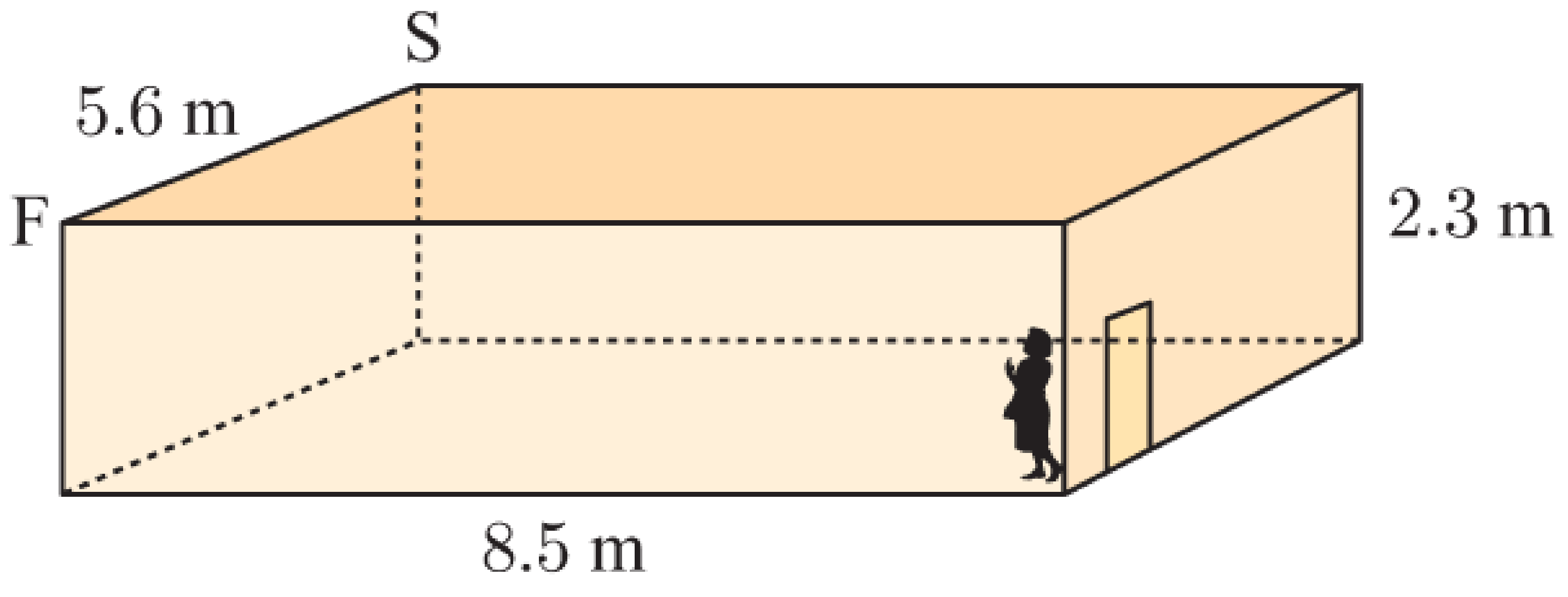
i CE **ii** CD.

b Find $D\widehat{C}E$.

19 Elizabeth is terrified of spiders. When she walks into a room, she notices one in the opposite corner S.

a If Elizabeth is 1.6 m tall, how far is the spider from her head?

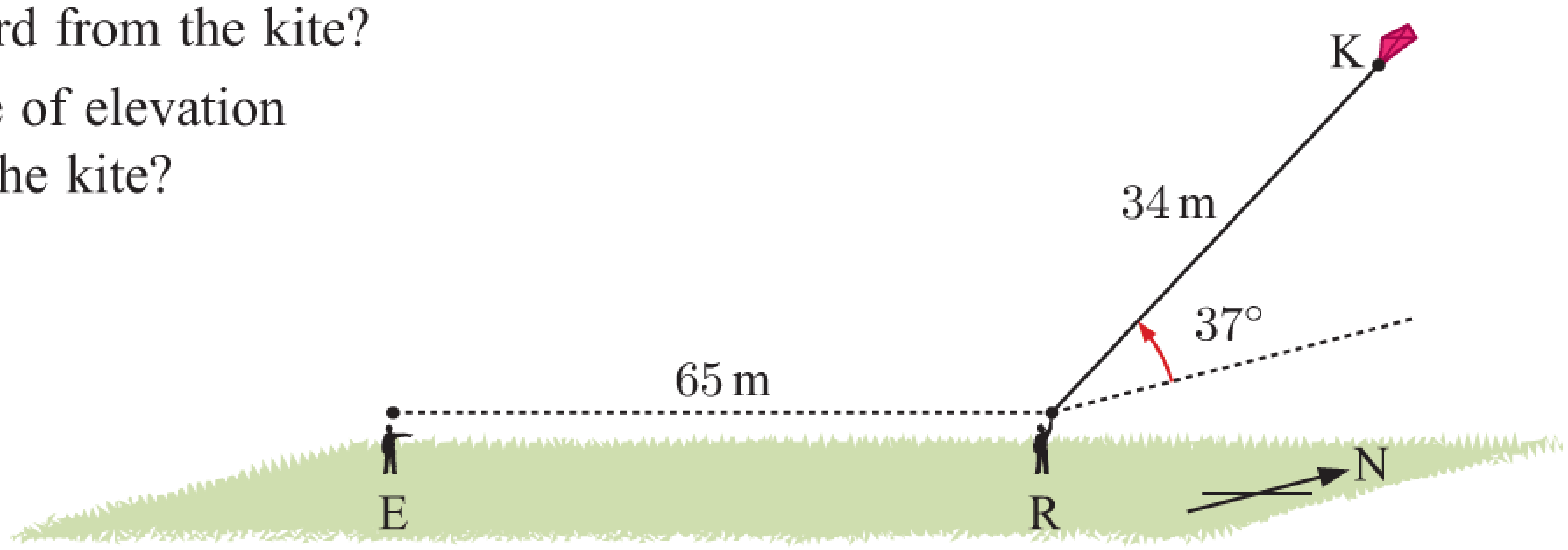
b The spider can see up to an angle of 42° from the direction it is facing. This spider is facing a fly at F. Can it see Elizabeth?



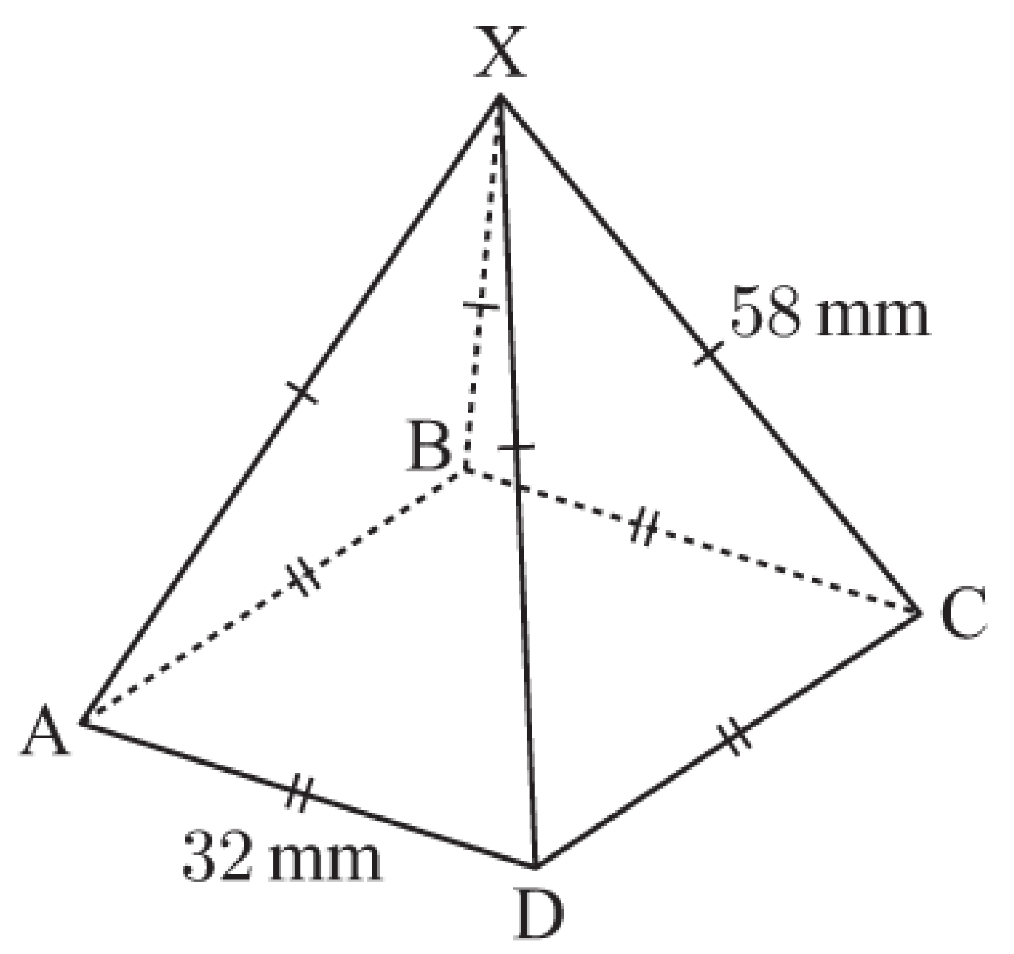
20 Rico is flying his kite with the aid of a southerly wind. He has let out 34 m of string, and the kite is at an angle of elevation of 37°. His friend Edward stands to the west, 65 m away.

a How far is Edward from the kite?

b What is the angle of elevation from Edward to the kite?

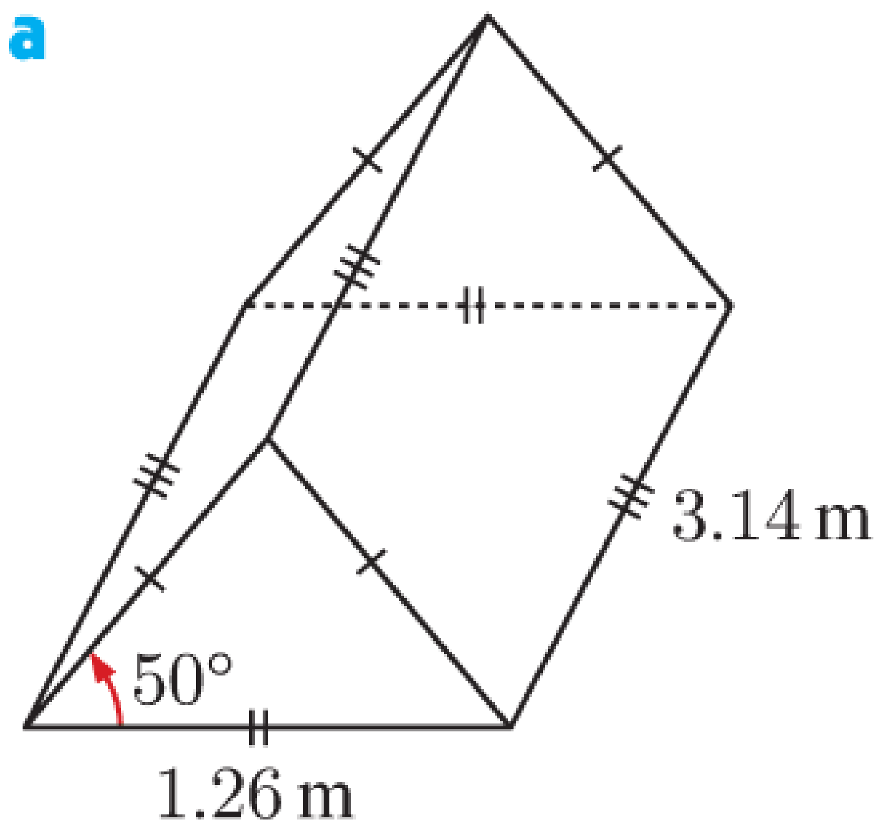


21 Find the angle between the slant edge [AX] and the base diagonal [AC].

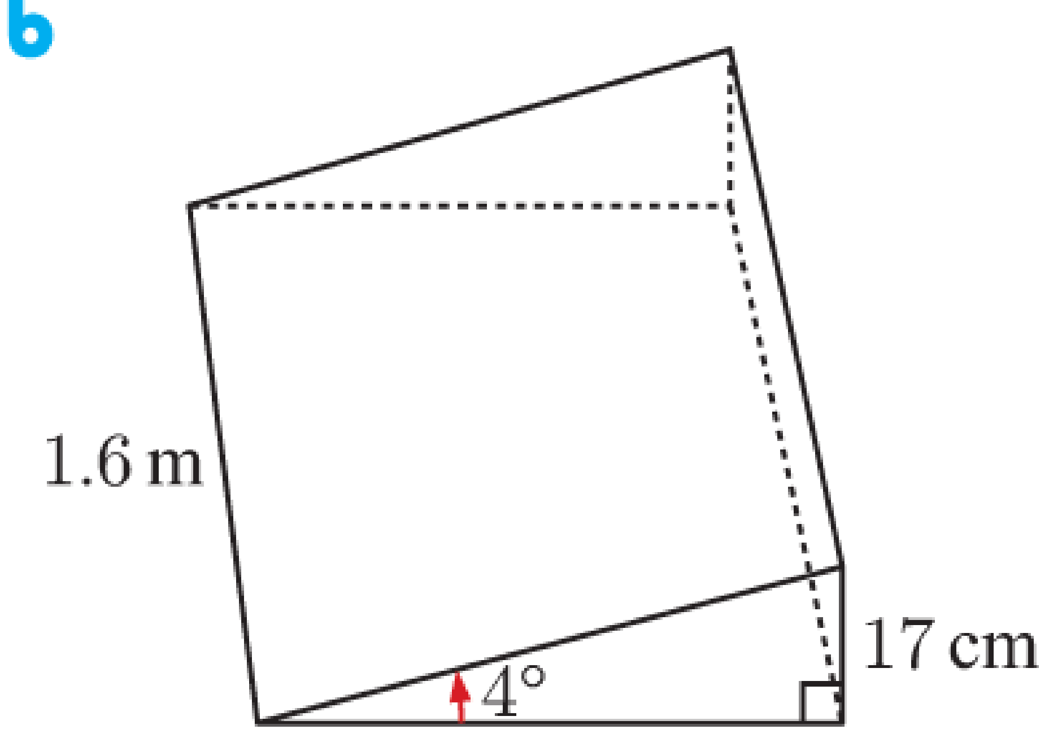


22 Find the volume of each solid:

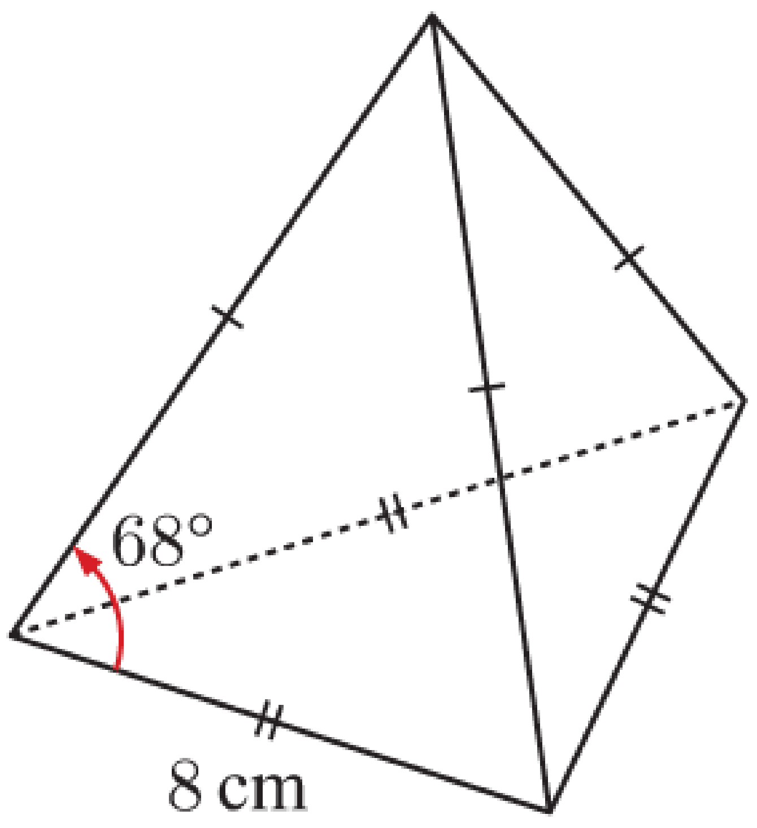
a



b



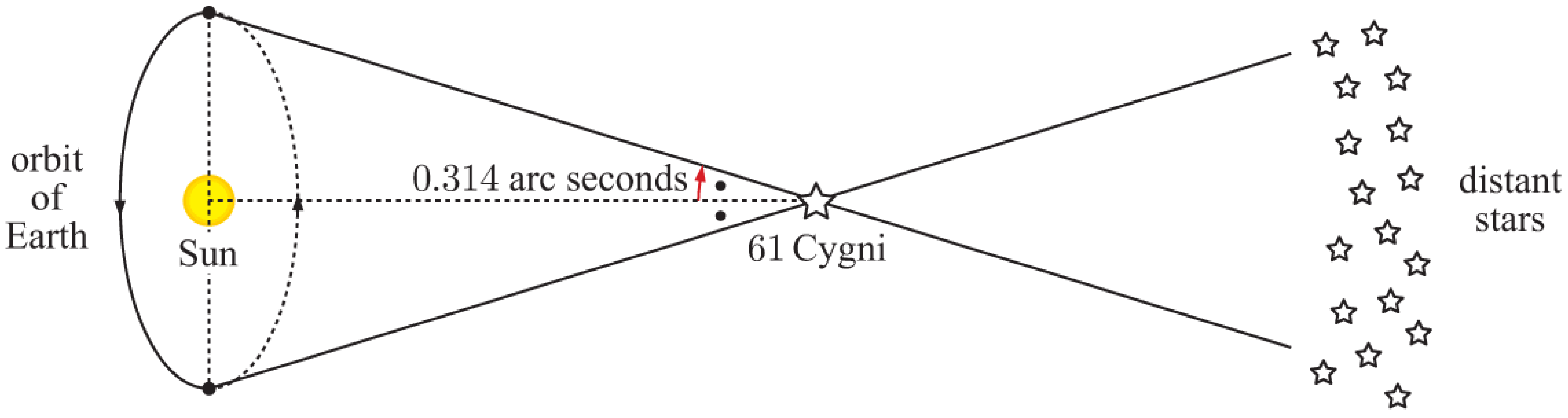
c



23 A **parallax** is the angle through which an object appears to move when viewed from different positions.

In 1838, the Prussian astronomer **Friedrich Wilhelm Bessel** (1784 - 1846) used the telescopes at the Königsberg Observatory to measure the parallax of the star 61 Cygni to be approximately 0.314 arc seconds, where one arc second is $\frac{1}{3600}$ of a degree. He did this by comparing the apparent positions of 61 Cygni during the year, relative to a fixed backdrop of distant stars.

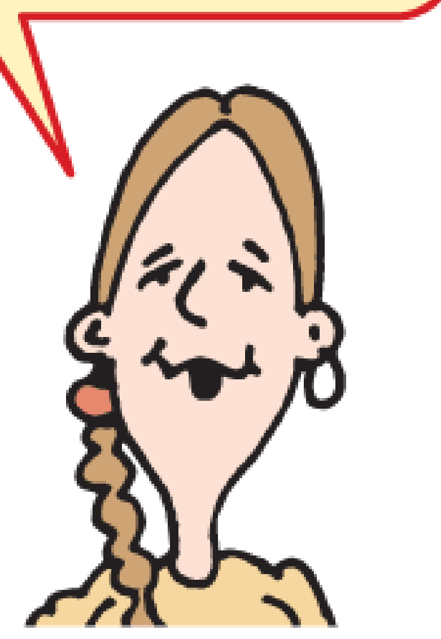
Friedrich Bessel



a Given that the radius of the Earth's orbit is $\approx 1.49\times 10^{11}$ m and that 1 light-year $\approx 9.461\times 10^{15}$ m, explain Bessel's calculation that 61 Cygni is about 10.3 light-years away.

b Modern estimates place 61 Cygni at about 11.4 light-years away. Calculate a more accurate value for the parallax of 61 Cygni.

1 parsec or "parallactic second" is the distance to a star with a parallax of 1 arc second measured across the Earth's orbit.
1 parsec $\approx$ 3.26 light-years.



E TRUE BEARINGS

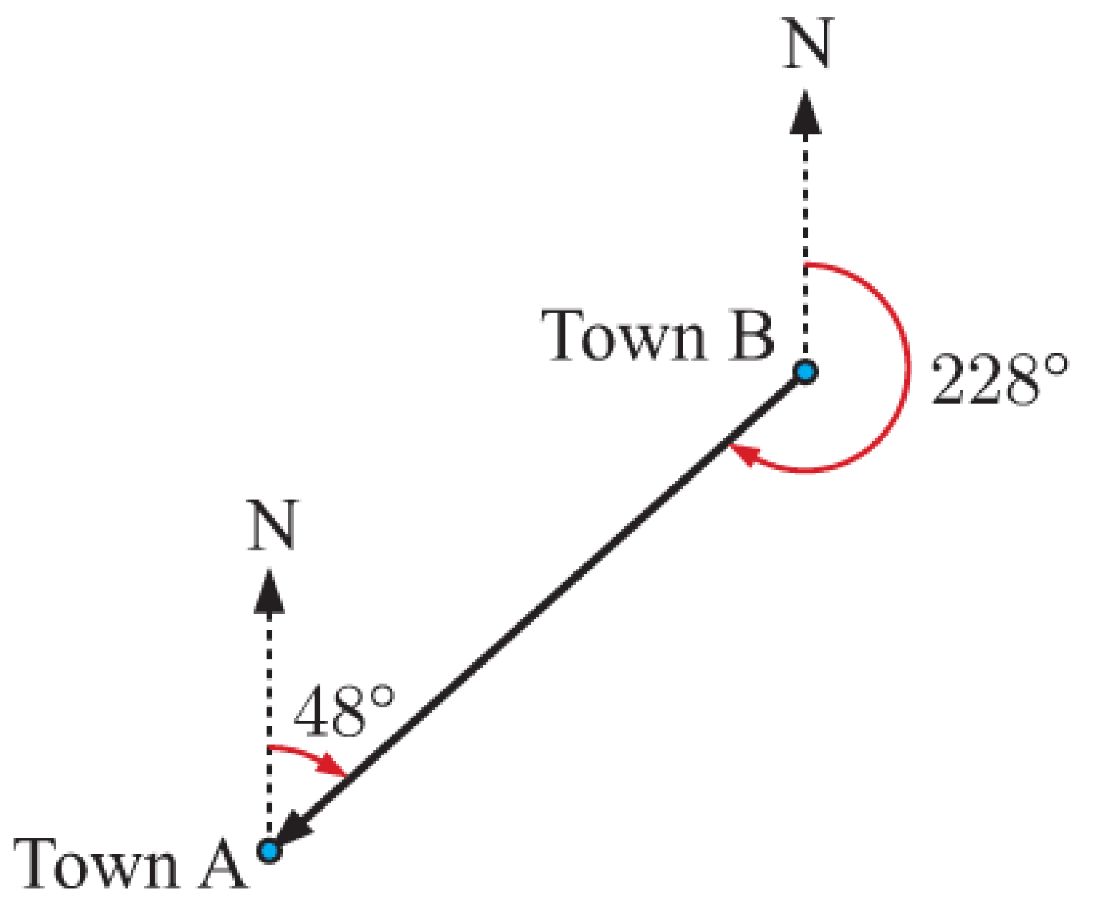
True bearings are used to describe the direction of one object from another. The direction is described by comparing it with the **true north direction**.

True bearings are measured **clockwise** from true north.
They are always written with 3 digits, plus decimals if necessary.

Suppose you are in town A and you want to go to town B. If you start facing true north, you need to turn 48° clockwise in order to face town B. We say that the **bearing of B from A** is 048°.

Now suppose you are in town B and want to go to town A. If you start facing true north, you need to turn 228° clockwise in order to face town A. The bearing of A from B is 228°.

Notice that the bearing of B from A and the bearing of A from B differ by 180°.



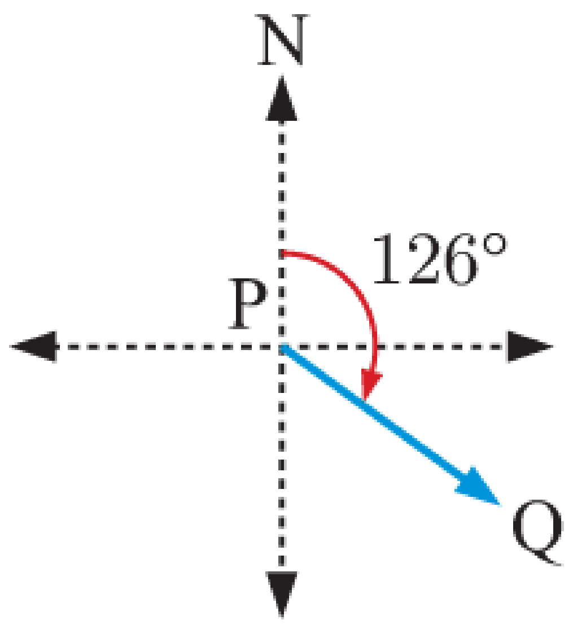
EXERCISE 7E

1 Draw a diagram showing that the bearing of B from A is:

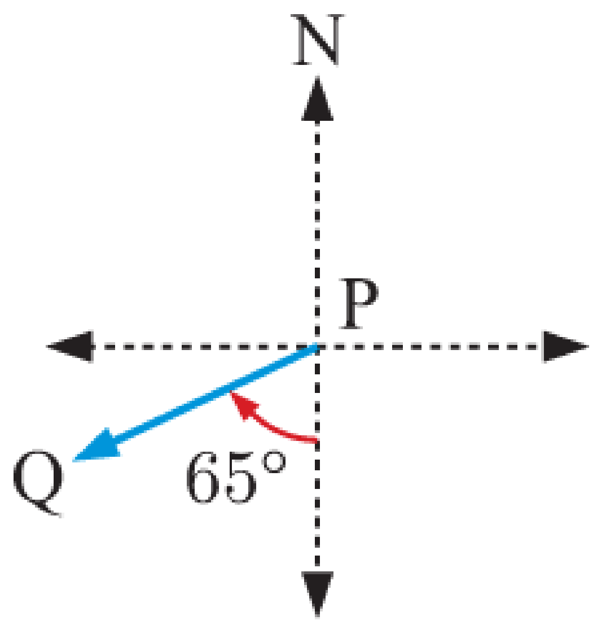
a 075° **b** 205° **c** 110° **d** 325°

2 Find the bearing of Q from P in each diagram:

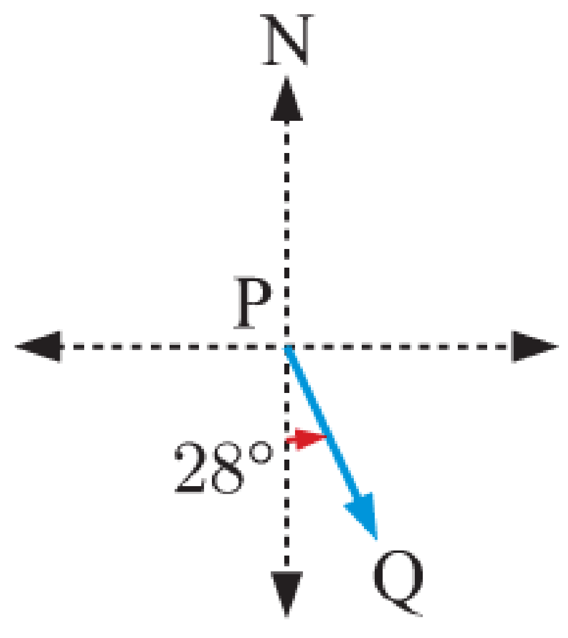
a



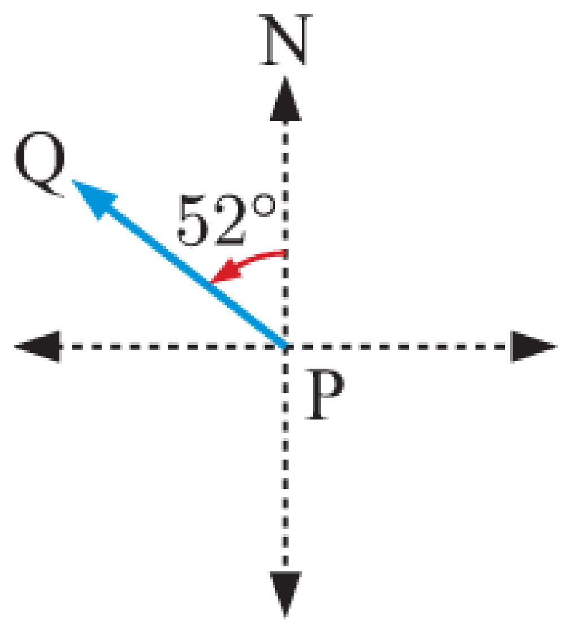
b



c

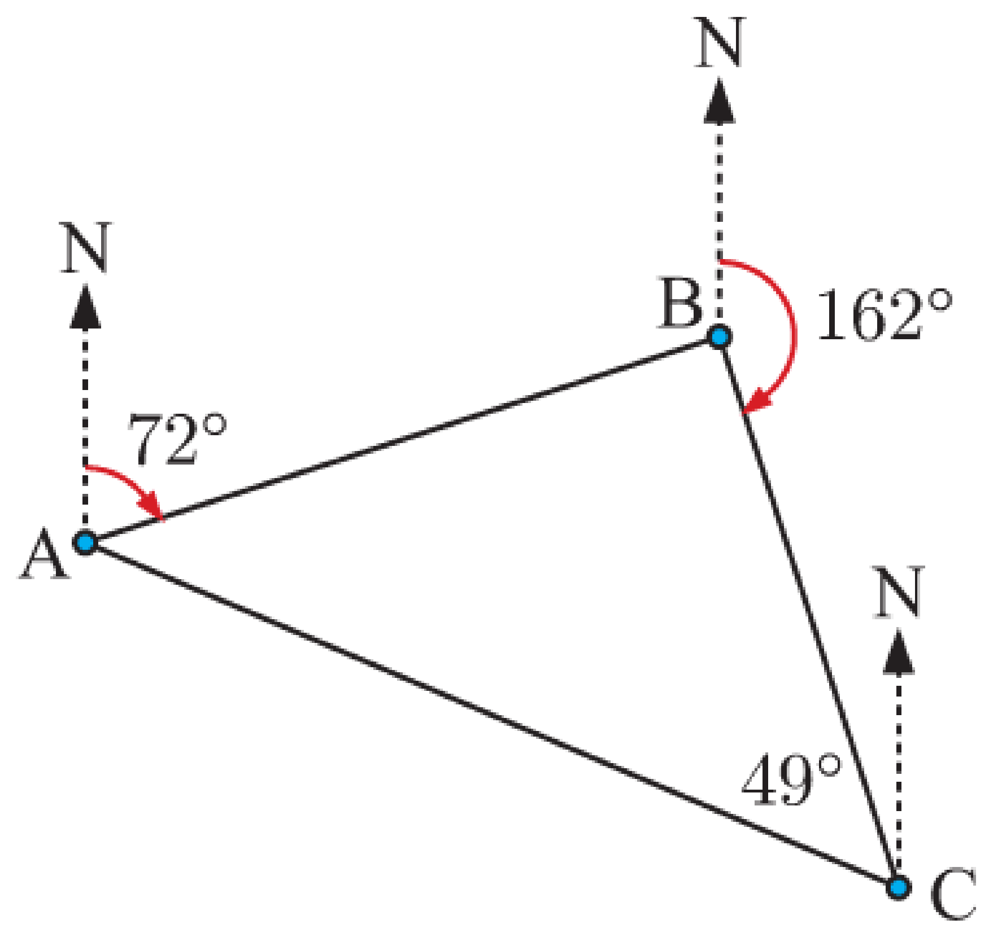


d



3 In the diagram given, find the bearing of:

a B from A **b** A from B **c** C from B
d B from C **e** C from A **f** A from C.



Example 11 🔊 **Self Tutor**

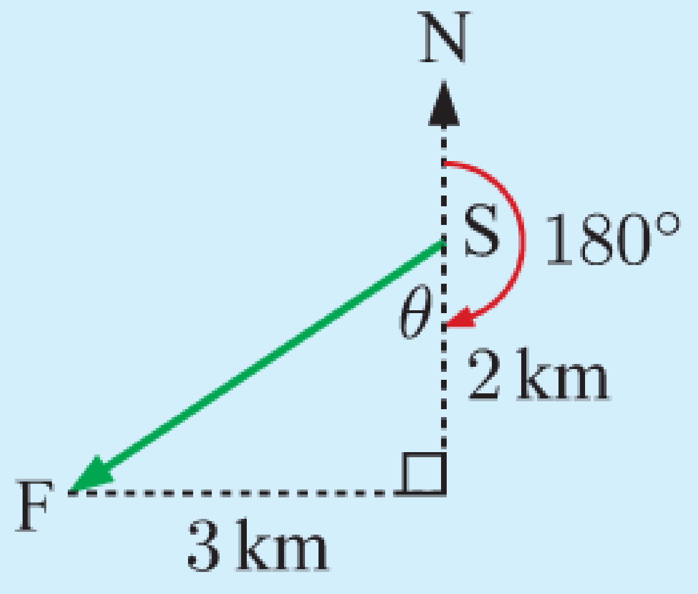
When Samantha goes jogging, she finishes 2 km south and 3 km west of where she started. Find Samantha's bearing from her starting point.

Suppose Samantha starts at S and finishes at F.

$\tan\theta = \frac{3}{2}$

$\therefore \quad \theta = \tan^{-1}\left(\frac{3}{2}\right) \approx 56.3°$

So, the bearing $\approx 180° + 56.3° \approx 236°$



4 When Walter drives to his sports club, he finishes 10 km east and 7 km south of where he started. Find Walter's bearing from his starting point.

5 Julia is swimming in the ocean. A current takes her 200 m north and 100 m west of where she started.

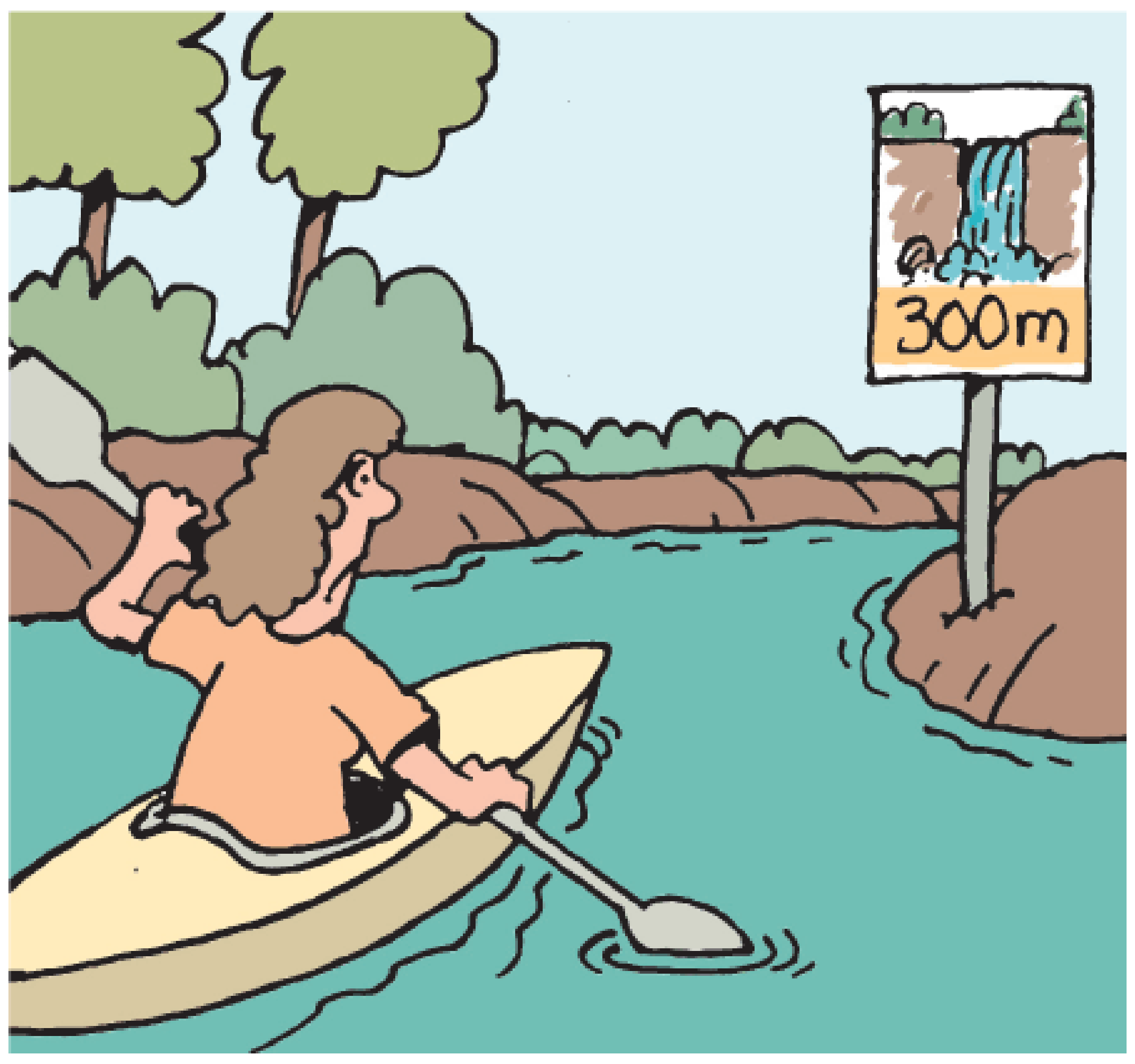
- **a** How far is Julia from her starting point?
- **b** Find Julia's bearing from her starting point.
- **c** In which direction is the starting point from where Julia is now?

6 Paul runs 1.5 km on the bearing 127°.

- **a** Draw a diagram of the situation.
- **b** How far east is Paul from his starting point?
- **c** How far south is Paul from his starting point?

7 Tiffany kayaks 4 km on the bearing 323°. How far west is Tiffany from her starting point?

8 A train travels on the bearing 072° until it is 12 km east of its starting point. How far did the train travel on this bearing?



Example 12

Self Tutor

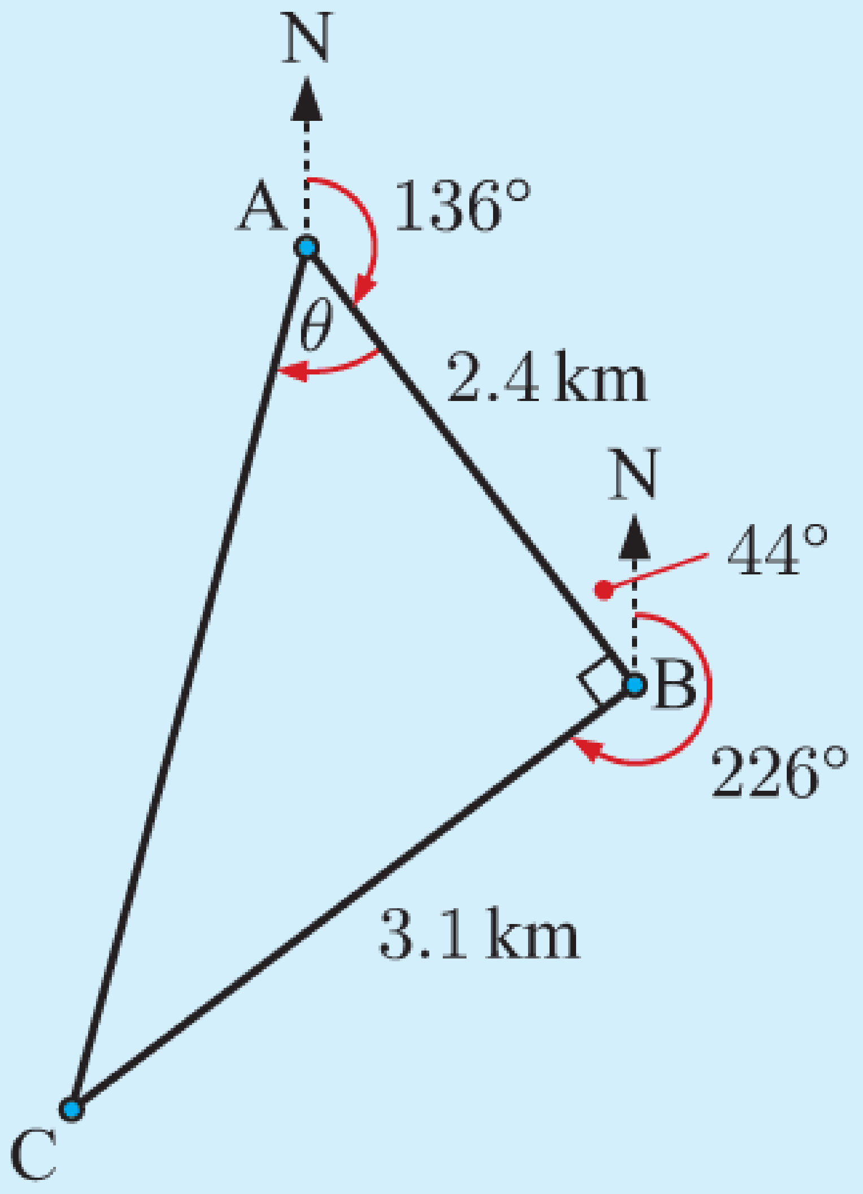
A courier departs from his depot A and drives on a 136° course for 2.4 km to an intersection B. He turns right at the intersection, and drives on a 226° course for 3.1 km to his destination C. Find:

a the distance of C from A **b** the bearing of C from A.

$\mathrm{A\widehat{B}N} = 180° - 136° = 44°$ {cointerior angles}

$\therefore\ \mathrm{A\widehat{B}C} = 360° - 44° - 226°$ {angles at a point}

$= 90°$



a $\mathrm{AC}^2 = 2.4^2 + 3.1^2$ {Pythagoras}

$\therefore\ \mathrm{AC} = \sqrt{2.4^2 + 3.1^2}$ {as AC > 0}

≈ 3.92 km

So, C is about 3.92 km from A.

b To find the bearing of C from A, we first need to find θ.

Now $\tan\theta = \dfrac{3.1}{2.4}$

$\therefore\ \theta = \tan^{-1}\left(\dfrac{3.1}{2.4}\right)$

$\therefore\ \theta \approx 52.3°$

The bearing of C from A is $136° + 52.3° \approx 188°$.

9 An orienteer runs 720 m in the direction 236° to a checkpoint, and then 460 m in the direction 146° to the finish. Find the:

- **a** direct distance from the starting point to the finishing point
- **b** bearing of the finishing point from the starting point.

10 A cruise ship sails from port P in the direction 112° for 13.6 km, and then in the direction 202° for 72 km. Find the distance and bearing of the cruise ship from P.

11 Yachts A and B depart from the same point. Yacht A sails 11 km on the bearing 034°. Yacht B sails 14 km on the bearing 124°. Find the distance and bearing of yacht B from yacht A.

12 An eagle is 2 km away on the bearing 293° from its nest. The eagle flies in a straight line for 5 km, and is now on the bearing 023° from its nest.

a On what bearing did the eagle fly? **b** How far north from its nest is the eagle now?

F THE ANGLE BETWEEN A LINE AND A PLANE

When the sun shines on the *gnomon* of a sundial, it casts a shadow onto the dial beneath it.

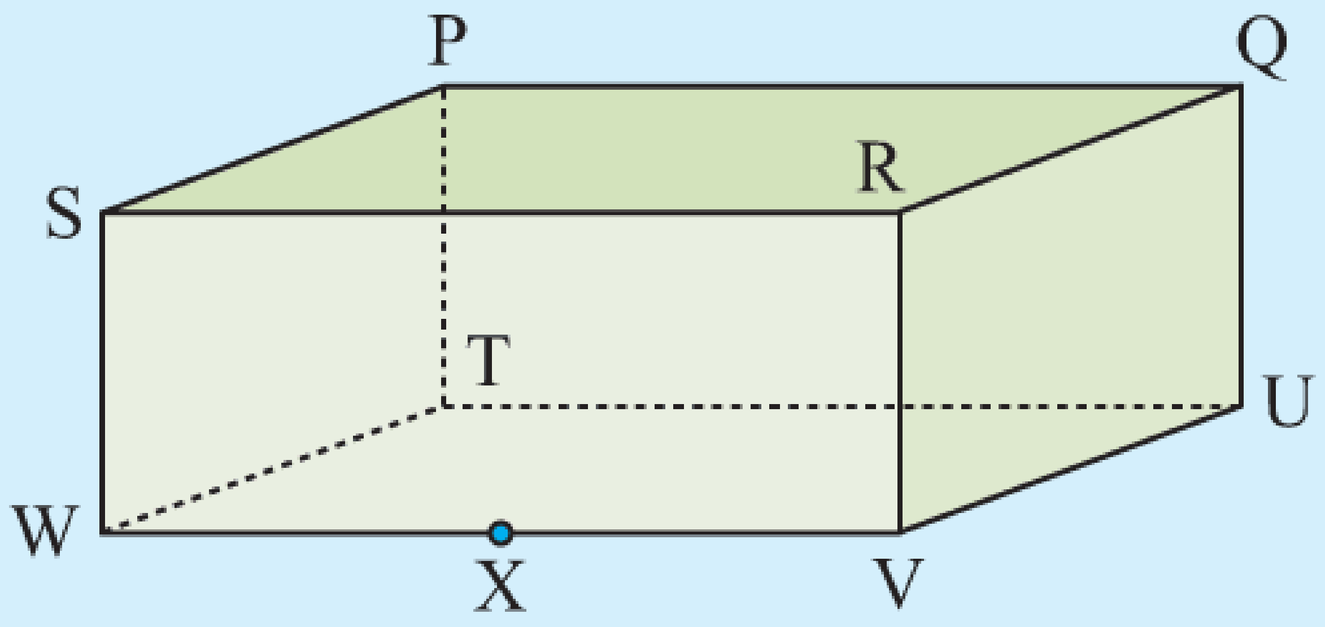
If the sun is directly overhead, its rays are *perpendicular* to the dial. The shadow formed is the **projection** of the gnomon onto the dial.

The **angle between a line and a plane** is the angle between the line and its **projection** on the plane.

Example 13 Self Tutor

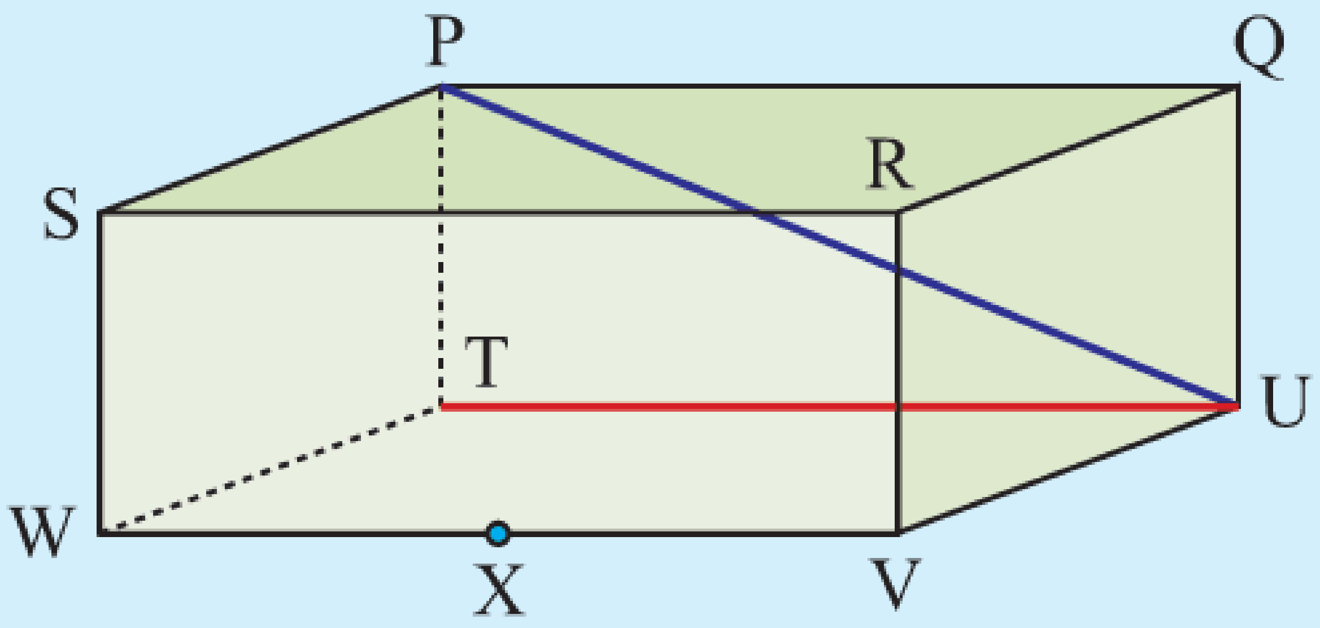
Name the angle between the following line segments and the base plane TUVW:

a [PU] **b** [QX]



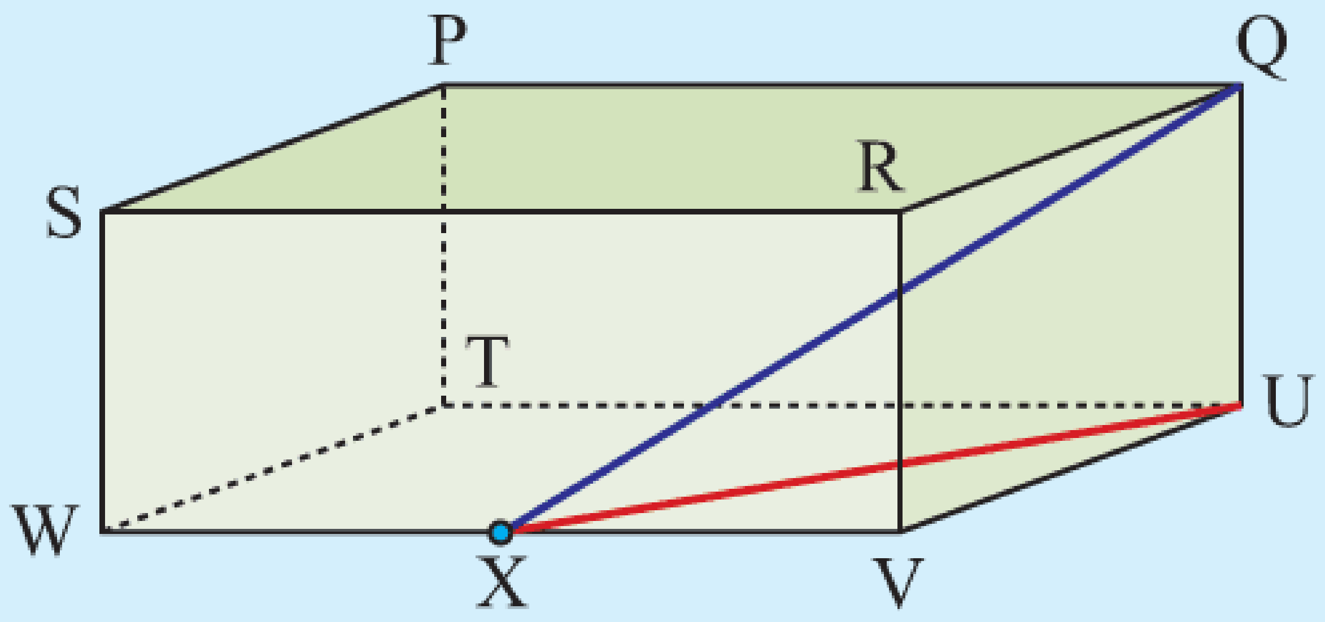
a The projection of [PU] onto the base plane is [TU].

$\therefore$ the required angle is $P\widehat{U}T$.



b The projection of [QX] onto the base plane is [UX].

$\therefore$ the required angle is $Q\widehat{X}U$.



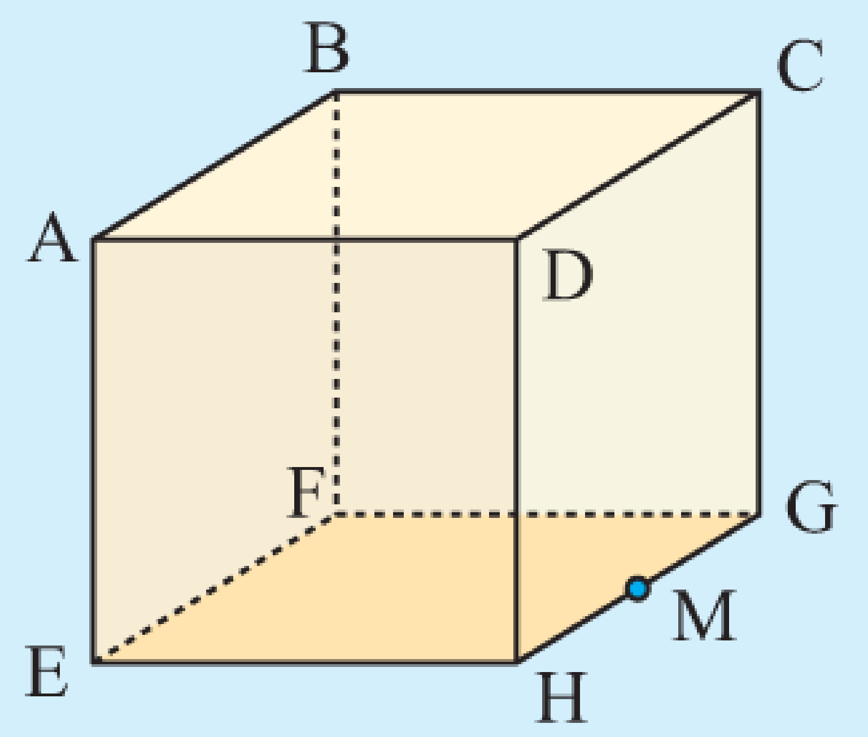
Example 14

Self Tutor

The diagram shows a cube with side length 12 cm. M is the midpoint of [GH].

Find the angle between the following line segments and the base plane EFGH:

a [AH] **b** [BM]



a The projection of [AH] onto the base plane is [EH].

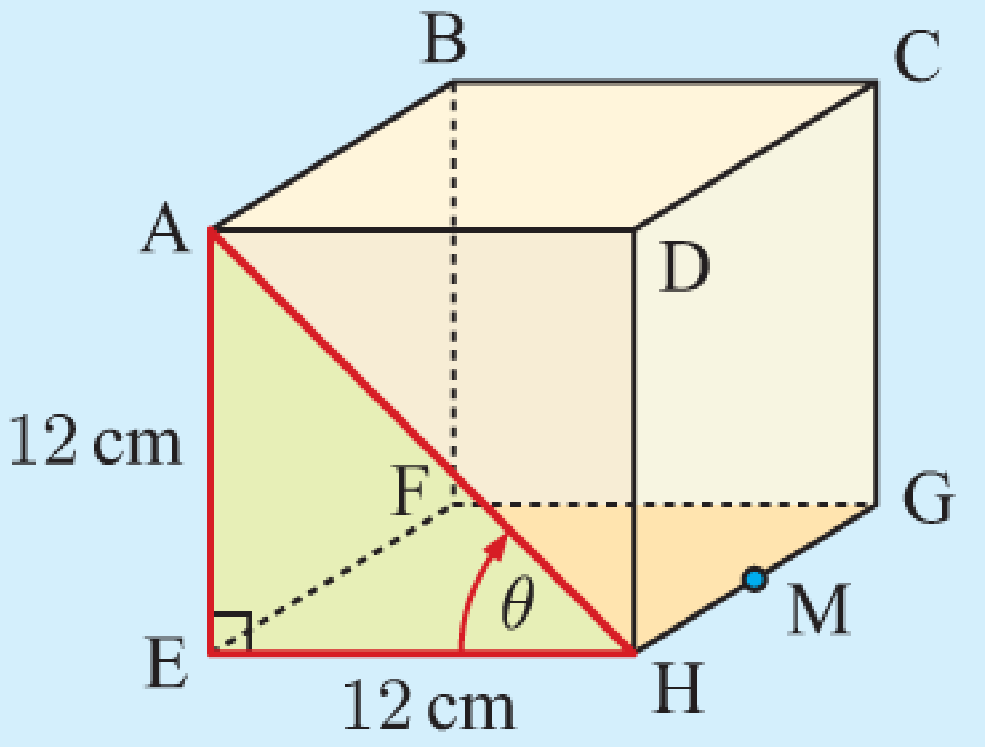
$\therefore$ the required angle is $A\widehat{H}E$.

$\tan\theta = \frac{12}{12}$ $\{\tan\theta = \frac{\text{OPP}}{\text{ADJ}}\}$

$\therefore \theta = \tan^{-1} 1$

$\therefore \theta = 45°$

The angle is 45°.



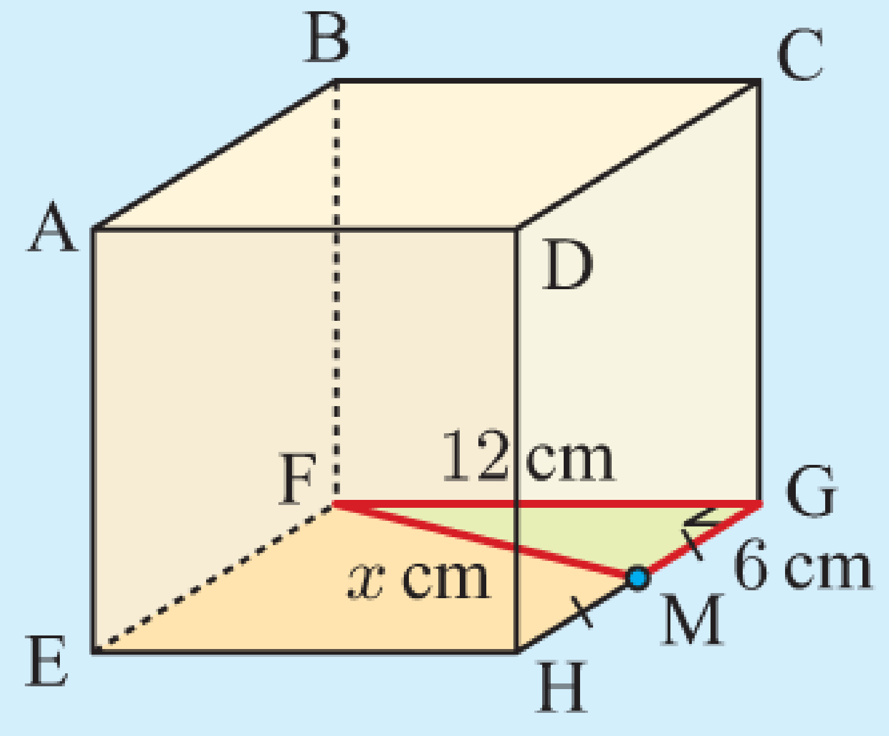
b The projection of [BM] onto the base plane is [FM].

$\therefore$ the required angle is $B\widehat{M}F$.

Let FM $= x$ cm.

Using Pythagoras in $\triangle$FGM,

$$x^2 = 12^2 + 6^2$$
$$\therefore x^2 = 180$$
$$\therefore x = \sqrt{180} \qquad \{\text{as } x > 0\}$$



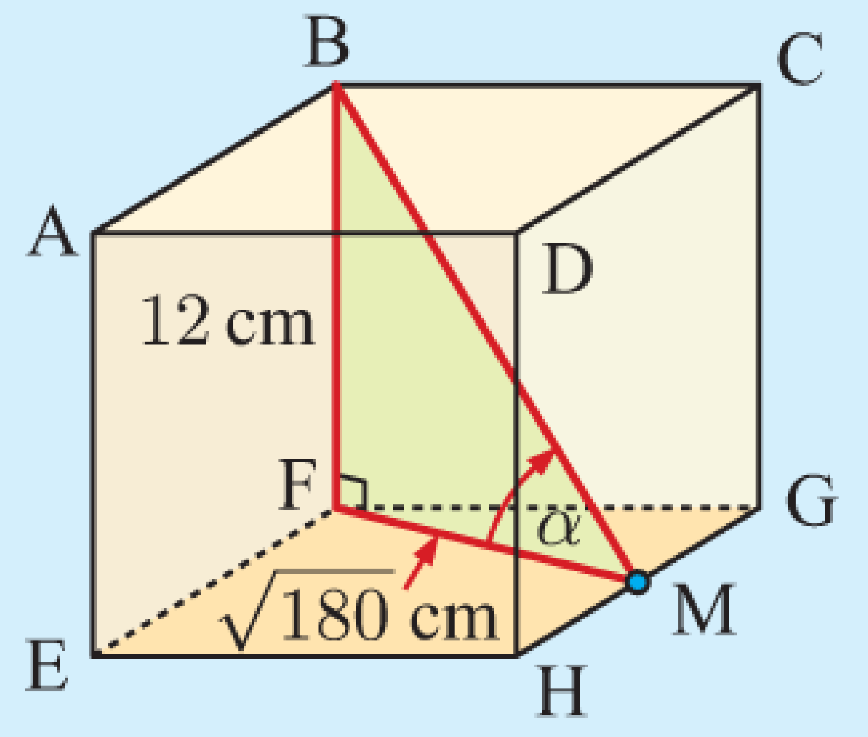
Let $B\widehat{M}F = \alpha$

$\therefore \tan\alpha = \frac{12}{\sqrt{180}}$ $\{\tan\alpha = \frac{\text{OPP}}{\text{ADJ}}\}$

$\therefore \alpha = \tan^{-1}\left(\frac{12}{\sqrt{180}}\right)$

$\therefore \alpha \approx 41.81°$

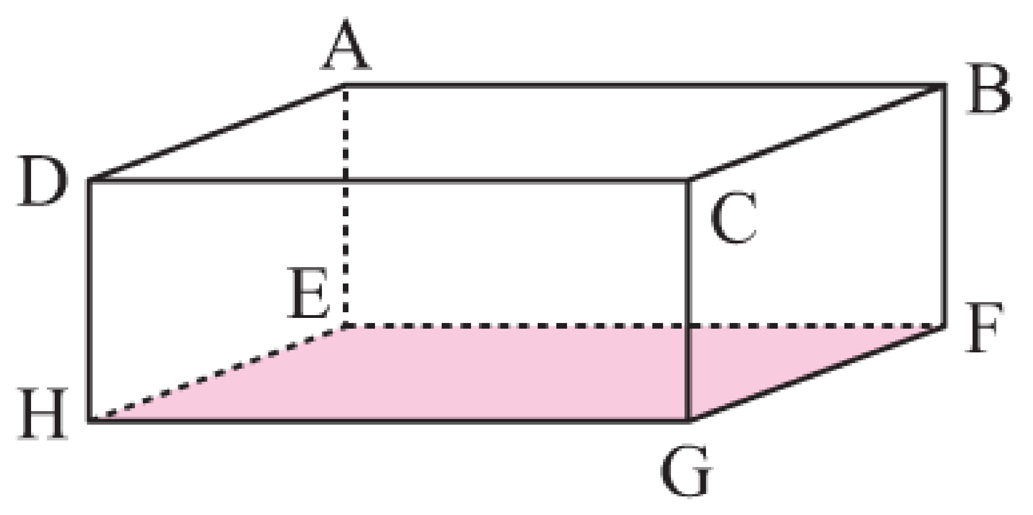
The angle is about 41.8°.



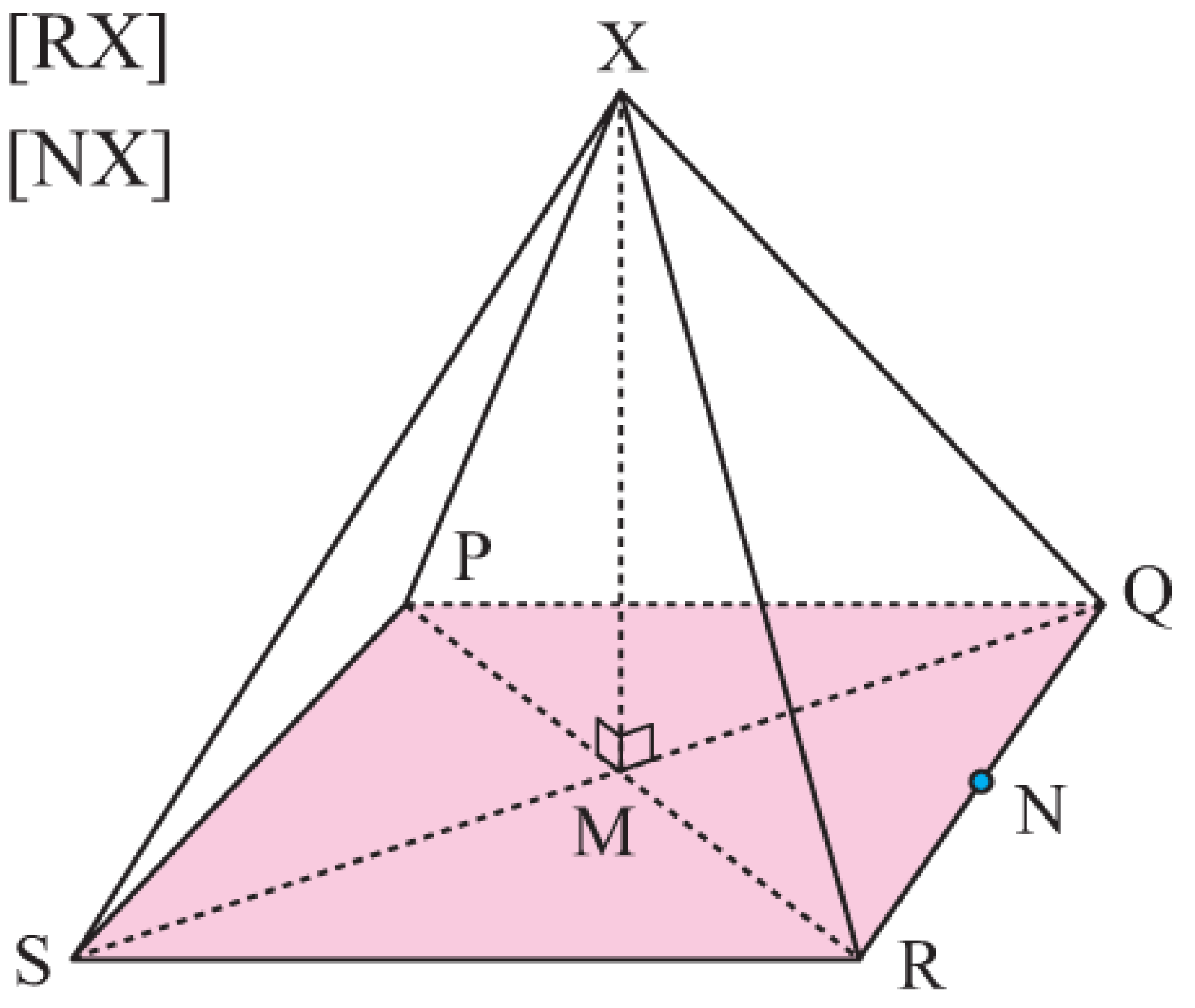
EXERCISE 7F

1 Name the projection of each line segment onto the base plane of the given figure:

a
- **i** [AH]
- **ii** [BE]
- **iii** [AG]
- **iv** [BH]

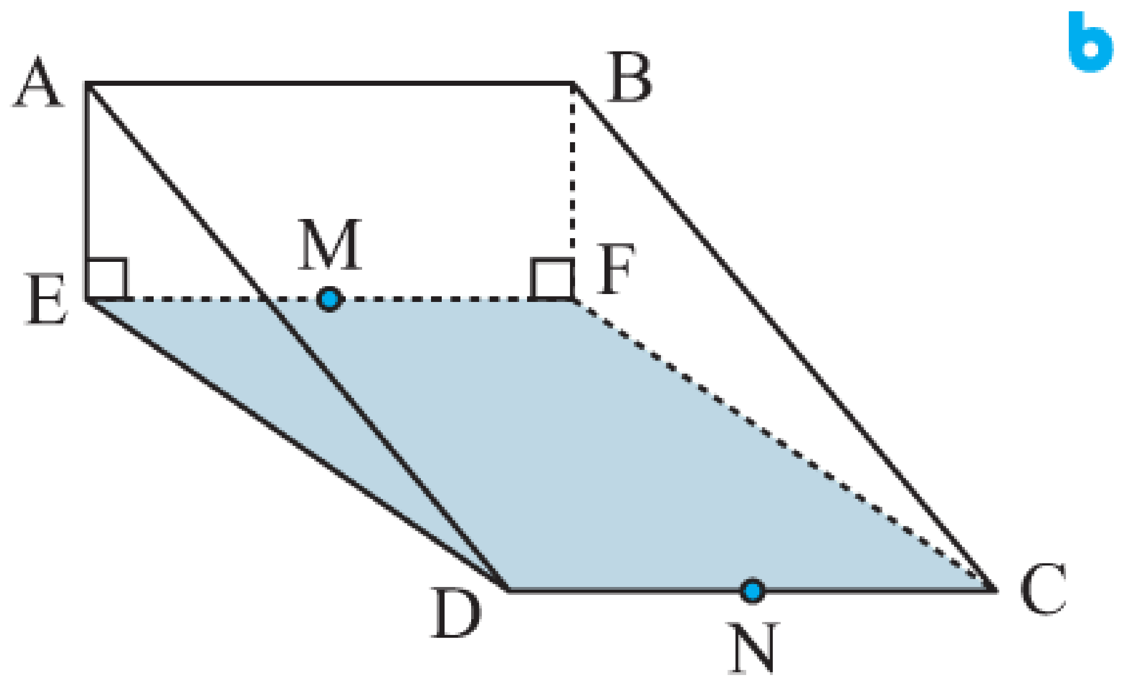


b
- **i** [RX]
- **ii** [NX]

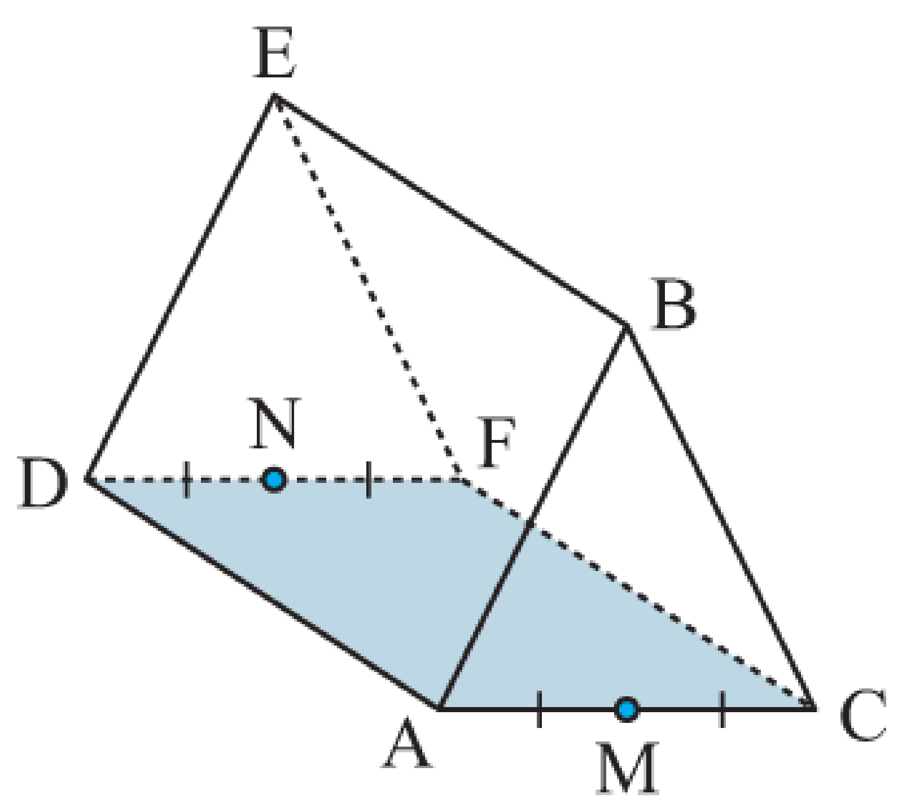


2 Name the angle between the following line segments and the base plane of the figure:

a **i** [AF]
ii [BM]
iii [AD]
iv [BN]

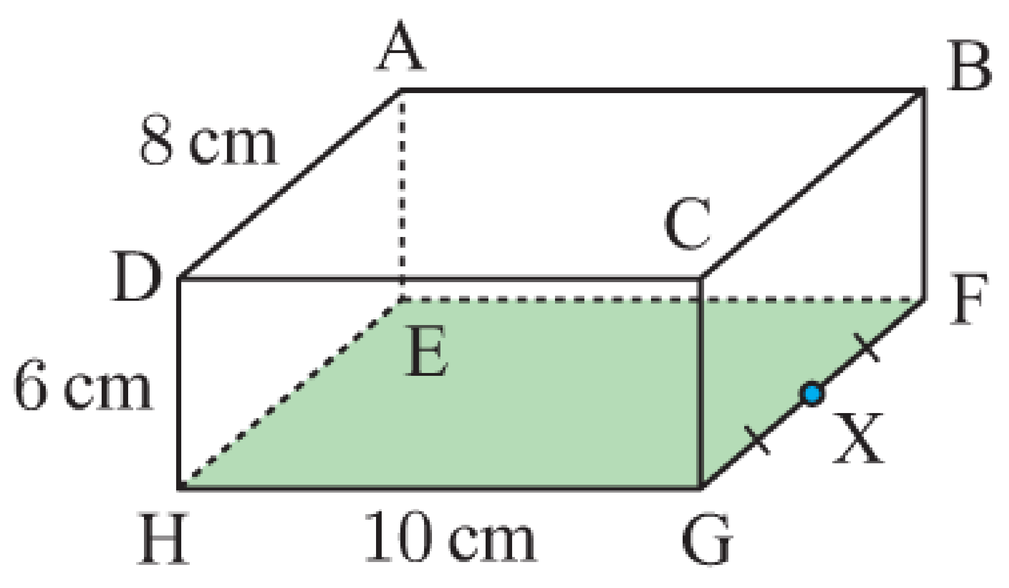


b **i** [AB]
ii [BN]
iii [AE]

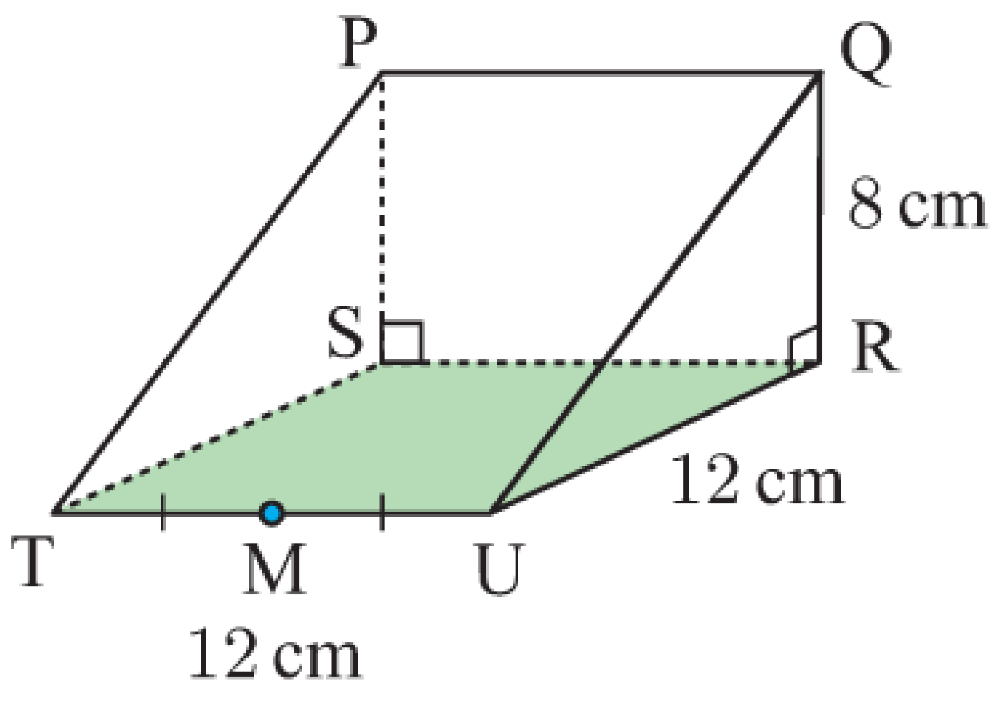


3 Find the angle between the following line segments and the base plane of the figure:

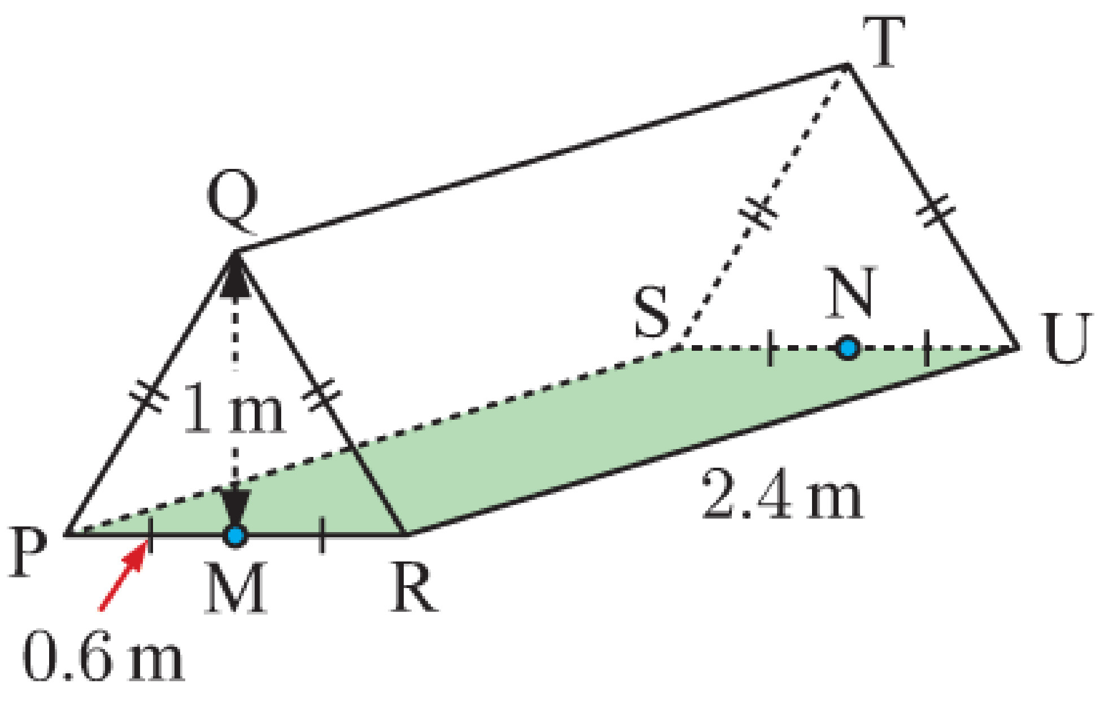
a **i** [CF]
ii [AG]
iii [BX]
iv [DX]



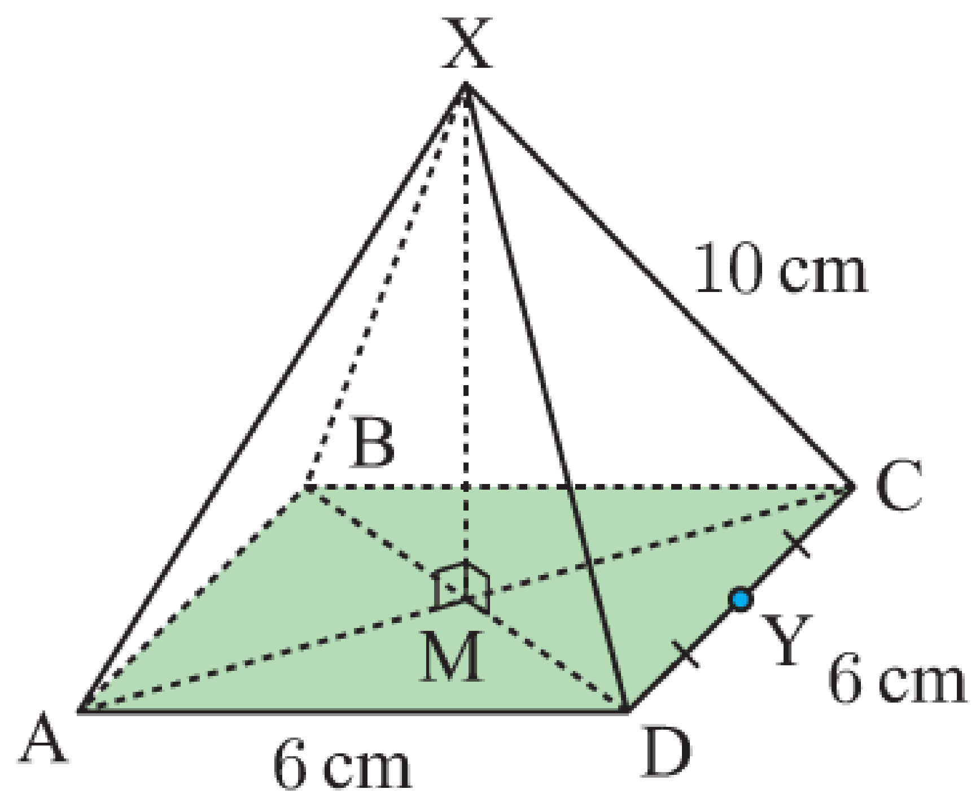
b **i** [PR]
ii [QU]
iii [PU]
iv [QM]



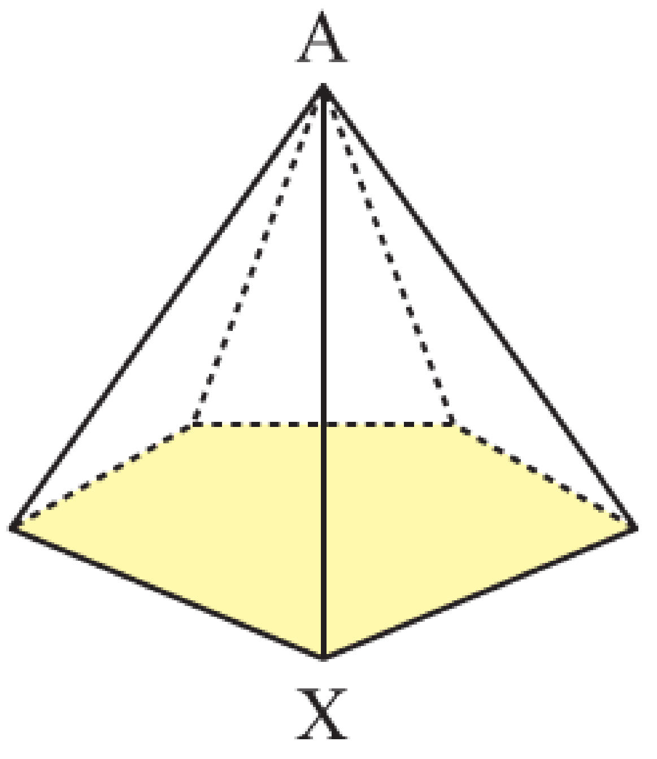
c **i** [QR]
ii [QU]
iii [QN]



d **i** [AX]
ii [XY]



4 The base of this pyramid is a regular pentagon. All sides of the pyramid have the same length.
Find the angle between [AX] and the base plane.



RESEARCH SUNDIALS

1 Who invented sundials?

2 The gnomon of a sundial is the tilted arm that extends above the dial.

- **a** At what time of day will the shadow of the gnomon be its mathematical projection on the dial?
- **b** Why does the longitude of a sundial matter?
- **c** How does the shadow help us tell the time at other times of the day?

3 Visit www.shadowspro.com to investigate sundials further. What other types of sundials are there? How do they differ?

4 In the 2018 ITV documentary *The Queen's Green Planet*, host Sir David Attenborough commented to Queen Elizabeth II of England that her sundial in the Buckingham Palace garden had been "neatly planted in the shade". The Queen asked her head gardener "Hadn't we thought of that? It wasn't in the shade originally, I'm sure. Maybe we could move it." But it was Sir David who responded "Depends if you want to tell the time or not!"

Jokes aside, the conversation highlights a very real problem that people faced for centuries: When the sky is overcast, one could neither use a sundial to tell the time, nor the stars to navigate.

What other inventions were made to help with timekeeping and navigation?

RESEARCH ASTROLABES

The **astrolabe** was invented around 200 BC. The Greek astonomer Hipparchus is often credited with its invention.

An astrolabe is an astronomical model of the celestial sphere. It was used primarily to take astronomical measurements such as the altitudes of astronomical bodies, but philosophers, astrologers, and sailors found many other uses for it.

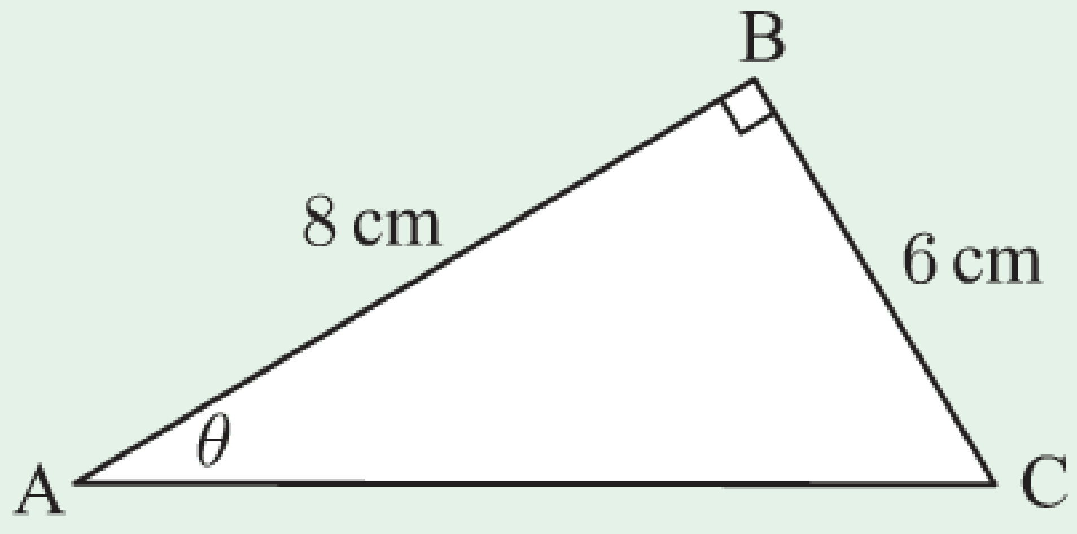
The astrolabe provided accurate measurements of the entire sky, such as the position of the sun, moon, and other heavenly bodies, and accurate times for sunrises, sunsets, and phases of the moon.

On land, we may be concerned about where we need to go, but at sea we also need to know where we are now. The astrolabe could be used to determine latitude and longitude, and measure altitude.

1. How was an astrolabe made?
2. How exactly does an astrolabe work?

REVIEW SET 7A

1



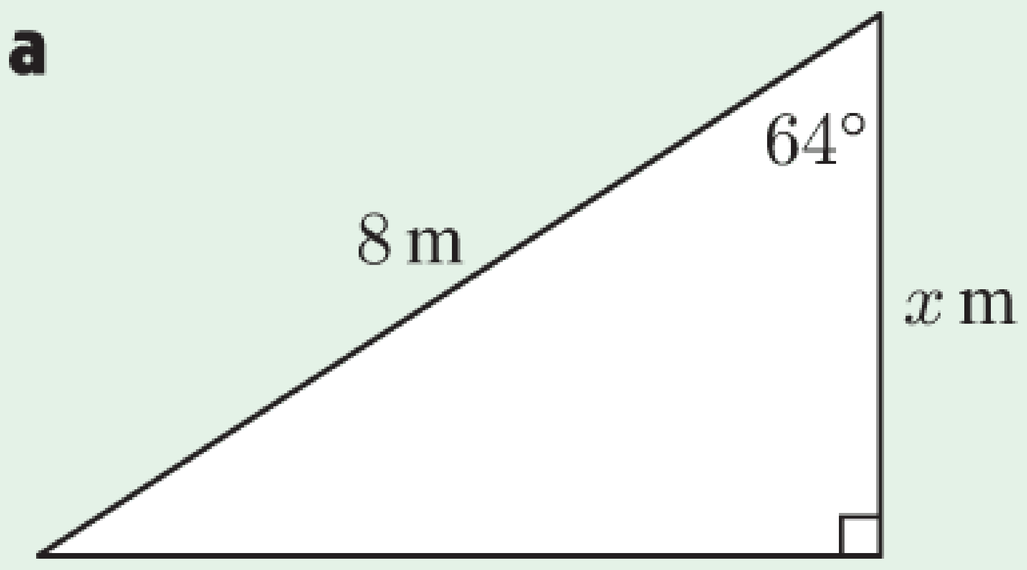
For the triangle alongside, find:

a the length of the hypotenuse

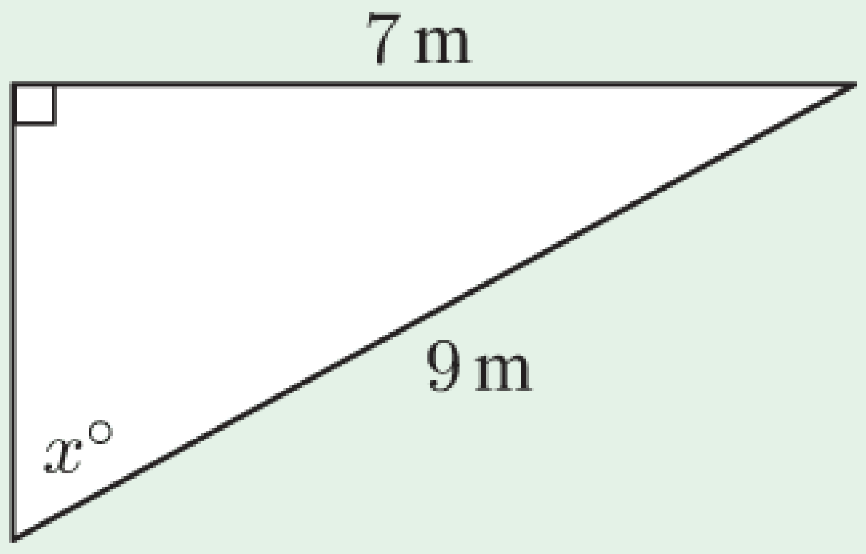
b $\sin\theta$ **c** $\cos\theta$ **d** $\tan\theta$.

2 Find x:

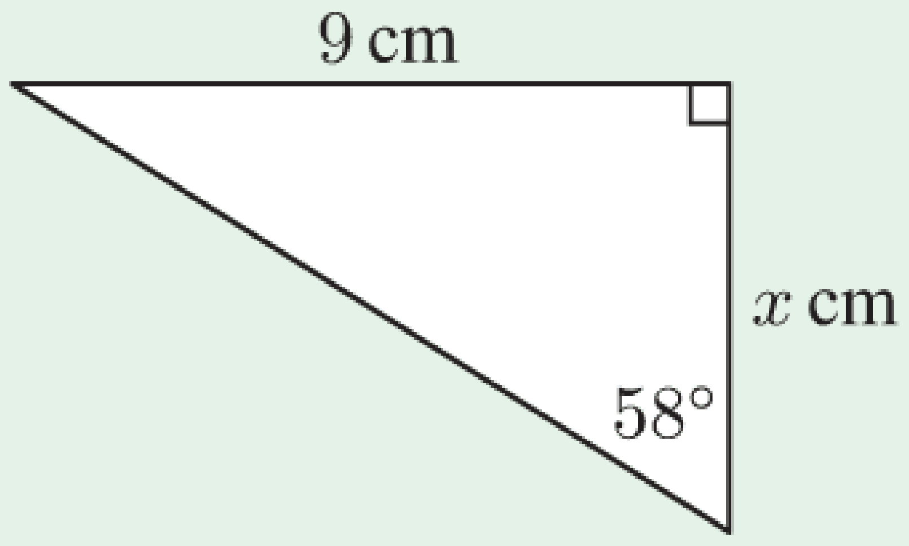
a



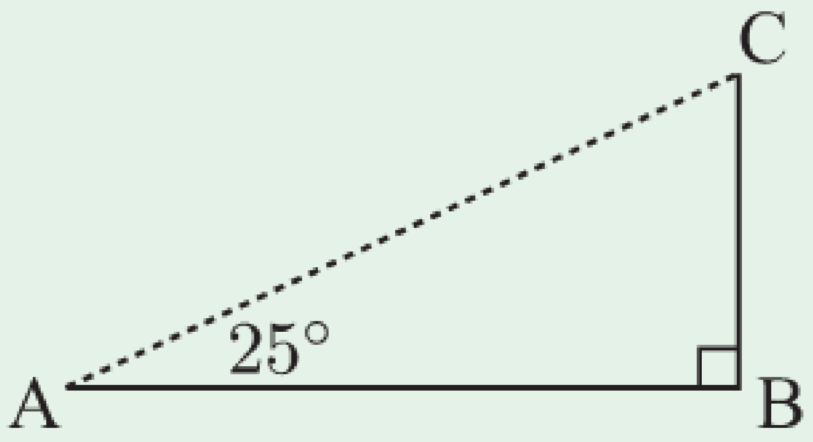
b



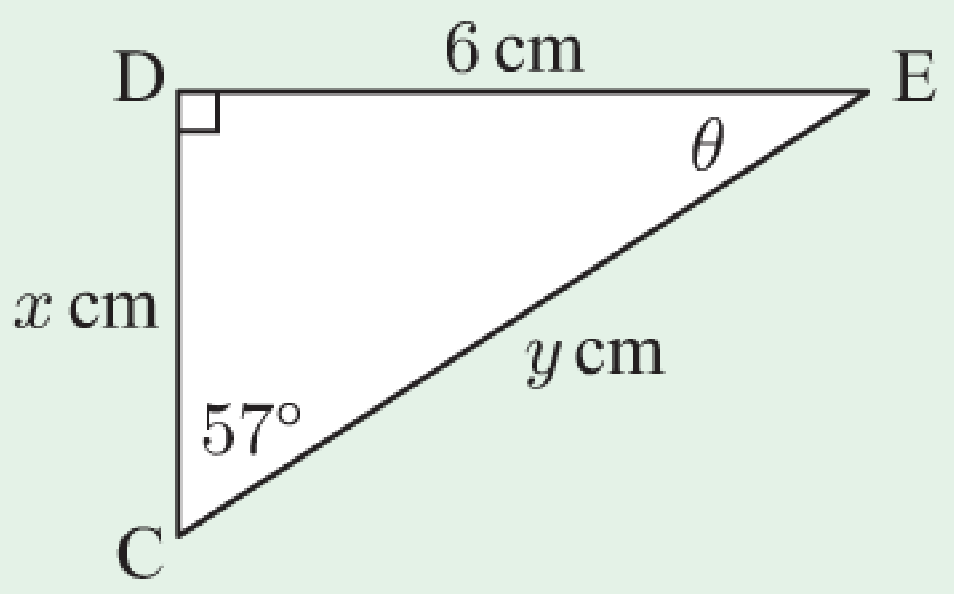
c



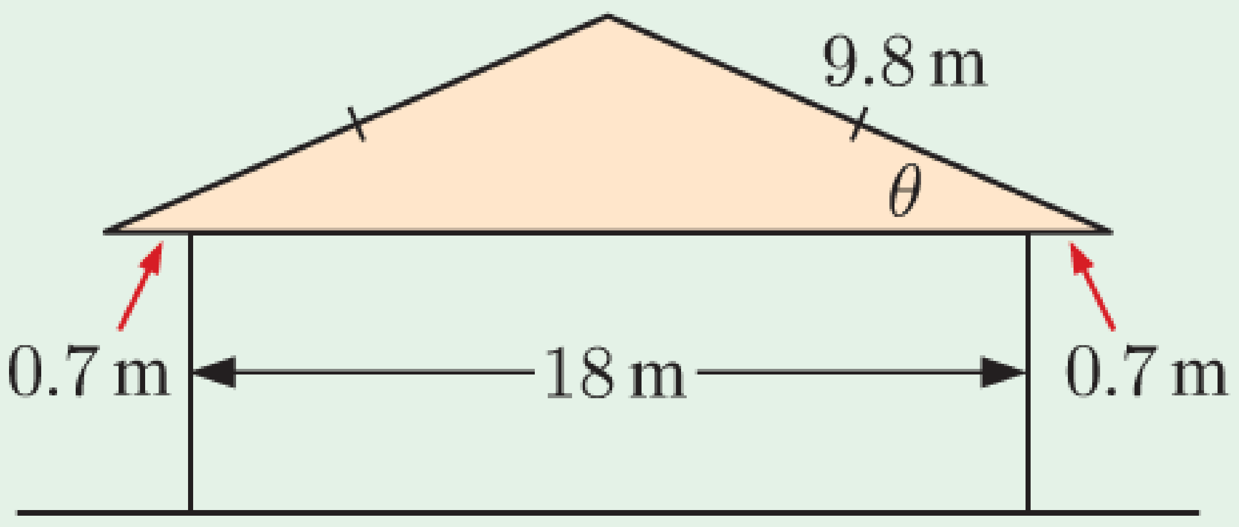
3 A 20 cm piece of wire was bent at B as shown alongside.
Find the area of triangle ABC.



4 Find the measure of all unknown sides and angles in triangle CDE:



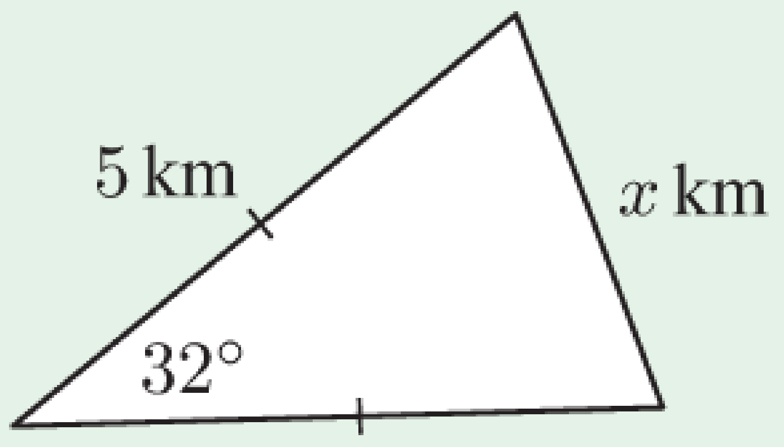
5 Find θ, the pitch of the roof.



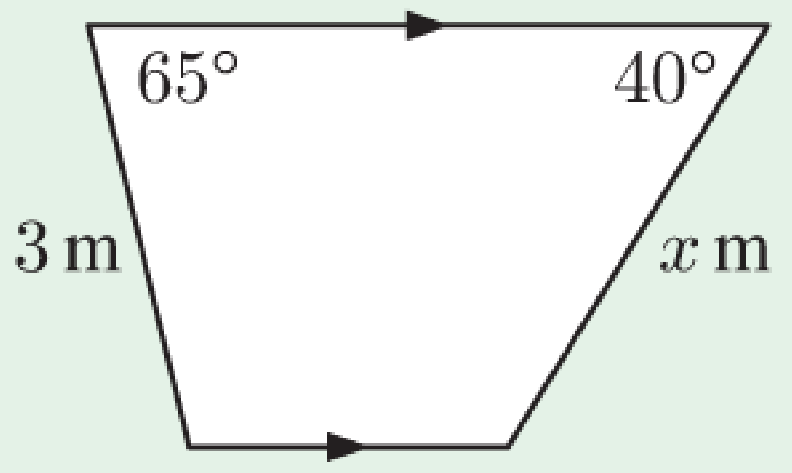
6 A rhombus has diagonals of length 15 cm and 8 cm. Find the larger angle of the rhombus.

7 Find, correct to 2 significant figures, the value of x:

a



b



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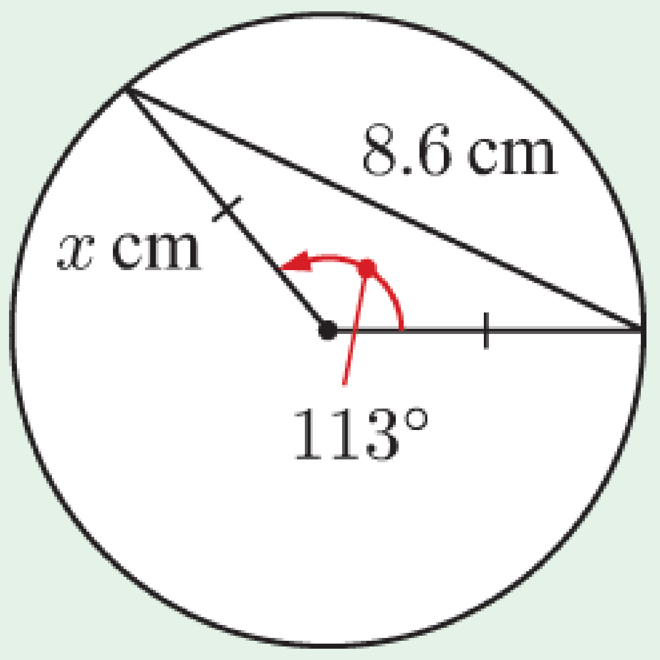
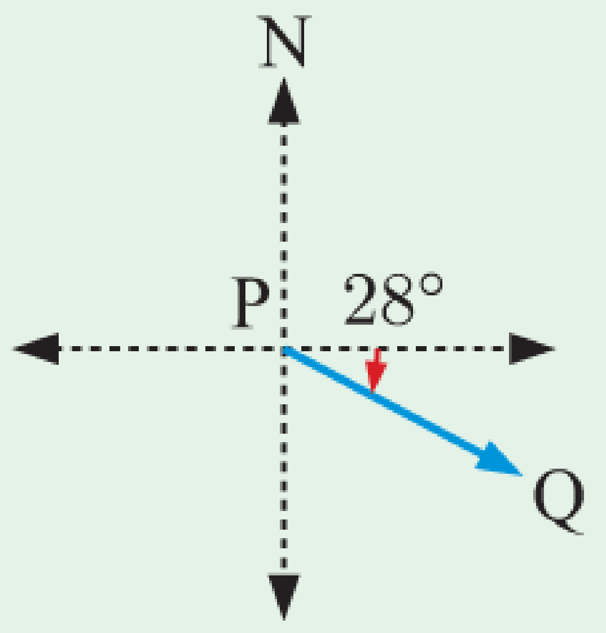


8 From a point 20 m horizontally from the base of a cylindrical lighthouse, the angle of elevation to the top of the lighthouse is 34°. Find the height of the lighthouse.

9 Find the bearing of Q from P in each diagram:

a



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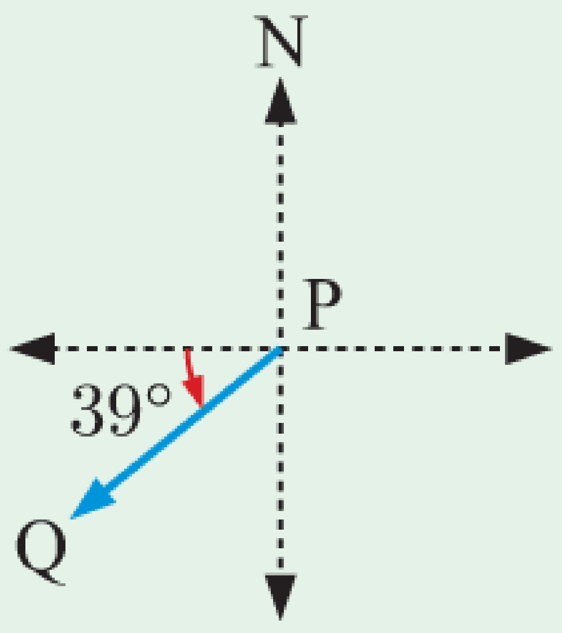
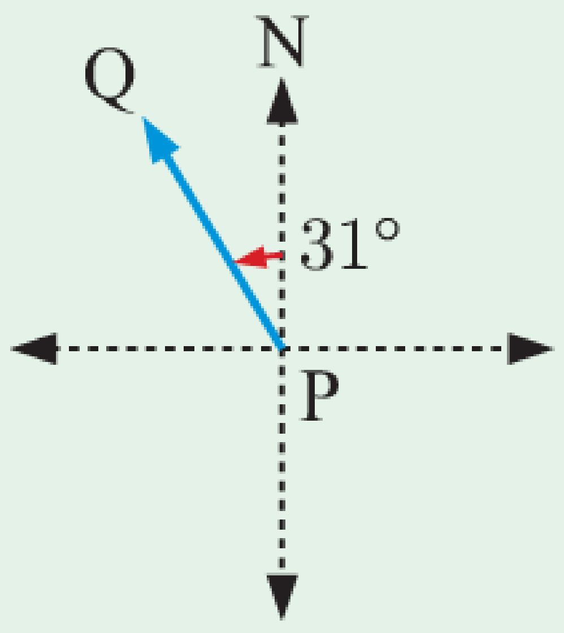
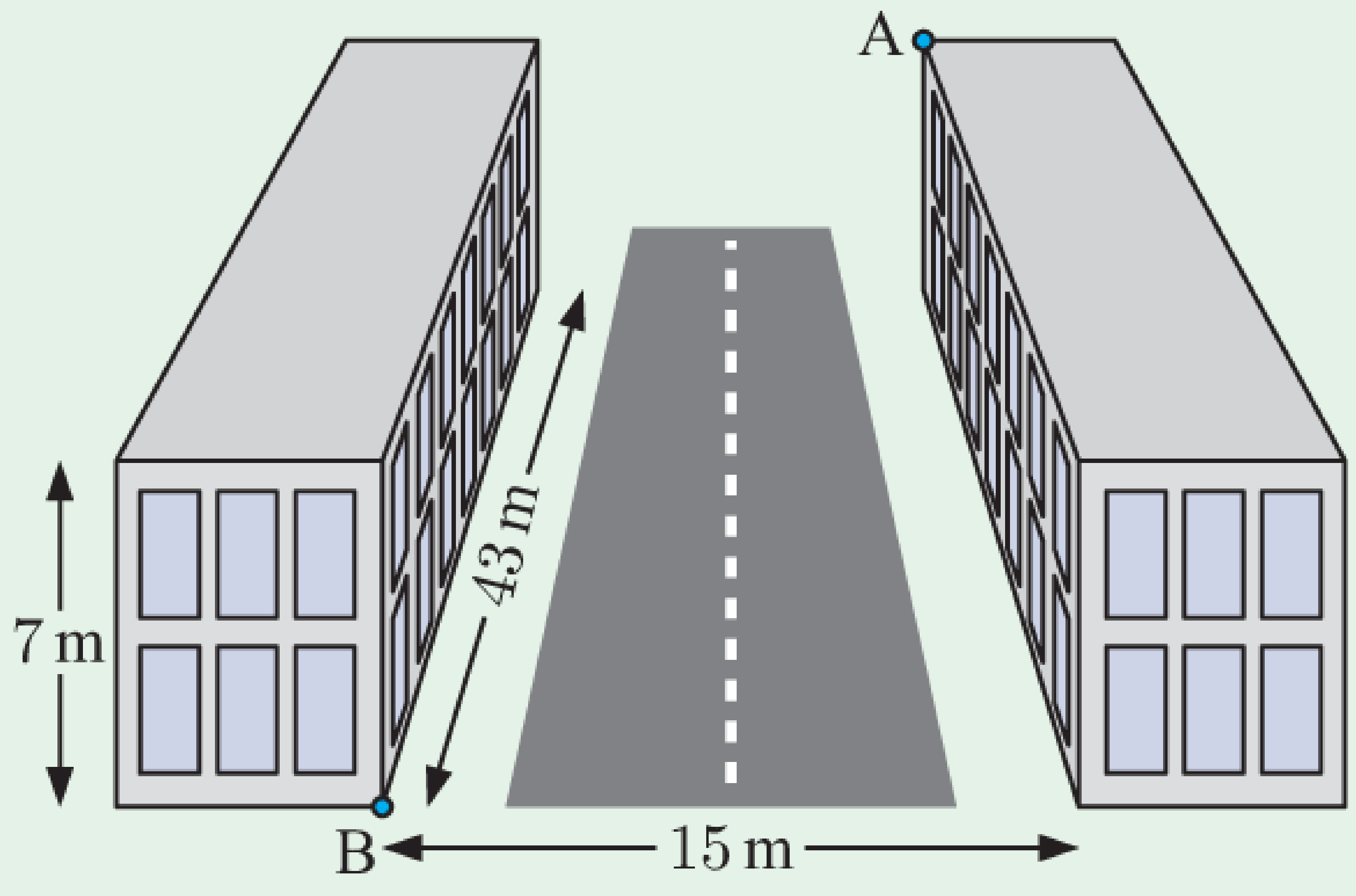


c

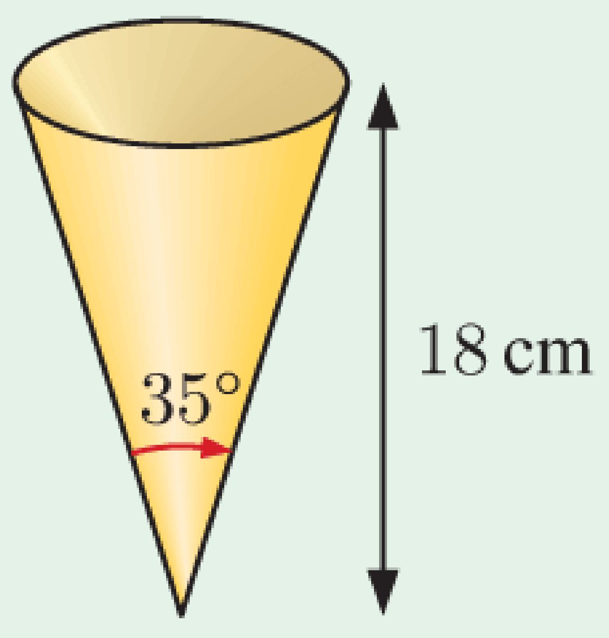


10 After a short flight, a helicopter is 12 km south and 5 km west of its helipad. Find the helicopter's distance and bearing from the helipad.

11 Two identical buildings stand parallel on opposite sides of a road. Find the angle of depression from A to B.

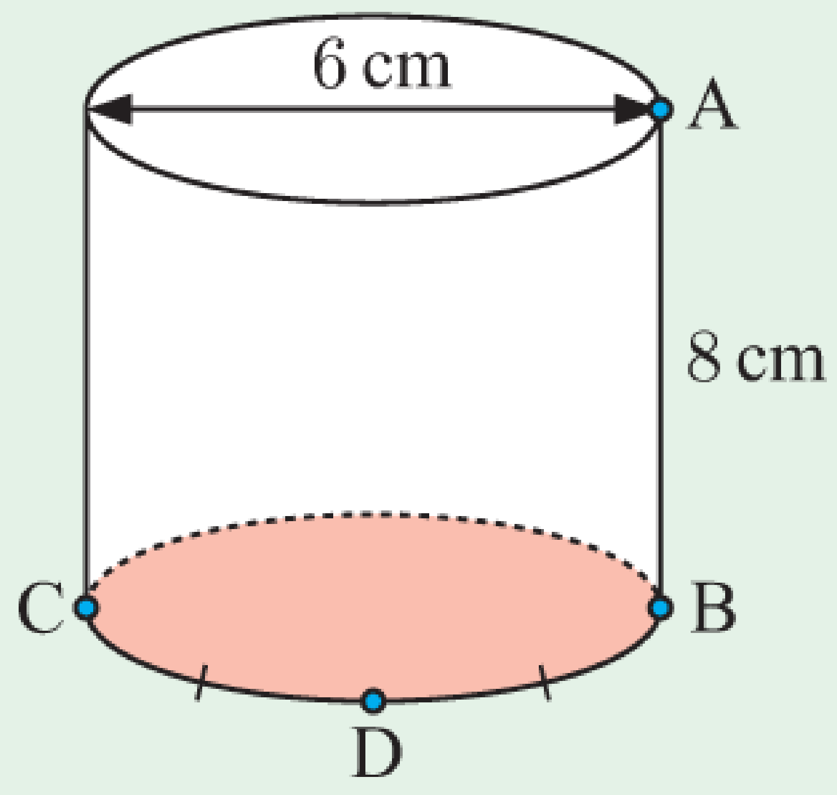


12 An open cone has a vertical angle measuring 35° and a height of 18 cm. Find the capacity of the cone in litres.



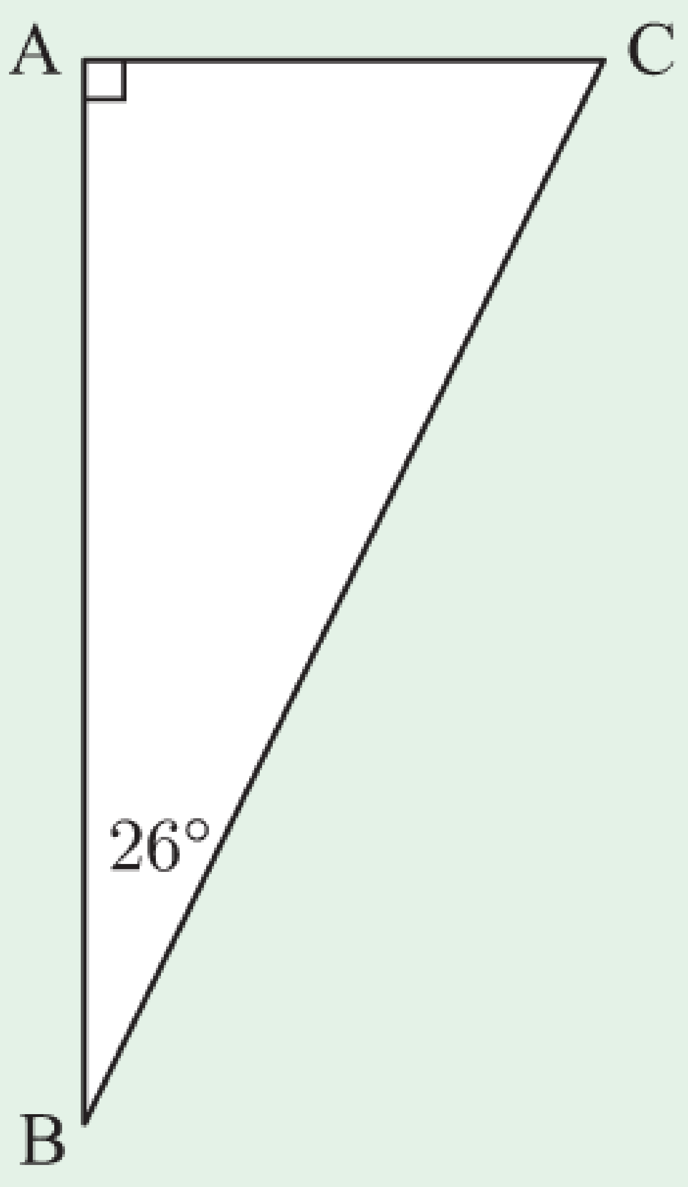
13 Find the angle between the following line segments and the base plane of the figure:

a [AC] **b** [AD]



REVIEW SET 7B

1 The right angled triangle alongside contains an angle of 26°.



a Use a ruler to measure the length of each side, to the nearest millimetre.

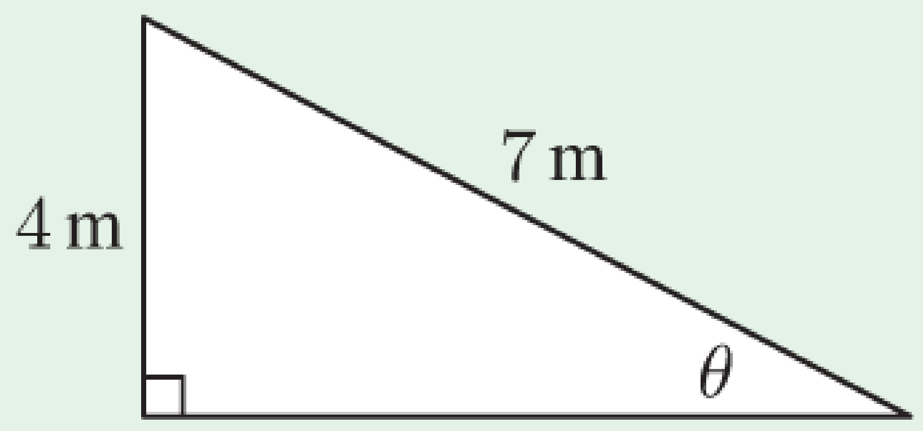
b Hence estimate the value, to 2 decimal places, of:

i sin 26° **ii** cos 26° **iii** tan 26°.

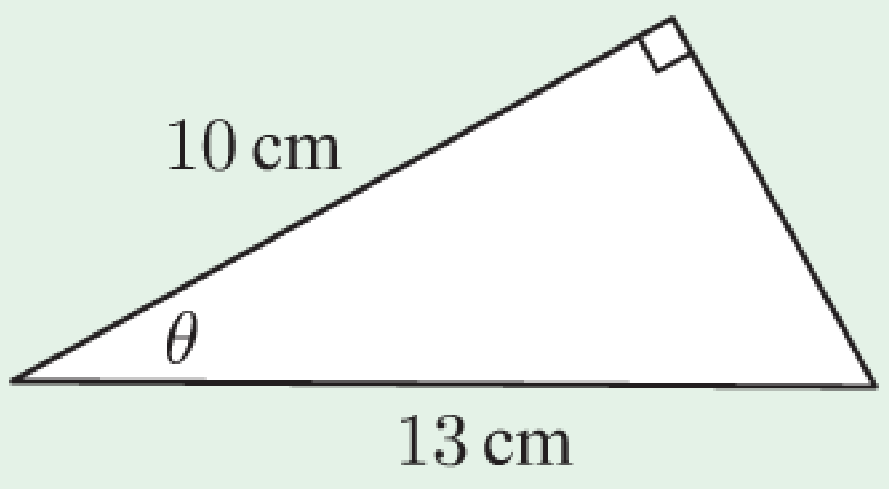
c Check your answers using a calculator.

2 Find the angle marked θ:

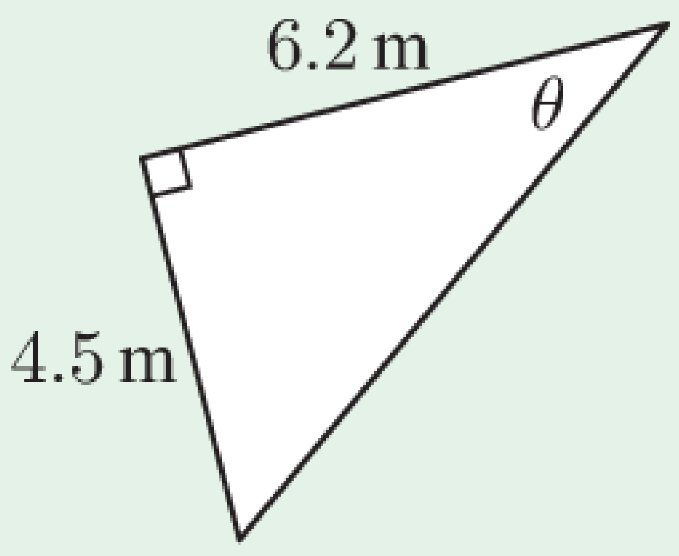
a



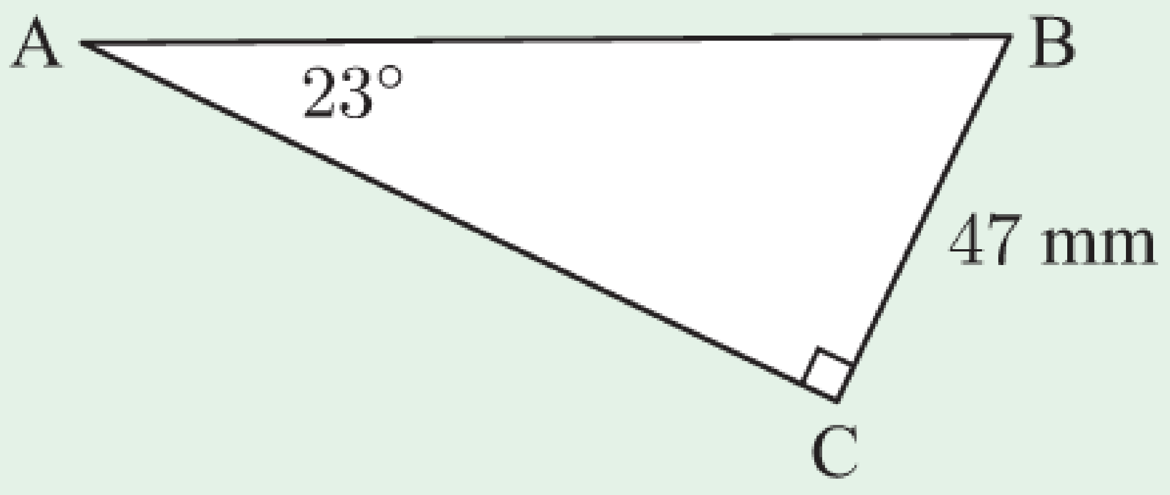
b



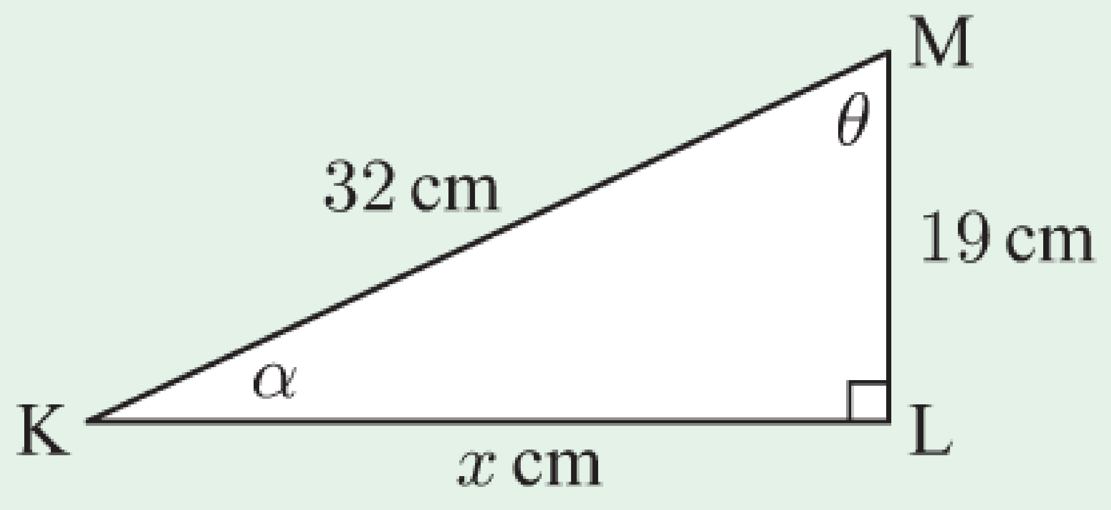
c



3 Find the lengths of the unknown sides:



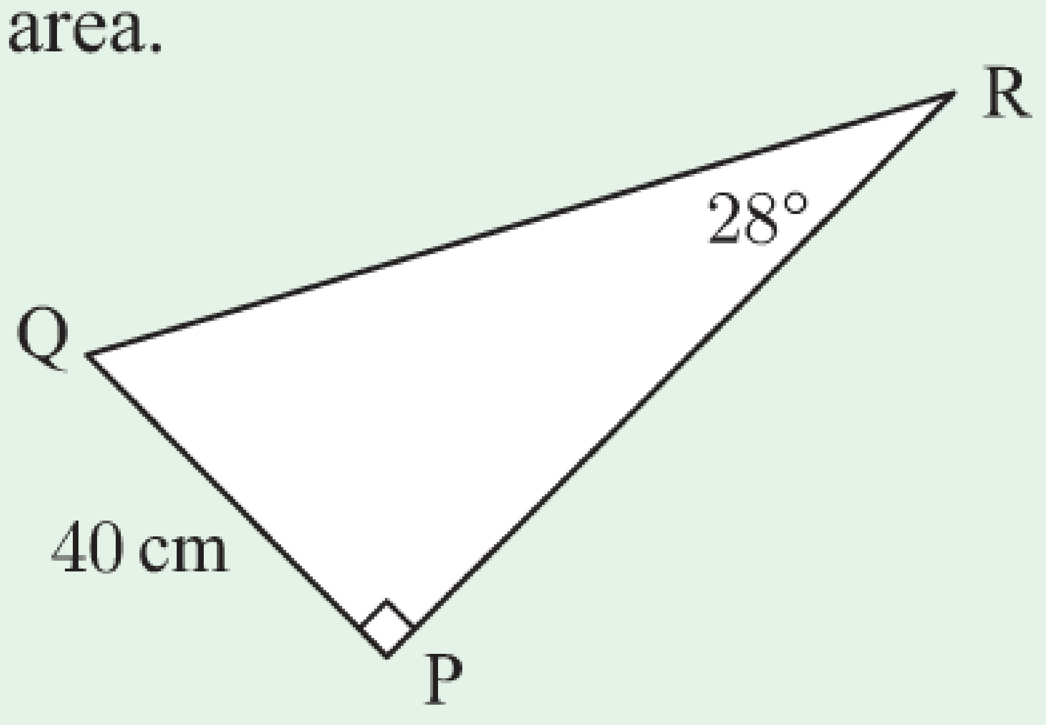
4 Find the measure of all unknown sides and angles in triangle KLM:



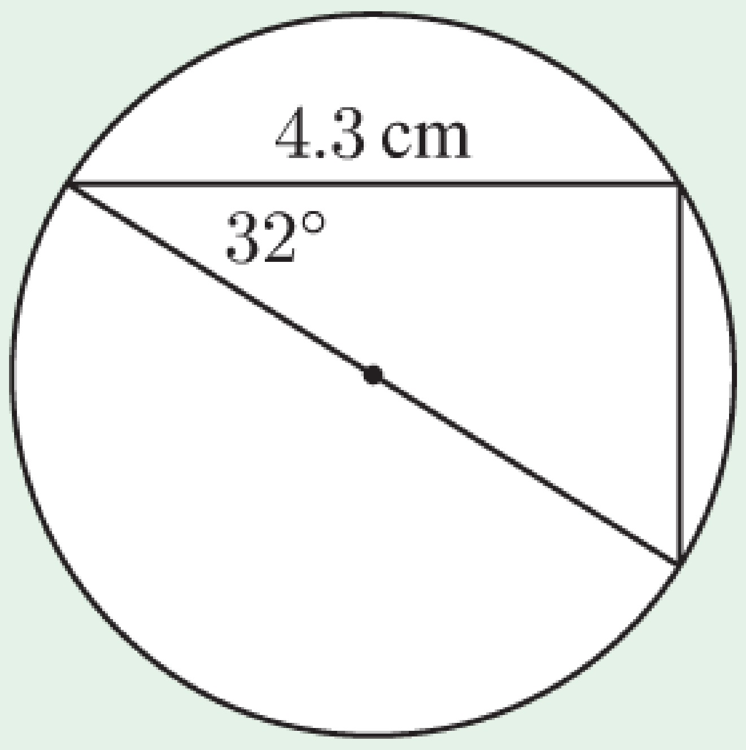
5 For the triangle PQR shown, find the:

a perimeter

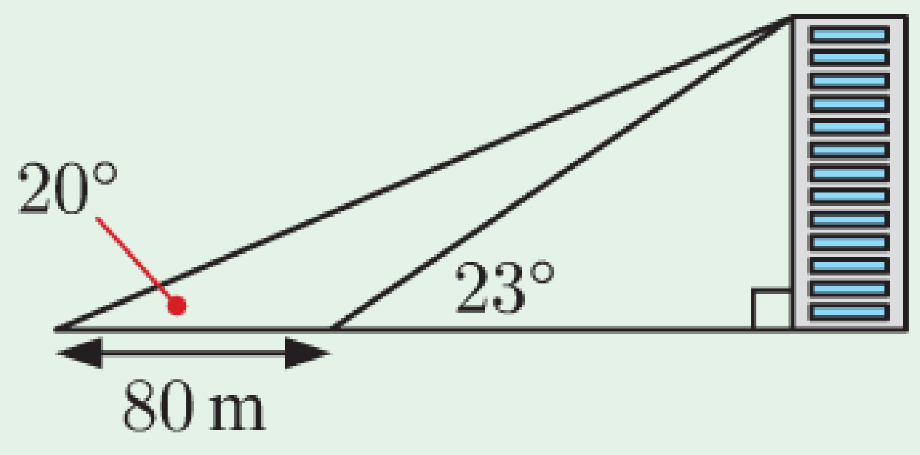
b area.



6 Find the radius of the circle:

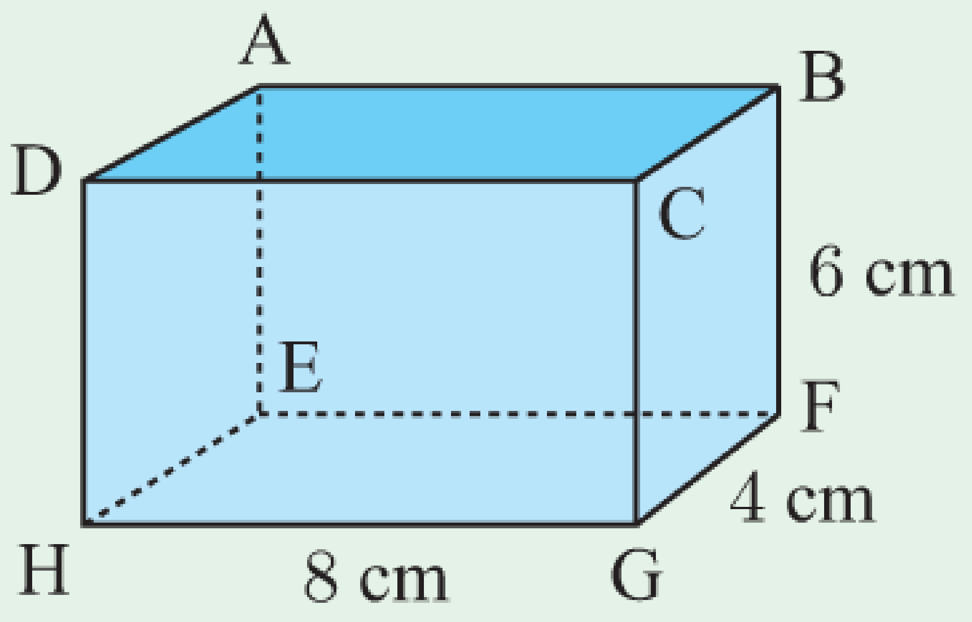


7 From a given point, the angle of elevation to the top of a tall building is 20°. After walking 80 m towards the building, the angle of elevation is now 23°. How tall is the building?



8 For the rectangular prism shown, find:

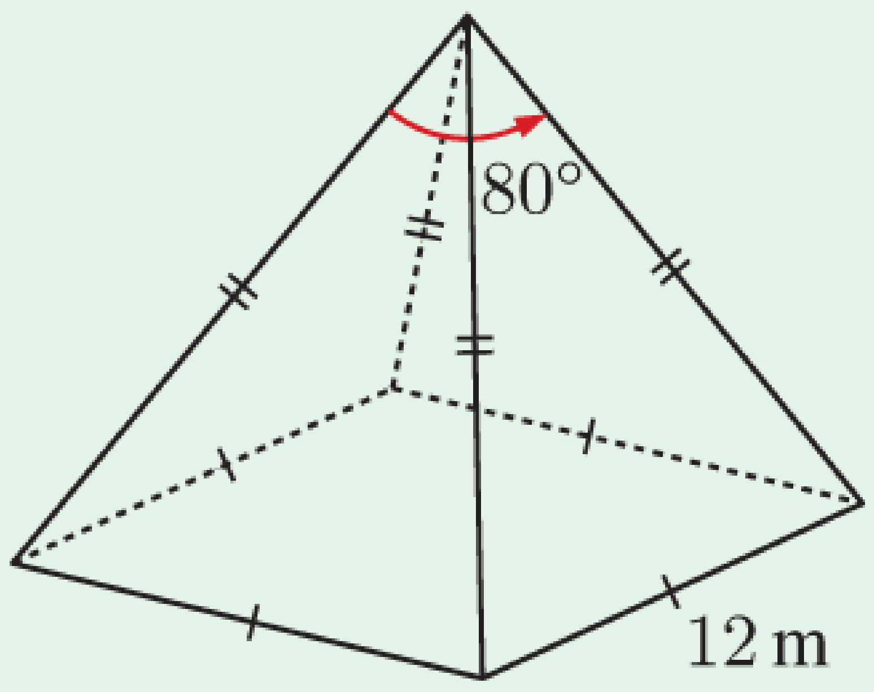
a $A\widehat{H}G$ **b** $D\widehat{F}H$.



9 Aaron ran 3 km on the bearing 213°, then 2.5 km on the bearing 303°. Find Aaron's distance and bearing from his starting point.

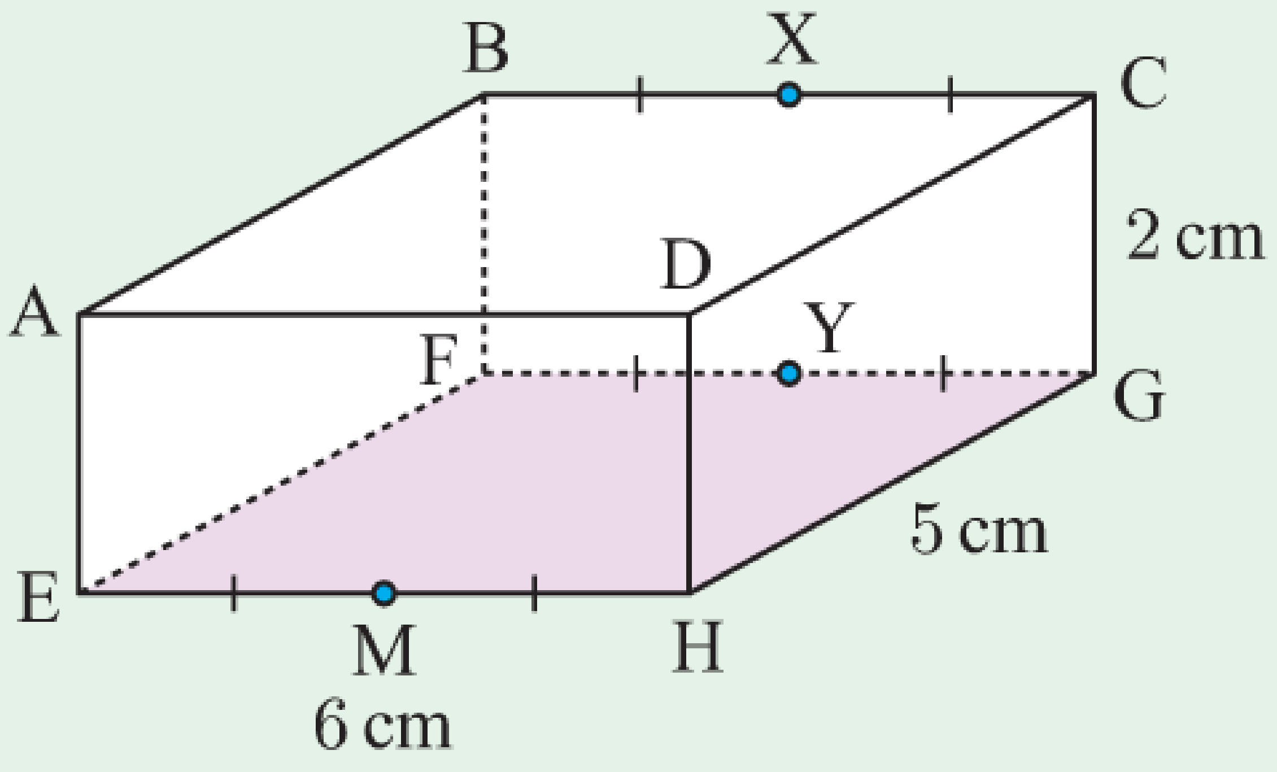
10 Joggers Amelia and Kristos depart from the same position at the same time. Amelia jogs in the direction 074°, and Kristos jogs in the direction 164°. After 30 minutes, Amelia has jogged 2 km further than Kristos, and her bearing from Kristos is 044°.
How far apart are the joggers at this time?

11 Find the volume of this pyramid:

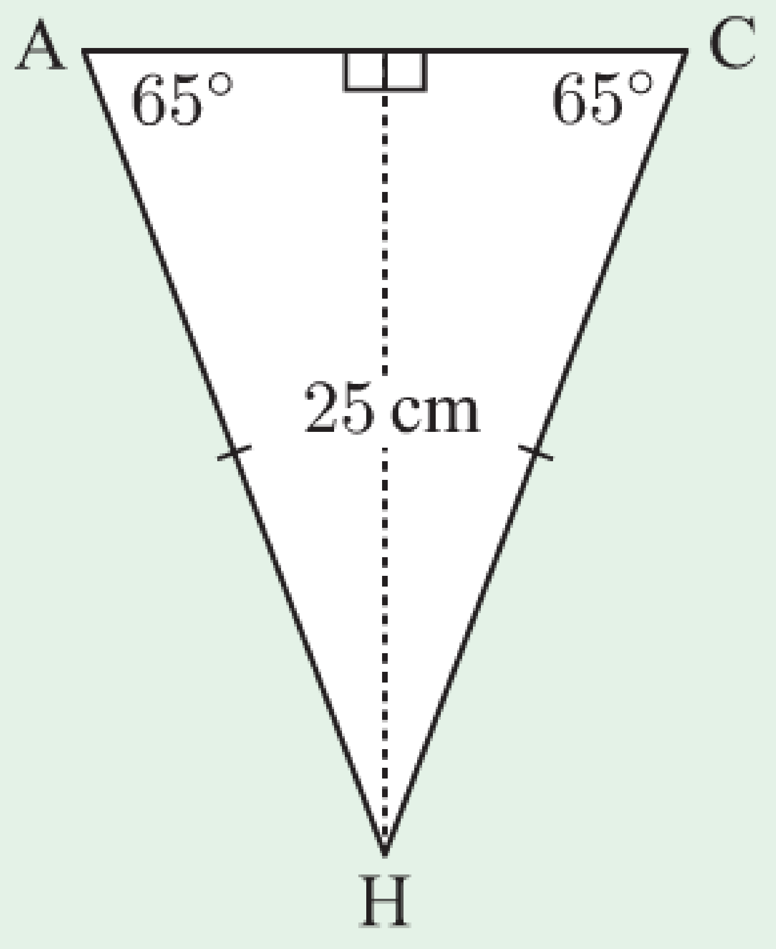


12 Find the angle between the following line segments and the base plane of the given figure:

a [BH] **b** [CM] **c** [XM]

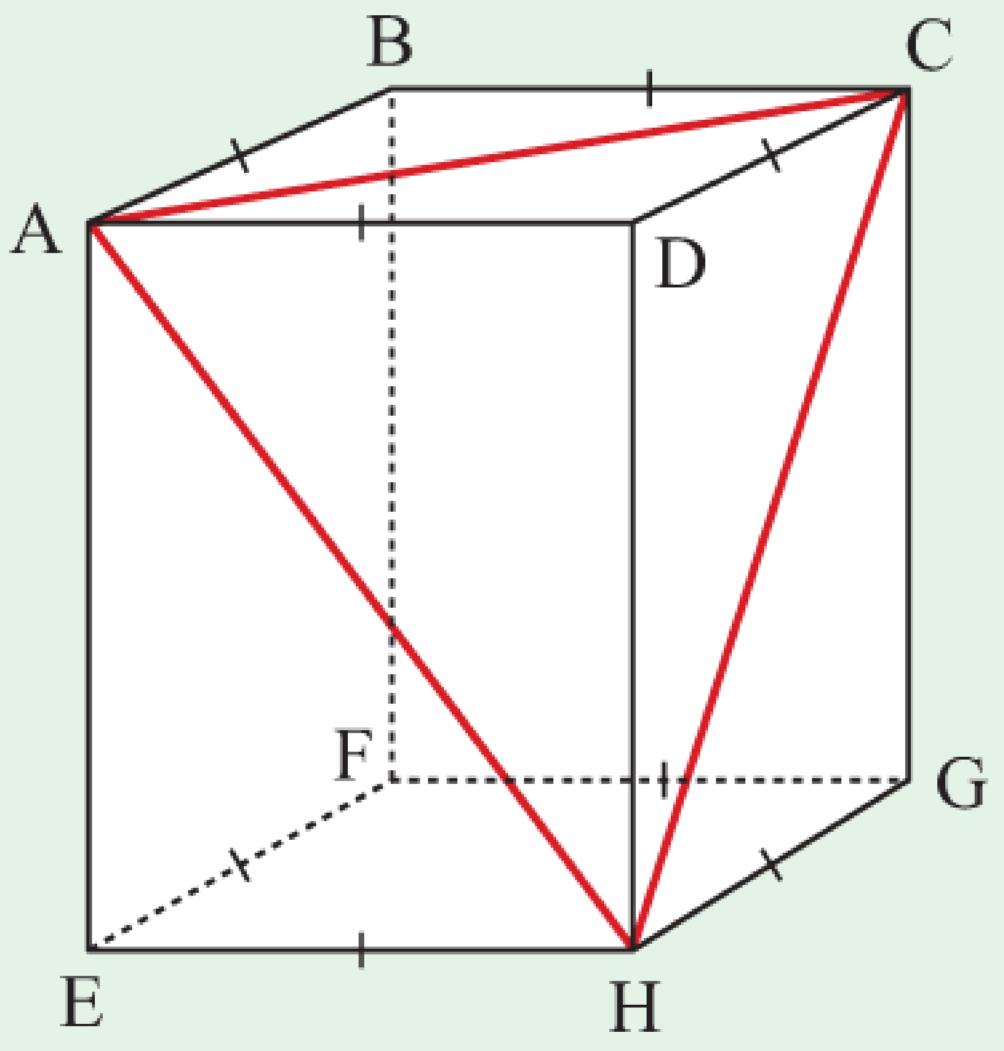


13 Triangle ACH is isosceles with altitude 25 cm and base angles $H\widehat{A}C = H\widehat{C}A = 65°$.



a Find the length:

i AH **ii** AC.



b Triangle ACH lies within the square-based rectangular prism shown. Find the volume of this prism.

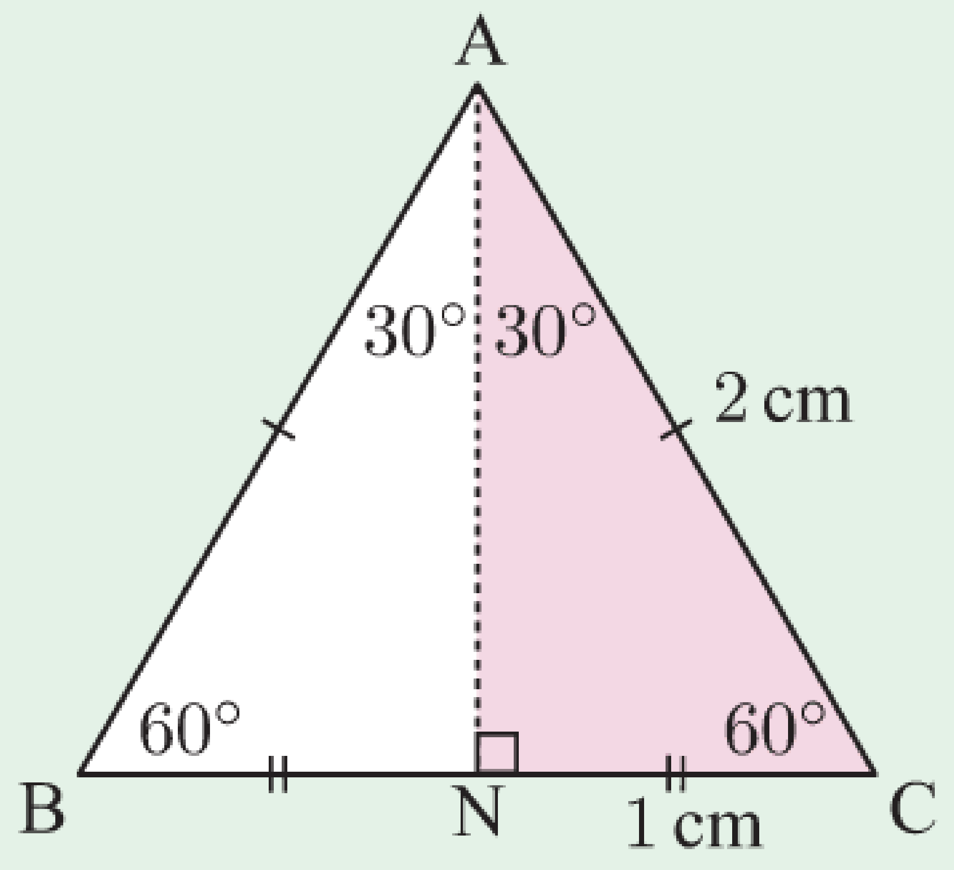
Chapter 8

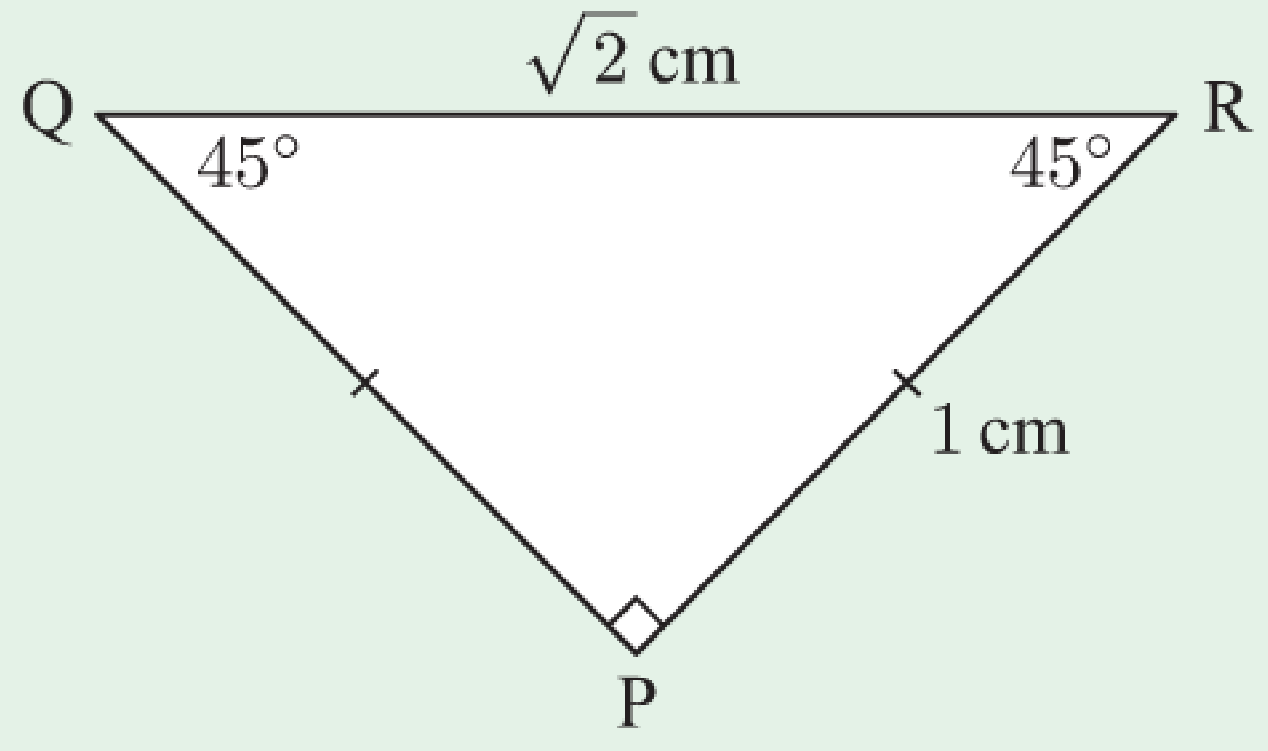
The unit circle and radian measure

Contents:

OPENING PROBLEM

Consider the triangles below:





Things to think about:

a Triangle ABC is an equilateral triangle with sides 2 cm long. Altitude [AN] bisects side [BC] and the vertical angle BAC.
Can you use this figure to find:
i $\sin 30°$ **ii** $\cos 60°$ **iii** $\cos 30°$ **iv** $\sin 60°$?

b Triangle PQR is a right angled isosceles triangle with hypotenuse $\sqrt{2}$ cm long.
Can you use this figure to find:
i $\cos 45°$ **ii** $\sin 45°$ **iii** $\tan 45°$?

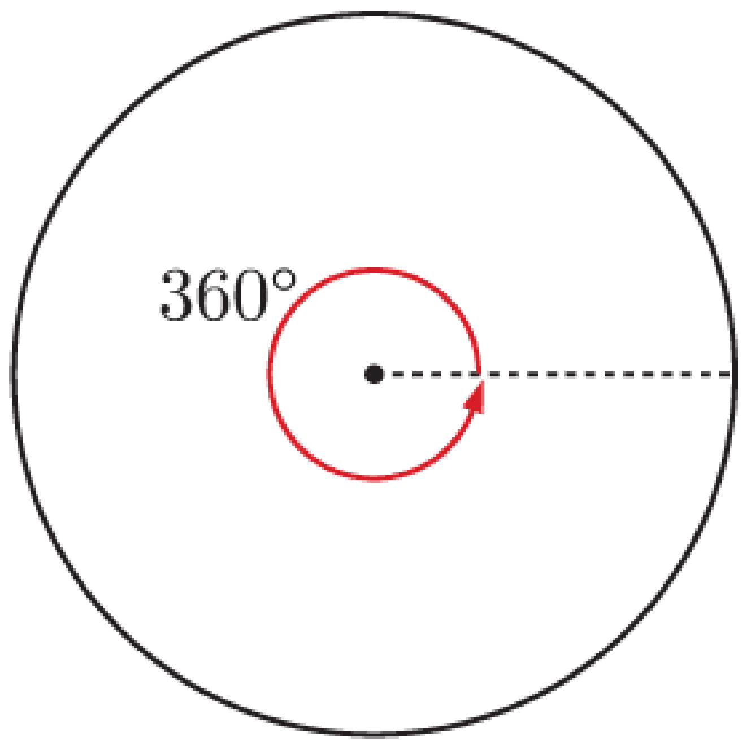
In this Chapter we build on our knowledge of angles and trigonometry. We consider:

- **radian** measure as an alternative to degrees
- the **unit circle** which helps us give meaning to the trigonometric ratios for *any* angle.

A RADIAN MEASURE

DEGREE MEASUREMENT OF ANGLES

We have seen previously that one full revolution makes an angle of 360°, and the angle on a straight line is 180°. Hence, one **degree**, 1°, can be defined as $\frac{1}{360}$th of one full revolution. This measure of angle is commonly used by surveyors and architects.



For greater accuracy we define one **minute**, $1'$, as $\frac{1}{60}$th of one degree and one **second**, $1''$, as $\frac{1}{60}$th of one minute. Obviously a minute and a second are very small angles.

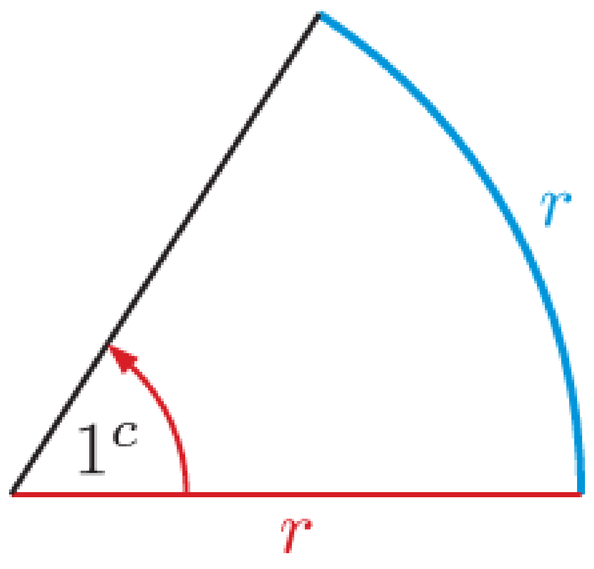
Most graphics calculators can convert fractions of angles measured in degrees into minutes and seconds. This is also useful for converting fractions of hours into minutes and seconds for time measurement, as one minute is $\frac{1}{60}$th of one hour, and one second is $\frac{1}{60}$th of one minute.

GRAPHICS CALCULATOR INSTRUCTIONS

RADIAN MEASUREMENT OF ANGLES

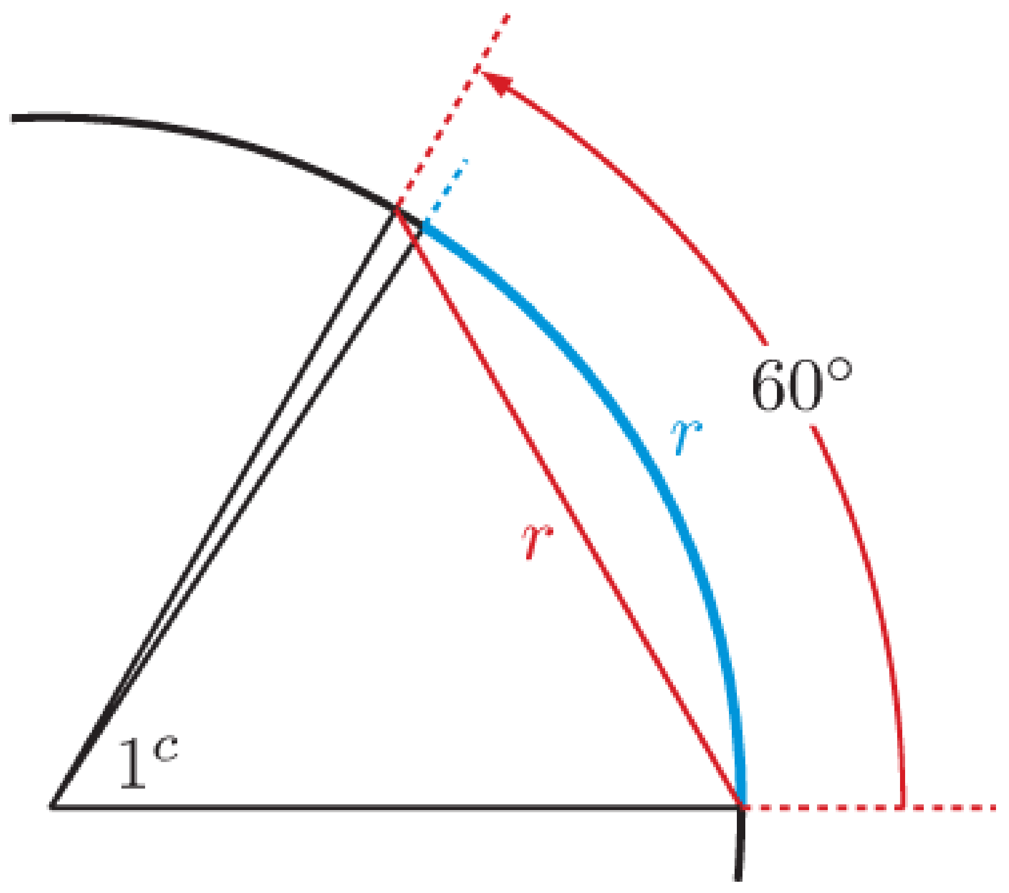
Around 1400 AD, the Persian mathematician **Al-Kashi** began measuring angles according to the length of the arc of a circle that the angle subtends. This idea was developed into **radian** measure by the Englishman **Roger Coates** in 1714. The word "radian" is an abbreviation of "radial angle".

Suppose the arc length formed by an angle is the same length as the radius. This angle is said to have a measure of 1 **radian** (1^c).



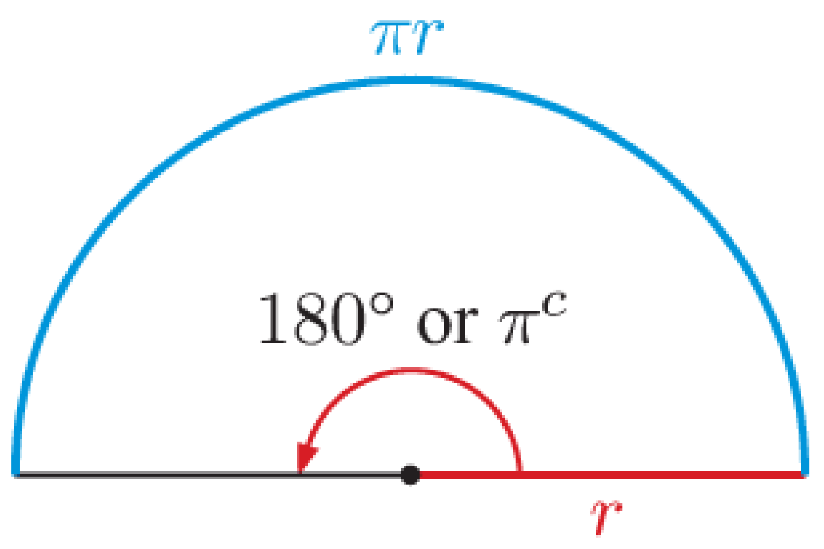
The symbol "c" is used for radian measure but is usually omitted. By contrast, the degree symbol is *always* used when the measure of an angle is given in degrees.

From the diagram to the right, it can be seen that 1^c is slightly smaller than $60°$. In fact, $1^c \approx 57.3°$.



DEGREE-RADIAN CONVERSIONS

Consider a semi-circle of radius r. The arc length is πr, so there are π radians in a semi-circle.

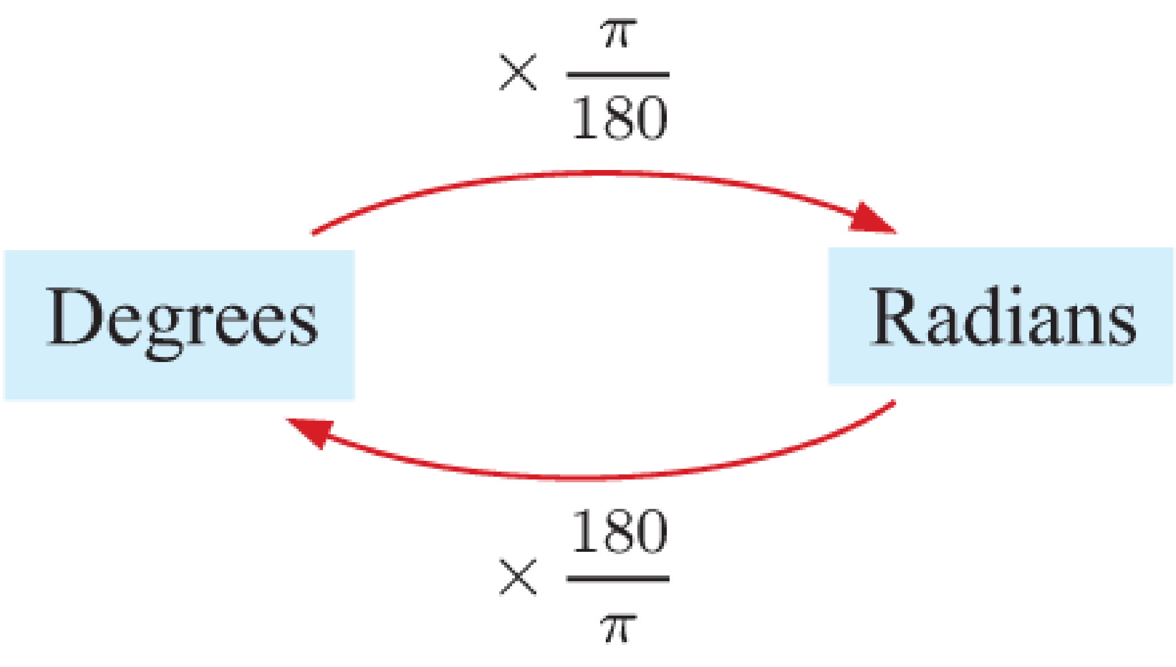


Therefore, $\boxed{\pi \text{ radians} \equiv 180°.}$

So, $1^c = \left(\frac{180}{\pi}\right)^\circ \approx 57.3°$ and $1° = \left(\frac{\pi}{180}\right)^c \approx 0.0175^c$.

To convert from degrees to radians, we multiply by $\frac{\pi}{180}$.

To convert from radians to degrees, we multiply by $\frac{180}{\pi}$.



Example 1 **Self Tutor**

Convert:

a $45°$ to radians, in terms of π **b** $126.5°$ to radians.

a $45° = \left(45 \times \frac{\pi}{180}\right)$ radians
$= \frac{\pi}{4}$ radians

b $126.5°$
$= \left(126.5 \times \frac{\pi}{180}\right)$ radians
≈ 2.21 radians

Angles in radians may be expressed either in terms of π or as decimals.

EXERCISE 8A

1 Convert to radians, in terms of π:

a 90° **b** 60° **c** 30° **d** 18° **e** 9°

f 135° **g** 225° **h** 270° **i** 360° **j** 720°

k 315° **l** 540° **m** 36° **n** 80° **o** 230°

2 Convert to radians, correct to 3 significant figures:

a 36.7° **b** 137.2° **c** 317.9° **d** 219.6° **e** 396.7°

Example 2 **Self Tutor**

Convert to degrees:

a $\frac{5\pi}{6}$ **b** 0.638 radians.

a $\frac{5\pi}{6}$

$= \left(\frac{5\pi}{6} \times \frac{180}{\pi}\right)^\circ$

$= 150^\circ$

b 0.638 radians

$= \left(0.638 \times \frac{180}{\pi}\right)^\circ$

$\approx 36.6^\circ$

3 Convert to degrees:

a $\frac{\pi}{5}$ **b** $\frac{3\pi}{5}$ **c** $\frac{3\pi}{4}$ **d** $\frac{\pi}{18}$ **e** $\frac{\pi}{9}$

f $\frac{7\pi}{9}$ **g** $\frac{\pi}{10}$ **h** $\frac{3\pi}{20}$ **i** $\frac{7\pi}{6}$ **j** $\frac{\pi}{8}$

4 Convert to degrees, correct to 2 decimal places:

a 2 **b** 1.53 **c** 0.867 **d** 3.179 **e** 5.267

5 Copy and complete, giving answers in terms of π:

a

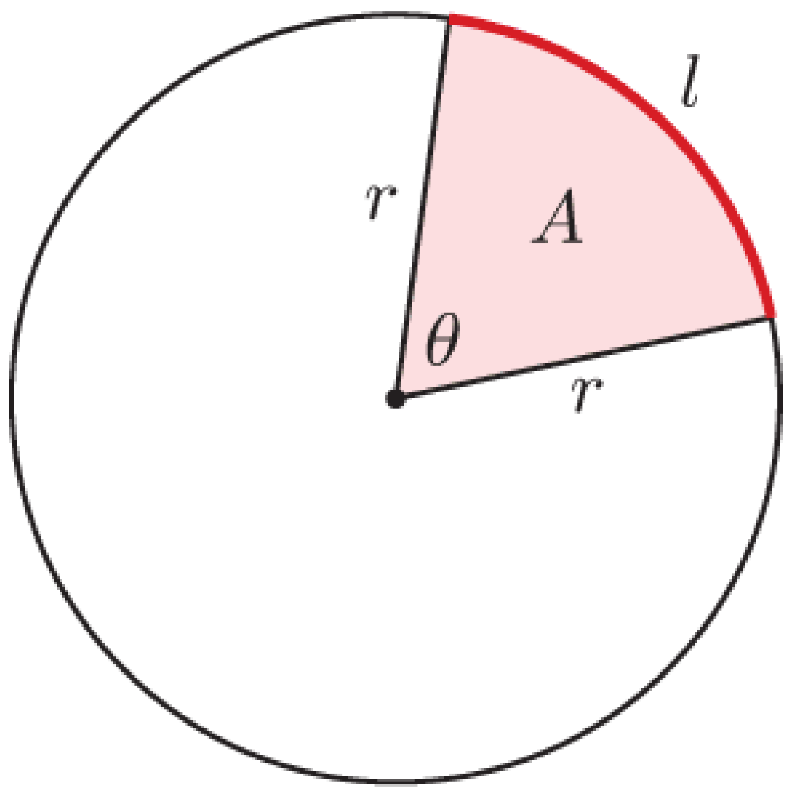
Degrees	0	45	90	135	180	225	270	315	360
Radians									

b

Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians													

B ARC LENGTH AND SECTOR AREA

You should have previously seen formulae for the length of an arc and the area of a sector, for an angle given in degrees.



For a sector with radius r and angle θ given in *degrees*,

$$\text{arc length} \quad \boldsymbol{l = \frac{\theta}{360} \times 2\pi r}$$

$$\text{area} \quad \boldsymbol{A = \frac{\theta}{360} \times \pi r^2}$$

However, if the angle θ is measured in radians, the formulae become much simpler.

- θ measures how many times longer the arc length is than the radius.

$$\therefore\ \theta = \frac{l}{r}$$

$$\therefore\ l = \theta r$$

- There are 2π radians in a circle so

$$\text{area of sector} = \frac{\theta}{2\pi} \times \text{area of circle}$$

$$\therefore\ A = \frac{\theta}{2\pi} \times \pi r^2$$

$$\therefore\ A = \tfrac{1}{2}\theta r^2$$

For a sector with radius r and angle θ given in *radians*:

- arc length $\boldsymbol{l = \theta r}$
- area $\boldsymbol{A = \frac{1}{2}\theta r^2}$

Example 3 **Self Tutor**

A sector has radius 12 cm and angle 3 radians. Find its:

a arc length **b** area

a $\text{arc length} = \theta r$

$= 3 \times 12$

$= 36$ cm

b $\text{area} = \frac{1}{2}\theta r^2$

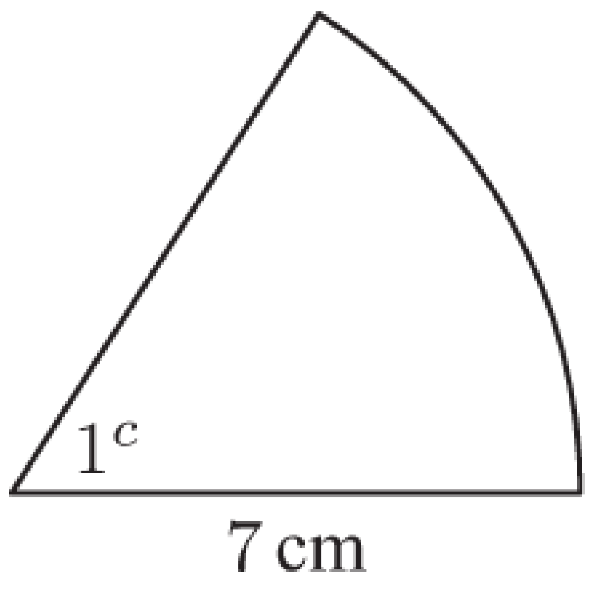
$= \frac{1}{2} \times 3 \times 12^2$

$= 216\ \text{cm}^2$

EXERCISE 8B

1 Find the arc length of each sector:

a



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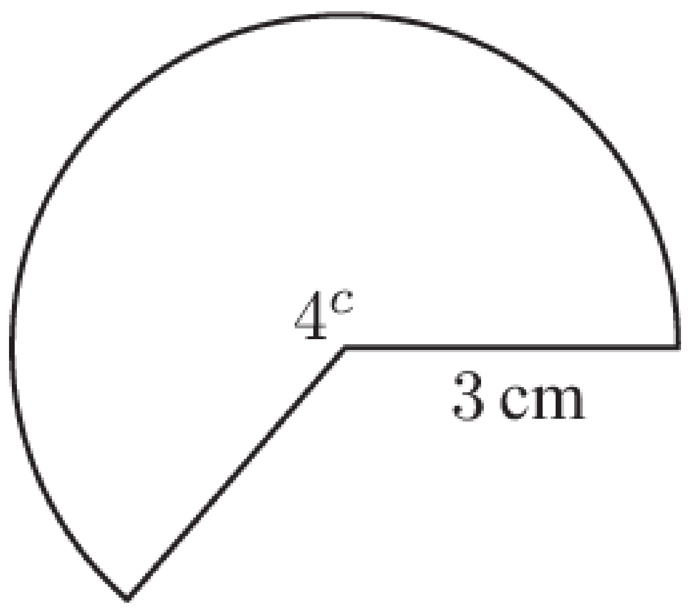

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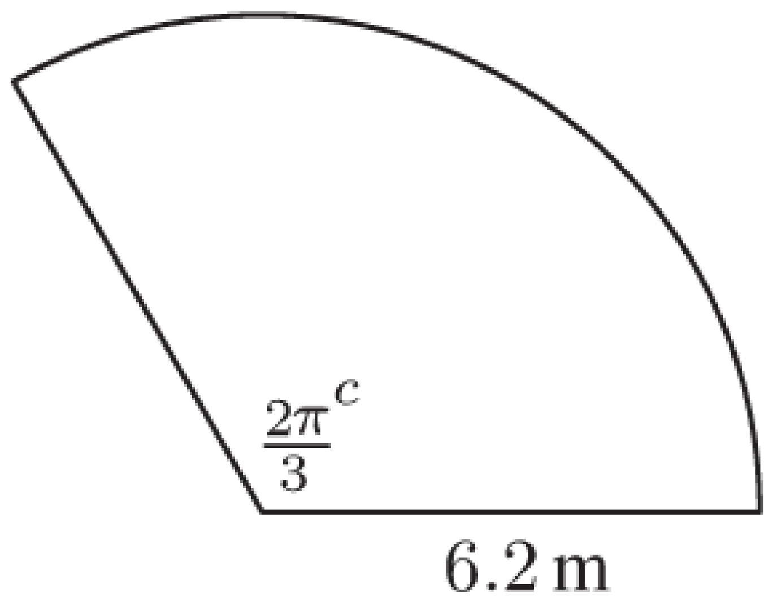
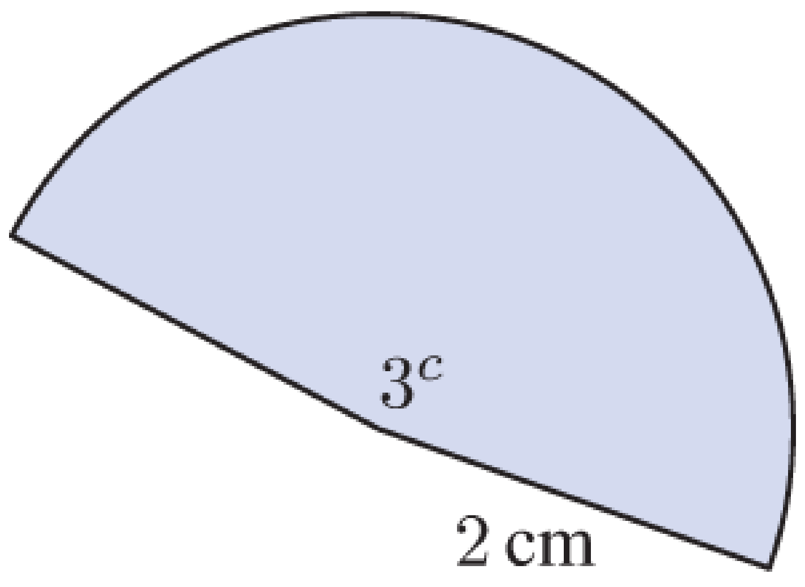


2 Find the area of each sector:

a



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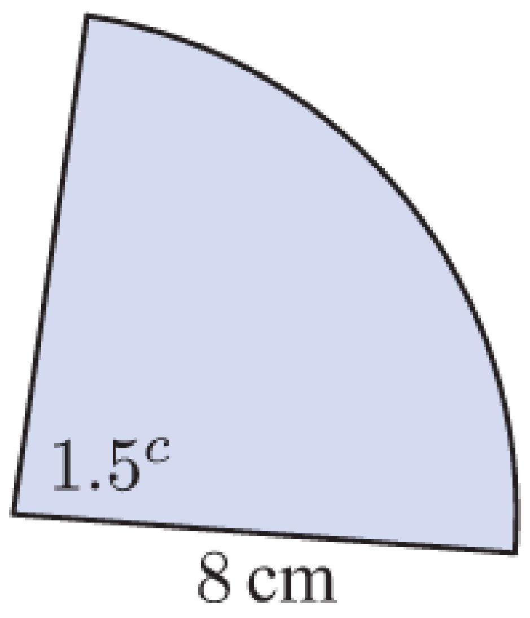

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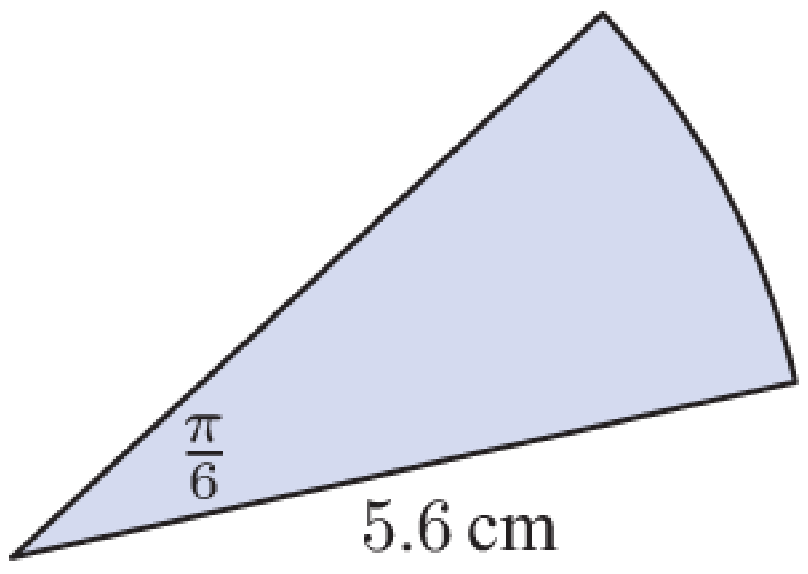


3 Find the arc length and area of a sector of a circle with:

a radius 9 cm and angle $\frac{7\pi}{4}$

b radius 4.93 cm and angle 4.67 radians.

Example 4 **Self Tutor**

A sector has radius 8.2 cm and arc length 12.3 cm. Find its:

a angle **b** area.

a $l = \theta r \quad \{\theta \text{ in radians}\}$

$\therefore\ \theta = \frac{l}{r} = \frac{12.3}{8.2} = 1.5$ radians

b $\text{area} = \frac{1}{2}\theta r^2$

$= \frac{1}{2} \times 1.5 \times 8.2^2$

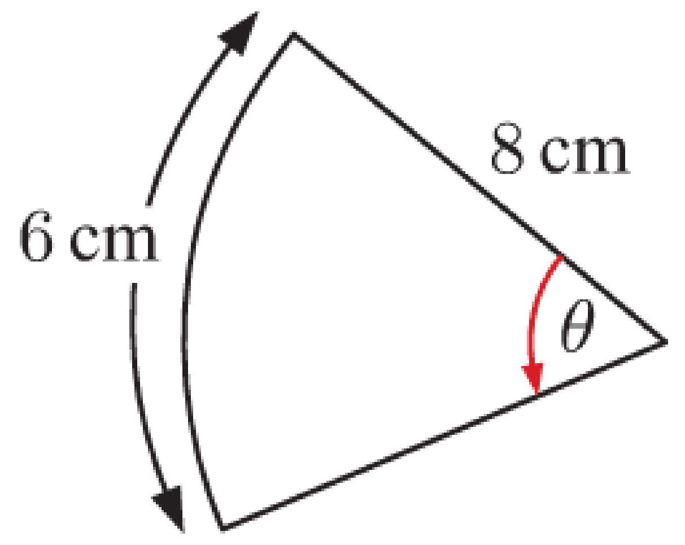
$= 50.43\ \text{cm}^2$

4 Find, in radians, the angle of a sector of:

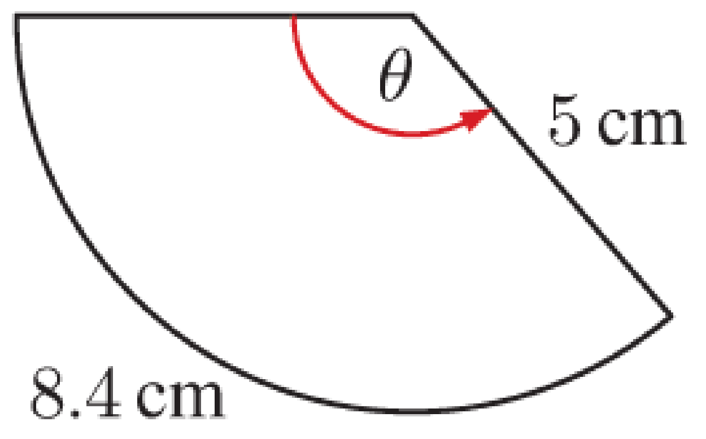
a radius 4.3 m and arc length 2.95 m **b** radius 10 cm and area 30 cm^2.

5 Find θ (in radians) for each of the following, and hence find the area of each figure:

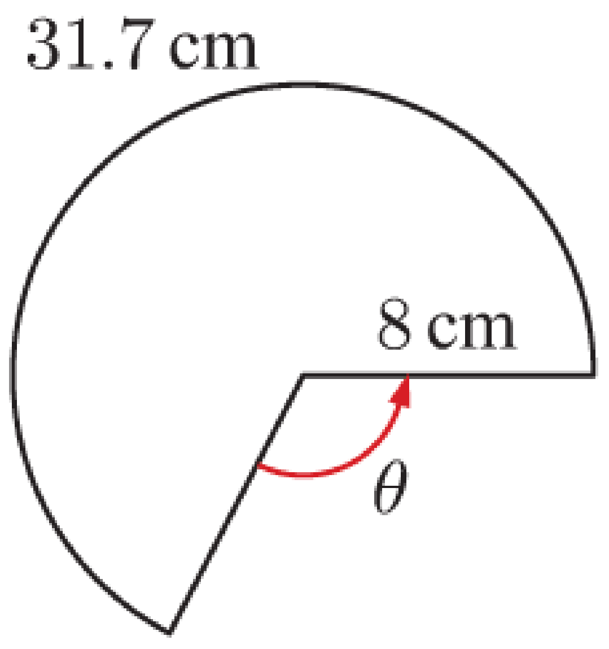
a



b



c



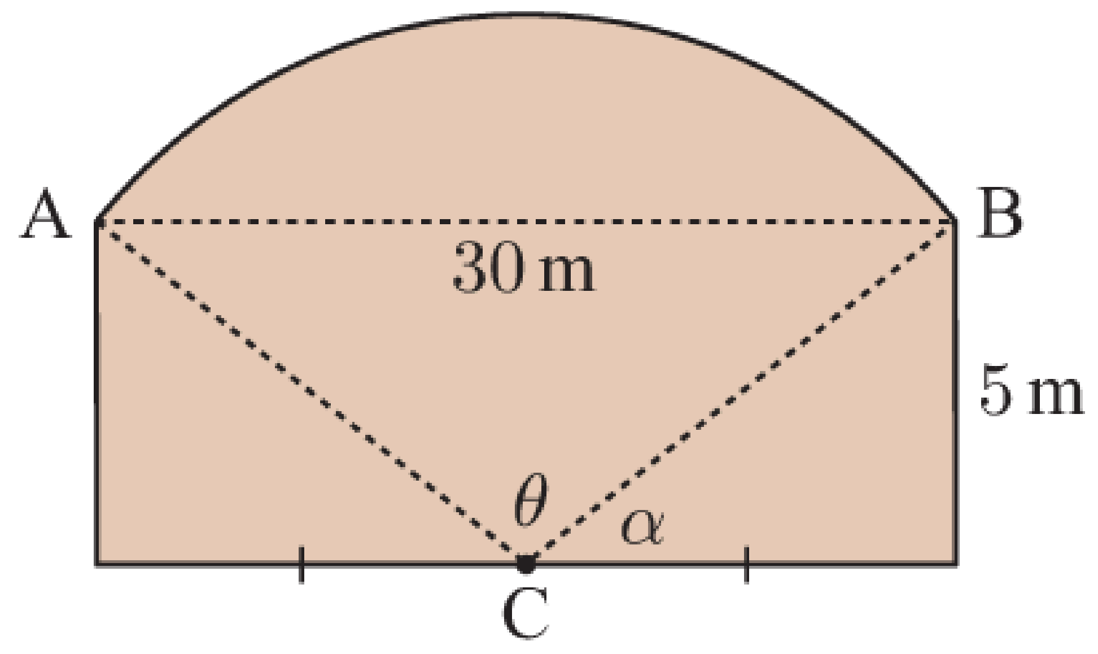
6 A sector has an angle of 1.88 radians and an arc length of 5.92 m. Find its:

a radius **b** area.

7 A sector has an angle of 1.19 radians and an area of 20.8 cm^2. Find its:

a radius **b** perimeter.

8



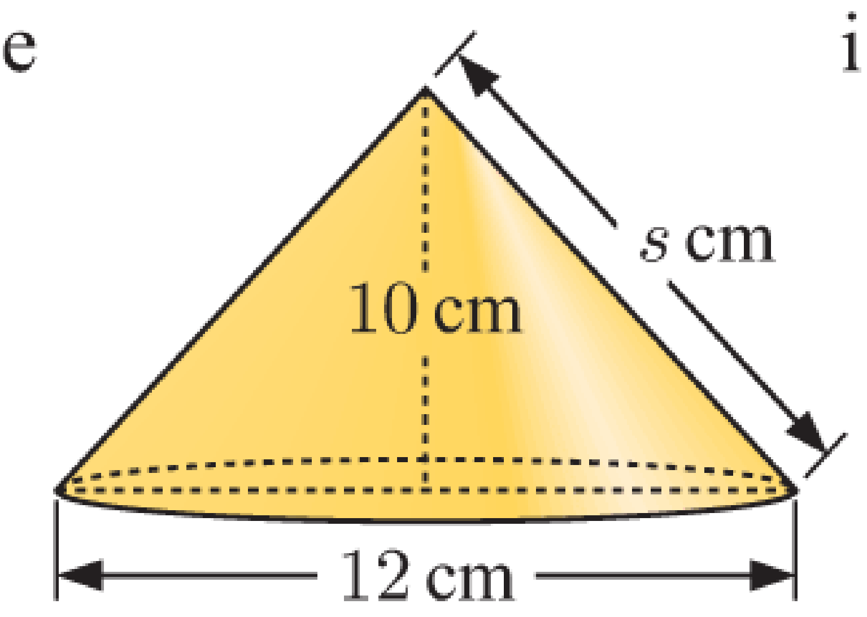
The end wall of a building has the shape illustrated, where the centre of arc AB is at C. Find:

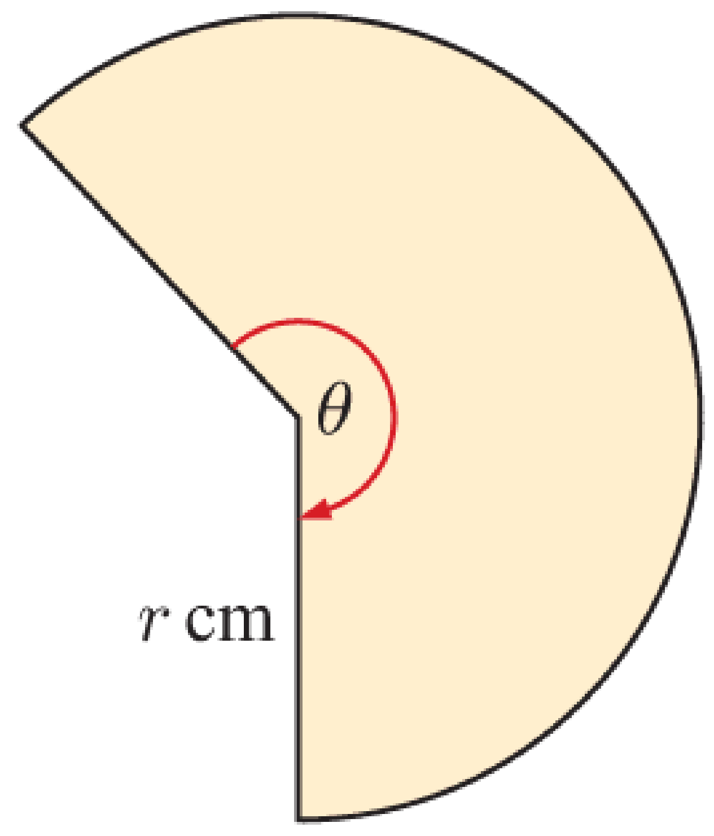
a α in radians to 4 significant figures

b θ in radians to 4 significant figures

c the area of the wall.

9 The cone is made from this sector:



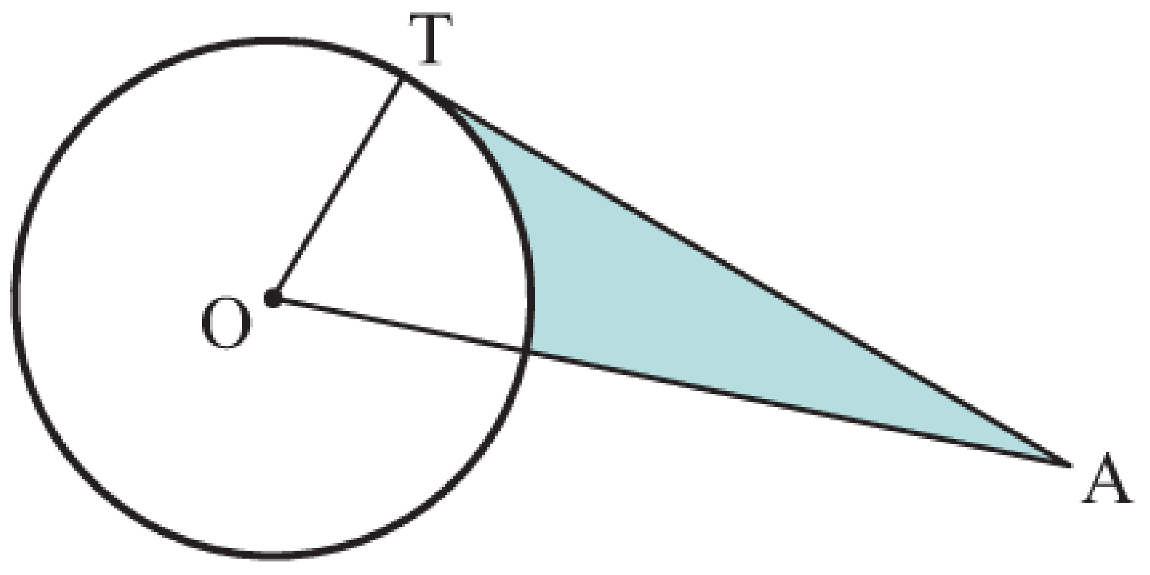


Find, correct to 3 significant figures:

a the slant length s cm **b** the value of r

c the arc length of the sector **d** the sector angle θ in radians.

10

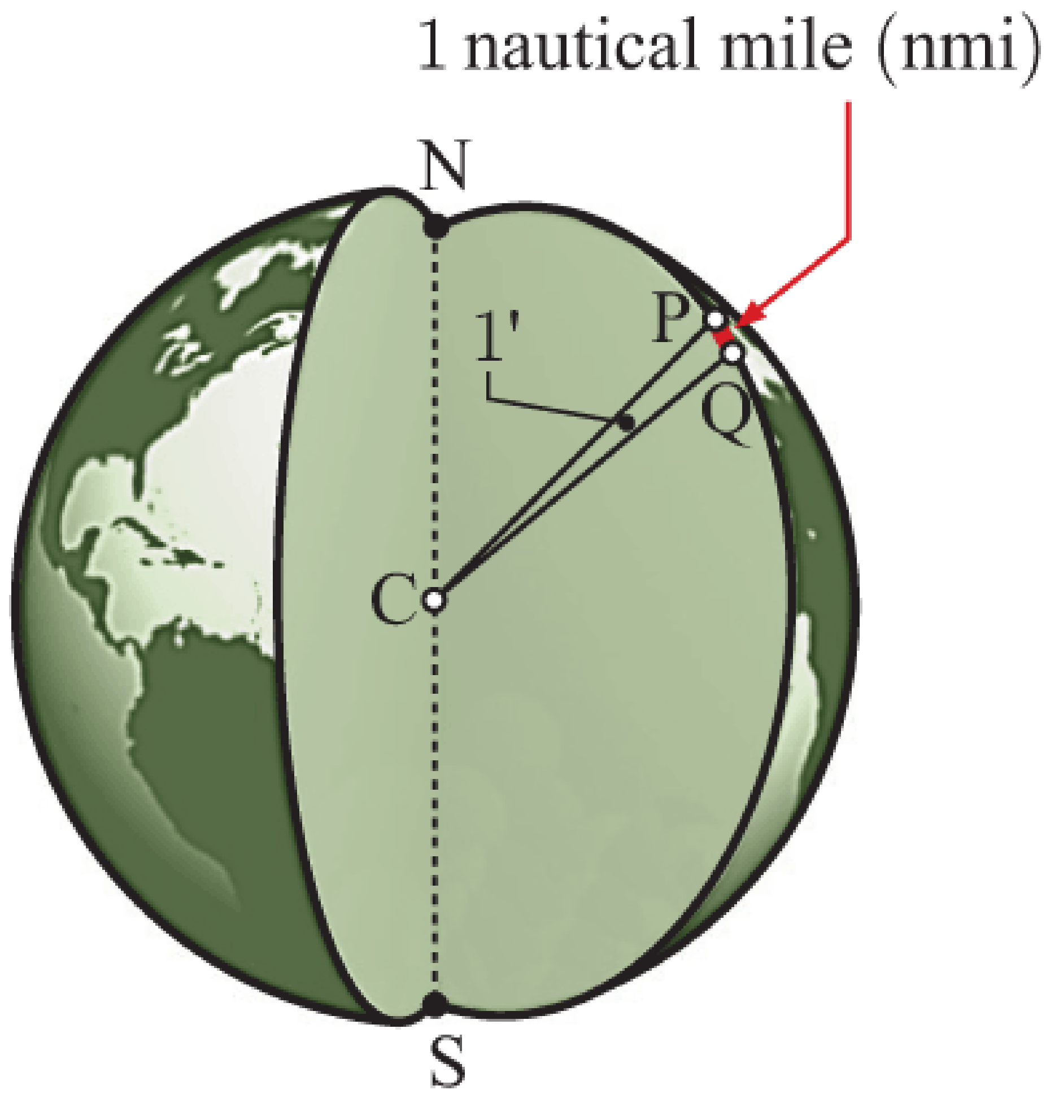


[AT] is a tangent to the given circle. OA = 13 cm and the circle has radius 5 cm. Find the perimeter of the shaded region.

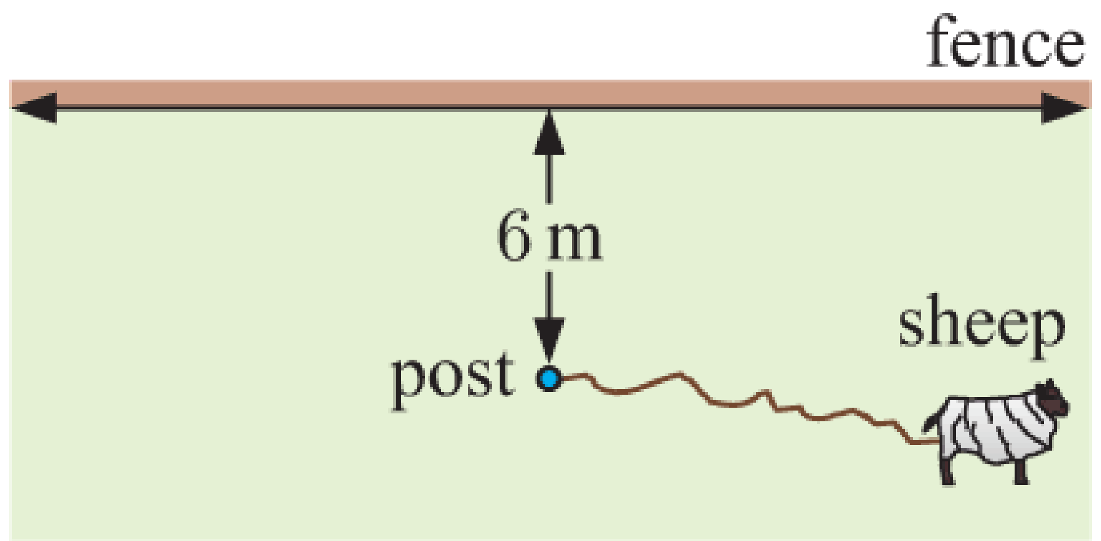
11 A **nautical mile** (nmi) is the distance on the Earth's surface that subtends an angle of 1 minute (or $\frac{1}{60}$th of a degree) of the Great Circle arc measured from the centre of the Earth. A **knot** is a speed of 1 nautical mile per hour.

a Given that the radius of the Earth is 6370 km, show that 1 nmi is approximately 1.853 km.

b Calculate how long it would take a plane to fly 2130 km from Perth to Adelaide if the plane can fly at 480 knots.



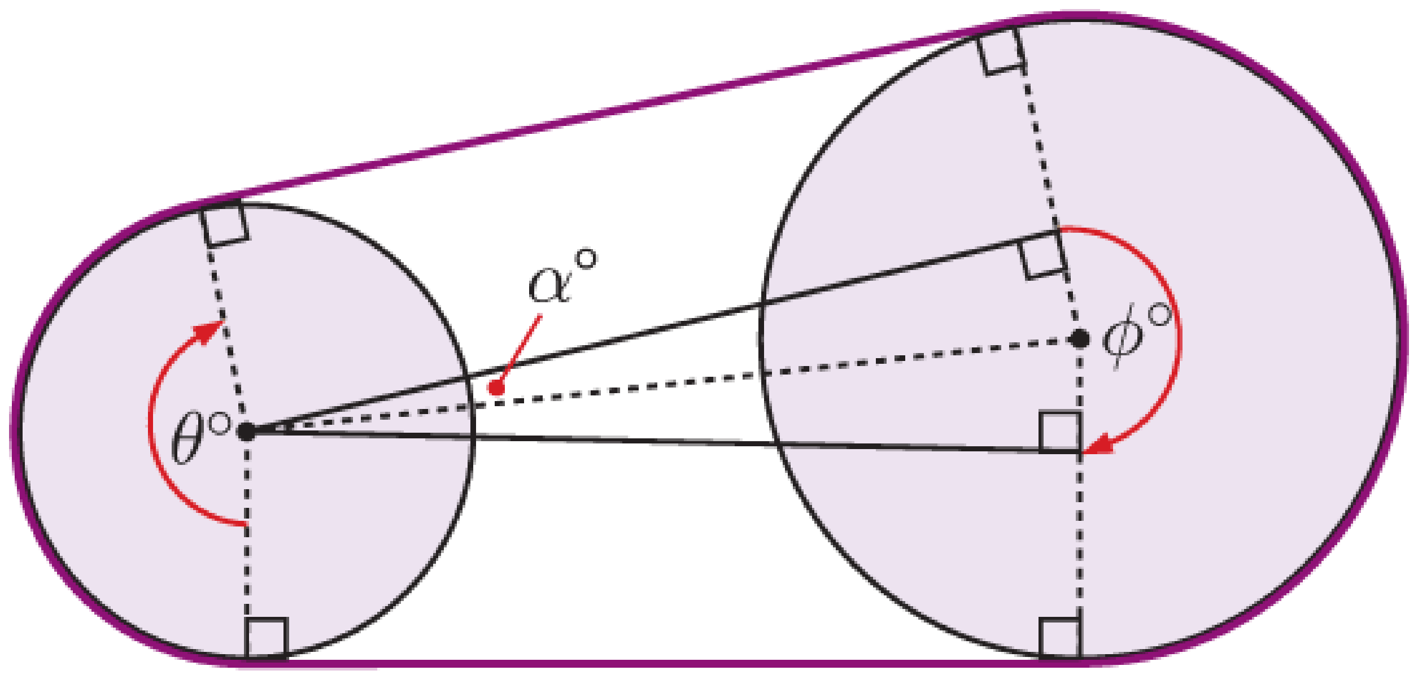
12 A sheep is tethered to a post which is 6 m from a long fence. The length of the rope is 9 m. Find the area which the sheep can feed on.



13 A belt fits tightly around two pulleys with radii 4 cm and 6 cm respectively. The distance between their centres is 20 cm.
Find, correct to 4 significant figures:

a α **b** θ **c** ϕ

d the length of the belt.

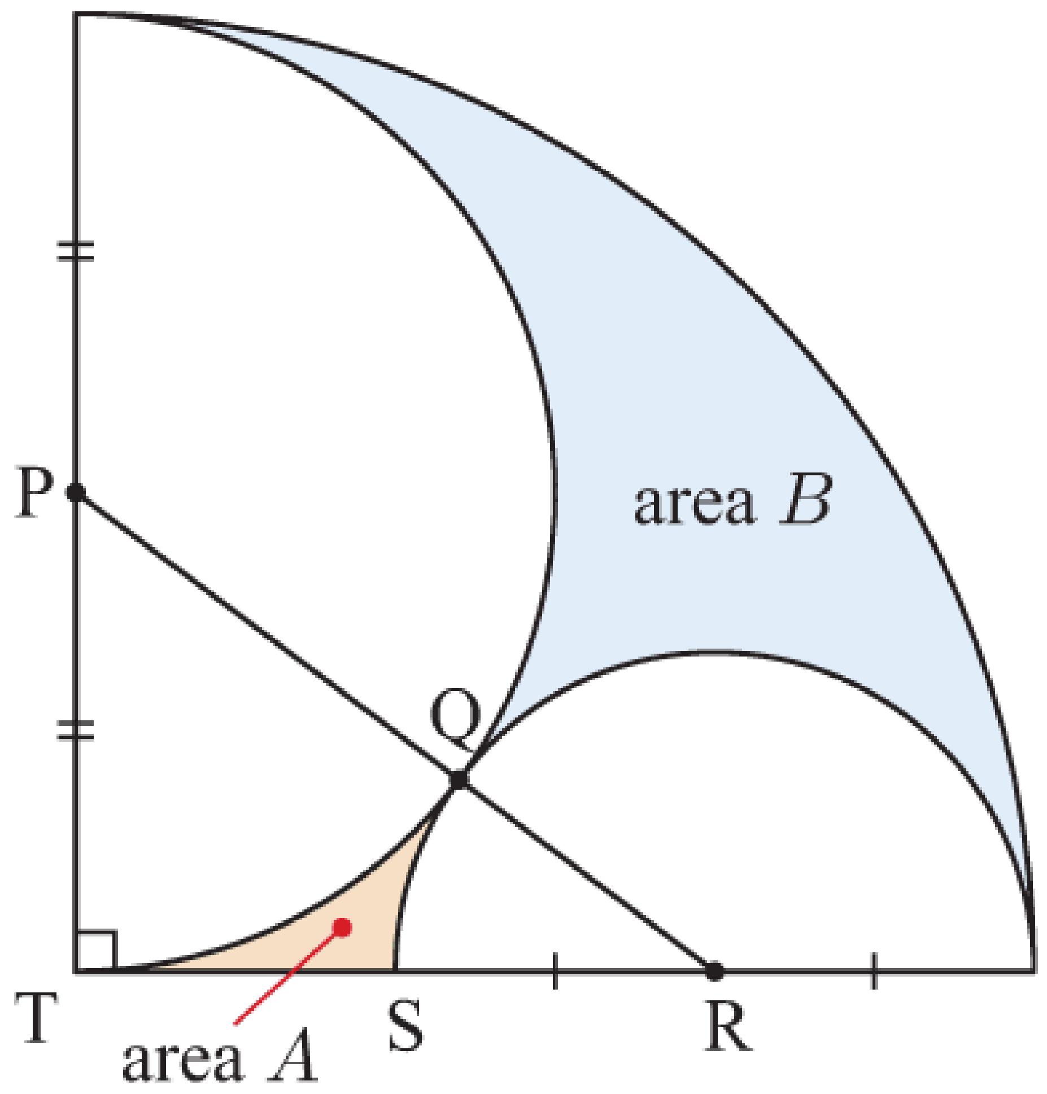


14 Two semi-circles touch each other within a quarter circle as shown. P, Q, and R are collinear. The radius of the quarter circle is 12 cm.

a Find the radius of the smaller semi-circle.

b Calculate the area of:

i A **ii** B.

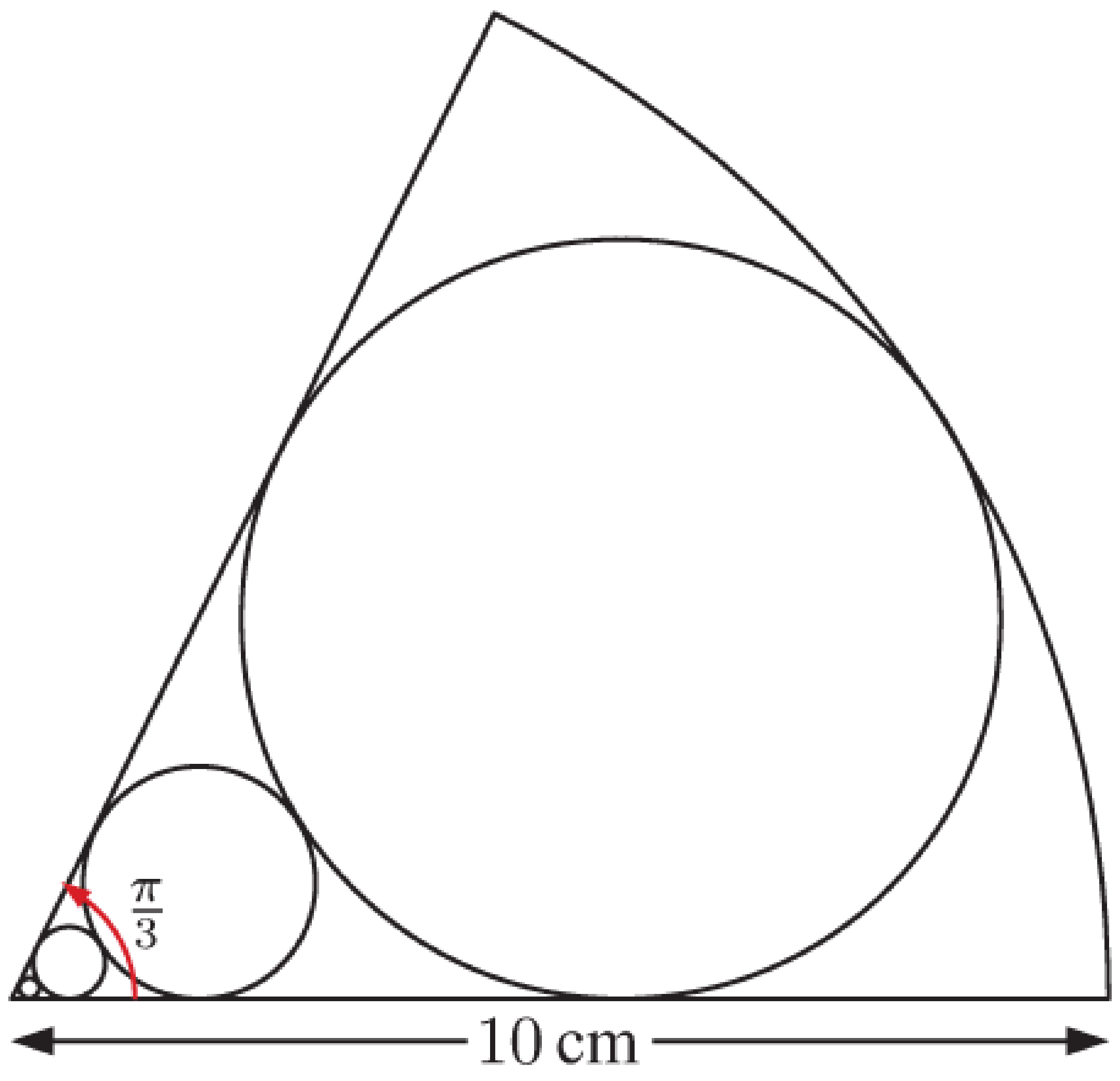


15 An infinite number of circles are drawn in a sector of a circle as shown.

a Show that the largest circle has radius $\frac{10}{3}$ cm.

b Find the total area of this infinite series of circles.

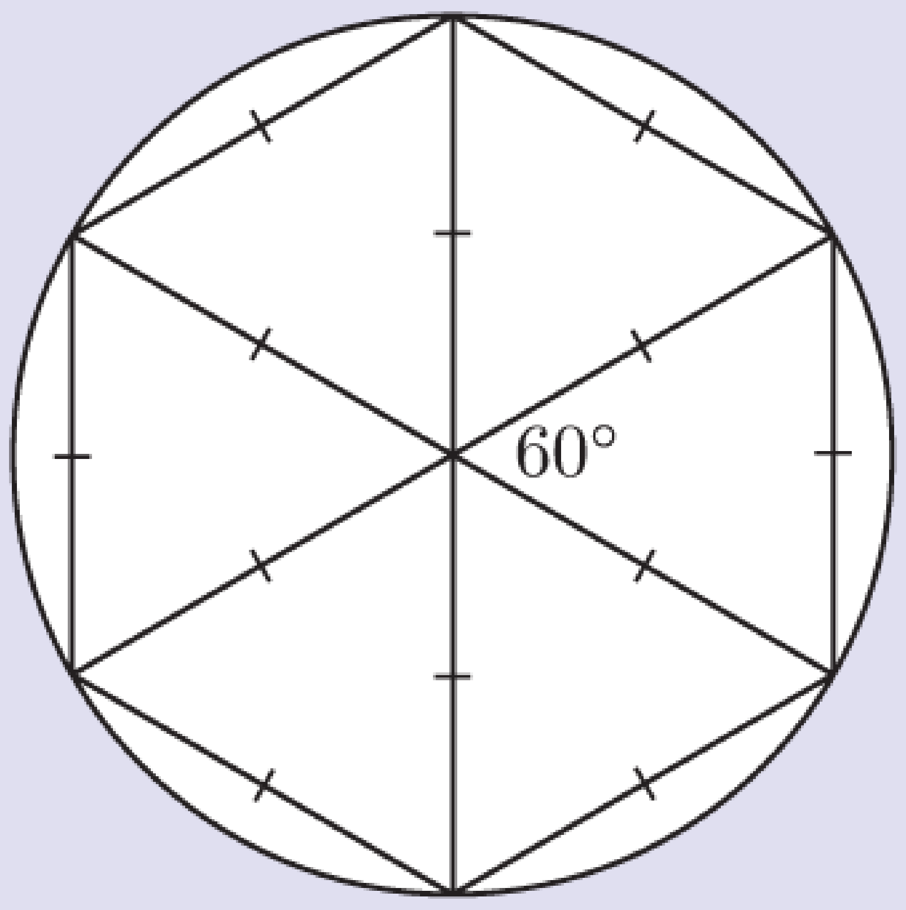
c What fraction of the sector is occupied by the circles?



THEORY OF KNOWLEDGE

There are several theories for why one complete turn was divided into 360 degrees:

- 360 is approximately the number of days in a year.
- The Babylonians used a counting system in base 60. If they drew 6 equilateral triangles within a circle as shown, and divided each angle into 60 subdivisions, then there were 360 subdivisions in one turn. The division of an hour into 60 minutes, and a minute into 60 seconds, is from this base 60 counting system.
- 360 has 24 divisors, including every integer from 1 to 10 except 7.

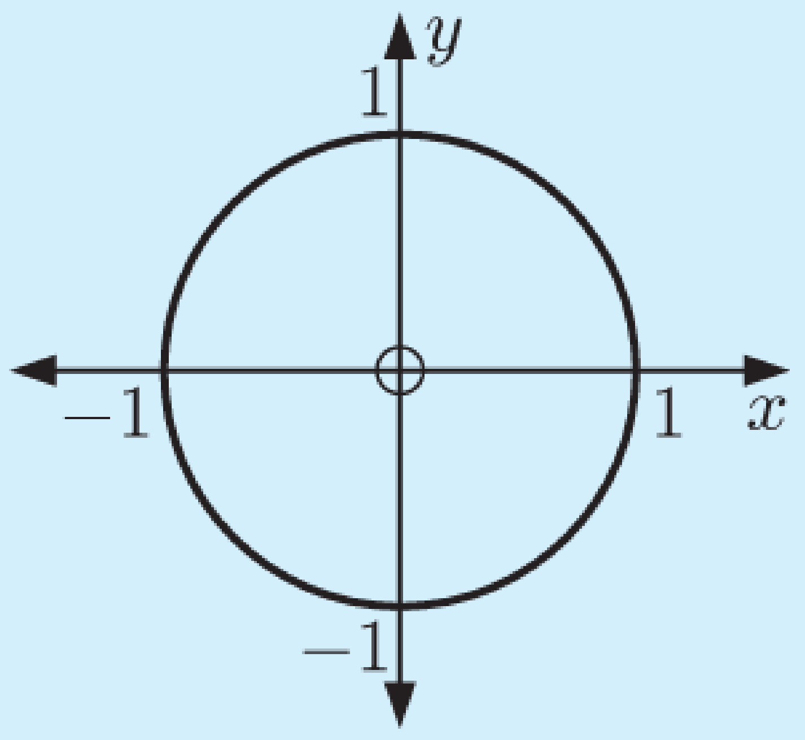


By contrast, we have seen how radians are convenient in simplifying formulae which relate angles with distances and areas.

1. Which angle measure do you think is more:
   - **a** *practical*
   - **b** *natural*
   - **c** *mathematical*?
2. What other measures of angle are there, and for what purpose were they defined?
3. Which temperature scale, Celsius, Kelvin, or Fahrenheit, do you think is more:
   - **a** *practical*
   - **b** *natural*?
4. What other measures have we defined as a way of convenience?
5. What things are done differently around the world, but would be useful to globally standardise? For example, why are there different power voltages in different countries? Why have they not been standardised?
6. What things do we measure in a particular way simply for reasons of history rather than practical purpose?

C THE UNIT CIRCLE

The **unit circle** is the circle with centre (0, 0) and radius 1 unit.

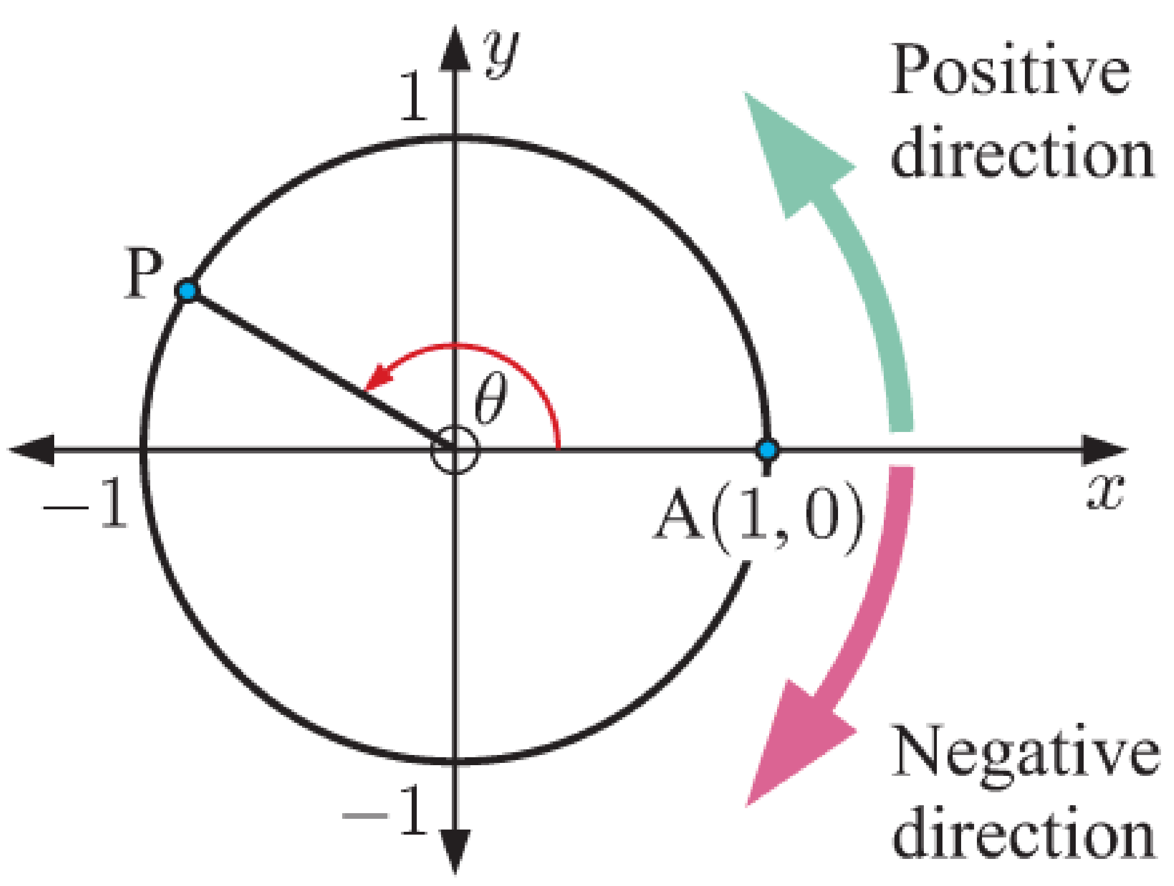


Applying the distance formula to a general point (x, y) on the circle, we find the equation of the unit circle is $x^2 + y^2 = 1$.

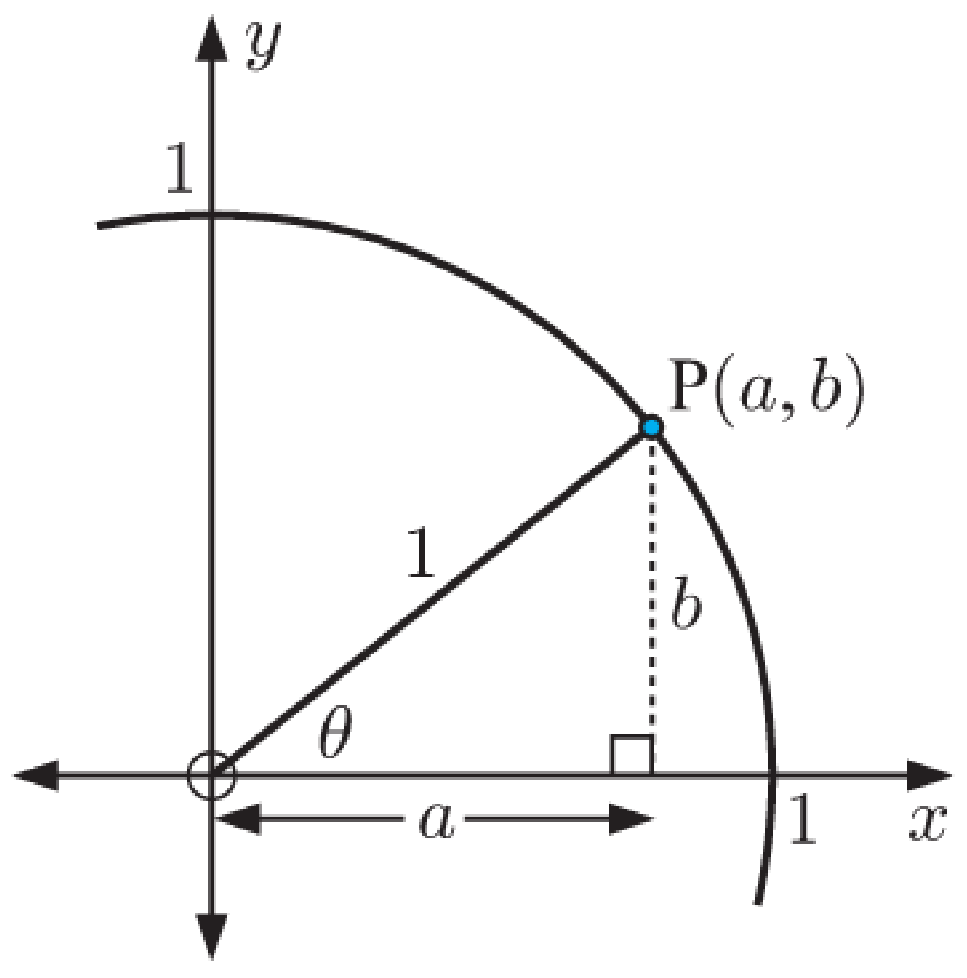
ANGLE MEASUREMENT

Suppose P lies anywhere on the unit circle, and A is (1, 0). Let θ be the angle measured anticlockwise from [OA] on the positive x-axis.

θ is **positive** for anticlockwise rotations and **negative** for clockwise rotations.



DEFINITION OF SINE AND COSINE



Consider a point P$(a,\ b)$ which lies on the unit circle in the first quadrant. [OP] makes an angle θ with the x-axis as shown.

Using right angled triangle trigonometry:

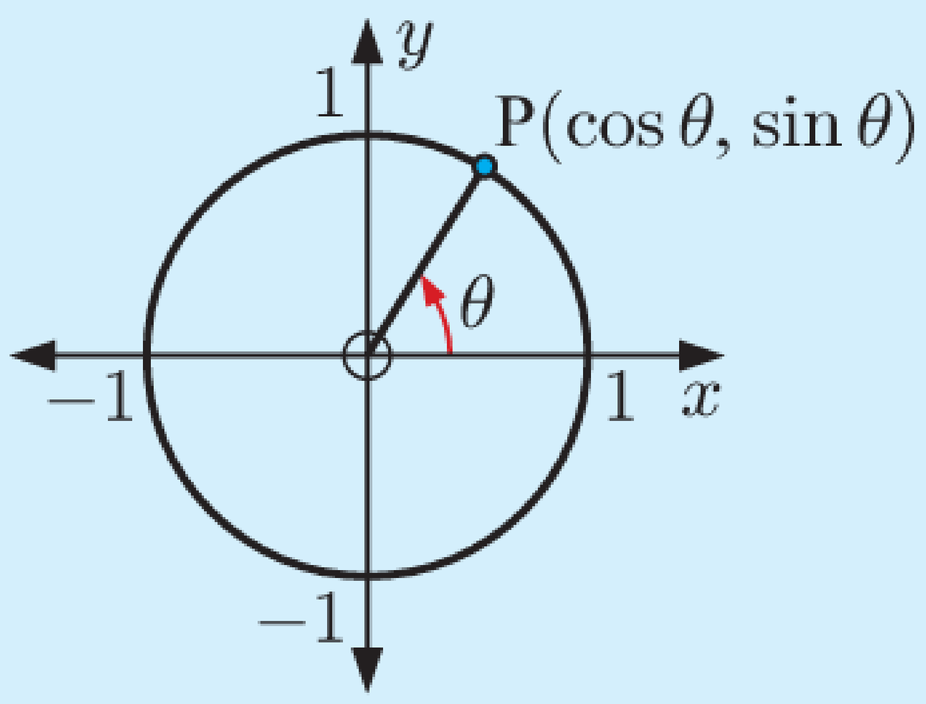
$$\cos\theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{a}{1} = a$$

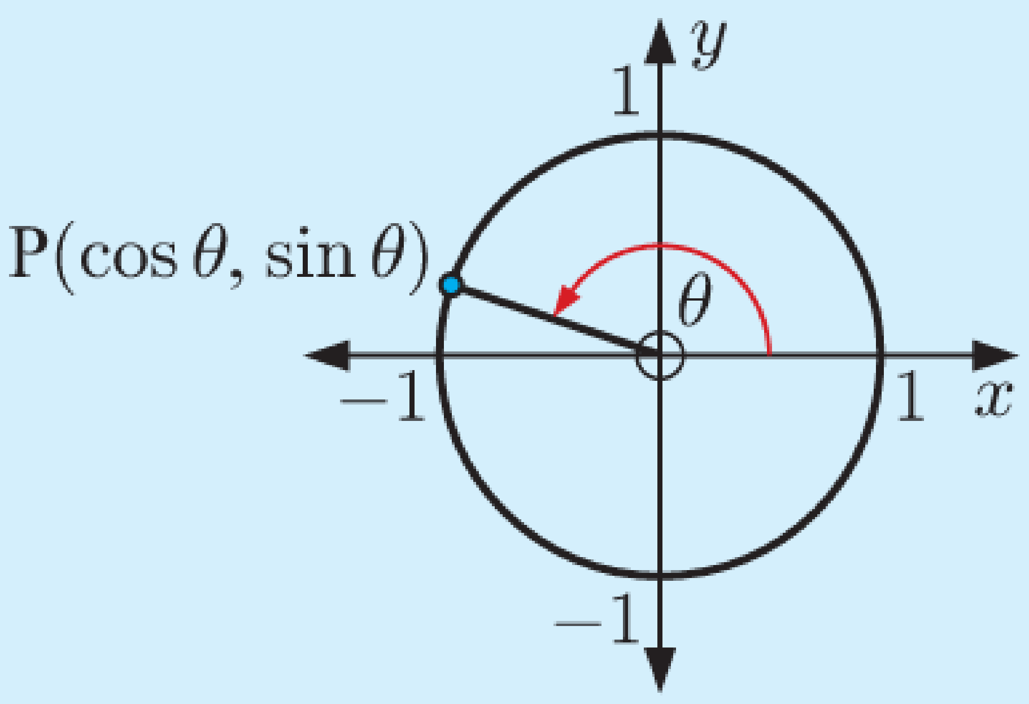
$$\sin\theta = \frac{\text{OPP}}{\text{HYP}} = \frac{b}{1} = b$$

More generally, we define:

If P is any point on the unit circle such that [OP] makes an angle θ measured anticlockwise from the positive x-axis:

- $\boldsymbol{\cos\theta}$ is the x-coordinate of P
- $\boldsymbol{\sin\theta}$ is the y-coordinate of P



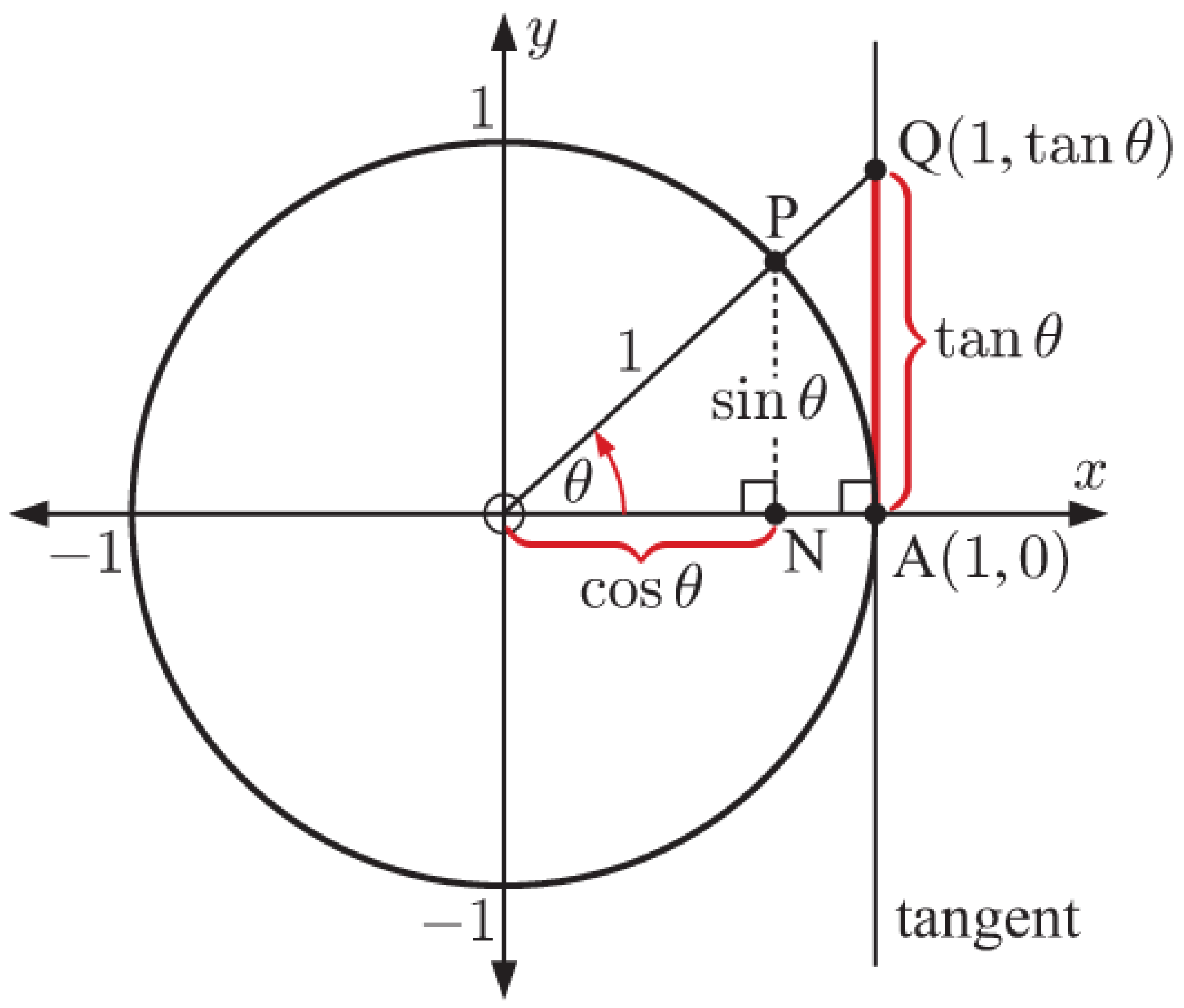


For all points on the unit circle, $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$, and $x^2 + y^2 = 1$. We therefore conclude:

For any angle θ:

- $\boldsymbol{-1 \leqslant \cos\theta \leqslant 1}$ and $\boldsymbol{-1 \leqslant \sin\theta \leqslant 1}$
- $\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$

DEFINITION OF TANGENT



Suppose we extend [OP] to meet the tangent from A$(1,\ 0)$.

We let the intersection between these lines be point Q.

Note that as P moves, so does Q.

The position of Q relative to A is defined as the **tangent function**.

Notice that triangles ONP and OAQ are equiangular and therefore similar.

Consequently $\dfrac{\text{AQ}}{\text{OA}} = \dfrac{\text{NP}}{\text{ON}}$ and hence $\dfrac{\text{AQ}}{1} = \dfrac{\sin\theta}{\cos\theta}$.

Under the definition that $\text{AQ} = \tan\theta$, $\boldsymbol{\tan\theta = \dfrac{\sin\theta}{\cos\theta}}$.

Since [OP] has gradient $\dfrac{\sin\theta}{\cos\theta}$, we can also say that $\tan\theta$ is the **gradient** of [OP].

INVESTIGATION THE TRIGONOMETRIC RATIOS

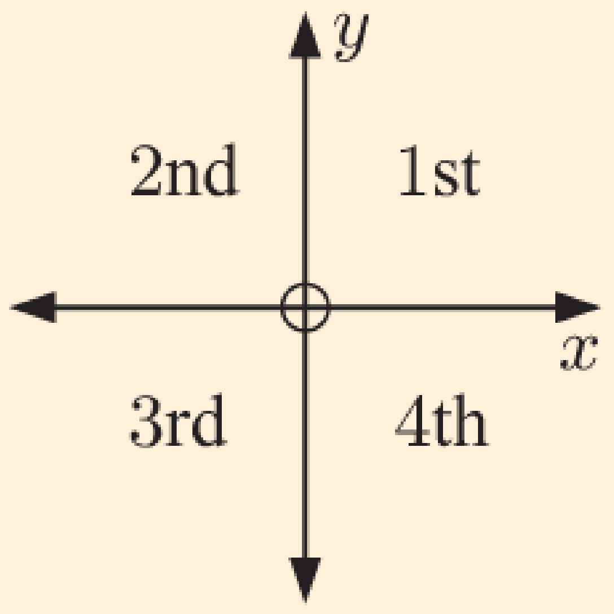
In this Investigation we explore the signs of the trigonometric ratios in each quadrant of the unit circle.

What to do:

THE UNIT CIRCLE

1 Click on the icon to run the Unit Circle software.
Drag the point P slowly around the circle.
Note the *sign* of each trigonometric ratio in each quadrant.

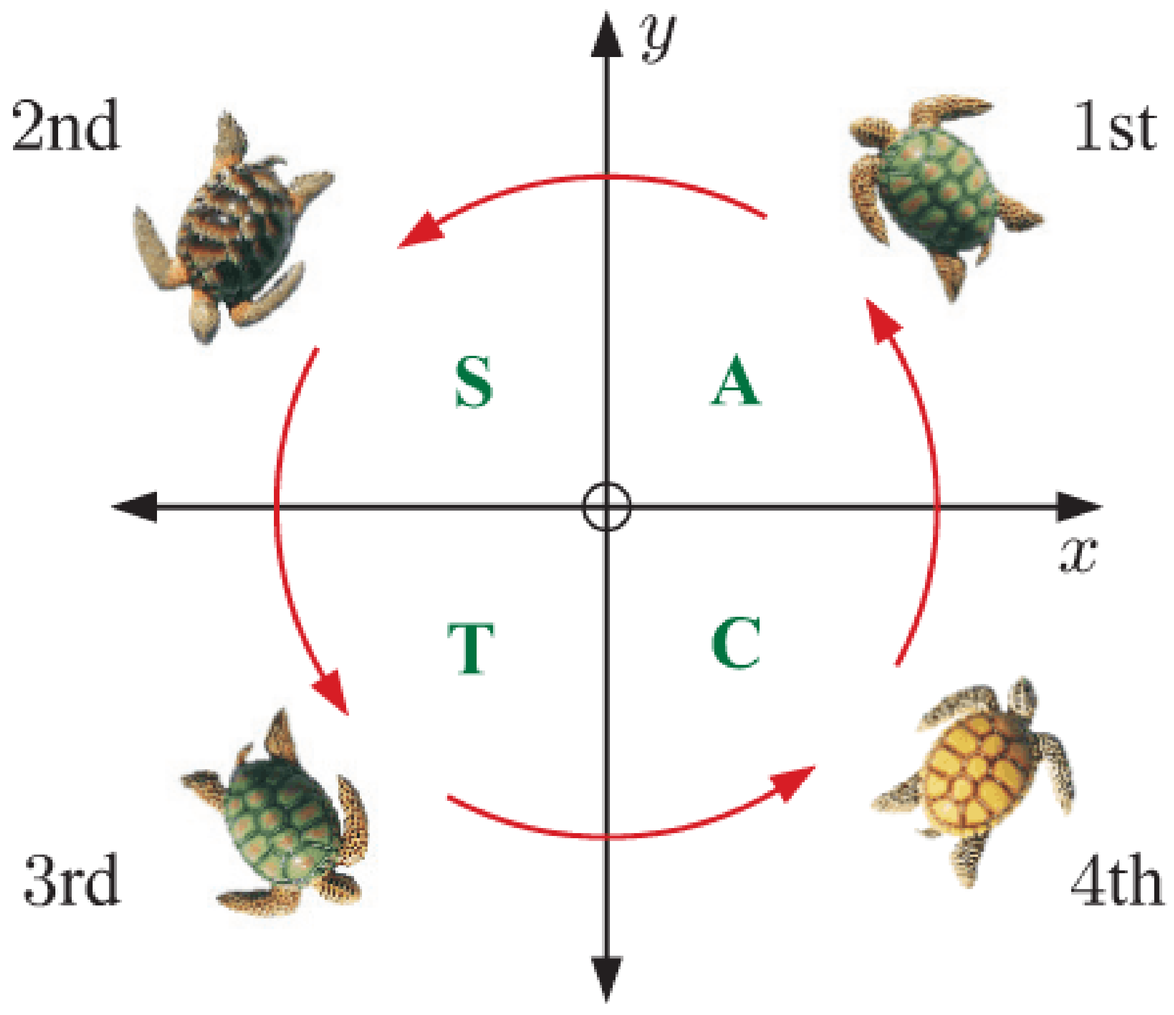
Quadrant	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	positive		
2			
3			
4			



2 Hence write down the trigonometric ratios which are *positive* for each quadrant.

From the **Investigation** you should have discovered that:

- $\sin\theta$, $\cos\theta$, and $\tan\theta$ are all positive in quadrant 1
- only $\sin\theta$ is positive in quadrant 2
- only $\tan\theta$ is positive in quadrant 3
- only $\cos\theta$ is positive in quadrant 4.



We can use a letter to show which trigonometric ratios are positive in each quadrant. The A stands for *all* of the ratios.

You might like to remember them using

All **S**illy **T**urtles **C**rawl.

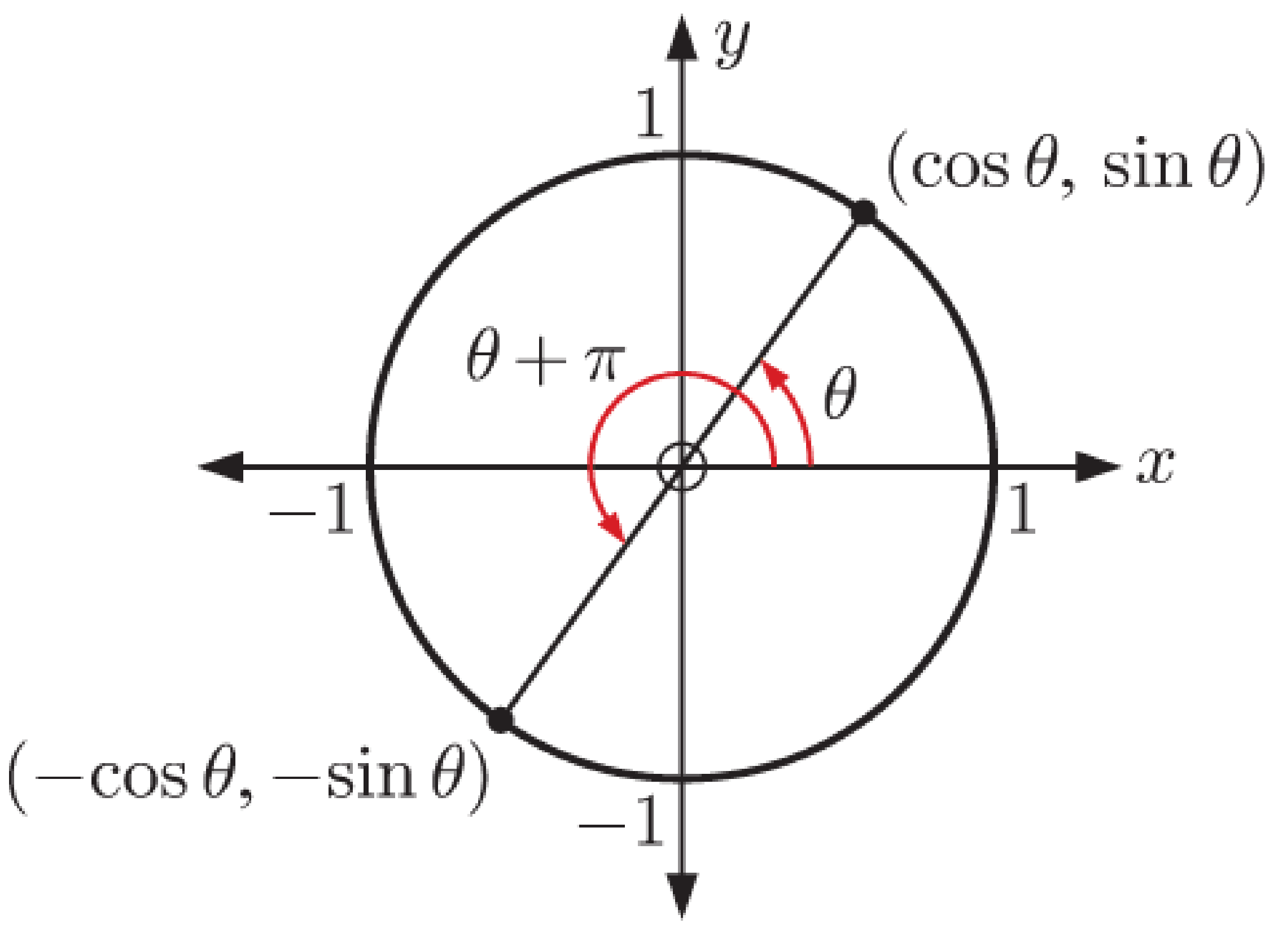
PERIODICITY OF TRIGONOMETRIC RATIOS

Since there are 2π radians in a full revolution, if we add any integer multiple of 2π to θ (in radians) then the position of P on the unit circle is unchanged.

For θ in radians and $k \in \mathbb{Z}$,

$$\boldsymbol{\cos(\theta + 2k\pi) = \cos\theta} \quad \text{and} \quad \boldsymbol{\sin(\theta + 2k\pi) = \sin\theta}.$$

We notice that for any point $(\cos\theta, \sin\theta)$ on the unit circle, the point directly opposite is $(-\cos\theta, -\sin\theta)$.



$$\therefore \quad \cos(\theta + \pi) = -\cos\theta$$
$$\sin(\theta + \pi) = -\sin\theta$$
$$\text{and} \quad \tan(\theta + \pi) = \frac{-\sin\theta}{-\cos\theta} = \tan\theta$$

For θ in radians and $k \in \mathbb{Z}$, $\boldsymbol{\tan(\theta + k\pi) = \tan\theta}$.

CALCULATOR USE

When using your calculator to find trigonometric ratios for angles, you must make sure your calculator is correctly set to either **degree** or **radian** mode. Click on the icon for instructions.

GRAPHICS CALCULATOR INSTRUCTIONS

EXERCISE 8C

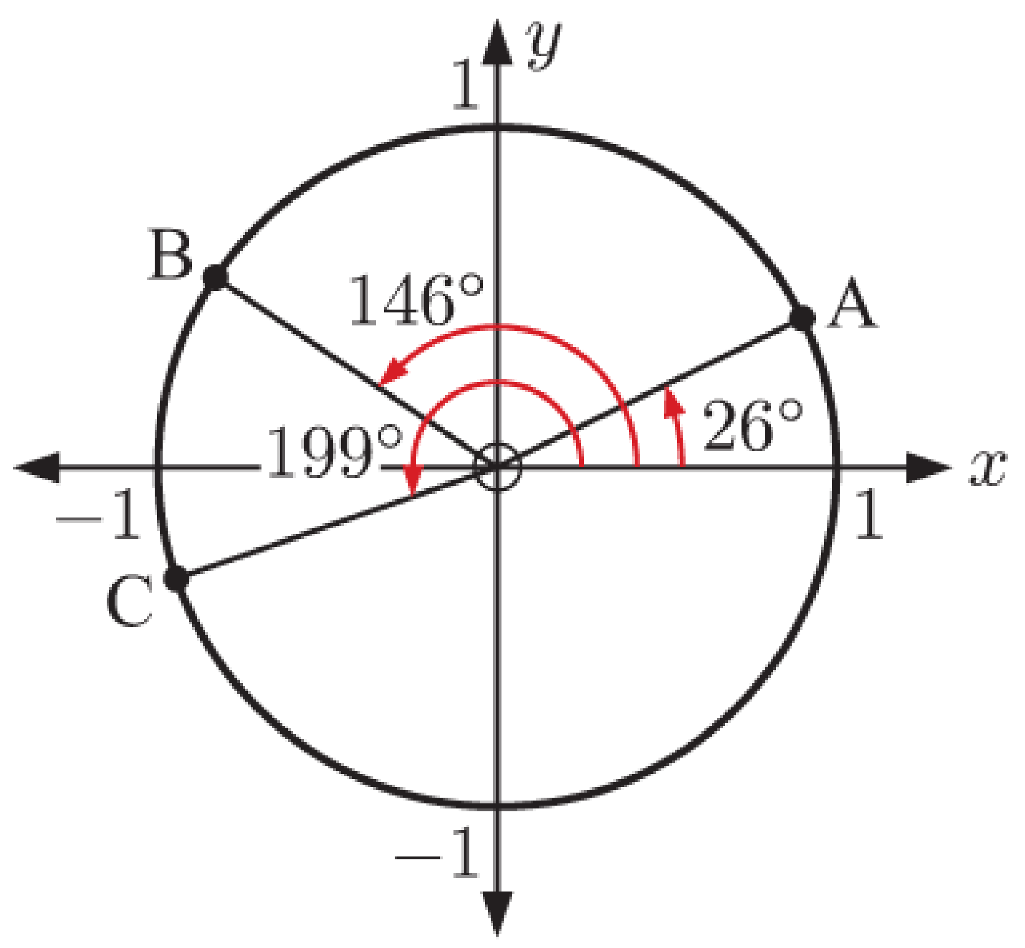
1 With the aid of a unit circle, complete the following table:

θ (degrees)	$0°$	$90°$	$180°$	$270°$	$360°$	$450°$
θ (radians)						
sine						
cosine						
tangent						

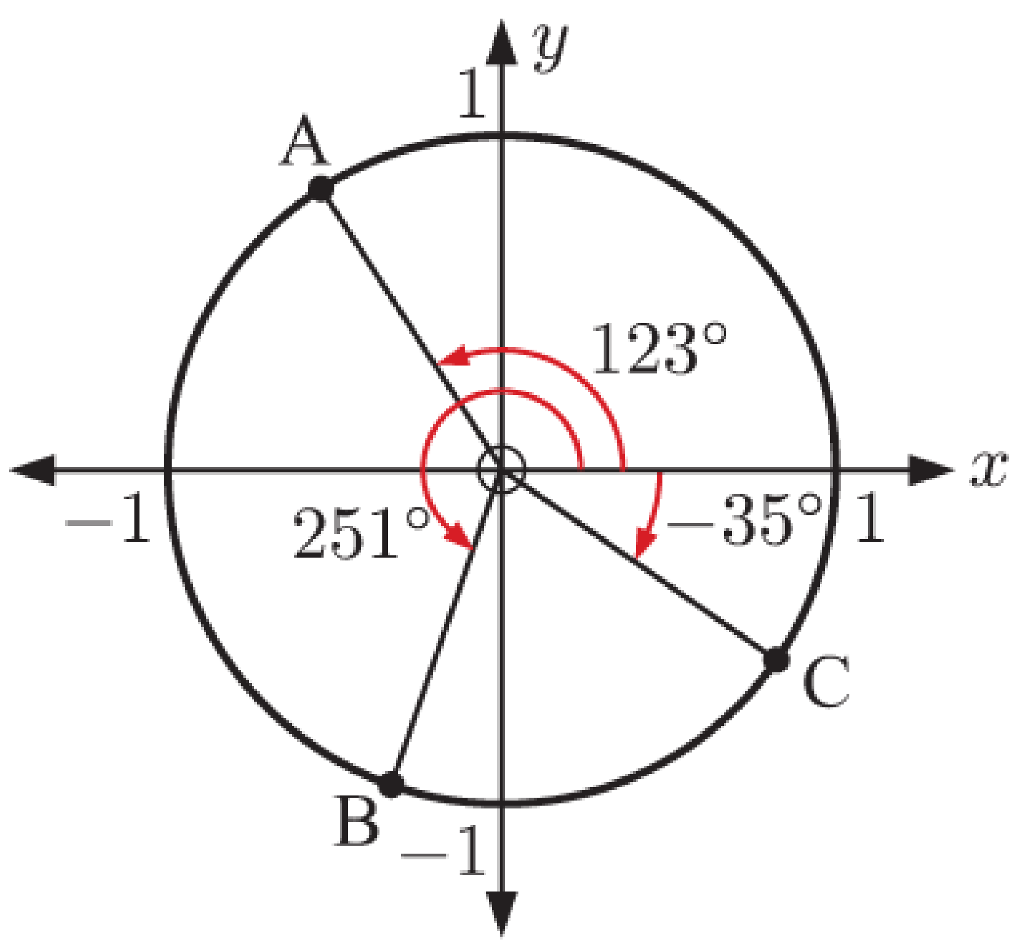
2 For each unit circle illustrated:

- **i** State the exact coordinates of points A, B, and C in terms of sine and cosine.
- **ii** Use your calculator to give the coordinates of A, B, and C correct to 3 significant figures.

a



b



3 **a** Use your calculator to evaluate:

- **i** $\frac{1}{\sqrt{2}}$
- **ii** $\frac{\sqrt{3}}{2}$

b Copy and complete the following table. Use your calculator to evaluate the trigonometric ratios, then **a** to write them exactly.

θ (degrees)	$30°$	$45°$	$60°$	$135°$	$150°$	$240°$	$315°$
θ (radians)							
sine							
cosine							
tangent							

4 Copy and complete:

Quadrant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	$0° < \theta < 90°$	$0 < \theta < \frac{\pi}{2}$	positive	positive	
2					
3					
4					

5 In which quadrants are the following true?

a $\cos\theta$ is positive. **b** $\cos\theta$ is negative.

c $\cos\theta$ and $\sin\theta$ are both negative. **d** $\cos\theta$ is negative and $\sin\theta$ is positive.

6 Explain why:

a $\cos 400° = \cos 40°$ **b** $\sin\frac{5\pi}{7} = \sin\frac{19\pi}{7}$ **c** $\tan\frac{13\pi}{8} = \tan\left(-\frac{11\pi}{8}\right)$

7 Which two of these have the same value?

A $\tan 15°$ **B** $\tan 50°$ **C** $\tan 200°$ **D** $\tan 230°$ **E** $\tan 300°$

8 Which two of these have the same value?

A $\sin 220°$ **B** $\sin\frac{2\pi}{9}$ **C** $\sin\left(-\frac{2\pi}{9}\right)$ **D** $\sin 120°$ **E** $\sin 40°$

9 **a** Use your calculator to evaluate:

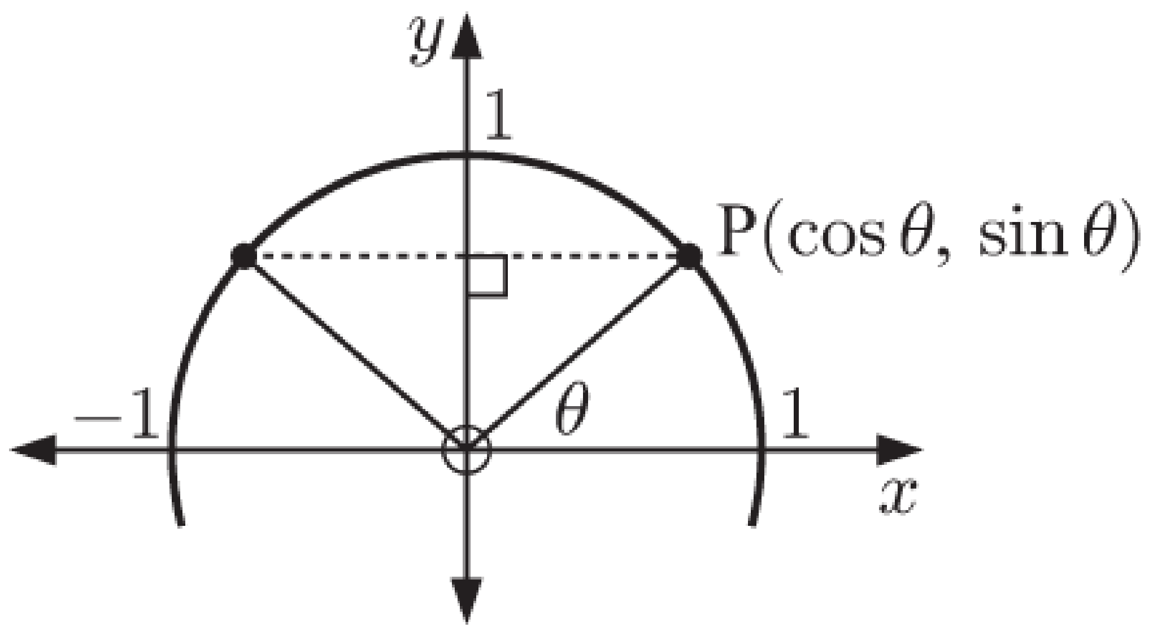
i $\sin 100°$ **ii** $\sin 80°$ **iii** $\sin 120°$ **iv** $\sin 60°$

v $\sin 150°$ **vi** $\sin 30°$ **vii** $\sin 45°$ **viii** $\sin 135°$

b Use the results from **a** to copy and complete: $\sin(180° - \theta) = ...$

c Write the rule you have just found in terms of radians.

d Justify your answer using the diagram alongside:



e Find the obtuse angle with the same sine as:

i 45° **ii** 51° **iii** $\frac{\pi}{3}$ **iv** $\frac{\pi}{6}$

10 **a** Use your calculator to evaluate:

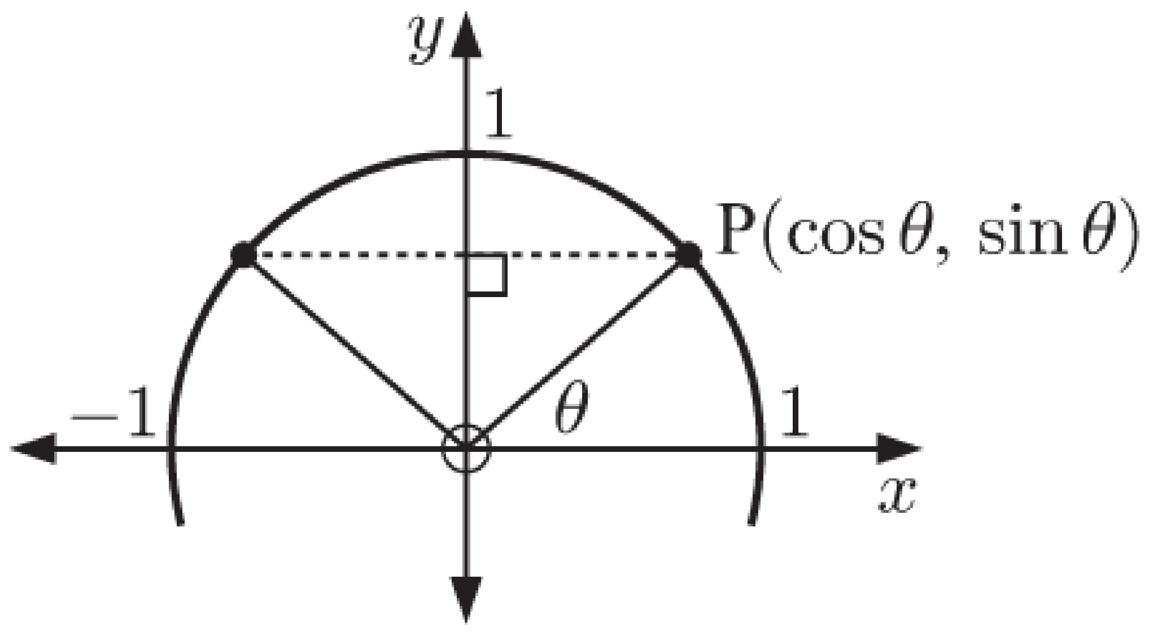
i $\cos 70°$ **ii** $\cos 110°$ **iii** $\cos 60°$ **iv** $\cos 120°$

v $\cos 25°$ **vi** $\cos 155°$ **vii** $\cos 80°$ **viii** $\cos 100°$

b Use the results from **a** to copy and complete: $\cos(180° - \theta) = ...$

c Write the rule you have just found in terms of radians.

d Justify your answer using the diagram alongside:



e Find the obtuse angle which has the negative cosine of:

i 40° **ii** 19° **iii** $\frac{\pi}{5}$ **iv** $\frac{2\pi}{5}$

11 Use the definition $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and your results from **9** and **10** to write $\tan(\pi - \theta)$ in terms of $\tan\theta$.

12 Without using your calculator, find:

a $\sin 137°$ if $\sin 43° \approx 0.6820$ **b** $\sin 59°$ if $\sin 121° \approx 0.8572$

c $\cos 143°$ if $\cos 37° \approx 0.7986$ **d** $\cos 24°$ if $\cos 156° \approx -0.9135$

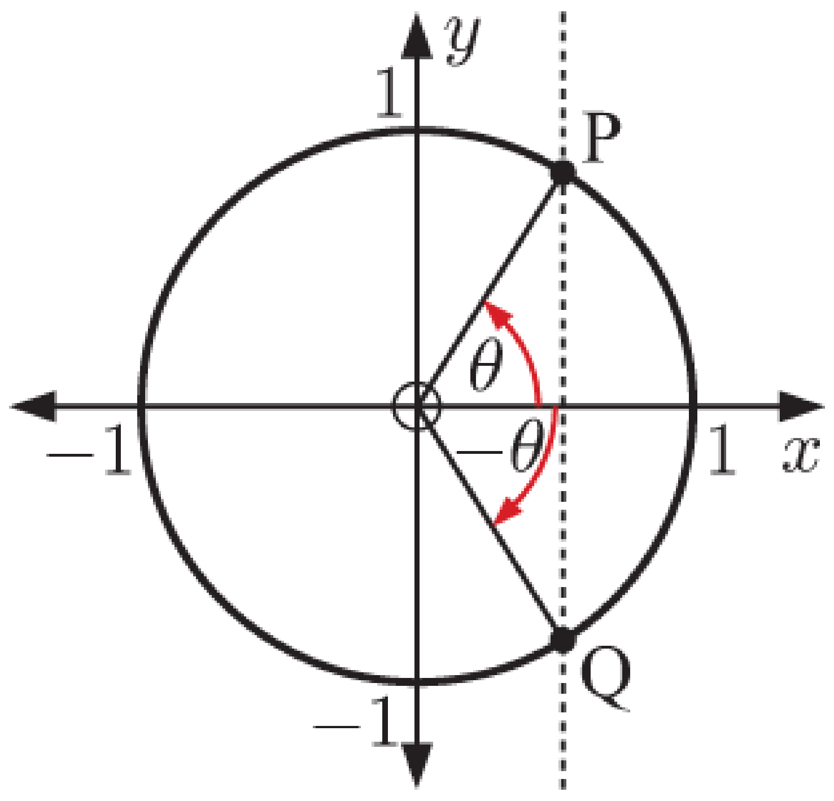
e $\sin 115°$ if $\sin 65° \approx 0.9063$ **f** $\cos 132°$ if $\cos 48° \approx 0.6691$

13 **a** Copy and complete:

θ (radians)	$\sin\theta$	$\sin(-\theta)$	$\cos\theta$	$\cos(-\theta)$
0.75				
1.772				
3.414				
6.25				
-1.17				

b What trigonometric formulae can be deduced from your results in **a**?

c The coordinates of P in the figure are $(\cos\theta, \sin\theta)$. By finding the coordinates of Q in terms of θ in *two different* ways, prove your formulae in **b**.



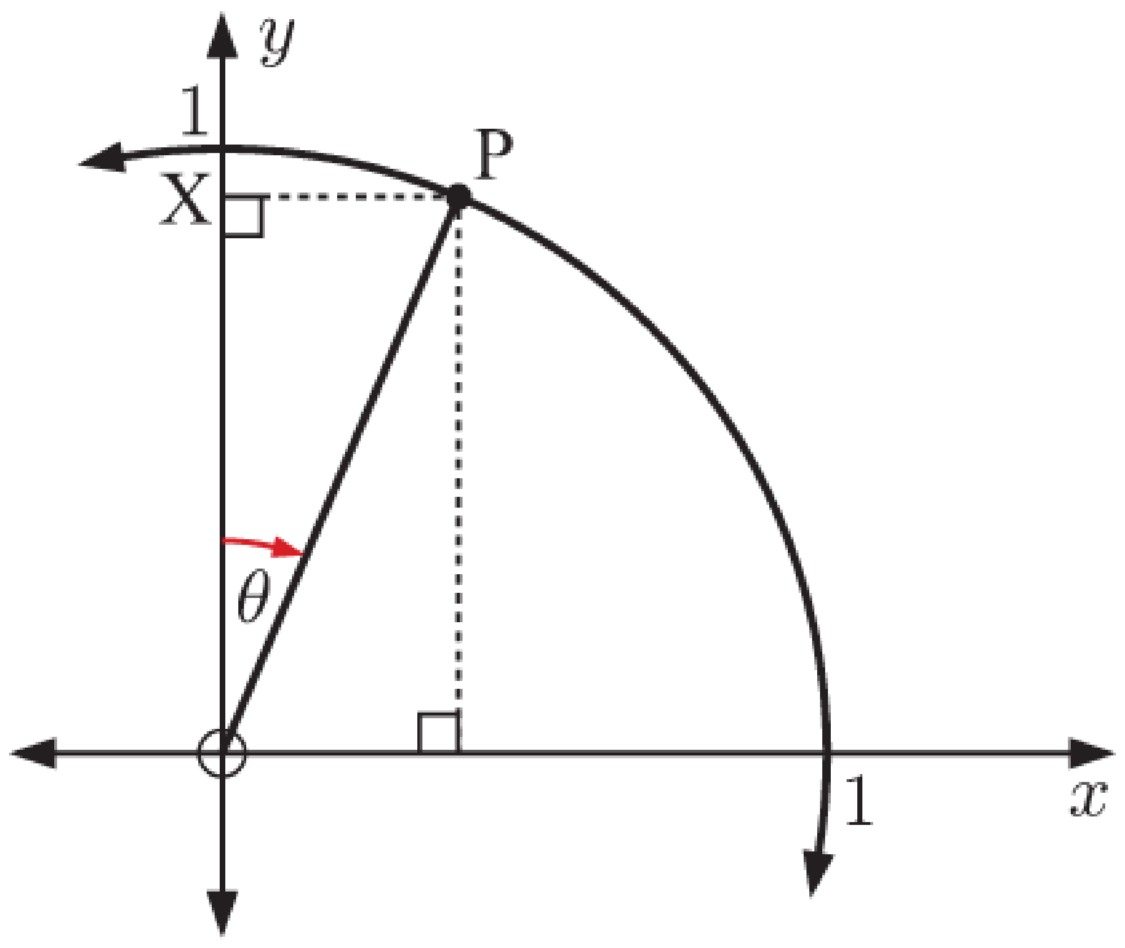
d Hence explain why

$$\cos(2\pi - \theta) = \cos\theta$$

and $$\sin(2\pi - \theta) = -\sin\theta.$$

e Write $\tan(2\pi - \theta)$ in terms of $\tan\theta$.

14 **a** Explain why P has coordinates $(\cos(\frac{\pi}{2} - \theta), \sin(\frac{\pi}{2} - \theta))$.



b Show that:

i $XP = \sin\theta$ ii $OX = \cos\theta$

c Hence, copy and complete:

i $\cos(\frac{\pi}{2} - \theta) = ...$ ii $\sin(\frac{\pi}{2} - \theta) = ...$

d Check your answer to **c** by calculating:

i $\cos\frac{\pi}{5}$ and $\sin\frac{3\pi}{10}$ ii $\sin\frac{\pi}{8}$ and $\cos\frac{3\pi}{8}$.

e Write $\tan(\frac{\pi}{2} - \theta)$ in terms of $\tan\theta$.

DISCUSSION IDENTITIES

In the previous Exercise you should have proven the following trigonometric **identities**:

	Degree form	*Radian form*
Supplementary angles	$\cos(180° - \theta) = -\cos\theta$ $\sin(180° - \theta) = \sin\theta$ $\tan(180° - \theta) = -\tan\theta$	$\cos(\pi - \theta) = -\cos\theta$ $\sin(\pi - \theta) = \sin\theta$ $\tan(\pi - \theta) = -\tan\theta$
Negative angles	$\cos(-\theta) = \cos\theta$ $\sin(-\theta) = -\sin\theta$ $\tan(-\theta) = -\tan\theta$	$\cos(-\theta) = \cos\theta$ $\sin(-\theta) = -\sin\theta$ $\tan(-\theta) = -\tan\theta$
Complementary angles	$\cos(90° - \theta) = \sin\theta$ $\sin(90° - \theta) = \cos\theta$ $\tan(90° - \theta) = \frac{1}{\tan\theta}$	$\cos(\frac{\pi}{2} - \theta) = \sin\theta$ $\sin(\frac{\pi}{2} - \theta) = \cos\theta$ $\tan(\frac{\pi}{2} - \theta) = \frac{1}{\tan\theta}$

- What do we mean by the word "identity"?
- Why are identities important?
- What other identities do we use?

D MULTIPLES OF $\frac{\pi}{6}$ AND $\frac{\pi}{4}$

Angles which are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ occur frequently in geometry, so it is important for us to write their trigonometric ratios exactly.

MULTIPLES OF $\frac{\pi}{4}$ OR 45°

Consider $\theta = 45°$.

Angle OPB also measures 45°, so triangle OBP is isosceles.

$\therefore$ we let $\text{OB} = \text{BP} = a$

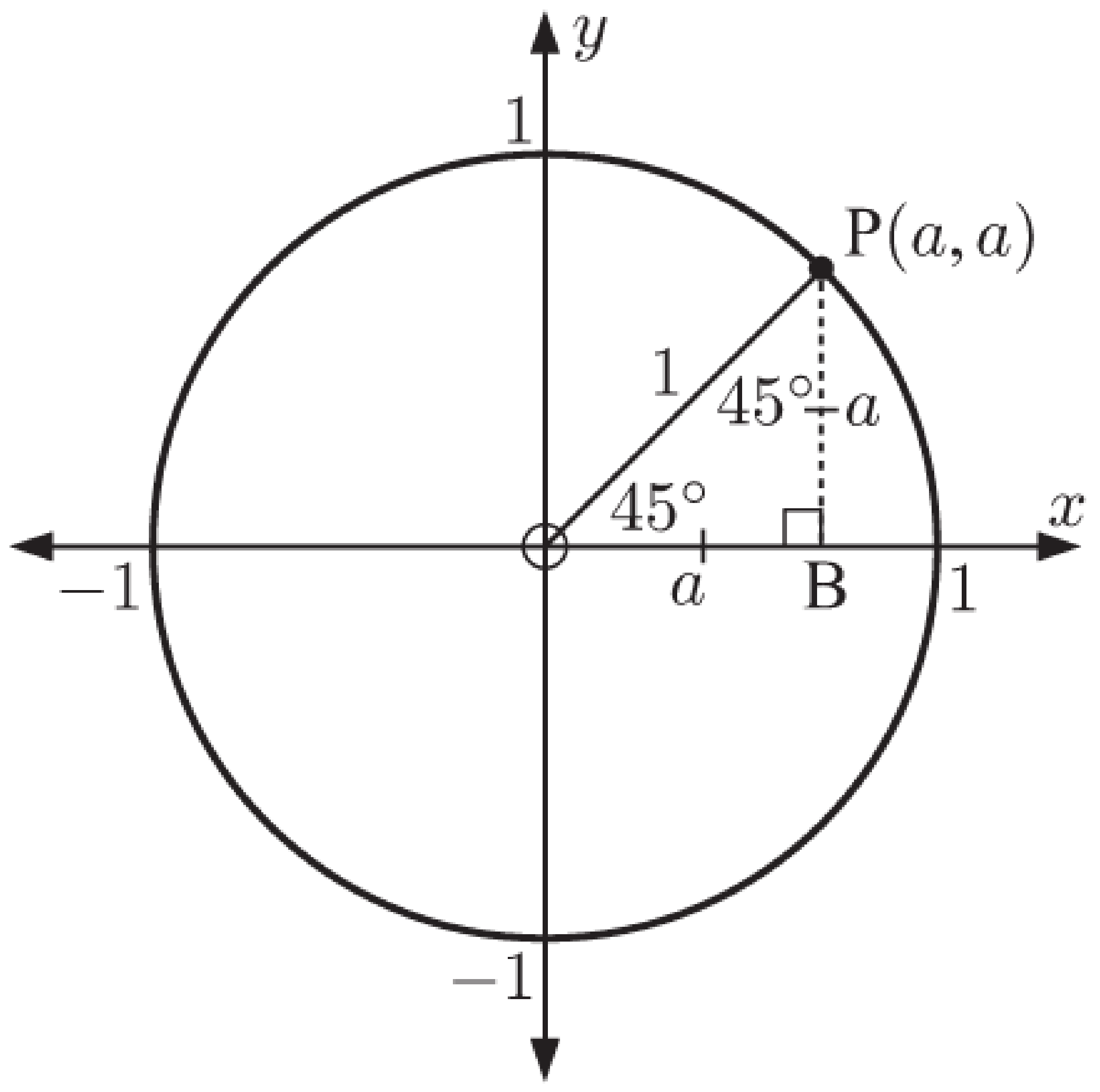
Now $a^2 + a^2 = 1^2$ {Pythagoras}

$\therefore a^2 = \frac{1}{2}$

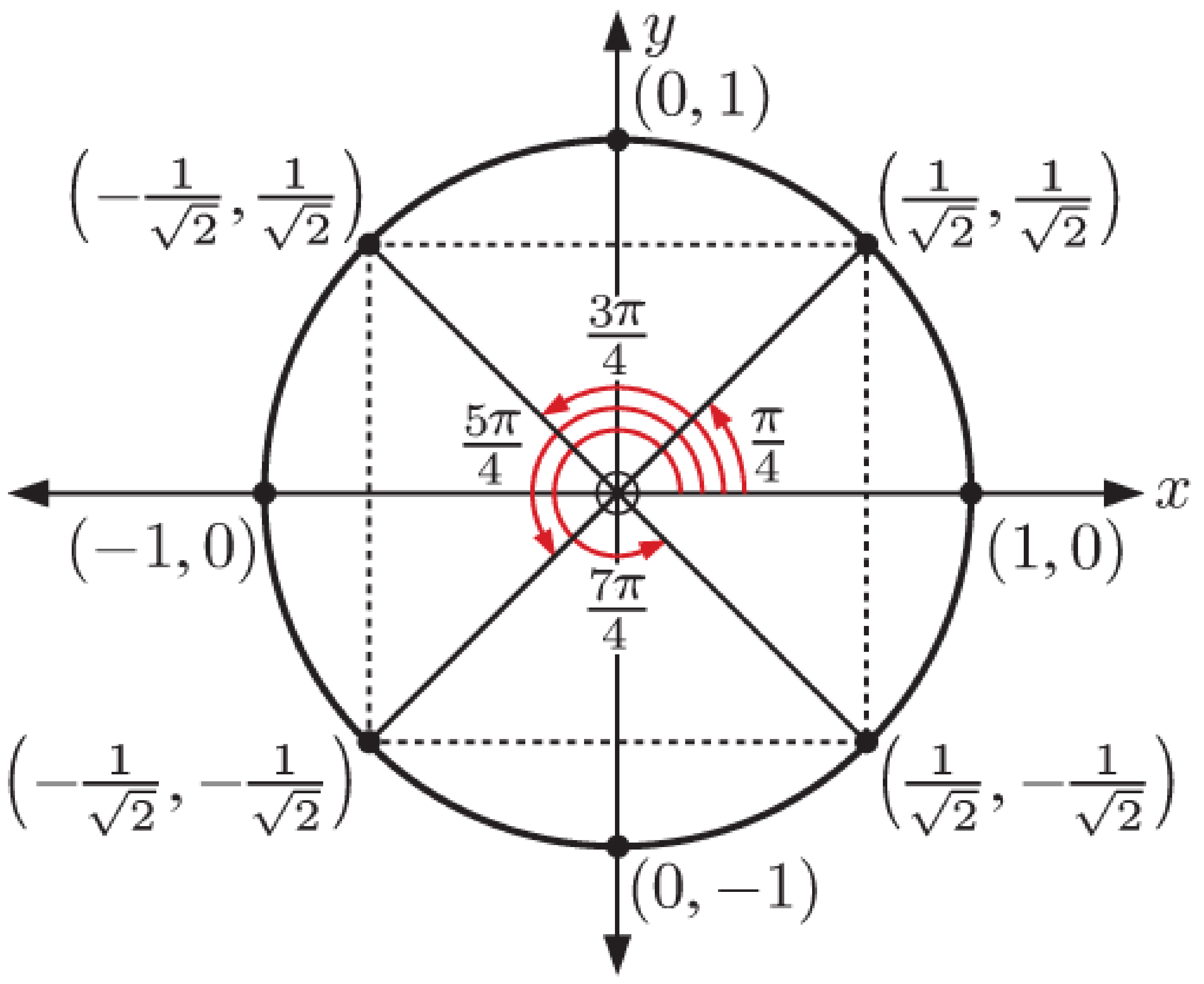
$\therefore a = \frac{1}{\sqrt{2}}$ {since $a > 0$}

$\therefore$ P is $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ where $\frac{1}{\sqrt{2}} \approx 0.707$.

So, $\boldsymbol{\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}}$ and $\boldsymbol{\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}}$



We can now find the coordinates of all points on the unit circle corresponding to multiples of $\frac{\pi}{4}$ by symmetry.



MULTIPLES OF $\frac{\pi}{6}$ OR 30°

Consider $\theta = 60°$.

Since $\text{OA} = \text{OP}$, triangle OAP is isosceles.

Now $\text{A}\widehat{\text{O}}\text{P} = 60°$, so the remaining angles are therefore also 60°. Triangle AOP is therefore equilateral.

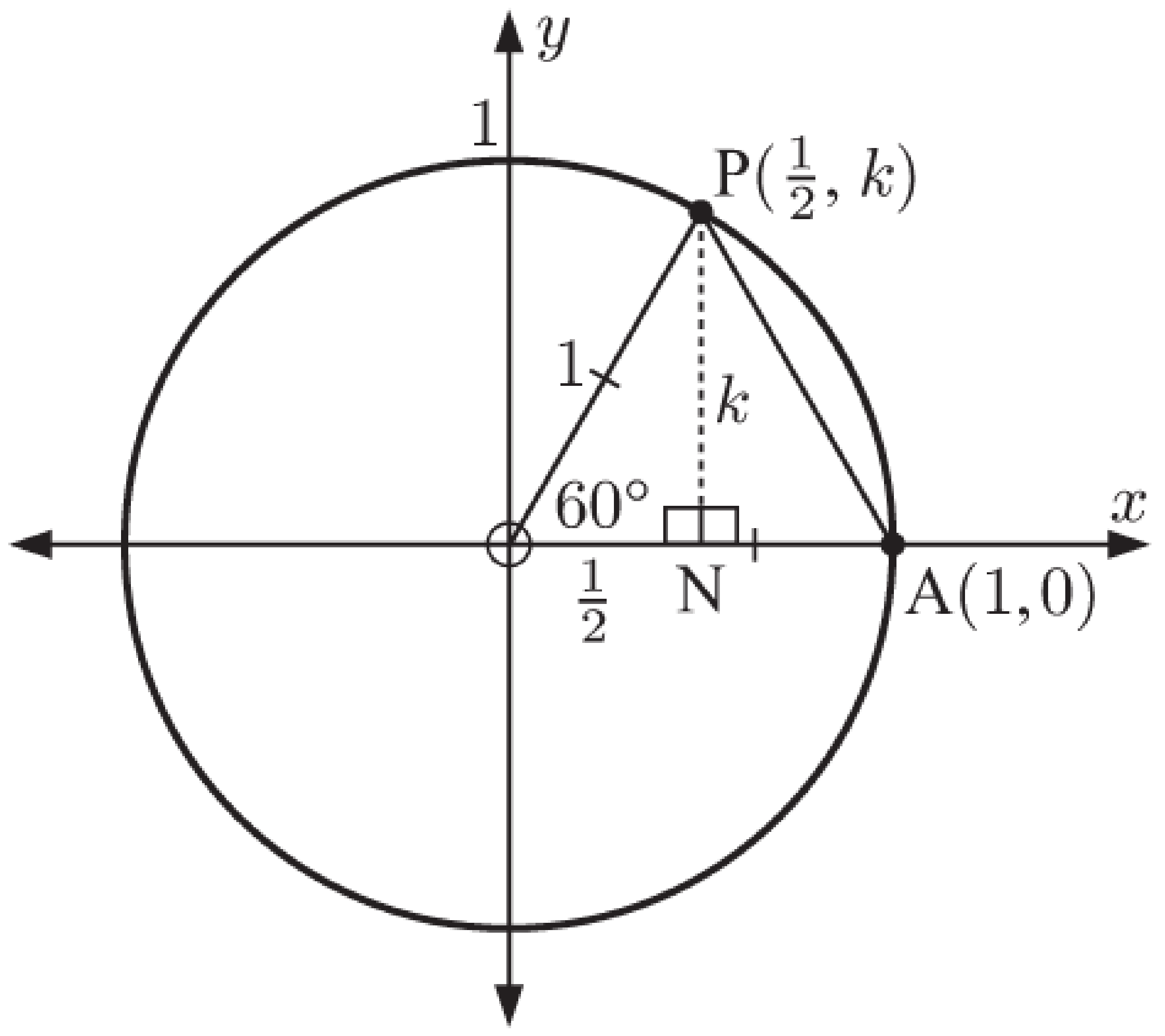
The altitude [PN] bisects base [OA], so $\text{ON} = \frac{1}{2}$.

If P is $(\frac{1}{2}, k)$, then $(\frac{1}{2})^2 + k^2 = 1$

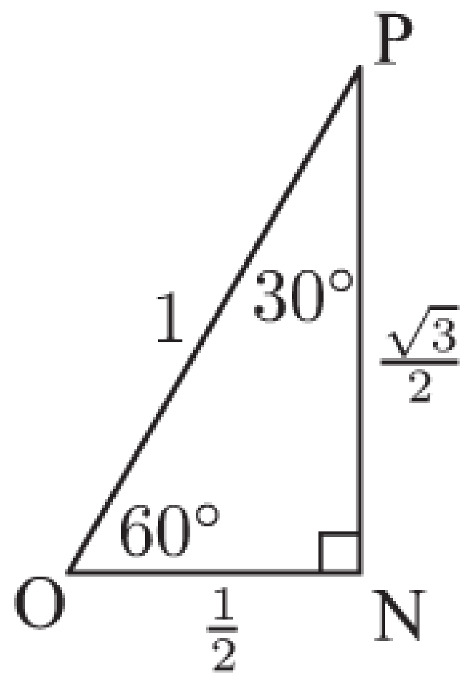
$\therefore k^2 = \frac{3}{4}$

$\therefore k = \frac{\sqrt{3}}{2}$ {since $k > 0$}

$\therefore$ P is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ where $\frac{\sqrt{3}}{2} \approx 0.866$.



So, $\boldsymbol{\cos\frac{\pi}{3} = \frac{1}{2}}$ and $\boldsymbol{\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}}$



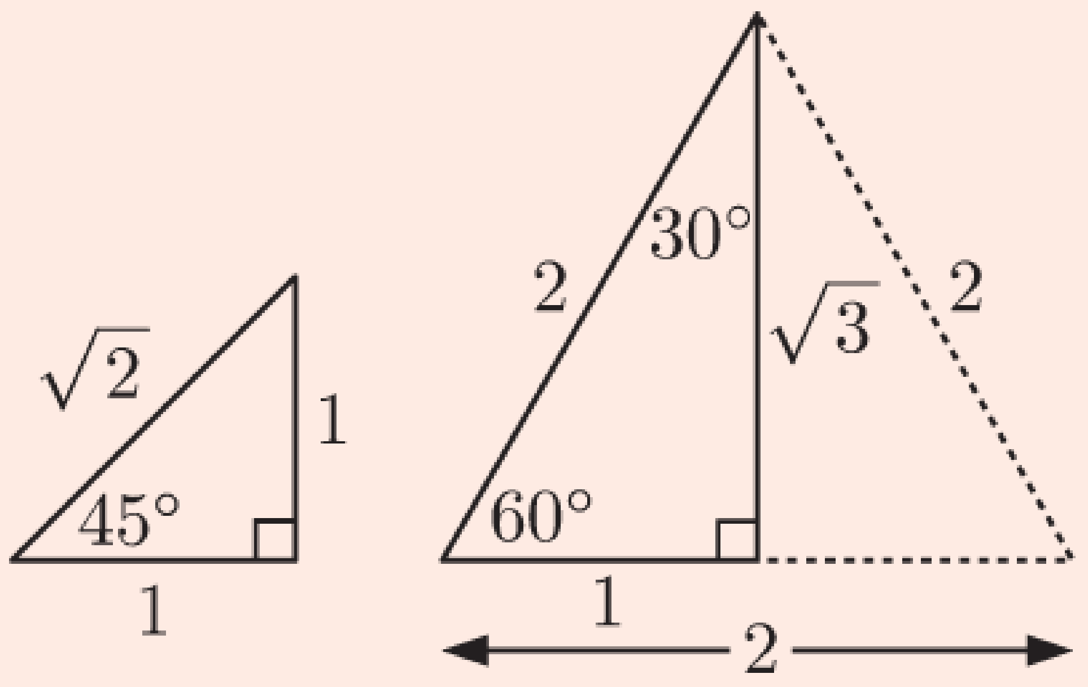
Now $\text{N}\widehat{\text{P}}\text{O} = \frac{\pi}{6} = 30°$. Hence $\boldsymbol{\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}}$ and $\boldsymbol{\sin\frac{\pi}{6} = \frac{1}{2}}$

DISCUSSION

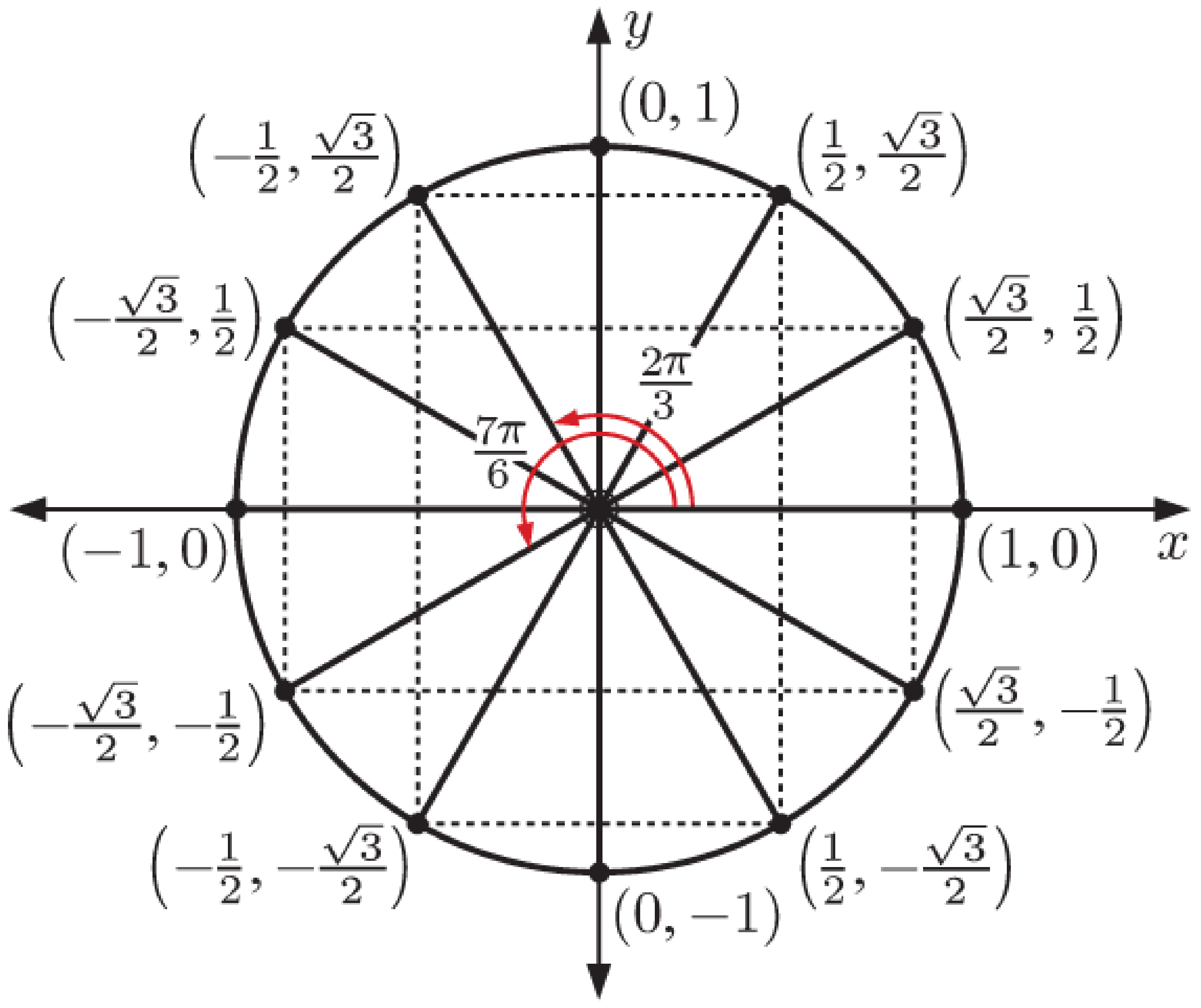
You should remember the values of cosine and sine for angles $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$.

However, if you forget, you can use these diagrams to quickly generate the results.

Discuss how you can do this.



We can now find the coordinates of all points on the unit circle corresponding to multiples of $\frac{\pi}{6}$ by symmetry.



SUMMARY

- For **multiples of** $\frac{\pi}{2}$, the coordinates of the points on the unit circle involve 0 and ± 1.
- For *other* **multiples of** $\frac{\pi}{4}$, the coordinates involve $\pm\frac{1}{\sqrt{2}}$.
- For *other* **multiples of** $\frac{\pi}{6}$, the coordinates involve $\pm\frac{1}{2}$ and $\pm\frac{\sqrt{3}}{2}$.

Example 5

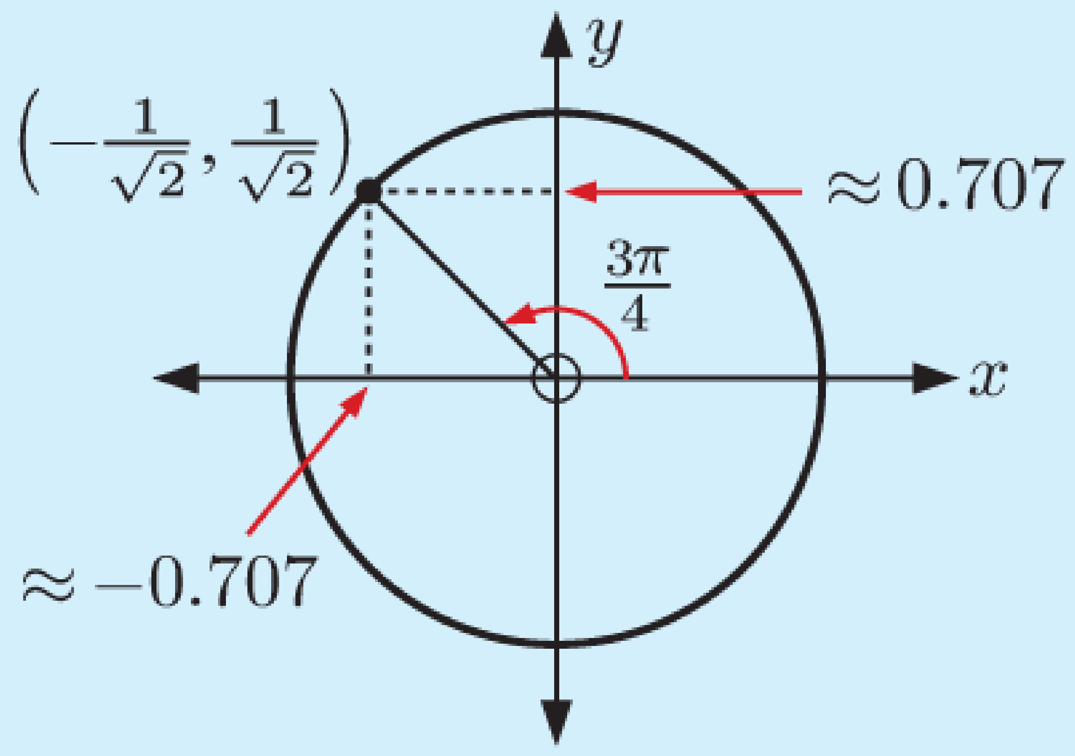
Self Tutor

Find the exact values of $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ for:

a $\alpha = \frac{3\pi}{4}$

b $\alpha = \frac{4\pi}{3}$

a

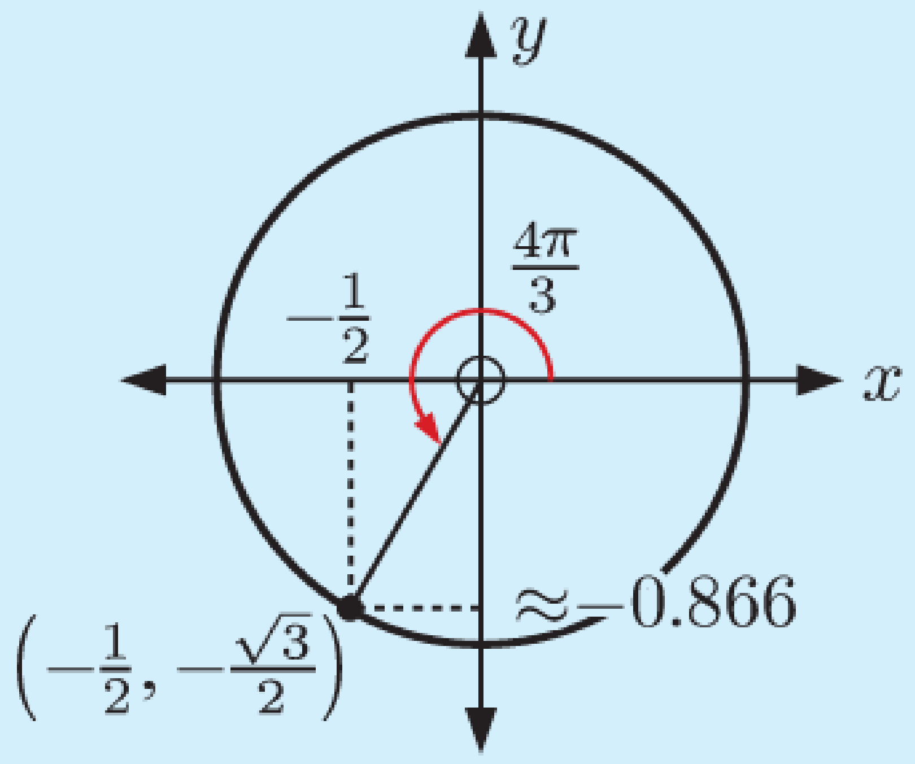


$$\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}$$

$$\tan\frac{3\pi}{4} = -1$$

b



$$\sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$\cos\frac{4\pi}{3} = -\frac{1}{2}$$

$$\tan\frac{4\pi}{3} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$

EXERCISE 8D

1 Use a unit circle diagram to find exact values for $\sin\theta$, $\cos\theta$, and $\tan\theta$ for θ equal to:

a $\frac{\pi}{4}$ **b** $\frac{3\pi}{4}$ **c** $\frac{7\pi}{4}$ **d** π **e** $-\frac{3\pi}{4}$

2 Use a unit circle diagram to find exact values for $\sin\beta$, $\cos\beta$, and $\tan\beta$ for β equal to:

a $\frac{\pi}{6}$ **b** $\frac{2\pi}{3}$ **c** $\frac{7\pi}{6}$ **d** $\frac{5\pi}{3}$ **e** $\frac{11\pi}{6}$

3 Find the exact values of:

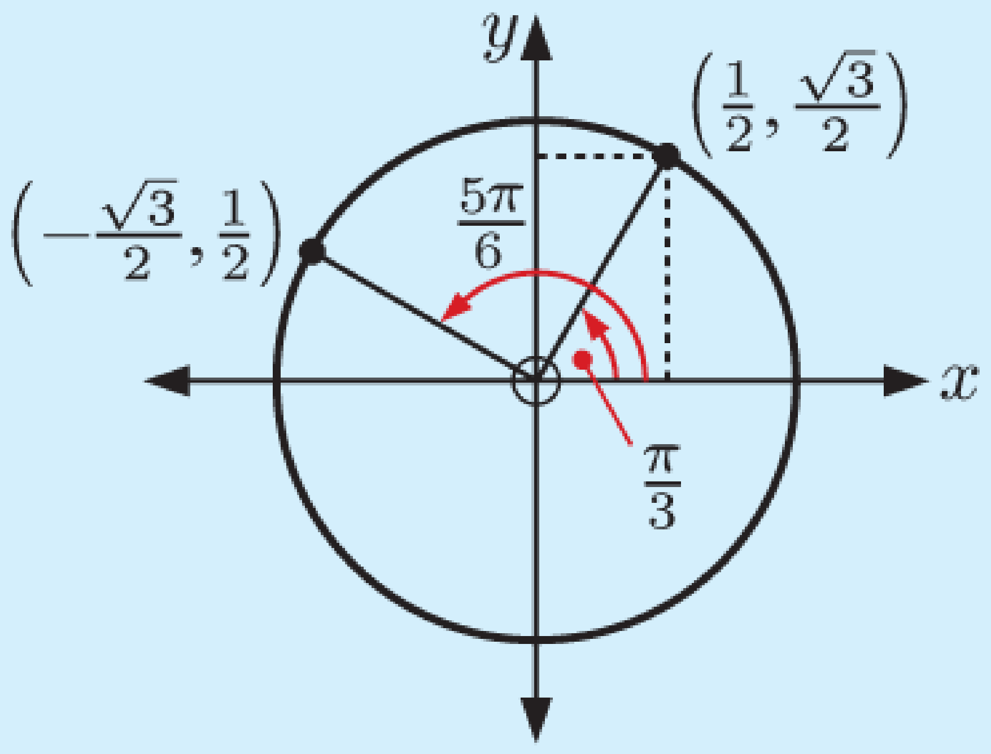
a $\cos\frac{2\pi}{3}$, $\sin\frac{2\pi}{3}$, and $\tan\frac{2\pi}{3}$ **b** $\cos\left(-\frac{\pi}{4}\right)$, $\sin\left(-\frac{\pi}{4}\right)$, and $\tan\left(-\frac{\pi}{4}\right)$

4 **a** Find the exact values of $\cos\frac{\pi}{2}$ and $\sin\frac{\pi}{2}$.

b What can you say about $\tan\frac{\pi}{2}$?

Example 6 **Self Tutor**

Without using a calculator, show that $8\sin\frac{\pi}{3}\cos\frac{5\pi}{6} = -6$.



$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

$$\begin{aligned}\therefore\ 8\sin\tfrac{\pi}{3}\cos\tfrac{5\pi}{6} &= 8\left(\tfrac{\sqrt{3}}{2}\right)\left(-\tfrac{\sqrt{3}}{2}\right)\\ &= 2(-3)\\ &= -6\end{aligned}$$

5 Without using a calculator, evaluate:

a $\sin^2\left(\frac{\pi}{3}\right)$ **b** $\sin\frac{\pi}{6}\cos\frac{\pi}{3}$ **c** $1-\cos^2\left(\frac{\pi}{6}\right)$

d $\sin^2\left(\frac{2\pi}{3}\right)-1$ **e** $\cos^2\left(\frac{\pi}{4}\right)-\sin\frac{7\pi}{6}$ **f** $\sin\frac{3\pi}{4}-\cos\frac{5\pi}{4}$

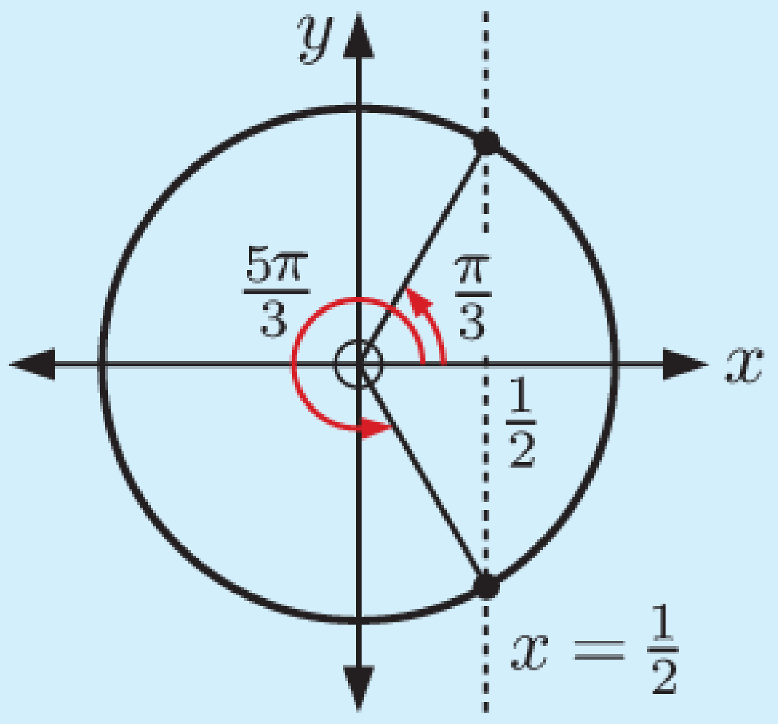
g $1-2\sin^2\left(\frac{7\pi}{6}\right)$ **h** $\cos^2\left(\frac{5\pi}{6}\right)-\sin^2\left(\frac{5\pi}{6}\right)$ **i** $\tan^2\left(\frac{\pi}{3}\right)-2\sin^2\left(\frac{\pi}{4}\right)$

j $2\tan\left(-\frac{5\pi}{4}\right)-\sin\frac{3\pi}{2}$ **k** $\dfrac{2\tan\frac{5\pi}{6}}{1-\tan^2\left(\frac{5\pi}{6}\right)}$ **l** $\dfrac{\cos\frac{\pi}{3}}{\sin\frac{4\pi}{3}+\tan\frac{\pi}{6}}$

Check all answers using your calculator.

Example 7 **Self Tutor**

Find all angles $0 \leqslant \theta \leqslant 2\pi$ with a cosine of $\frac{1}{2}$.



Since the cosine is $\frac{1}{2}$, we draw the vertical line $x = \frac{1}{2}$.

Because $\frac{1}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

6 Find all angles between 0 and 2π with:

a a sine of $\frac{1}{2}$
b a sine of $\frac{\sqrt{3}}{2}$
c a cosine of $\frac{1}{\sqrt{2}}$
d a cosine of $-\frac{1}{2}$
e a cosine of $-\frac{1}{\sqrt{2}}$
f a sine of $-\frac{\sqrt{3}}{2}$

7 Find all angles between 0 and 2π (inclusive) which have a tangent of:

a 1
b -1
c $\sqrt{3}$
d 0
e $\frac{1}{\sqrt{3}}$
f $-\sqrt{3}$

8 Find all angles between 0 and 4π with:

a a cosine of $\frac{\sqrt{3}}{2}$
b a sine of $-\frac{1}{2}$
c a sine of -1

9 Find θ if $0 \leqslant \theta \leqslant 2\pi$ and:

a $\cos\theta = \frac{1}{2}$
b $\sin\theta = \frac{\sqrt{3}}{2}$
c $\cos\theta = -1$
d $\sin\theta = 1$
e $\cos\theta = -\frac{1}{\sqrt{2}}$
f $\sin^2\theta = 1$
g $\cos^2\theta = 1$
h $\cos^2\theta = \frac{1}{2}$
i $\tan\theta = -\frac{1}{\sqrt{3}}$
j $\tan^2\theta = 3$

10 Find *all* values of θ for which $\tan\theta$ is:

a zero
b undefined.

E THE PYTHAGOREAN IDENTITY

From the equation of the unit circle $x^2 + y^2 = 1$, we obtain the **Pythagorean identity**:

For any angle θ, $\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$.

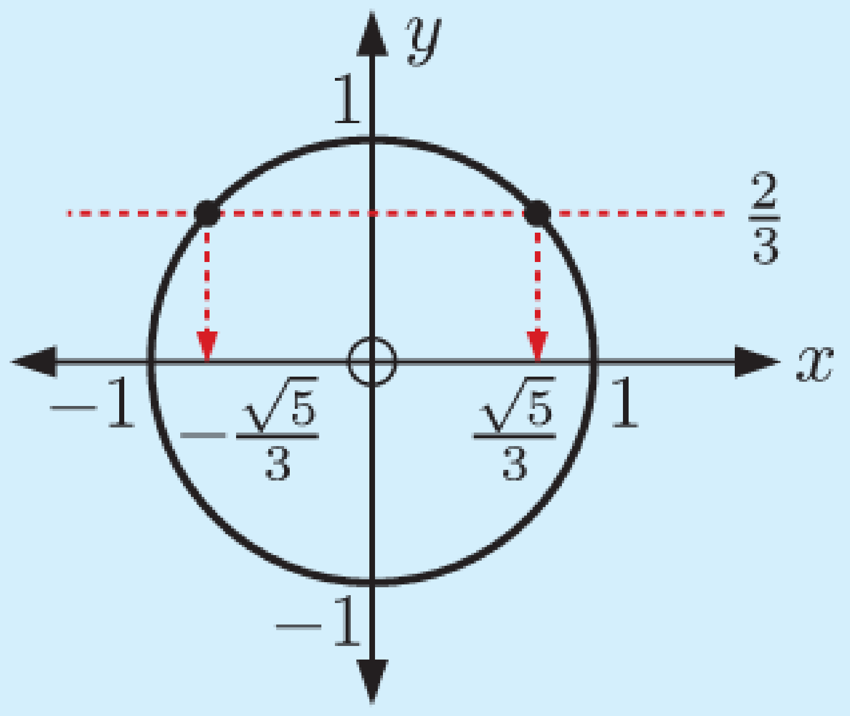
We can use this identity to find one trigonometric ratio from another.

Example 8

Self Tutor

Find the possible exact values of $\cos\theta$ for $\sin\theta = \frac{2}{3}$. Illustrate your answers.

$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore \cos^2\theta + \left(\tfrac{2}{3}\right)^2 = 1$$
$$\therefore \cos^2\theta = \tfrac{5}{9}$$
$$\therefore \cos\theta = \pm\tfrac{\sqrt{5}}{3}$$



EXERCISE 8E

1 Find the possible exact values of $\cos\theta$ for:

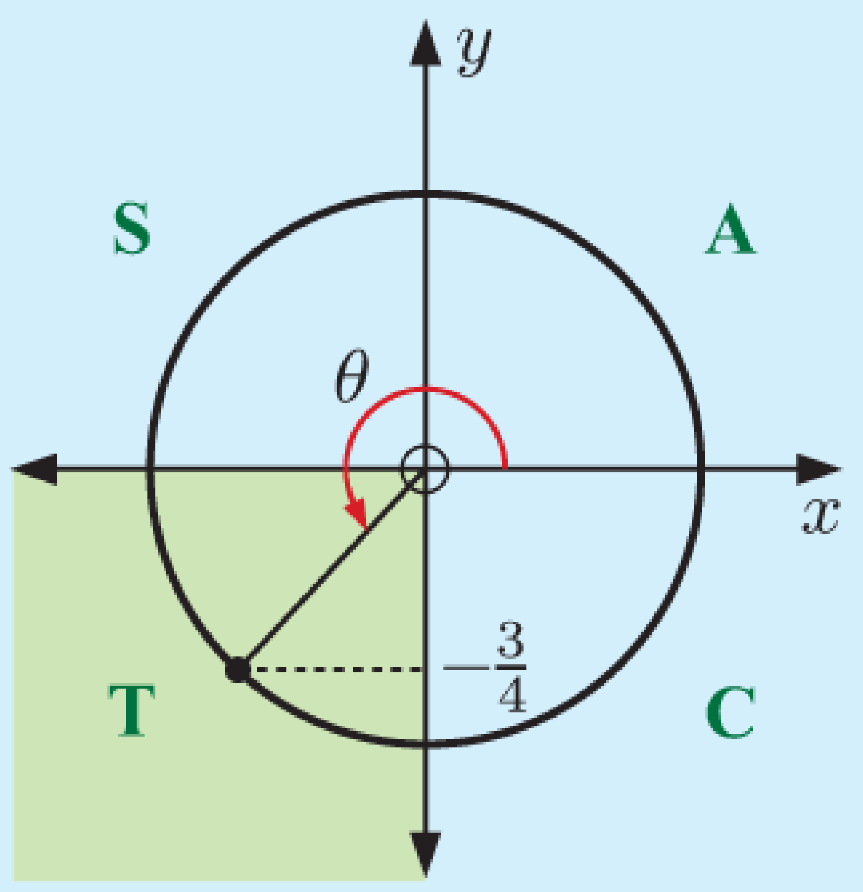
a $\sin\theta = \frac{1}{2}$
b $\sin\theta = -\frac{1}{3}$
c $\sin\theta = 0$
d $\sin\theta = -1$

2 Find the possible exact values of $\sin\theta$ for:

a $\cos\theta = \frac{4}{5}$
b $\cos\theta = -\frac{3}{4}$
c $\cos\theta = 1$
d $\cos\theta = 0$

Example 9 Self Tutor

If $\sin\theta = -\frac{3}{4}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\cos\theta$ and $\tan\theta$. Give exact values.



Now $\cos^2\theta + \sin^2\theta = 1$

$\therefore\ \cos^2\theta + \frac{9}{16} = 1$

$\therefore\ \cos^2\theta = \frac{7}{16}$

$\therefore\ \cos\theta = \pm\frac{\sqrt{7}}{4}$

But $\pi < \theta < \frac{3\pi}{2}$, so θ is a quadrant 3 angle.

$\therefore$ $\cos\theta$ is negative.

$\therefore\ \cos\theta = -\frac{\sqrt{7}}{4}$ and $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{-\frac{3}{4}}{-\frac{\sqrt{7}}{4}} = \dfrac{3}{\sqrt{7}}$

3 Without using a calculator, find:

- **a** $\sin\theta$ if $\cos\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$
- **b** $\cos\theta$ if $\sin\theta = \frac{2}{5}$ and $\frac{\pi}{2} < \theta < \pi$
- **c** $\cos\theta$ if $\sin\theta = -\frac{3}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$
- **d** $\sin\theta$ if $\cos\theta = -\frac{5}{13}$ and $\pi < \theta < \frac{3\pi}{2}$.

4 Find $\tan\theta$ exactly given:

- **a** $\sin\theta = \frac{1}{3}$ and $\frac{\pi}{2} < \theta < \pi$
- **b** $\cos\theta = \frac{1}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$
- **c** $\sin\theta = -\frac{1}{\sqrt{3}}$ and $\pi < \theta < \frac{3\pi}{2}$
- **d** $\cos\theta = -\frac{3}{4}$ and $\frac{\pi}{2} < \theta < \pi$.

Example 10 Self Tutor

If $\tan\theta = -2$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\sin\theta$ and $\cos\theta$. Give exact answers.

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = -2$

$\therefore\ \sin\theta = -2\cos\theta$

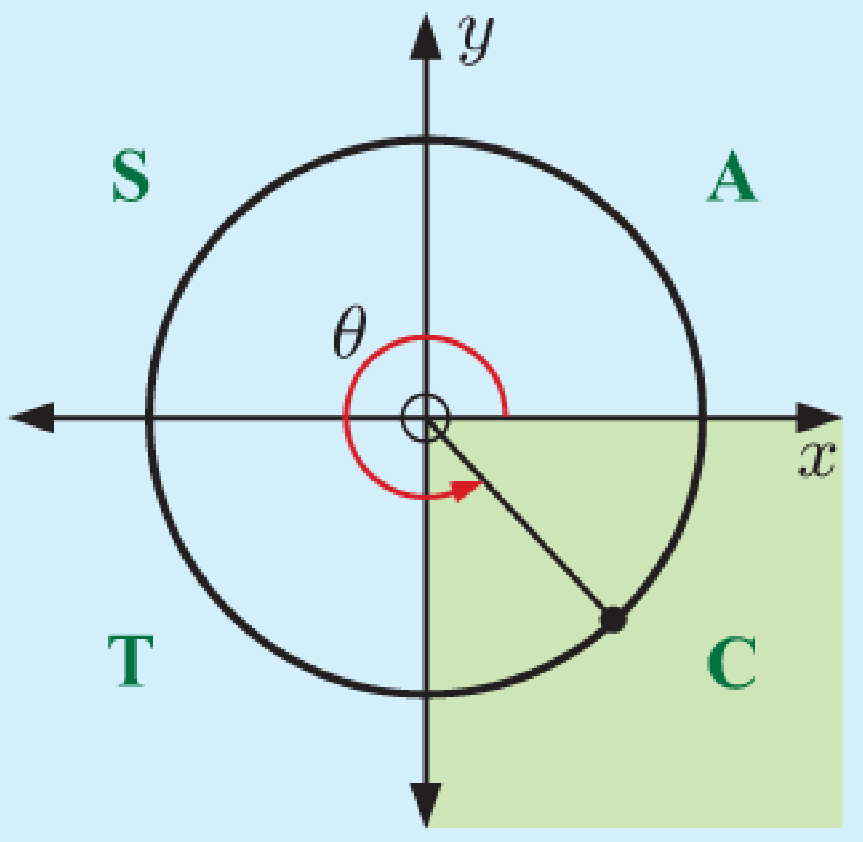
Now $\sin^2\theta + \cos^2\theta = 1$

$\therefore\ (-2\cos\theta)^2 + \cos^2\theta = 1$

$\therefore\ 4\cos^2\theta + \cos^2\theta = 1$

$\therefore\ 5\cos^2\theta = 1$

$\therefore\ \cos\theta = \pm\frac{1}{\sqrt{5}}$



But $\frac{3\pi}{2} < \theta < 2\pi$, so θ is a quadrant 4 angle. $\cos\theta$ is positive and $\sin\theta$ is negative.

$\therefore\ \cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = -\frac{2}{\sqrt{5}}$.

5 Find exact values for $\sin\theta$ and $\cos\theta$ given that:

- **a** $\tan\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$
- **b** $\tan\theta = -\frac{4}{3}$ and $\frac{\pi}{2} < \theta < \pi$
- **c** $\tan\theta = \frac{\sqrt{5}}{3}$ and $\pi < \theta < \frac{3\pi}{2}$
- **d** $\tan\theta = -\frac{12}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$

6 Suppose $\tan\theta = k$ where k is a constant and $\pi < \theta < \frac{3\pi}{2}$. Write expressions for $\sin\theta$ and $\cos\theta$ in terms of k.

F FINDING ANGLES

In **Exercise 8C** you should have proven that:

For θ in degrees:

- $\sin(180^\circ - \theta) = \sin\theta$
- $\cos(180^\circ - \theta) = -\cos\theta$
- $\cos(360^\circ - \theta) = \cos\theta$
- $\sin(360^\circ - \theta) = -\sin\theta$

For θ in radians:

- $\sin(\pi - \theta) = \sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\cos(2\pi - \theta) = \cos\theta$
- $\sin(2\pi - \theta) = -\sin\theta$

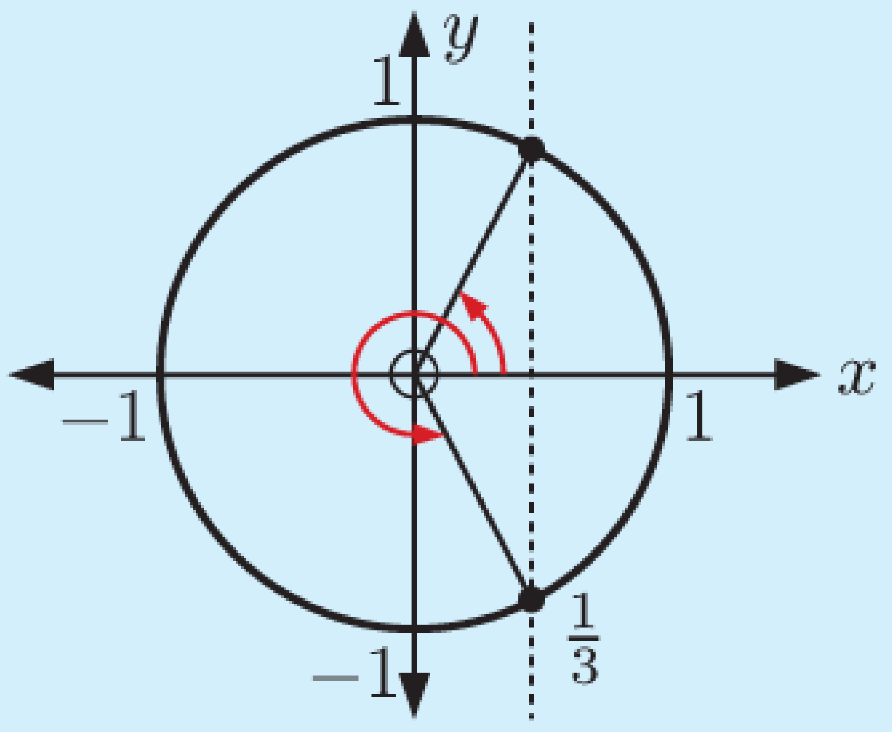
We need results such as these, and also the periodicity of the trigonometric ratios, to find angles which have a particular sine, cosine, or tangent.

Example 11 **Self Tutor**

Find the two angles θ on the unit circle, with $0^\circ \leqslant \theta \leqslant 360^\circ$, such that:

a $\cos\theta = \frac{1}{3}$ **b** $\sin\theta = \frac{3}{4}$ **c** $\tan\theta = 2$

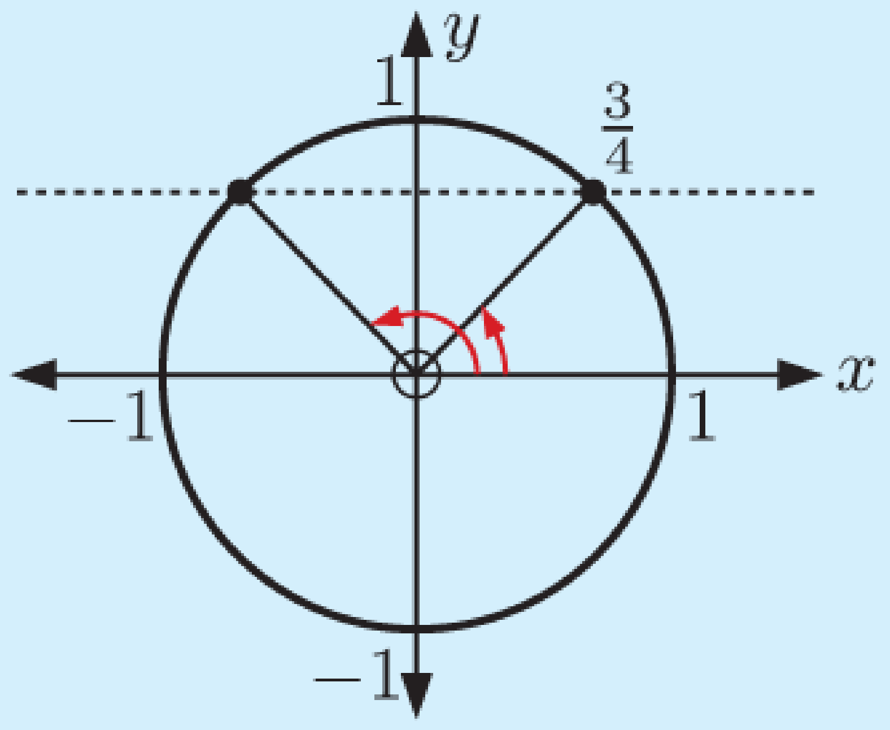
a Using technology, $\cos^{-1}\left(\frac{1}{3}\right) \approx 70.53^\circ$



$\therefore\ \theta \approx 70.53^\circ$ or $360^\circ - 70.53^\circ$

$\therefore\ \theta \approx 70.5^\circ$ or 289.5°

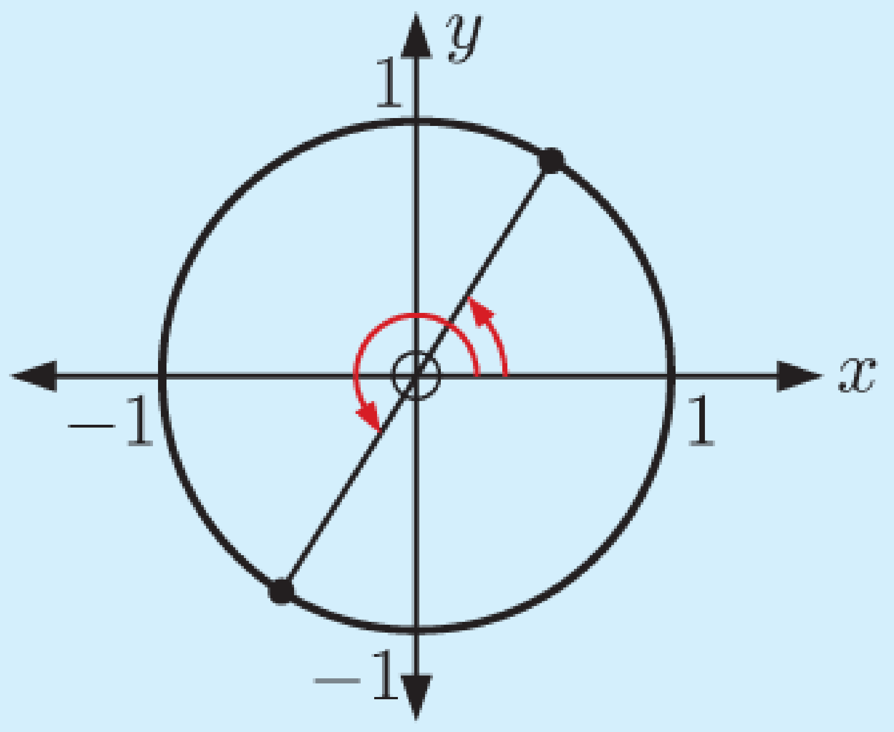
b Using technology, $\sin^{-1}\left(\frac{3}{4}\right) \approx 48.59^\circ$



$\therefore\ \theta \approx 48.59^\circ$ or $180^\circ - 48.59^\circ$

$\therefore\ \theta \approx 48.6^\circ$ or 131.4°

c Using technology, $\tan^{-1}(2) \approx 63.43^\circ$



$\therefore\ \theta \approx 63.43^\circ$ or $180^\circ + 63.43^\circ$

$\therefore\ \theta \approx 63.4^\circ$ or 243.4°

For positive $\cos\theta$, $\sin\theta$, or $\tan\theta$, your calculator will give the *acute* angle θ.

EXERCISE 8F

1 Find two angles θ on the unit circle, with $0^\circ \leqslant \theta \leqslant 360^\circ$, such that:

a $\tan\theta = 4$ **b** $\cos\theta = 0.83$ **c** $\sin\theta = \frac{3}{5}$

d $\cos\theta = 0$ **e** $\tan\theta = 6.67$ **f** $\cos\theta = \frac{2}{17}$

2 Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\tan\theta = \frac{1}{3}$ **b** $\cos\theta = \frac{3}{7}$ **c** $\sin\theta = 0.61$

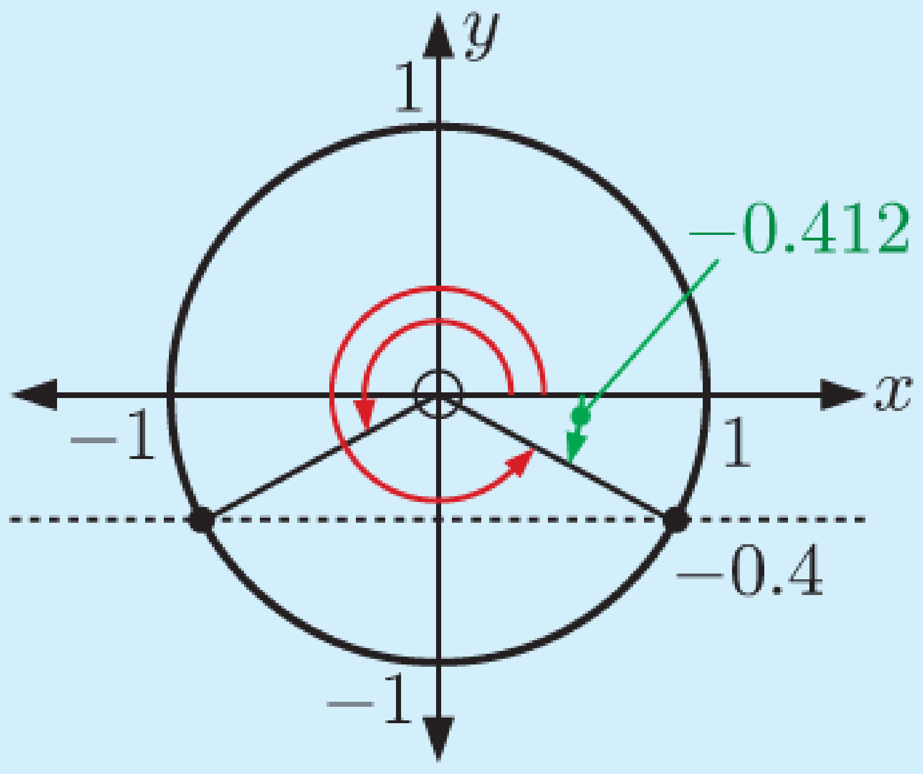
d $\cos\theta = \frac{1}{4}$ **e** $\tan\theta = 0.114$ **f** $\sin\theta = \frac{1}{6}$

Example 12 **Self Tutor**

Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\sin\theta = -0.4$ **b** $\cos\theta = -\frac{2}{3}$ **c** $\tan\theta = -\frac{1}{3}$

a Using technology, $\sin^{-1}(-0.4) \approx -0.412$



If $\sin\theta$ or $\tan\theta$ is negative, your calculator will give θ in the domain $-\frac{\pi}{2} < \theta < 0$.

If $\cos\theta$ is negative, your calculator will give the *obtuse* angle θ.

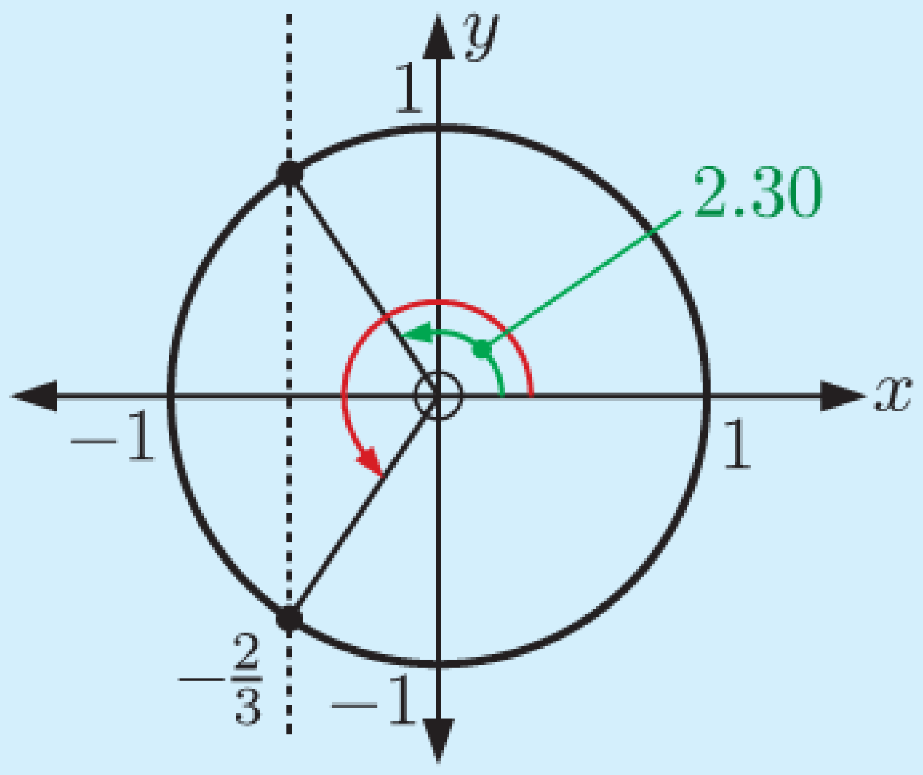
The angles given by your calculator are shown in green.

But $0 \leqslant \theta \leqslant 2\pi$

$\therefore\ \theta \approx \pi + 0.412$ or $2\pi - 0.412$

$\therefore\ \theta \approx 3.55$ or 5.87

b Using technology, $\cos^{-1}\left(-\frac{2}{3}\right) \approx 2.30$

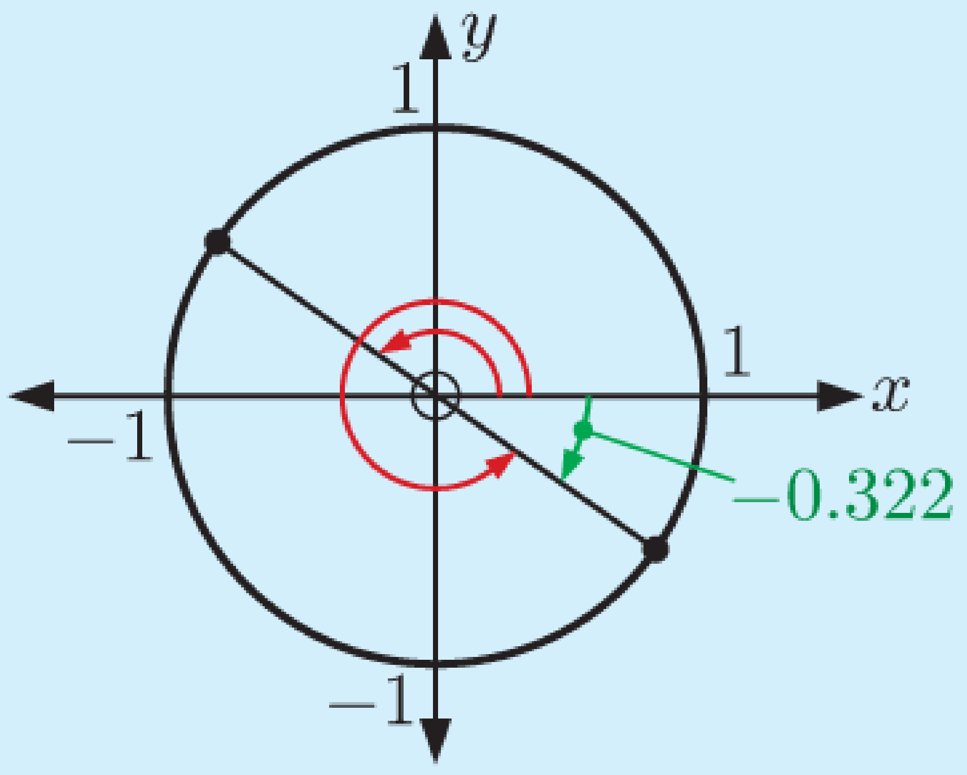


But $0 \leqslant \theta \leqslant 2\pi$

$\therefore\ \theta \approx 2.30$ or $2\pi - 2.30$

$\therefore\ \theta \approx 2.30$ or 3.98

c Using technology, $\tan^{-1}\left(-\frac{1}{3}\right) \approx -0.322$



But $0 \leqslant \theta \leqslant 2\pi$

$\therefore\ \theta \approx \pi - 0.322$ or $2\pi - 0.322$

$\therefore\ \theta \approx 2.82$ or 5.96

3 Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\cos\theta = -\frac{1}{4}$ **b** $\sin\theta = 0$ **c** $\tan\theta = -3.1$ **d** $\sin\theta = -0.421$

e $\tan\theta = 1.2$ **f** $\cos\theta = 0.7816$ **g** $\sin\theta = \frac{1}{11}$ **h** $\cos\theta = -\frac{1}{\sqrt{3}}$

4 Find all θ such that $-180° \leqslant \theta \leqslant 180°$ and:

a $\cos\theta = -\frac{1}{10}$ **b** $\sin\theta = \frac{4}{5}$ **c** $\tan\theta = -\frac{3}{2}$

d $\cos\theta = 0.8$ **e** $\tan\theta = -\frac{5}{6}$ **f** $\sin\theta = -\frac{7}{11}$

5 **a** Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that $\cos\theta = \frac{3}{10}$.

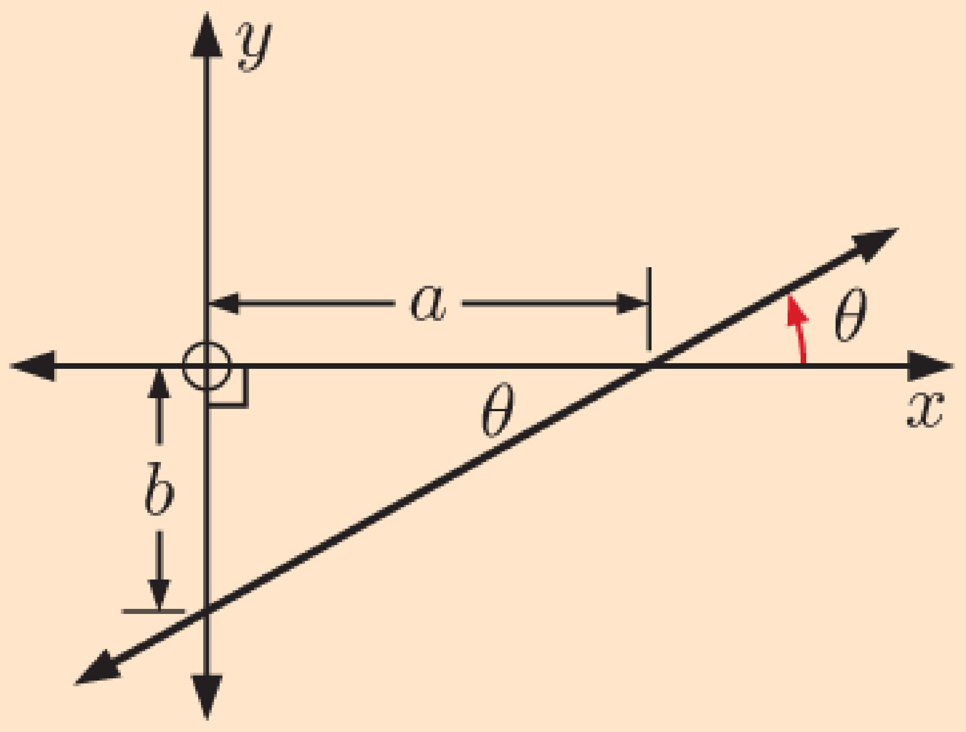
b For each value of θ, find $\sin\theta$ and $\tan\theta$ exactly.

G THE EQUATION OF A STRAIGHT LINE

If a straight line makes an angle of θ with the positive x-axis then its gradient is $m = \tan\theta$.

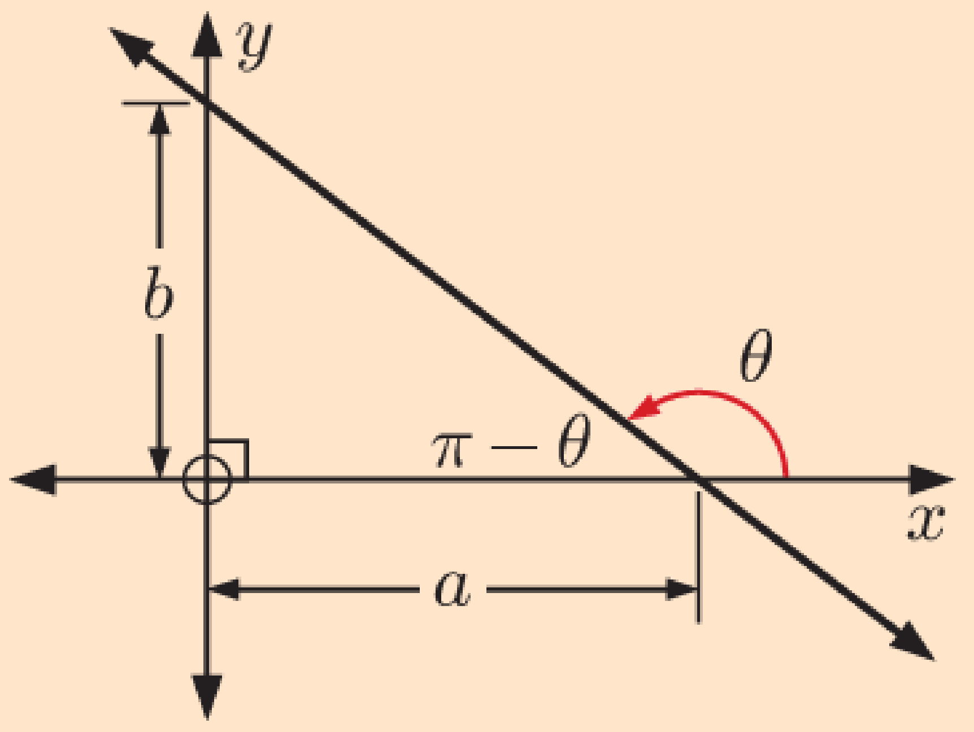
Proof:

- For $m \geqslant 0$:



$$\text{Gradient } m = \frac{0 - -b}{a - 0} = \frac{b}{a} = \tan\theta$$

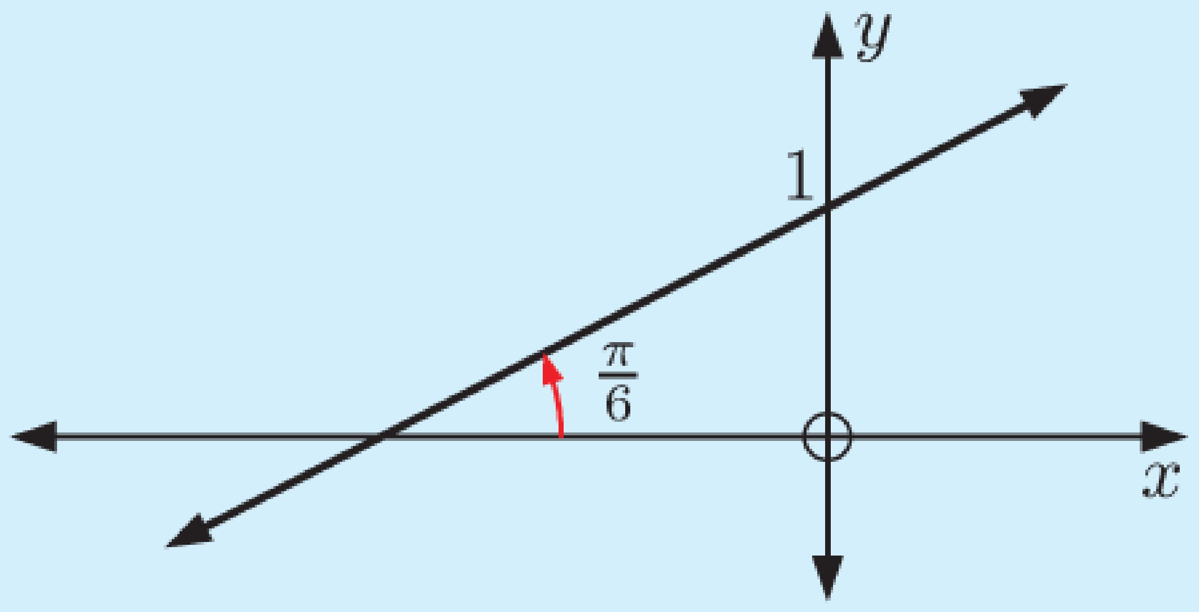
- For $m < 0$:



$$\text{Gradient } m = \frac{0 - b}{a - 0} = -\frac{b}{a} = -\tan(\pi - \theta) = \tan\theta$$

Example 13 **Self Tutor**

Find the equation of the given line:



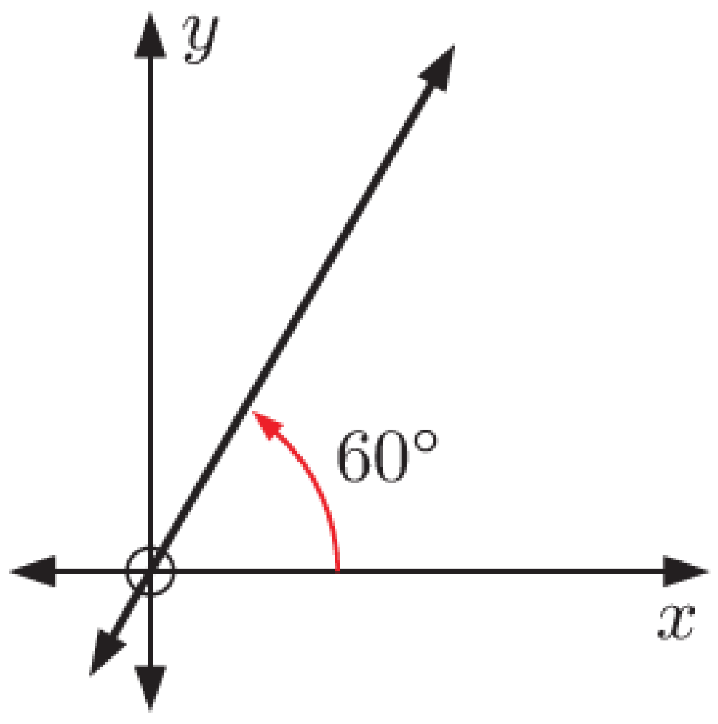
The line has gradient $m = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$ and y-intercept 1.

$\therefore$ the line has equation $y = \frac{1}{\sqrt{3}}x + 1$.

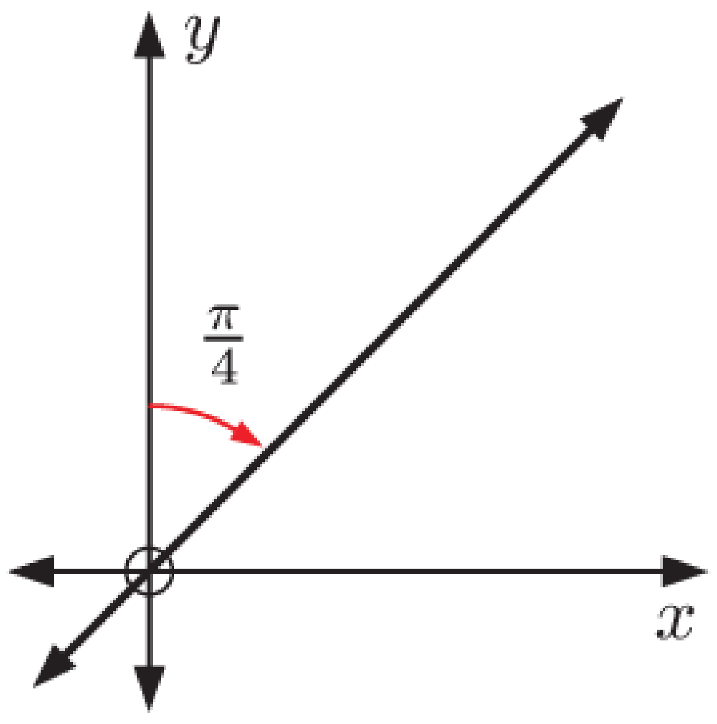
EXERCISE 8G

1 Find the equation of each line:

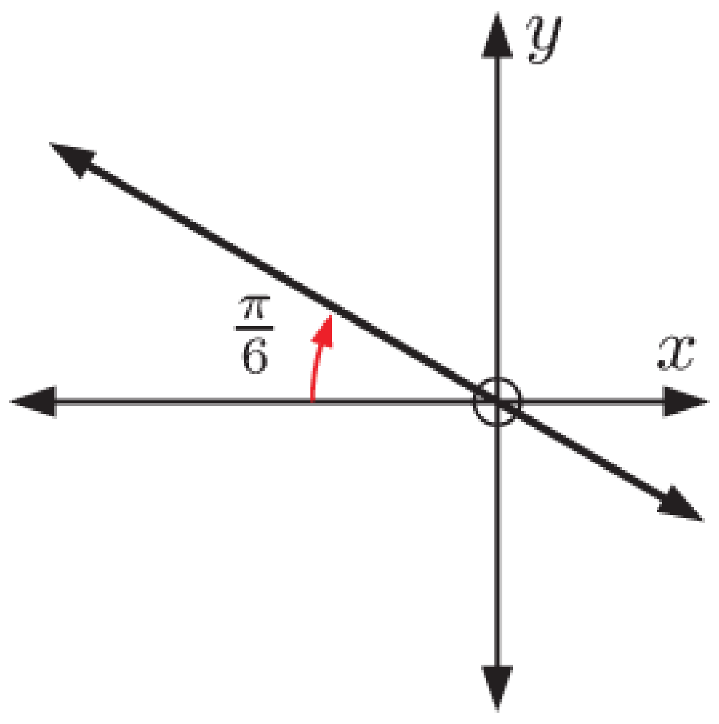
a



b

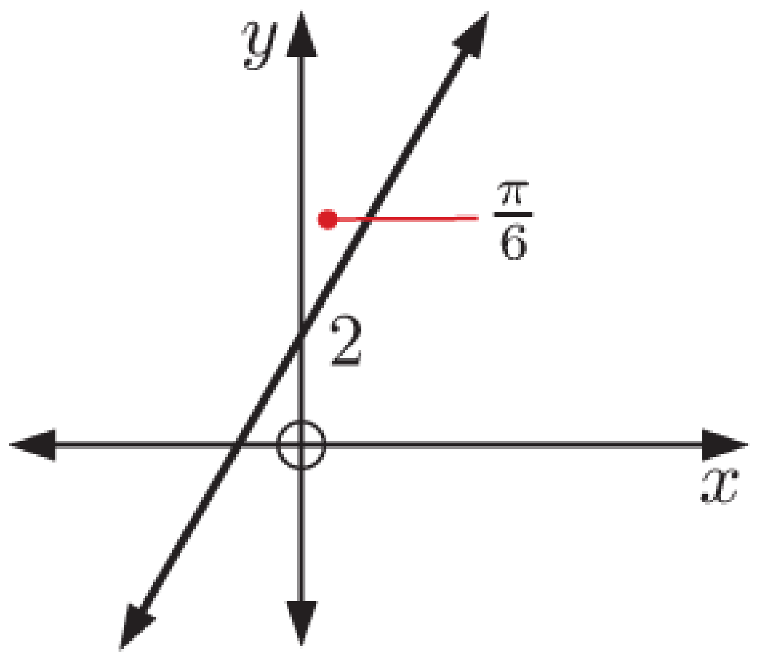


c

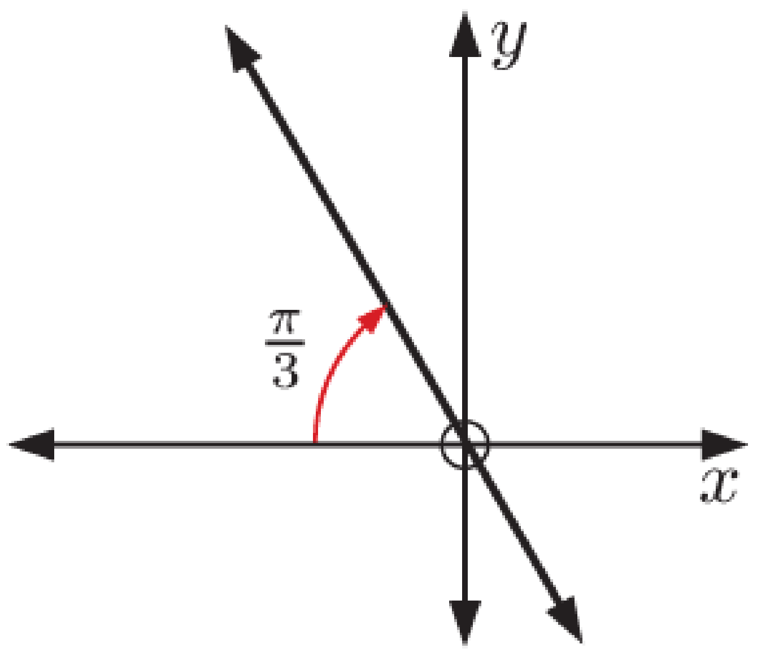


2 Find the equation of each line:

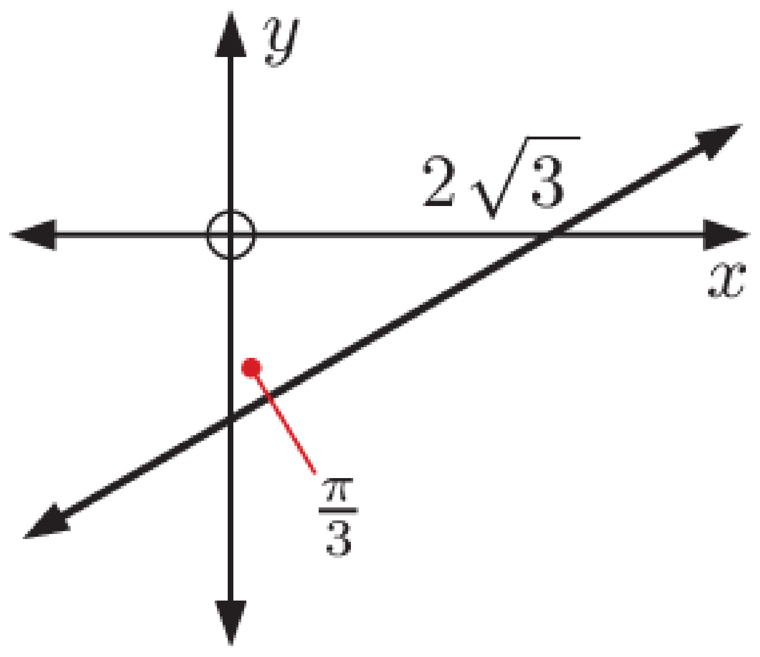
a



b

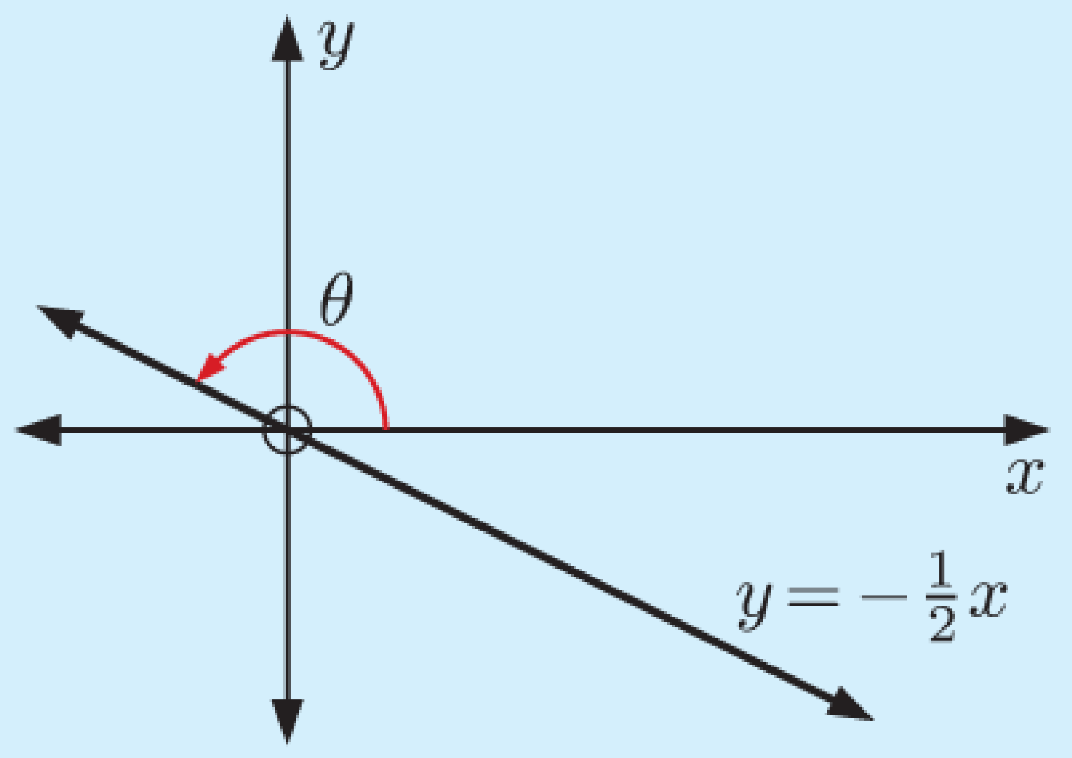


c



Example 14 **Self Tutor**

Find, in radians, the measure of θ:



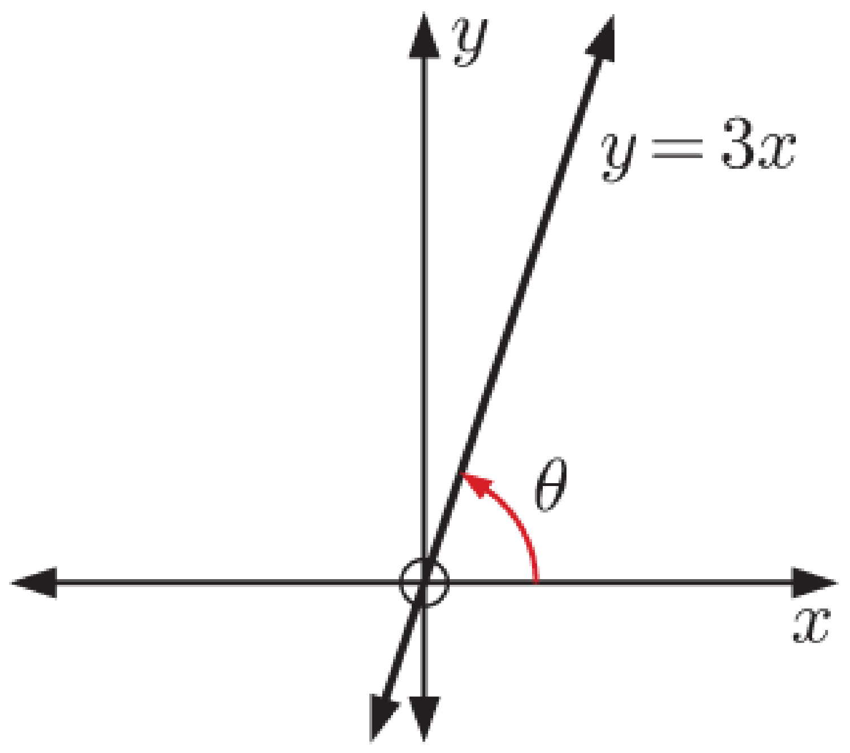
The line has gradient $-\frac{1}{2}$, so $\tan\theta = -\frac{1}{2}$.

Using technology, $\tan^{-1}(-\frac{1}{2}) \approx -0.464$

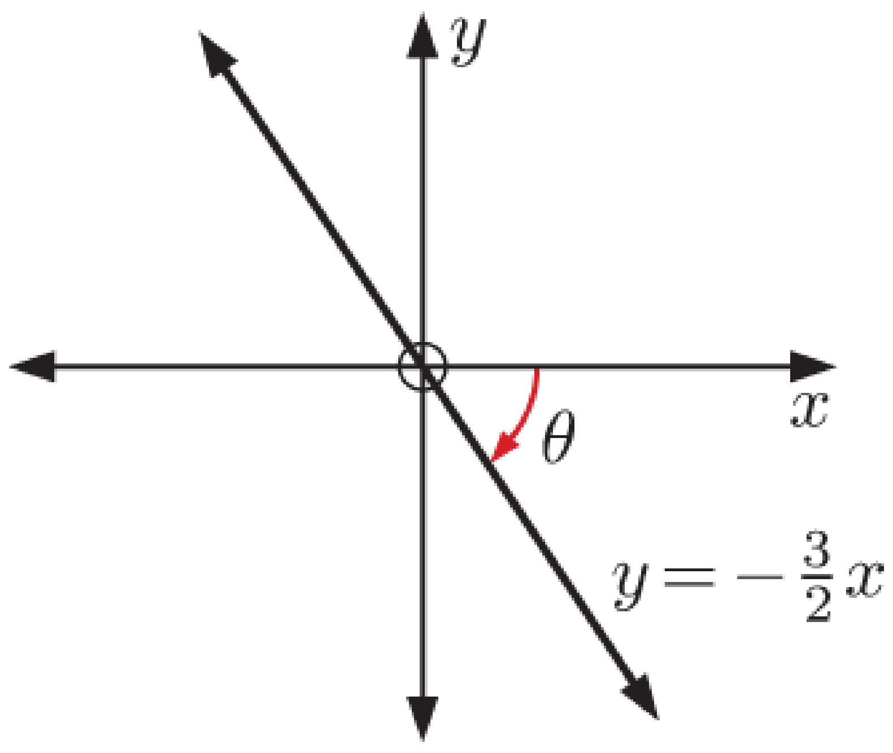
But $0 < \theta < \pi$, so $\theta \approx \pi - 0.464 \approx 2.68$

3 Find, in radians, the measure of θ:

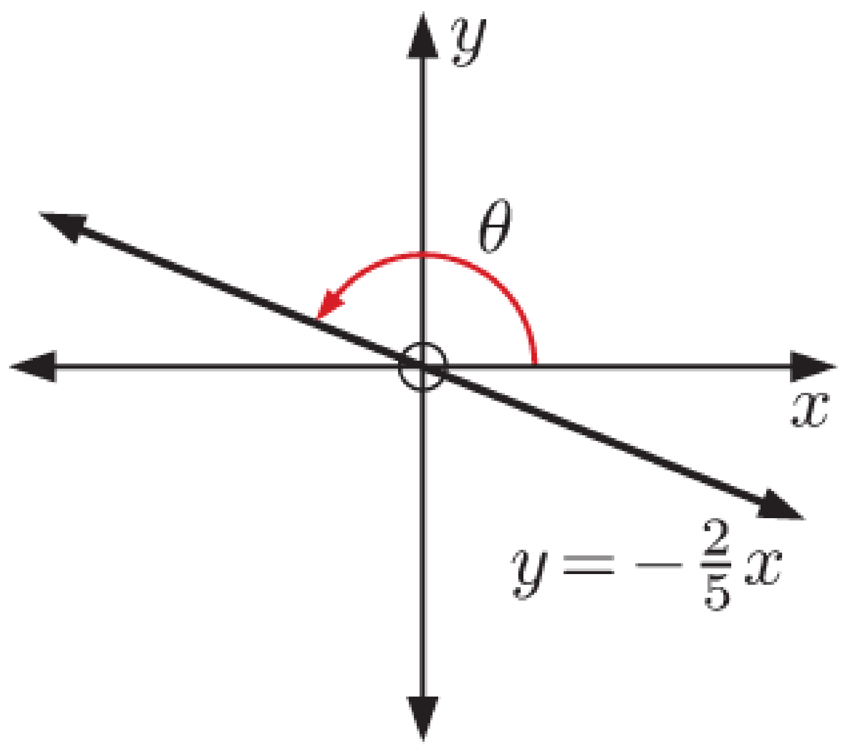
a



b

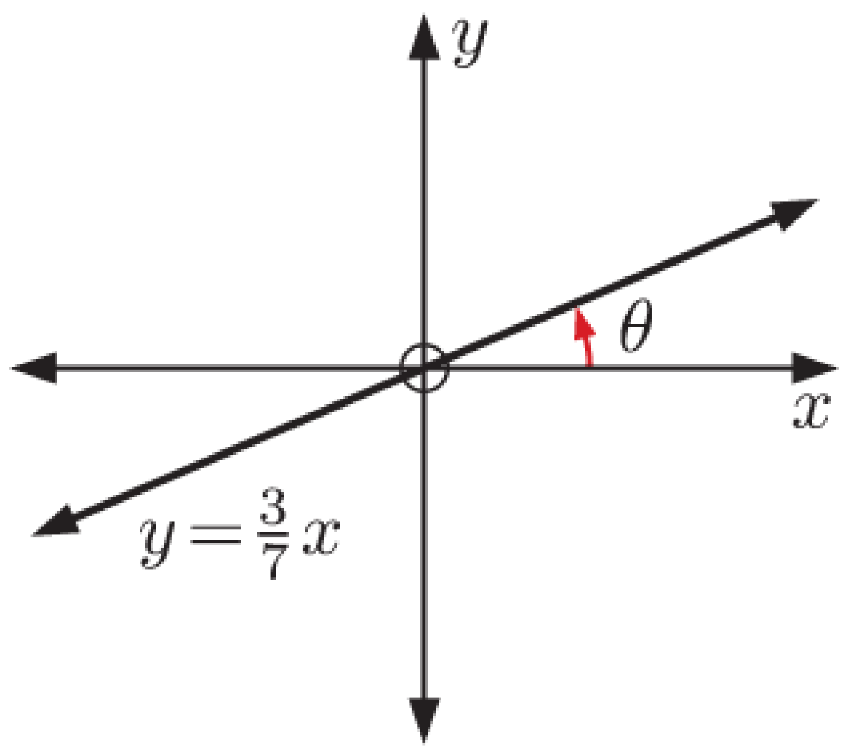


c

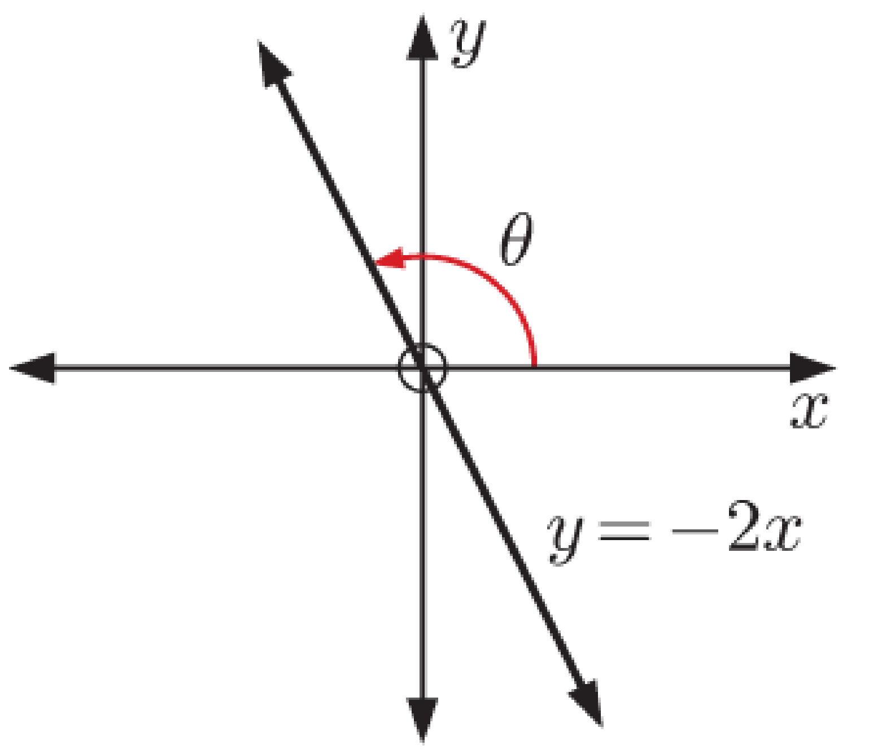


4 Find, in degrees, the measure of θ:

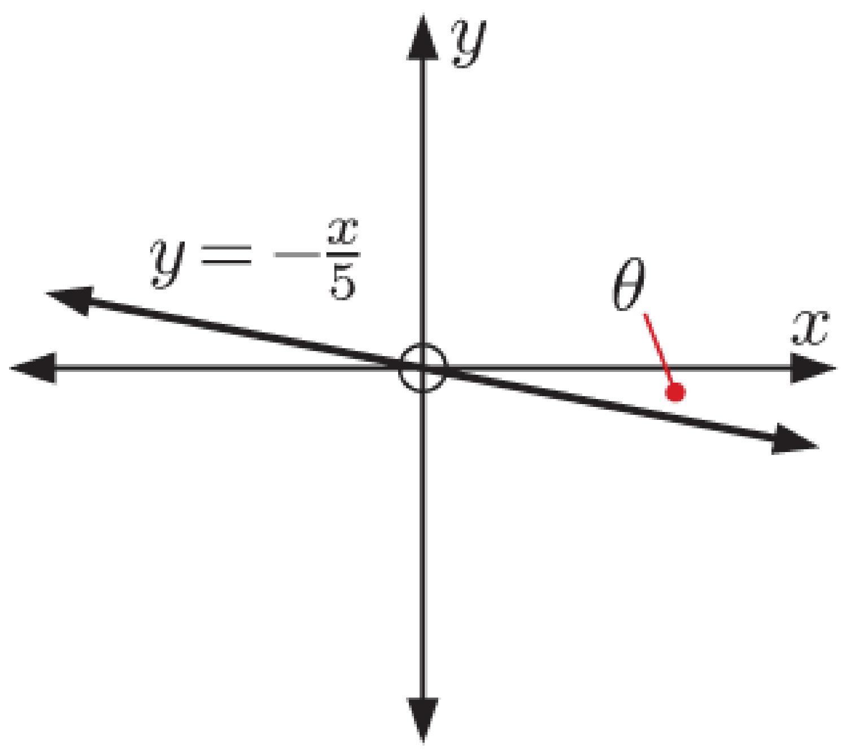
a



b



c



REVIEW SET 8A

1 Convert to radians in terms of π:

a 120° **b** 225° **c** 150° **d** 540°

2 Illustrate the quadrants where $\sin\theta$ and $\cos\theta$ have the same sign.

3 Determine the coordinates of the point on the unit circle corresponding to an angle of:

a 320° **b** 163° **c** 0.68^c **d** $\frac{11\pi}{6}$

4 Find the arc length of a sector with angle 1.5 radians and radius 8 cm.

5 Find the acute angles that have the same:

a sine as $\frac{2\pi}{3}$ **b** sine as 165° **c** cosine as 276°.

6 Find:

a $\sin 159°$ if $\sin 21° \approx 0.358$

b $\cos 92°$ if $\cos 88° \approx 0.035$

c $\cos 75°$ if $\cos 105° \approx -0.259$

d $\tan(-133°)$ if $\tan 47° \approx 1.072$.

7 Use a unit circle diagram to find:

a $\cos 360°$ and $\sin 360°$

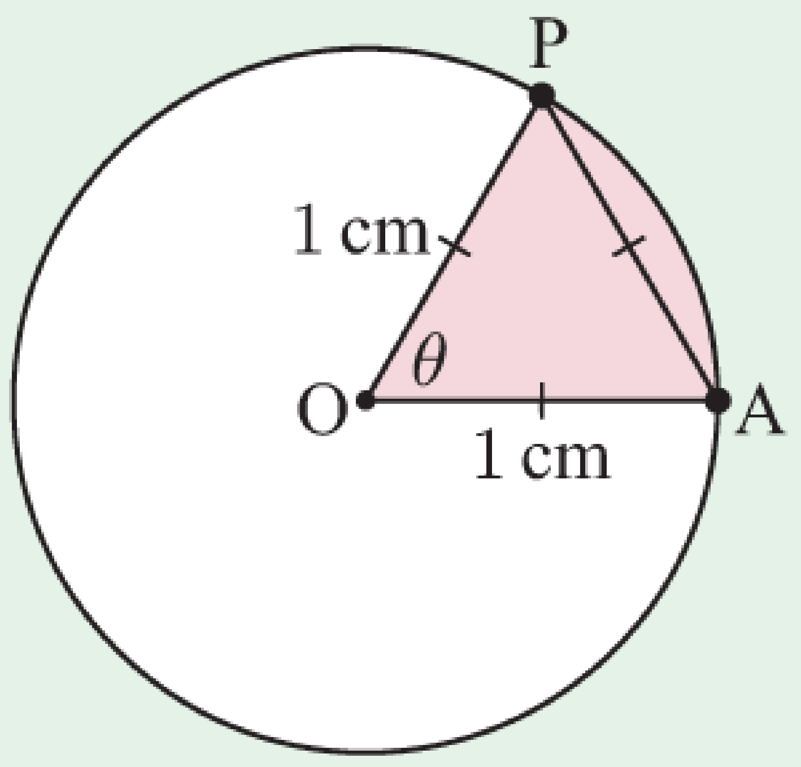
b $\cos(-\pi)$ and $\sin(-\pi)$.

8 Find exact values for $\sin\theta$, $\cos\theta$, and $\tan\theta$ for θ equal to:

a $\frac{2\pi}{3}$

b $\frac{8\pi}{3}$

9



a State the value of θ in:

- **i** degrees
- **ii** radians.

b State the arc length AP.

c State the area of the minor *sector* OAP.

10 If $\sin x = -\frac{1}{4}$ and $\pi < x < \frac{3\pi}{2}$, find $\tan x$ exactly.

11 If $\cos\theta = \frac{3}{4}$, find the possible values of $\sin\theta$.

12 Evaluate:

a $2\sin\frac{\pi}{3}\cos\frac{\pi}{3}$

b $\tan^2\left(\frac{\pi}{4}\right) - 1$

c $\cos^2\left(\frac{\pi}{6}\right) - \sin^2\left(\frac{\pi}{6}\right)$

13 Given $\tan x = -\frac{3}{2}$ and $\frac{3\pi}{2} < x < 2\pi$, find:

a $\cos x$

b $\sin x$.

14 Suppose $\cos\theta = \frac{\sqrt{11}}{\sqrt{17}}$ and θ is acute. Find the exact value of $\tan\theta$.

15 Explain how to use the unit circle to find θ when $\cos\theta = -\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$.

16 Find two angles on the unit circle with $0 \leqslant \theta \leqslant 2\pi$, such that:

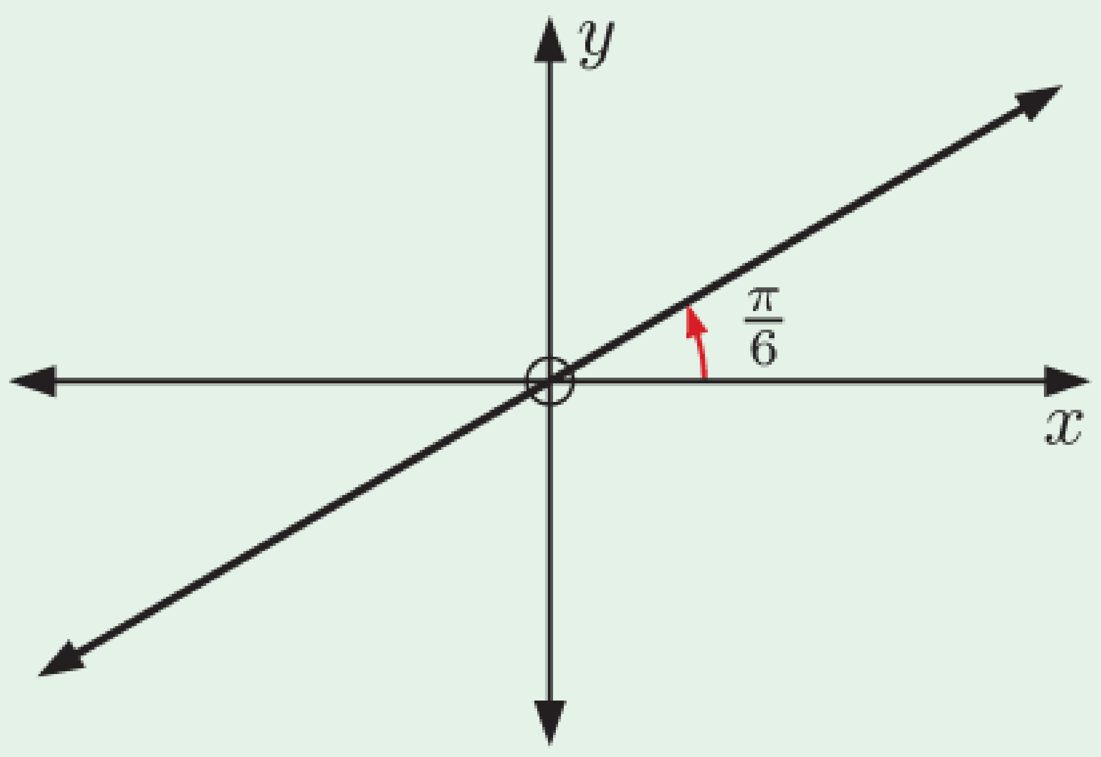
a $\cos\theta = \frac{2}{3}$

b $\sin\theta = -\frac{1}{4}$

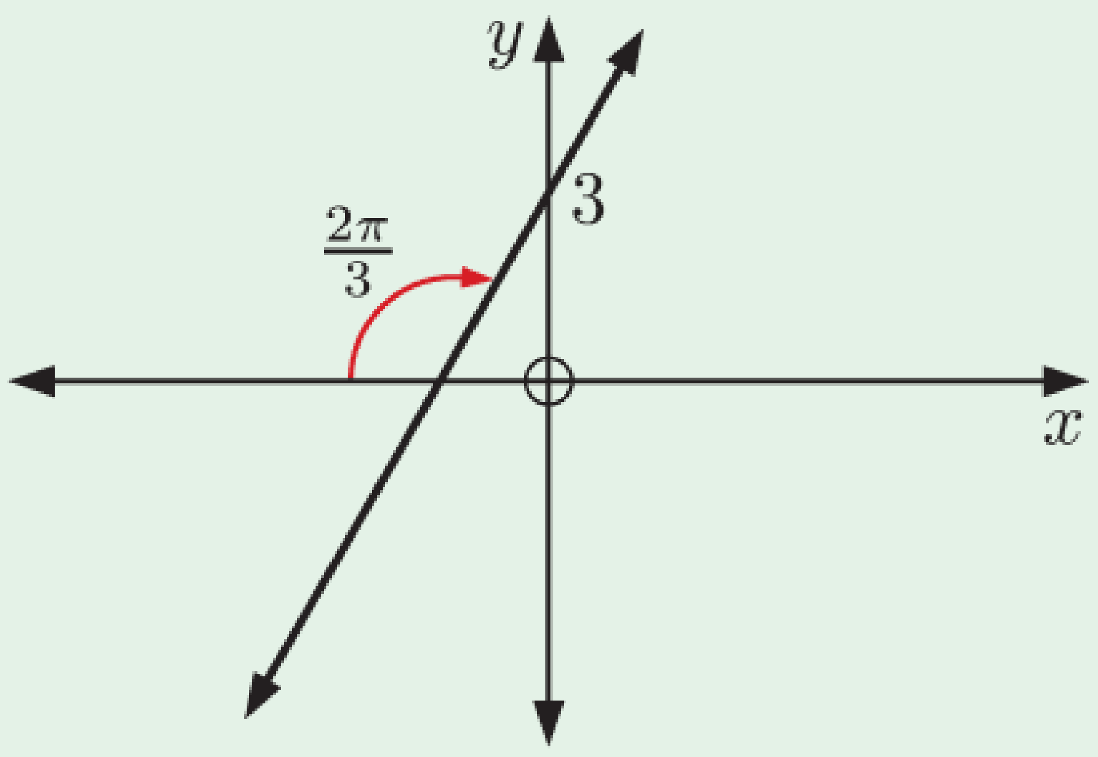
c $\tan\theta = 3$

17 Find the equation of each line:

a



b



REVIEW SET 8B

1 Convert to degrees, to 2 decimal places:

a $\frac{2\pi}{5}$

b 1.46

c 0.435^c

d -5.271

2 Determine the area of a sector with angle $\frac{5\pi}{12}$ and radius 13 cm.

3 Find the angle [OA] makes with the positive x-axis if the x-coordinate of the point A on the unit circle is -0.222.

4 Find the radius and area of a sector of perimeter 36 cm with an angle of $\frac{2\pi}{3}$.

5 A sector has perimeter 21 cm and area 27 cm^2. Find the radius of the sector.

6 Use a unit circle diagram to find:

a $\cos\frac{3\pi}{2}$ and $\sin\frac{3\pi}{2}$ **b** $\cos\left(-\frac{\pi}{2}\right)$ and $\sin\left(-\frac{\pi}{2}\right)$

7 Suppose $m = \sin p$, where p is acute. Write an expression in terms of m for:

a $\sin(\pi - p)$ **b** $\sin(p + 2\pi)$ **c** $\cos p$ **d** $\tan p$

8 Find all angles between 0° and 360° which have:

a a cosine of $-\frac{\sqrt{3}}{2}$ **b** a sine of $\frac{1}{\sqrt{2}}$ **c** a tangent of $-\sqrt{3}$

9 Find θ for $0 \leqslant \theta \leqslant 2\pi$ if:

a $\cos\theta = -1$ **b** $\sin^2\theta = \frac{3}{4}$

10 Find the obtuse angles which have the same:

a sine as 47° **b** sine as $\frac{\pi}{15}$ **c** cosine as 186°

11 Find the perimeter and area of a sector with radius 11 cm and angle 63°.

12 Show that $\cos\frac{3\pi}{4} - \sin\frac{3\pi}{4} = -\sqrt{2}$.

13 If $\cos\theta = -\frac{3}{4}$, $\frac{\pi}{2} < \theta < \pi$ find the exact value of:

a $\sin\theta$ **b** $\tan\theta$ **c** $\cos(\pi - \theta)$

14 Without using a calculator, evaluate:

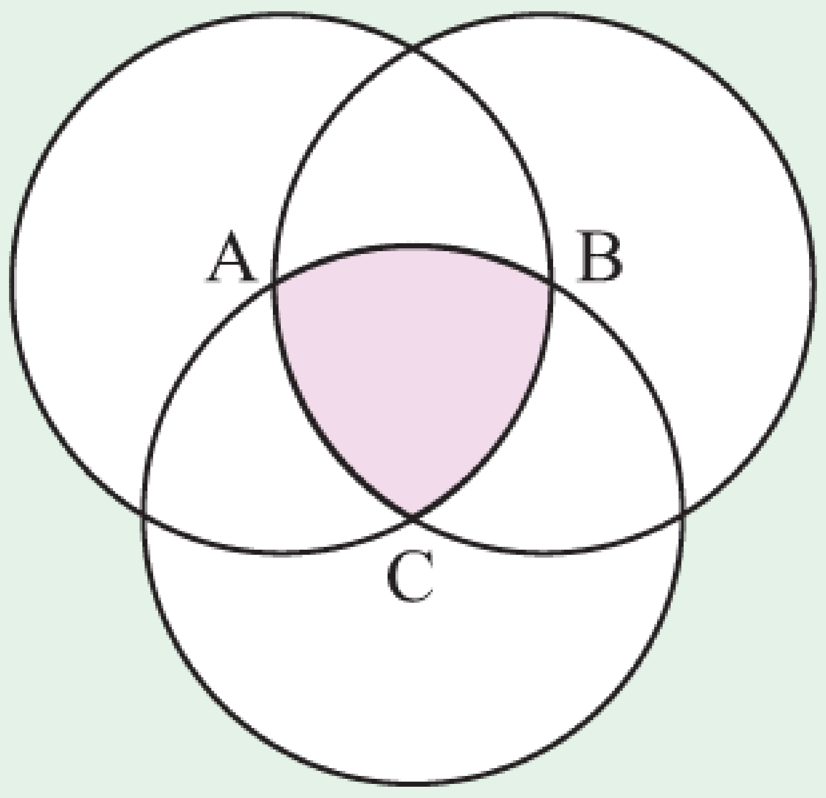
a $\tan^2 60° - \sin^2 45°$ **b** $\cos^2\left(\frac{\pi}{4}\right) + \sin\frac{\pi}{2}$

c $\cos\frac{5\pi}{3} - \tan\frac{5\pi}{4}$ **d** $\tan^2\left(\frac{2\pi}{3}\right)$

15 Use a unit circle diagram to show that $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$ for $\frac{\pi}{2} < \theta < \pi$.

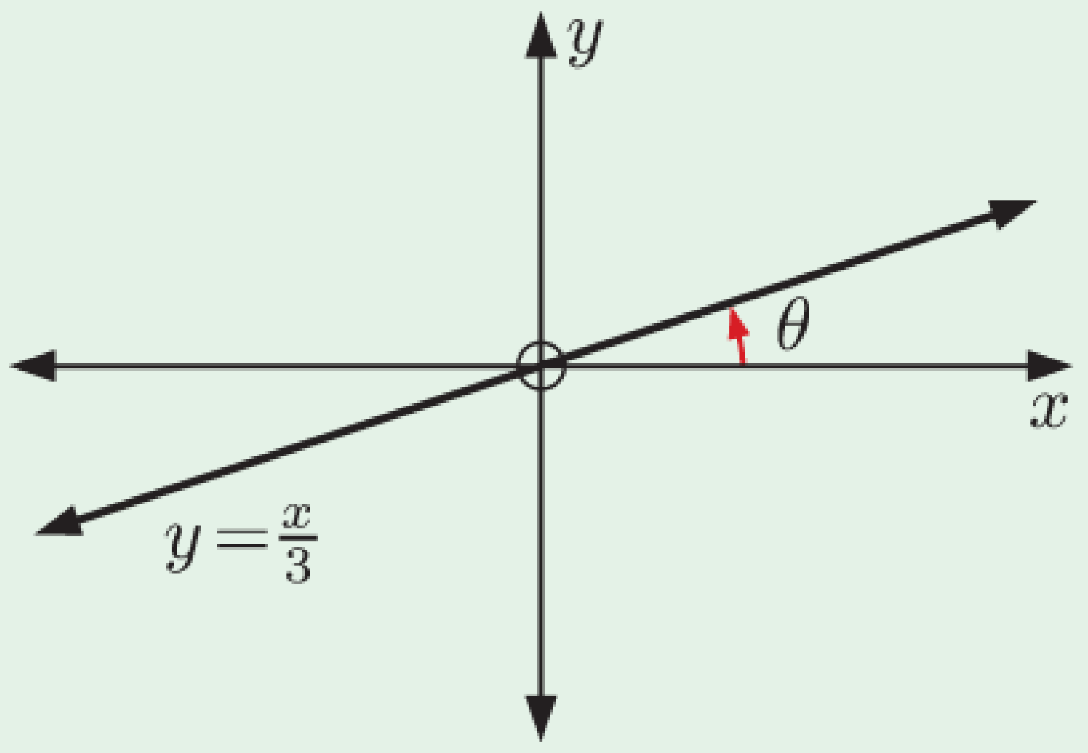
16 Three circles with radius r are drawn as shown, each with its centre on the circumference of the other two circles. A, B, and C are the centres of the three circles.

Prove that an expression for the area of the shaded region is $A = \frac{r^2}{2}(\pi - \sqrt{3})$.



17 Find, in radians, the measure of θ:

a



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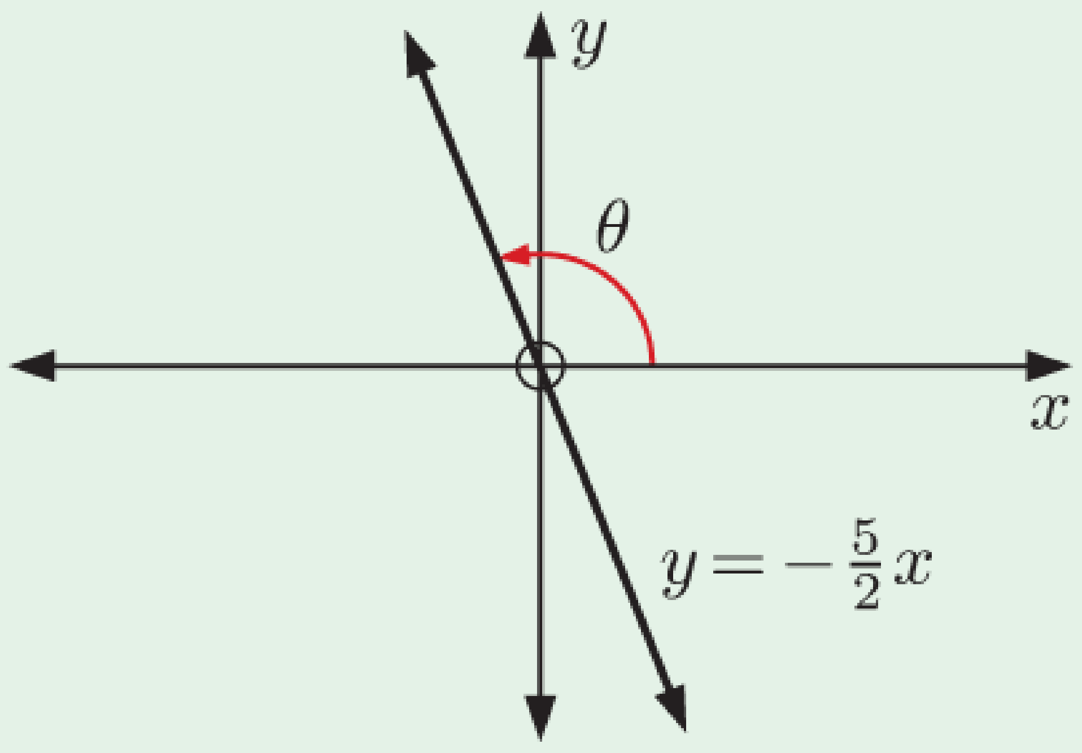


Chapter 9

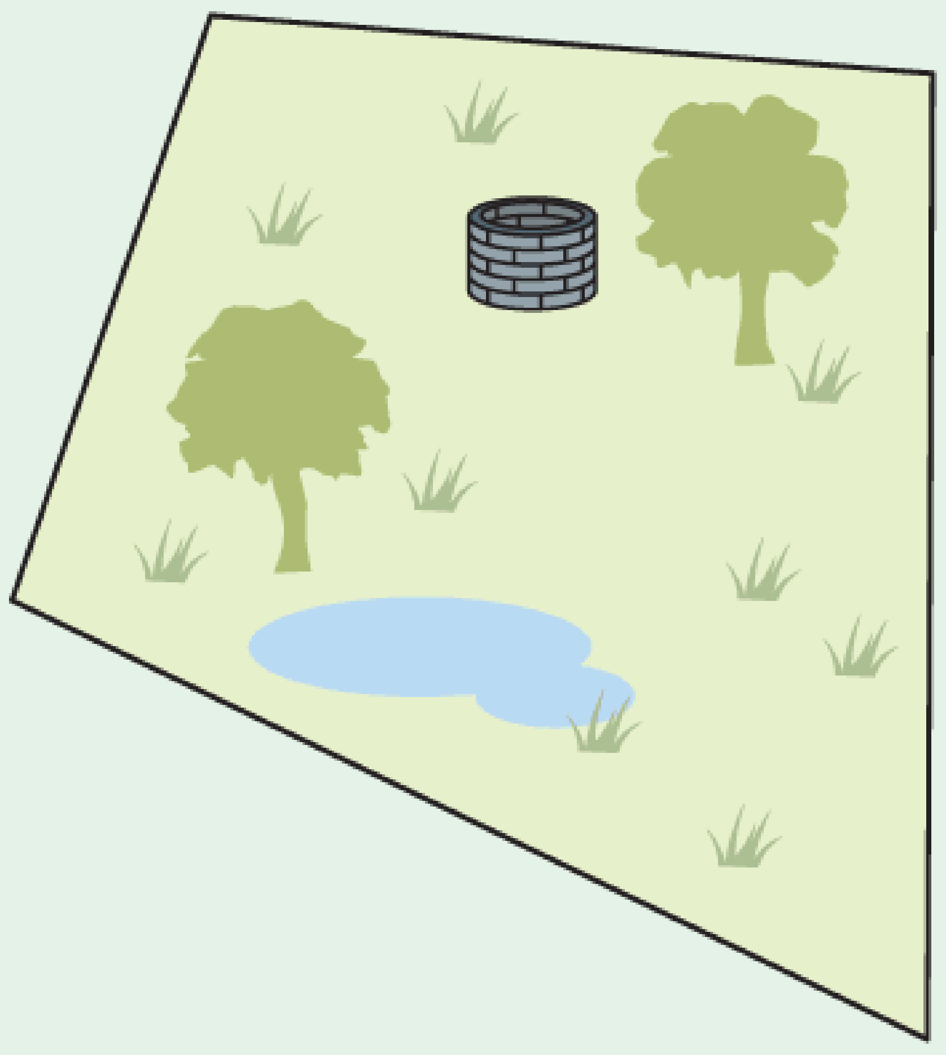
Non-right angled triangle trigonometry

Contents:

OPENING PROBLEM

Robert works for the City Council. He has been asked to find the area of its central park, so he has arrived with a measuring wheel to measure some lengths.

Robert has just realised that while the park is a quadrilateral, it is not a rectangle.



Things to think about:

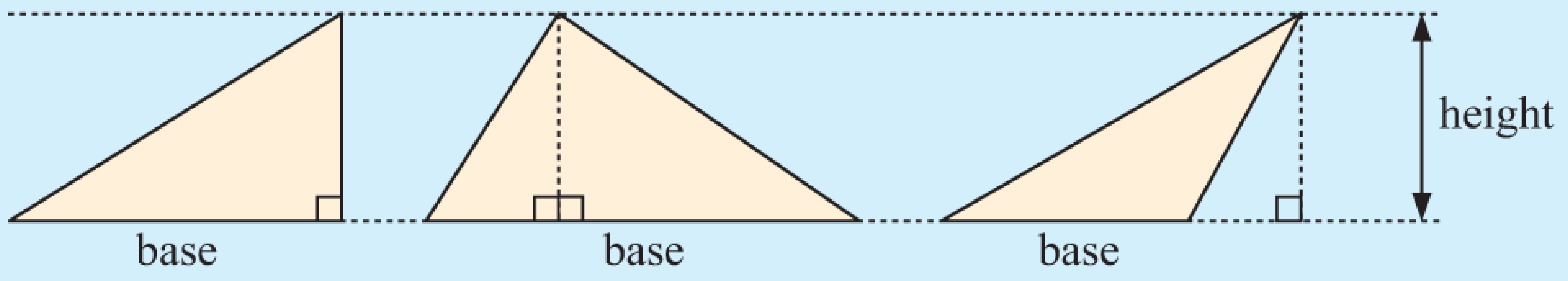
- **a** Does Robert need to measure angles in order to find the area of the park? If not, what lengths does he need to measure? Are the four side lengths of the quadrilateral sufficient?
- **b** Will Robert be able to find the angles at the corners of the park using length measurements alone?

In this Chapter we will see how trigonometry can be used to solve problems involving non-right angled triangles.

A THE AREA OF A TRIANGLE

We have seen in previous years that the area of any triangle can be calculated using:

$$\textbf{Area} = \tfrac{1}{2}\ \textbf{base} \times \textbf{height}$$



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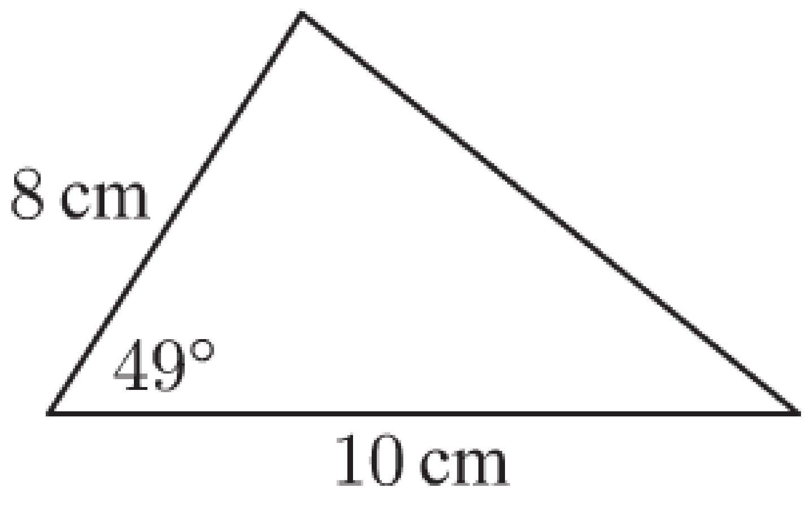
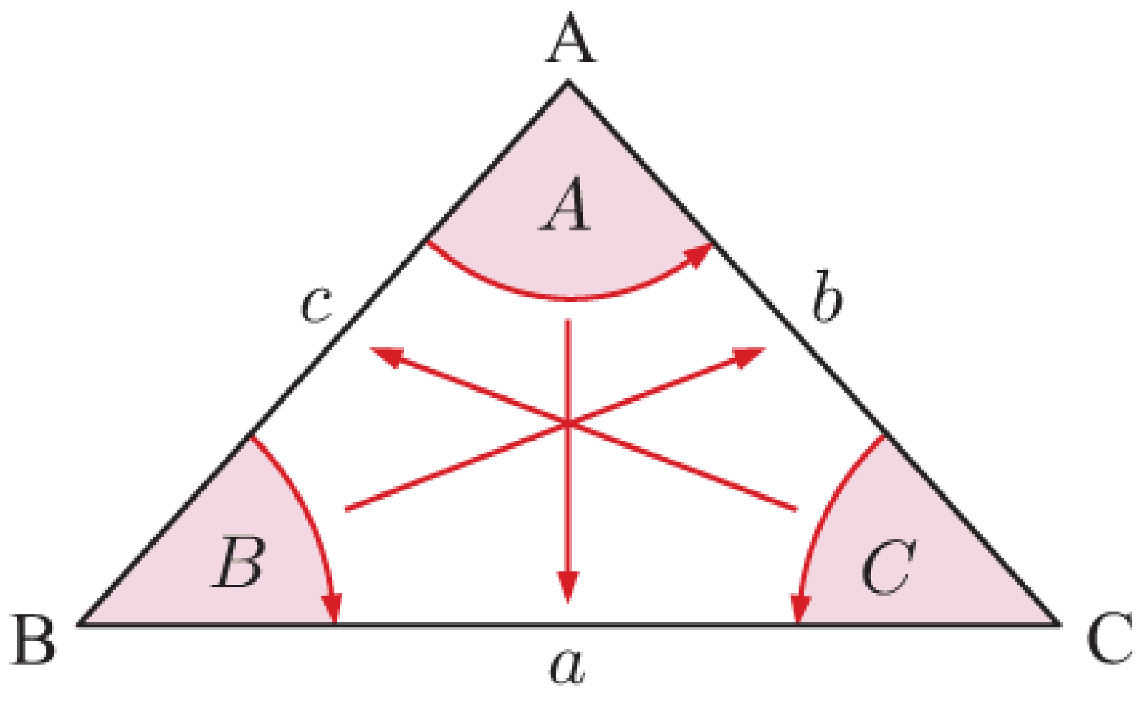
However, if we do not know the perpendicular height of a triangle, we can use trigonometry to calculate the area.

To do this we need to know two sides of the triangle and the **included angle** between them. For example, in the triangle alongside the angle 49° is *included* between the sides of length 8 cm and 10 cm.



CONVENTION FOR LABELLING TRIANGLES

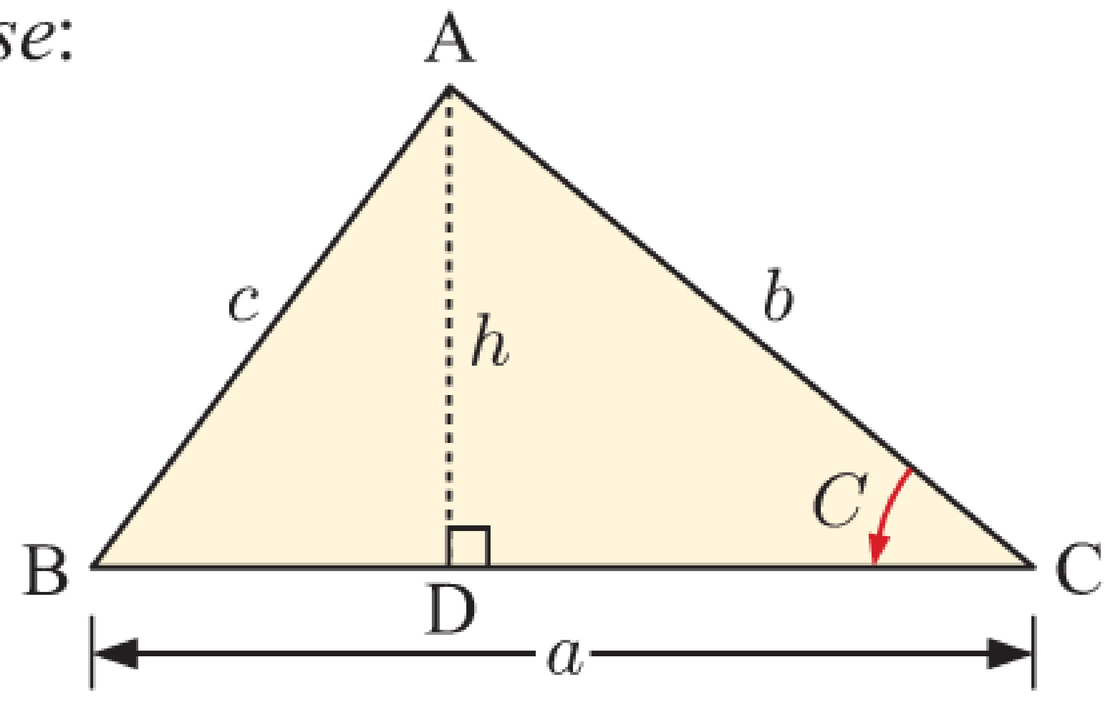
For triangle ABC, the angles at vertices A, B, and C are labelled A, B, and C respectively. The sides opposite these angles are labelled a, b, and c respectively.



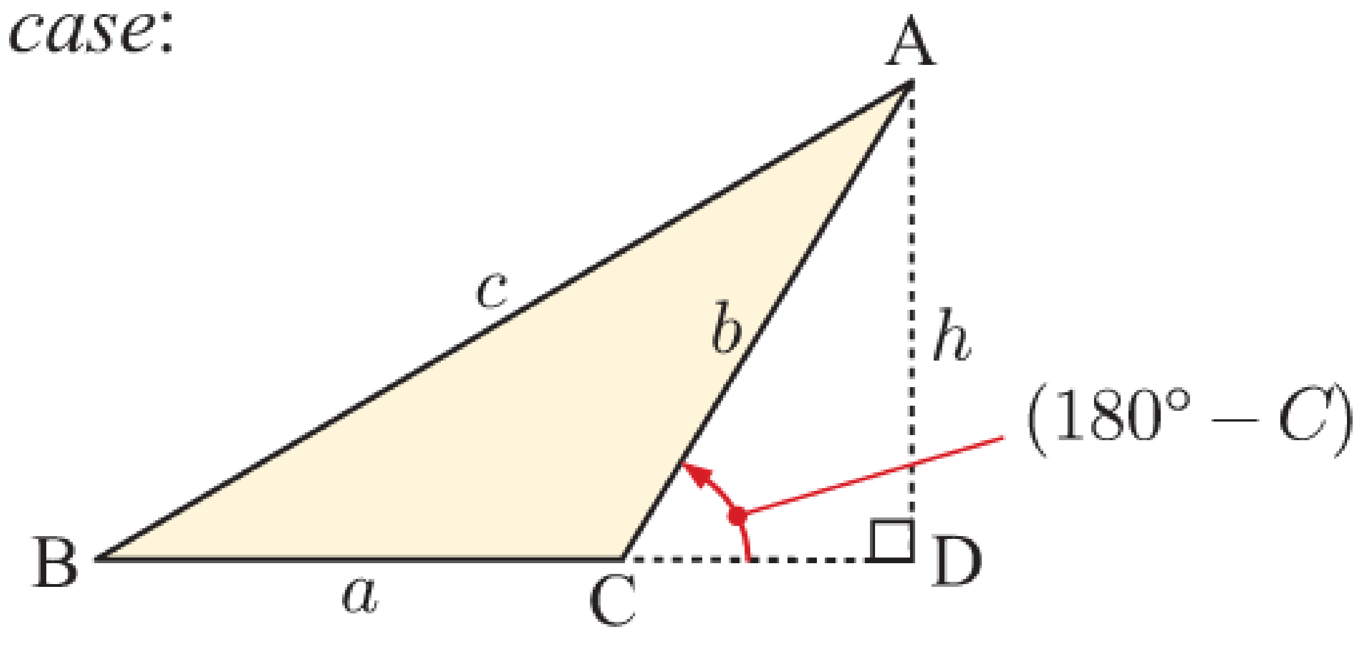
CALCULATING THE AREA OF A TRIANGLE

Any triangle that is not right angled must be either acute or obtuse. We will consider both cases:

Acute case:



Obtuse case:



In both triangles the altitude h is constructed from A to D on [BC] (extended if necessary).

Acute case: $\sin C = \dfrac{h}{b}$

$\therefore \quad h = b\sin C$

Obtuse case: $\sin(180° - C) = \dfrac{h}{b}$

$\therefore \quad h = b\sin(180° - C)$

But $\sin(180° - C) = \sin C$

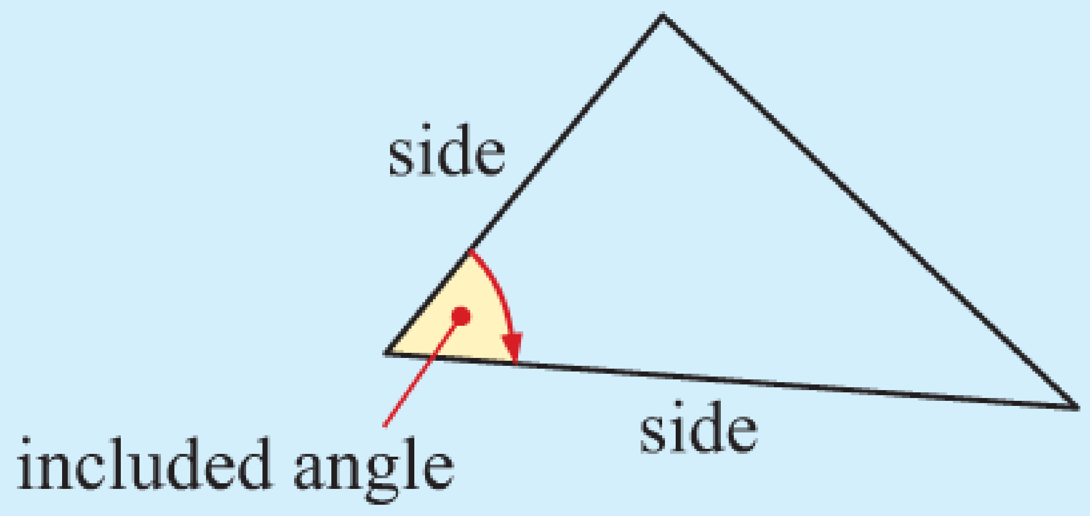
$\therefore \quad h = b\sin C$

So, since area $= \frac{1}{2}ah$, we now have **Area** $= \frac{1}{2}\boldsymbol{ab}\,\mathbf{sin}\,\boldsymbol{C}$.

Using different altitudes we can show that the area is also $\frac{1}{2}bc\sin A$ or $\frac{1}{2}ac\sin B$.

Given the lengths of two sides of a triangle, and the size of the included angle between them, the area of the triangle is

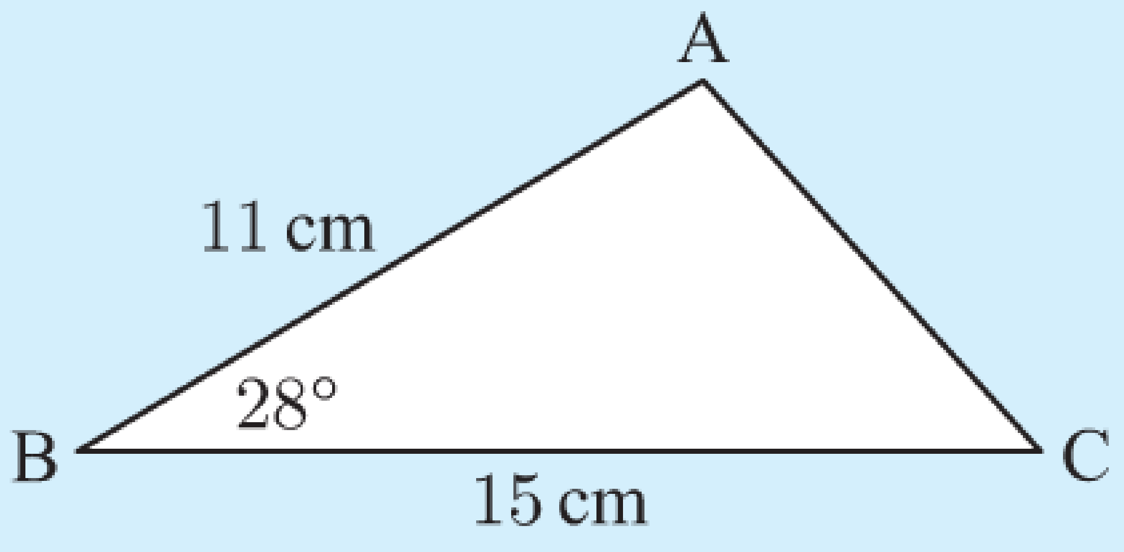
half of the product of two sides and the sine of the included angle.



Example 1

Self Tutor

Find the area of triangle ABC.



Area $= \frac{1}{2}ac\sin B$

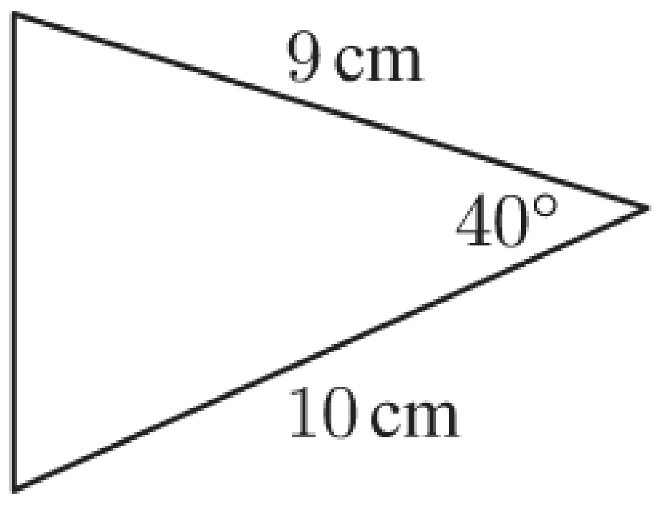
$= \frac{1}{2} \times 15 \times 11 \times \sin 28°$

$\approx 38.7\ \text{cm}^2$

EXERCISE 9A

1 Find the area of:

a



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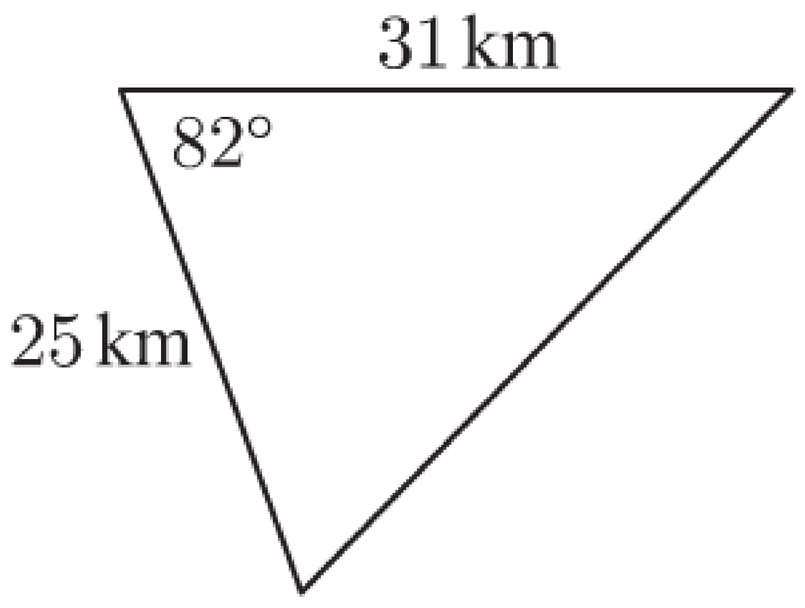

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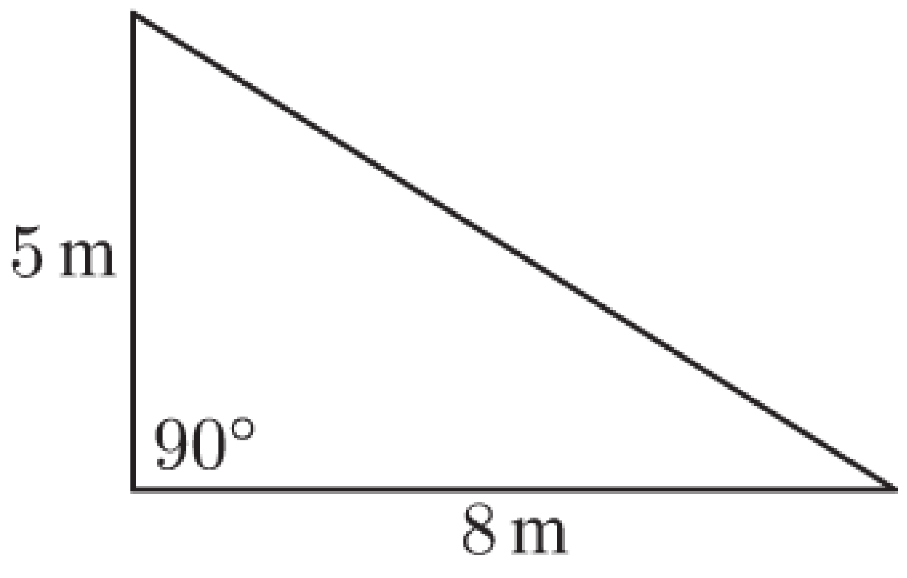
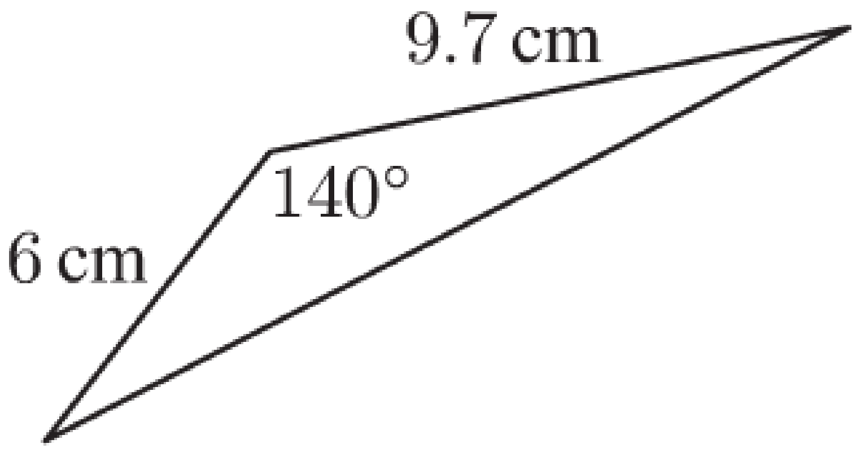


2 Find the area of:

a



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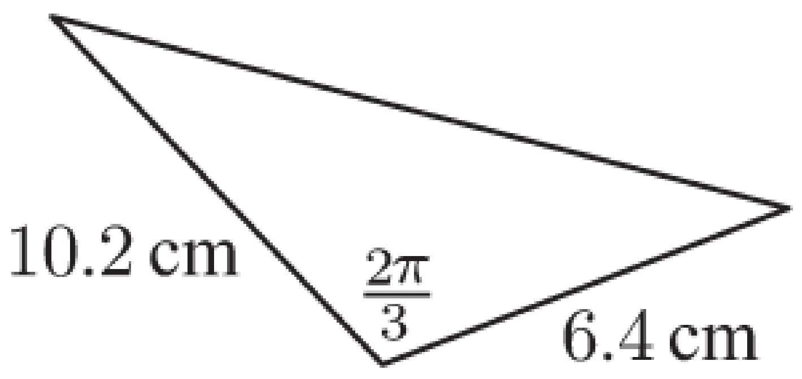

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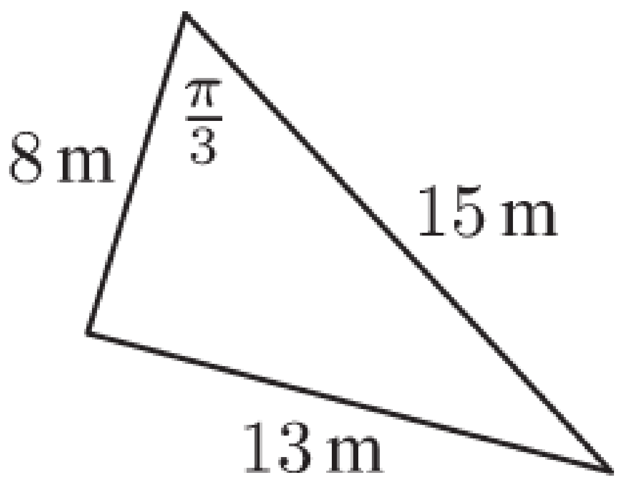
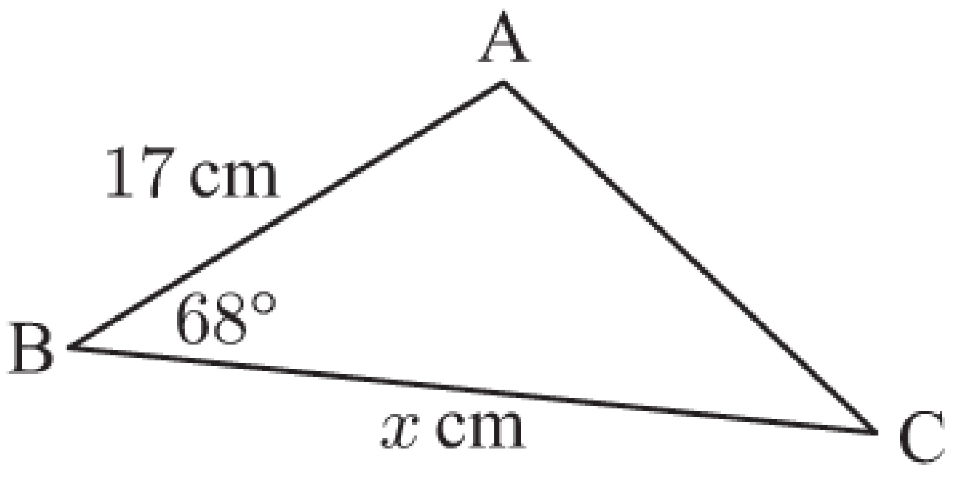


3 Triangle ABC has area 150 cm^2.
Find the value of x.



4 Calculate the area of:

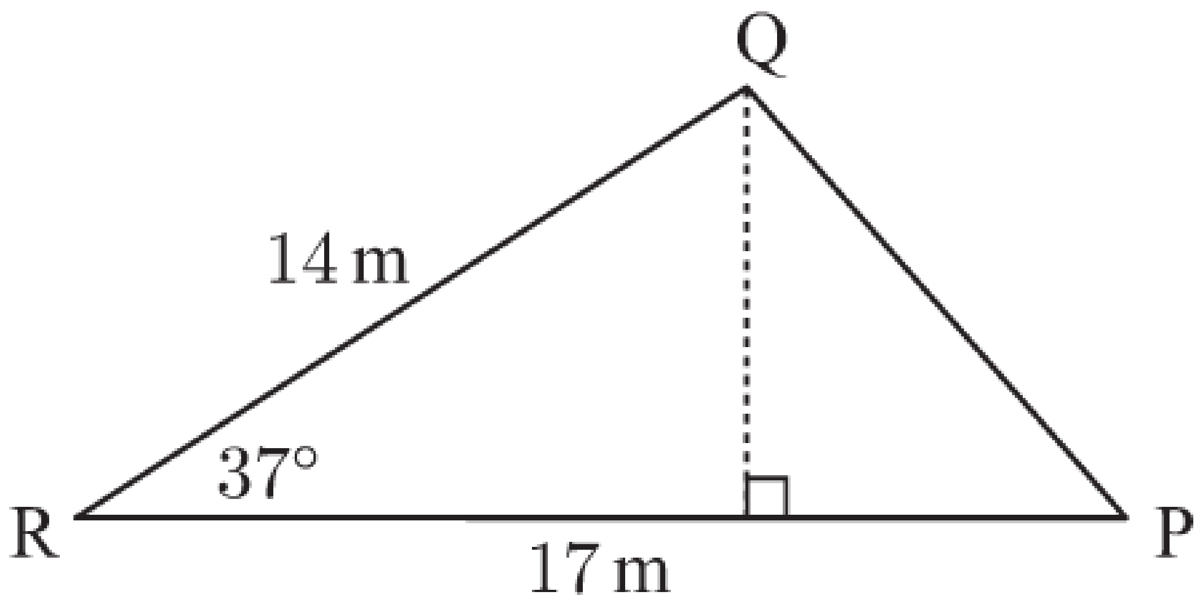
a an isosceles triangle with equal sides of length 21 cm and an included angle of 49°

b an equilateral triangle with sides of length 57 cm.

5 A parallelogram has two adjacent sides with lengths 4 cm and 6 cm respectively. If the included angle measures 52°, find the area of the parallelogram.

6 A rhombus has sides of length 12 cm and an angle of 72°. Find its area.

7



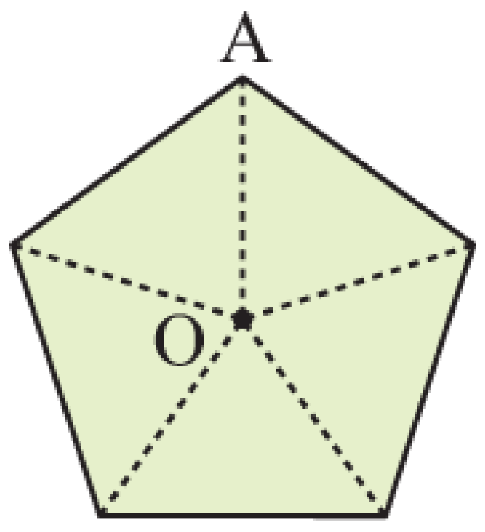
a Find the area of triangle PQR to 3 decimal places.

b Hence find the length of the altitude from Q to [RP].

8 Find the area of a regular hexagon with sides of length 12 cm.

9 A rhombus has area 50 cm^2 and an internal angle of size 63°. Find the length of its sides.

10 A regular pentagonal garden plot has centre of symmetry O and an area of 338 m^2. Find the distance OA.



Example 2 **Self Tutor**

A triangle has two sides with lengths 10 cm and 11 cm, and an area of 50 cm^2. Determine the possible measures of the included angle. Give your answers accurate to 1 decimal place.

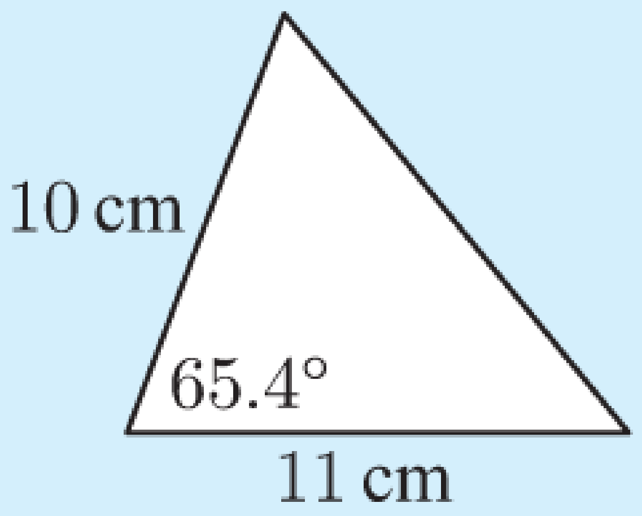
If the included angle is θ, then $\frac{1}{2} \times 10 \times 11 \times \sin\theta = 50$

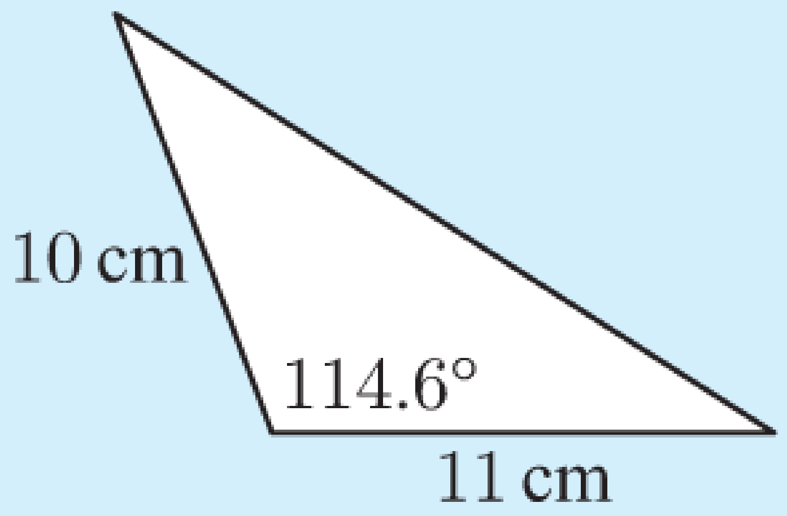
$$\therefore \quad \sin\theta = \frac{50}{55}$$

Now $\sin^{-1}\left(\frac{50}{55}\right) \approx 65.4°$

$\therefore \quad \theta \approx 65.4°$ or $180° - 65.4°$

$\therefore \quad \theta \approx 65.4°$ or $114.6°$





The two different possible angles are 65.4° and 114.6°.

11 Find the possible values of the included angle of a triangle with:

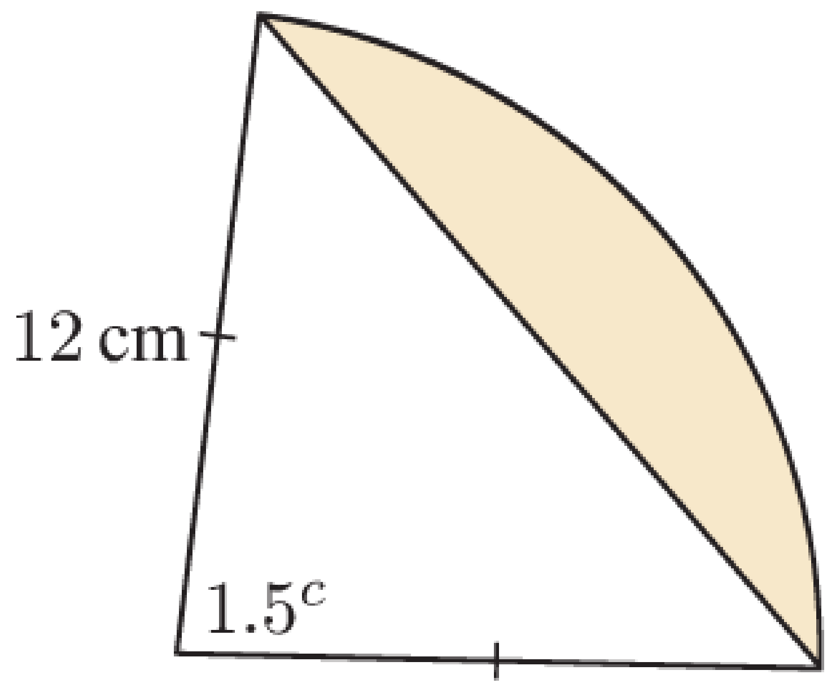
- **a** sides of length 5 cm and 8 cm, and area 15 cm^2
- **b** sides of length 45 km and 53 km, and area 800 km^2.

12 The Australian 50 cent coin has the shape of a regular dodecagon, which is a polygon with 12 sides.

Eight of these 50 cent coins will fit exactly on an Australian \$5 note as shown. What fraction of the \$5 note is *not* covered?

13 Find the shaded area:

a



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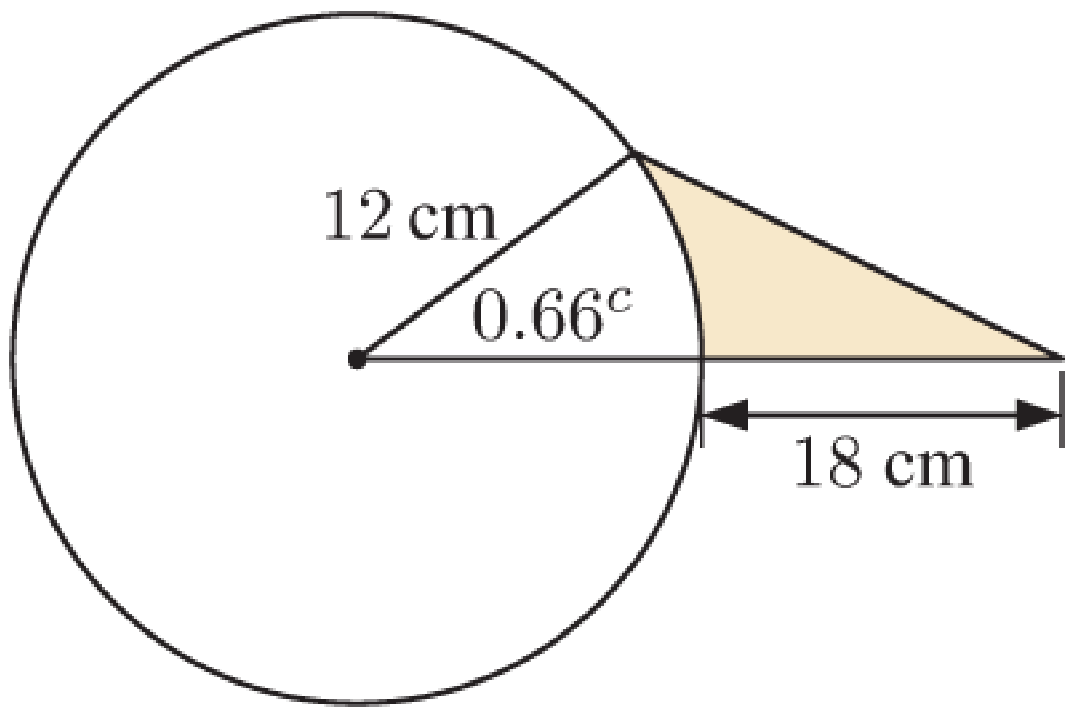

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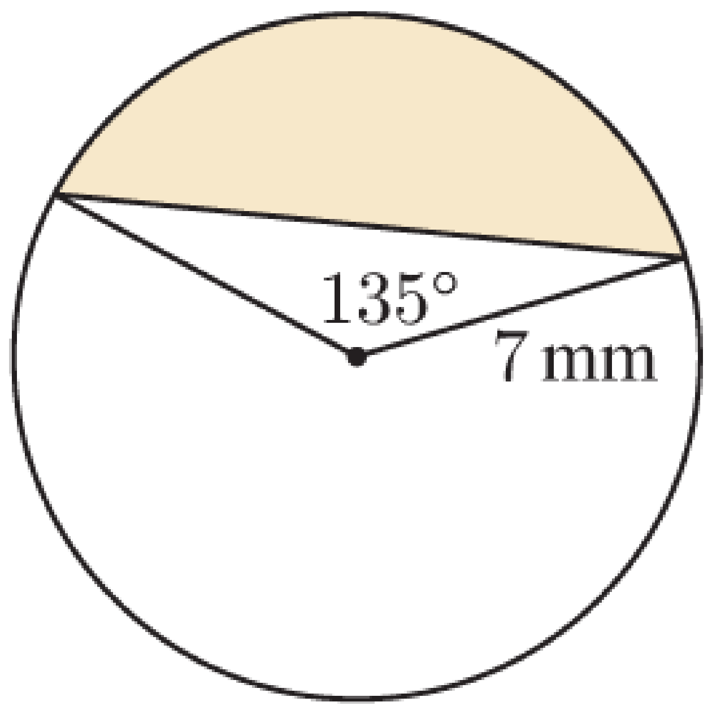

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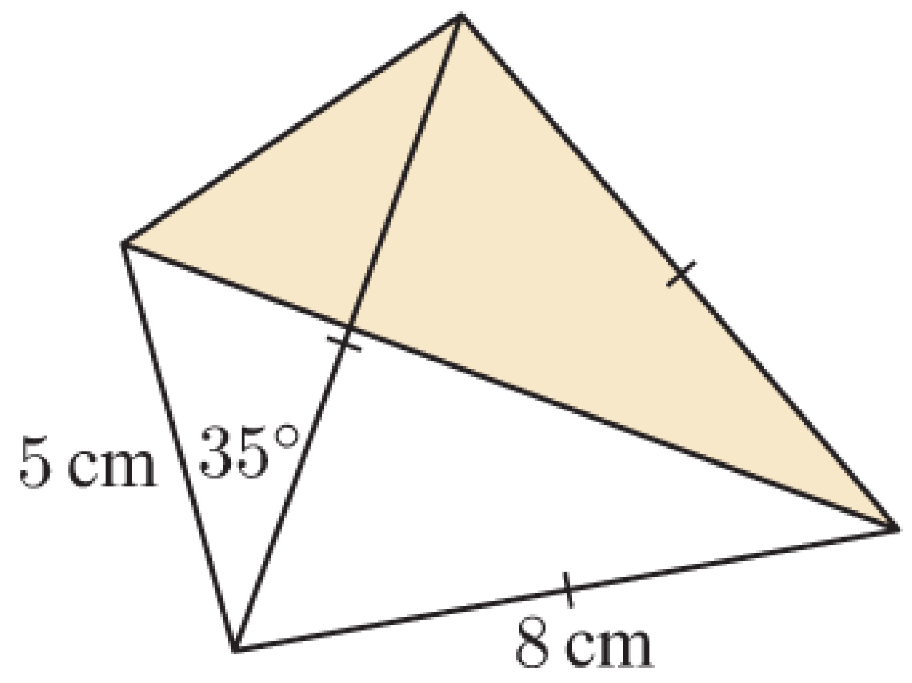
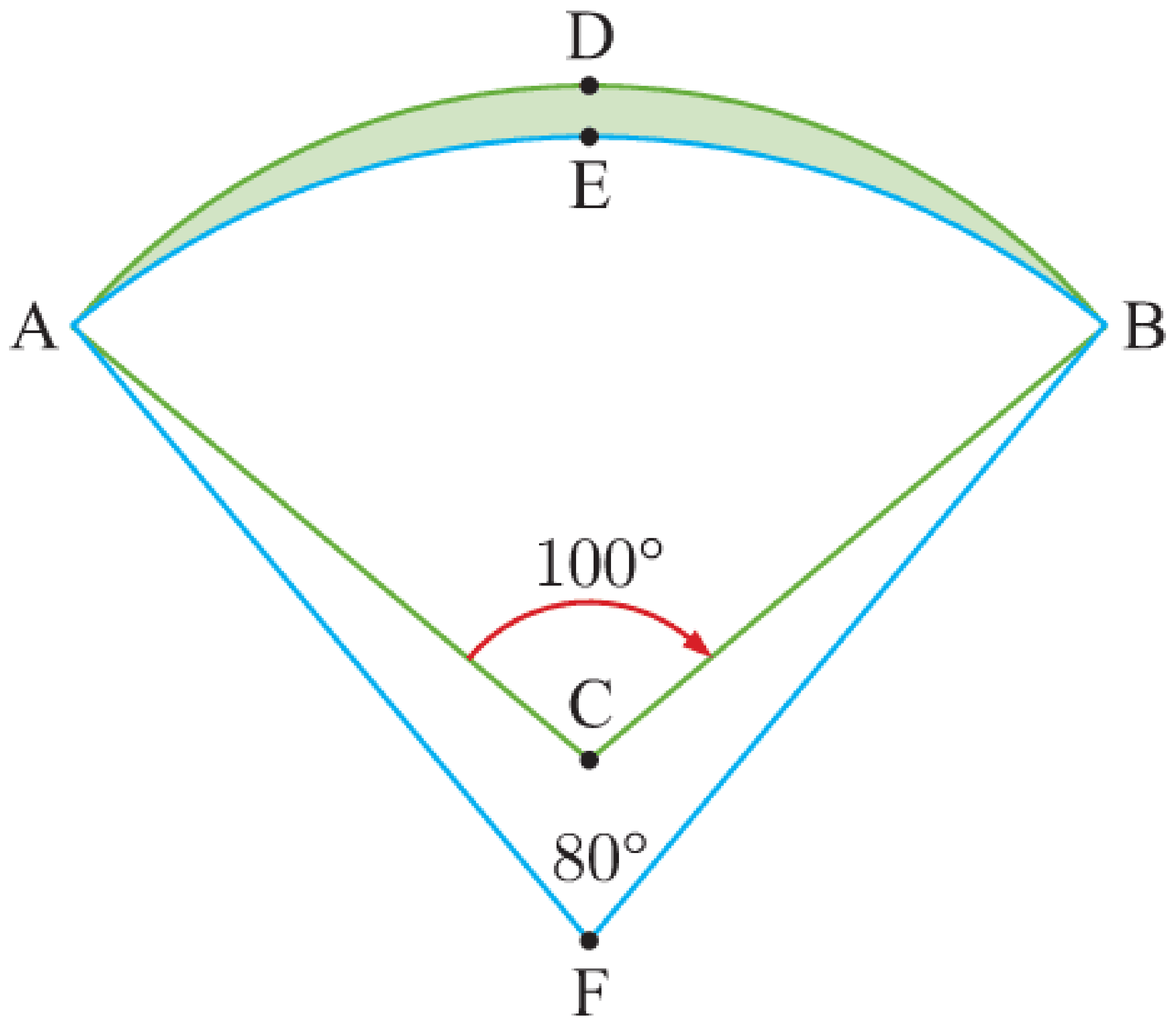


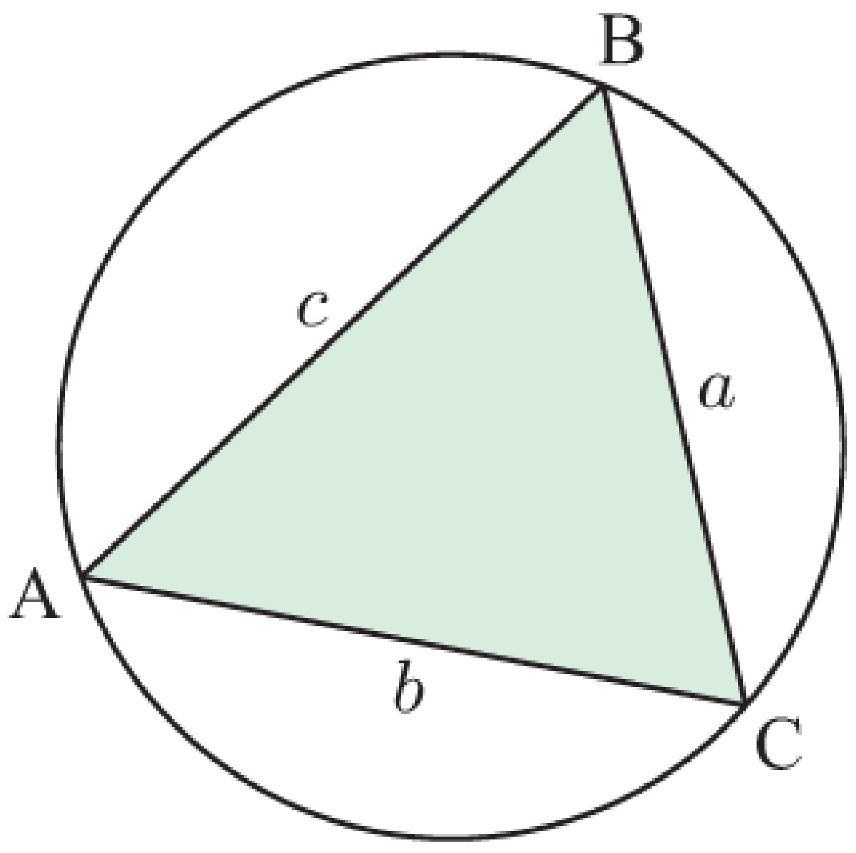
14 ADB is an arc of the circle with centre C and radius 7.3 cm. AEB is an arc of the circle with centre F and radius 8.7 cm.

Find the shaded area.



15 The acute angled triangle ABC has vertices on a circle with radius r.

Show that the area of the triangle is $\dfrac{abc}{4r}$.

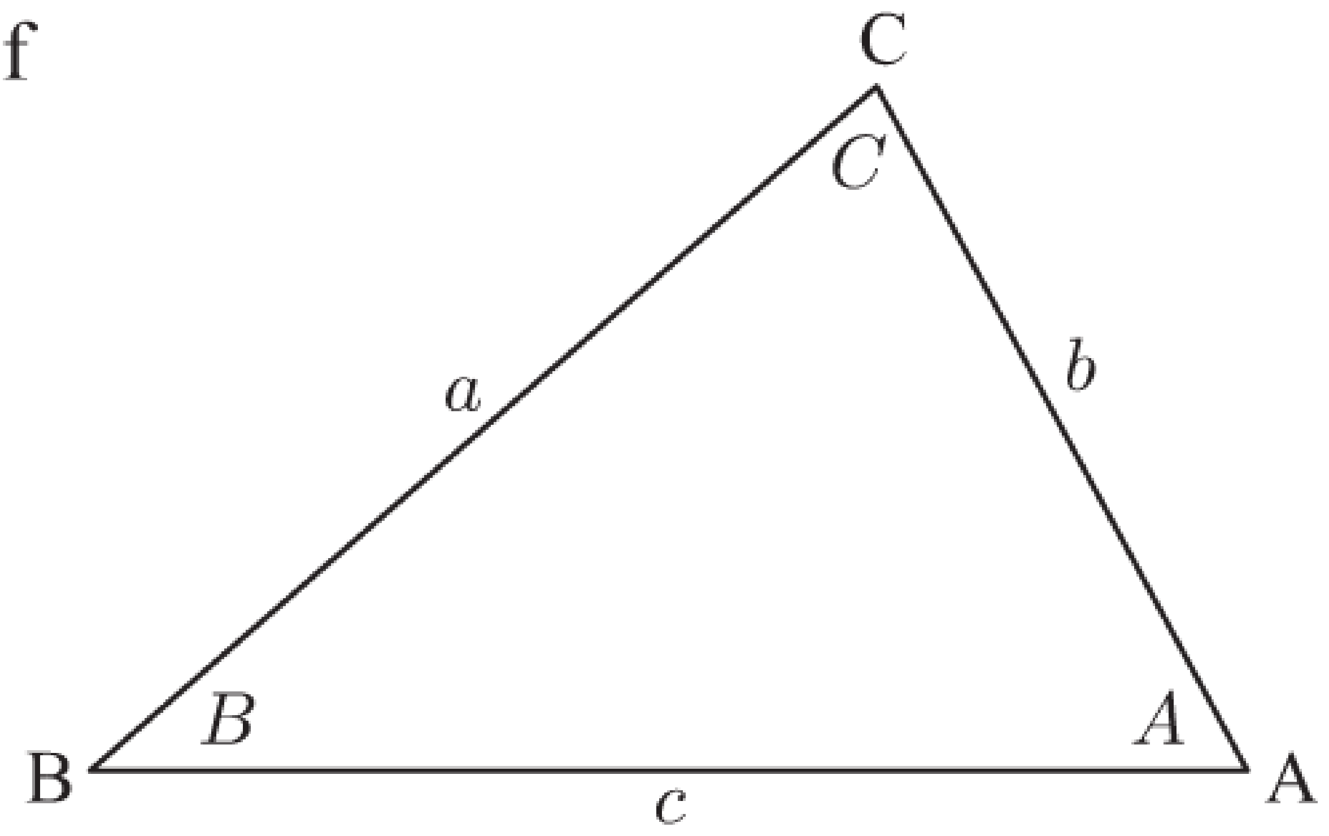


B THE COSINE RULE

The **cosine rule** relates the three sides of a triangle and one of the angles.

In any $\triangle$ABC:

$$\begin{aligned} \boldsymbol{a^2} &\boldsymbol{= b^2 + c^2 - 2bc\cos A} \\ \textit{or} \quad \boldsymbol{b^2} &\boldsymbol{= a^2 + c^2 - 2ac\cos B} \\ \textit{or} \quad \boldsymbol{c^2} &\boldsymbol{= a^2 + b^2 - 2ab\cos C} \end{aligned}$$



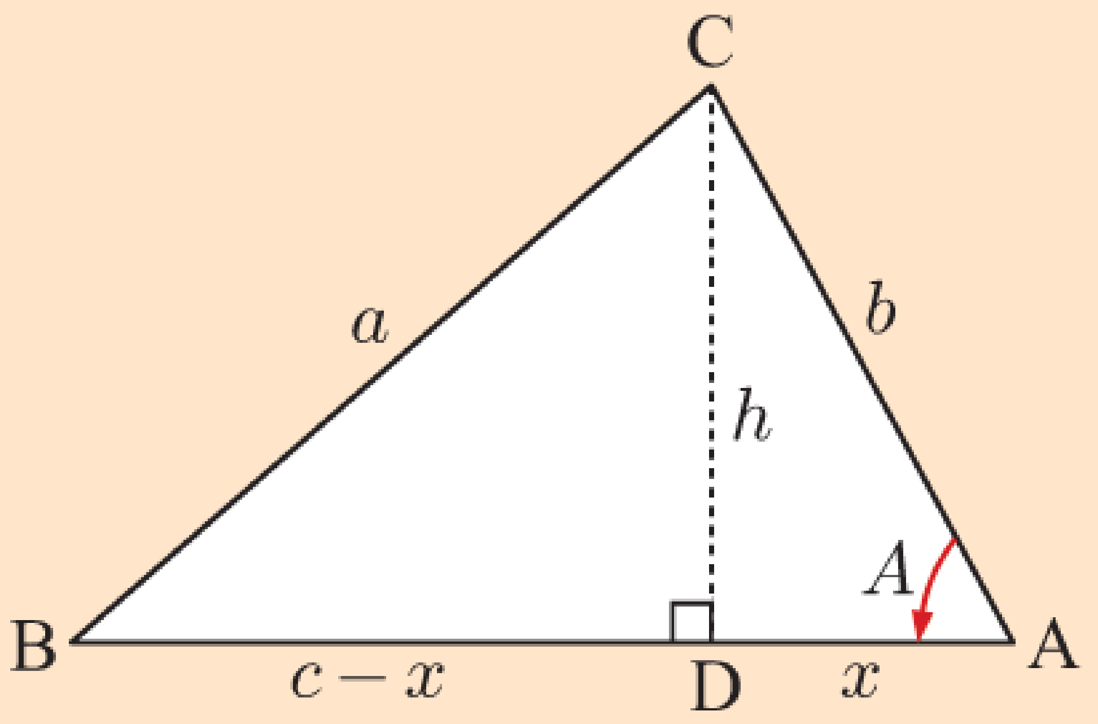
Proof: First, consider an **acute angled triangle** ABC.

Draw the altitude from C to [AB].

Let AD $= x$ and let CD $= h$.

Applying Pythagoras in $\triangle$BCD,

$$a^2 = h^2 + (c - x)^2$$
$$\therefore \quad a^2 = h^2 + c^2 - 2cx + x^2$$



Applying Pythagoras in $\triangle$ADC gives $h^2 + x^2 = b^2$

$$\therefore \quad h^2 = b^2 - x^2$$
$$\therefore \quad a^2 = b^2 + c^2 - 2cx$$

In $\triangle$ADC, $\cos A = \dfrac{x}{b}$

$$\therefore \quad x = b\cos A$$
$$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos A$$

Now consider an **obtuse angled triangle** ABC with the obtuse angle at A.

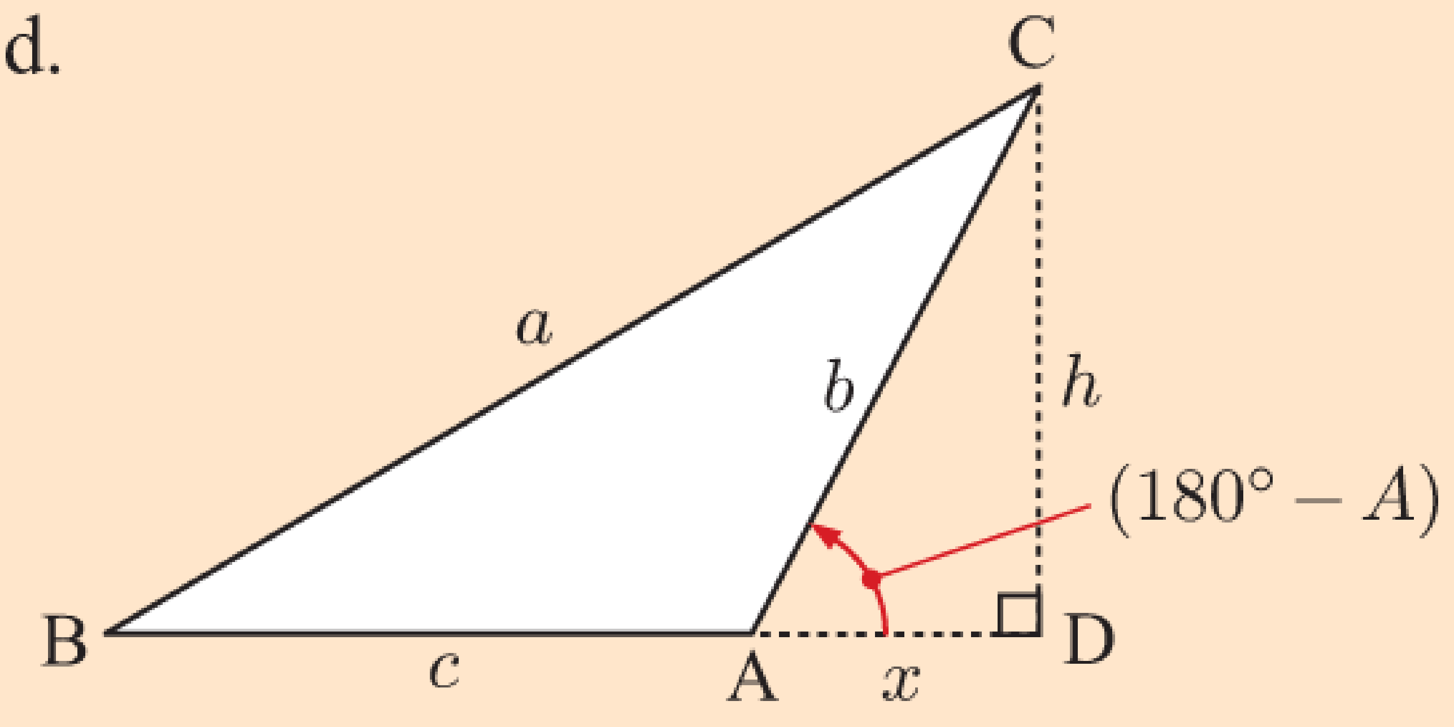
It still has two acute angles at B and C for which the proof above is valid.

Draw the altitude from C to [AB] extended.

Let AD $= x$ and let CD $= h$.

Applying Pythagoras in $\triangle$BCD,

$$a^2 = h^2 + (c + x)^2$$
$$\therefore \quad a^2 = h^2 + c^2 + 2cx + x^2$$



Applying Pythagoras in $\triangle$ADC gives

$$h^2 = b^2 - x^2$$
$$\therefore \quad a^2 = b^2 + c^2 + 2cx$$

In $\triangle$ADC, $\cos(180^\circ - A) = \dfrac{x}{b}$

$$\therefore \quad -\cos A = \frac{x}{b}$$
$$\therefore \quad x = -b\cos A$$
$$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos A$$

The other variations of the cosine rule are developed by rearranging the vertices of $\triangle$ABC.

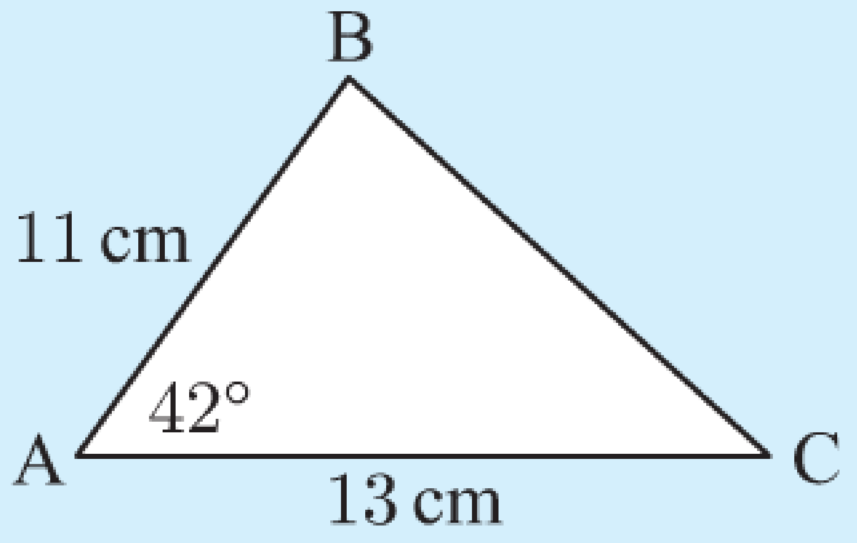
Note that if $A = 90^\circ$ then $\cos A = 0$, and $a^2 = b^2 + c^2 - 2bc\cos A$ reduces to $a^2 = b^2 + c^2$, which is the Pythagorean Rule.

There are two situations in which the cosine rule can be used.

- If we are given **two sides** and an **included angle**, the cosine rule can be used to find the length of the third side.

Example 3 **Self Tutor**

Find, correct to 2 decimal places, the length of [BC].



By the cosine rule:

$$BC^2 = 11^2 + 13^2 - 2 \times 11 \times 13 \times \cos 42^\circ$$

$$\therefore\ BC = \sqrt{(11^2 + 13^2 - 2 \times 11 \times 13 \times \cos 42^\circ)}$$

$$\therefore\ BC \approx 8.80$$

$\therefore$ [BC] is about 8.80 cm in length.

- If we are given **all three sides** of a triangle, the cosine rule can be used to find any of the angles. To do this, we rearrange the original cosine rule formulae:

$$\boldsymbol{\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}}$$

We then use the inverse cosine function $\cos^{-1}$ to evaluate the angle.

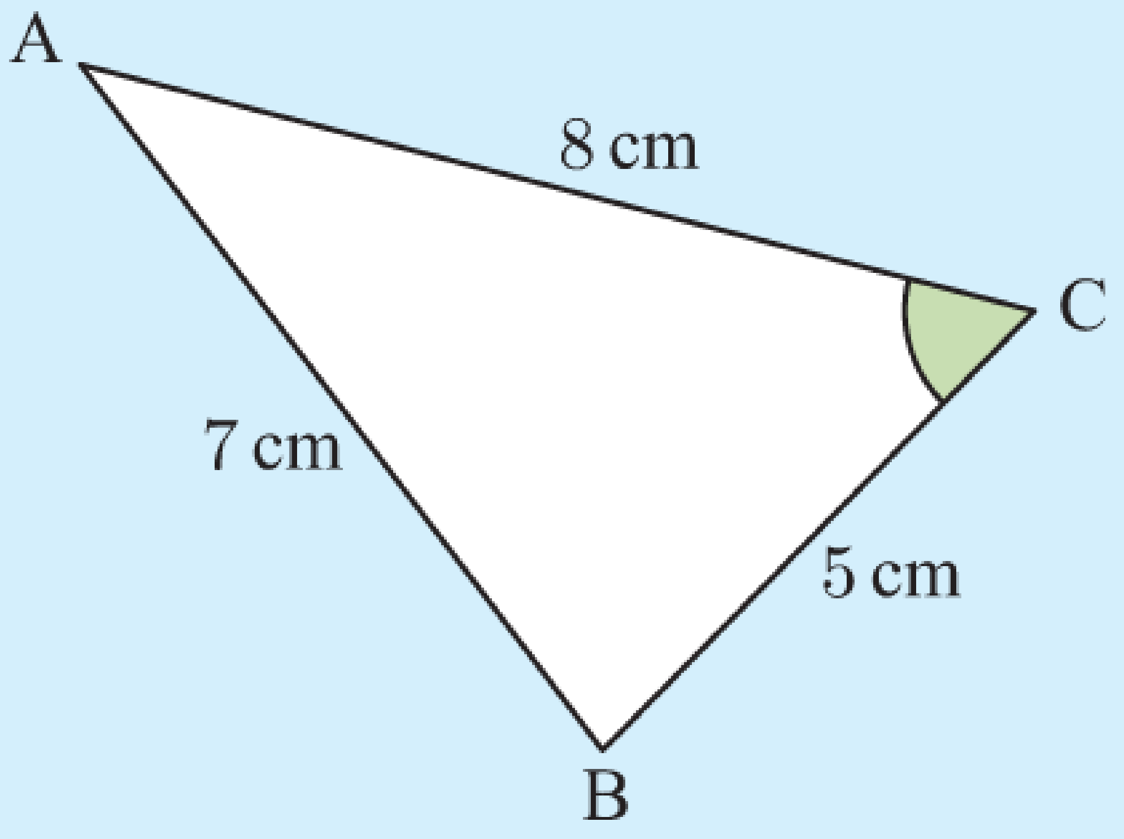
Example 4 **Self Tutor**

In triangle ABC, AB = 7 cm, BC = 5 cm, and CA = 8 cm.

a Find the measure of $B\widehat{C}A$.

b Find the area of triangle ABC.

a



By the cosine rule:

$$\cos C = \frac{(5^2 + 8^2 - 7^2)}{(2 \times 5 \times 8)}$$

$$\therefore\ C = \cos^{-1}\left(\frac{5^2 + 8^2 - 7^2}{2 \times 5 \times 8}\right)$$

$$\therefore\ C = \cos^{-1}\left(\tfrac{1}{2}\right)$$

$\therefore\ C = 60^\circ$ So, $B\widehat{C}A$ measures 60°.

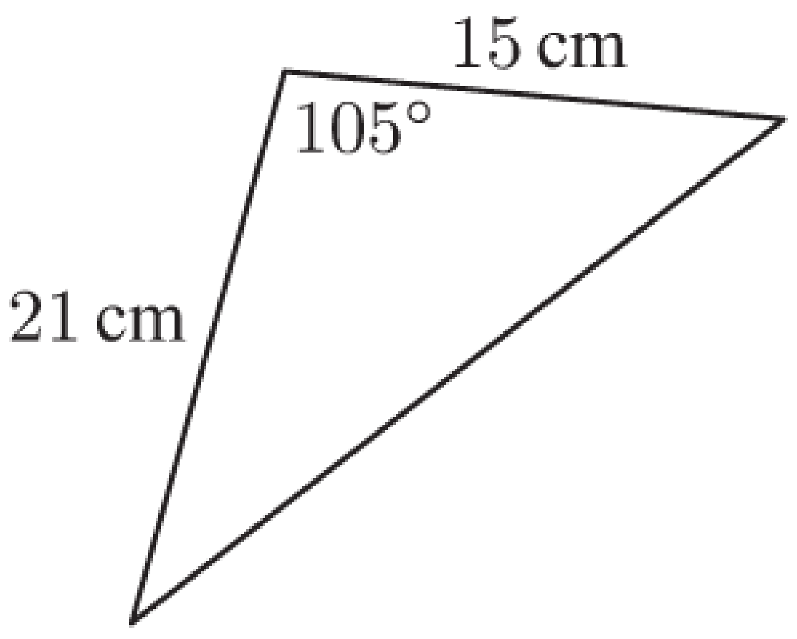
b The area of $\triangle ABC = \frac{1}{2} \times 8 \times 5 \times \sin 60^\circ$

$\approx 17.3\ \text{cm}^2$

EXERCISE 9B

1 Find the length of the remaining side in each triangle:

a



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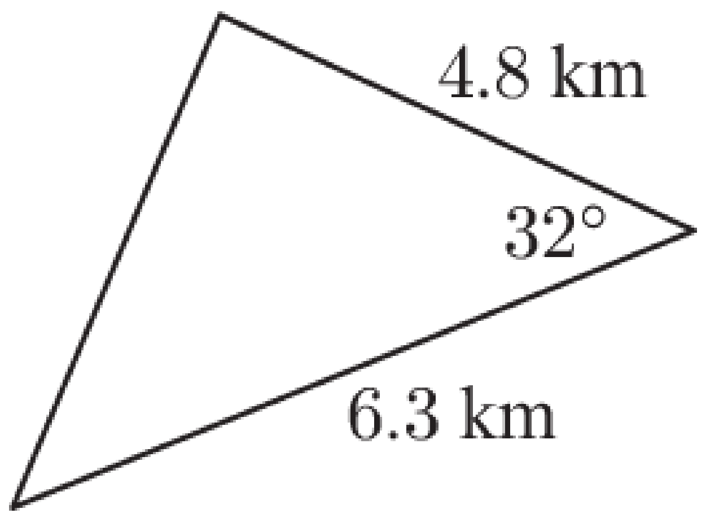

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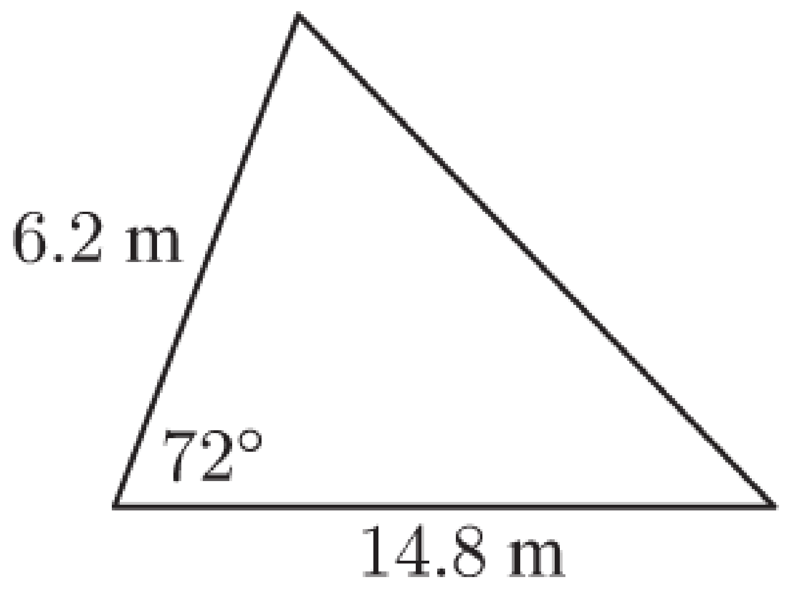
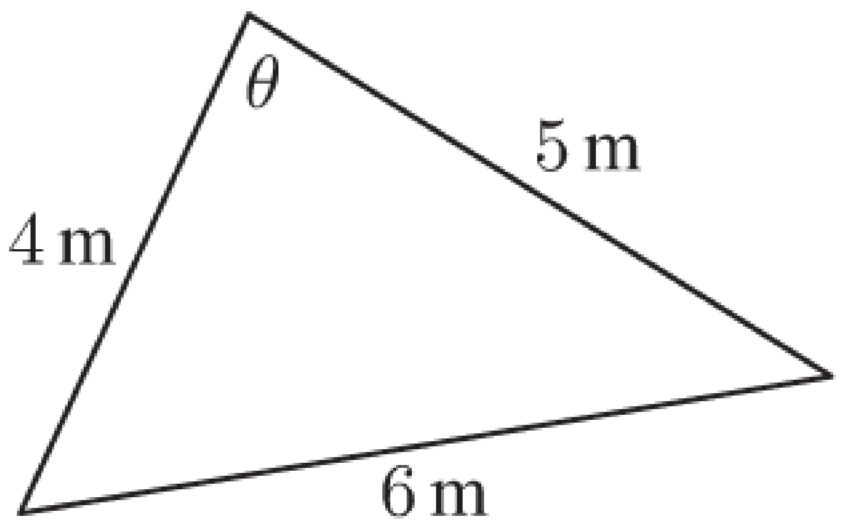
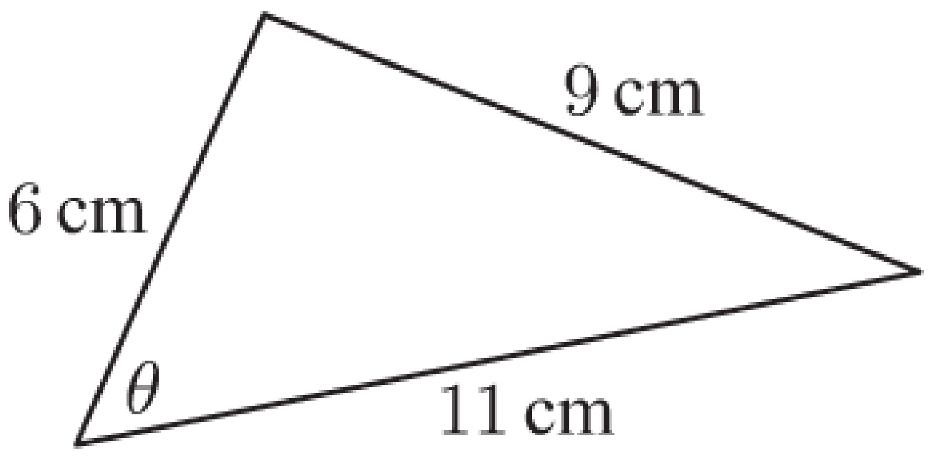


2 Find the measure of the angle marked θ:

a



b



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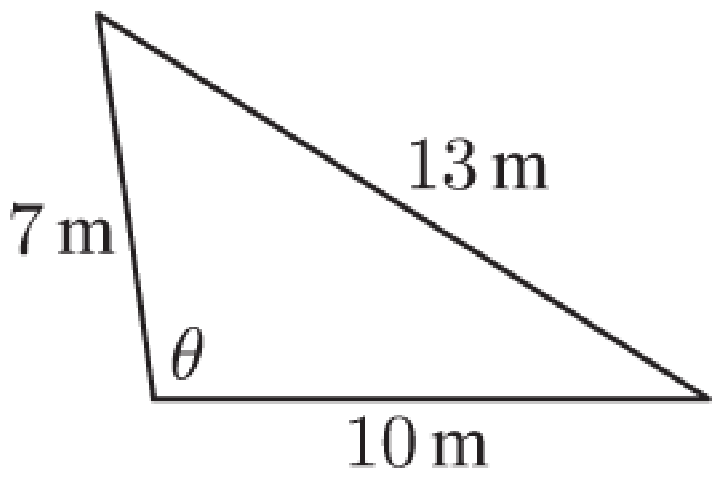
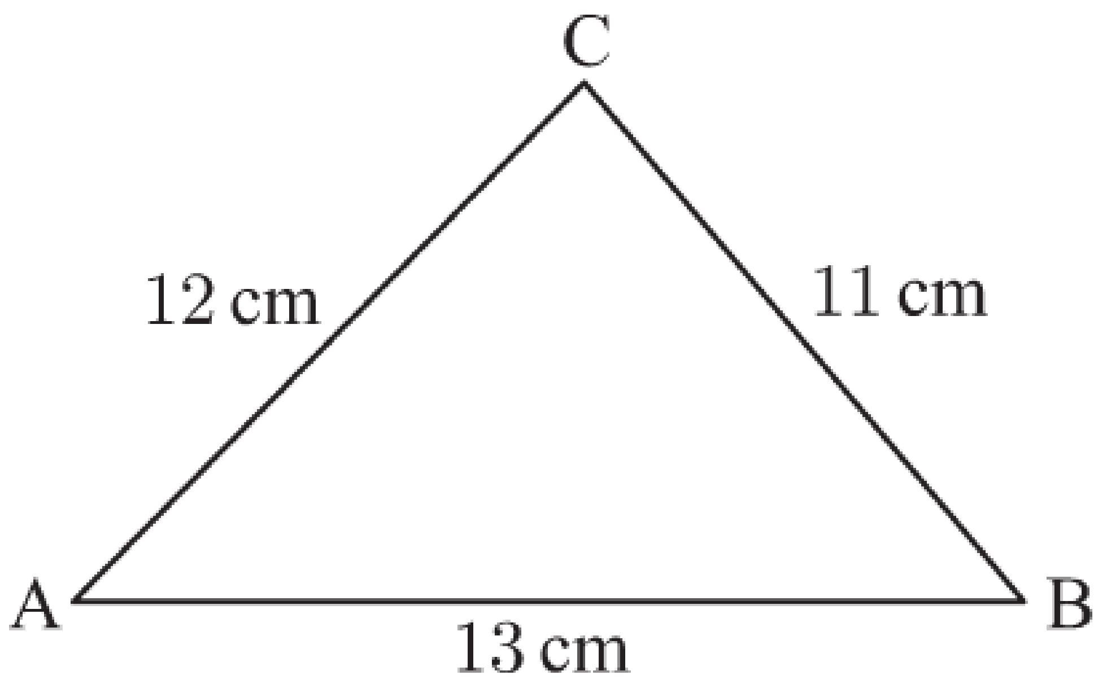
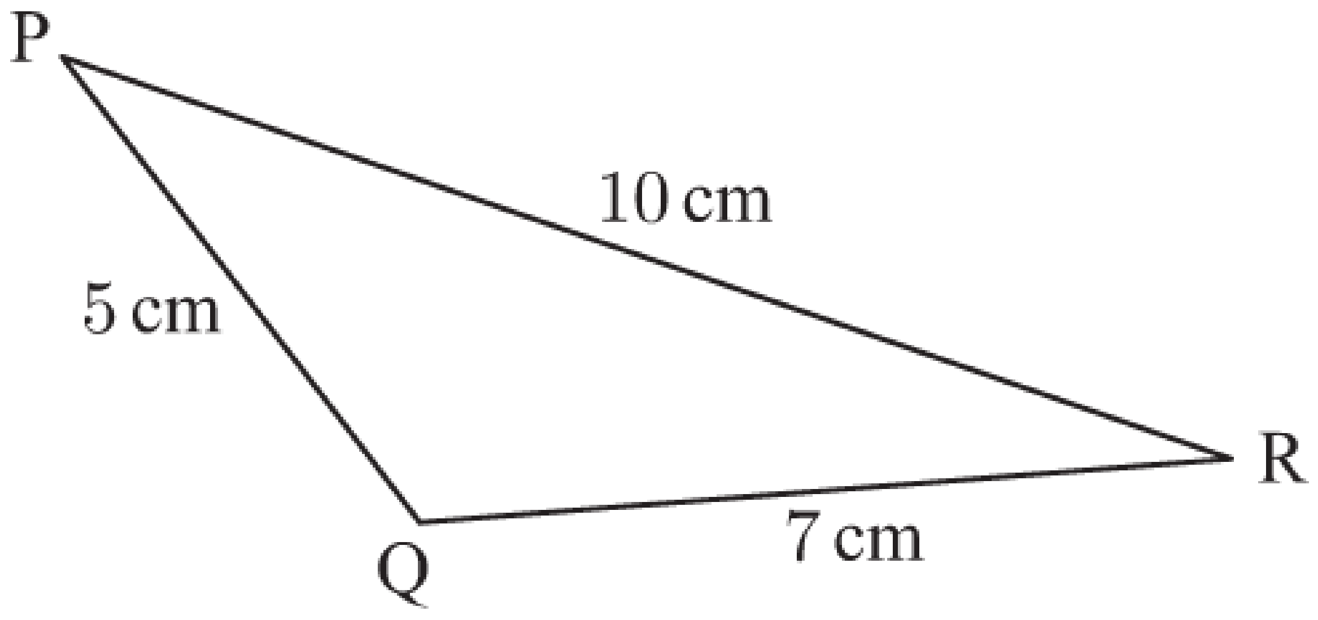


3 Find the measure of all angles of:



4 **a** Find the measure of obtuse $P\widehat{Q}R$.

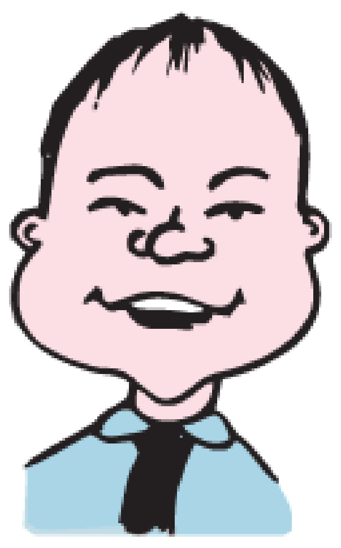
b Hence find the area of $\triangle PQR$.



5 **a** Find the smallest angle of a triangle with sides 11 cm, 13 cm, and 17 cm.

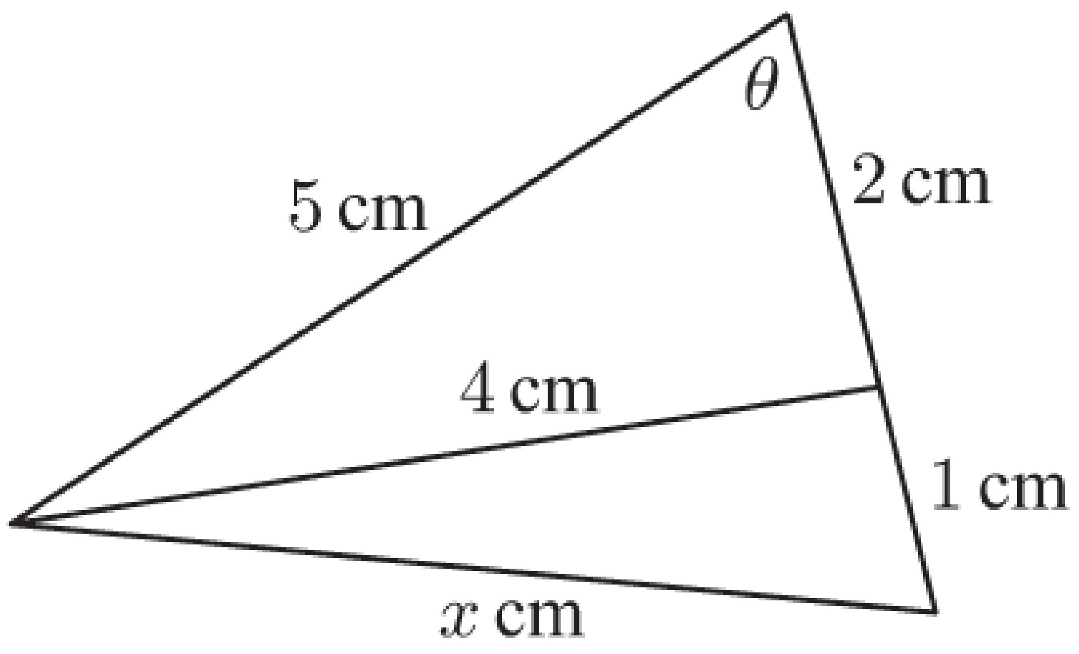
b Find the largest angle of a triangle with sides 4 cm, 7 cm, and 9 cm.

The smallest angle is always opposite the shortest side.



6 **a** Find $\cos\theta$ but not θ.

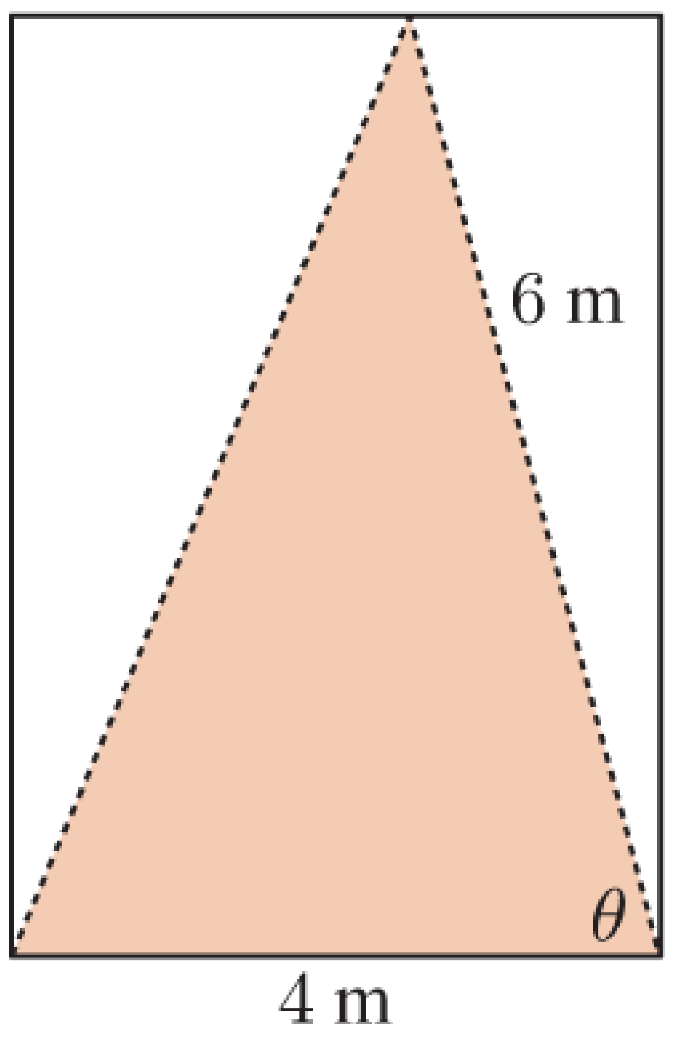
b Hence find the value of x.



7 A triangular sail is to be cut from a section of cloth. Two of the sides must have lengths 4 m and 6 m as illustrated. The total area for the sail must be 11.6 m^2, the maximum allowed for the boat to race in its class.

a Find the angle θ between the two sides of given length.

b Find the length of the third side of the sail.



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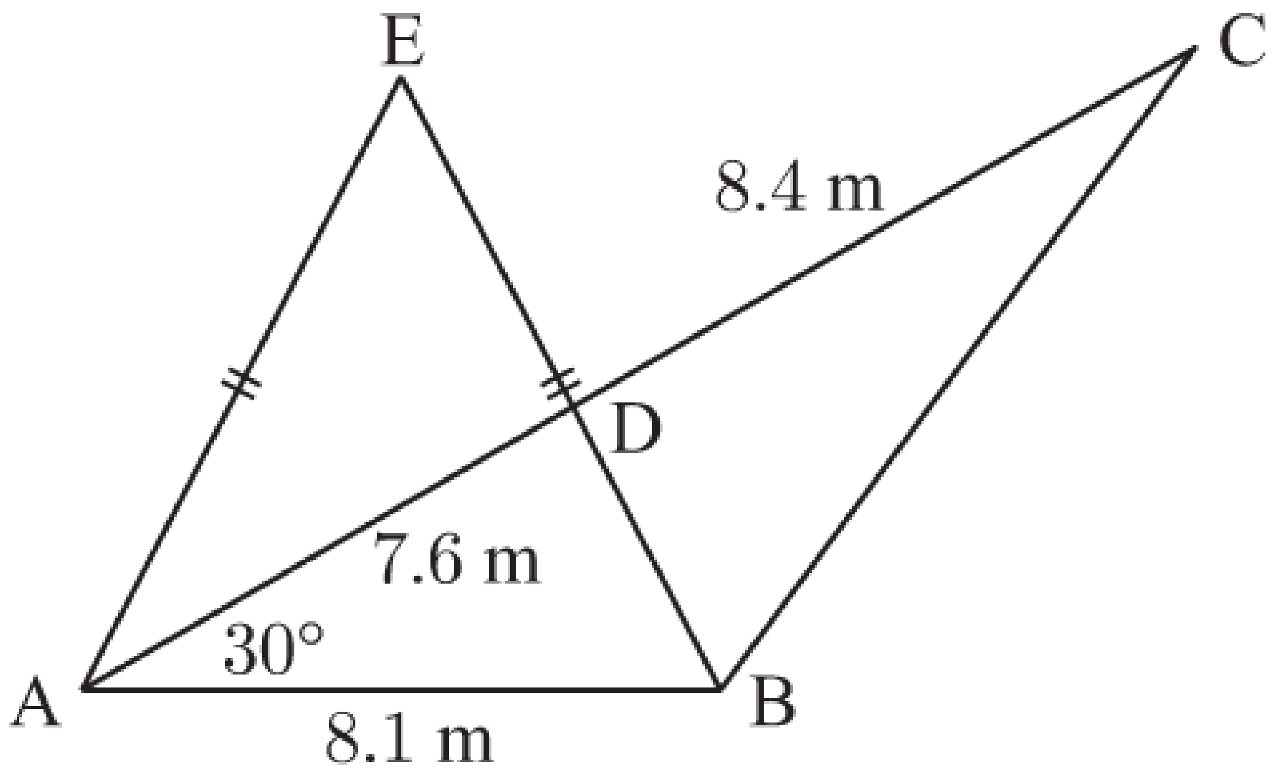


Consider the figure shown.

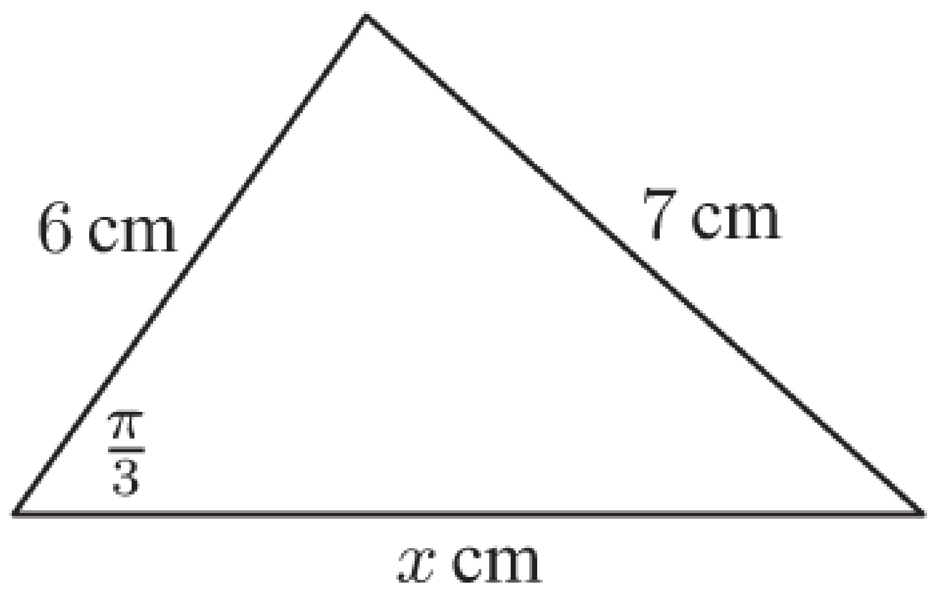
a Find the lengths of [DB] and [BC].

b Calculate the measures of $A\widehat{B}E$ and $D\widehat{B}C$.

c Find the area of $\triangle$BCD.

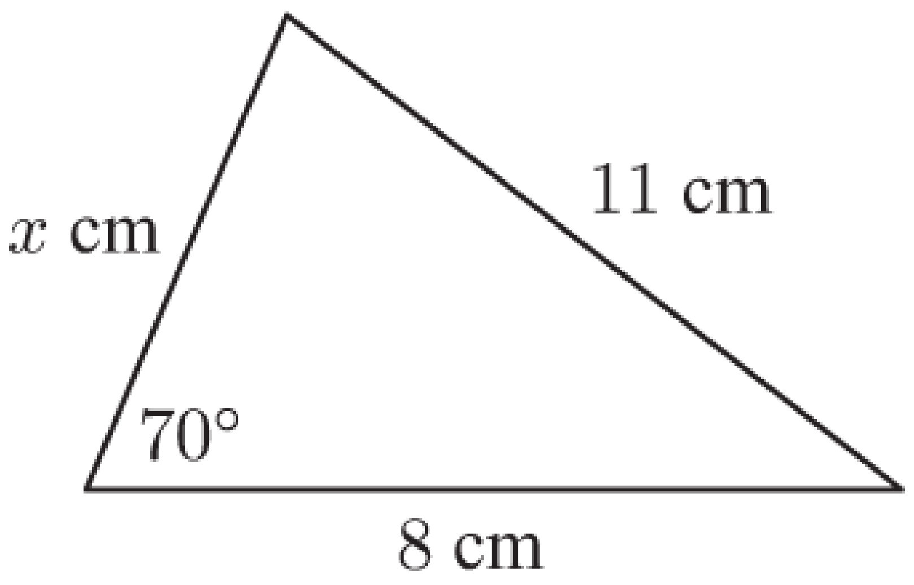
9 **a** Show that $x^2 - 6x - 13 = 0$.

b Hence find the exact value of x.

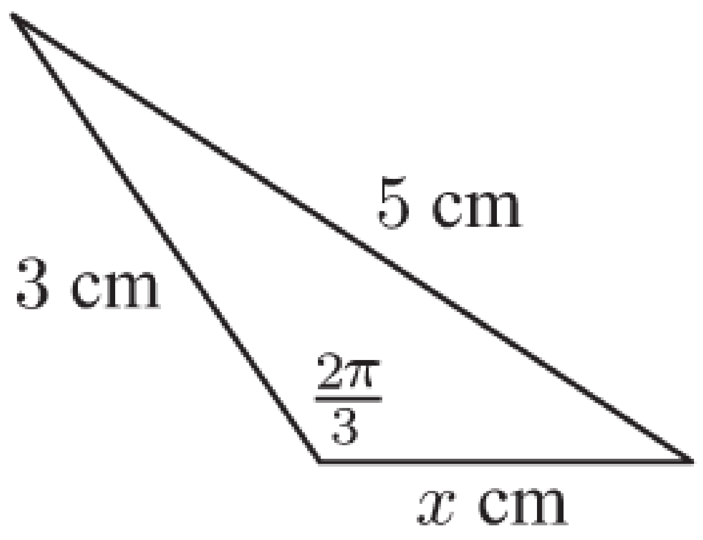


10 Find the value of x:

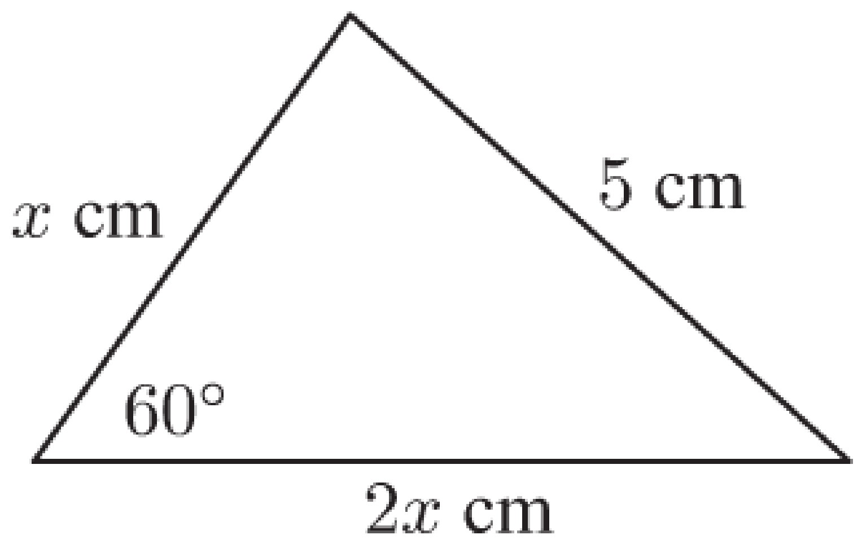
a



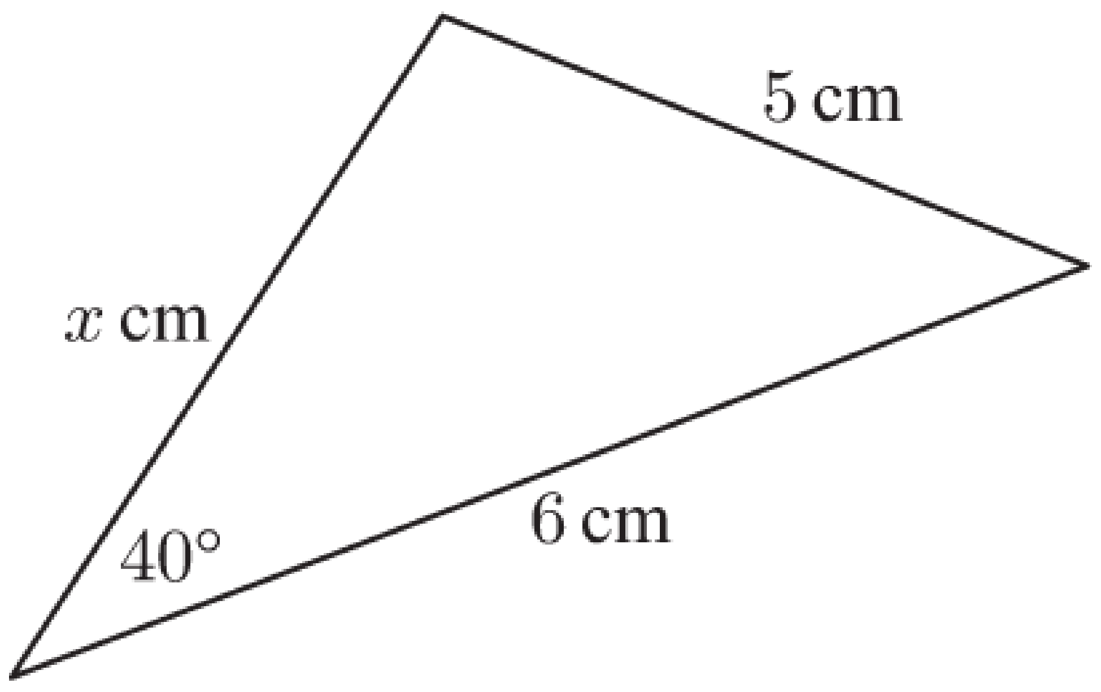
b



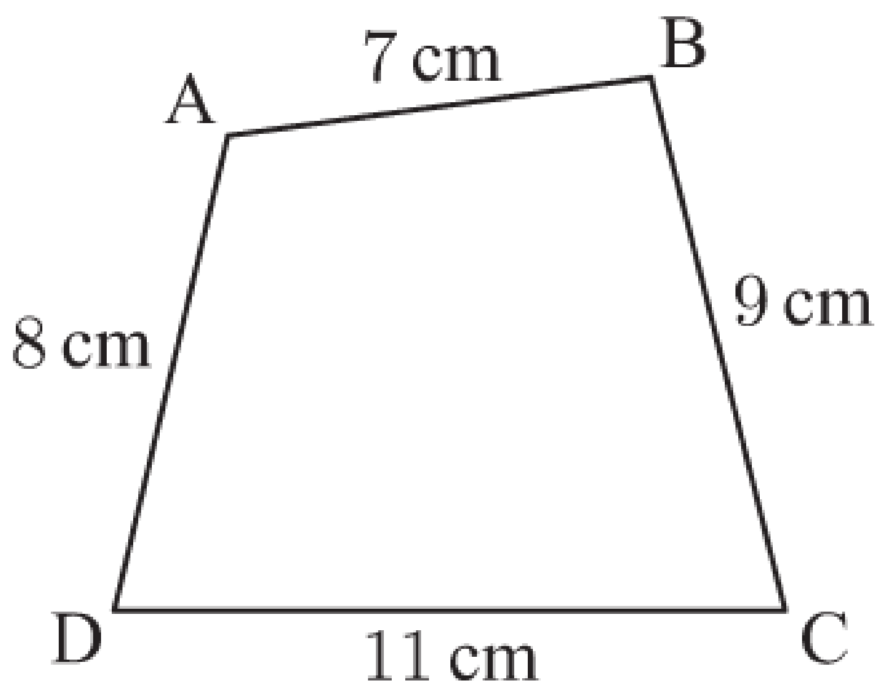
c



11 Find the possible values for x.

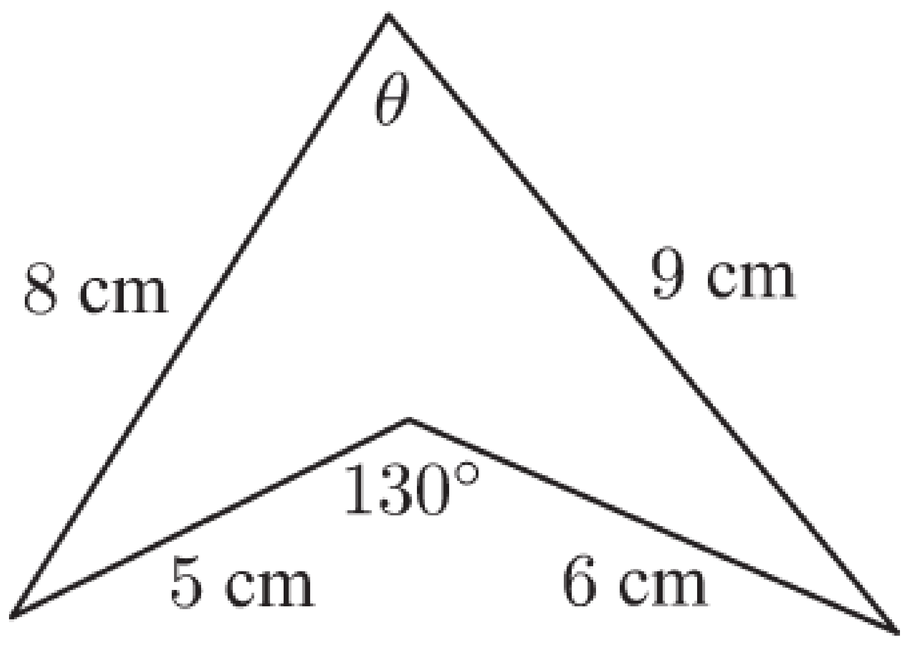


12



In quadrilateral ABCD, the diagonal [AC] has length 12 cm. Find the length of the other diagonal [BD].

13 Find the angle θ:

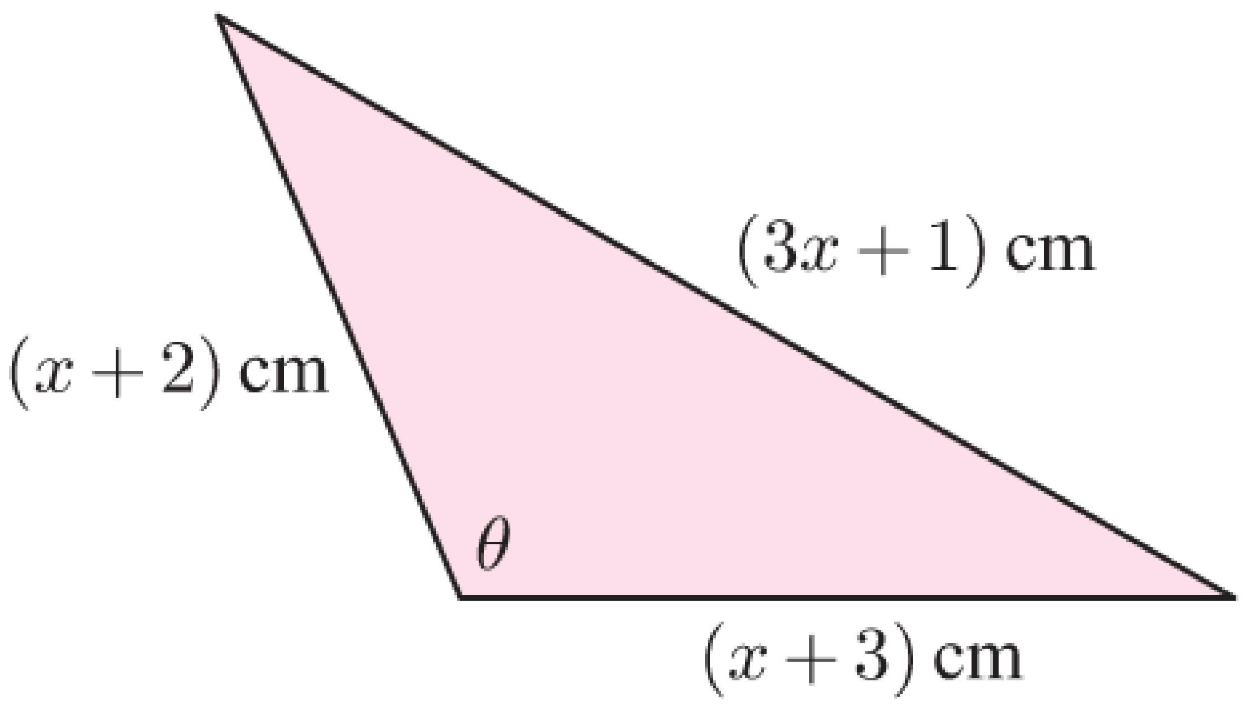


14 ABC is an equilateral triangle with sides 10 cm long. P is a point within the triangle which is 5 cm from A and 6 cm from B. How far is P from C?

15 In the diagram alongside, $\cos\theta = -\frac{1}{5}$.

a Find x.

b Hence find the exact area of the triangle.



16 The parallel sides of a trapezium have lengths 5 cm and 8 cm. The other two sides have lengths 6 cm and 4 cm. Find the angles of the trapezium, to the nearest degree.

C THE SINE RULE

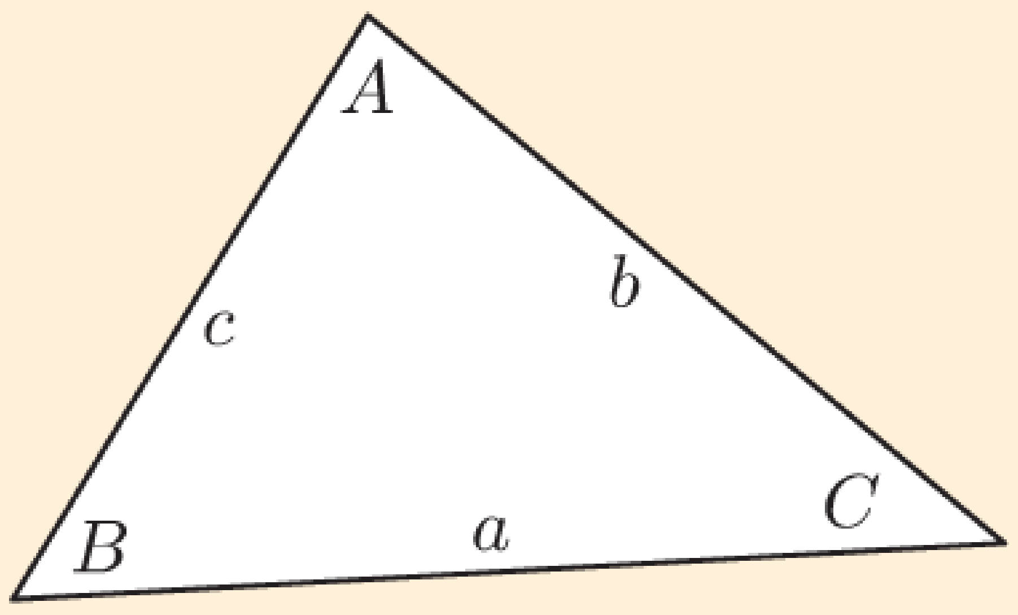
The **sine rule** is a set of equations which connects the lengths of the sides of any triangle with the sines of the angles of the triangle. The triangle does not have to be right angled for the sine rule to be used.

INVESTIGATION 1 THE SINE RULE

You will need: Paper, scissors, ruler, protractor

What to do:

1. Cut out a large triangle. Label the sides a, b, and c, and the opposite angles A, B, and C.
2. Use your ruler to measure the length of each side.
3. Use your protractor to measure the size of each angle.
4. Copy and complete this table:

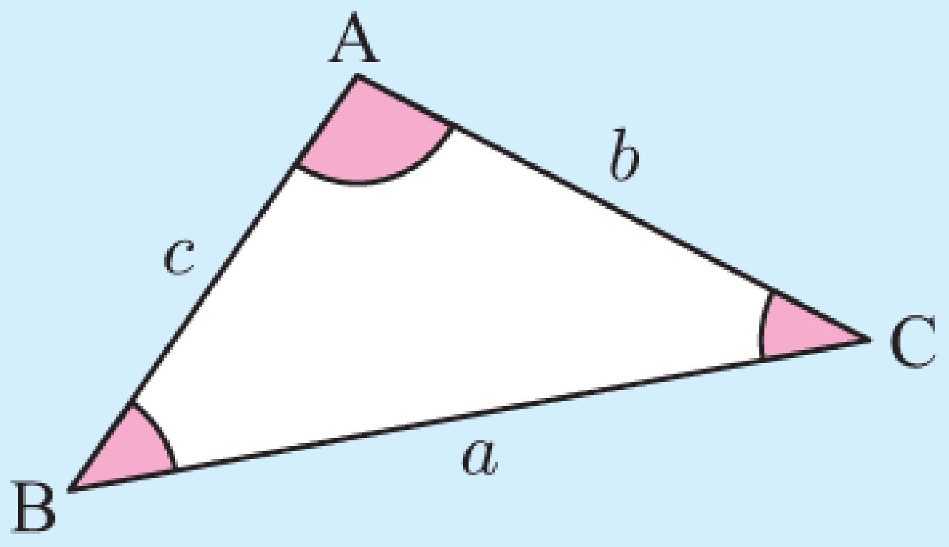


a	b	c	A	B	C	$\frac{\sin A}{a}$	$\frac{\sin B}{b}$	$\frac{\sin C}{c}$

5. Comment on your results.

In any triangle ABC with sides a, b, and c units in length, and opposite angles A, B, and C respectively,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \textit{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



Proof: The area of any triangle ABC is given by $\frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$.

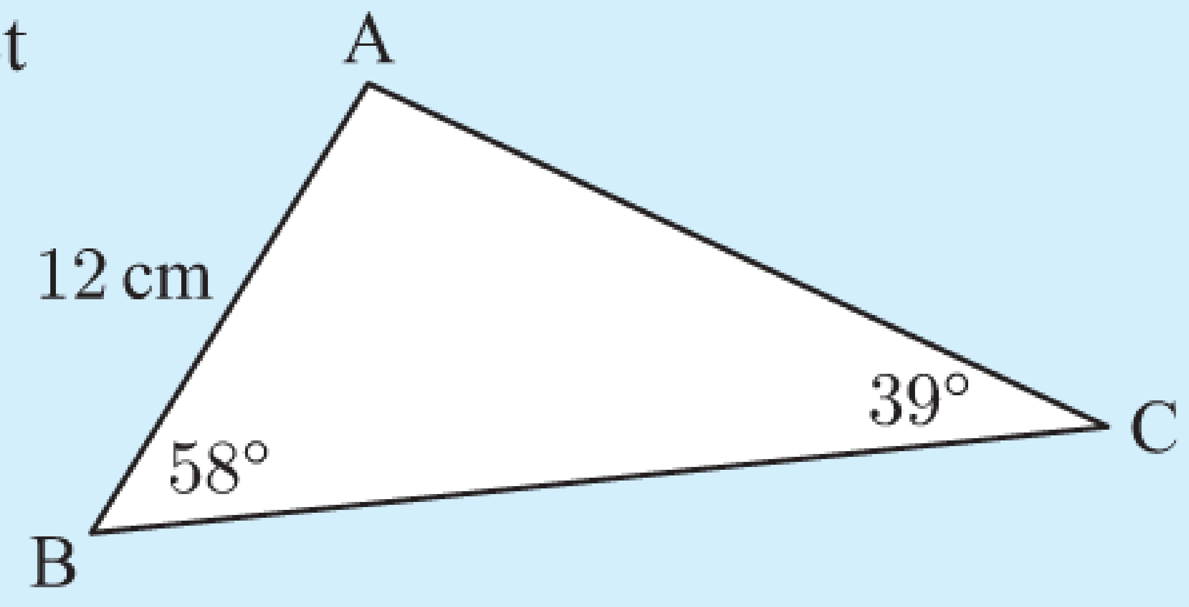
Dividing each expression by $\frac{1}{2}abc$ gives $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

FINDING SIDE LENGTHS

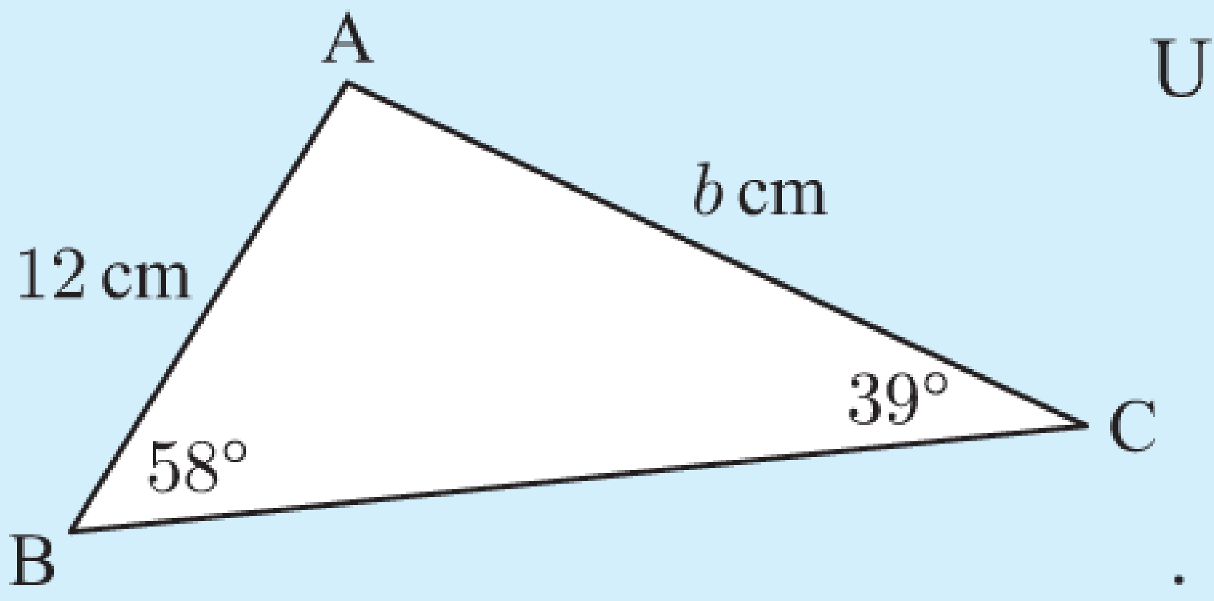
If we are given two angles and one side of a triangle we can use the sine rule to find another side length.

Example 5 **Self Tutor**

Find the length of [AC] correct to 2 decimal places.



If necessary, you can use the angle sum of a triangle = 180° to find the third angle.



Using the sine rule, $\dfrac{b}{\sin 58°} = \dfrac{12}{\sin 39°}$

$$\therefore \quad b = \frac{12 \times \sin 58°}{\sin 39°}$$

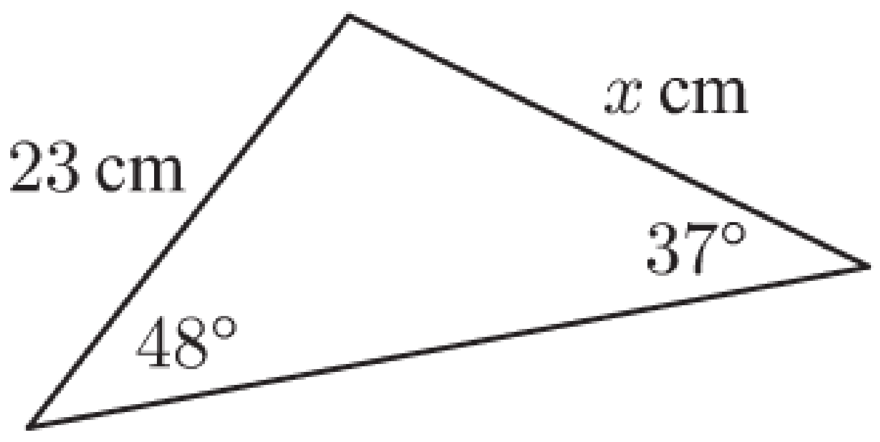
$$\therefore \quad b \approx 16.17$$

$\therefore$ [AC] is about 16.17 cm long.

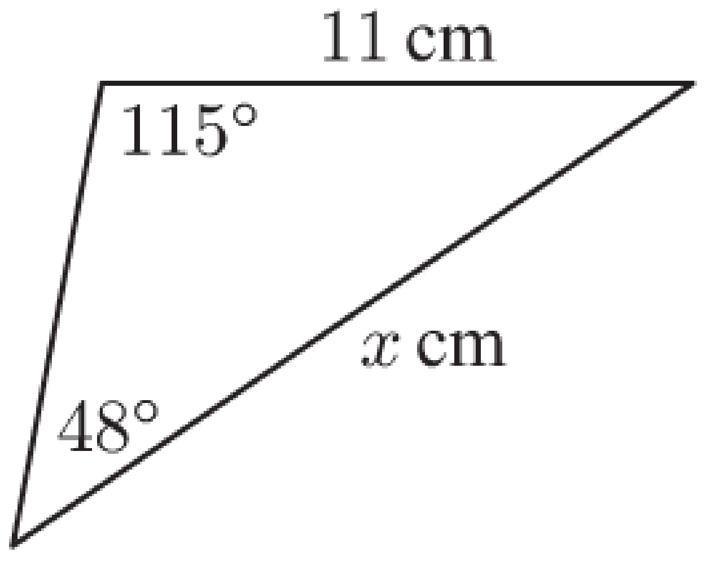
EXERCISE 9C.1

1 Find the value of x:

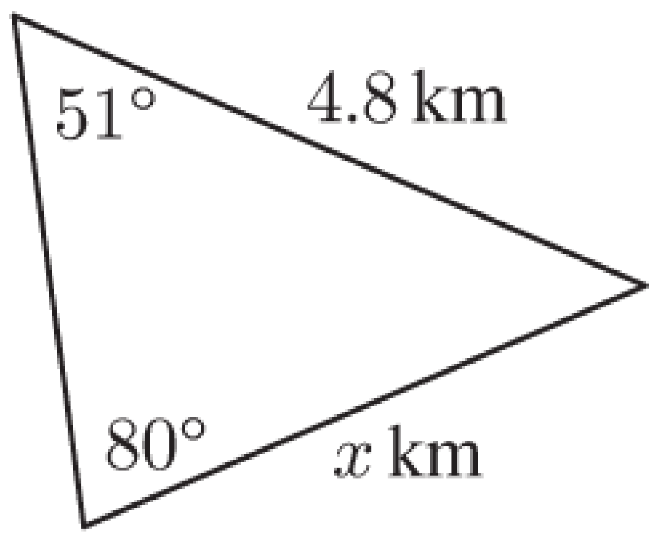
a



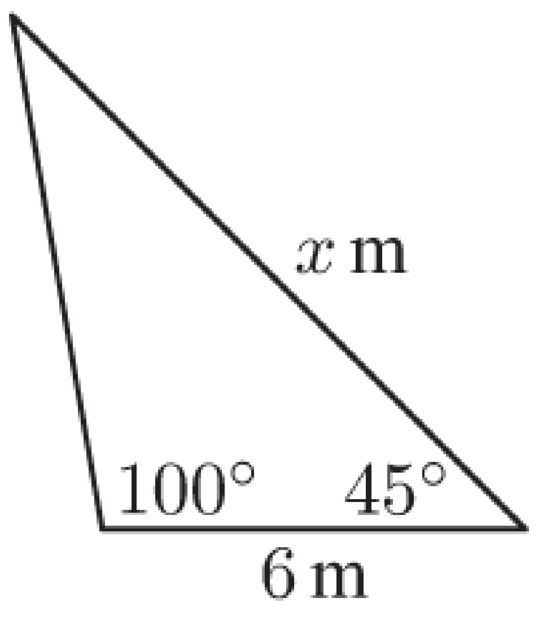
b



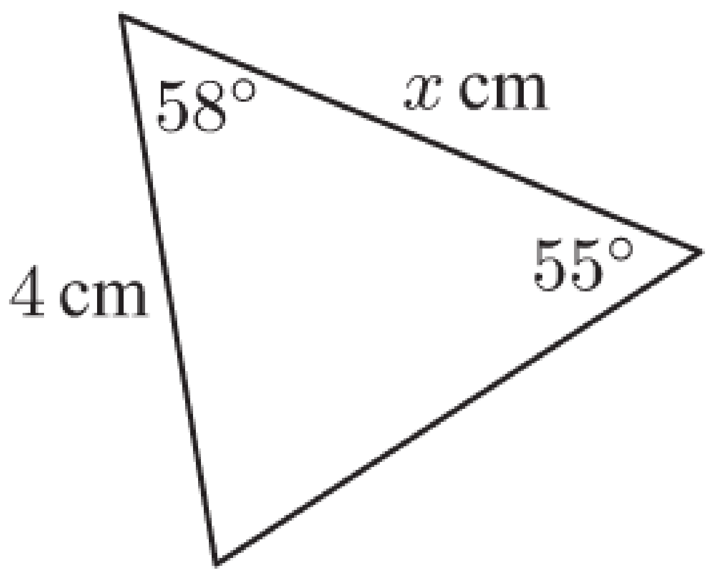
c



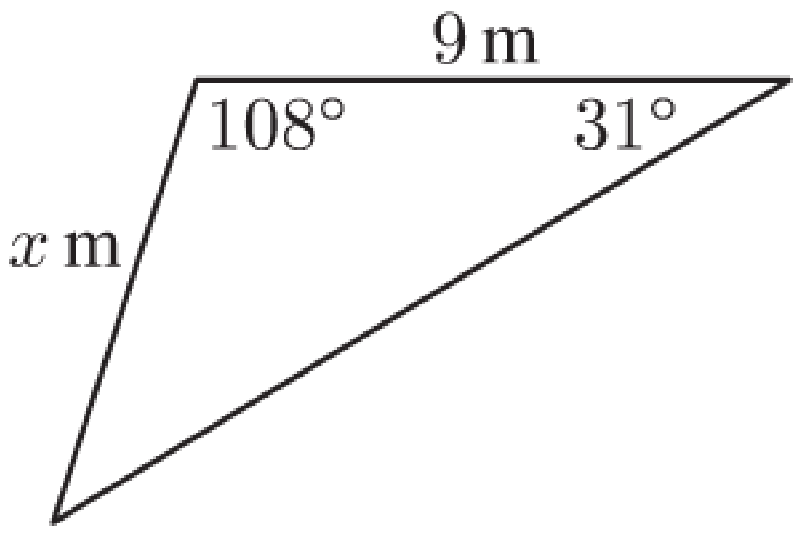
d



e



f

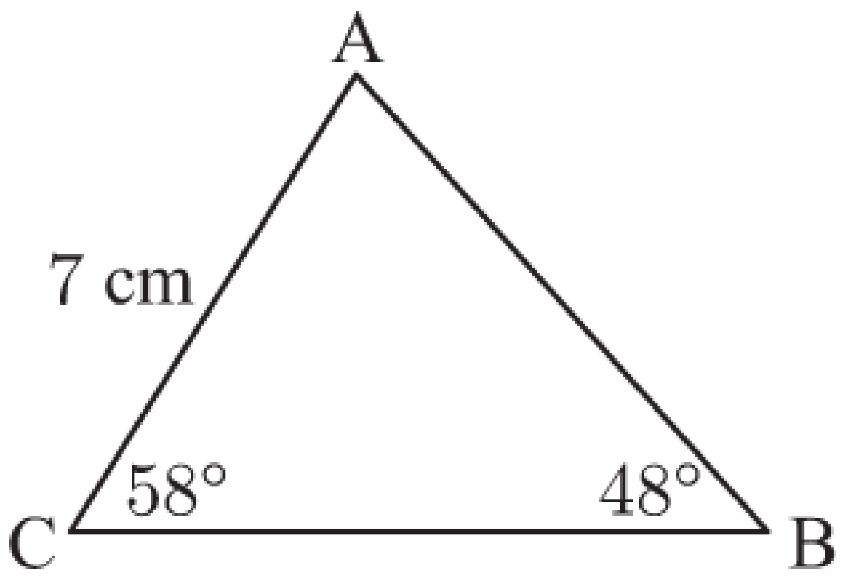


2 Consider triangle ABC.

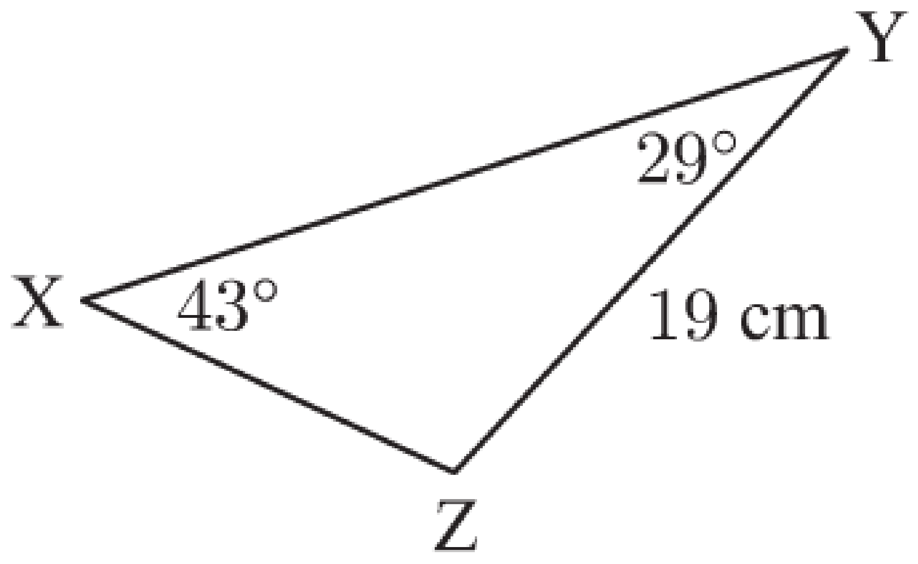
- **a** Given $A = 63°$, $B = 49°$, and $b = 18$ cm, find a.
- **b** Given $A = 82°$, $C = 25°$, and $c = 34$ cm, find b.
- **c** Given $B = 21°$, $C = 48°$, and $a = 6.4$ cm, find c.

3 Find *all* unknown sides and angles of:

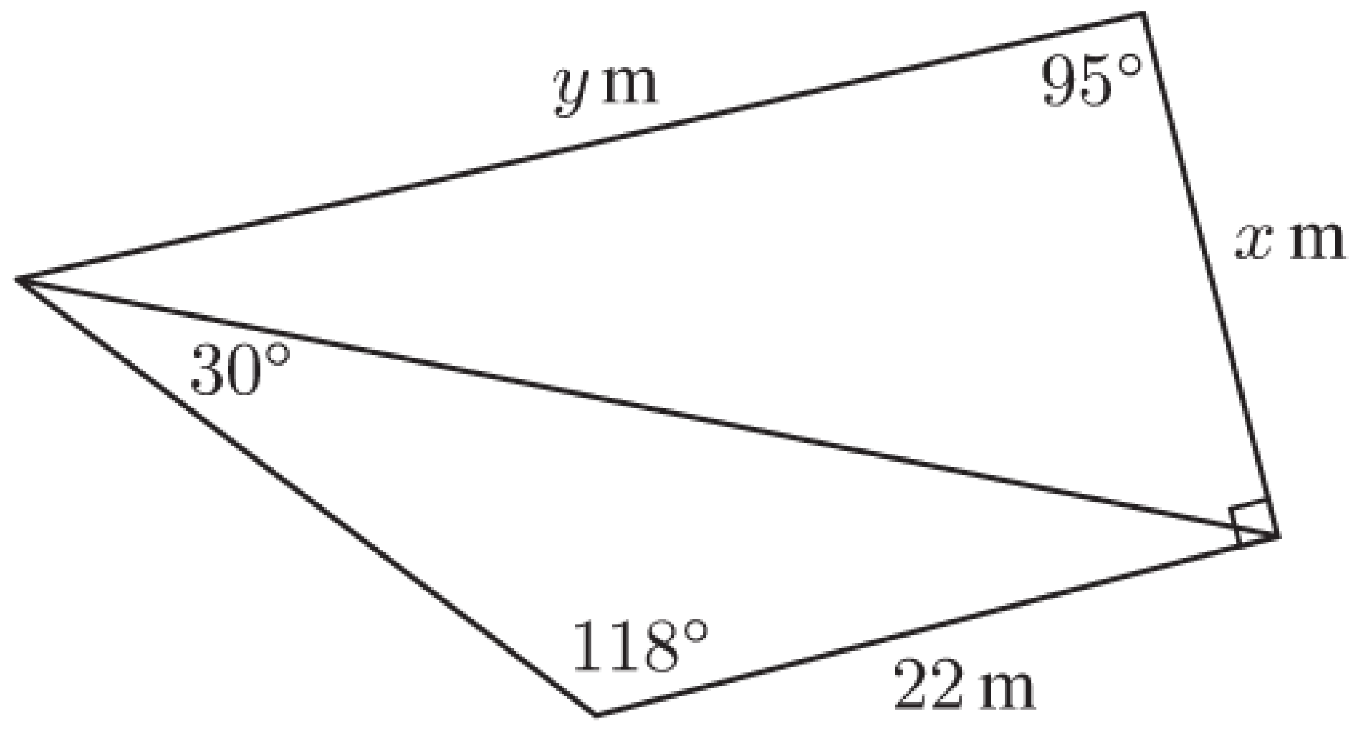
a



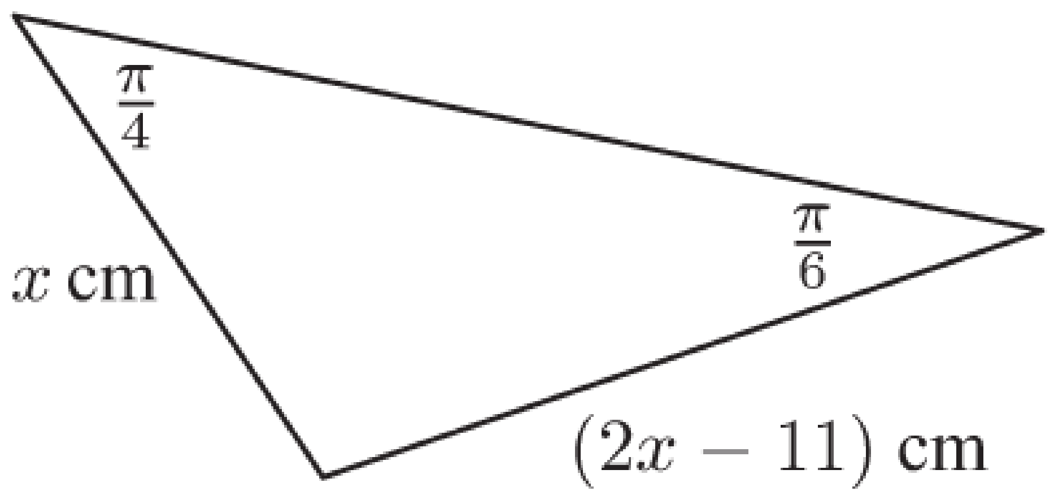
b



4 Find x and y in the given figure.



5 Find the exact value of x, giving your answer in the form $a + b\sqrt{2}$ where $a, b \in \mathbb{Q}$.



FINDING ANGLES

Finding angles using the sine rule is complicated because there may be two possible answers. For example, if $\sin\theta = \frac{1}{2}$ then θ could be 30° or 150°. We call this situation an **ambiguous case**.

You can click on the icon to obtain an interactive demonstration of the ambiguous case, or else you can work through the following **Investigation**.

DEMO

INVESTIGATION 2 THE AMBIGUOUS CASE

You will need a blank sheet of paper, a ruler, a protractor, and a compass for the tasks that follow. In each task you will be required to construct triangles from given information.

What to do:

1 Draw AB = 10 cm. Construct an angle of 30° at point A. Using B as the centre, draw an arc of a circle with radius 6 cm. Let C denote the point where the arc intersects the ray from A. How many different possible points C are there, and therefore how many different triangles ABC may be constructed?

2 Repeat the procedure from **1** three times, starting with AB = 10 cm and constructing an angle of 30° at point A. When you draw the arc with centre B, use the radius:

a 5 cm **b** 3 cm **c** 12 cm

3 Using your results from **1** and **2**, discuss the possible number of triangles you can obtain given two sides and a non-included angle.

You should have discovered that when you are given two sides and a non-included angle, you could get two triangles, one triangle, or it may be impossible to draw any triangles at all.

Now consider the calculations involved in each of the cases in the **Investigation**.

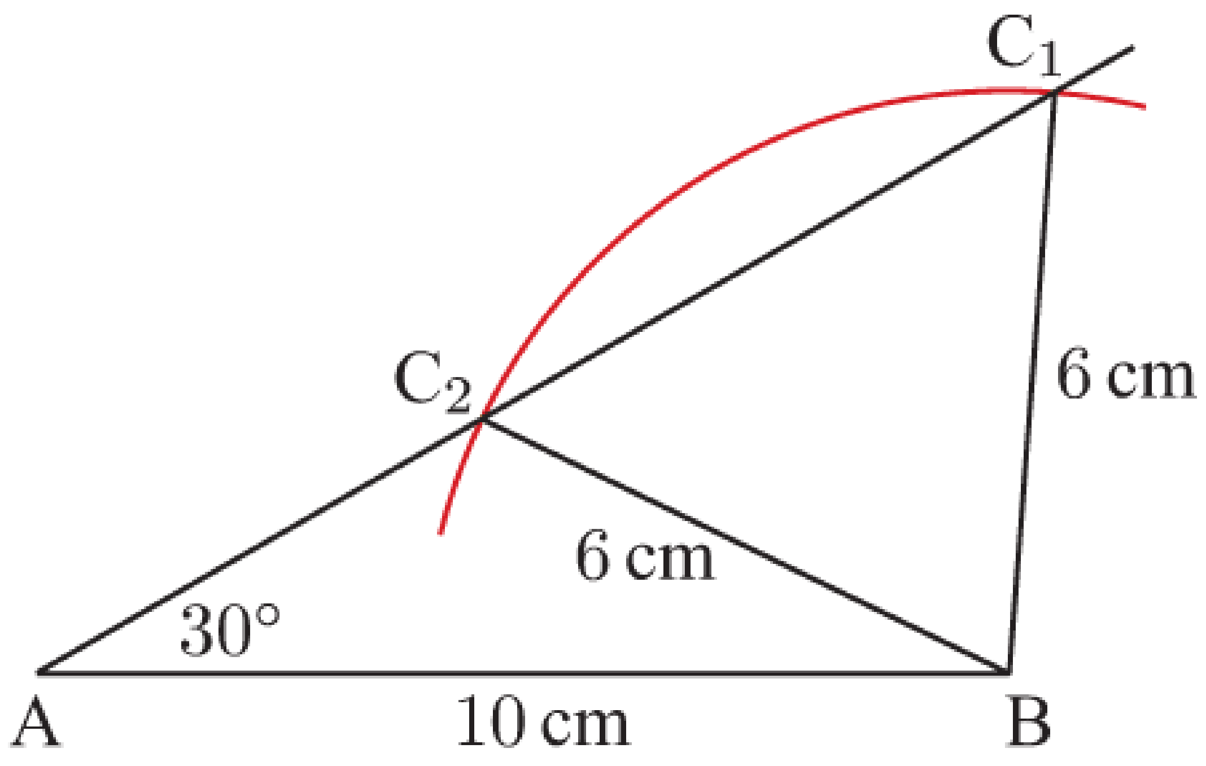
Case 1: Given: $c = 10$ cm, $a = 6$ cm, $A = 30°$

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\therefore \sin C = \frac{c \sin A}{a}$$

$$\therefore \sin C = \frac{10 \times \sin 30°}{6} \approx 0.8333$$

$$\therefore C \approx 56.44° \text{ or } 180° - 56.44° = 123.56°$$

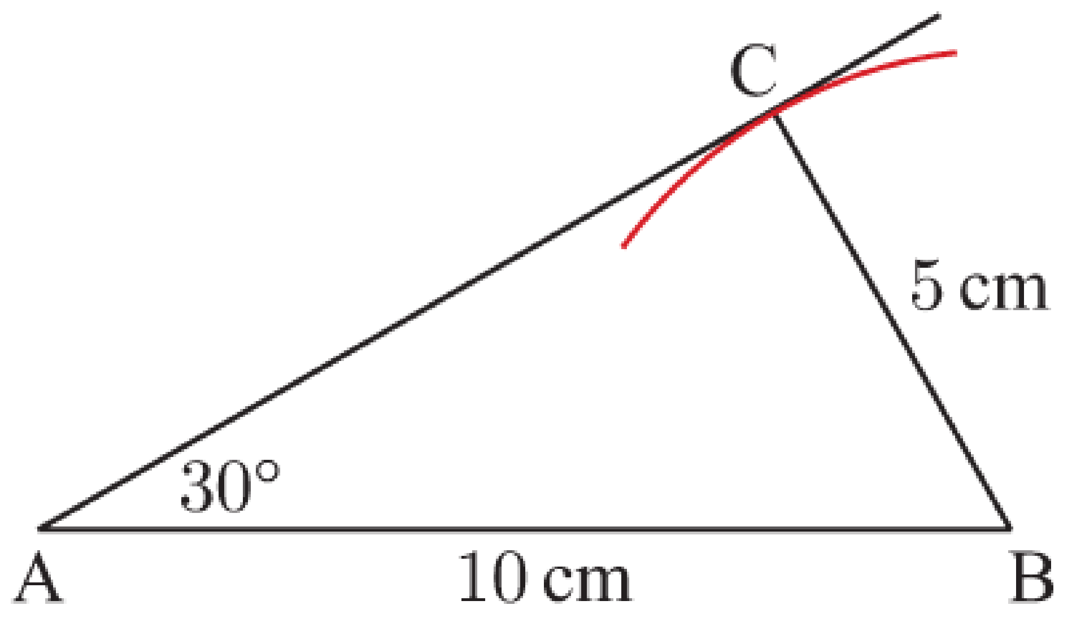


Case 2: Given: $c = 10$ cm, $a = 5$ cm, $A = 30°$

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\therefore \sin C = \frac{c \sin A}{a}$$

$$\therefore \sin C = \frac{10 \times \sin 30°}{5} = 1$$



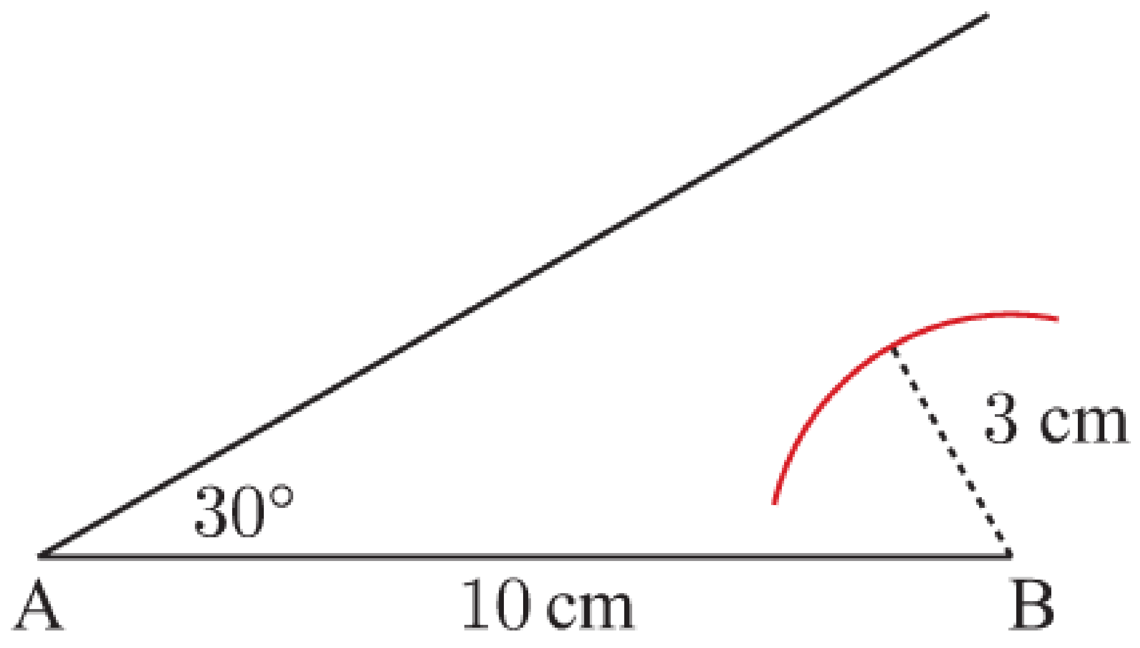
There is only one possible solution for C in the range from $0°$ to $180°$, and that is $C = 90°$. Only one triangle is therefore possible. Complete the solution of the triangle yourself.

Case 3: Given: $c = 10$ cm, $a = 3$ cm, $A = 30°$

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\therefore \sin C = \frac{c \sin A}{a}$$

$$\therefore \sin C = \frac{10 \times \sin 30°}{3} \approx 1.6667$$



There is no angle that has a sine value > 1, so no triangles can be drawn to match the information given.

Case 4: Given: $c = 10$ cm, $a = 12$ cm, $A = 30°$

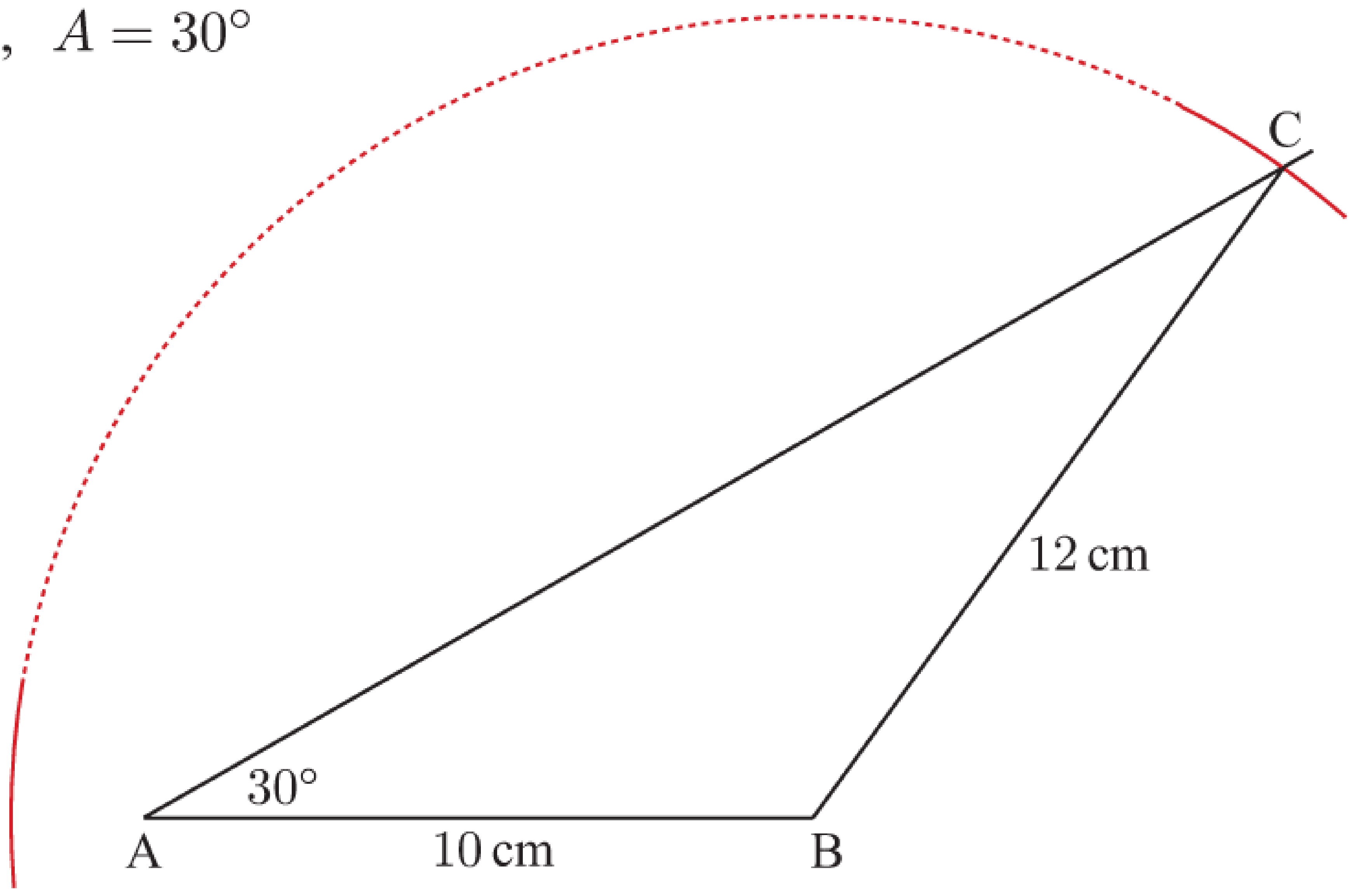
$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

$$\therefore \sin C = \frac{c \sin A}{a}$$

$$\therefore \sin C = \frac{10 \times \sin 30°}{12} \approx 0.4167$$

$$\therefore C \approx 24.62° \quad \textit{or}$$

$$180° - 24.62° = 155.38°$$



However, in this case only one of these two angles is valid. Since $A = 30°$, C cannot possibly equal $155.38°$ because $30° + 155.38° > 180°$.

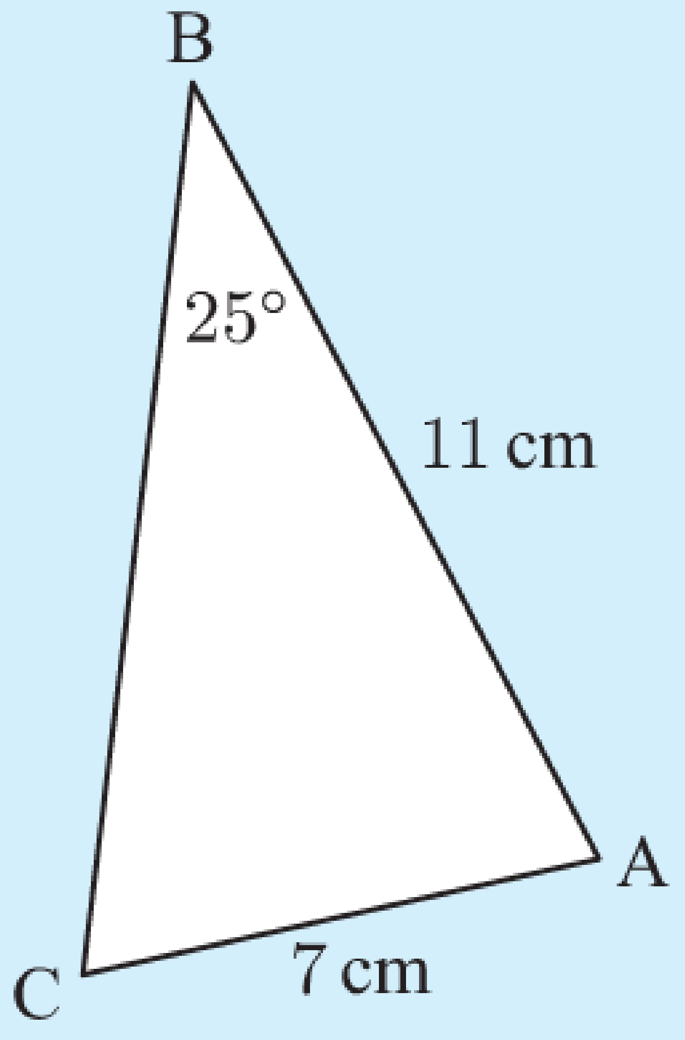
Therefore, there is only one possible solution, $C \approx 24.62°$.

Conclusion: Each situation using the sine rule with two sides and a non-included angle must be examined very carefully.

In particular, if the given angle is acute and opposite the shorter of the two given sides, then two different triangles are possible.

Example 6 **Self Tutor**

Find the measure of angle C in triangle ABC if AC = 7 cm, AB = 11 cm, and angle B measures 25°.



$$\frac{\sin C}{c} = \frac{\sin B}{b} \quad \{\text{sine rule}\}$$

$$\therefore \frac{\sin C}{11} = \frac{\sin 25°}{7}$$

$$\therefore \sin C = \frac{11 \times \sin 25°}{7}$$

$$\therefore C = \sin^{-1}\left(\frac{11 \times \sin 25°}{7}\right) \text{ or its supplement}$$

{as C may be obtuse}

$$\therefore C \approx 41.6° \text{ or } 180° - 41.6°$$

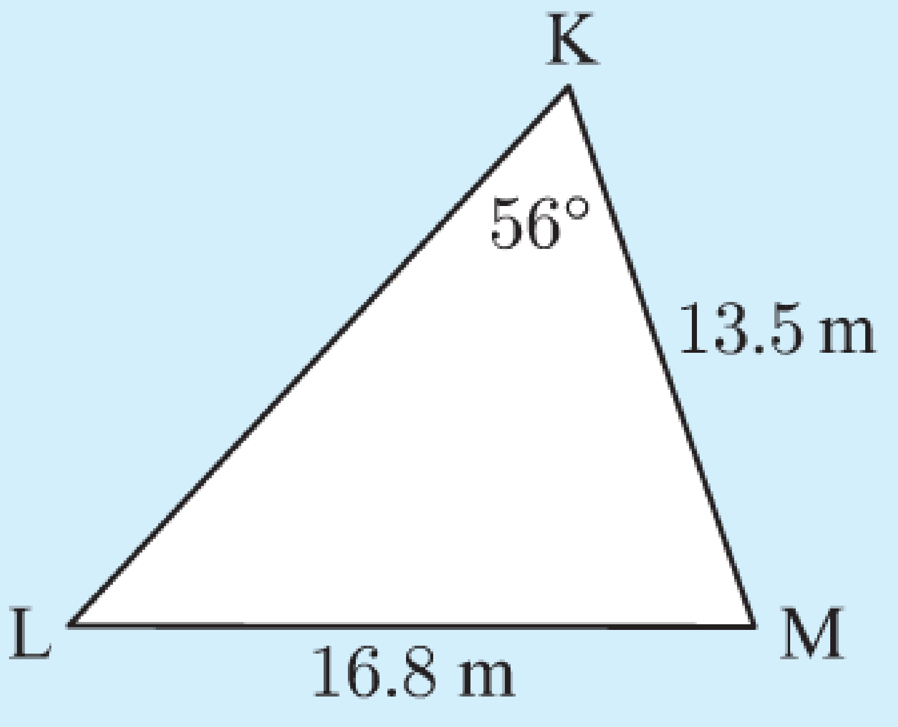
$$\therefore C \approx 41.6° \text{ or } 138.4°$$

$\therefore$ C measures 41.6° if angle C is acute, or 138.4° if angle C is obtuse.

In this case there is insufficient information to determine the actual shape of the triangle. There are two possible triangles.

Example 7 **Self Tutor**

Find the measure of angle L in triangle KLM given that angle K measures 56°, LM = 16.8 m, and KM = 13.5 m.



$$\frac{\sin L}{13.5} = \frac{\sin 56°}{16.8} \quad \{\text{sine rule}\}$$

$$\therefore \sin L = \frac{13.5 \times \sin 56°}{16.8}$$

$$\therefore L = \sin^{-1}\left(\frac{13.5 \times \sin 56°}{16.8}\right) \text{ or its supplement}$$

$$\therefore L \approx 41.8° \text{ or } 180° - 41.8°$$

$$\therefore L \approx 41.8° \text{ or } 138.2°$$

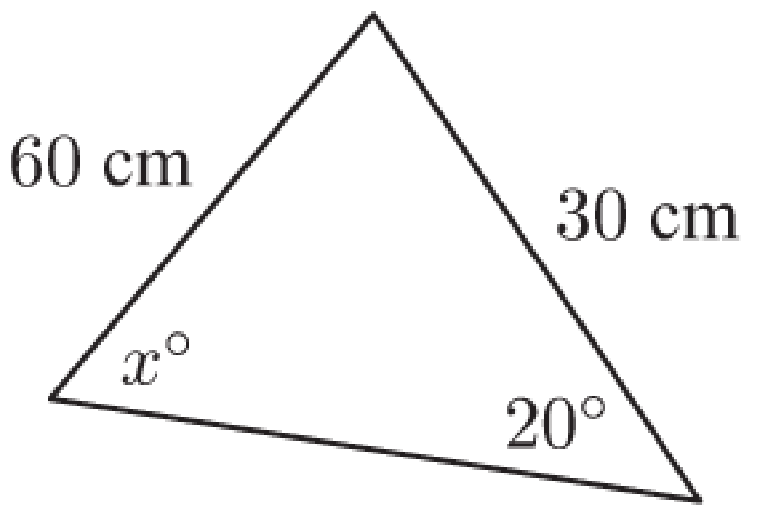
We reject $L \approx 138.2°$, since $138.2° + 56° > 180°$ which is impossible in a triangle.

$\therefore$ $L \approx 41.8°$, a unique solution in this case.

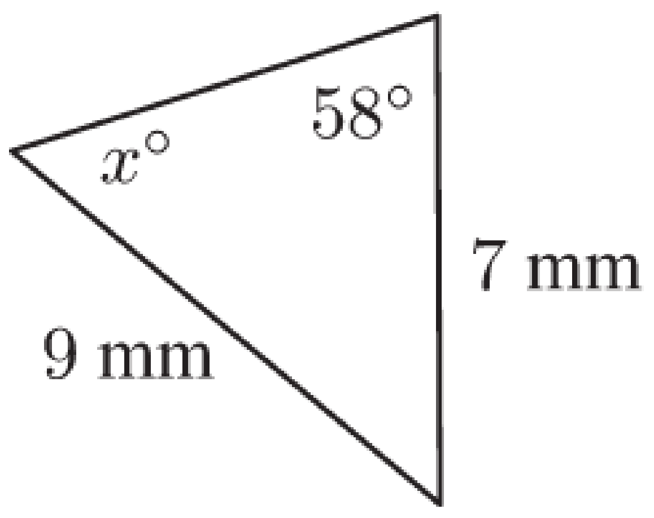
EXERCISE 9C.2

1 Find the value of x:

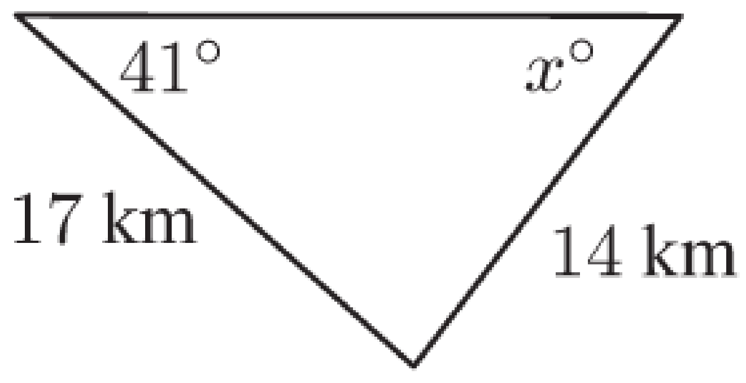
a



b



c

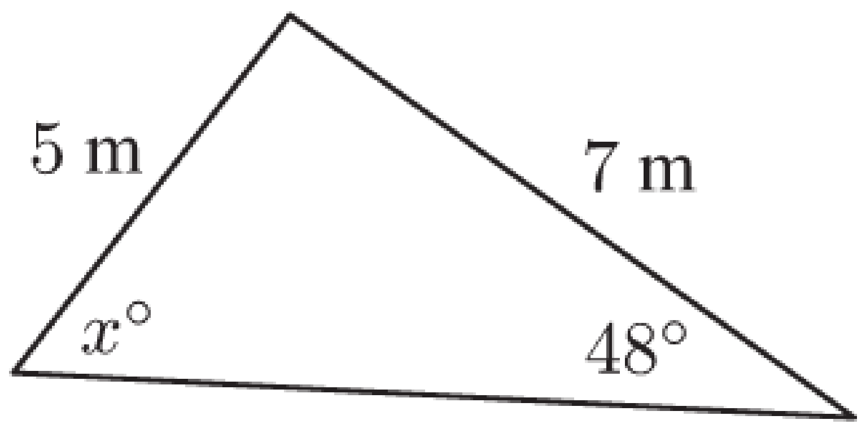


2 Triangle ABC has angle $B = 40°$ and side lengths $b = 8$ cm and $c = 11$ cm. Find the two possible measures of angle C.

3 Consider triangle ABC.

- **a** Given $a = 8$ cm, $b = 11$ cm, and $A\widehat{B}C = 45°$, find the measure of $B\widehat{A}C$.
- **b** Given $a = 32$ cm, $b = 23$ cm, and $B\widehat{A}C = 42°$, find the measure of $A\widehat{B}C$.
- **c** Given $c = 30$ m, $b = 36$ m, and $A\widehat{B}C = 37°$, find the measure of $A\widehat{C}B$.
- **d** Given $a = 8.4$ cm, $b = 10.3$ cm, and $A\widehat{B}C = 63°$, find the measure of $B\widehat{A}C$.
- **e** Given $b = 22.1$ cm, $c = 16.5$ cm, and $A\widehat{C}B = 38°$, find the measure of $A\widehat{B}C$.
- **f** Given $a = 3.1$ km, $c = 4.3$ km, and $B\widehat{A}C = 18°$, find the measure of $A\widehat{C}B$.

4 Unprepared for class, Mr Whiffen asks his students to find the value of x in the diagram shown.

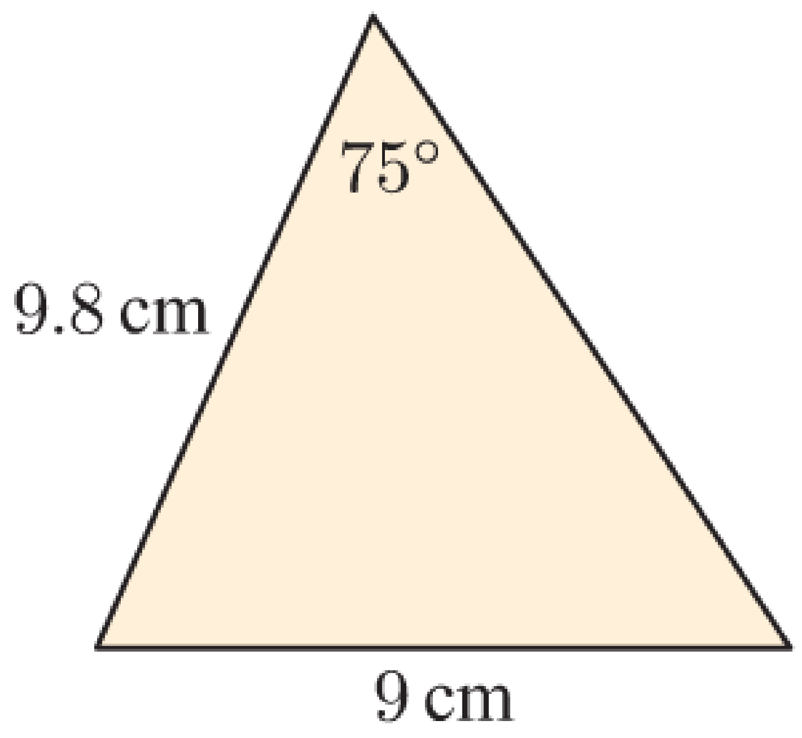


- **a** Show that Mr Whiffen's question cannot be solved.
- **b** Explain what this means about the triangle Mr Whiffen created.

5 In triangle ABC, $A\widehat{B}C = 30°$, $AC = 9$ cm, and $AB = 7$ cm.

- **a** Find the measure of: **i** $A\widehat{C}B$ **ii** $B\widehat{A}C$
- **b** Hence find the area of the triangle.

6 Is it possible to have a triangle with the measurements shown? Explain your answer.



7 In triangle PQR, $P\widehat{R}Q = 50°$, $PR = 11$ m, and $PQ = 9$ m.

- **a** Show that there are two possible measures of $P\widehat{Q}R$.
- **b** Sketch triangle PQR for each case.
- **c** For each case, find:
  - **i** the measure of $Q\widehat{P}R$ **ii** the area of the triangle **iii** the perimeter of the triangle.

D PROBLEM SOLVING WITH TRIGONOMETRY

If we are given a problem involving a triangle, we must first decide which rule is best to use.

If the triangle is right angled then the trigonometric ratios or Pythagoras' theorem can be used. For some problems we can add an extra line or two to the diagram to create a right angled triangle.

However, if we do not have a right angled triangle then we usually have to choose between the sine and cosine rules. In these cases the following checklist may be helpful:

Use the **cosine rule** when given:
- three sides
- two sides and an included angle.

Use the **sine rule** when given:
- one side and two angles
- two sides and a non-included angle.

In situations where either rule could be used to find an angle, the cosine rule should be used to avoid the ambiguous case.

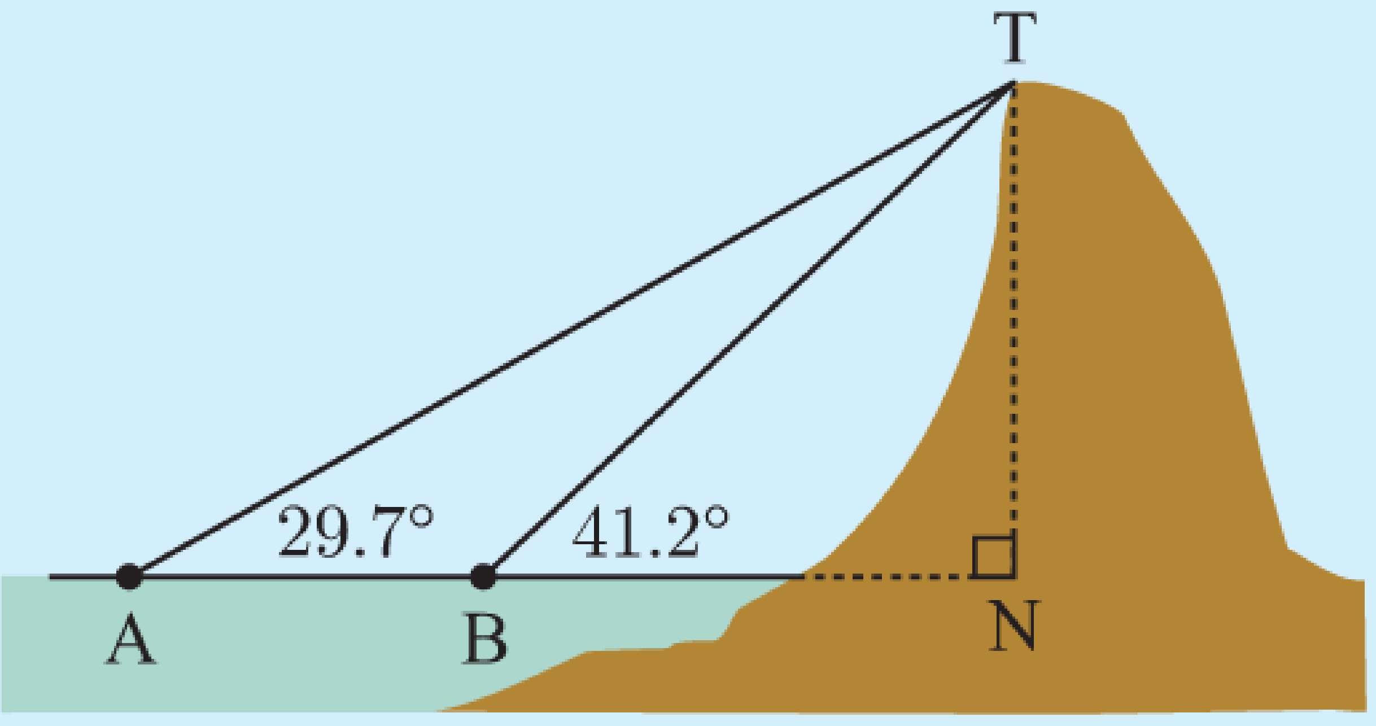
Example 8

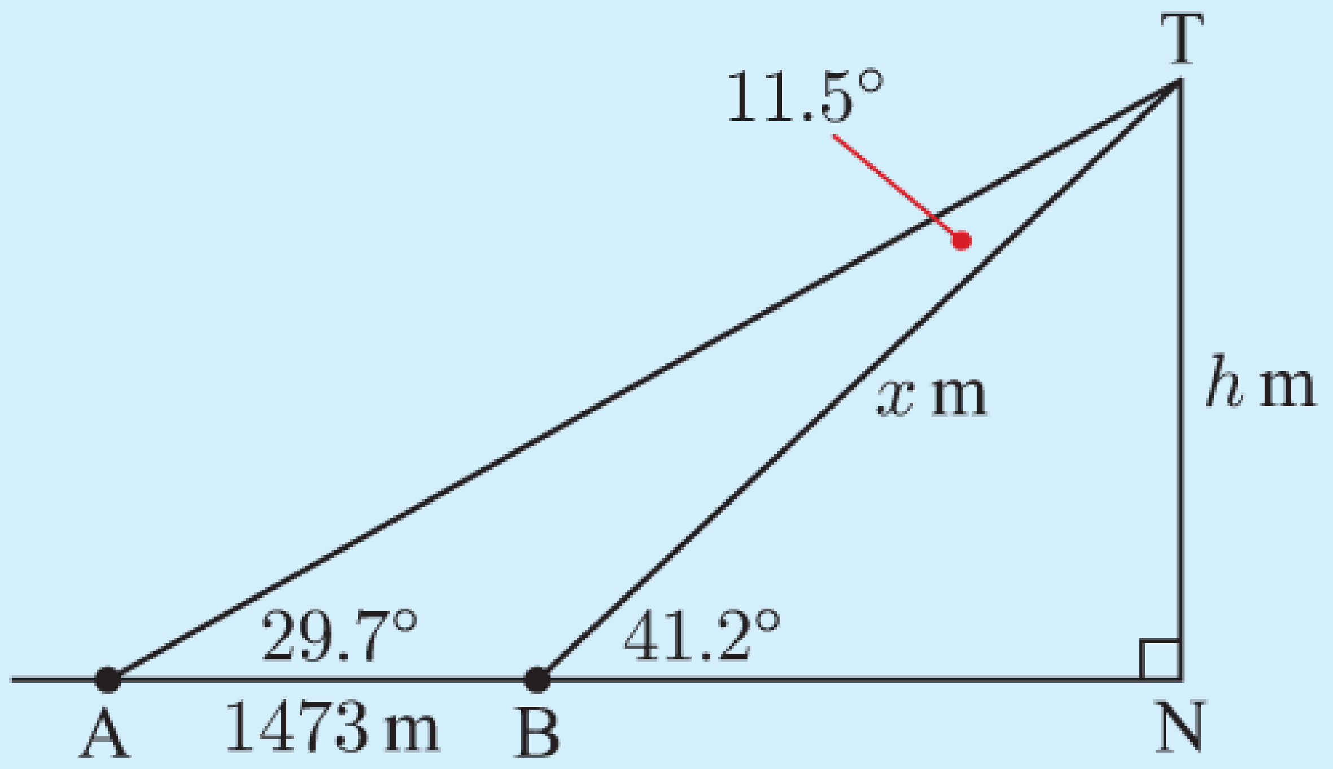
Self Tutor

The angles of elevation to the top of a mountain are measured from two beacons A and B at sea.

The measurements are shown on the diagram.

If the beacons are 1473 m apart, how high is the mountain?





Let BT be x m and NT be h m.

$$\begin{aligned} A\widehat{T}B &= 41.2° - 29.7° \quad \{\text{exterior angle of } \triangle\text{BNT}\} \\ &= 11.5° \end{aligned}$$

We find x in $\triangle$ABT using the sine rule:

$$\frac{x}{\sin 29.7°} = \frac{1473}{\sin 11.5°}$$

$$\therefore \quad x = \frac{1473}{\sin 11.5°} \times \sin 29.7°$$

$$\approx 3660.62$$

Now, in $\triangle$BNT, $\sin 41.2° = \frac{h}{x} \approx \frac{h}{3660.62}$

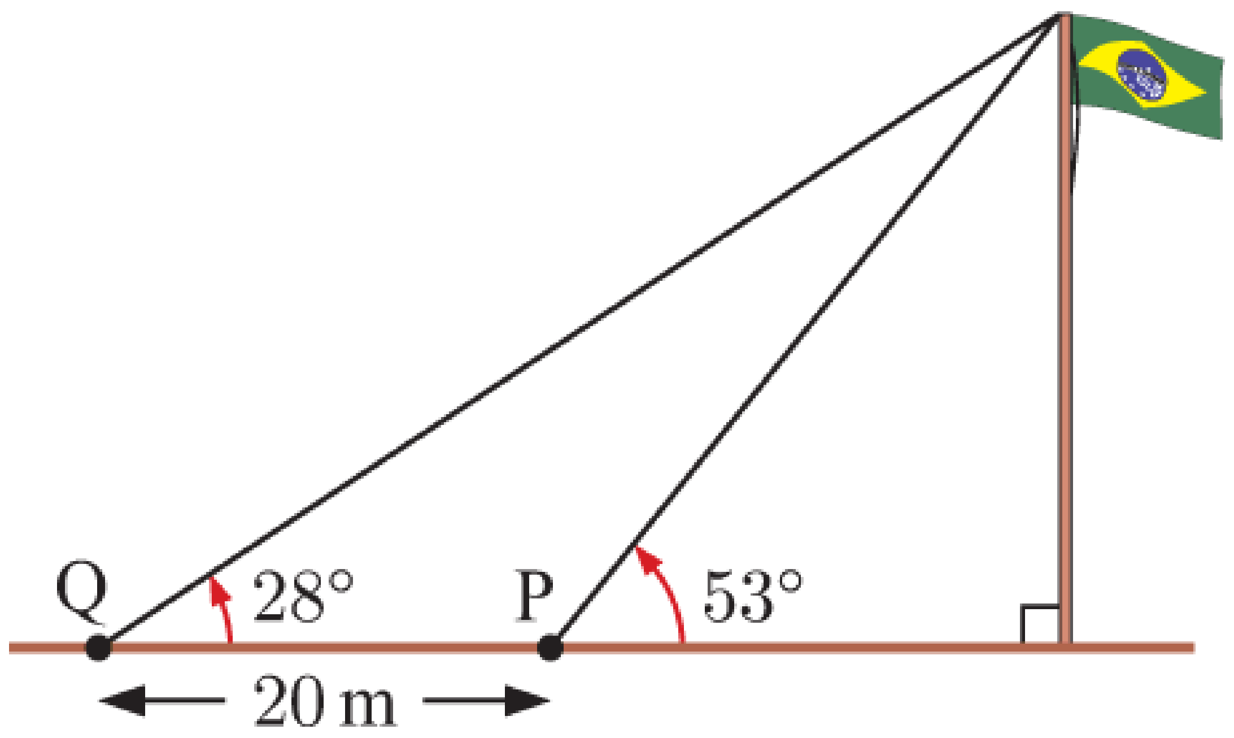
$$\therefore \quad h \approx \sin 41.2° \times 3660.62$$

$$\approx 2410$$

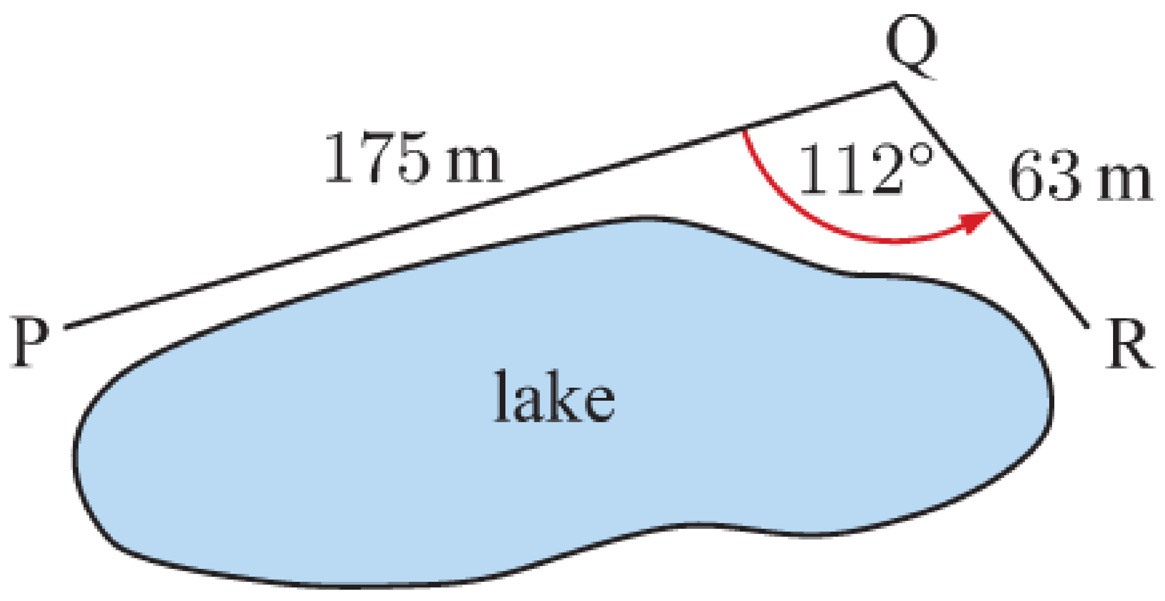
The mountain is about 2410 m high.

EXERCISE 9D

1 Rodrigo wishes to determine the height of a flagpole. He takes a sighting to the top of the flagpole from point P. He then moves 20 metres further away from the flagpole to point Q, and takes a second sighting. The information is shown in the diagram alongside. How high is the flagpole?

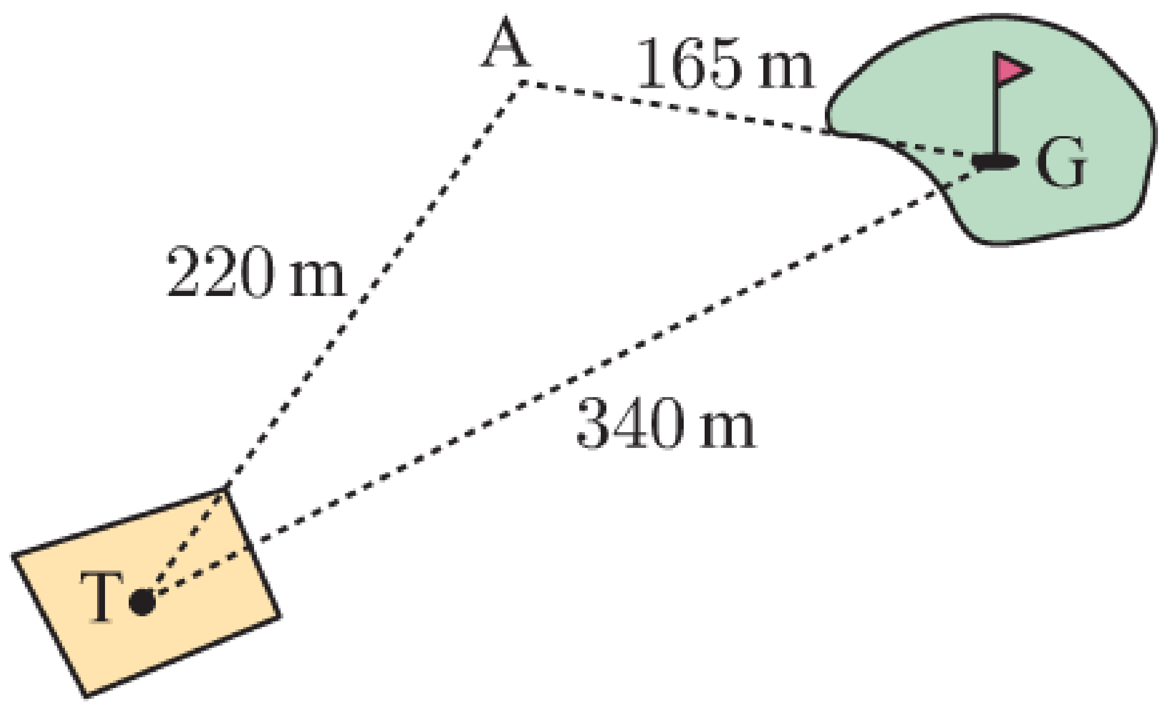


2 To get from P to R, a park ranger has to walk along a path to Q and then to R.
What is the distance in a straight line from P to R?



3 An orienteer runs for $4\frac{1}{2}$ km, then turns through an angle of 32° and runs for another 6 km. How far is she from her starting point?

4 A golfer played his tee shot a distance of 220 m to point A. He then played a 165 m six iron to the green. If the distance from tee to green is 340 m, determine the angle the golfer was off line with his tee shot.



5 A helicopter A observes two ships B and C. B is 23.8 km from the helicopter and C is 31.9 km from it. The angle of view $B\widehat{A}C$ from the helicopter to B and C, is 83.6°. How far are the ships apart?

6 Hikers Ritva and Esko leave point P at the same time. Ritva walks 4 km on the bearing 040°, then a further 6 km on the bearing 155° to get to their campsite.
Esko hikes to the camp site directly from P.

a How far does Esko hike?

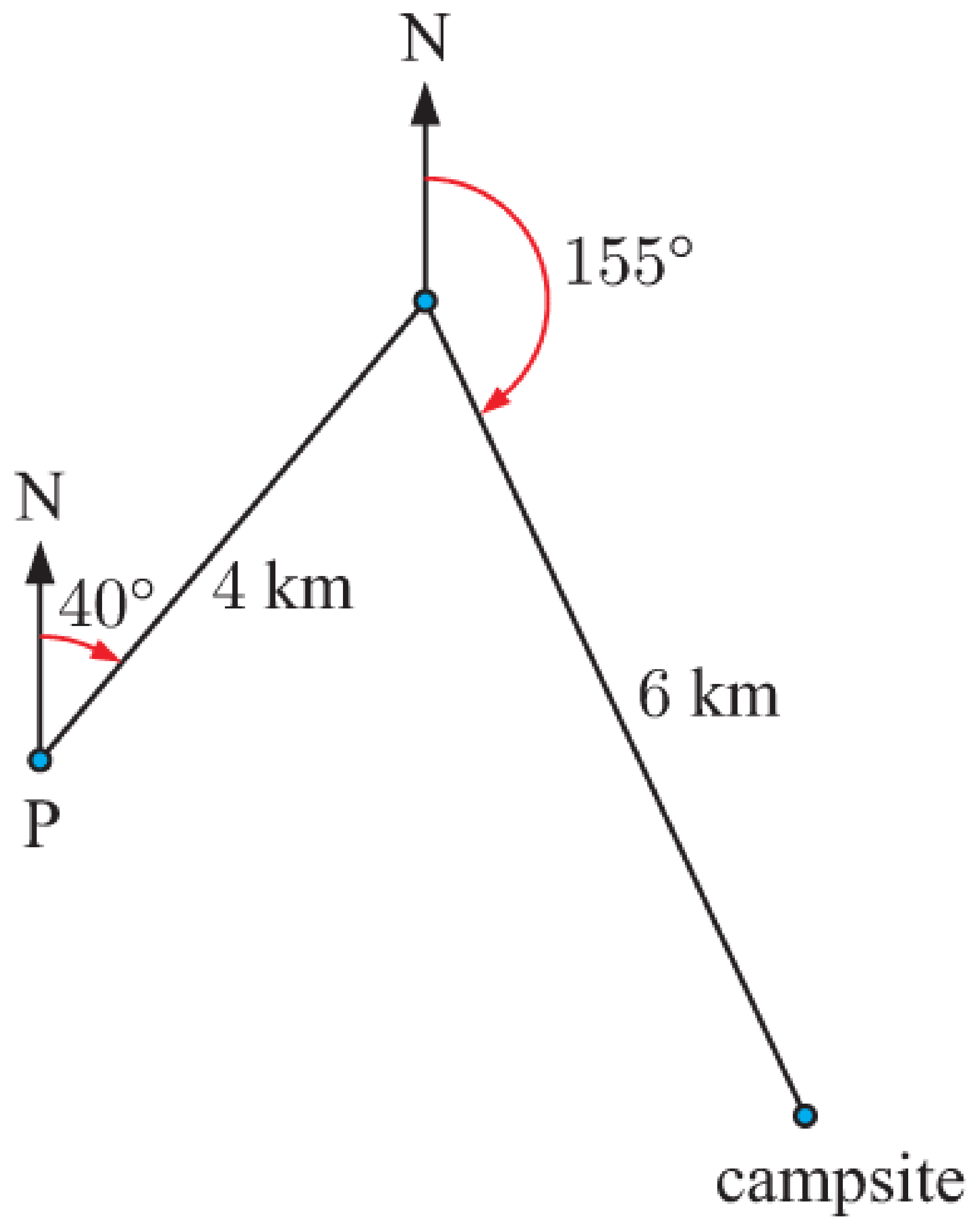
b In which direction does Esko hike?

c Ritva hikes at 5 km h^{-1} and Esko hikes at 3 km h^{-1}.

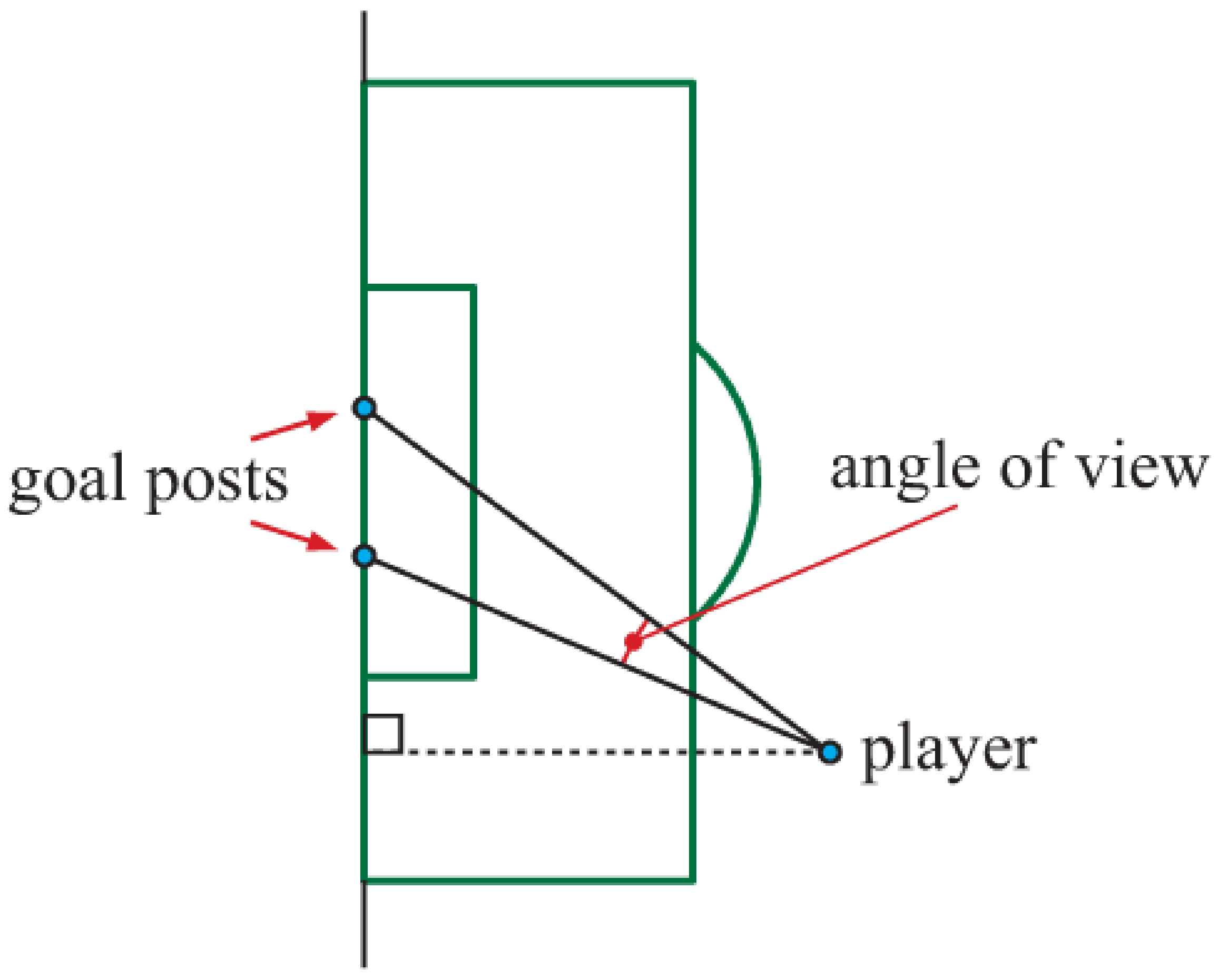
  i Who will arrive at the camp site first?

  ii How long will this person need to wait before the other person arrives?

d On what bearing should the hikers walk from the camp site to return directly to P?



7 A football goal is 5 metres wide. When a player is 26 metres from one goal post and 23 metres from the other, he shoots for goal. What is the angle of view of the goal that the player sees?

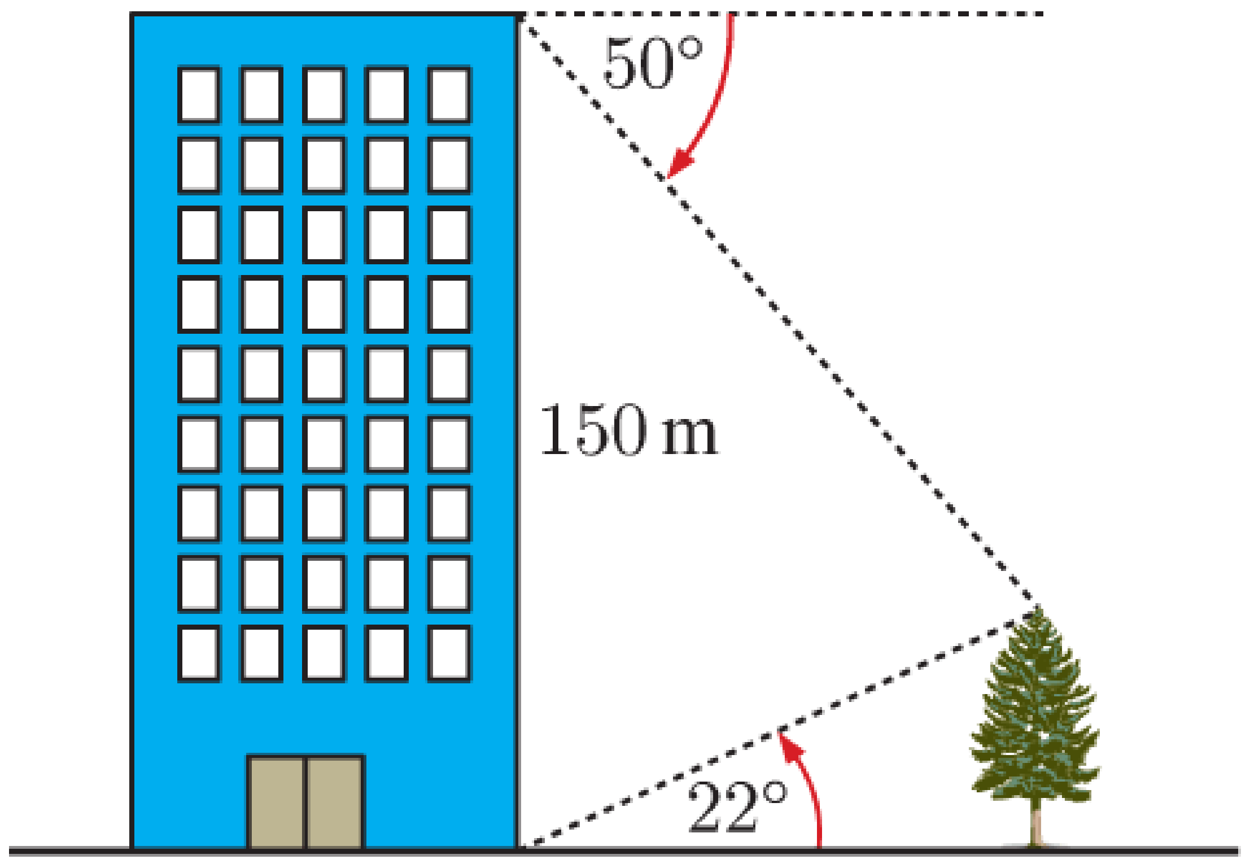


8 A tower 42 metres high stands on top of a hill. From a point some distance from the base of the hill, the angle of elevation to the top of the tower is 13.2°, and the angle of elevation to the bottom of the tower is 8.3°. Find the height of the hill.

9 From the foot of a building I have to look 22° upwards to sight the top of a tree. From the top of the building, 150 metres above ground level, I have to look down at an angle of 50° below the horizontal to sight the tree top.

a How high is the tree?

b How far from the building is this tree?

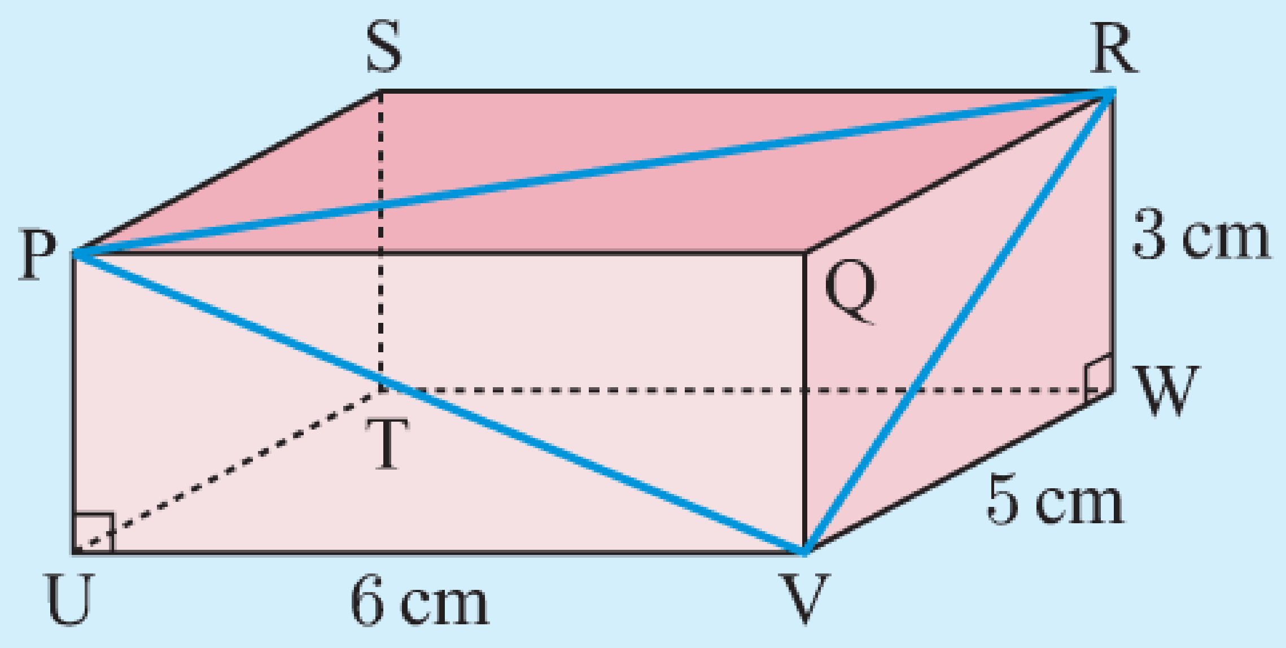


10 Two yachts are sailing close to a dangerous reef. The captain of the *Porpoise* notices a lighthouse 2.4 km away on the bearing 223°. The captain of the *Queen Maria* measures the lighthouse as 2.1 km away. He also observes the *Porpoise* to the right of the lighthouse, with an angle of 53° between them.

- **a** Display this information on a diagram.
- **b** Find the distance between the yachts.
- **c** Find the bearing of the *Queen Maria* from the *Porpoise*.

Example 9 **Self Tutor**

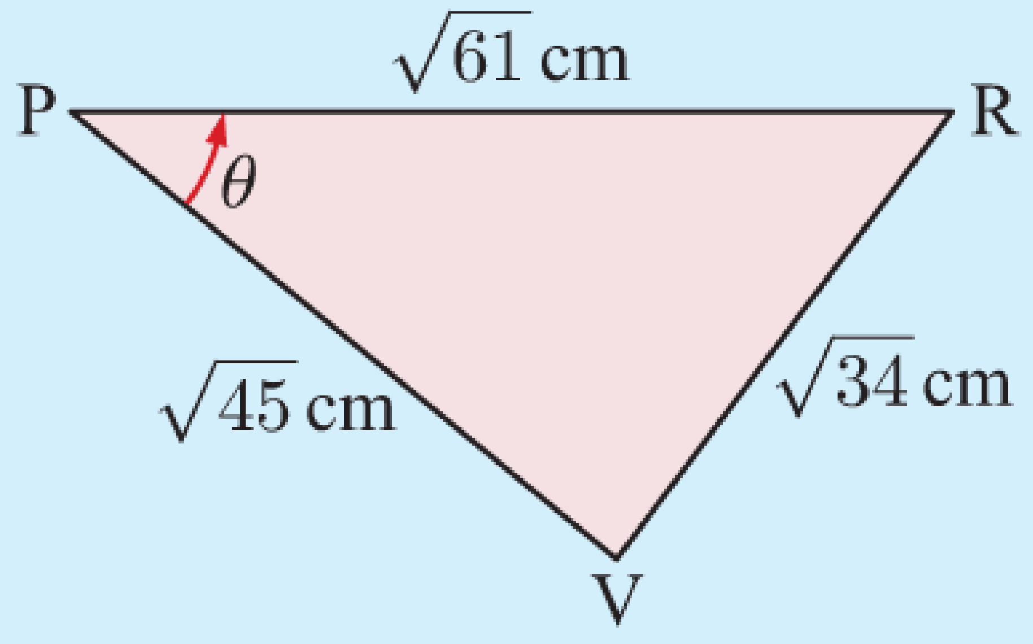
Find the measure of $R\widehat{P}V$.



In $\triangle$RVW, $RV = \sqrt{5^2 + 3^2} = \sqrt{34}$ cm. {Pythagoras}

In $\triangle$PUV, $PV = \sqrt{6^2 + 3^2} = \sqrt{45}$ cm. {Pythagoras}

In $\triangle$PQR, $PR = \sqrt{6^2 + 5^2} = \sqrt{61}$ cm. {Pythagoras}



By rearrangement of the cosine rule,

$$\cos\theta = \frac{(\sqrt{61})^2 + (\sqrt{45})^2 - (\sqrt{34})^2}{2\sqrt{61}\sqrt{45}}$$

$$= \frac{61 + 45 - 34}{2\sqrt{61}\sqrt{45}}$$

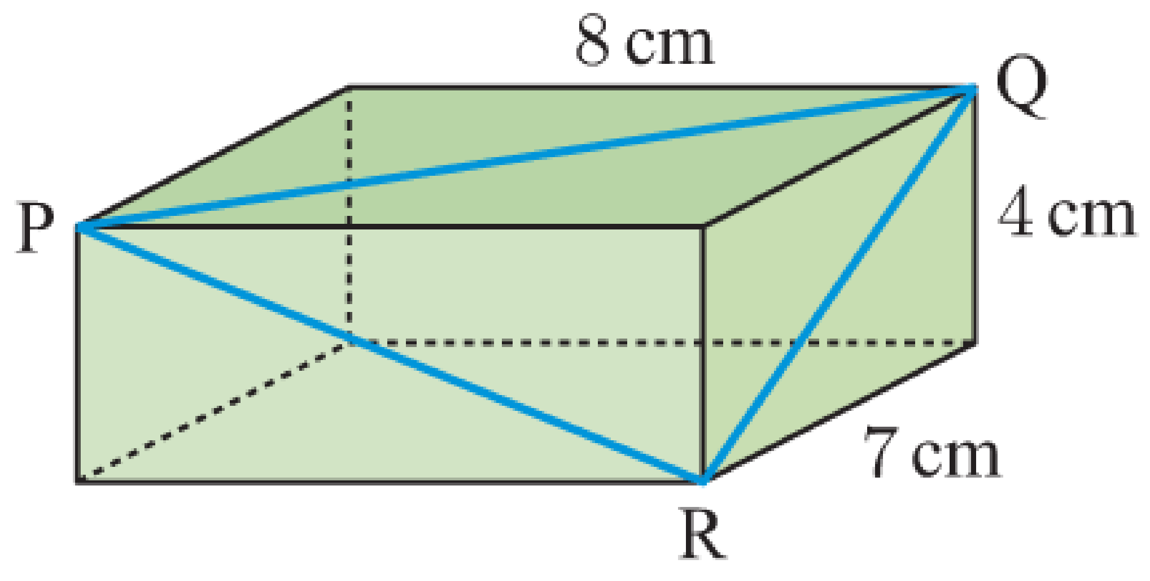
$$= \frac{72}{2\sqrt{61}\sqrt{45}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{36}{\sqrt{61}\sqrt{45}}\right) \approx 46.6°$$

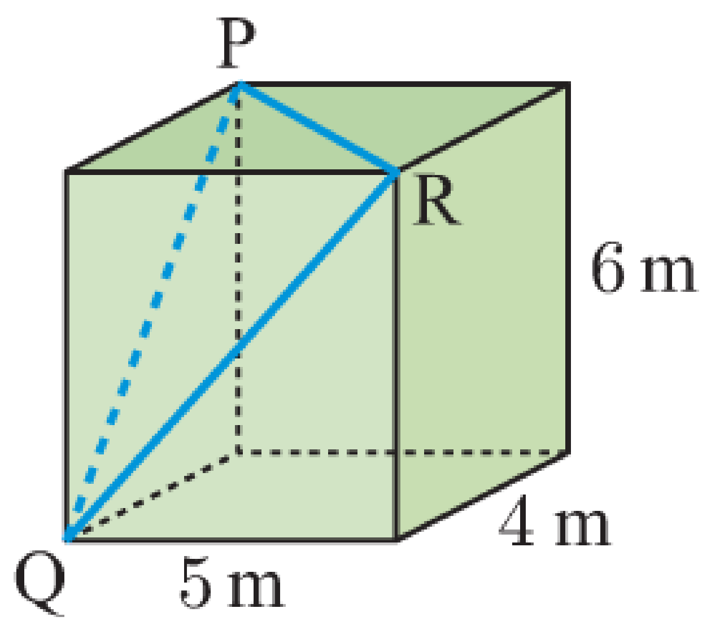
$\therefore$ $R\widehat{P}V$ measures about 46.6°.

11 Find the measure of $P\widehat{Q}R$:

a



b



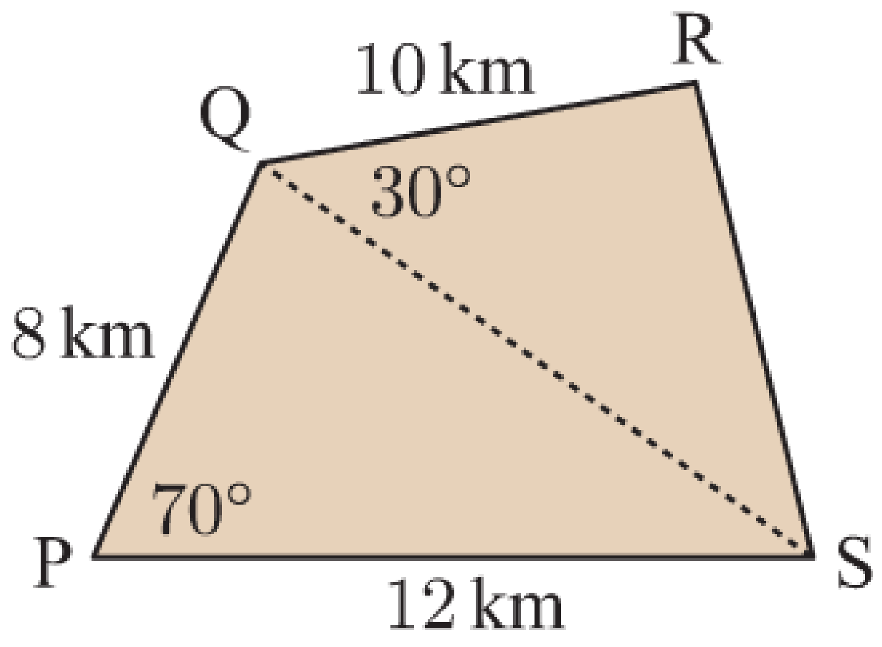
12 Two observation posts A and B are 12 km apart. A third observation post C is located 15 km from A such that $C\widehat{B}A$ is 67°. Find the measure of $C\widehat{A}B$.

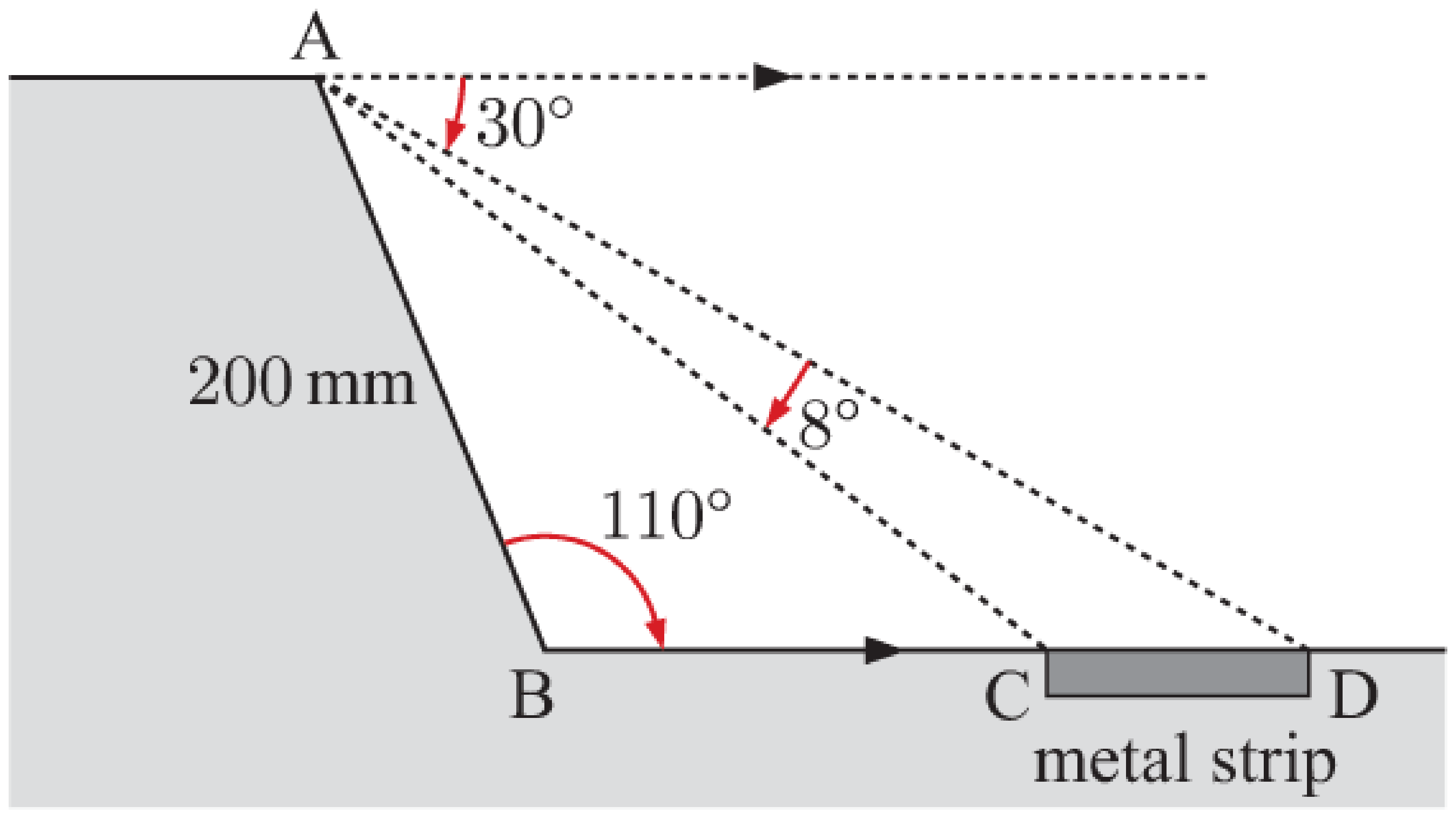
13 Thabo and Palesa start at point A. They each walk in a straight line at an angle of 120° to one another. Thabo walks at 6 km h^{-1} and Palesa walks at 8 km h^{-1}. How far apart are they after 45 minutes?

14 Stan and Olga are considering buying a sheep farm. A surveyor has supplied them with the given accurate sketch. Find the area of the property, giving your answer in:

a km^2

b hectares.

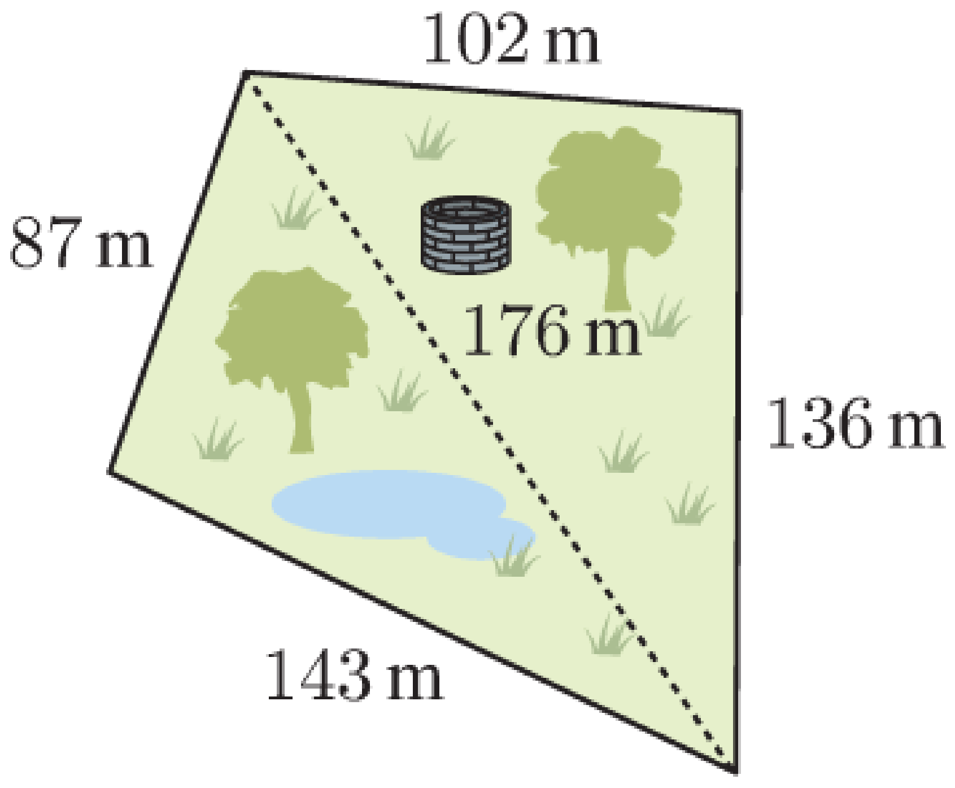


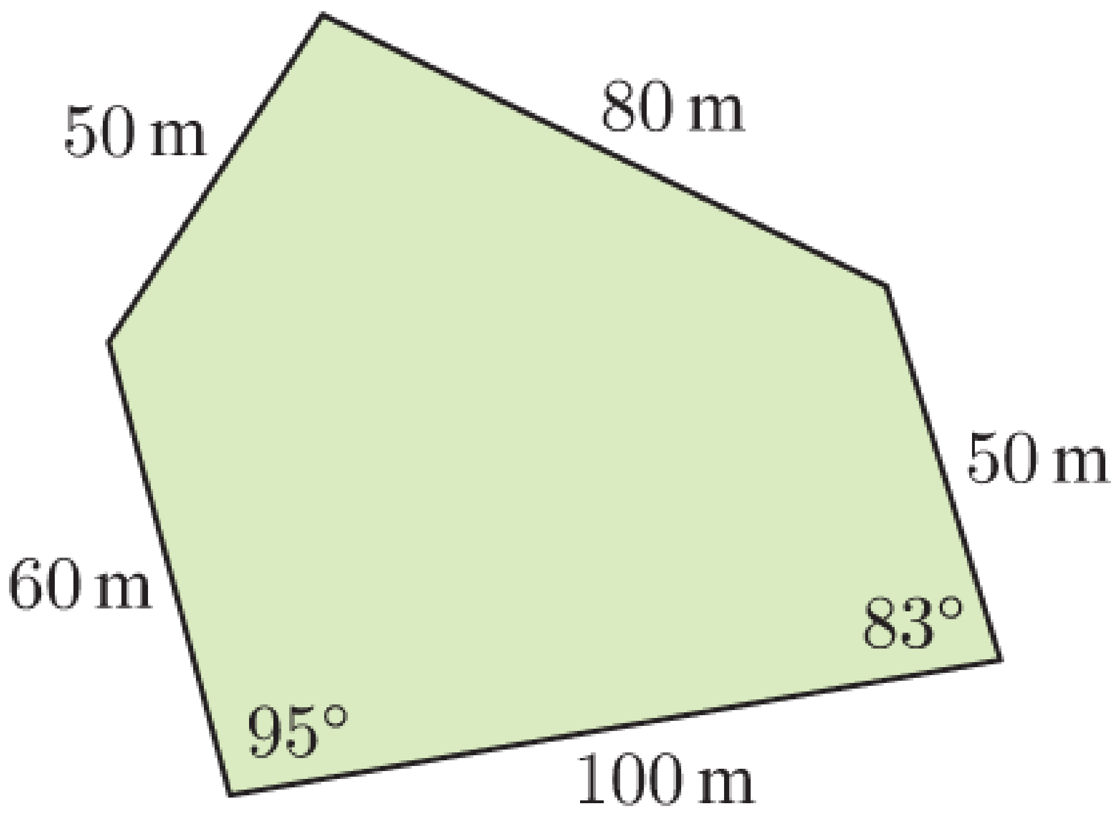
15 

The cross-section design of the kerbing for a driverless-bus roadway is shown opposite. The metal strip is inlaid into the concrete and is used to control the direction and speed of the bus. Find the width of the metal strip.

16 Robert from the City Council has made some measurements of the park in the **Opening Problem**. They are summarised in the diagram.

Find the area of the park and the angle at each corner.



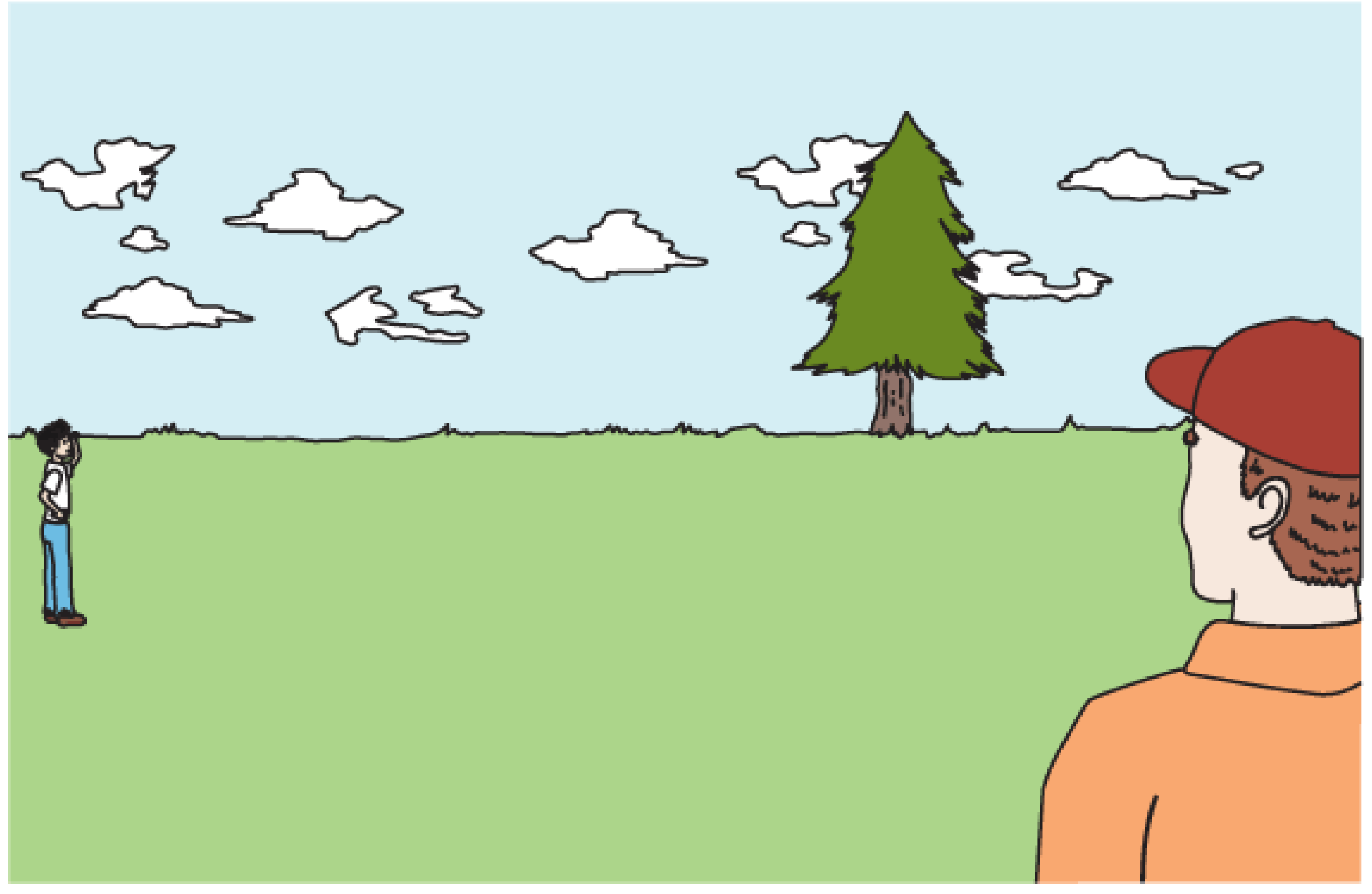
17 

A surveyor has produced this plan of a property. Find its area.

18 A Chinese restaurant and a fish and chip shop are 7 km apart. The Chinese restaurant offers free delivery within 5 km, and the fish and chip shop offers free delivery within 3 km. Find the area of the region which receives free delivery from both locations.

19 One angle of a triangular garden measures 60°. The garden has perimeter 36 m and area $30\sqrt{3}$ m^2. Find the measure of the remaining two angles of the garden.

20 Sam and Markus are standing on level ground 100 metres apart. A large tree is due north of Markus and on the bearing 065° from Sam. The top of the tree appears at an angle of elevation of 25° to Sam and 15° to Markus. Find the height of the tree.



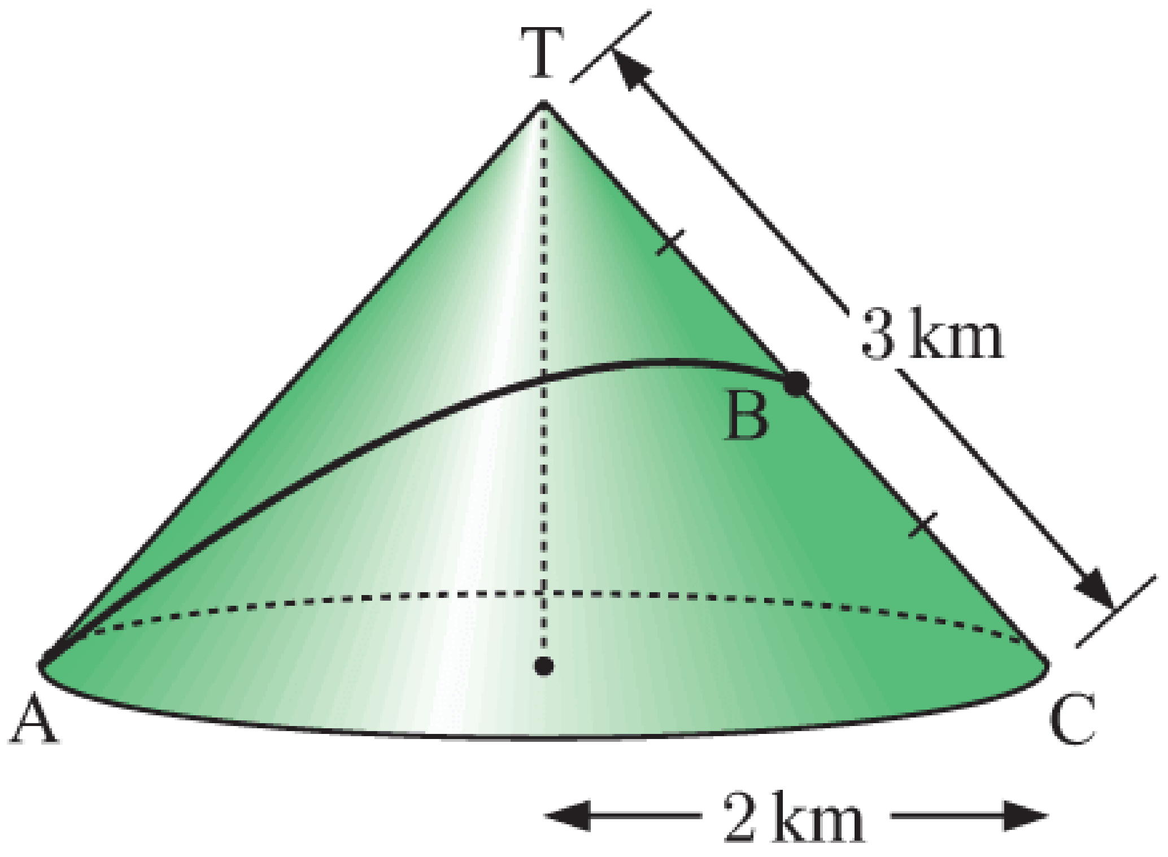
21 A mountain is a perfect cone with base radius 2 km and slant height 3 km.

From the southernmost point A on the base, a path leads up around the side of the mountain to B, the point on the northern slope which is halfway up the slope from the base C. A and C are diametrically opposite.

The path leading from A to B is the shortest possible distance from A to B around the mountainside. Find:

a the length of the path from A to B

b the length of the part of the path from B to the point where the path is horizontal.

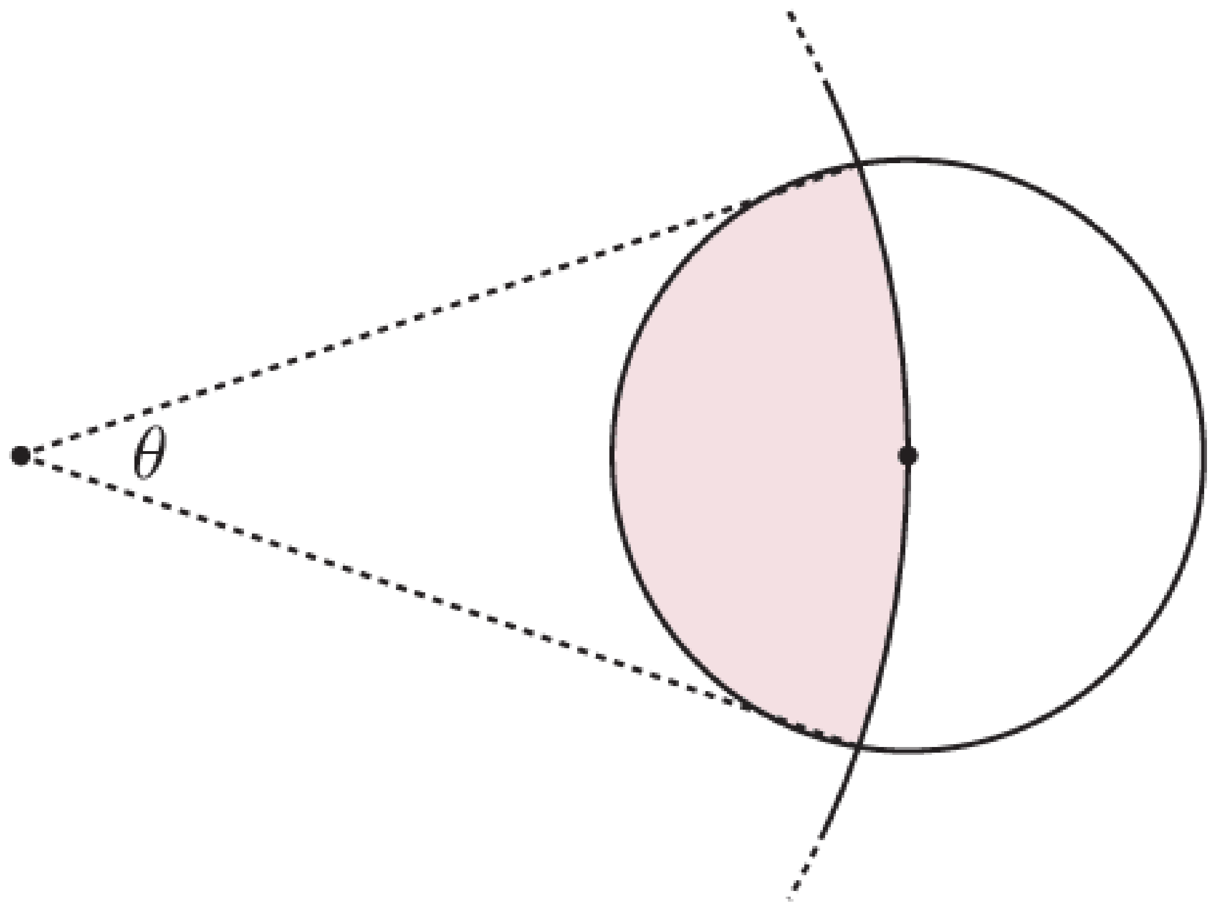


22 Two circles have radii in the ratio 3 : 1. The larger circle passes through the centre of the smaller circle as shown.

The shaded area is π cm^2.

a Show that $\theta = 4\sin^{-1}\left(\frac{1}{6}\right)$.

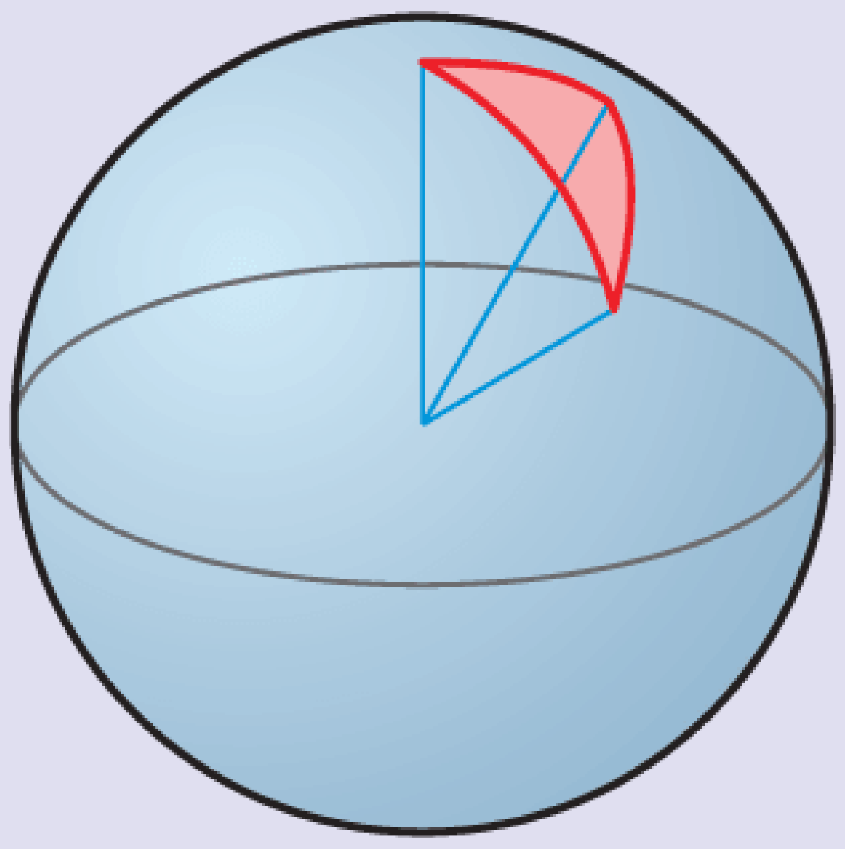
b Find, to 3 decimal places, the radius of the smaller circle.



THEORY OF KNOWLEDGE

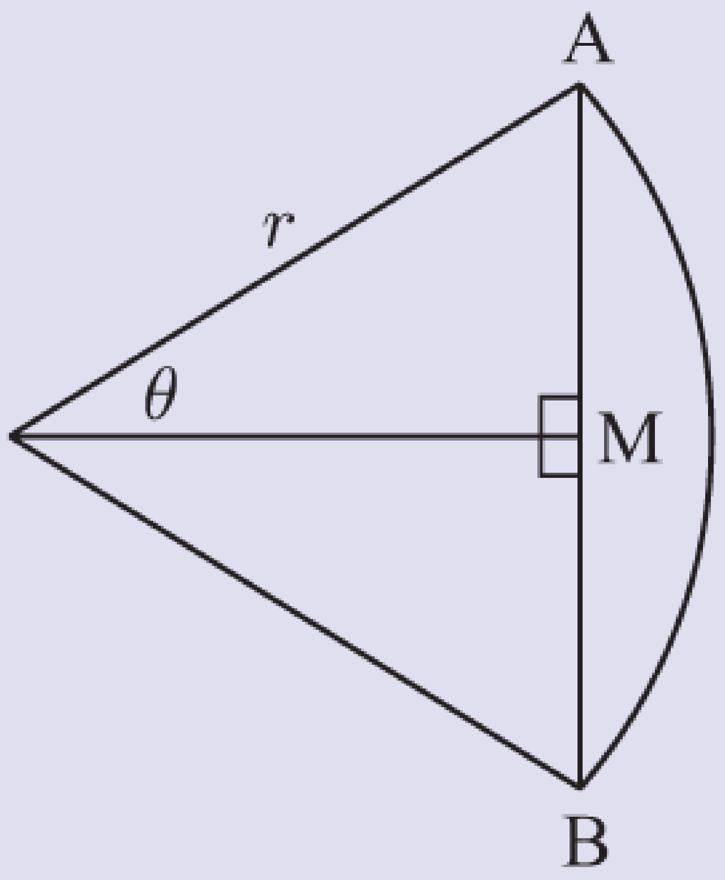
Trigonometry appears to be one of the most useful disciplines of mathematics. Its study has been driven by the need to solve real-world problems throughout history.

The study of trigonometry began when Greek, Babylonian, and Arabic astronomers needed to calculate the positions of stars and planets. These early mathematicians considered the trigonometry of spherical triangles, which are triangles on the surface of a sphere formed by three great arcs.



Trigonometric functions were developed by Hipparchus around 140 BC, and then by Ptolemy and Menelaus around 100 AD.

Around 500 AD, Hindu mathematicians published a table called the *Aryabhata*. It was a table of lengths of half chords, which are the lengths AM $= r\sin\theta$ in the diagram. This is trigonometry of triangles in a plane, as we study in schools today.



1 How does society and culture affect mathematical knowledge?

2 Should congruence and similarity, or the work of Pythagoras, be considered part of modern trigonometry?

3 A man walks south from the North Pole and turns 90° left when he reaches the equator. He walks for a while and then turns 90° left to walk back to the North Pole.

  a Has the man walked in a triangle?

  b Is the angle sum of a triangle always equal to 180°?

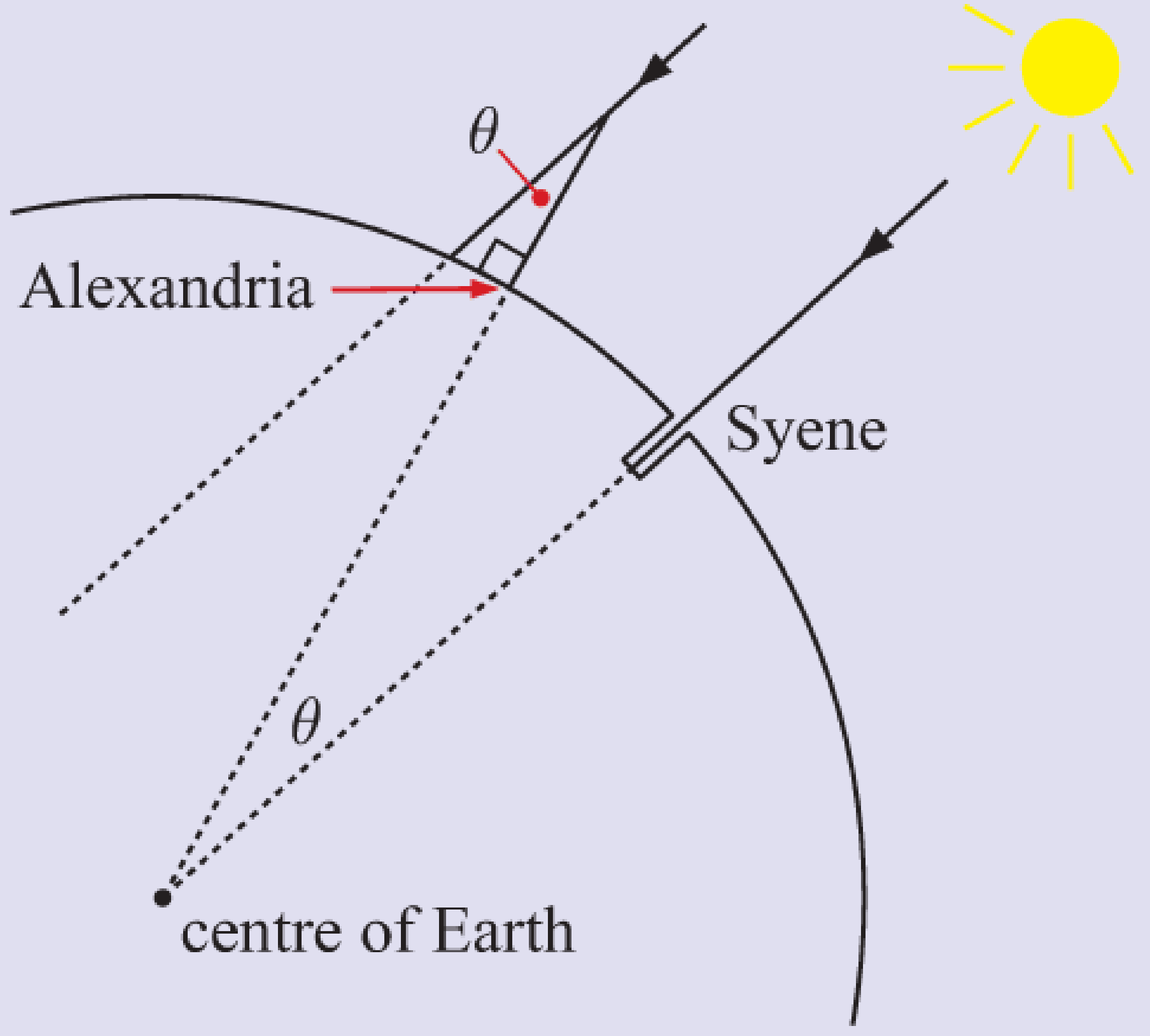
4 Is a spherical triangle more or less complicated than a triangle in a plane?

5 How does the research of the ancient astronomers relate to modern problems of satellites, telecommunications, and GPS navigation?

Erathosthenes of Cyrene (276 BC - 194 BC) deduced that the Earth is round using two observations made at noon on the summer solstice:

- In Syene (now Aswan, Egypt), sunlight illuminated the bottom of a well.
- In Alexandria, 5000 stadia ($\approx$ 880 km) away, a perfectly vertical rod cast a shadow.

Using the lengths of the rod and its shadow, Erathosthenes calculated the angular separation of Syene and Alexandria. He hence calculated the circumference of the Earth within 10% of its actual value.

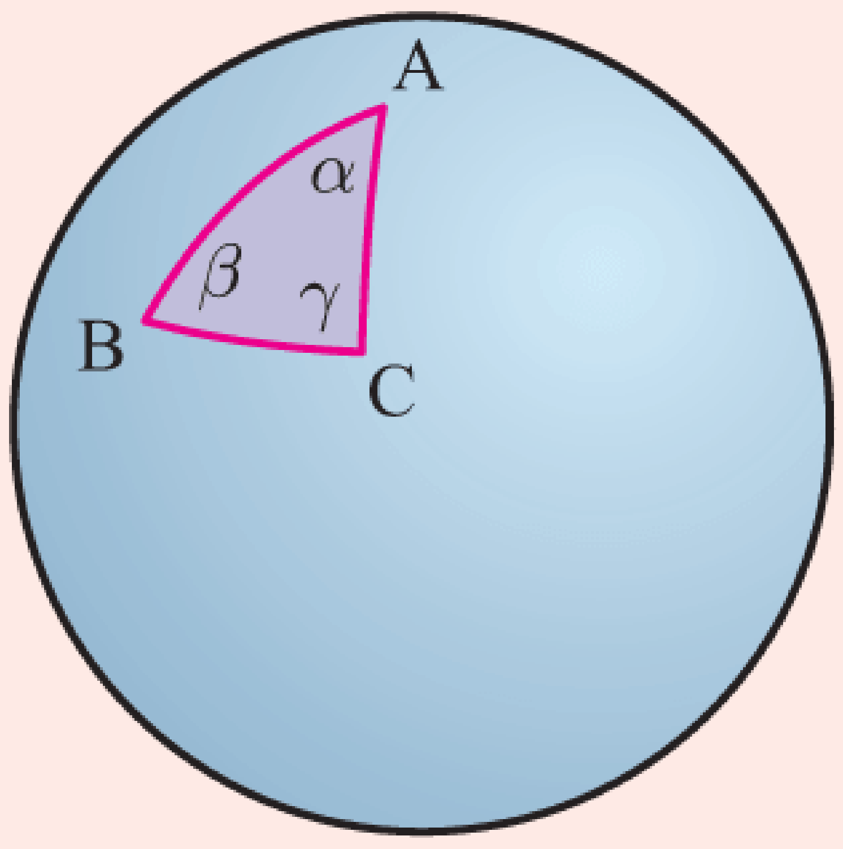


6 Why did a "flat Earth" theory persist for so long, despite ancient astronomers knowing the Earth was round? What lessons can be learned from this in protecting and promoting knowledge?

ACTIVITY

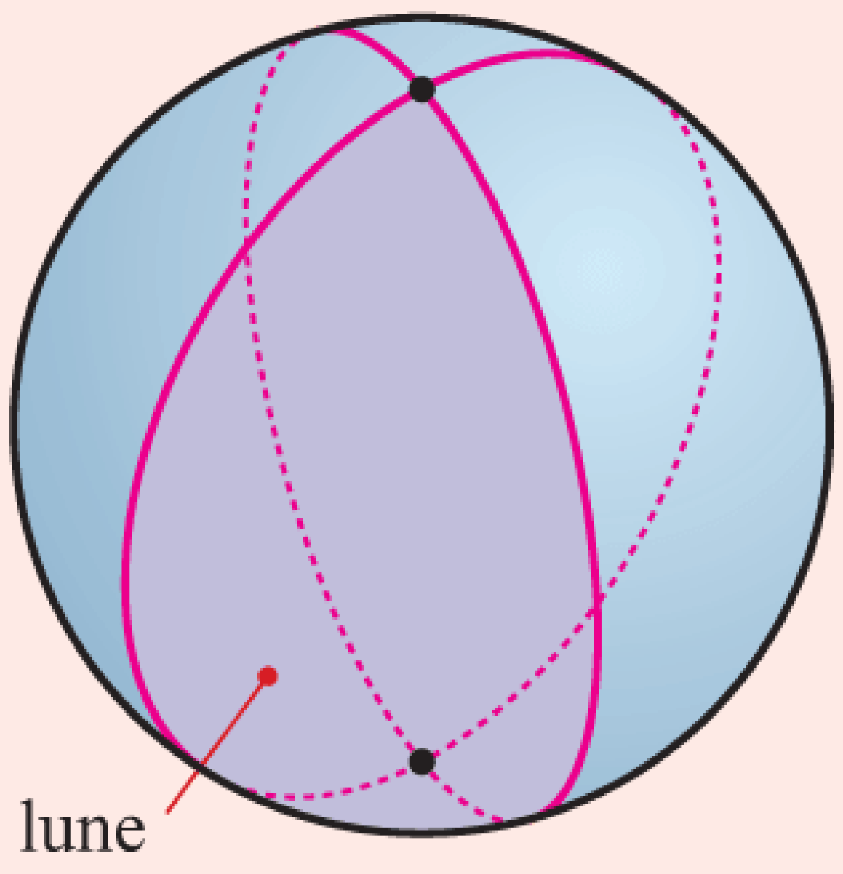
THE AREA OF A SPHERICAL TRIANGLE

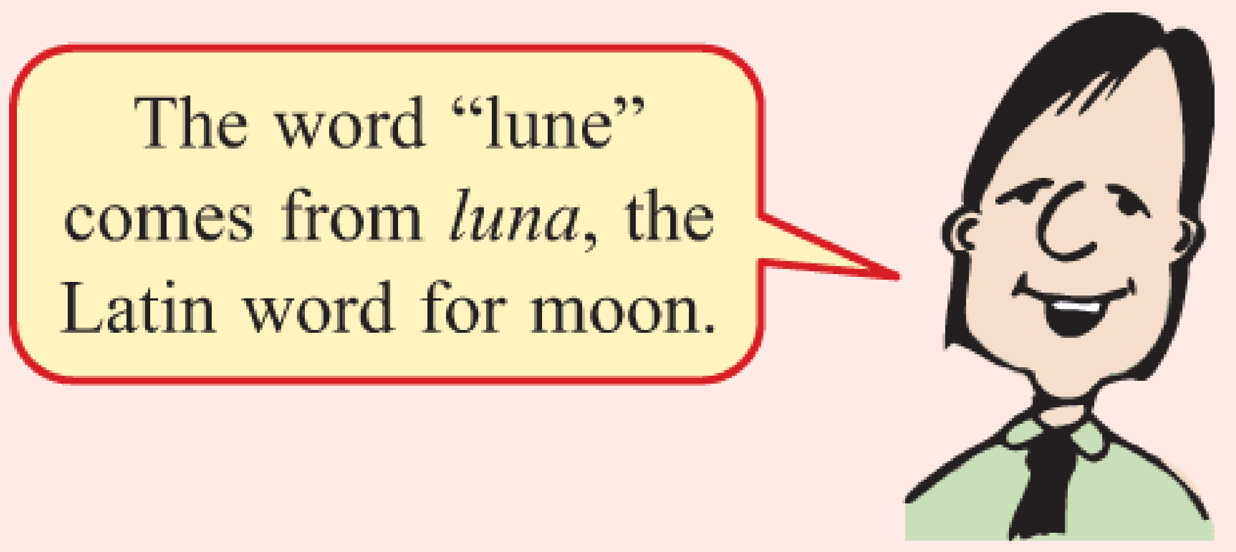
In this Activity we derive a formula for the area of a spherical triangle ABC with angles α, β, and γ.



To achieve this we need some definitions:

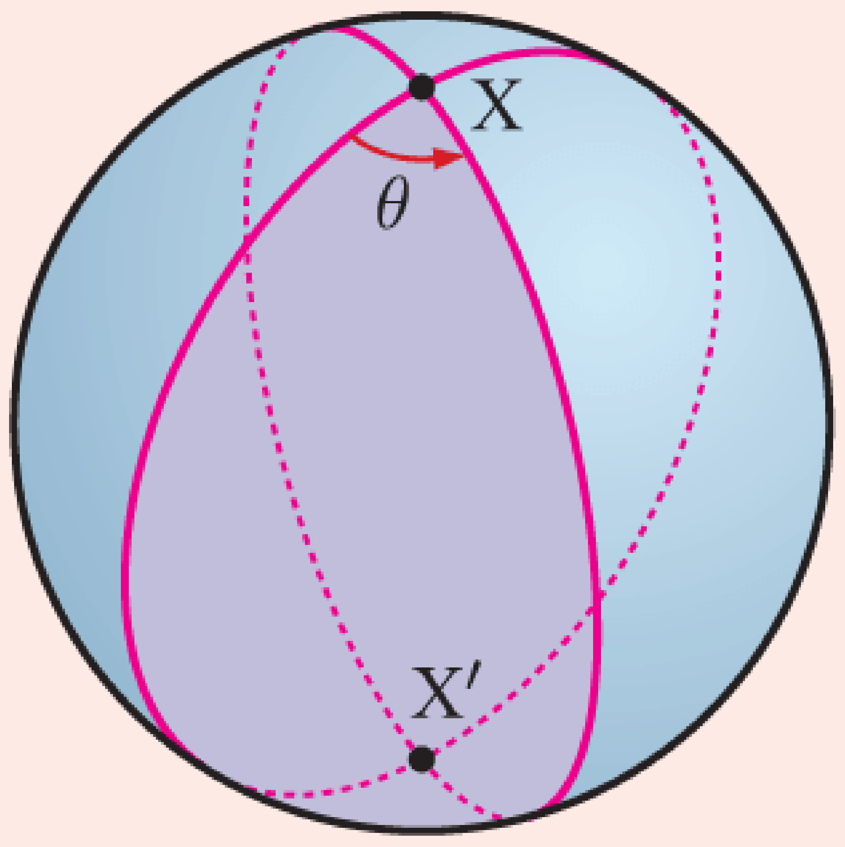
- A **great circle** is any circle drawn on a sphere whose centre is the centre of the sphere.
- A **spherical lune** is an area on a sphere bounded by two half great circles which meet at diametrically opposite points.





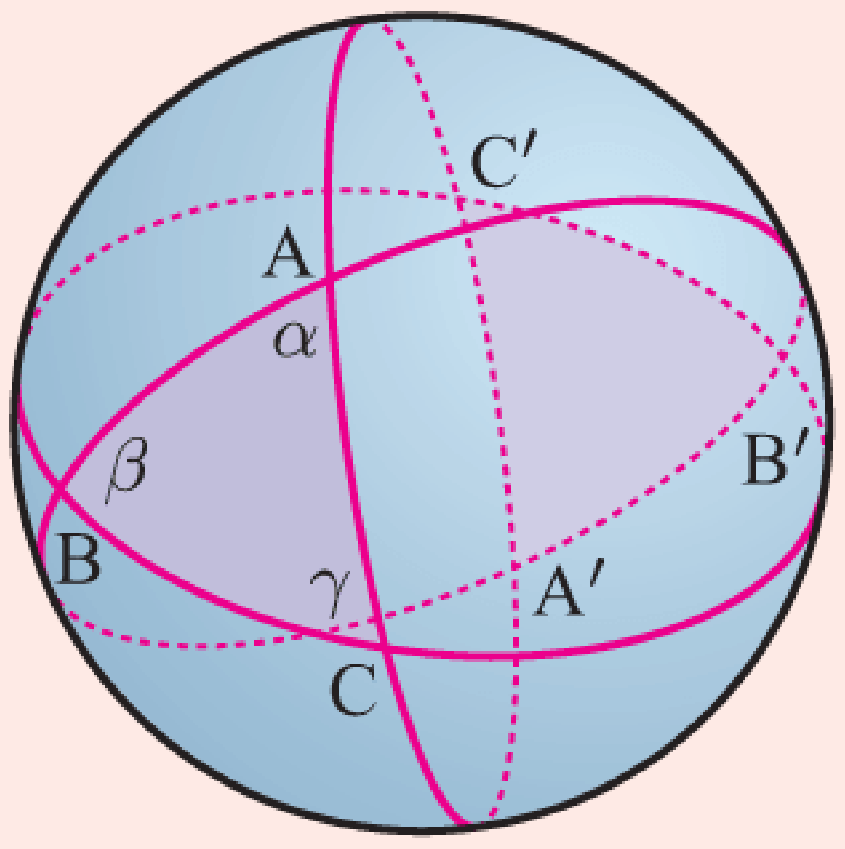
What to do:

1 Two great circles on a sphere with radius r meet at X and X′. They form a lune with angle θ as shown. Explain why the surface area $S_{\text{X},\,\theta}$ of the lune is given by $S_{\text{X},\,\theta} = 2r^2\theta$.



2 Consider a spherical triangle ABC with angles α, β, and γ. Suppose the arcs AB, AC, and BC are extended to form great circles.

- **a** Explain why the great circles will form a congruent spherical triangle A′B′C′ on the other side of the sphere.
- **b** By considering the areas of lunes, explain why
$$4\pi r^2 = 2S_{\text{A},\,\alpha} + 2S_{\text{B},\,\beta} + 2S_{\text{C},\,\gamma} - 4A$$
where A is the area of the spherical triangle ABC.
- **c** Hence show that the area of the spherical triangle is given by $A = (\alpha + \beta + \gamma - \pi)r^2$.



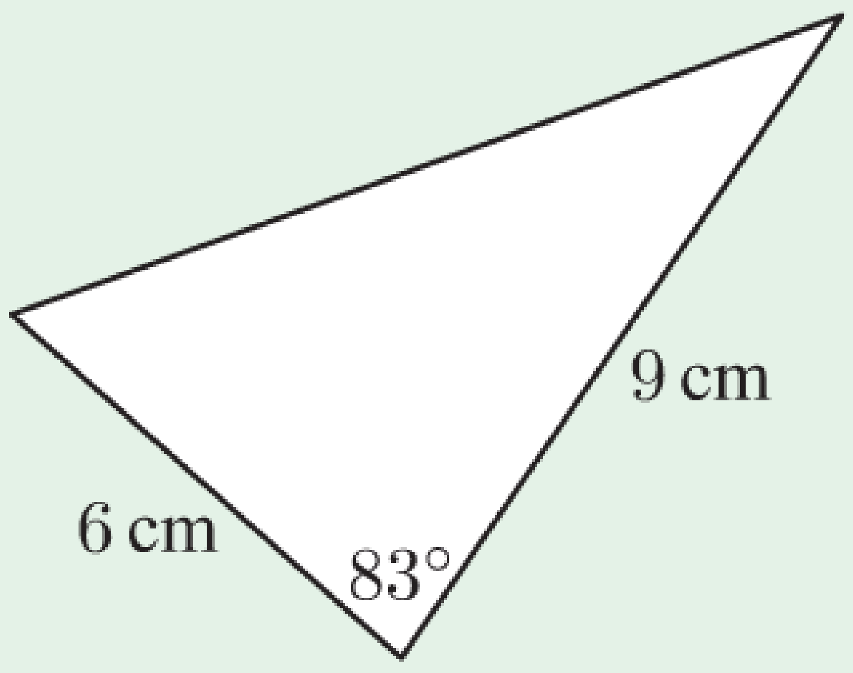
3 Explain why the area formula verifies that the angle sum of a spherical triangle is greater than 180°.

4 Is it possible for two spherical triangles on a given sphere to be similar but not congruent? Explain your answer.

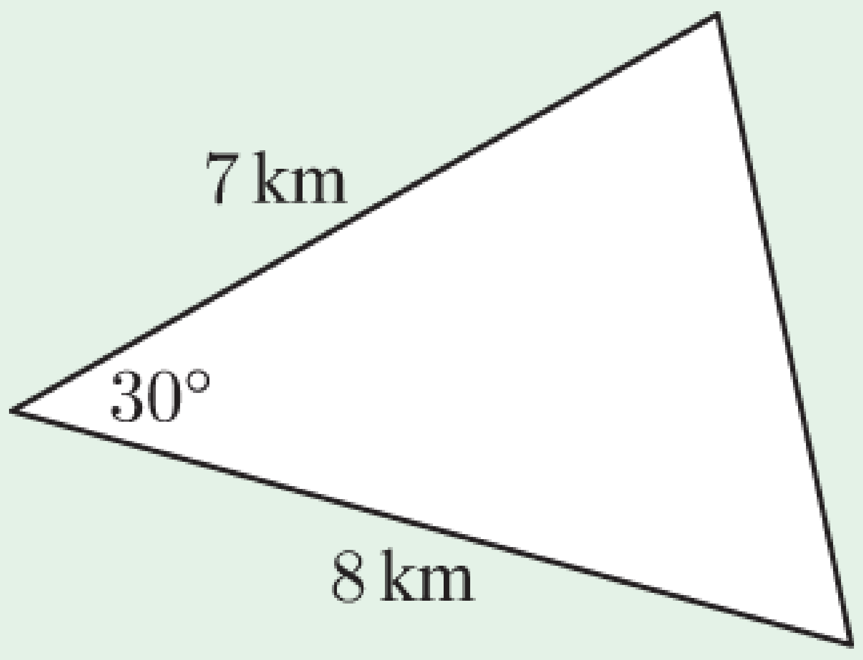
REVIEW SET 9A

1 Find the area of:

a



b



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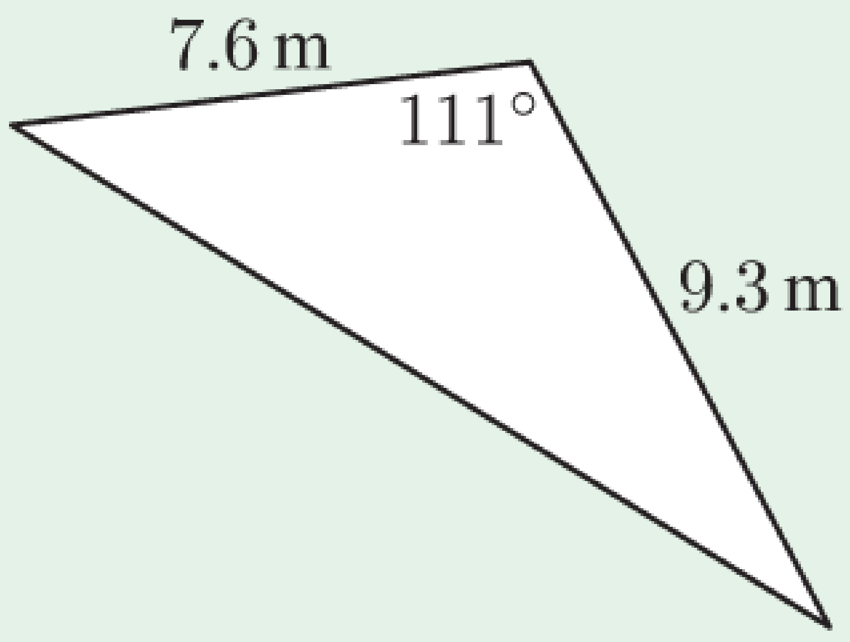
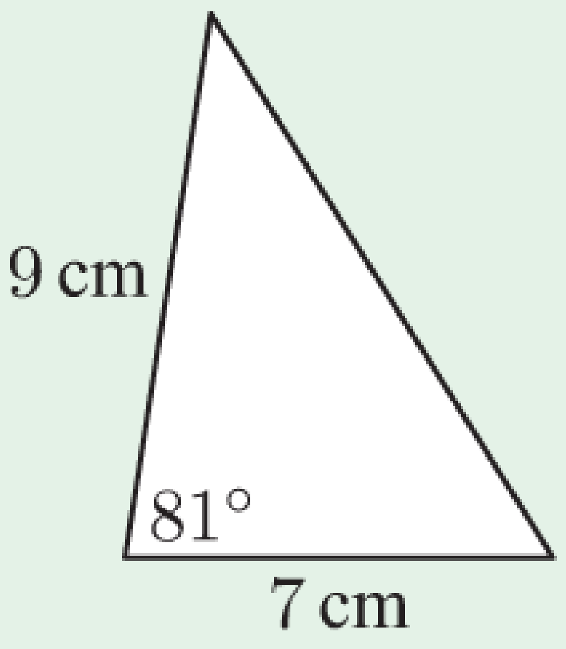


2 A rhombus has sides of length 5 cm and an angle of 65°. Find its area.

3 Find the length of the remaining side in each triangle:

a



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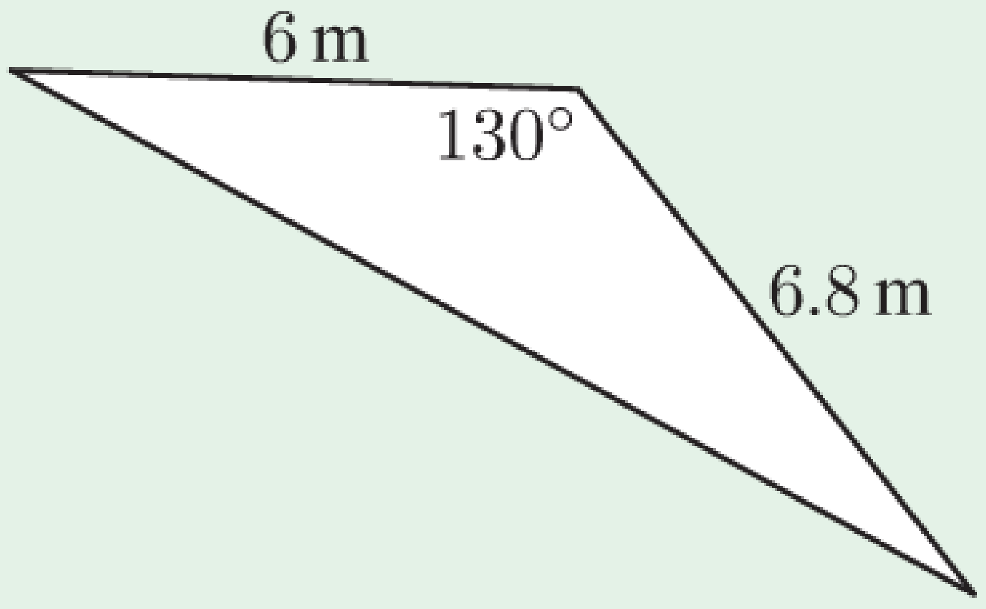
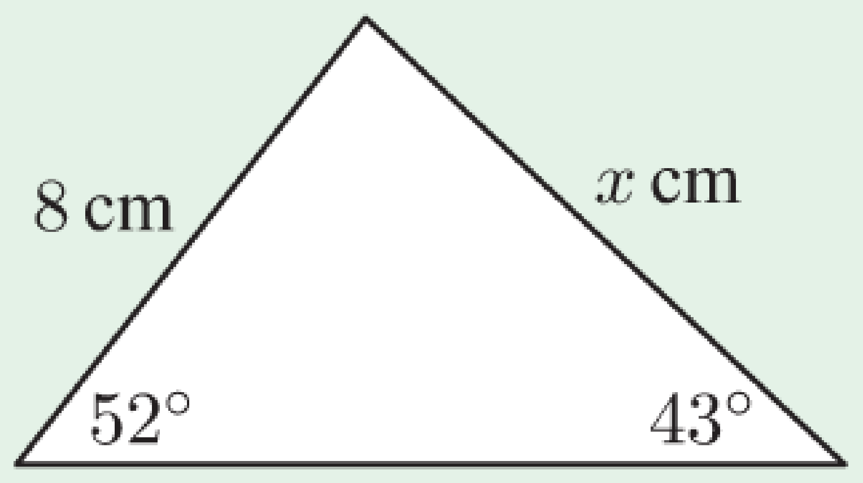
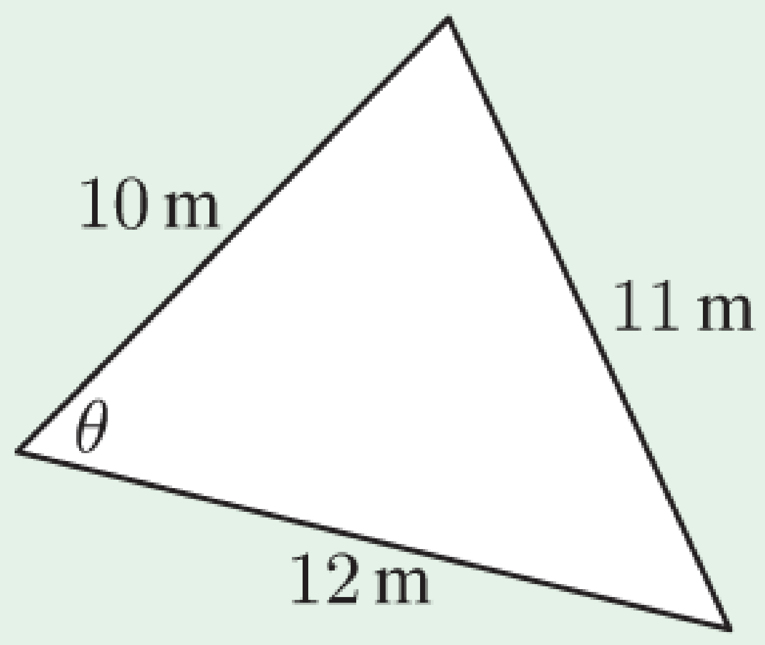


4 Find the unknown in:

a



b



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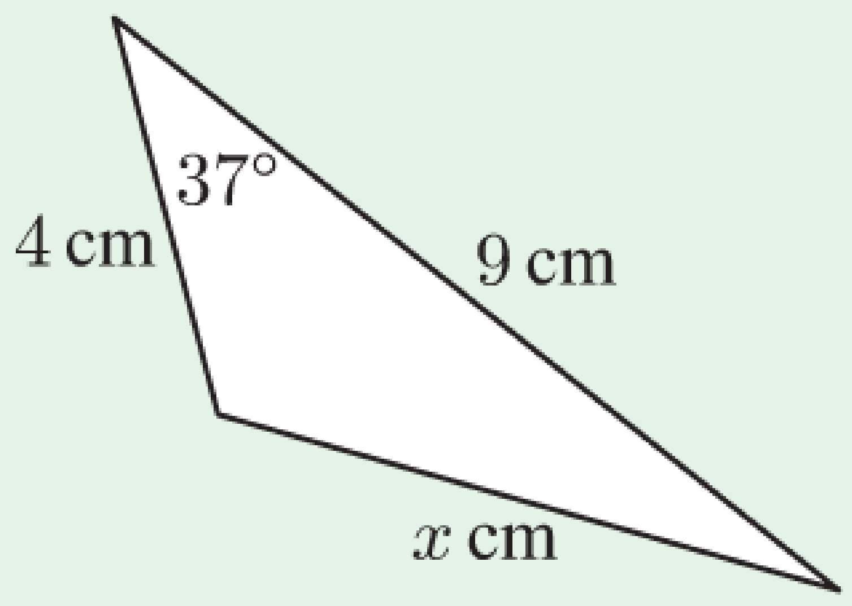
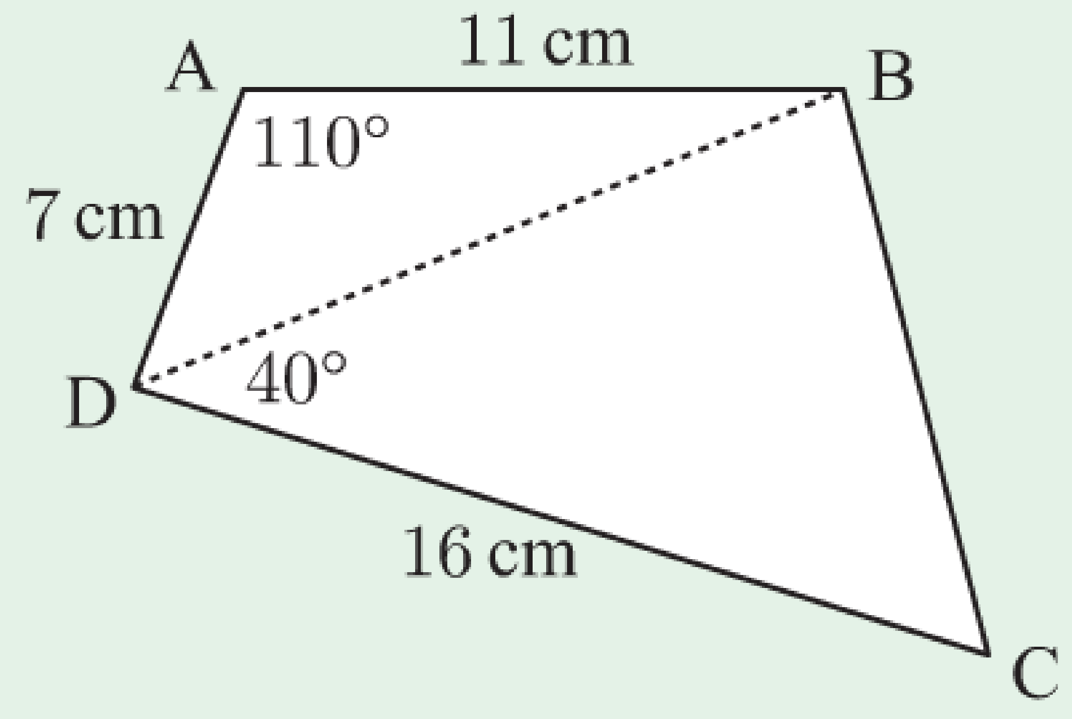
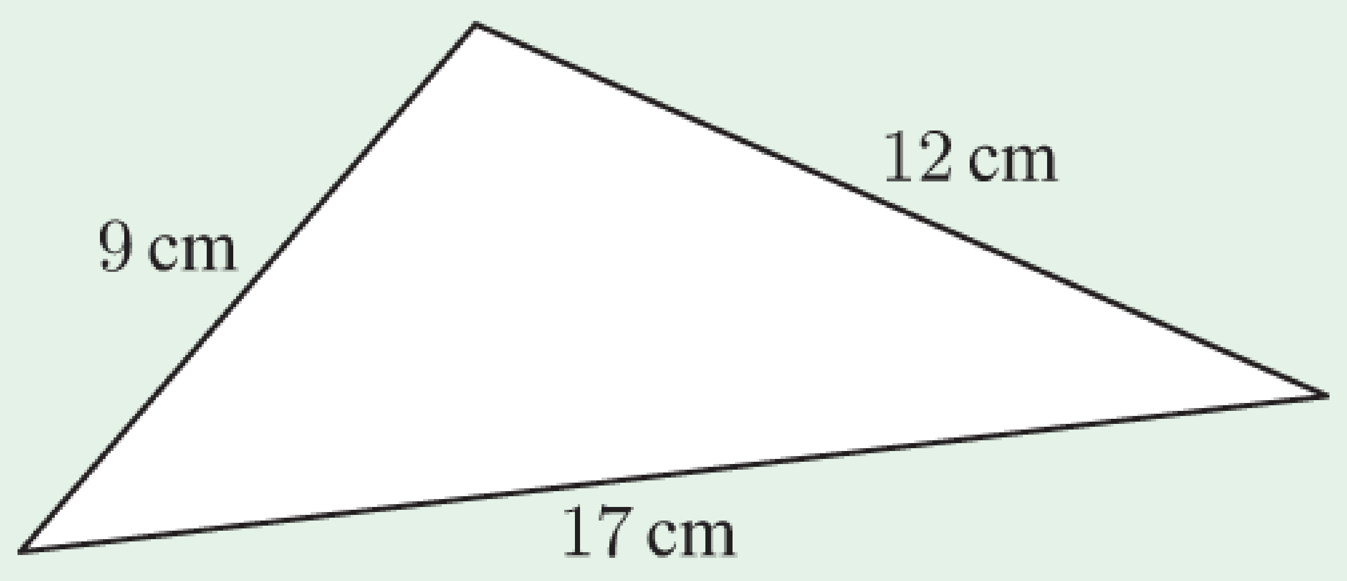


5 Find the area of quadrilateral ABCD:



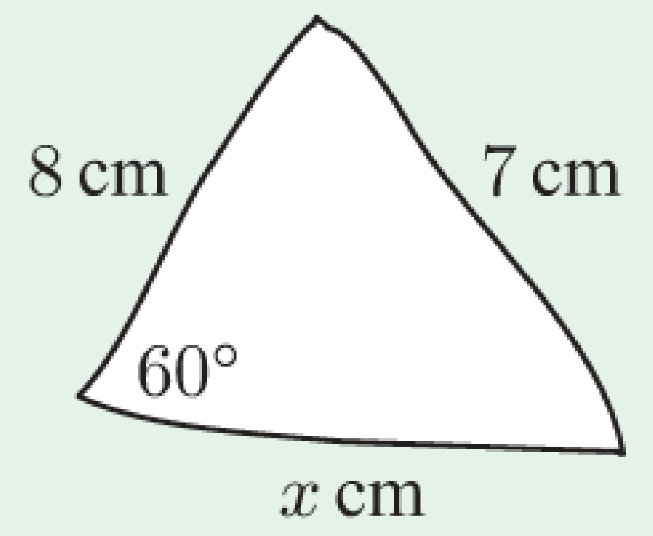
6 Find the area of this triangle.



7 Kady was asked to draw the illustrated triangle exactly.

a Use the cosine rule to find x.

b What should Kady's response be?

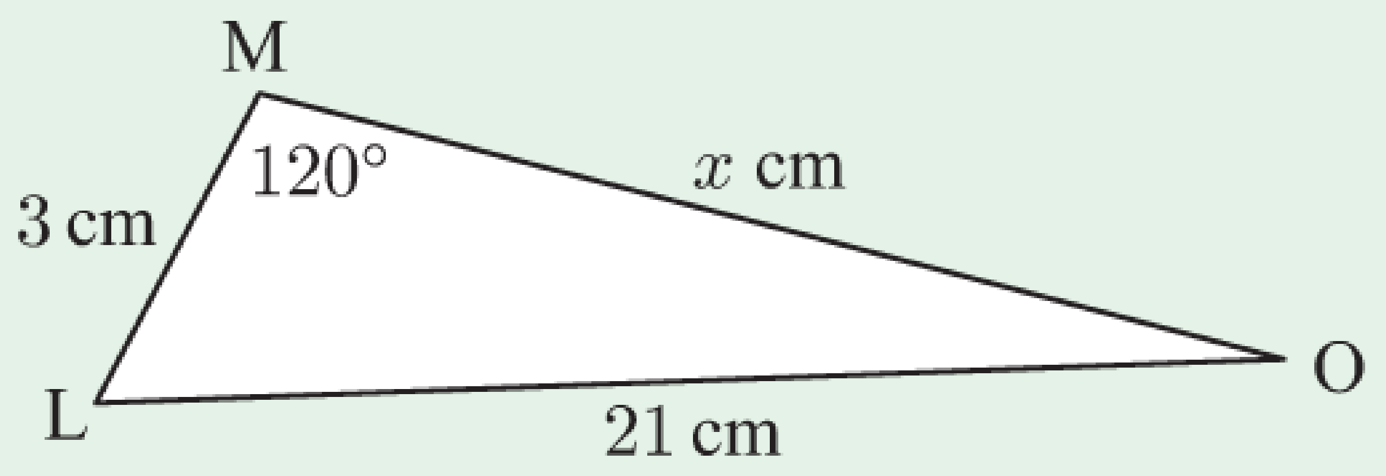


8 Triangle LMO has $L\widehat{M}O = 120°$, $LM = 3$ cm, $LO = 21$ cm, and $MO = x$ cm.

a Show that $x^2 + 3x - 432 = 0$.

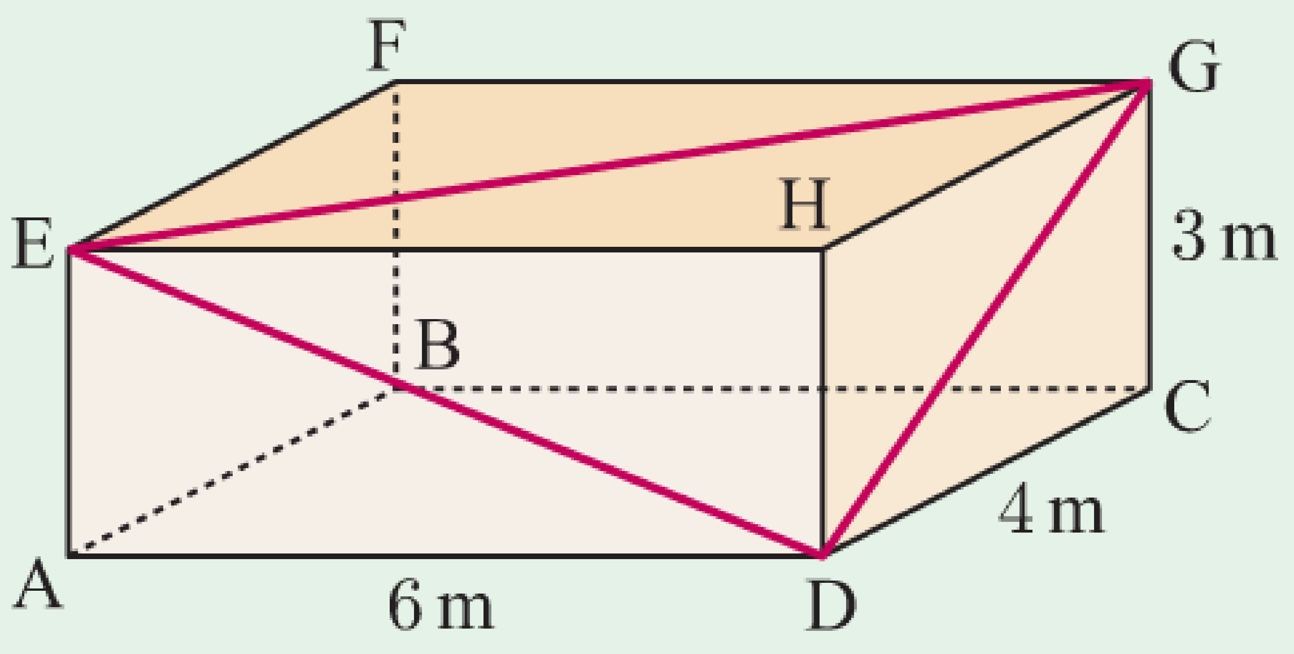
b Find x correct to 3 significant figures.

c Find the perimeter of triangle LMO.

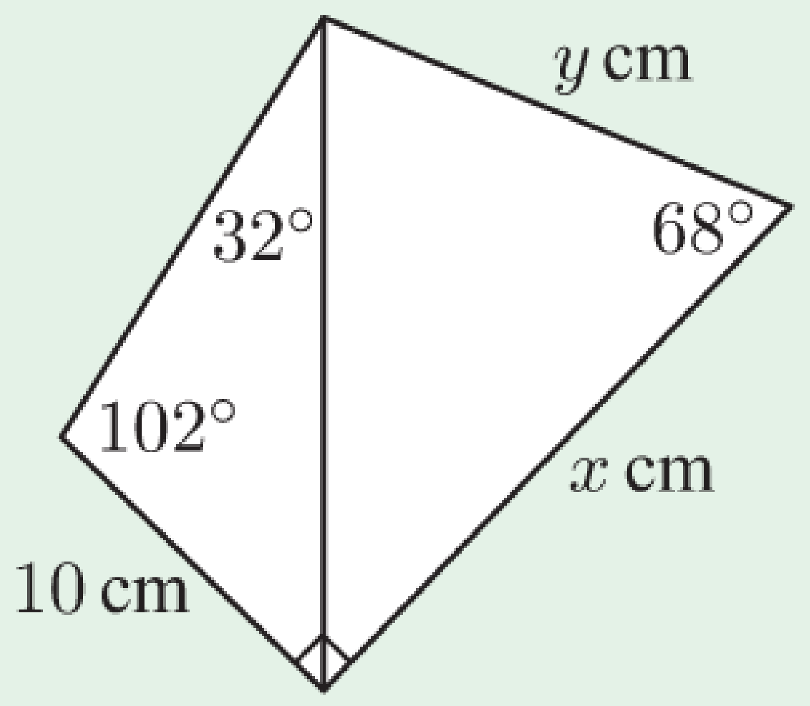


9 Two angles of a triangle have size 35° and 82°, and the area of the triangle is 40 cm^2. Find the length of each side of the triangle.

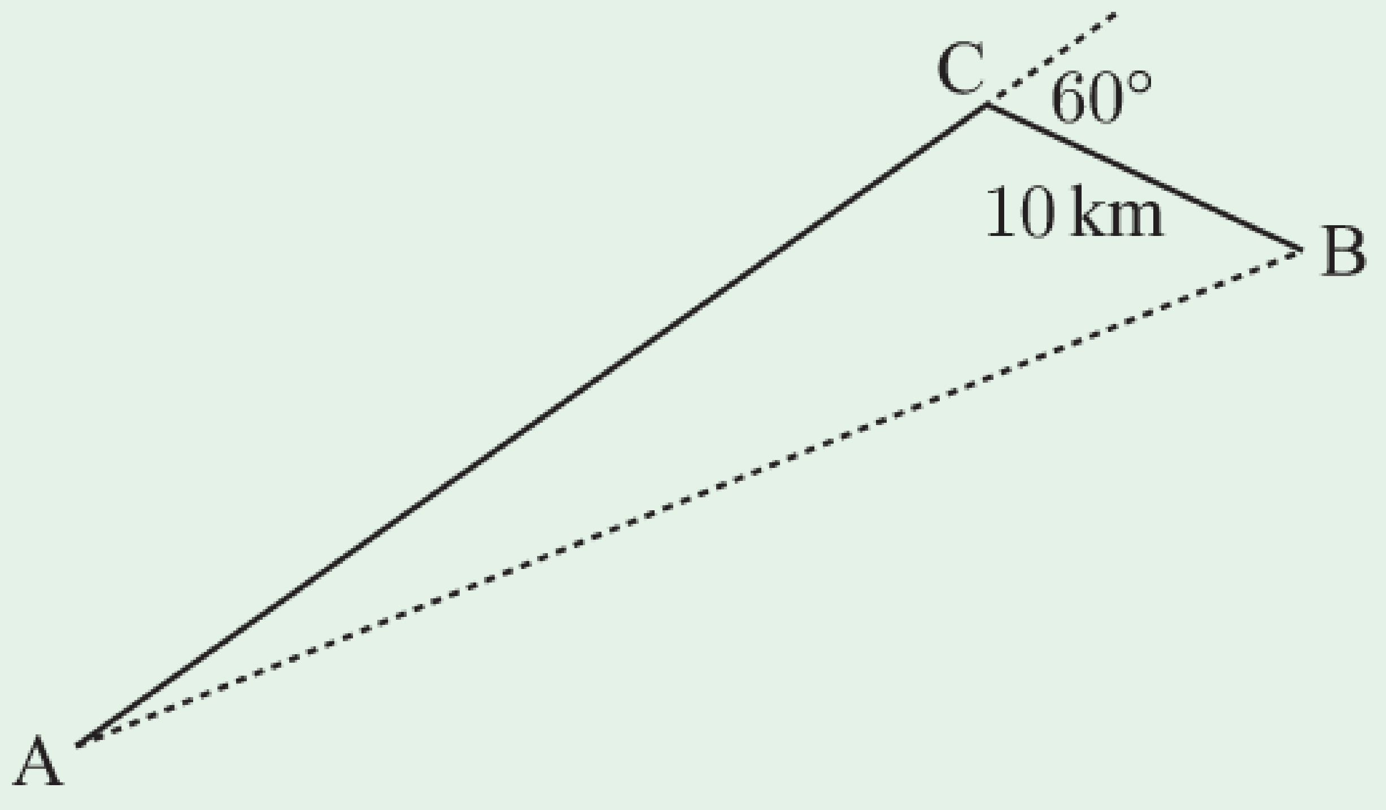
10 Find the measure of $E\widehat{D}G$:



11 Find x and y in this figure.



12



A boat was supposed to be sailing directly from A to B. However, it travelled in a straight line to C before the captain realised he was off course. He turned the boat through an angle of 60°, then travelled another 10 km to B. The trip would have been 4 km shorter if the boat had gone straight from A to B. How far did the boat travel?

13 In triangle ABC, $A\widehat{C}B = 42^\circ$, $AB = 5$ cm, and $AC = 7$ cm.

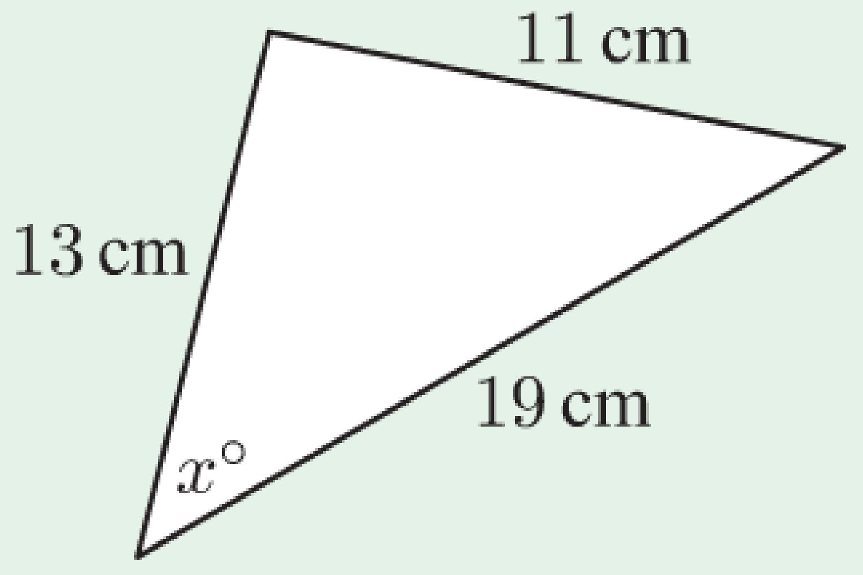
- **a** Find the two possible measures of $A\widehat{B}C$.
- **b** Find the area of triangle ABC in each case.

14 Dune buggies X and Y are 500 m apart, and the bearing of Y from X is 215°. X travels 200 m due east and Y travels 100 m due north. How far apart are the dune buggies now?

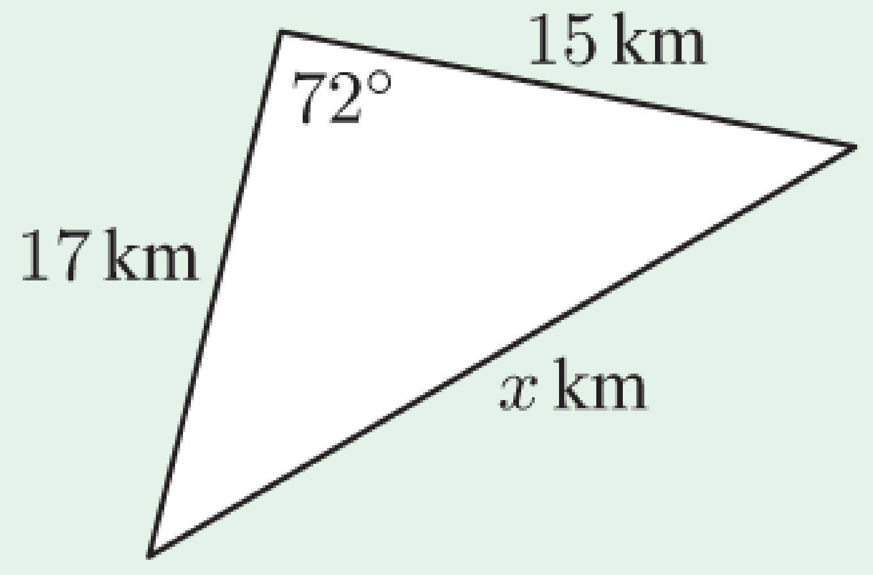
REVIEW SET 9B

1 Find the value of x:

a



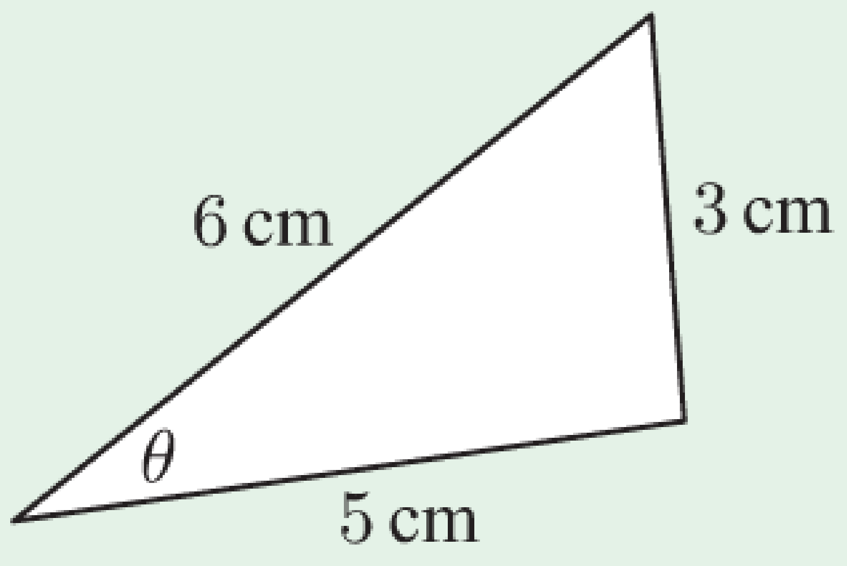
b



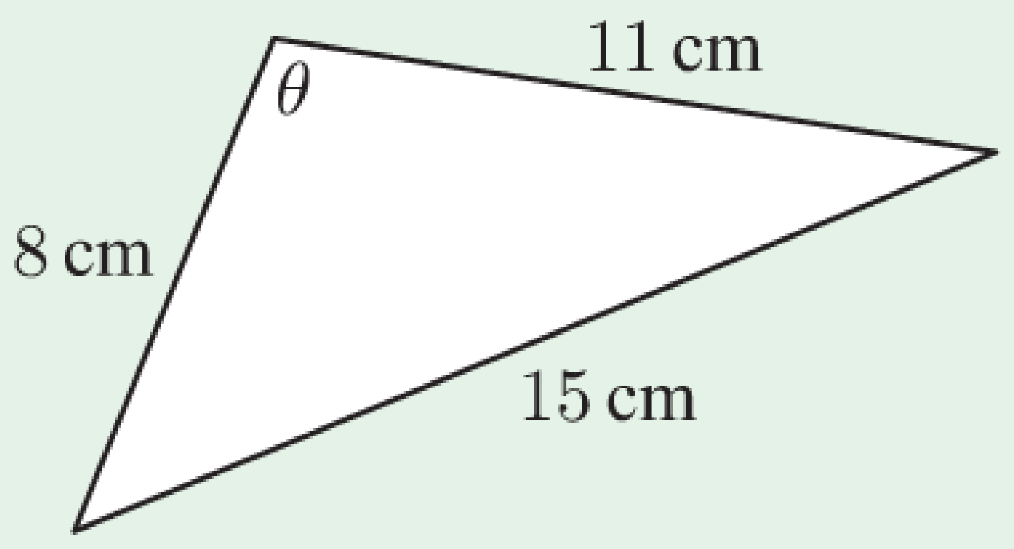
2 A triangle has two sides with lengths 11.3 cm and 19.2 cm, and an area of 80 cm^2. Find the possible measures of the included angle.

3 Find the measure of the angle marked θ:

a

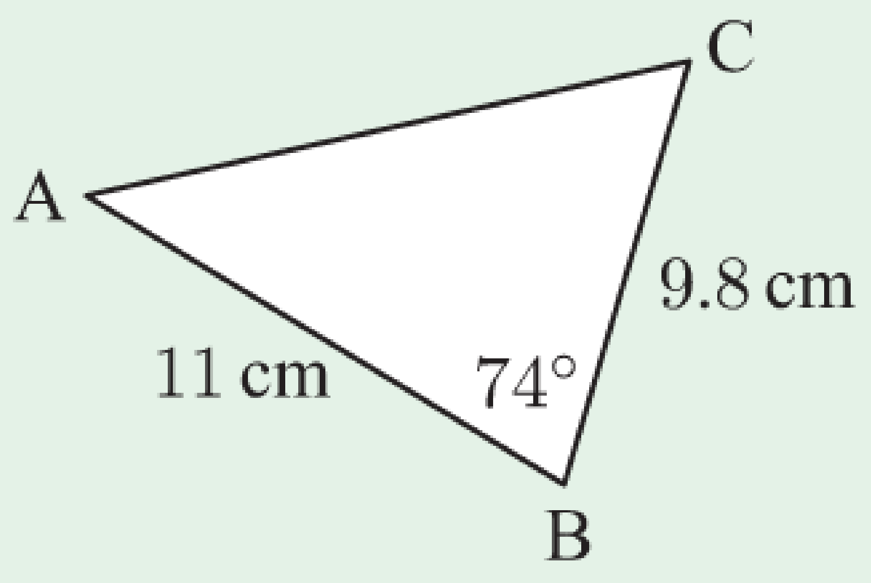


b



4 Find any unknown sides and angles in:

a



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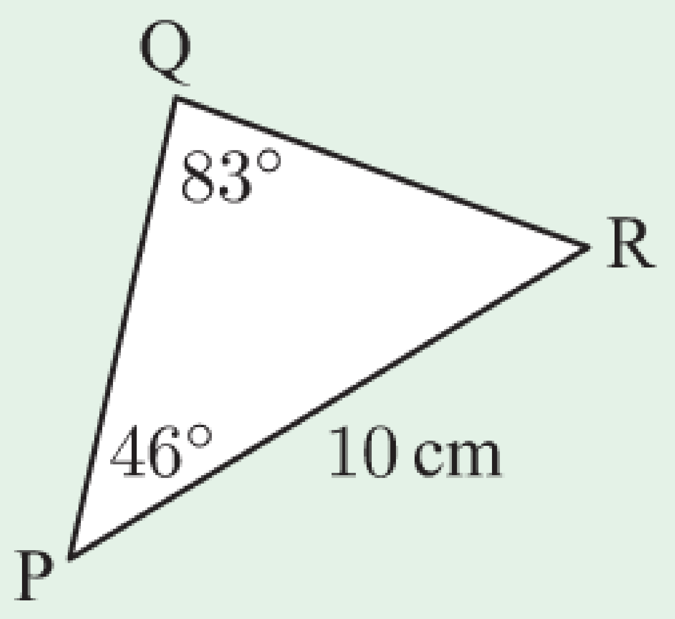
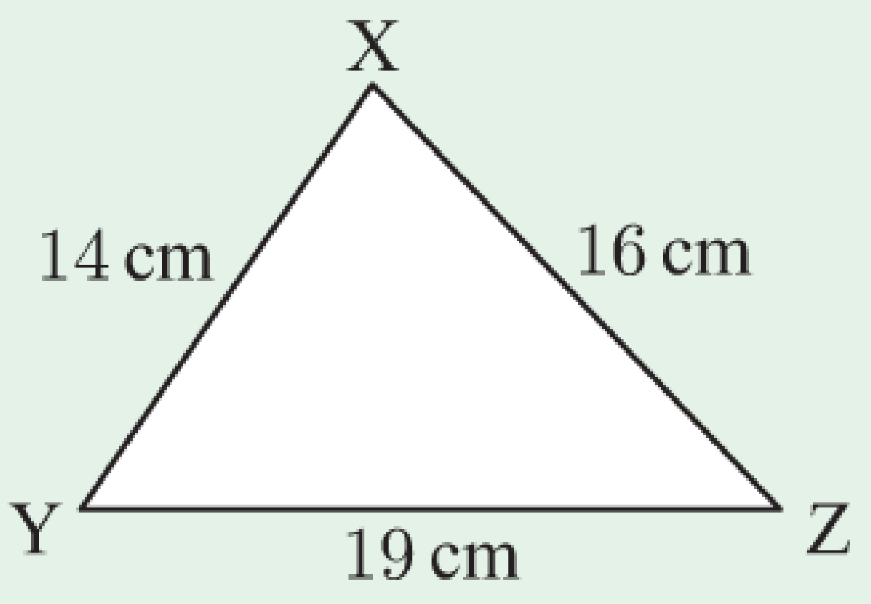
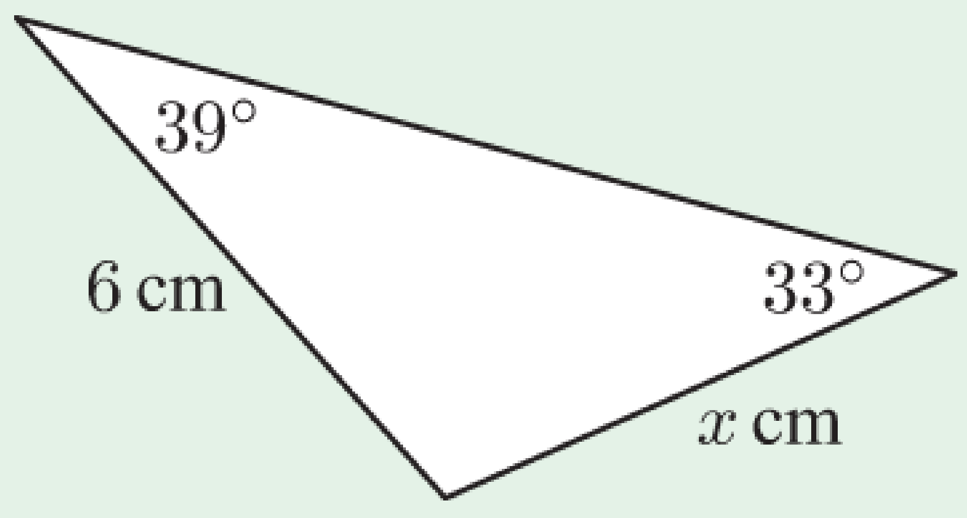


c

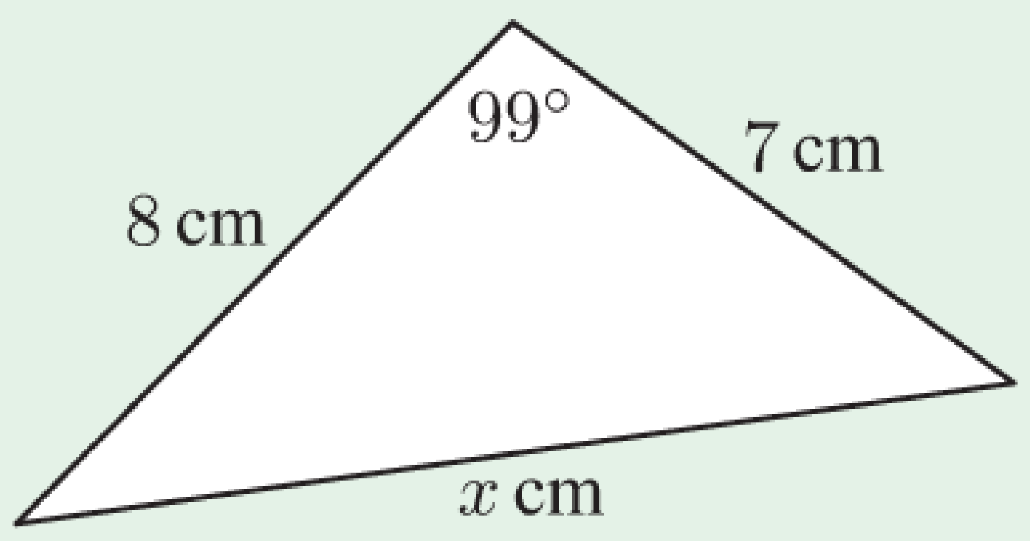


5 Find x in each triangle:

a



b



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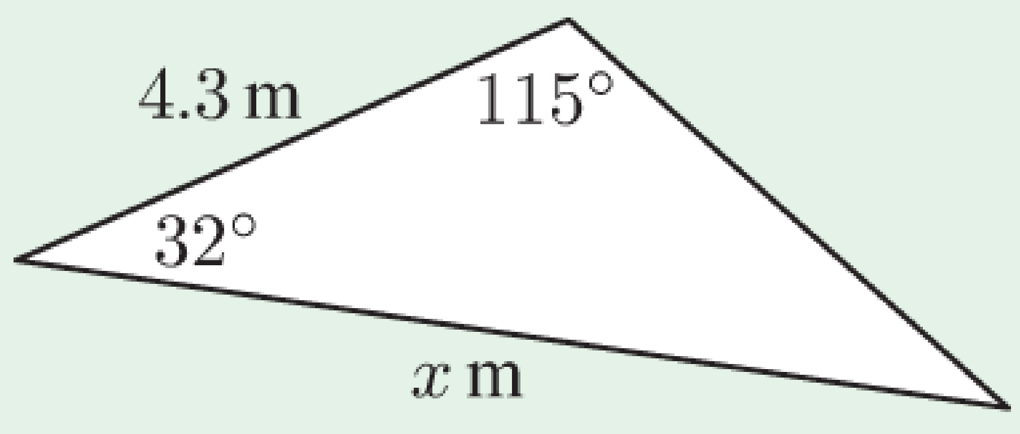
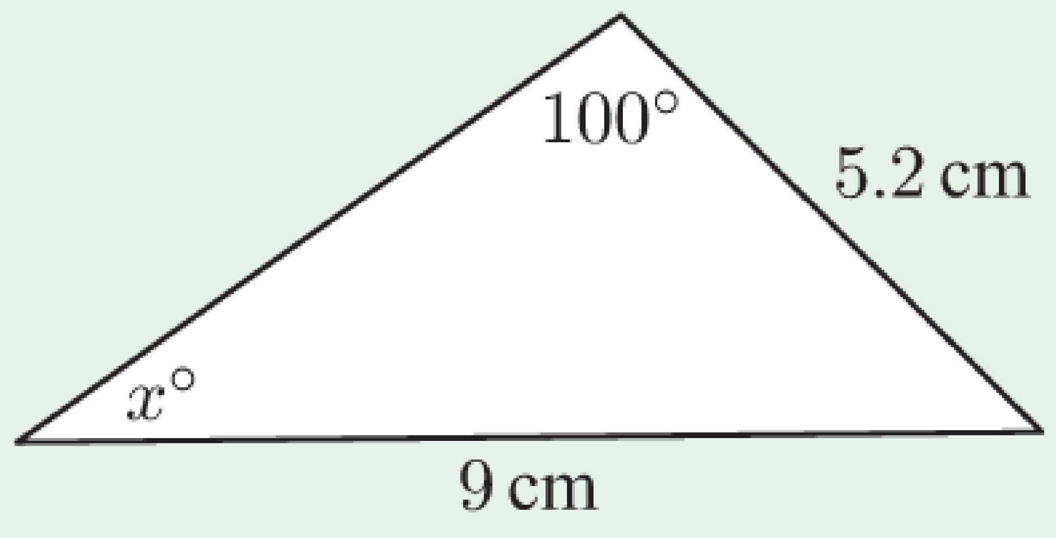


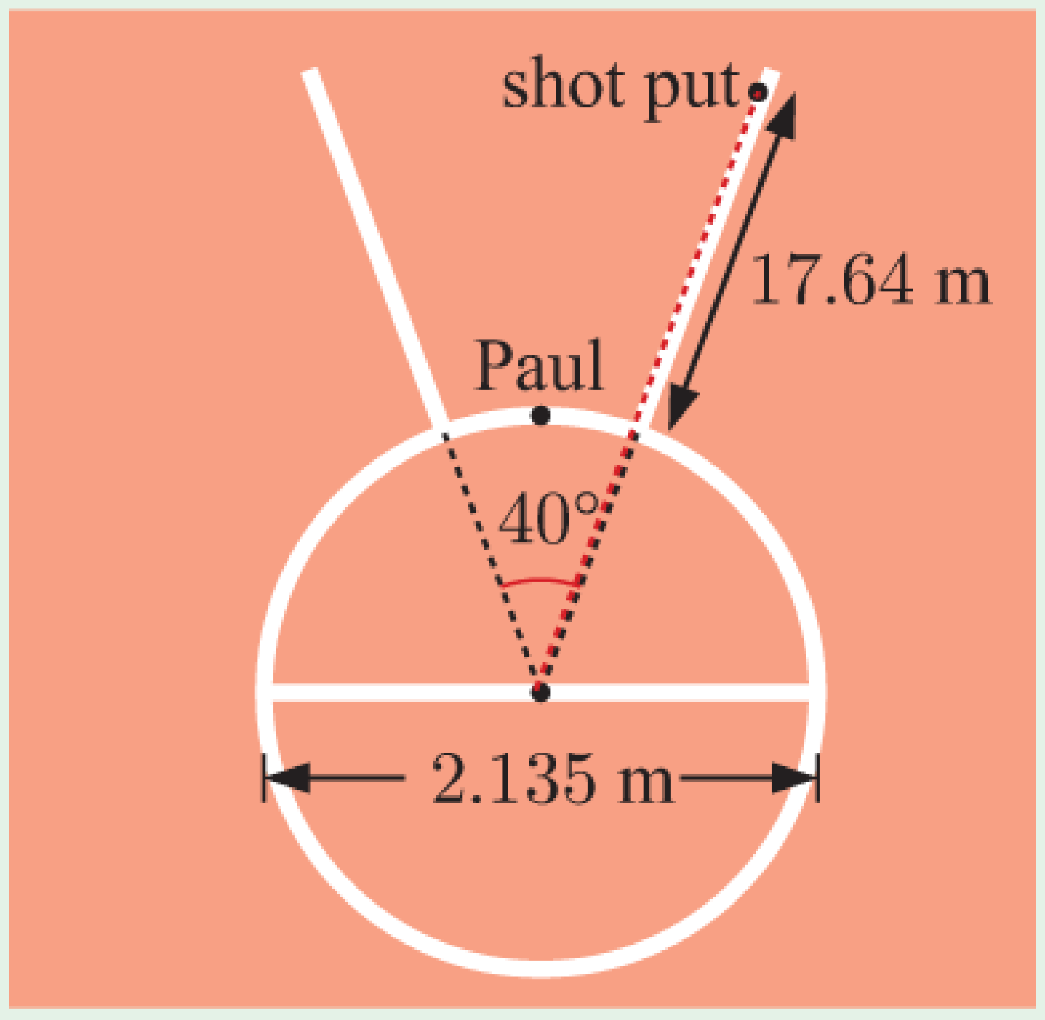
d



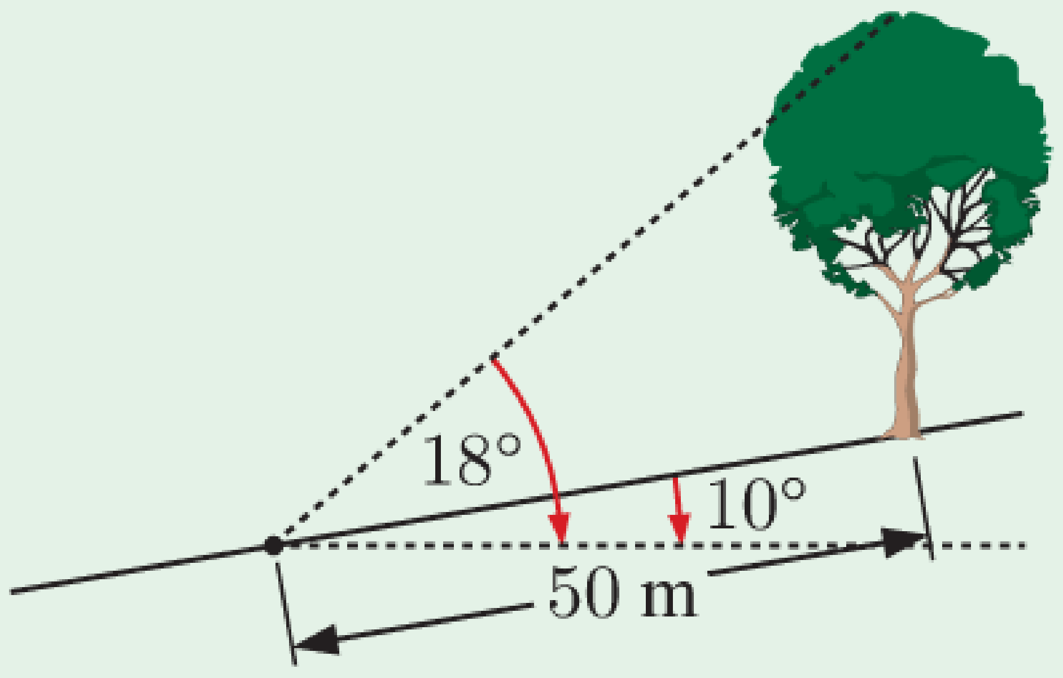
6 Paul "puts" a shot put from the front of a throwing circle with diameter 2.135 m. It only just lands inside the 40° throwing boundaries.

The official measurement goes from the shot to the nearest point of the throwing circle, and reads 17.64 m.

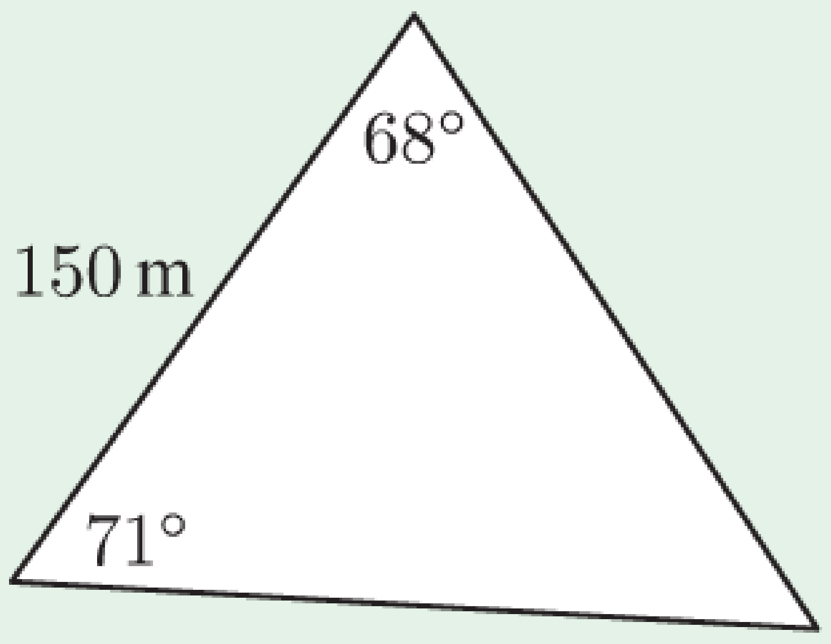
How far did Paul actually put the shot?



7 A vertical tree is growing on the side of a hill with gradient 10° to the horizontal. From a point 50 m downhill from the tree, the angle of elevation to the top of the tree is 18°. Find the height of the tree.

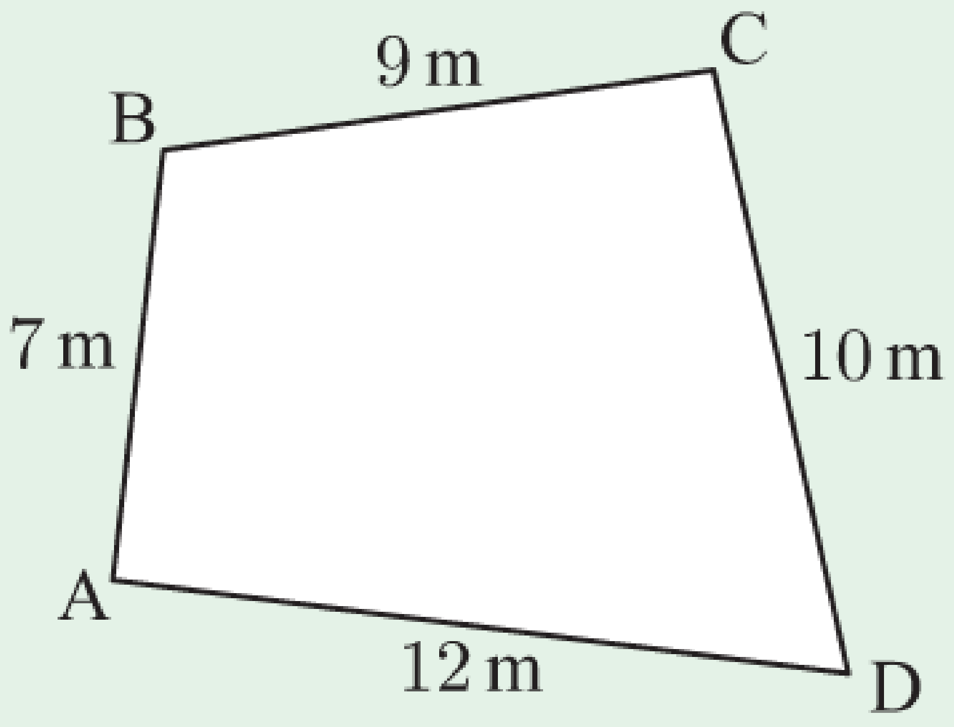


8 Find the perimeter and area of this triangle.



9 Peter, Sue, and Alix are sea-kayaking. Peter is 430 m from Sue on the bearing 113°. Alix is on the bearing 210° and is 310 m from Sue. Find the distance and bearing of Peter from Alix.

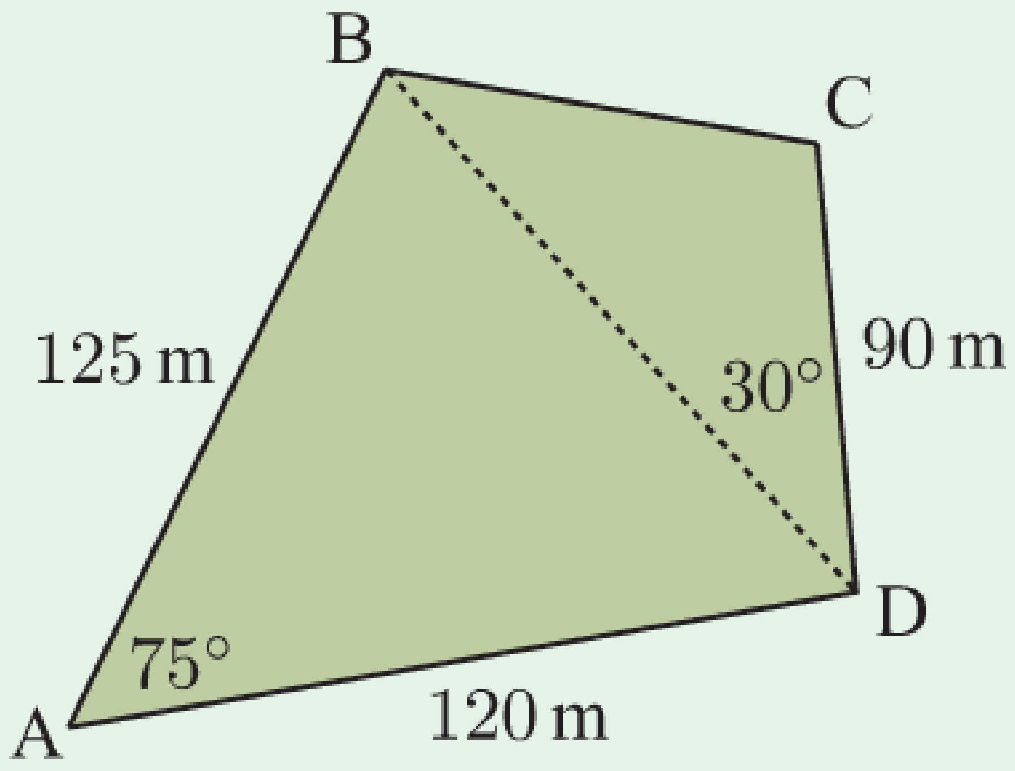
10 In quadrilateral ABCD, $A\widehat{B}C = 105°$. Find the measure of the other three angles.



11 Find the measure of angle Q in triangle PQR given that $Q\widehat{P}R = 47°$, $QR = 11$ m, and $PR = 9.6$ m.

12 Anke and Lucas are considering buying a block of land. The land agent supplies them with the given accurate sketch. Find the area of the property, giving your answer in:

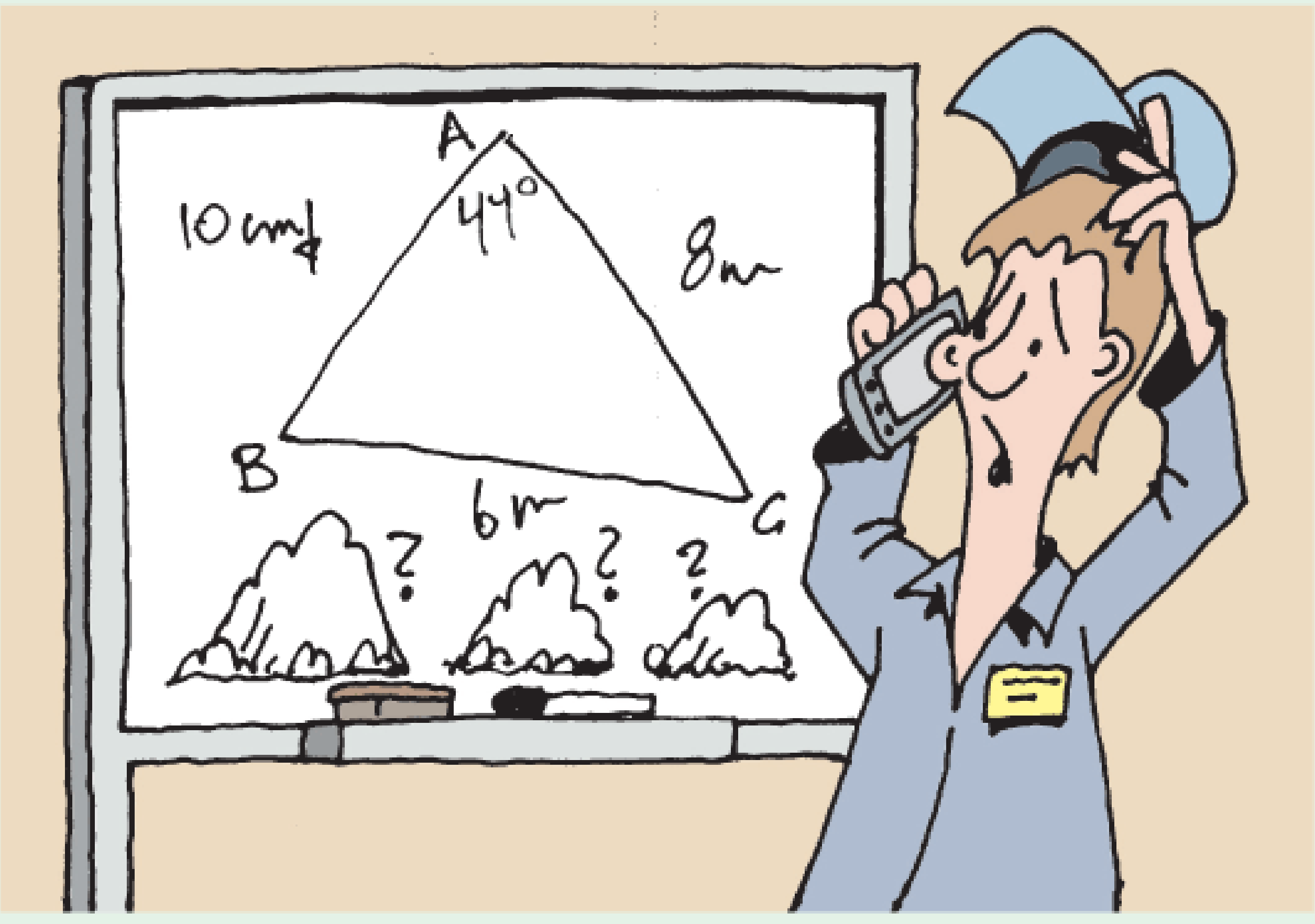
a m^2 **b** hectares.



13 Soil contractor Frank was given the following dimensions over the telephone:

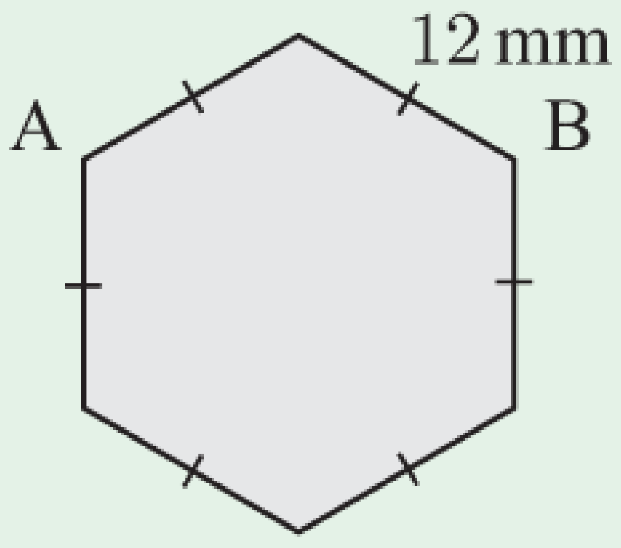
The triangular garden plot ABC has $C\widehat{A}B$ measuring 44°, [AC] is 8 m long, and [BC] is 6 m long. Frank needs to supply soil for the plot to a depth of 10 cm.

- **a** Explain why Frank needs extra information from his client.
- **b** What is the maximum volume of soil that could be needed if his client is unable to supply the necessary information?

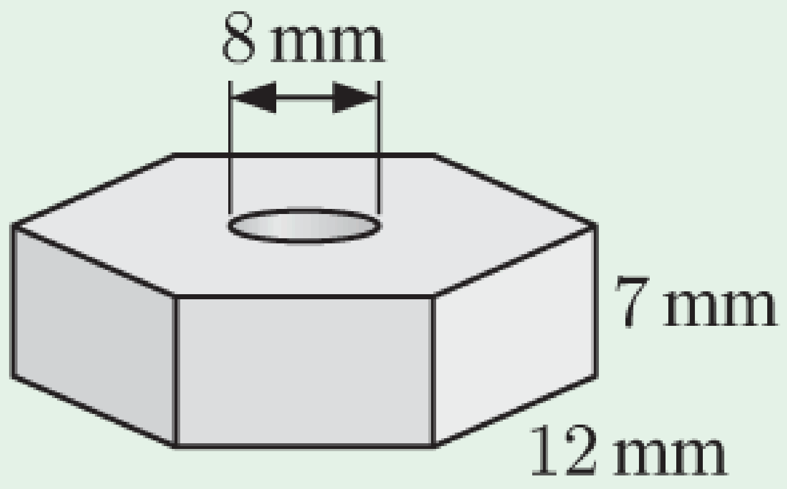


14 **a** Consider the regular hexagon shown. Find:

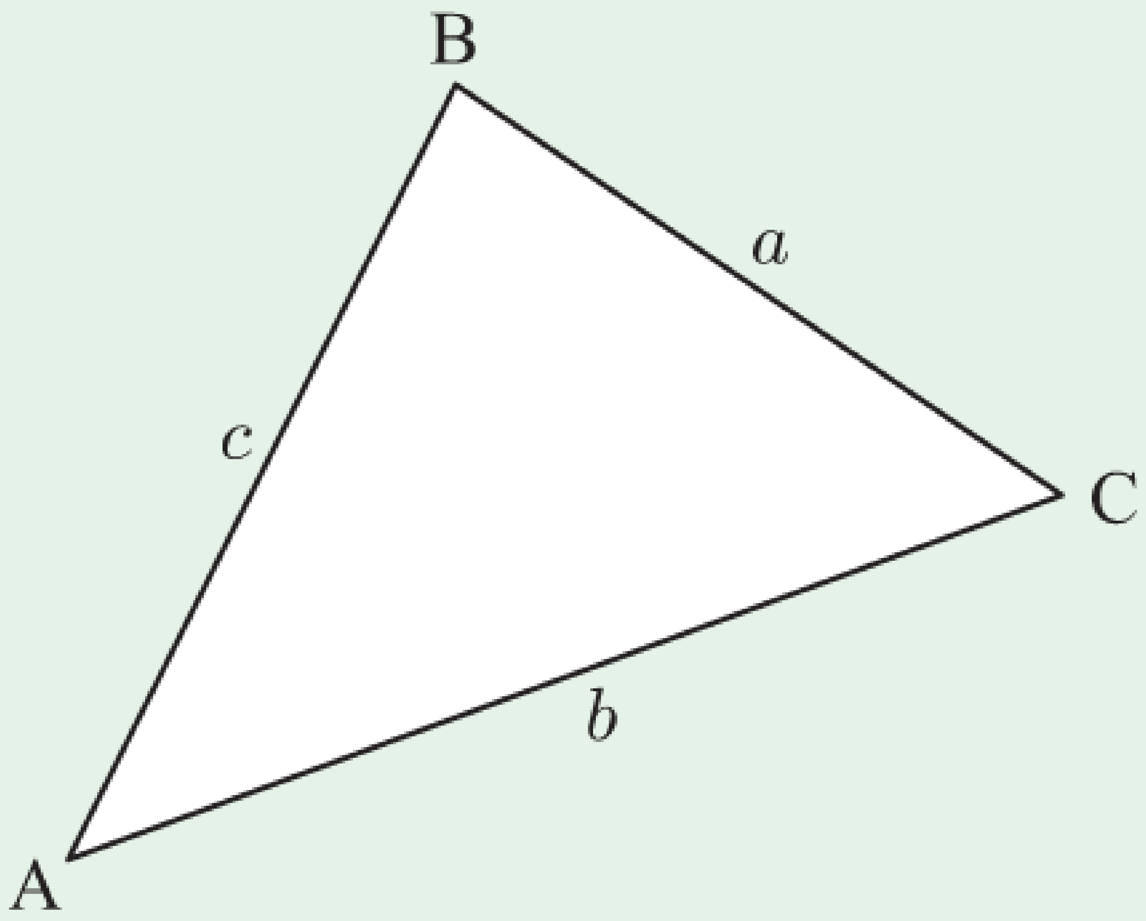
- **i** the length AB
- **ii** the area of the hexagon.



b Find the volume of this metal nut.



15 Over 2000 years ago, the Greek mathematician **Heron** or **Hero** discovered a formula for finding the area of a triangle with sides a, b, and c. Heron's formula is $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{a+b+c}{2}$ is the **semi-perimeter** of the triangle.



- **a** Show that

$$A^2 = \frac{b^2c^2}{4}\left(1 + \frac{b^2+c^2-a^2}{2bc}\right)\left(1 - \frac{b^2+c^2-a^2}{2bc}\right).$$

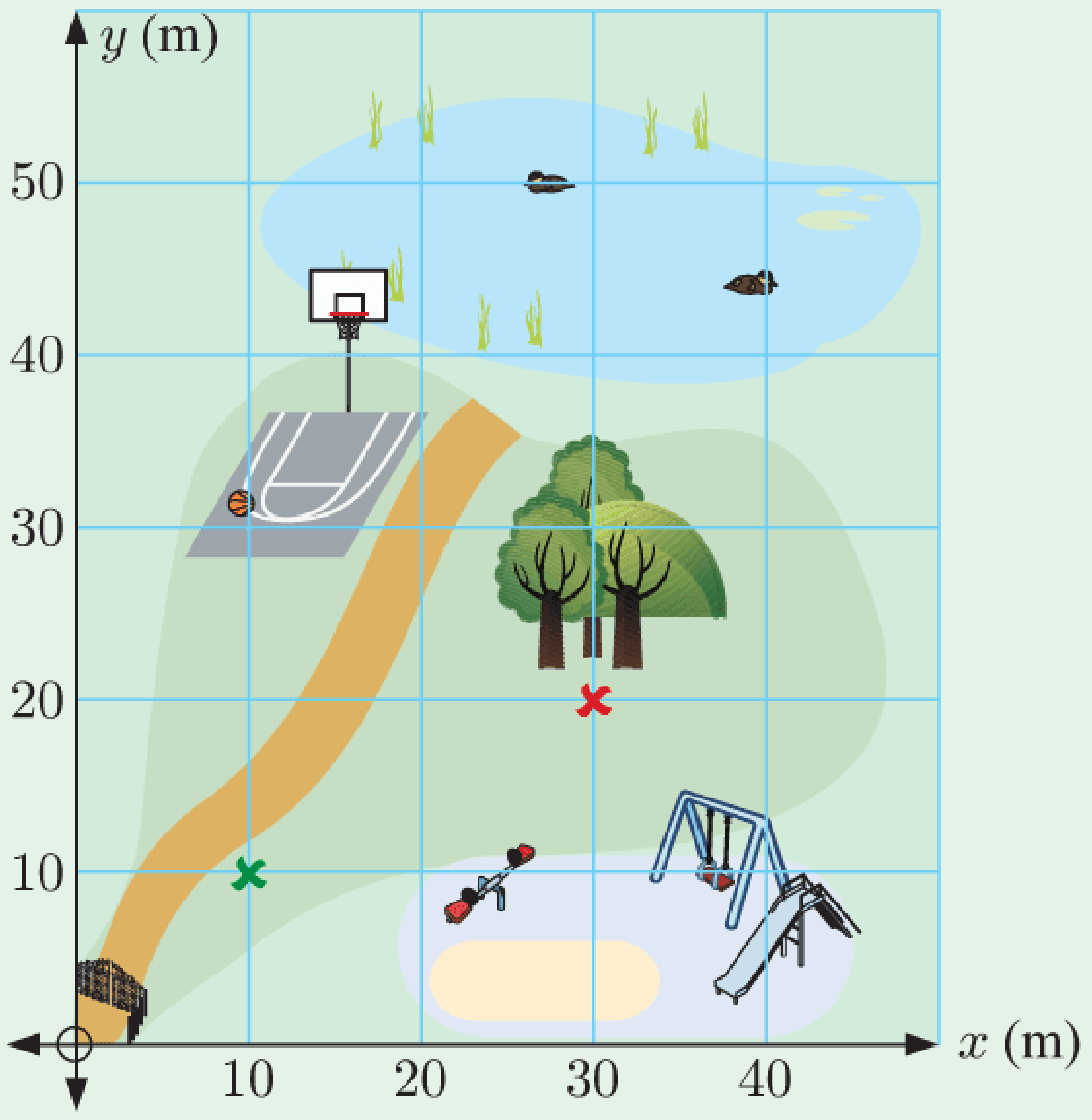
- **b** Hence prove Heron's formula.

Chapter 10

Points in space

Contents:

OPENING PROBLEM



The map of a park is shown alongside.

To help us describe the location of objects in the park, we can place a 2-dimensional coordinate grid over the map, so that the origin is at the park's entrance, and each grid unit represents 10 metres.

Ayla is currently at the location marked with an ×. A bird is sitting in a tree, 10 metres above the ground. It is directly above Ayla.

Things to think about:

a How can we extend our coordinate system to describe the location of the bird?

b The bird spies a worm in the garden at ×.

- **i** How far is the bird from the worm?
- **ii** If the bird flies in a straight line to the worm, at what angle to the ground will it fly?

A POINTS IN SPACE

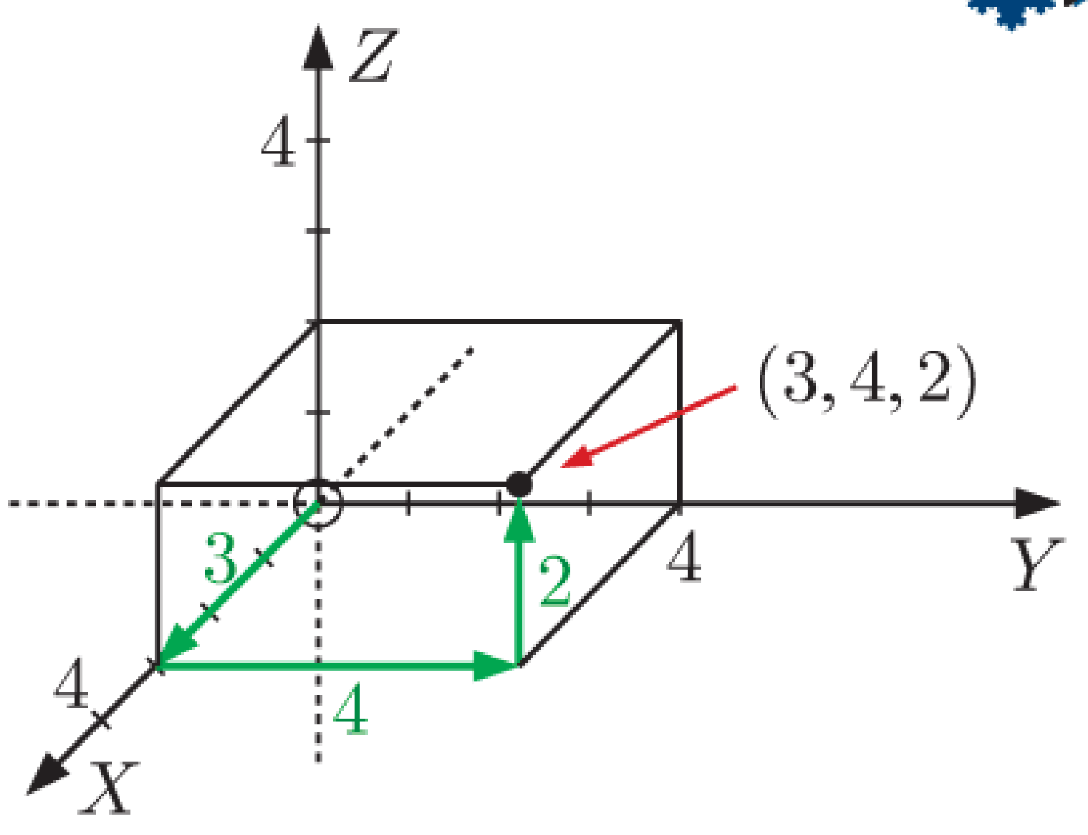
3-D POINT PLOTTER

In 3-dimensional coordinate geometry, we specify an origin O, and three mutually perpendicular axes called the X-axis, the Y-axis, and the Z-axis.

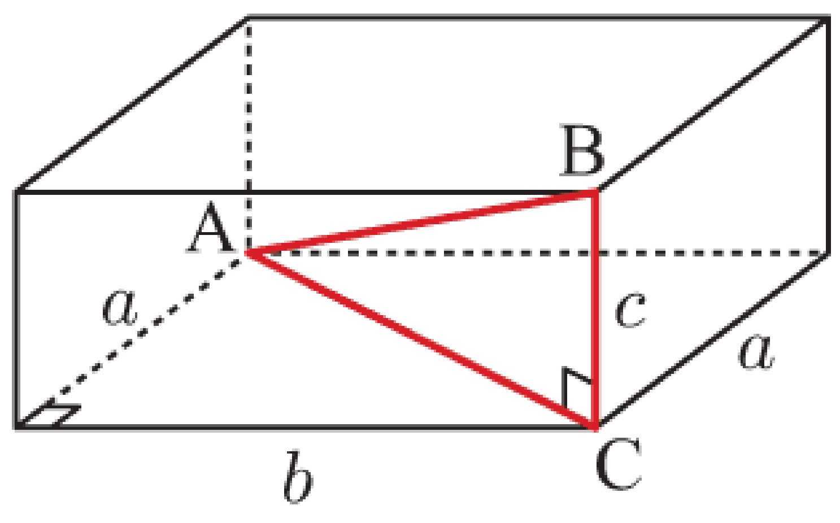
Any point in space can then be specified using an ordered triple in the form (x, y, z).

We generally suppose that the Y and Z-axes are in the plane of the page, and the X-axis is coming out of the page as shown.

The point $(3, 4, 2)$ is found by starting at the origin O$(0, 0, 0)$, moving 3 units along the X-axis, 4 units in the Y-direction, and then 2 units in the Z-direction.



We see that $(3, 4, 2)$ is located on the corner of a rectangular prism opposite O.



Now consider the rectangular prism illustrated, in which A is opposite B.

$$\text{AC}^2 = a^2 + b^2 \qquad \{\text{Pythagoras}\}$$

$$\text{and} \quad \text{AB}^2 = \text{AC}^2 + c^2 \qquad \{\text{Pythagoras}\}$$

$$\therefore \text{AB}^2 = a^2 + b^2 + c^2$$

$$\therefore \text{AB} = \sqrt{a^2 + b^2 + c^2} \qquad \{\text{AB} > 0\}$$

Suppose A is (x_1, y_1, z_1) and B is (x_2, y_2, z_2).

- The **distance** $\text{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.
- The **midpoint** of [AB] is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

Example 1 **Self Tutor**

Consider $A(4, -3, 1)$ and $B(-2, 4, -1)$. Find:

a the distance AB **b** the midpoint of [AB].

a
$$\begin{aligned} AB &= \sqrt{(-2-4)^2 + (4 - -3)^2 + (-1-1)^2} \\ &= \sqrt{(-6)^2 + 7^2 + (-2)^2} \\ &= \sqrt{36+49+4} \\ &= \sqrt{89} \text{ units} \end{aligned}$$

b The midpoint is $\left(\frac{4+-2}{2}, \frac{-3+4}{2}, \frac{1+-1}{2}\right)$,

which is $(1, \frac{1}{2}, 0)$.

EXERCISE 10A

1 On separate axes, plot the points:

a $(5, 0, 0)$ **b** $(0, -1, 0)$ **c** $(0, 0, 2)$ **d** $(4, 3, 0)$

e $(2, 0, -1)$ **f** $(0, 4, -3)$ **g** $(3, 1, 1)$ **h** $(2, 4, 2)$

i $(3, -2, 5)$ **j** $(-3, 3, 3)$ **k** $(-4, 3, -2)$ **l** $(-1, -4, -2)$

PRINTABLE 3-D PLOTTER PAPER

2 For each pair of points, find:

i the distance AB **ii** the midpoint of [AB].

a $A(0, 0, 0)$ and $B(6, -4, 2)$ **b** $A(4, 1, 0)$ and $B(0, 1, -2)$

c $A(1, -1, 2)$ and $B(5, -3, 0)$ **d** $A(-2, 0, 5)$ and $B(-6, 7, 3)$

e $A(-1, 5, 2)$ and $B(4, 1, -1)$ **f** $A(2, 6, -3)$ and $B(-5, 3, 2)$

3 Determine whether triangle ABC is scalene, isosceles, or equilateral:

a A is $(2, 1, -3)$, B is $(-5, 5, 3)$, and C is $(-2, 3, 6)$

b A is $(3, -1, 5)$, B is $(-1, -4, 0)$, and C is $(2, 7, -3)$

4 Suppose A is $(3, 1, -2)$, B is $(-6, 7, 13)$, and C is $(5, 9, -4)$. Show that triangle ABC is right angled.

5 Consider the points $P(1, 4, -1)$, $Q(6, -8, 7)$, and $R(-5, -2, -9)$. Let M be the midpoint of [PQ], and N be the midpoint of [QR].

a Find the coordinates of M and N.

b Show that [MN] is half the length of [PR].

6 Suppose A is $(3, 5, -2)$, and $M(-2, 8, \frac{1}{2})$ is the midpoint of [AB]. Find the coordinates of B.

7 The line segments [PQ], [QR], and [PR] have midpoints $(1, 2, \frac{3}{2})$, $(\frac{5}{2}, 3, \frac{9}{2})$, and $(-\frac{1}{2}, 6, 3)$ respectively. Find the coordinates of P, Q, and R.

8 The distance from $P(2, 4, -3)$ to $Q(k, -1, -2)$ is 7 units. Find the possible values of k.

9 Find the relationship between x, y, and z if the point $P(x, y, z)$:

a is always 3 units from $O(0, 0, 0)$

b is always 1 unit from $A(2, 5, 4)$.

Comment on your answer in each case.

10 Illustrate and describe these sets:

a $\{(x, y, z) \mid x = 3\}$

b $\{(x, y, z) \mid y = 2, \ z = -1\}$

c $\{(x, y, z) \mid x^2 + y^2 = 4, \ z = 0\}$

d $\{(x, y, z) \mid x^2 + y^2 + z^2 = 9\}$

e $\{(x, y, z) \mid 0 \leqslant x \leqslant 4, \ 0 \leqslant y \leqslant 1, \ z = 2\}$

f $\{(x, y, z) \mid 0 \leqslant x \leqslant 3, \ 0 \leqslant y \leqslant 5, \ 0 \leqslant z \leqslant 2\}$.

B MEASUREMENT

We can use the measurement formulae studied in **Chapter 6** to find the surface area and volume of solids in 3-dimensional space.

Example 2 **Self Tutor**

A square-based pyramid has base coordinates $O(0, 0, 0)$, $A(4, 0, 0)$, $B(4, 4, 0)$, and $C(0, 4, 0)$. The apex of the pyramid is $D(2, 2, 5)$.

a Verify that the apex lies directly above the centre of the base.

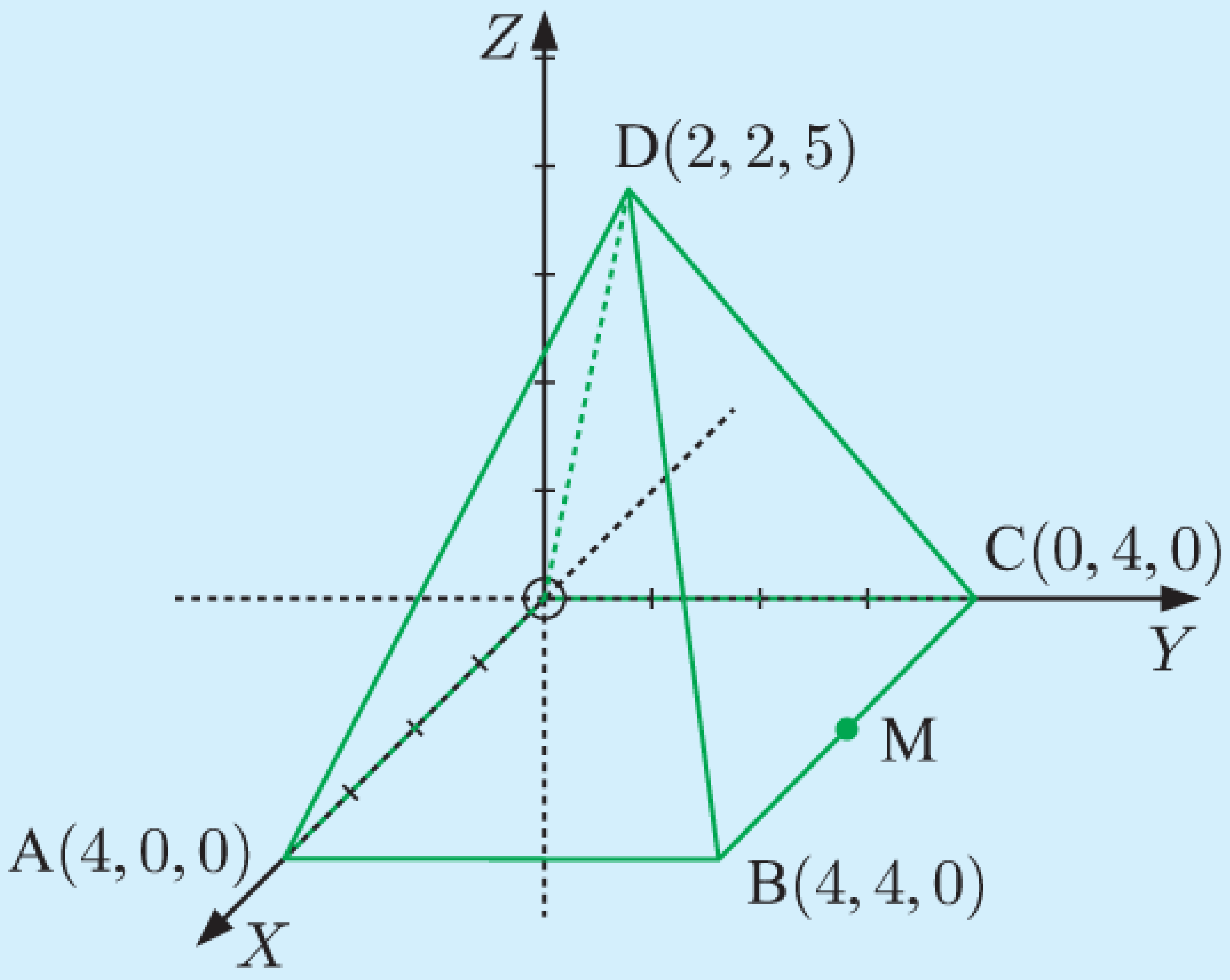
b Find the volume of the pyramid.

c Suppose M is the midpoint of [BC].

  i Find the coordinates of M.

  ii Find the exact length of [MD].

  iii Hence find the surface area of the pyramid.



a To find the centre of the base, we locate the midpoints of the diagonals.

The midpoint of [OB] is $\left(\frac{0+4}{2}, \frac{0+4}{2}, \frac{0+0}{2}\right)$ which is $(2, 2, 0)$.

The midpoint of [AC] is $\left(\frac{4+0}{2}, \frac{0+4}{2}, \frac{0+0}{2}\right)$ which is $(2, 2, 0)$.

$\therefore$ the centre of the base is $(2, 2, 0)$.

$\therefore$ the apex $(2, 2, 5)$ lies directly above the centre of the base.

b
$$\begin{aligned}\text{Volume} &= \tfrac{1}{3}(\text{area of base} \times \text{height})\\ &= \tfrac{1}{3} \times 4 \times 4 \times 5\\ &= \tfrac{80}{3} \text{ units}^3\end{aligned}$$

c **i** M is $\left(\frac{4+0}{2}, \frac{4+4}{2}, \frac{0+0}{2}\right)$ which is $(2, 4, 0)$.

ii
$$\begin{aligned}\text{MD} &= \sqrt{(2-2)^2 + (2-4)^2 + (5-0)^2}\\ &= \sqrt{0^2 + (-2)^2 + 5^2}\\ &= \sqrt{29} \text{ units}\end{aligned}$$

iii Area of triangle BCD $= \frac{1}{2} \times 4 \times \sqrt{29}$

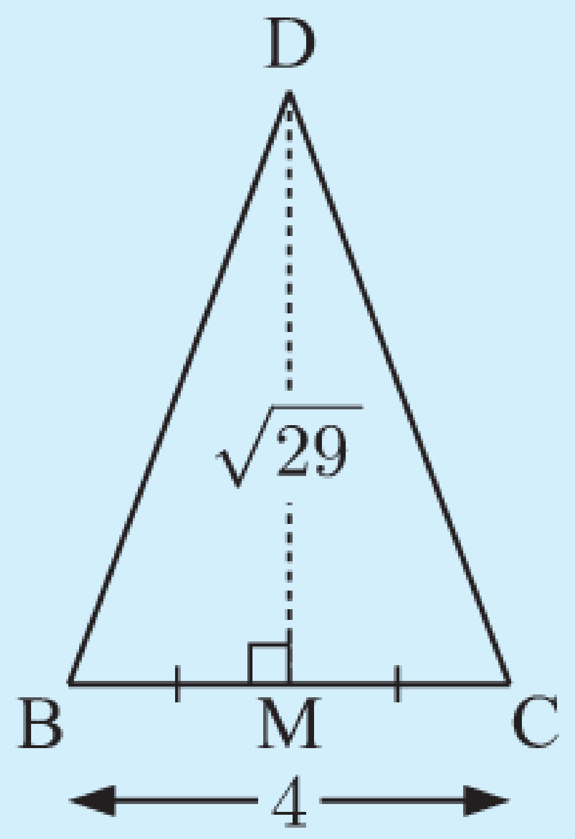
$= 2\sqrt{29}$ units^2

$\therefore$ surface area of pyramid

$=$ area of base $+$ area of 4 triangular faces

$= (4 \times 4 \; + \; 4 \times 2\sqrt{29})$ units^2

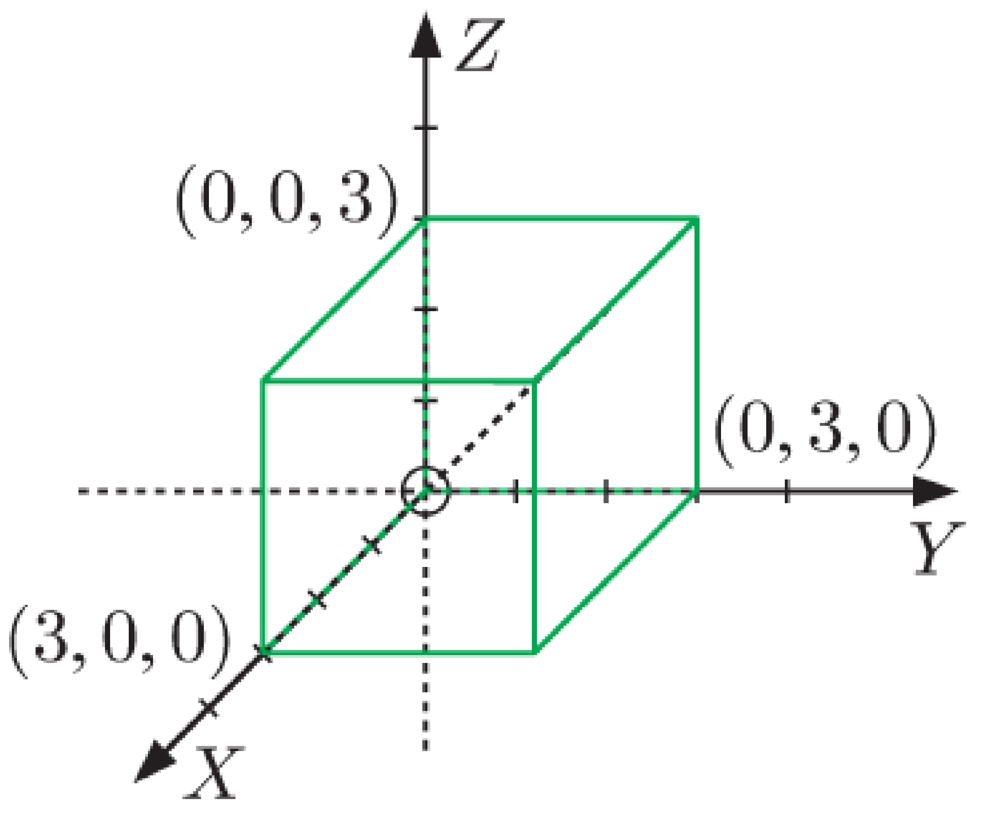
≈ 59.1 units^2



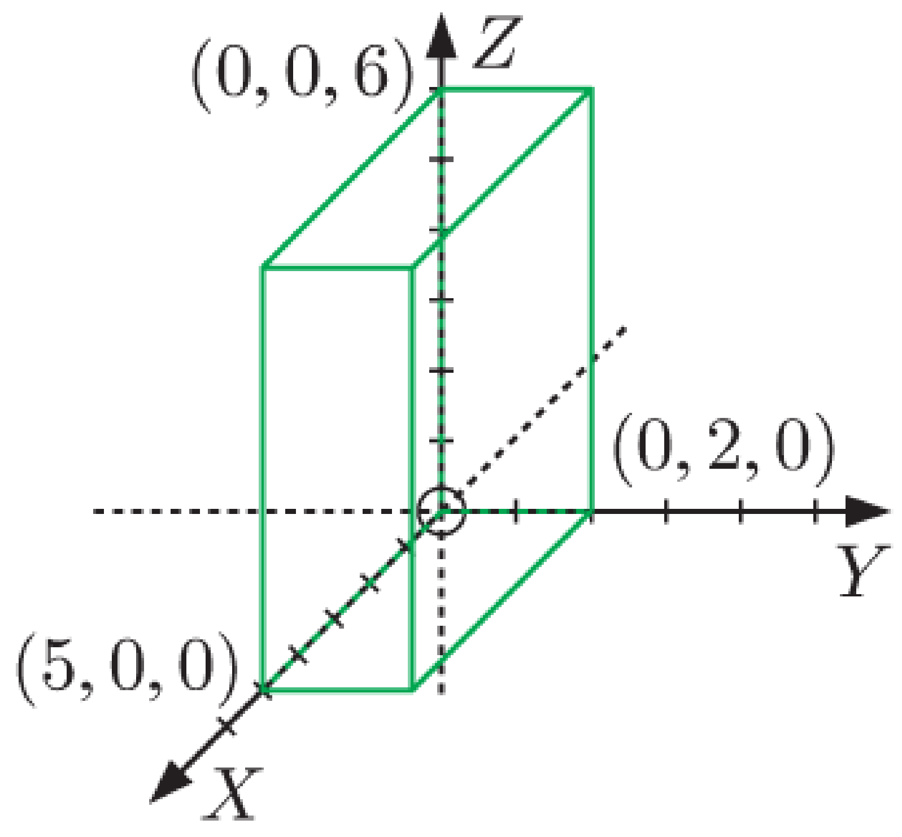
EXERCISE 10B

1 Find the volume of each rectangular prism:

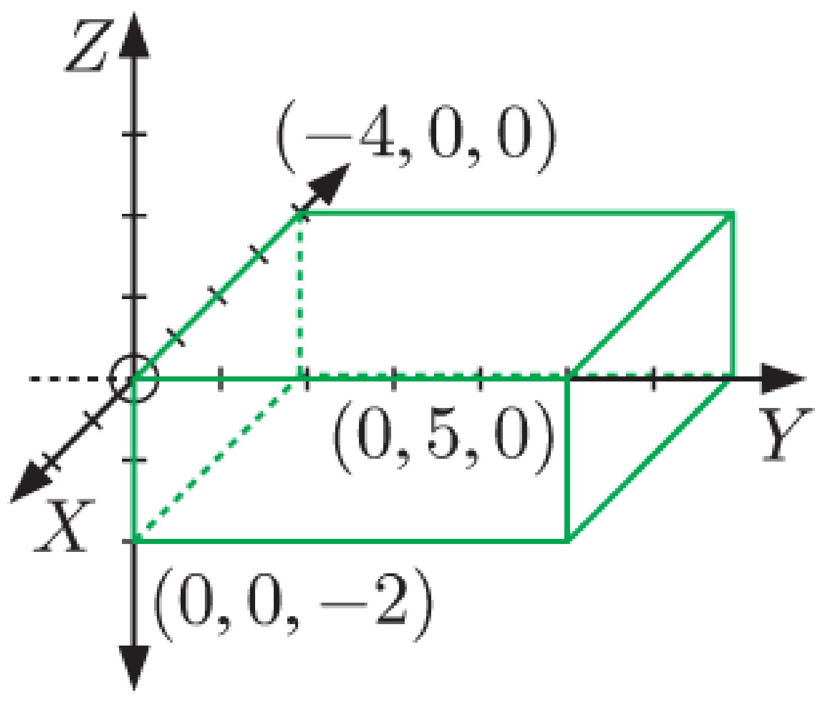
a



b

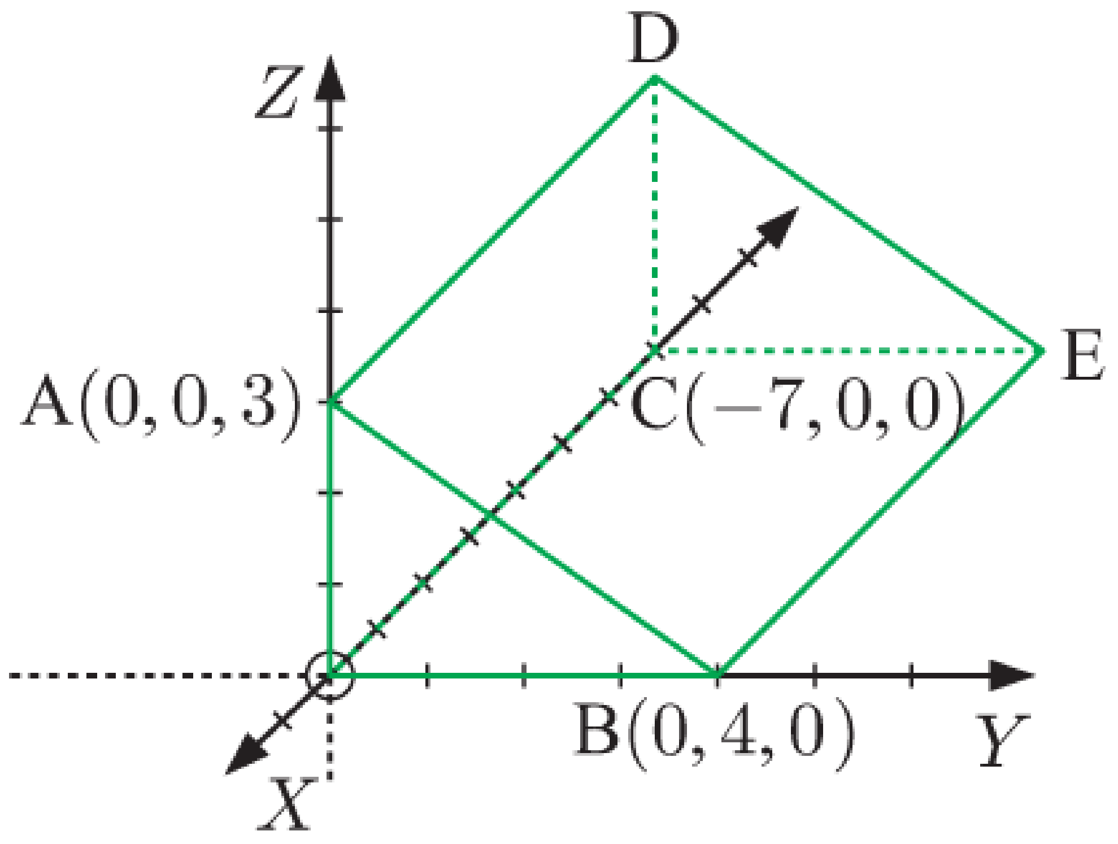


c



2 Consider the triangular-based prism alongside.

- **a** State the coordinates of D and E.
- **b** Find the volume of the prism.
- **c** Find the length of [AB].
- **d** Hence find the surface area of the prism.



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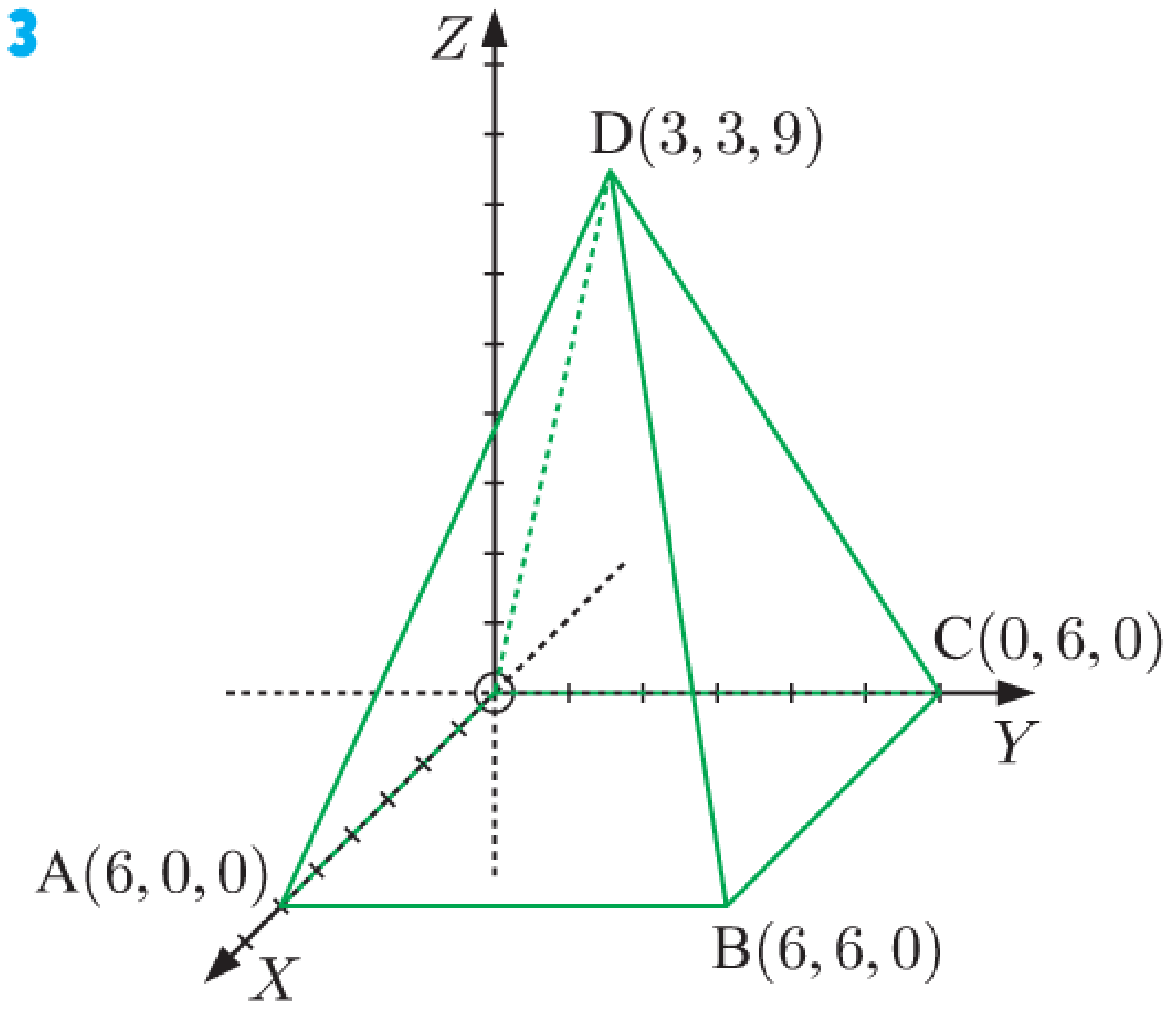


A square-based pyramid has base coordinates O(0, 0, 0), A(6, 0, 0), B(6, 6, 0), and C(0, 6, 0). The apex of the pyramid is D(3, 3, 9).

- **a** Verify that the apex lies directly above the centre of the base.
- **b** Find the volume of the pyramid.
- **c** Suppose M is the midpoint of [AB].
  - **i** Find the coordinates of M.
  - **ii** Find the length of [MD].
  - **iii** Hence find the surface area of the pyramid.

4 Find the volume and surface area of a rectangular-based pyramid with base coordinates O(0, 0, 0), A(10, 0, 0), B(10, 18, 0), and C(0, 18, 0), and apex (5, 9, 12).

5 The base of a cone lies in the X-Y plane, and is centred at the origin.

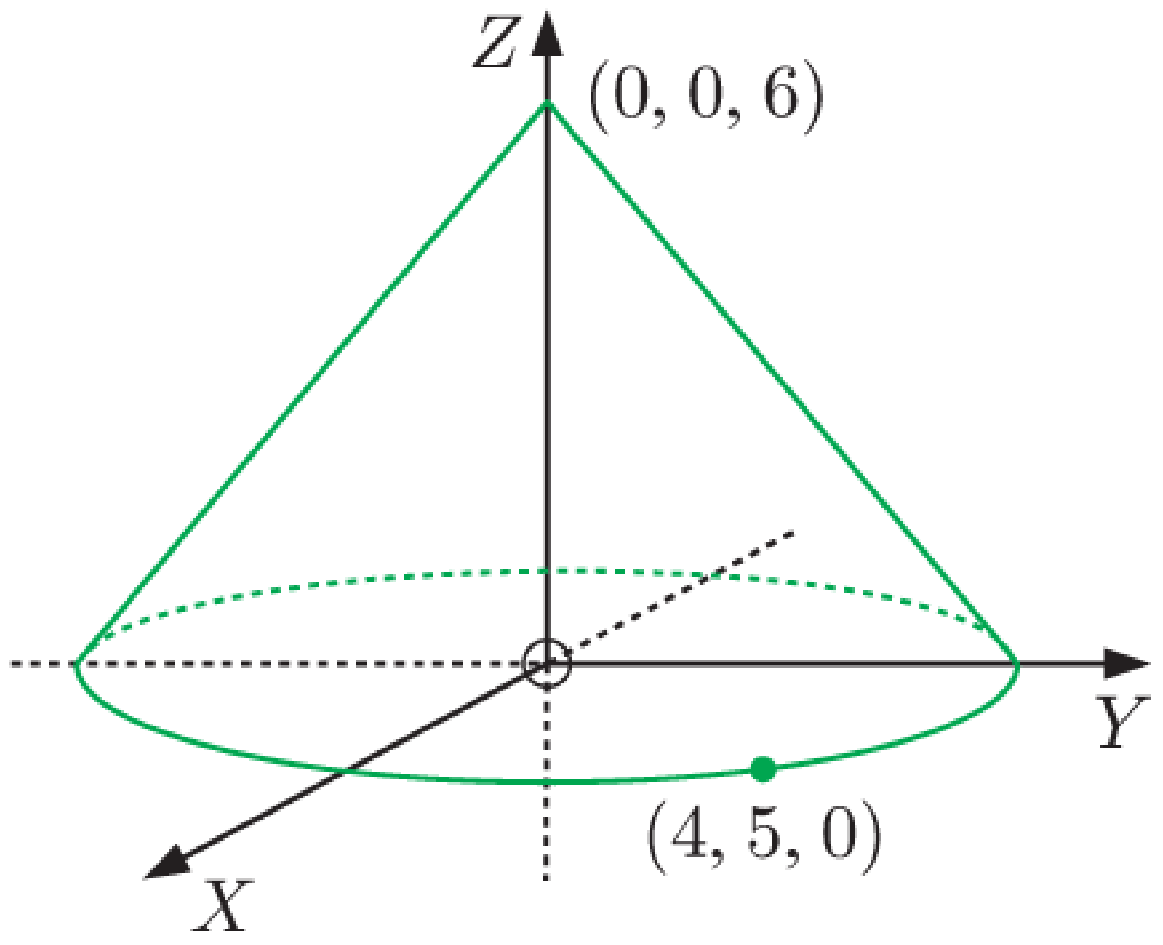
The point $(4, 5, 0)$ lies on the edge of the base, and the apex of the cone is $(0, 0, 6)$.

a Find the base radius of the cone.

b Find the exact volume of the cone.

c Find the slant height of the cone.

d Hence find the surface area of the cone.



6 The point $(-4, -6, 10)$ lies on the surface of a sphere with centre $(2, 3, -1)$.

a Find the radius of the sphere.

b Hence find the volume of the sphere.

7 A sphere has diameter [PQ], where P is $(-1, 1, 2)$, and Q is $(-5, 7, -8)$.

a Locate the centre of the sphere.

b Find the radius of the sphere.

c Hence find the volume and surface area of the sphere.

8 The base of a cylinder lies in the X-Y plane, and is centred at $(1, -3, 0)$. The point $(-2, -2, 0)$ lies on the edge of the base, and the cylinder has volume 40π units^3.

a Find the height of the cylinder.

b The point $(3, k, 2)$ lies on the curved surface of the cylinder. Find the possible values of k.

c Find the surface area of the cylinder.

9 A square-based pyramid has coordinates A$(4, 0, 2)$, B$(-1, 3, 2)$, C$(1, -5, 2)$, and D. The apex lies directly above the centre of the base, and the pyramid has surface area 210 units^2.

a Find the coordinates of D.

b Find the height of the pyramid, correct to 1 decimal place.

c Hence find the volume of the pyramid.

10 The map for a park is shown alongside, where each grid unit represents 10 m.

A large tent is to be constructed on the park for a festival. It will be a rectangular-based pyramid with base ABCD indicated on the map. The apex of the tent is 15 m above the centre of the base.

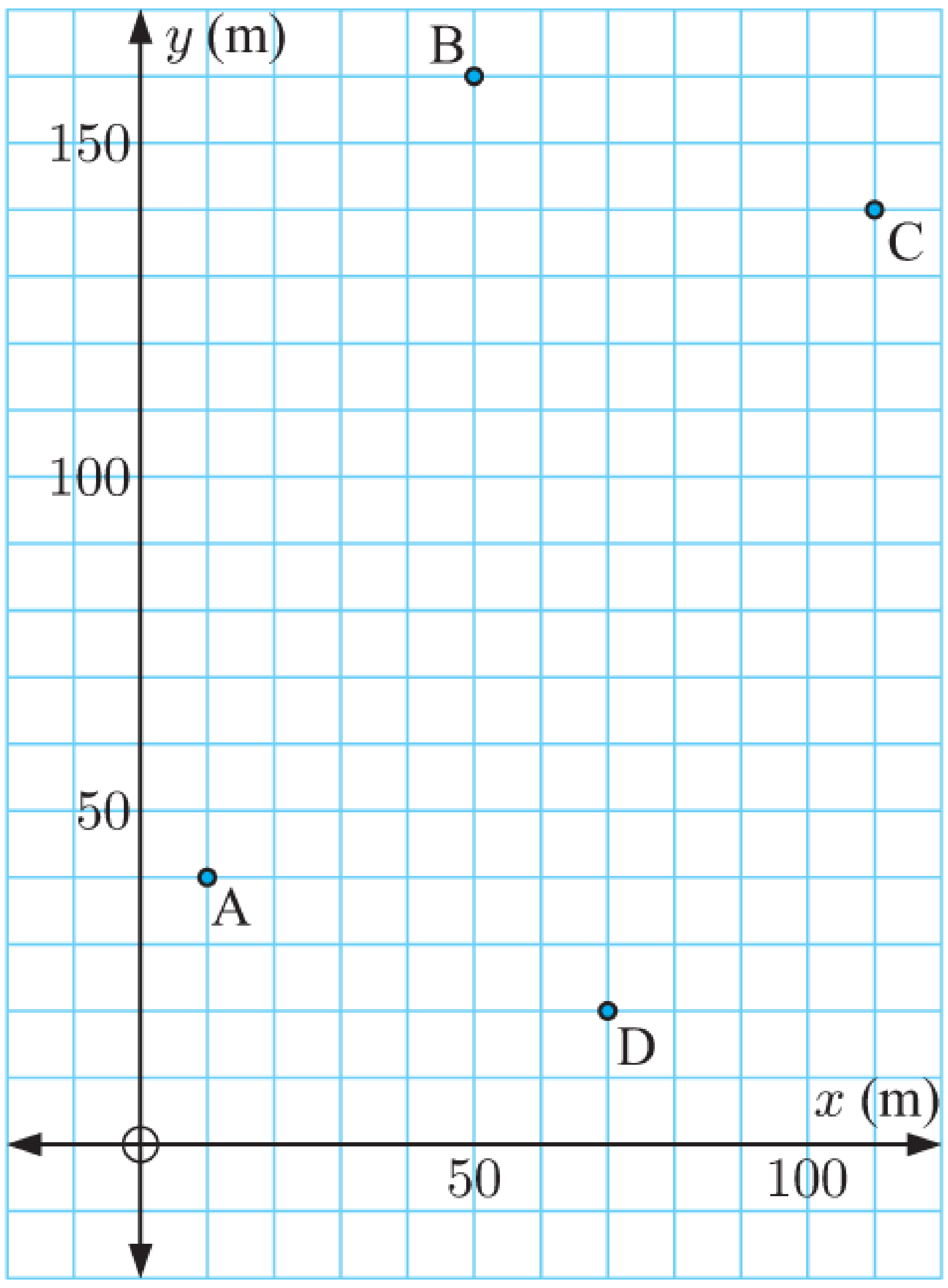
a Suppose ground level has Z-coordinate 0. Find the 3-dimensional coordinates of:

  i each corner of the base

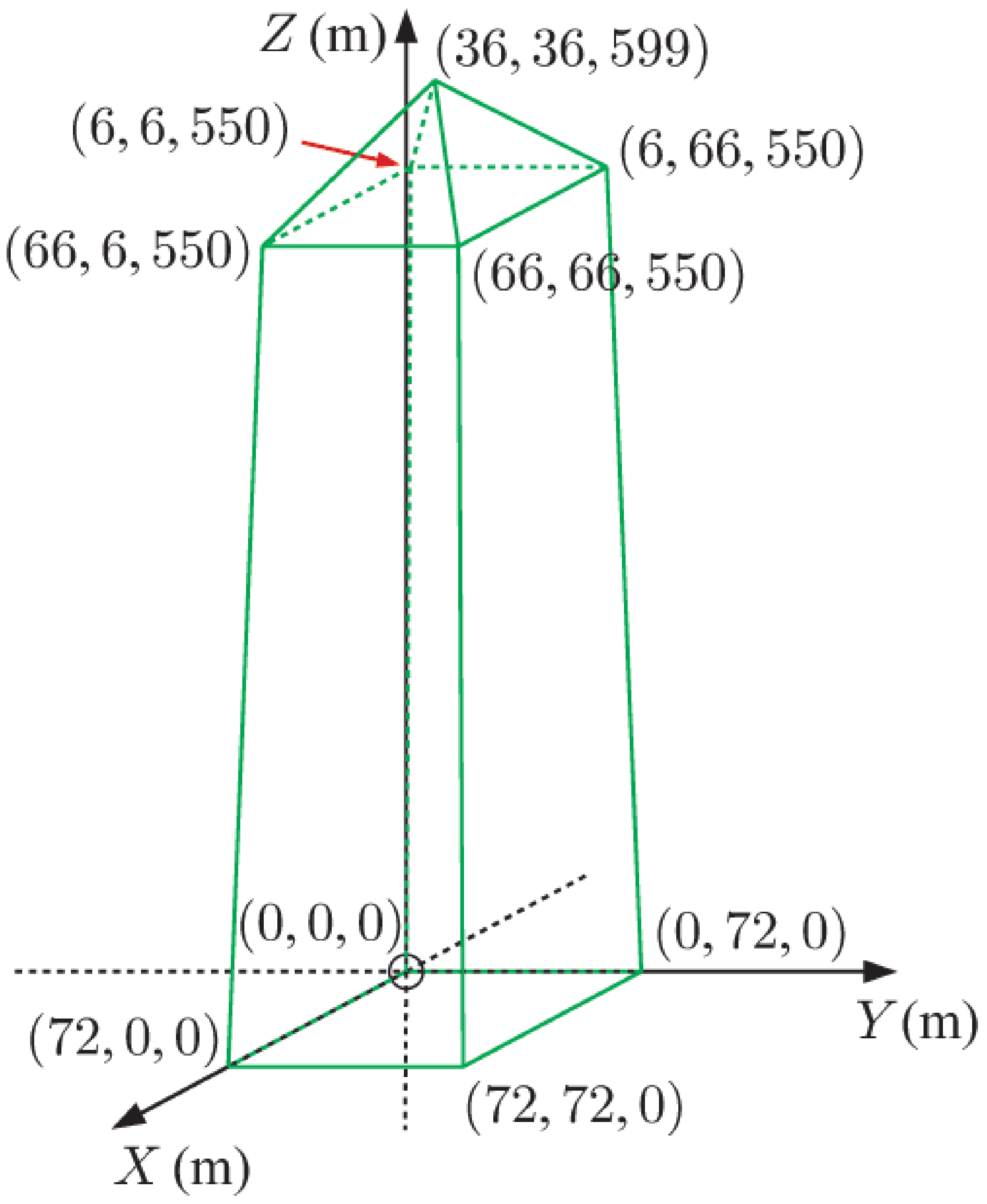
  ii the apex of the tent.

b Find the volume of air inside the tent.

c Find the amount of material needed for the tent. Do not include the floor.

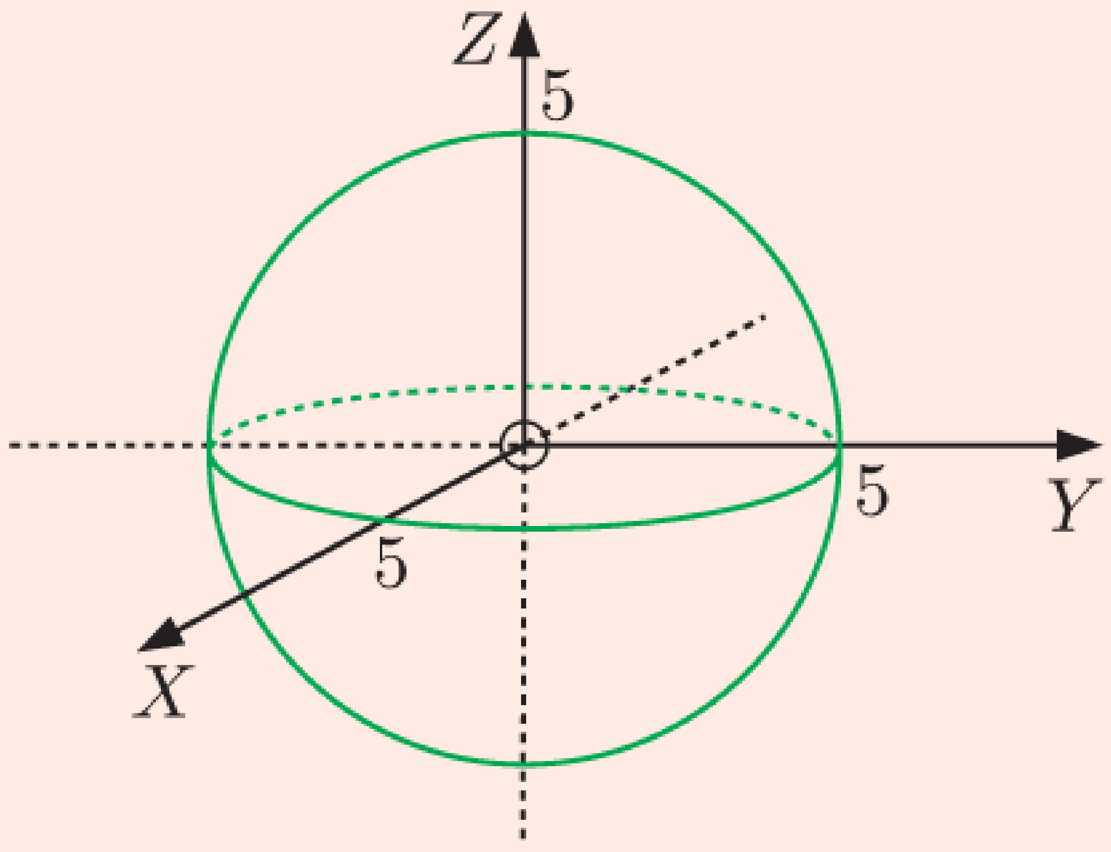


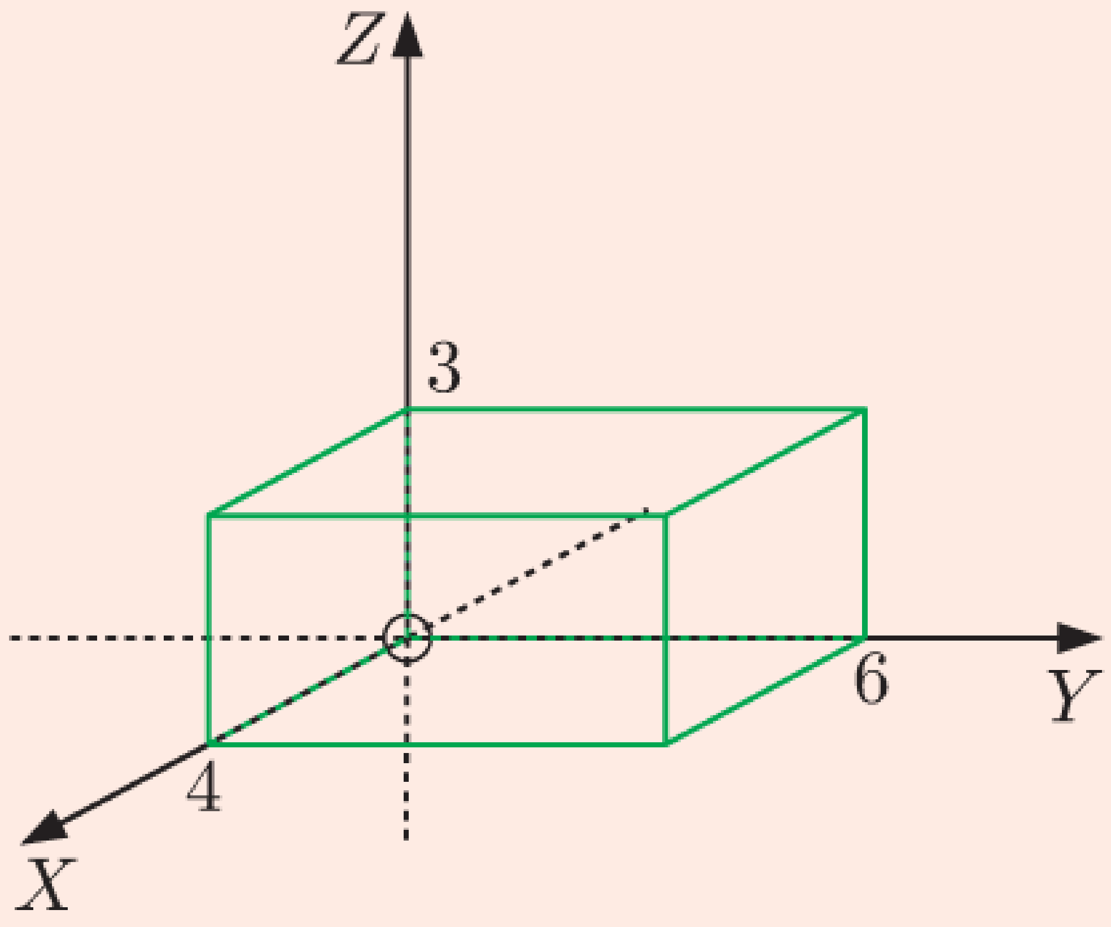
11 The Ping An Finance Centre in Shenzhen, China, is the fourth tallest building in the world.
Find the volume of this building.

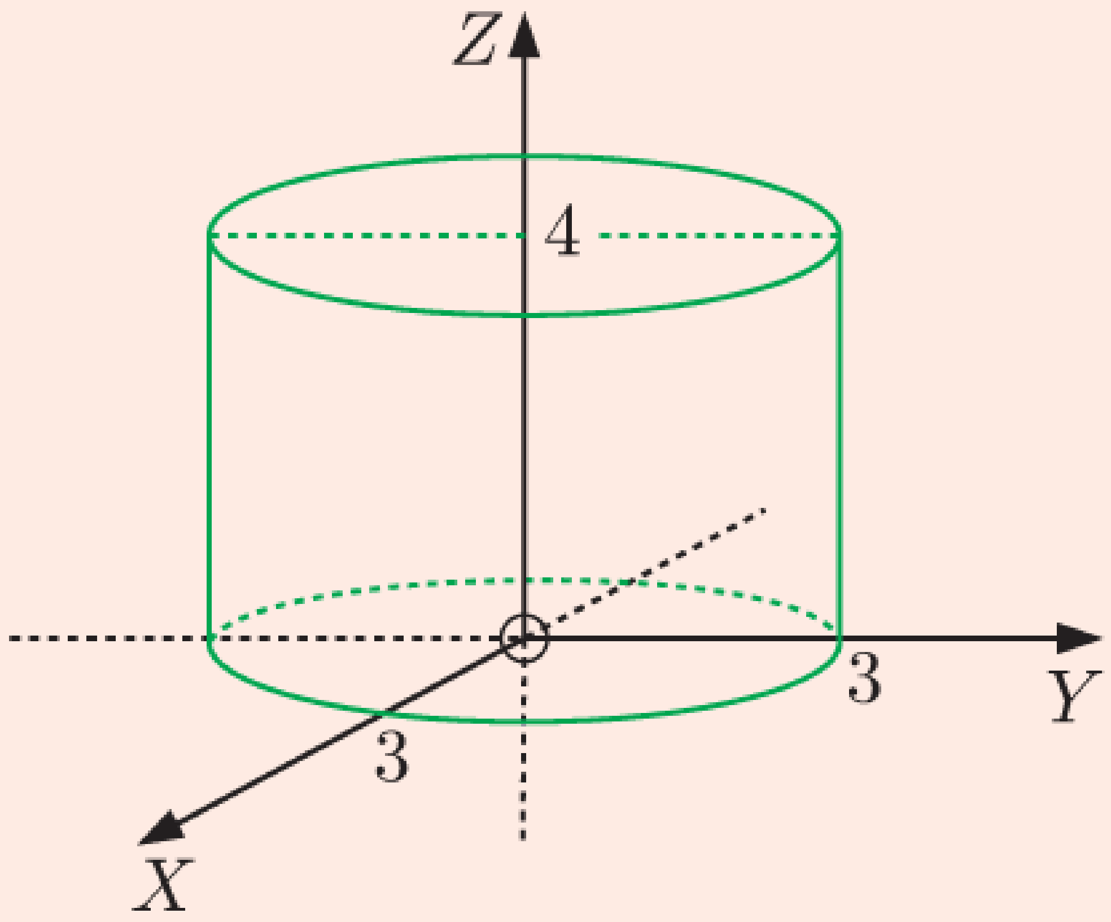


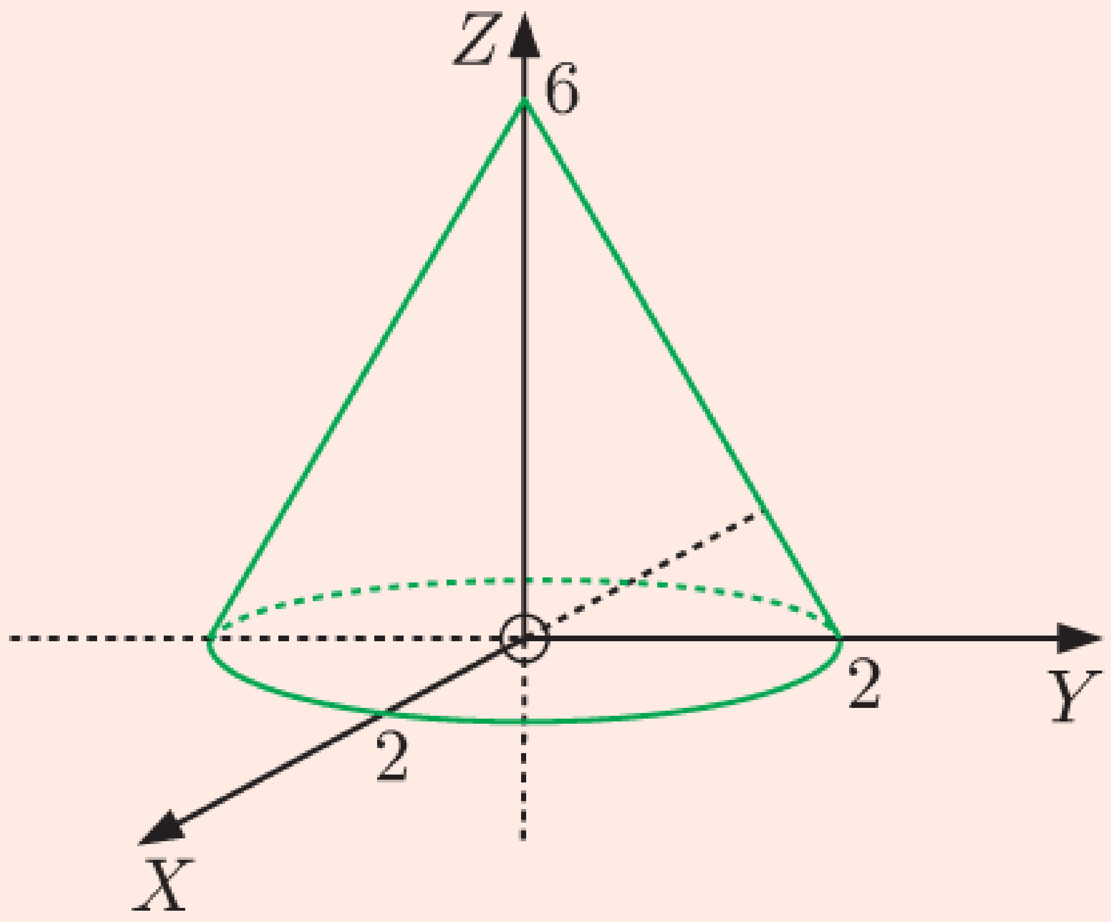
DISCUSSION

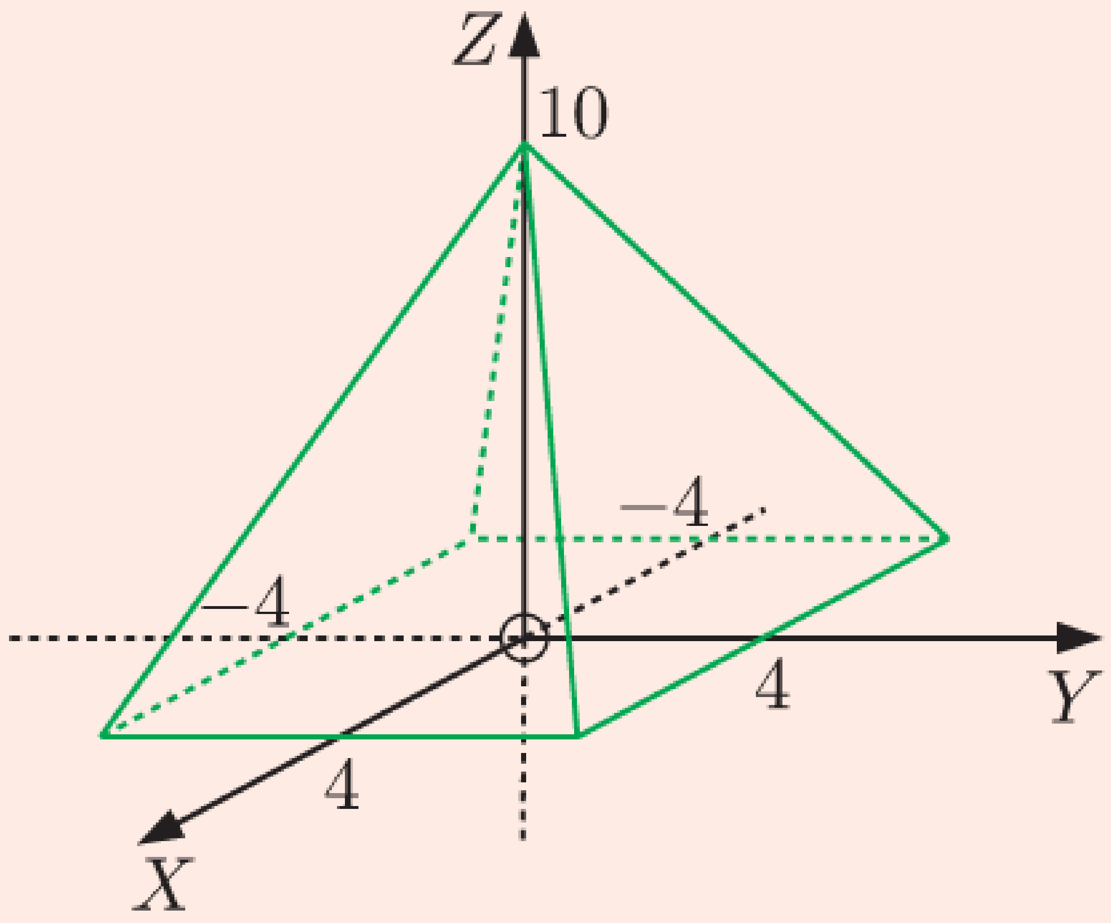
Describe an algebraic test to determine whether a given point (a, b, c) lies inside each solid:

- 

- 

- 

- 

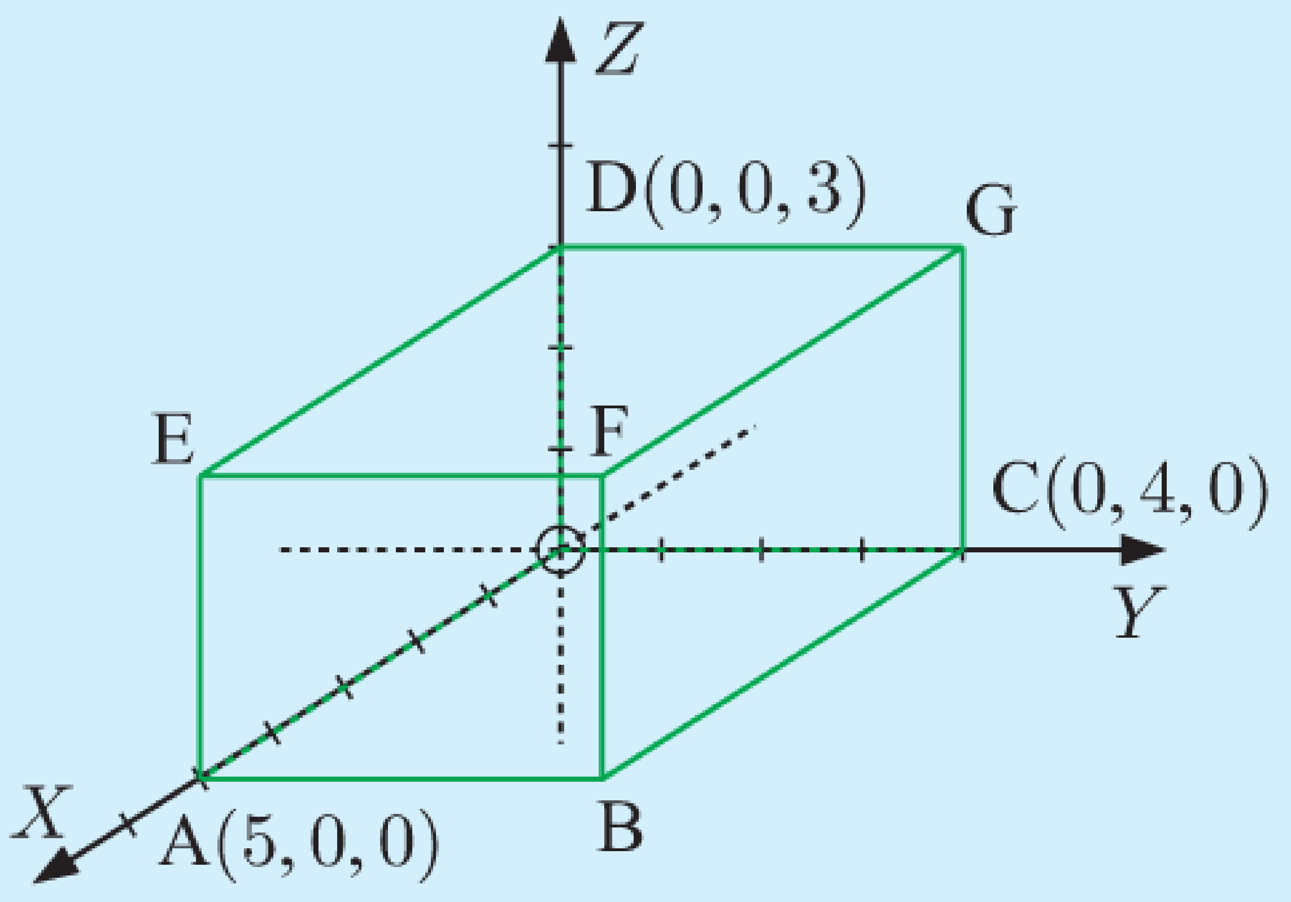
- 

C TRIGONOMETRY

The trigonometry techniques we have studied can also be applied to figures in 3-dimensional space.

Example 3 **Self Tutor**

Find the angle between the line segment [EC] and the base plane ABCO.



The required angle is $A\widehat{C}E$.

Now $AE = 3$ units

and $AC = \sqrt{(0-5)^2 + (4-0)^2 + (0-0)^2}$

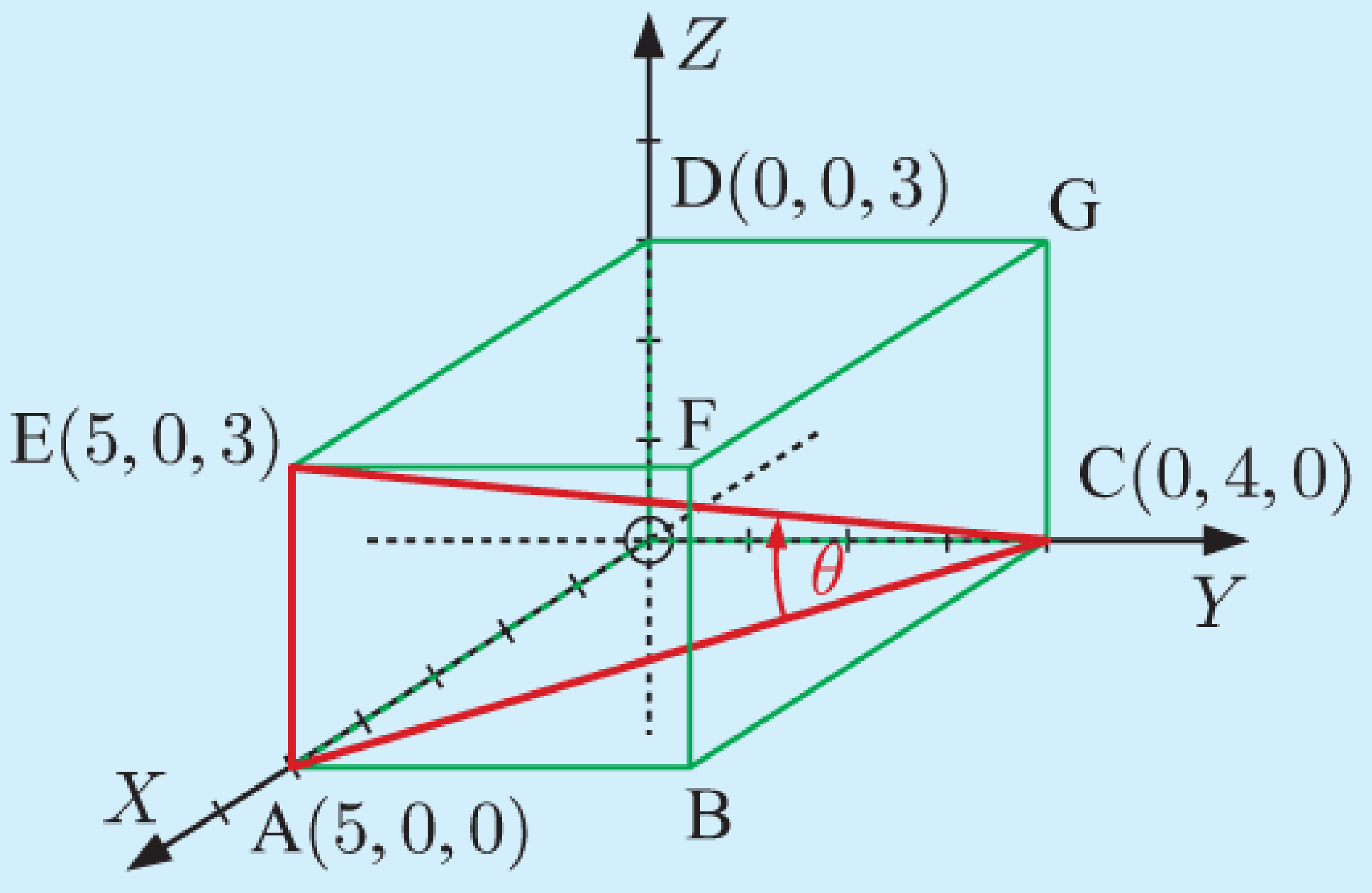
$= \sqrt{(-5)^2 + 4^2}$

$= \sqrt{41}$ units

$\therefore \tan\theta = \frac{3}{\sqrt{41}}$

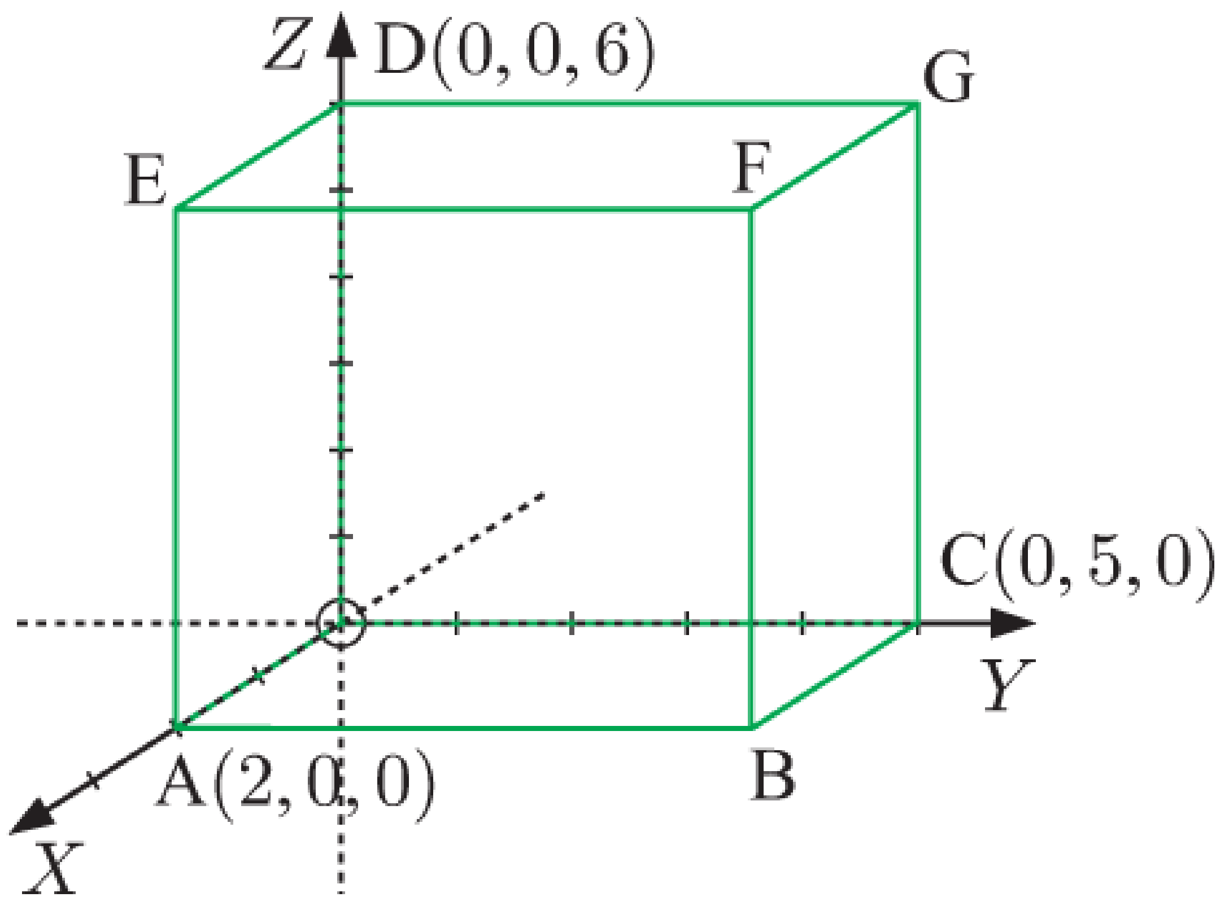
$\therefore \theta = \tan^{-1}\left(\frac{3}{\sqrt{41}}\right) \approx 25.1°$

The angle is about 25.1°.



EXERCISE 10C

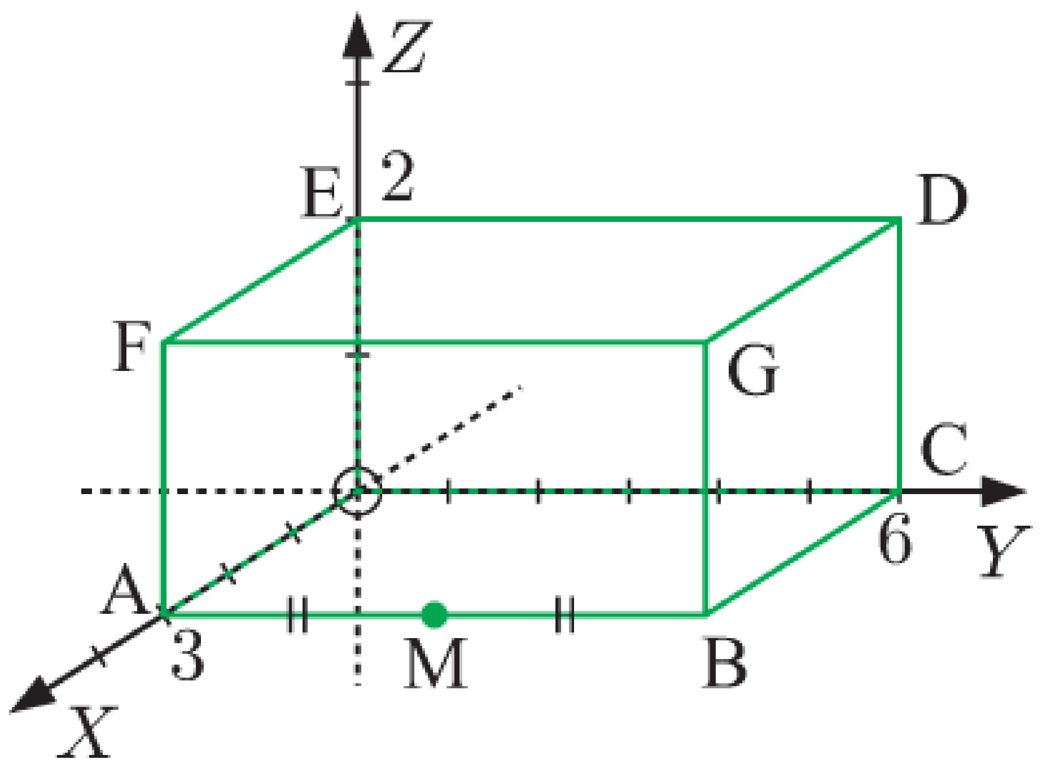
1



Find the angle between the following line segments and the base plane ABCO:

a [BE] **b** [AG]

2



Find:

a the midpoint M of [AB]

b $A\widehat{D}O$

c $B\widehat{M}D$.

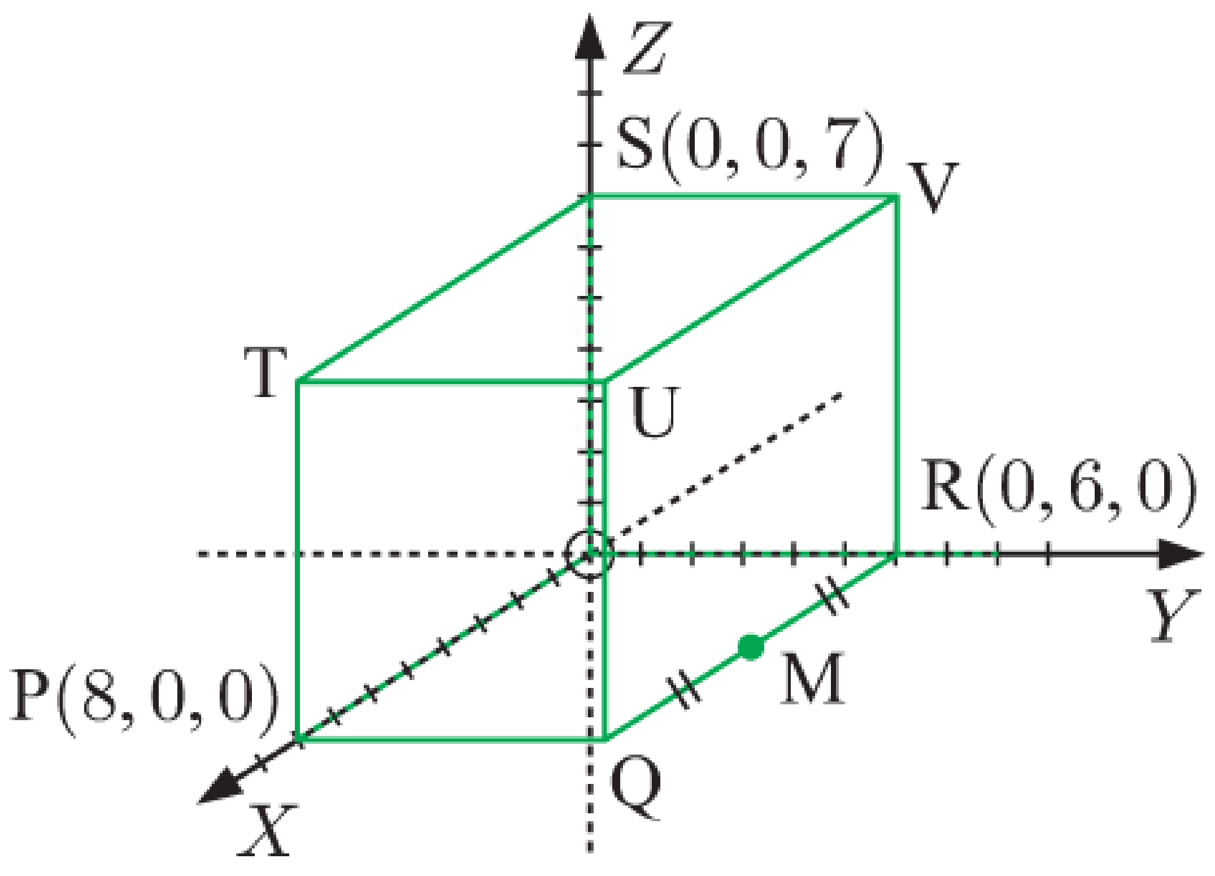
3 M is the midpoint of [QR].

a Find the coordinates of M.

b Find the measure of $Q\widehat{M}T$.

c Find the angle between the following line segments and the base plane:

i [QS] **ii** [TM]



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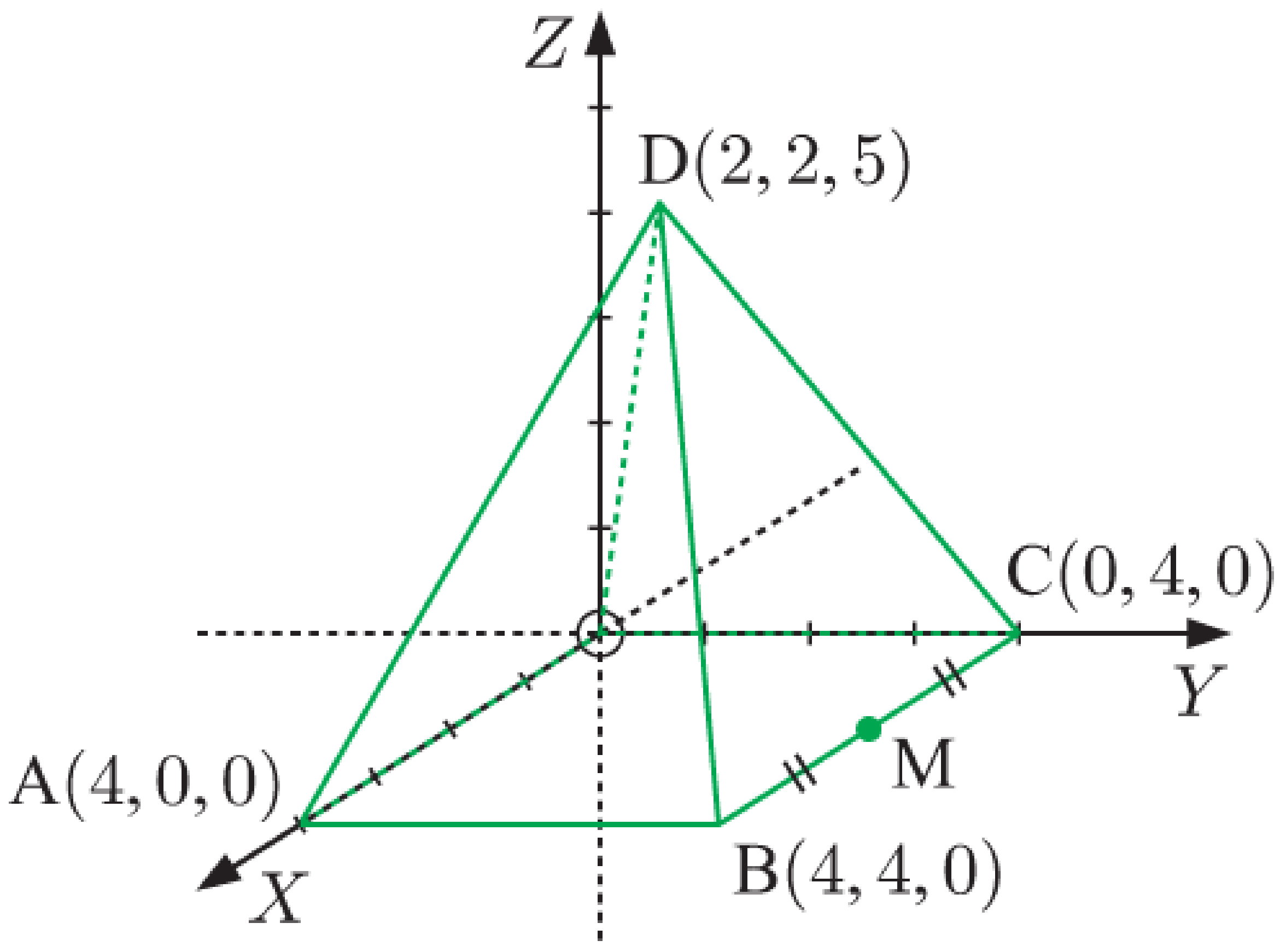


a State the coordinates of M.

b Find the angle between the following line segments and the base plane of the pyramid:

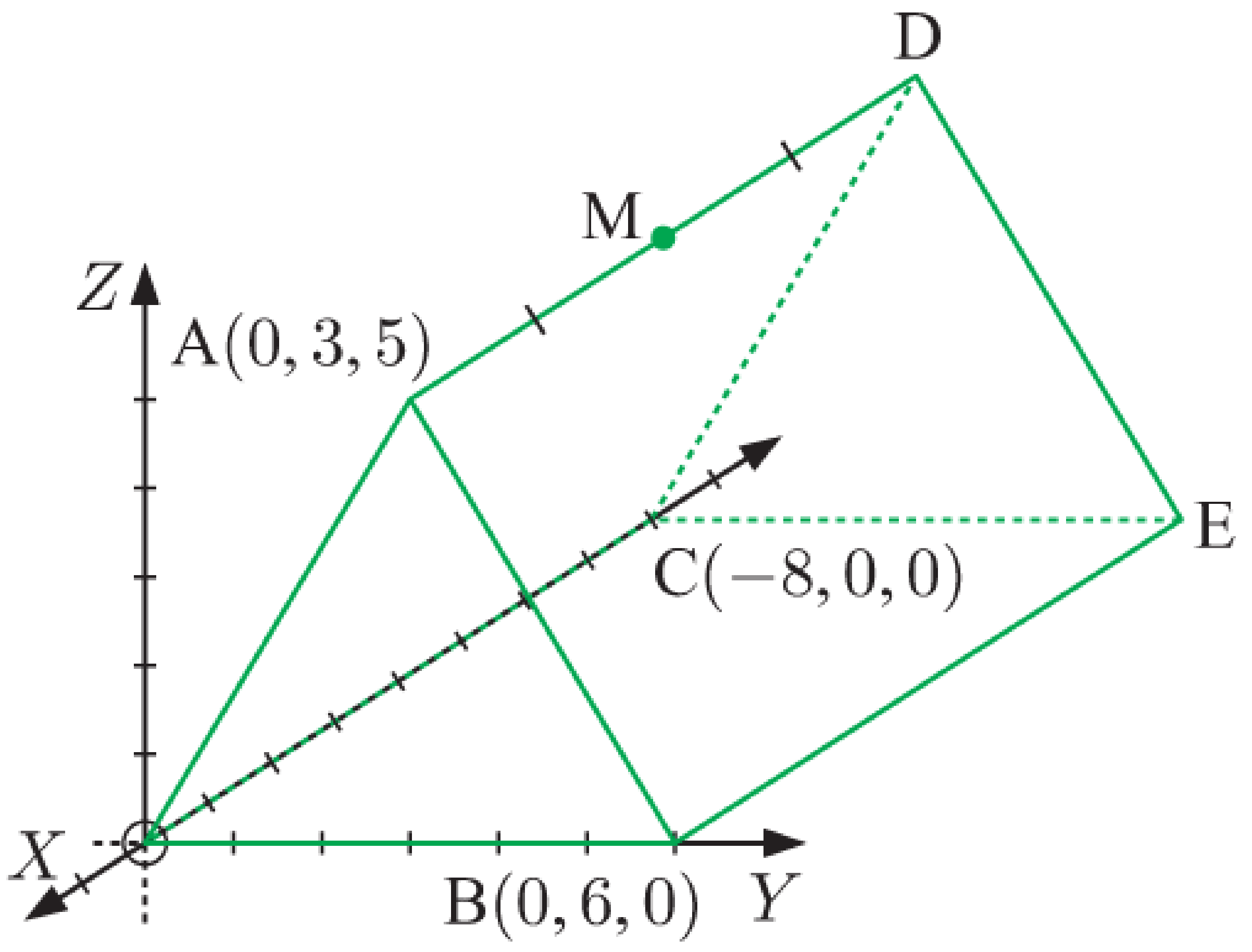
i [DM] **ii** [DA]

5 **a** State the coordinates of M.

b Find the angle between the following line segments and the base plane:

i [AE] **ii** [MB]

c Find the angle $A\widehat{B}M$.



Example 4 **Self Tutor**

Given P(1, 4, 5), Q(−2, 0, 1), and R(0, −2, 6), find the measure of $P\widehat{Q}R$.

$$PQ = \sqrt{(-2-1)^2 + (0-4)^2 + (1-5)^2} = \sqrt{(-3)^2 + (-4)^2 + (-4)^2} = \sqrt{41} \text{ units}$$

$$PR = \sqrt{(0-1)^2 + (-2-4)^2 + (6-5)^2} = \sqrt{(-1)^2 + (-6)^2 + 1^2} = \sqrt{38} \text{ units}$$

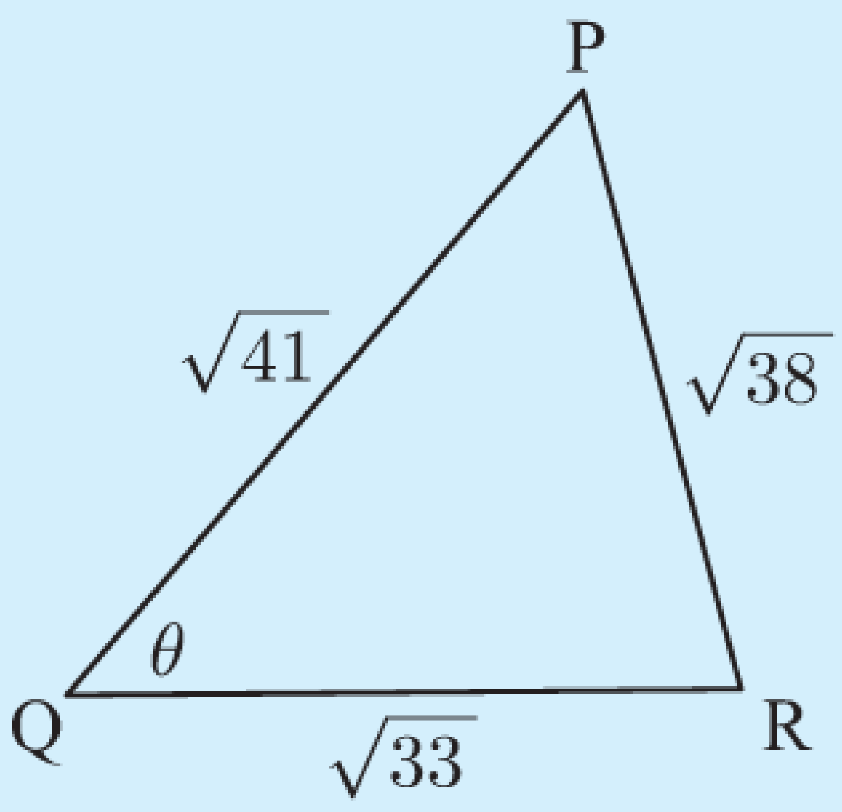
$$QR = \sqrt{(0--2)^2 + (-2-0)^2 + (6-1)^2} = \sqrt{2^2 + (-2)^2 + 5^2} = \sqrt{33} \text{ units}$$

By the cosine rule,

$$\cos\theta = \frac{(\sqrt{41})^2 + (\sqrt{33})^2 - (\sqrt{38})^2}{2 \times \sqrt{41} \times \sqrt{33}}$$

$$\therefore \quad \cos\theta = \frac{41 + 33 - 38}{2 \times \sqrt{41} \times \sqrt{33}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{36}{2 \times \sqrt{41} \times \sqrt{33}}\right) \approx 60.7°$$



So, $P\widehat{Q}R$ is about 60.7°.

6 Consider the points $A(-1, 0, 2)$, $B(4, 1, 1)$, and $C(-2, 2, 0)$.

a Find the length of:

i [AB] **ii** [AC] **iii** [BC].

b Find the measure of $A\widehat{B}C$.

7 Find the measure of $P\widehat{Q}R$ given:

a $P(0, 2, -2)$, $Q(3, -1, -4)$, $R(4, 0, -1)$ **b** $P(-3, 2, 1)$, $Q(5, 0, 1)$, $R(-3, -1, 2)$.

8 Consider the points $A(-1, -5, 2)$, $B(0, -2, 3)$, and $C(4, 0, -2)$.

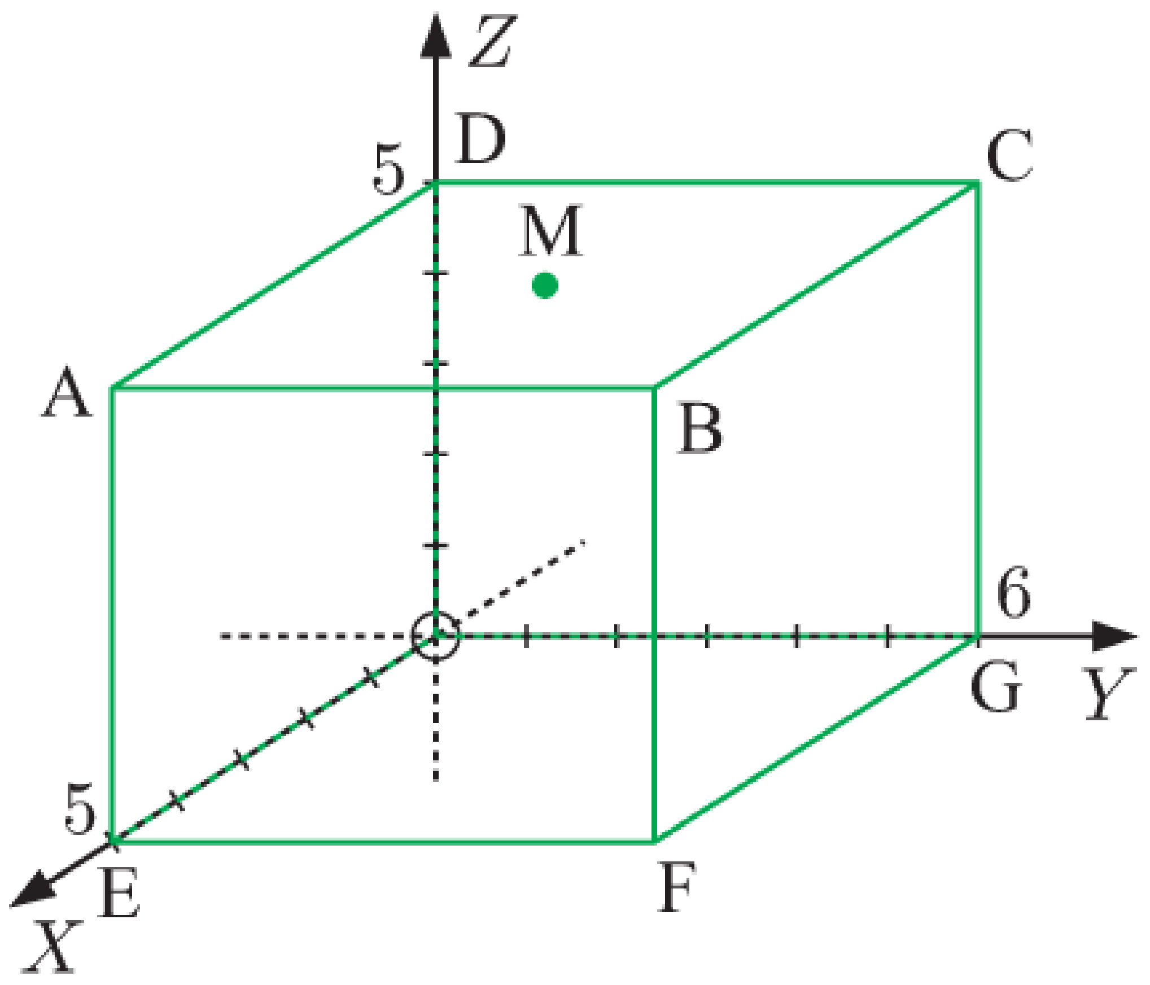
a Find the measure of $B\widehat{A}C$. **b** Hence find the area of triangle ABC.

9 Suppose A is $(1, 2, 3)$, B is $(2, 5, k)$, C is $(5, 1, 0)$, and $B\widehat{A}C = 60°$. Find k.

10 **a** Find the coordinates of M, the centre of the face ABCD.

b Calculate the angle:

i $O\widehat{E}M$ **ii** $A\widehat{M}G$ **iii** $O\widehat{M}F$.



11 Answer the **Opening Problem** on page **234**.

12 The radar from an airport's control centre is shown alongside. The grid units are kilometres.

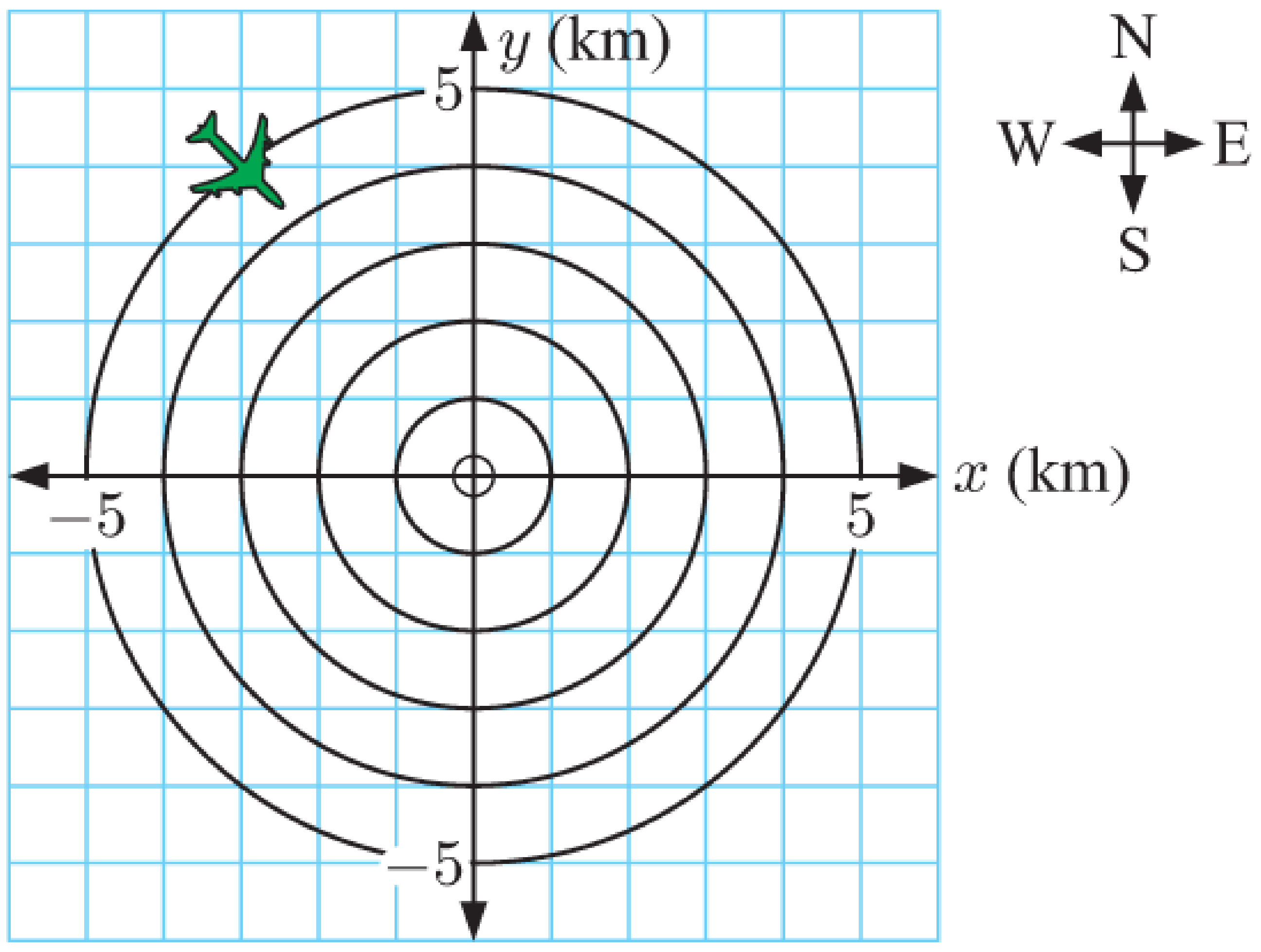
An aeroplane with altitude 500 m appears on the radar at $(-3, 4)$. It is approaching a runway 1 km east of the control centre.

a Find the 3-dimensional coordinates of the plane.

b How far is the plane from the control centre?

c Find the true bearing of the runway from the plane.

d Find the angle of the plane's descent.



13 Explorers Gabriel, Jack, and Malina are located at $(2, 3, 1)$, $(-1, -3, 1.2)$, and $(0, 4, 0.7)$ respectively. The units are kilometres and $Z = 0$ represents sea level.

a How far apart are Jack and Malina?

b Write the explorers in order of altitude, from highest to lowest.

c Find the angle of:

i elevation from Gabriel to Jack

ii depression from Gabriel to Malina.

THEORY OF KNOWLEDGE EUCLID'S POSTULATES

Euclid was one of the great mathematical thinkers of ancient times. He founded a school in Alexandria during the reign of Ptolemy I, which lasted from 323 BC until 284 BC.

Euclid's most famous mathematical writing is a set of 13 books called *Elements*. It is the first attempt to create a complete, systematic study of geometry as pure mathematics, and has been a major source of information for the study of geometric techniques, logic, and reasoning.

Elements is sometimes regarded as the most influential textbook ever written, consisting of definitions, postulates, theorems, constructions, and proofs. It was first printed in 1482 in Venice, making it one of the first mathematical works ever printed.

The foundation of Euclid's work is his set of postulates, which are the assumptions or **axioms** used to prove further results. Euclid's postulates are:

THE ELEMENTS
OF GEOMETRIE
of the moſt auncient Philoſopher
EVCLIDE
of Megara.

Faithfully (now firſt) tranſlated into the Engliſhe toung, by H. Billingſley, *Citizen of London. Whereunto are annexed certaine Scholies, Annotations, and Inuentions, of the beſt Mathematiciens, both of time paſt, and in this our age.*

With a very fruitfull Præface made by M. I. Dee, *ſpecifying the chiefe Mathematicall Sciẽces, what they are, and whereunto commodious: where, alſo, are diſcloſed certaine new Secrets Mathematicall and Mechanicall, vntill theſe our daies, greatly miſſed.*

Imprinted at London by *Iohn Daye.*

1. Any two points can be joined by a straight line.
2. Any straight line segment can be extended indefinitely in a straight line.
3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as centre.
4. All right angles are congruent.
5. **Parallel postulate**: If two lines intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

1 Can an axiom be proven? Is an axiom necessarily true?

2 Consider the first postulate.

- **a** What is a straight line? How do you know that a line is straight?
- **b** Is straightness more associated with shortest distance or with shortest time? Does light travel in a straight line?
- **c** Is straightness a matter of perception? Does it depend on the reference frame of the observer?

VIDEO

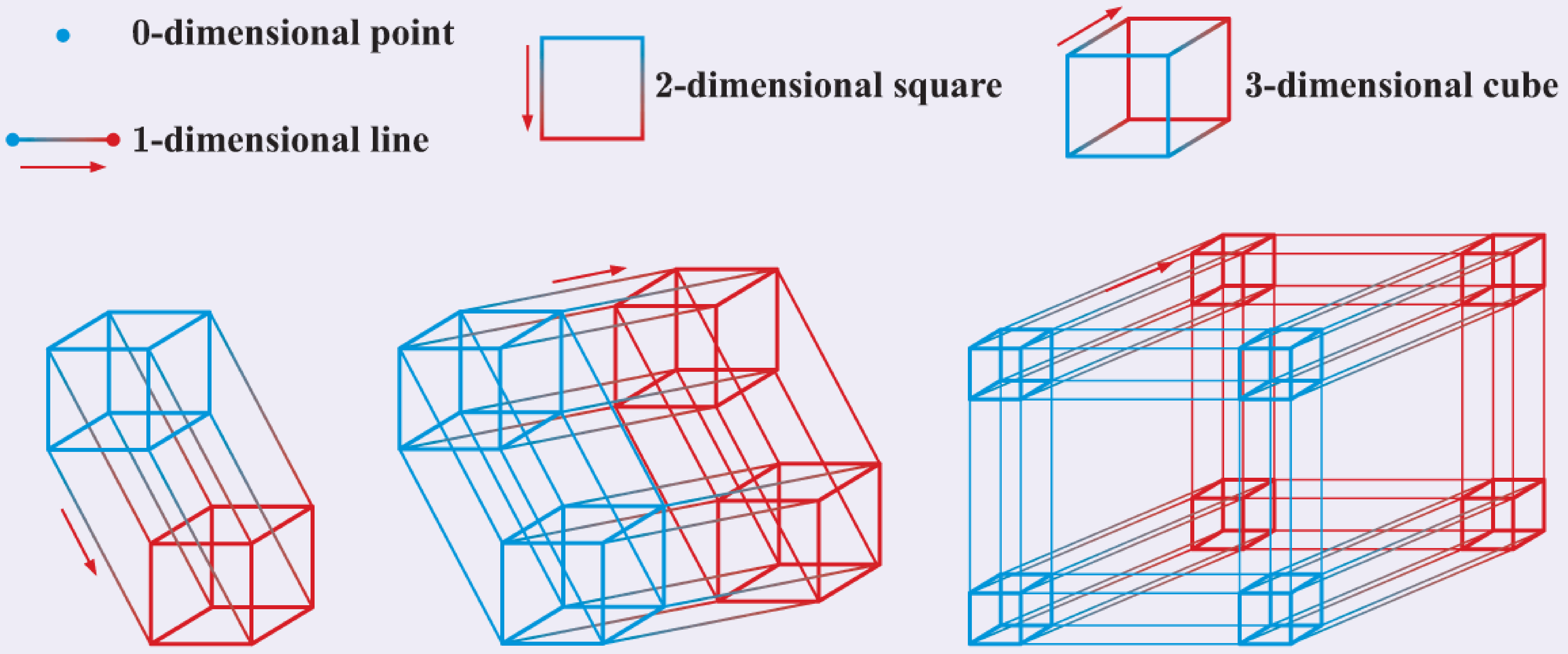
3 For hundreds of years, many people believed the world to be flat. It was then discovered the world was round, so that if you travelled for long enough in a particular direction, you would return to the same place, but at a different time.

- **a** How do we define *direction*?
- **b** Is a three-dimensional vector sufficient to describe a direction in space-time?
- **c** Can any straight line segment be extended indefinitely in a straight line?

4 Comment on the definition:

> A straight line is an infinite set of points in a particular direction.

5 We are used to representing 3-dimensional objects such as a cube on 2-dimensional paper. This process can be extended to give visual representation to "cubes" in higher dimensions:



4-dimensional cube **5-dimensional cube** **6-dimensional cube**

If the extra dimensions do not represent physical space, is there purpose to giving them physical representation?

6 Can the universe be completely described by a finite-dimensional space?

REVIEW SET 10A

1 On separate axes, plot the points:

a $(3, 1, 0)$ **b** $(-4, 0, 2)$ **c** $(2, -3, -1)$

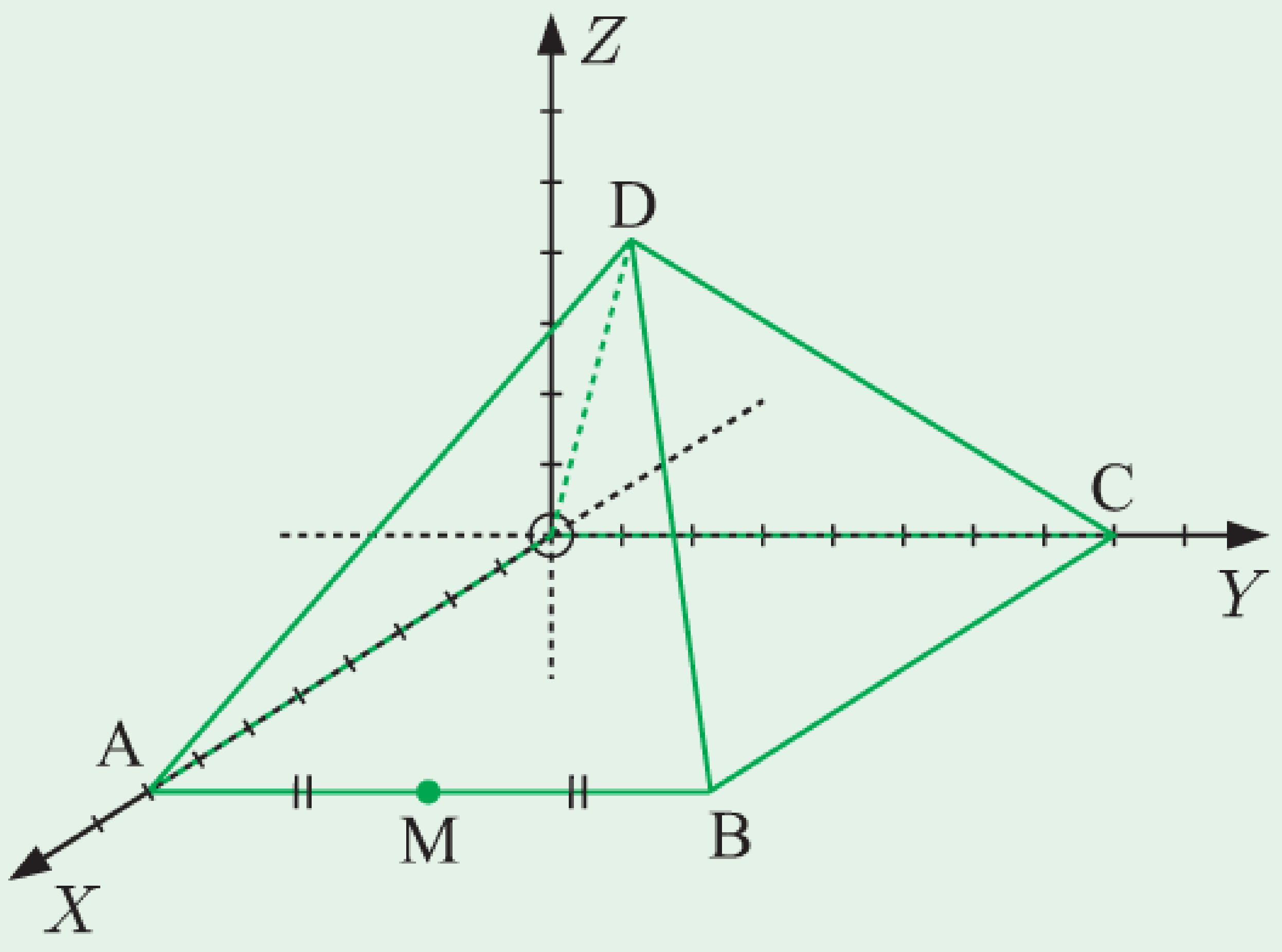
2 For each pair of points, find:

i the distance PQ **ii** the midpoint of [PQ].

a $P(1, -2, 0)$, $Q(-3, -6, 2)$ **b** $P(-3, 1, 6)$, $Q(-2, -7, 1)$

3 Suppose A is $(-2, 5, 1)$, B is $(3, 5, -3)$, and C is $(0, -1, 2)$. Determine whether triangle ABC is scalene, isosceles, or equilateral.

4



A square-based pyramid has base coordinates $O(0, 0, 0)$, $A(8, 0, 0)$, $B(8, 8, 0)$, and $C(0, 8, 0)$. The apex of the pyramid is $D(4, 4, 6)$. M is the midpoint of [AB].

a Find the volume of the pyramid.

b Find the coordinates of M.

c Find the length MD.

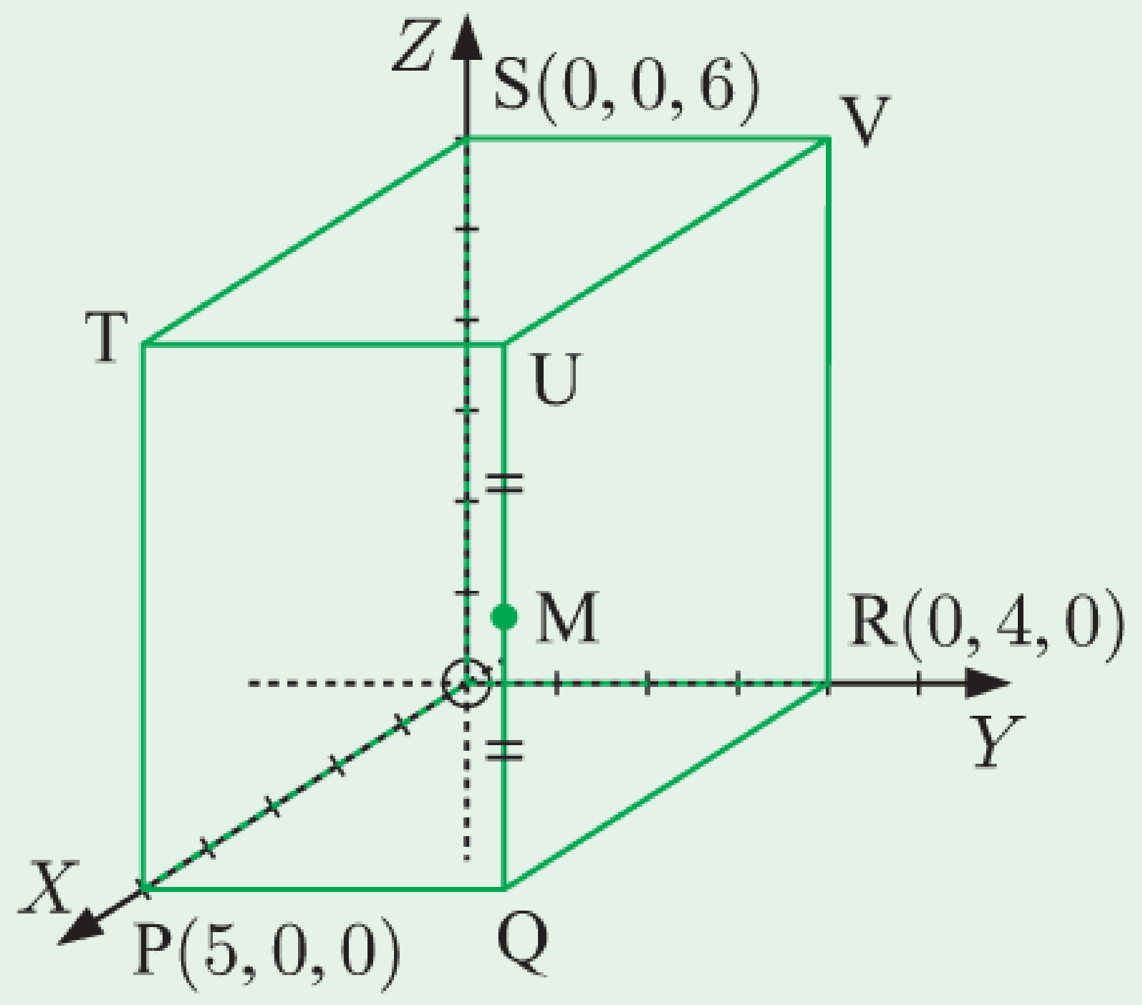
d Hence find the surface area of the pyramid.

e Find the measure of $M\widehat{D}B$.

5 The base of a hemisphere lies in the X-Y plane, and is centred at the origin. The point $(2, -5, 0)$ lies on the edge of the base.

- **a** Find the radius of the hemisphere.
- **b** Find the volume and surface area of the hemisphere.

6



In this rectangular prism, M is the midpoint of [UQ].

- **a** Find the coordinates of M.
- **b** Find angle $\text{U}\widehat{\text{M}}\text{S}$.
- **c** Find the angle between the following line segments and the base plane:
  - **i** [PV]
  - **ii** [OM]

7 Given A$(-1, 2, 5)$, B$(0, 3, 1)$, and C$(2, -4, 0)$, find the measure of $\text{B}\widehat{\text{A}}\text{C}$.

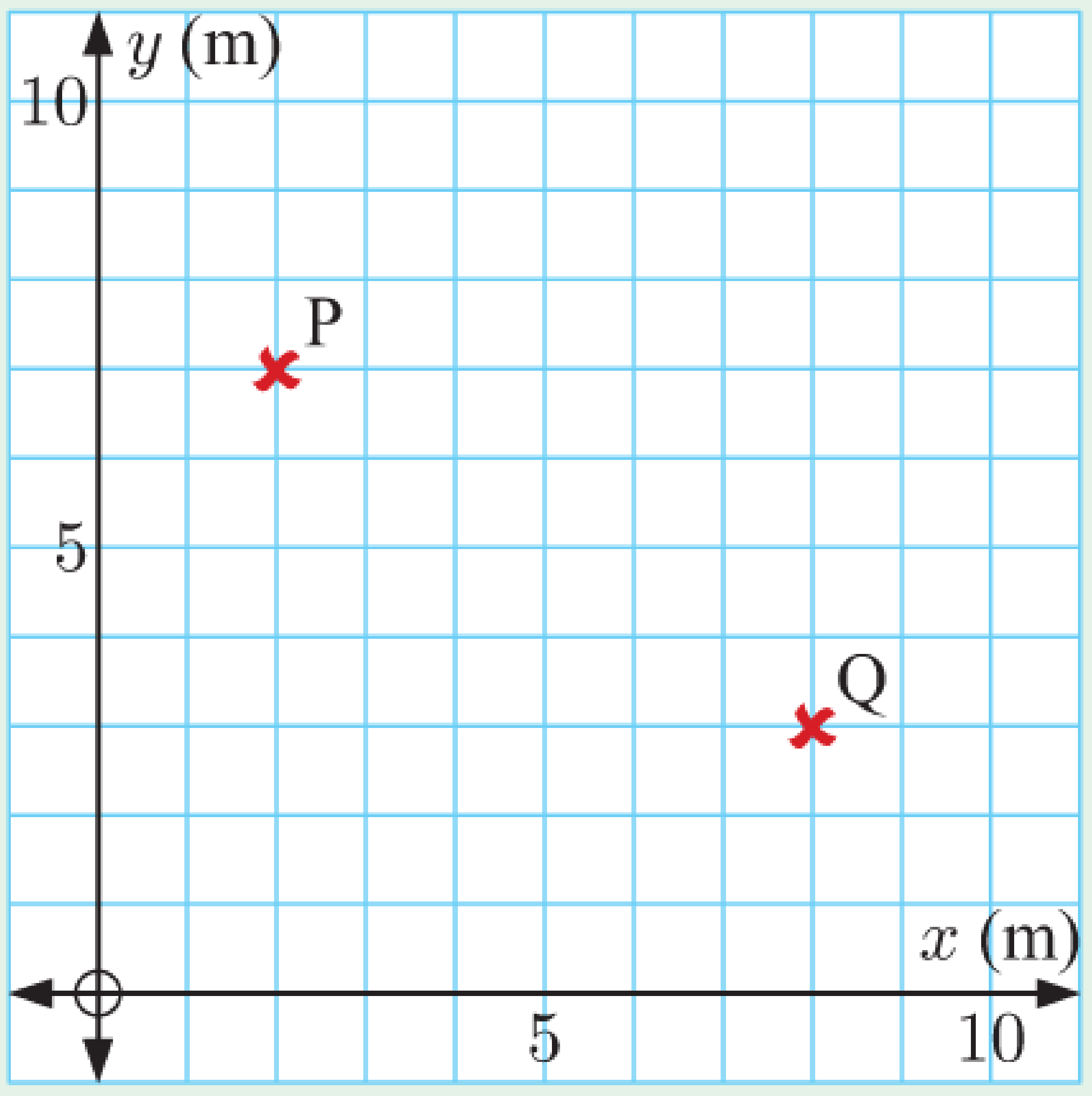
8 Consider the points A$(8, -7, 2)$, B$(-1, k, 8)$, and C$(1, -3, 2)$. CA = CB and $k > 0$.

- **a** Find k.
- **b** For the circle centred at C and passing through A and B, find the area of the minor sector CAB.

9 An archaeological dig site has been divided up using grid lines at 1 metre intervals, to make it easier to describe locations on the site.

Fossils have been found at P, 2.5 m underground, and at Q, 2.9 m underground.

- **a** Suppose ground level has Z-coordinate 0. Find the 3-dimensional coordinates of each fossil.
- **b** Find the distance between the fossils.
- **c** Find the angle of depression from P to Q.



REVIEW SET 10B

1 For each pair of points, find:

- **i** the distance AB
- **ii** the midpoint of [AB].

- **a** A$(-3, 0, 5)$, B$(-1, 6, 4)$
- **b** A$(-7, 4, 6)$, B$(-2, 1, -1)$

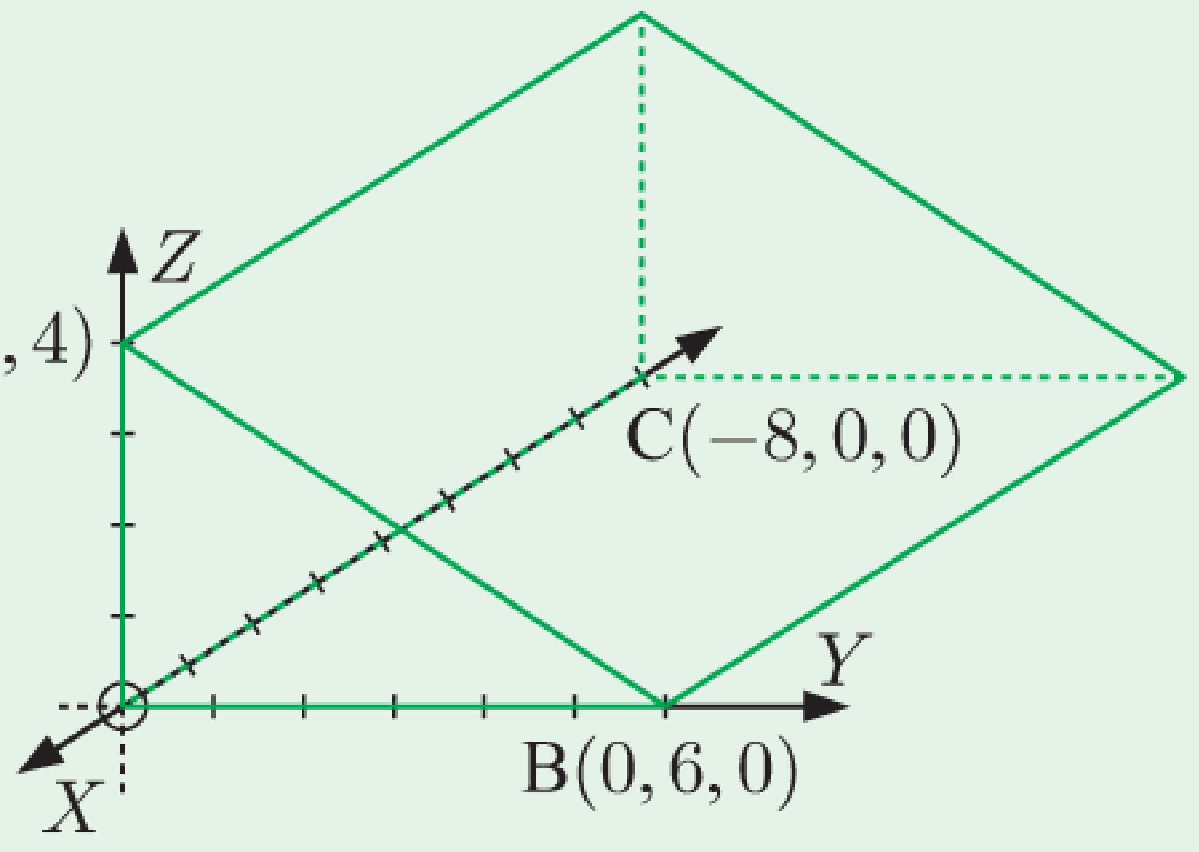
2 Suppose P is $(-5, 0, 1)$, Q is $(-2, -2, 2)$, and R is $(-1, 5, -1)$.

- **a** Show that triangle PQR is right angled.
- **b** Find the measure of $\text{P}\widehat{\text{Q}}\text{R}$.

3 The distance from $(4, -2, 1)$ to $(1, 3, k)$ is 8 units. Find the possible values of k.

4 Consider the triangular-based prism shown.

- **a** Find the volume of the prism.
- **b** Find the length of [AB].
- **c** Hence find the surface area of the prism.



5 A sphere has diameter [PQ], where P is $(4, -2, 3)$ and Q is $(-6, 2, -5)$.

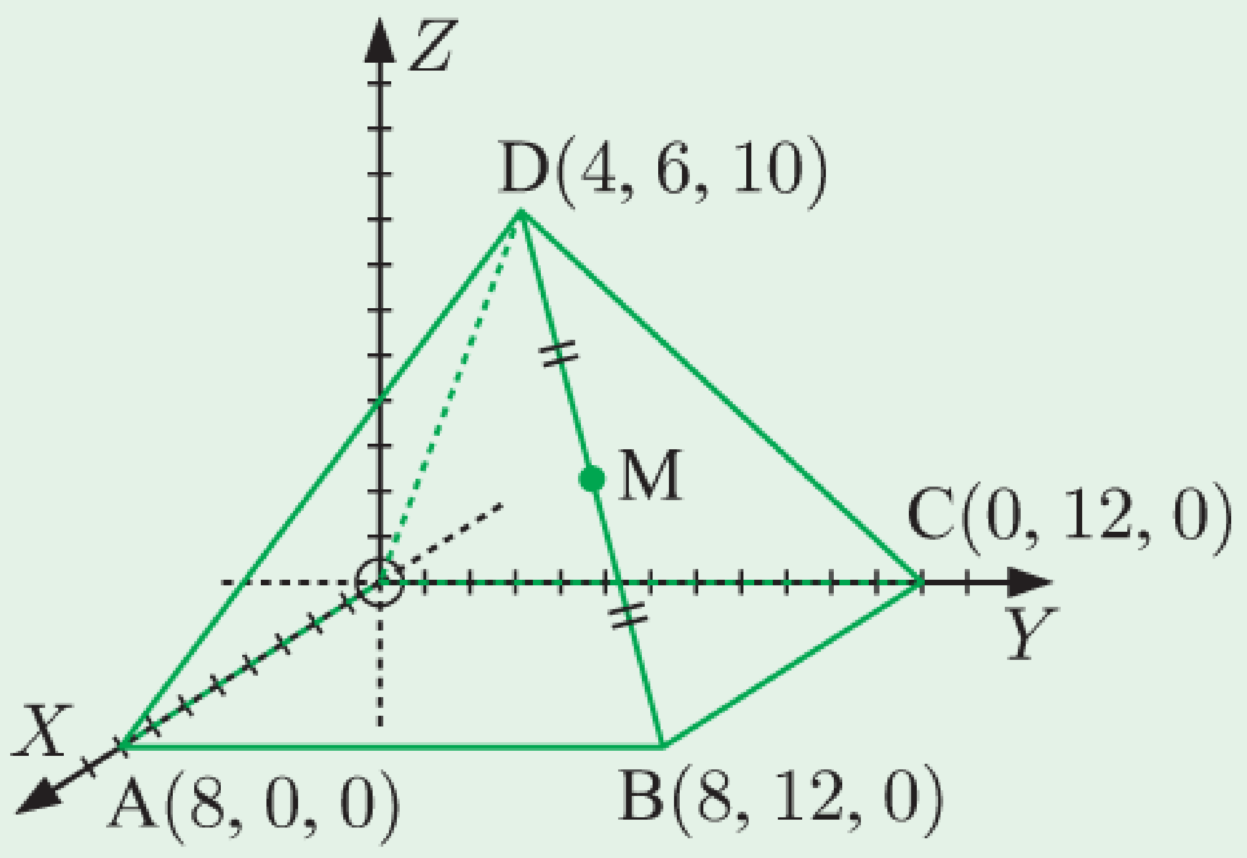
- **a** Find the coordinates of the centre of the sphere.
- **b** Find the radius of the sphere.
- **c** Hence find the volume and surface area of the sphere.

6 Consider the points $P(-2, 1, 3)$, $Q(0, -1, 4)$, and $R(-3, 2, 0)$.

- **a** Find the measure of $P\widehat{R}Q$.
- **b** Hence find the area of triangle PQR.

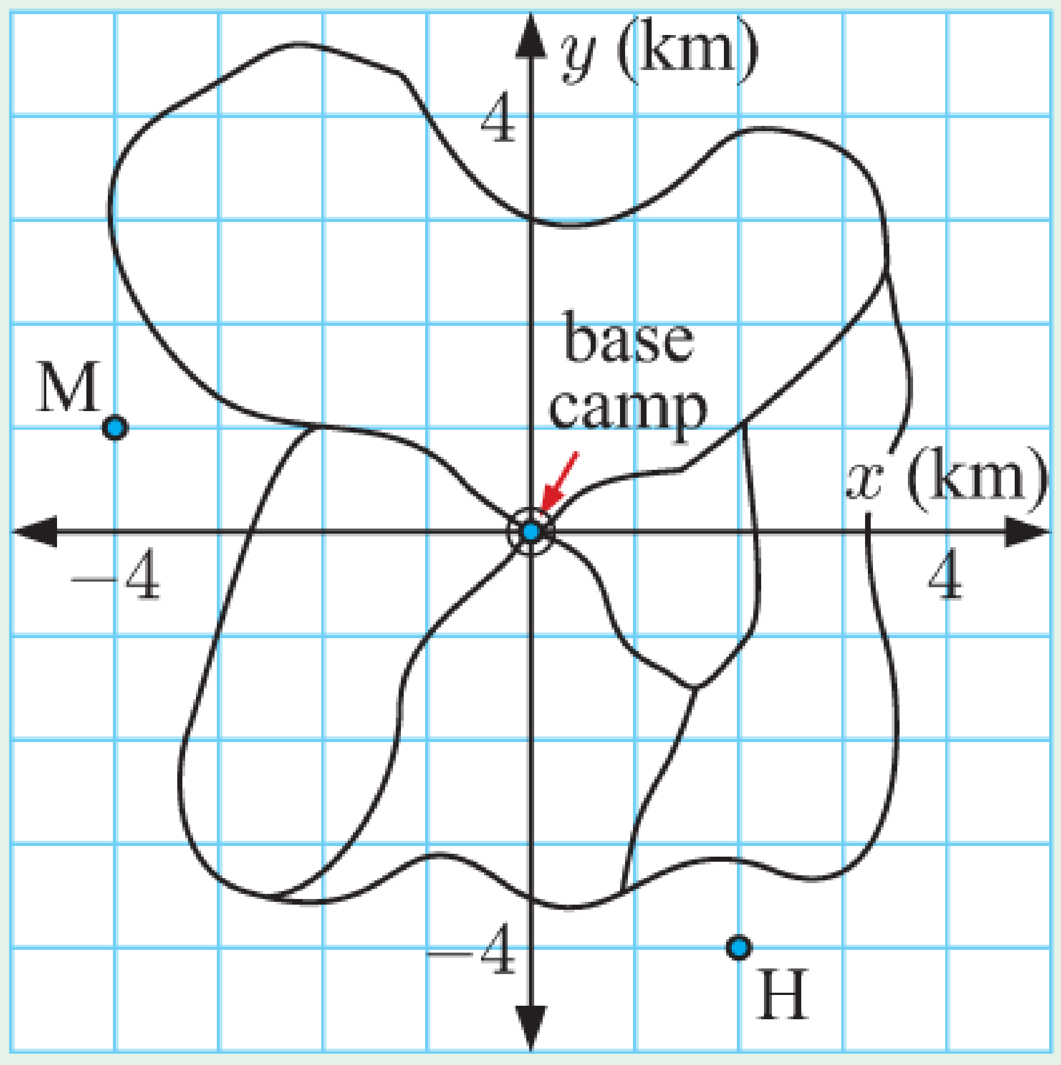
7 In the rectangular-based pyramid alongside, M is the midpoint of [BD].

- **a** Find the coordinates of M.
- **b** Find $A\widehat{D}C$.
- **c** Find the angle between the following line segments and the base plane:
  - **i** [DA]
  - **ii** [MC]



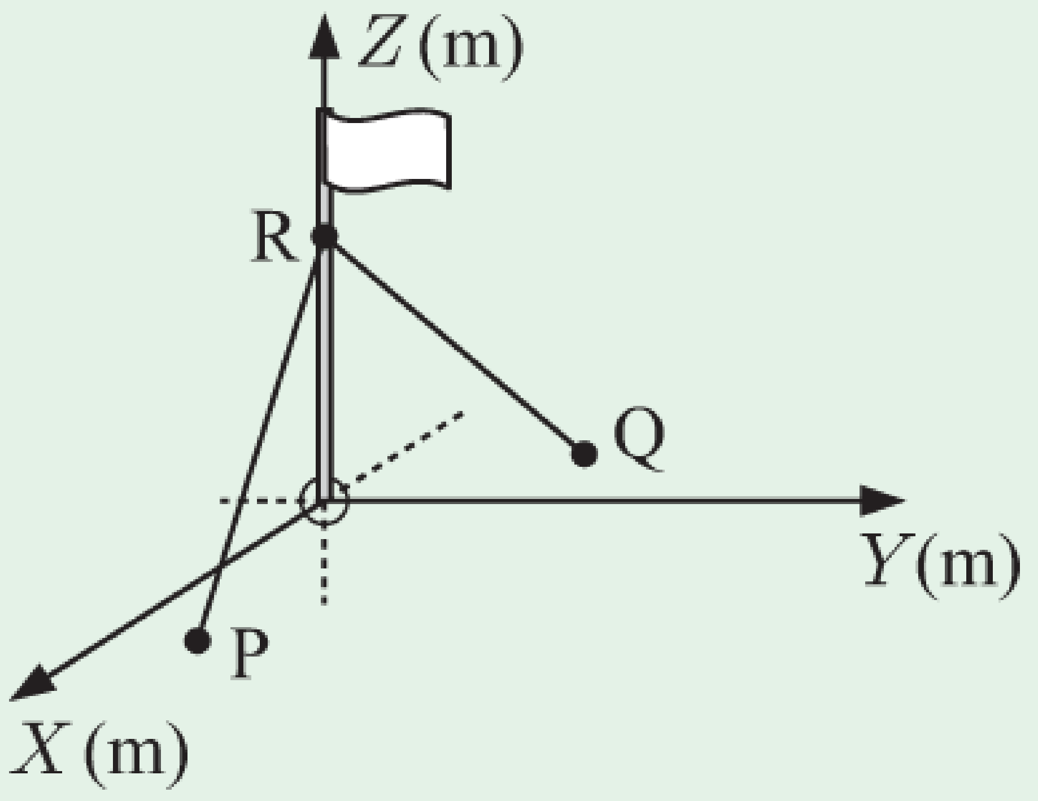
8 On the terrain map shown, the grid units are kilometres. The hiker is at a viewing platform H, 200 m above the base camp O.

- **a** State the 3-dimensional coordinates of the hiker.
- **b** How far is the hiker from the base camp?
- **c** From the viewing platform, the hiker can see the summit of a mountain at M. The mountain top is 500 m above base camp level.
  - **i** Find the 3-dimensional coordinates of the mountain top.
  - **ii** Find the distance between the hiker and the mountain top.
  - **iii** Find the angle of elevation from the hiker to the mountain top.



9 A flagpole is located on the Z-axis. It is supported by wires fixed to the ground at $P(3, 1, 0)$ and $Q(-1, 2, 0)$. The wires meet at R, 3 m up the flagpole.

- **a** Find the coordinates of R.
- **b** Find the angle each wire makes with the flagpole.
- **c** Find the angle at which the wires meet.



Chapter 11

Probability

Contents:

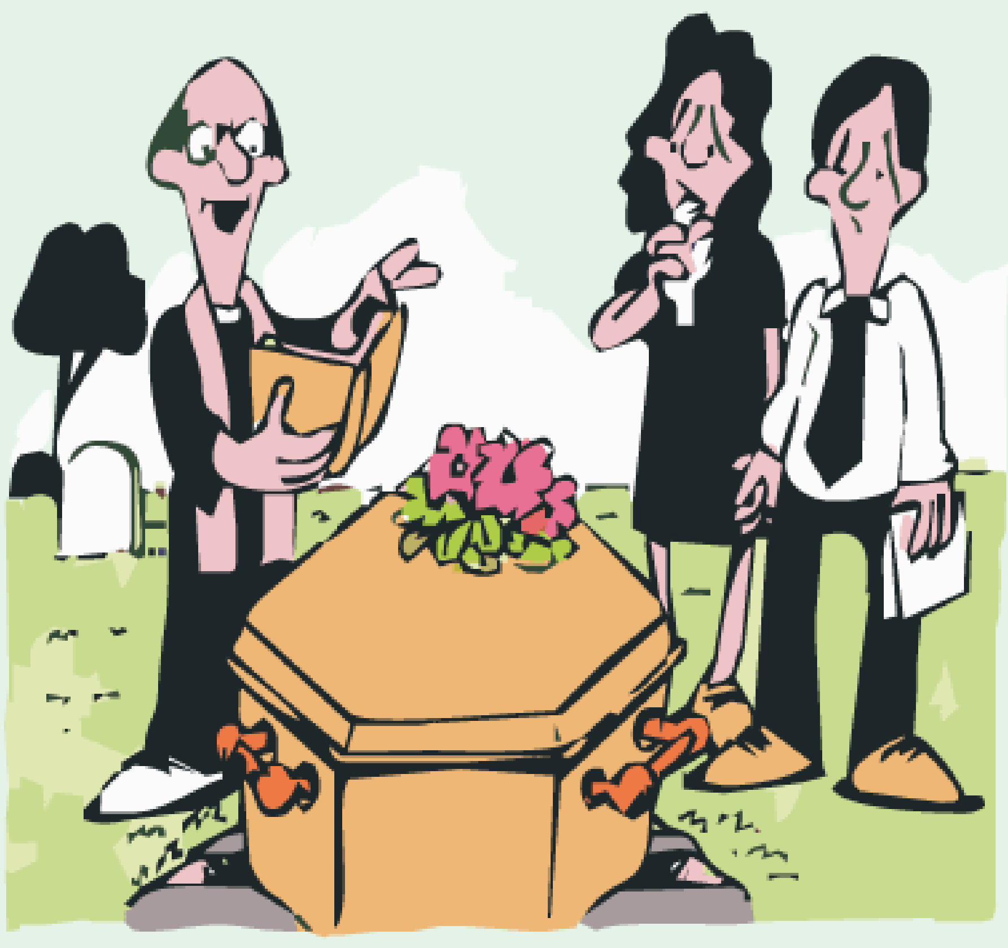
OPENING PROBLEM

In the late 17th century, English mathematicians compiled and analysed mortality tables which showed the number of people who died at different ages. From these tables they could estimate the probability that a person would be alive at a future date. This led to the establishment of the first life insurance company in 1699.

Life insurance companies use statistics on **life expectancy** and **death rates** to calculate the premiums to charge people who insure with them.

The **life table** shown is from Australia. It shows the number of people out of 100 000 births who survive to different ages, and the expected years of remaining life at each age.

For example, we can see that out of 100 000 births, 98 052 males are expected to survive to the age of 20, and from that age the survivors are expected to live a further 54.35 years.



LIFE TABLE

Male			Female		
Age	*Number surviving*	*Expected remaining life*	*Age*	*Number surviving*	*Expected remaining life*
0	100 000	73.03	0	100 000	79.46
5	98 809	68.90	5	99 307	75.15
10	98 698	63.97	10	99 125	70.22
15	98 555	59.06	15	98 956	65.27
20	98 052	54.35	20	98 758	60.40
25	97 325	49.74	25	98 516	55.54
30	96 688	45.05	30	98 278	50.67
35	96 080	40.32	35	98 002	45.80
40	95 366	35.60	40	97 615	40.97
45	94 323	30.95	45	96 997	36.22
50	92 709	26.45	50	95 945	31.59
55	89 891	22.20	55	94 285	27.10
60	85 198	18.27	60	91 774	22.76
65	78 123	14.69	65	87 923	18.64
70	67 798	11.52	70	81 924	14.81
75	53 942	8.82	75	72 656	11.36
80	37 532	6.56	80	58 966	8.38
85	20 998	4.79	85	40 842	5.97
90	8416	3.49	90	21 404	4.12
95	2098	2.68	95	7004	3.00
99	482	2.23	99	1953	2.36

Things to think about:

- **a** Can you use the life table to estimate how many years you can expect to live?
- **b** Can you estimate the chance that a new-born boy or girl will reach the age of 15?
- **c** Can the table be used to estimate the chance that:
  - **i** a 15 year old boy *will* reach age 75
  - **ii** a 15 year old girl *will not* reach age 75?
- **d** In general, do males or females live longer?
- **e** An insurance company sells policies to people to insure them against death over a 30-year period. If the person dies during this period, the beneficiaries receive the agreed payout figure. Why are such policies cheaper to take out for a 20 year old than for a 50 year old?
- **f** How many of your classmates would you expect to be alive and able to attend a 30 year class reunion?
- **g** How do you think life tables would compare between countries?

In the real world, we cannot predict with certainty what will happen in the future. Understanding the **chance** or likelihood of something happening is extremely useful for us to make decisions.

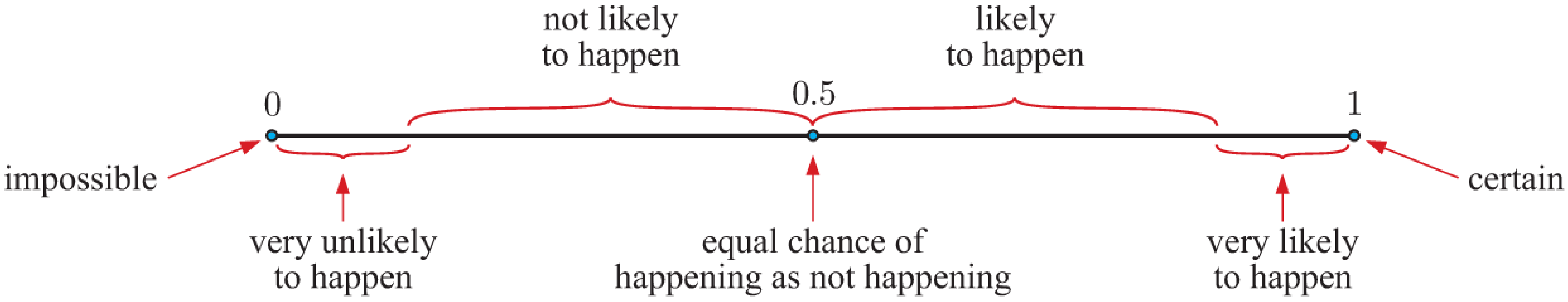
In mathematics, the chance of an event occurring is assigned a number between 0 and 1 inclusive. We call this number a **probability**.

An **impossible** event has 0% chance of happening, and is assigned the probability 0.

A **certain** event has 100% chance of happening, and is assigned the probability 1.

All other events can be assigned a probability between 0 and 1.

This number line shows how we could interpret different probabilities:



We can determine probabilities based on:

- the results of an experiment
- what we theoretically expect to happen.

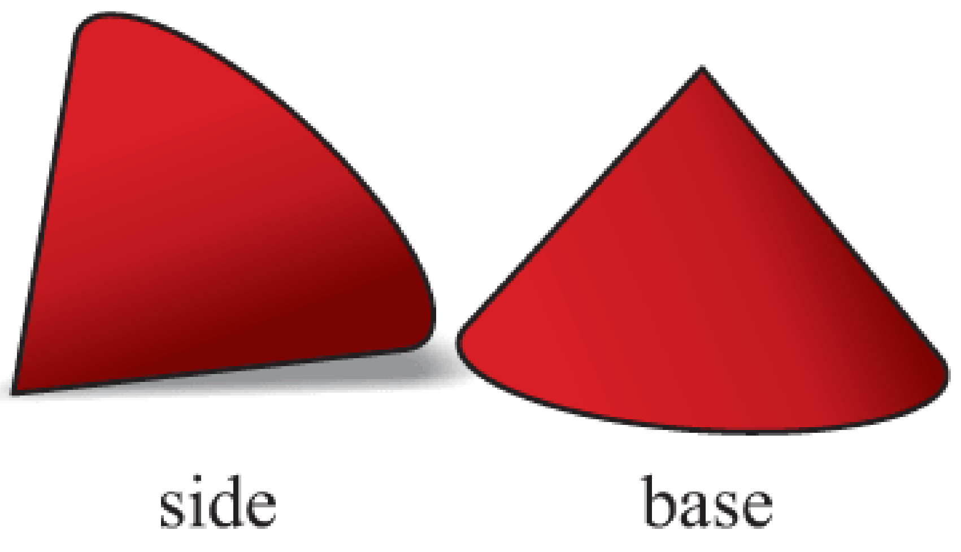
Probability theory is applied in physical and biological sciences, economics, politics, sport, quality control, production planning, and many other areas.

A EXPERIMENTAL PROBABILITY

In experiments involving chance we use the following terms to talk about what we are doing and the results we obtain:

- The **number of trials** is the total number of times the experiment is repeated.
- The **outcomes** are the different results possible for one trial of the experiment.
- The **frequency** of a particular outcome is the number of times that this outcome is observed.
- The **relative frequency** of an outcome is the frequency of that outcome expressed as a fraction or percentage of the total number of trials.

For example, when a small plastic cone was tossed into the air 279 times it fell on its *side* 183 times and on its *base* 96 times.



We say:

- the number of trials is 279
- the outcomes are *side* and *base*
- the frequencies of *side* and *base* are 183 and 96 respectively
- the relative frequencies of *side* and *base* are $\frac{183}{279} \approx 0.656$ and $\frac{96}{279} \approx 0.344$ respectively.

In the absence of any further data, the relative frequency of each event is our best estimate of the probability of that event occurring.

experimental probability = relative frequency

In this case: $P(\textit{side}) \approx$ the experimental probability the cone will land on its side when tossed
≈ 0.656
$P(\textit{base}) \approx$ the experimental probability the cone will land on its base when tossed
≈ 0.344

INVESTIGATION 1 DICE ROLLING EXPERIMENT

You will need:

At least one normal six-sided die with numbers 1 to 6 on its faces. Several dice would be useful to speed up the experiment.

WORKSHEET

What to do:

1 List the possible outcomes for the uppermost face when the die is rolled.

2 Discuss what you would expect the relative frequency of rolling a 2 to be when a die is rolled many times.

3 Roll a die 20 times, and count the number of times a 2 is rolled. Hence, calculate the relative frequency of rolling a 2.

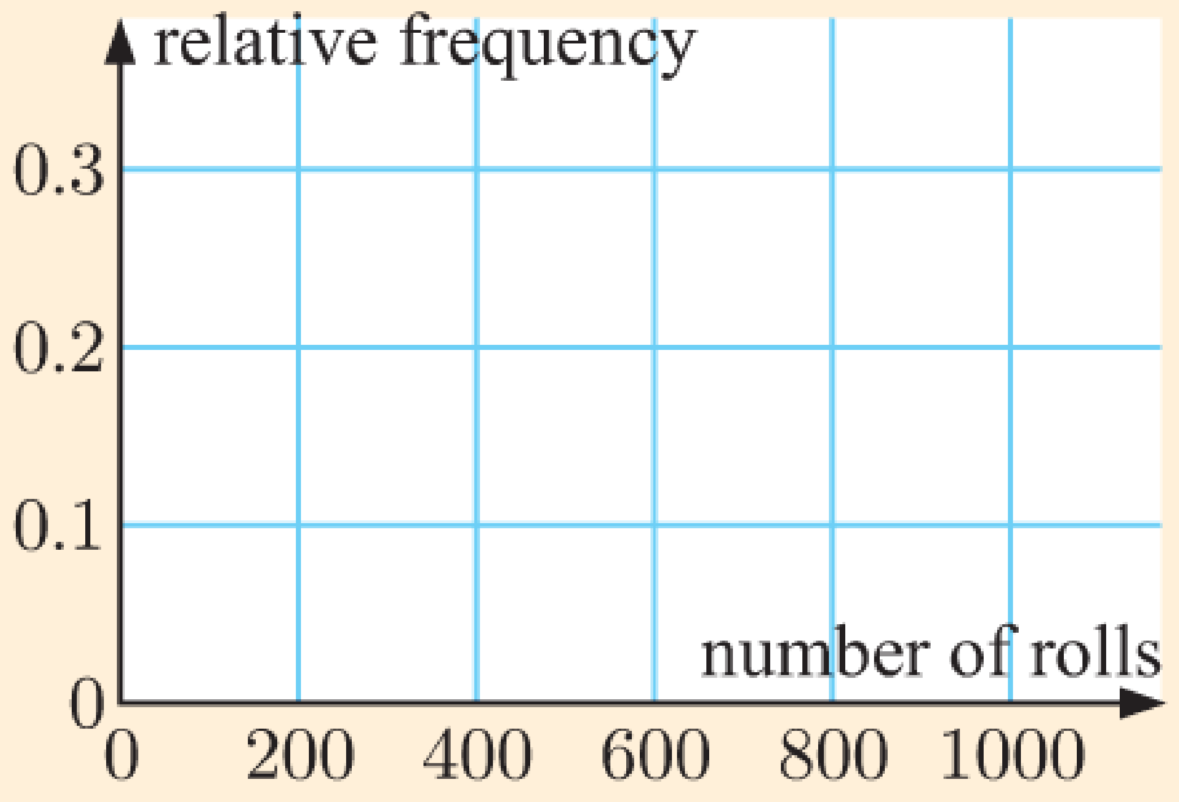
4 Pool your results with another student, so in total you have data for 40 rolls. Calculate the relative frequency of rolling a 2 for 40 rolls.

5 Use the simulation to roll a die 60, 100, 200, 300, 500, and 1000 times. In each case, calculate the relative frequency of rolling a 2.

SIMULATION

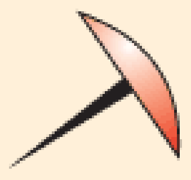
6 Plot a graph of relative frequency against the number of rolls. What do you notice?

7 What do you think will happen to the relative frequency of rolling a 2 as the number of rolls increases?



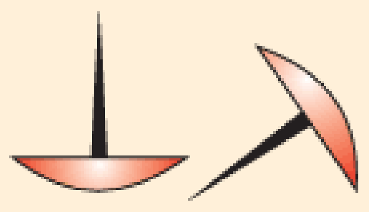
The larger the number of trials, the more confident we are that the estimated probability is accurate.

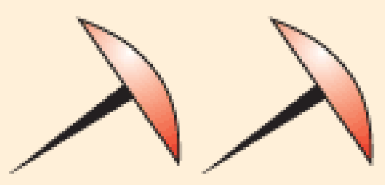
INVESTIGATION 2 TOSSING DRAWING PINS

If a drawing pin tossed in the air finishes  we say it has finished on its *back*. If it finishes  we say it has finished on its *side*.

If two drawing pins are tossed simultaneously, the possible results are:


two backs


back and side


two sides

What to do:

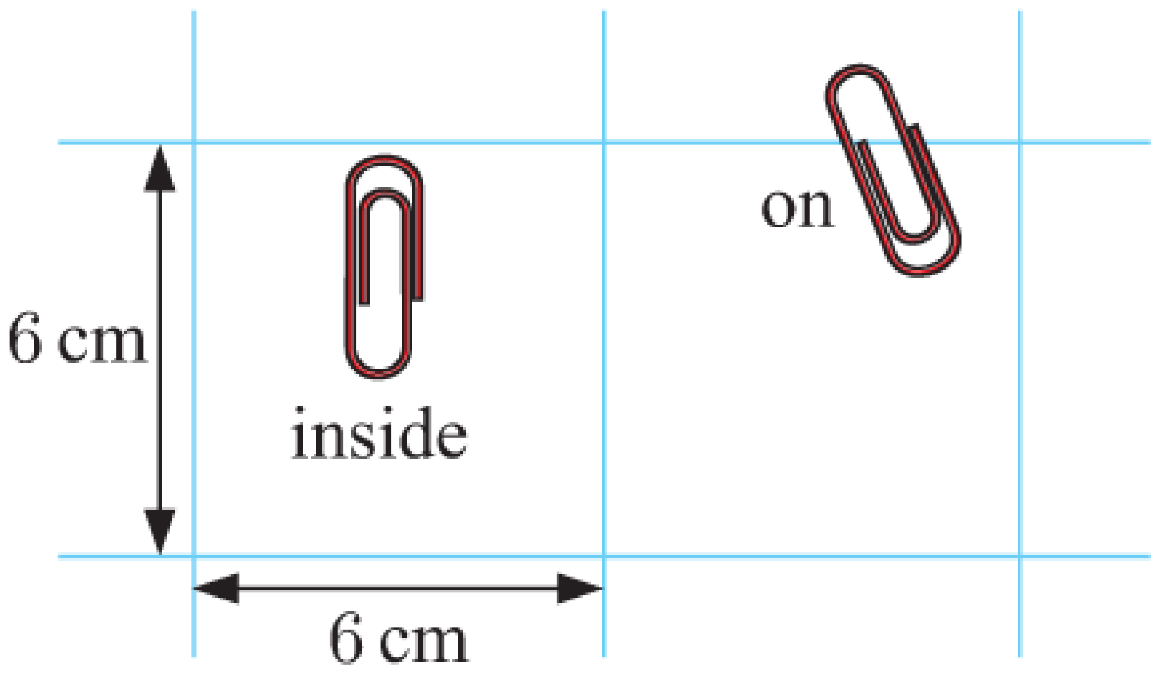
1. Obtain two drawing pins of the same shape and size. Toss the pair 80 times and record the outcomes in a table.
2. Obtain relative frequencies (experimental probabilities) for each of the three outcomes.
3. Pool your results with four other people using drawing pins with the same shape. Hence obtain experimental probabilities from 400 tosses.
4. Which gives the more reliable probability estimates, your results or the whole group's? Explain your answer.

In some situations, such as in the **Investigation** above, experimentation is the only way of obtaining probabilities.

EXERCISE 11A

1 When a batch of 145 paper clips was dropped onto 6 cm by 6 cm squared paper, it was observed that 113 fell completely inside squares and 32 landed on a grid line. Find, to 2 decimal places, the experimental probability of a clip falling:

a inside a square **b** on a line.



2

Length	*Frequency*
0 - 19	17
20 - 39	38
40 - 59	19
60+	4

Jose surveyed the length of TV commercials (in seconds). Find, to 3 decimal places, the experimental probability that the next TV commercial will last:

a 20 to 39 seconds **b** at least one minute

c between 20 and 59 seconds (inclusive).

3 Betul records the number of phone calls she receives over a period of consecutive days.

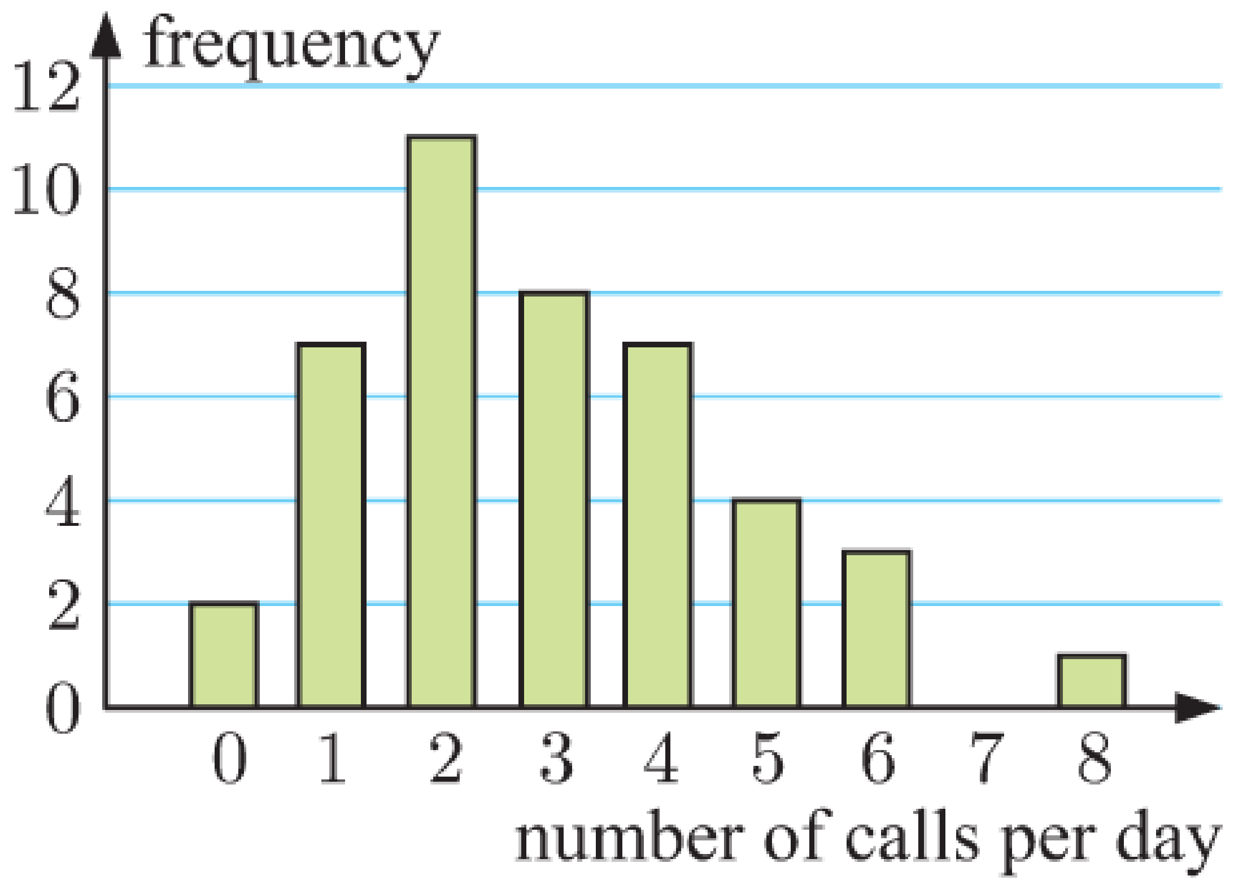
a For how many days did the survey last?

b Estimate the probability that tomorrow Betul will receive:

  i no phone calls

  ii 5 or more phone calls

  iii less than 3 phone calls.



4 Pat does a lot of travelling in her car, and she keeps records on how often she fills her car with petrol. The table alongside shows the frequencies of the number of days between refills. Estimate the probability that:

a there is a four day gap between refills

b there is at least a four day gap between refills.

Days between refills	*Frequency*
1	37
2	81
3	48
4	17
5	6
6	1

Example 1 **Self Tutor**

The table below shows the number of short-term visitors coming to Australia in the period April - June 2018, and the main reason for their visit.

Short-Term Visitors to Australia

Main reason for journey	*April 2018*	*May 2018*	*June 2018*
Convention/conference	8300	14 800	8800
Business	27 200	33 900	31 900
Visiting friends/relatives	77 500	52 700	59 900
Holiday	159 300	119 300	156 500
Employment	4200	4300	5500
Education	9800	7900	12 500
Other	35 200	28 000	33 200
Total	321 500	260 900	308 300

a Estimate the probability that a person who visits in June is on holiday.

b Estimate the probability that a person who came to Australia in the period April - June 2018 arrived in May.

c Lars arrived in Australia in April, May, or June 2018 to visit his brother. Estimate the probability that he arrived in April.

a $$\text{P(on holiday in June)} \approx \frac{156\,500 \xleftarrow{} \text{number on holiday in June}}{308\,300 \xleftarrow{} \text{total number for June}}$$

$$\approx 0.508$$

b $321\,500 + 260\,900 + 308\,300 = 890\,700$ short-term visitors arrived during the three months.

$$\therefore \quad \text{P(arrives in May)} \approx \frac{260\,900}{890\,700} \approx 0.293$$

c $77\,500 + 52\,700 + 59\,900 = 190\,100$ people came to Australia to visit friends or relatives during this period.

$$\therefore \quad \text{P(arrived in April)} \approx \frac{77\,500 \xleftarrow{} \text{number visiting friends or relatives in April}}{190\,100 \xleftarrow{} \text{total number visiting friends or relatives over April, May, and June}}$$

$$\approx 0.408$$

5 The table shows data from a survey conducted at five schools to study the rate of smoking among 15 year old students.

School	*Number of 15 year olds*		*Number of smokers*	
	Male	*Female*	*Male*	*Female*
A	45	51	10	11
B	36	42	9	6
C	52	49	13	13
D	28	33	9	10
E	40	39	7	4
Total	201	214	48	44

a Estimate the probability that a randomly chosen female 15 year old student at school **C** is a smoker.

b Estimate the probability that a randomly chosen 15 year old student at school **E** is *not* a smoker.

c If a 15 year old is chosen at random from the five schools, estimate the probability that he or she is a smoker.

6 This table describes the complaints received by a telecommunications ombudsman concerning internet services over a four year period.

Reason	2014/15	2015/16	2016/17	2017/18
Access	585	1127	2545	1612
Billing	1822	2102	3136	3582
Contracts	242	440	719	836
Credit control	3	44	118	136
Customer Service	12	282	1181	1940
Disconnection	n/a	n/a	n/a	248
Faults	86	79	120	384
Privacy	93	86	57	60
Provision	172	122	209	311
Total	3015	4282	8085	9109

Find the probability that a complaint received:

- **a** in 2016/17 was about customer service
- **b** at any time during the 4 year period was related to billing
- **c** in 2017/18 did *not* relate to either billing or faults.

7 This table provides data on the daily maximum temperatures in Barcelona during summer.

Summer Temperatures in Barcelona	*Month*		
	June	*July*	*Aug*
Mean days max. $\geqslant 40°C$	0.3	1.2	0.7
Mean days max. $\geqslant 35°C$	3.0	5.8	5.3
Mean days max. $\geqslant 30°C$	9.4	12.3	12.0

- **a** Estimate the probability that on an August day in Barcelona, the maximum temperature will be:
  - **i** 35°C or higher
  - **ii** less than 30°C.
- **b** Estimate the probability that on any summer day in Barcelona, the temperature will be 30°C or higher.
- **c** It is a 40°C summer day in Barcelona. Estimate the probability that the month is July.

B TWO-WAY TABLES

Two-way tables are tables which compare two categorical variables.

For example, the teachers at a school were asked which mode of transport they used to travel to school. Their responses are summarised in the table below. The variables are *gender* and *mode of transport*.

	Car	Bicycle	Bus
Male	37	10	10
Female	30	5	13

13 female teachers catch the bus to school.

In the following Example we will see how these tables can be used to estimate probabilities. To help us, we extend the table to include totals for each row and column.

Example 2 **Self Tutor**

People exiting a new ride at a theme park were asked whether they liked or disliked the ride. The results are shown in the two-way table alongside.

	Child	*Adult*
Liked the ride	55	28
Disliked the ride	17	30

Use this table to estimate the probability that a randomly selected person who went on the ride:

a liked the ride **b** is a child *and* disliked the ride **c** is an adult *or* disliked the ride.

We extend the table to include totals for each row and column.

	Child	*Adult*	*Total*
Liked the ride	55	28	83
Disliked the ride	17	30	47
Total	72	58	130

a 83 out of the 130 people surveyed liked the ride.

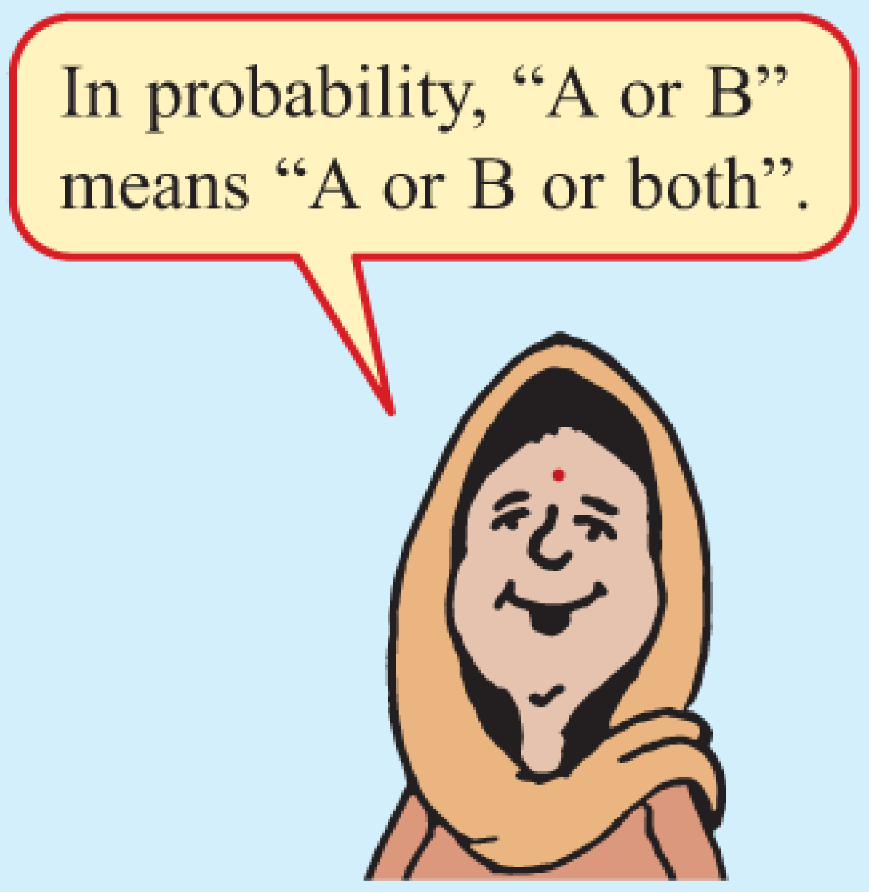
$\therefore$ P(liked the ride) $\approx \frac{83}{130} \approx 0.638$

b 17 of the 130 people surveyed are children who disliked the ride.

$\therefore$ P(child *and* disliked the ride) $\approx \frac{17}{130} \approx 0.131$

c $28+30+17=75$ of the 130 people are adults or people who disliked the ride.

$\therefore$ P(adults *or* disliked the ride) $\approx \frac{75}{130} \approx 0.577$



EXERCISE 11B

1 The types of ticket used to attend a basketball game were recorded as people entered the stadium. The results are shown alongside.

	Adult	*Child*
Season ticket holder	1824	779
Not a season ticket holder	3247	1660

a What was the total attendance for the match?

b One person is randomly selected to sit on the home team's bench. Find the probability that the person selected is:

i a child **ii** not a season ticket holder **iii** an adult season ticket holder.

2 Students at a school were asked whether they played a sport.

- In the junior school, 131 students played a sport and 28 did not.
- In the middle school, 164 students played a sport and 81 did not.
- In the senior school, 141 students played a sport and 176 did not.

a Copy and complete this table.

	Junior	*Middle*	*Senior*	*Total*
Sport				
No sport				
Total				

b Find the probability that a randomly selected student:

i plays sport **ii** plays sport and is in the junior school

iii does not play sport and is in middle school or higher.

3 A small hotel in London has kept a record of all room bookings made for the year. The results are summarised in the two-way table.

	Single	*Double*	*Family*
Peak season	225	420	98
Off-peak season	148	292	52

a Estimate the probability that the next randomly selected booking will be:

i in the peak season
ii a single room in the off-peak season
iii a single or a double room
iv during the peak season or a family room.

b A randomly selected booking is in the off-peak season. Estimate the probability that it is a family room.

c A randomly selected booking is *not* a single room. Estimate the probability that it is in the peak season.

C SAMPLE SPACE AND EVENTS

The **sample space** U is the set of all possible outcomes of an experiment.

An **event** is a set of outcomes in the sample space that have a particular property.

You should notice that we are applying the set theory we studied in **Chapter 2**:

- the sample space is the **universal set** U
- the outcomes are the **elements** of the sample space
- events are **subsets** of the sample space.

We can therefore **list** the outcomes in the sample space and in events using **set notation**, and illustrate them with a **Venn diagram**.

COMPLEMENTARY EVENTS

Two events are **complementary** if exactly one of the events *must* occur.
If A is an event, then A' is the complementary event of A, or "not A".

Example 3 **Self Tutor**

A normal six-sided die is rolled once. Let A be the event that a prime number is rolled.

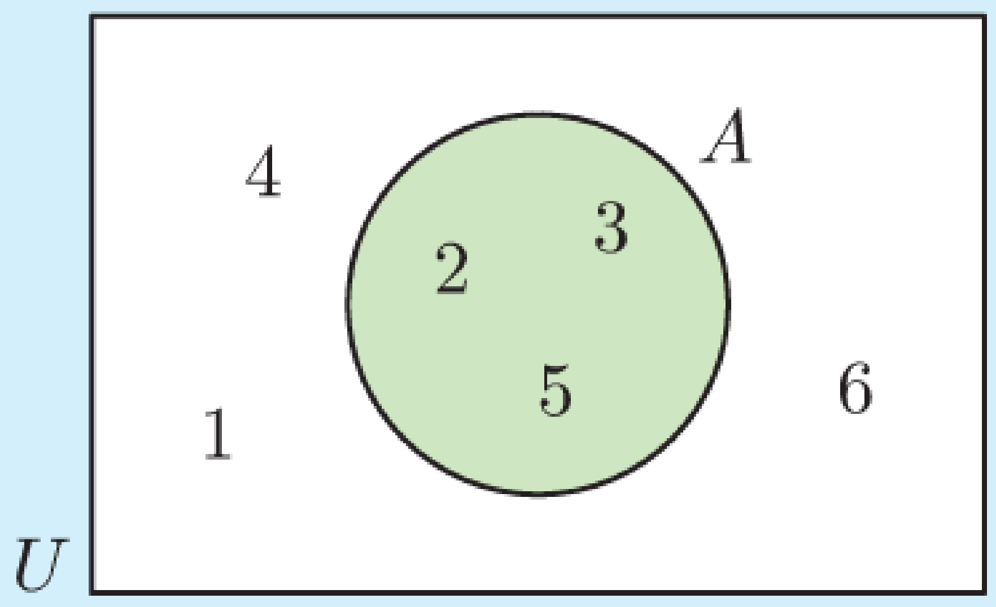
a Use set notation to list the outcomes in:

i the sample space U **ii** A **iii** A'.

b Draw a Venn diagram to illustrate the sample space.

a **i** $U = \{1, 2, 3, 4, 5, 6\}$
ii $A = \{2, 3, 5\}$
iii $A' = \{1, 4, 6\}$

b



2-DIMENSIONAL GRIDS AND TREE DIAGRAMS

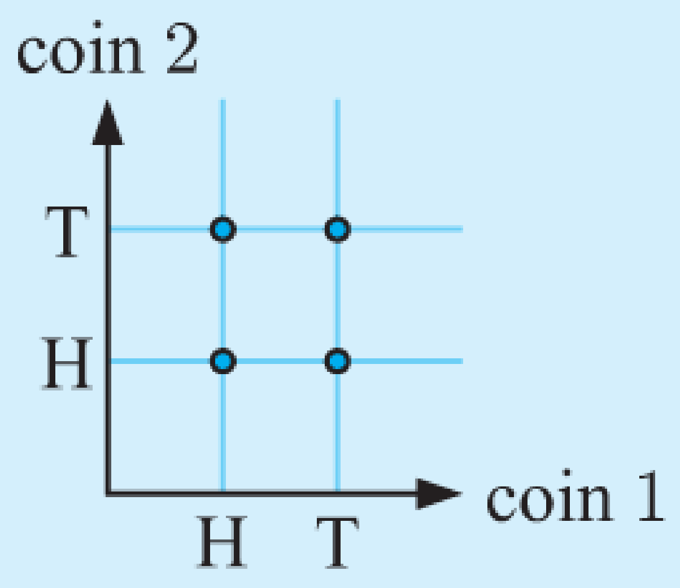
When an experiment involves more than one operation we can still list the sample space. However, it is often more efficient to illustrate the sample space on a **2-dimensional grid** or using a **tree diagram**.

Example 4 Self Tutor

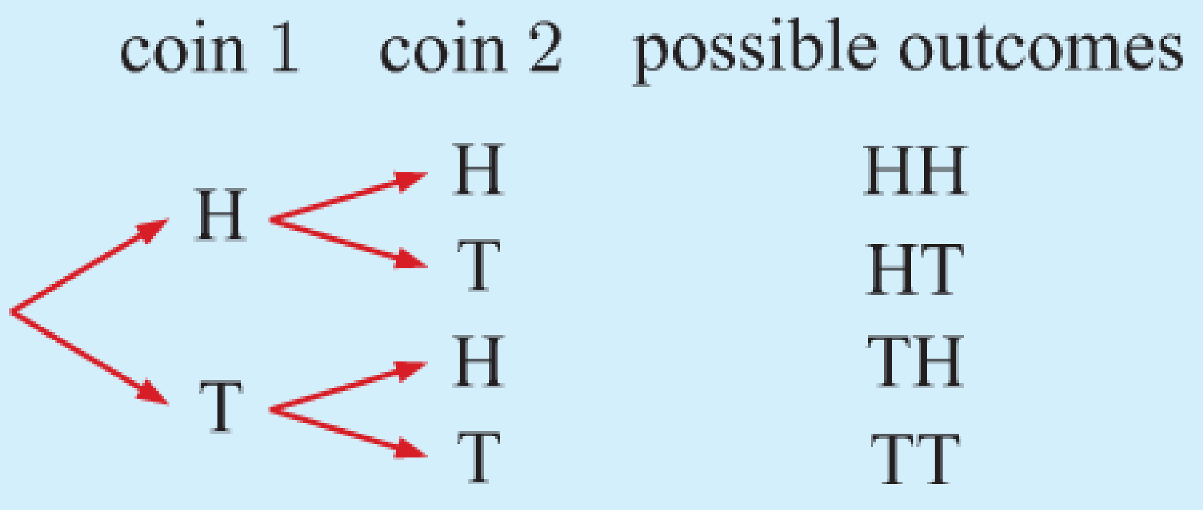
Illustrate the possible outcomes when two coins are tossed using:

a a 2-dimensional grid **b** a tree diagram.

a



b



Notice in the Example that each outcome in the sample space {HH, HT, TH, TT} is represented by:

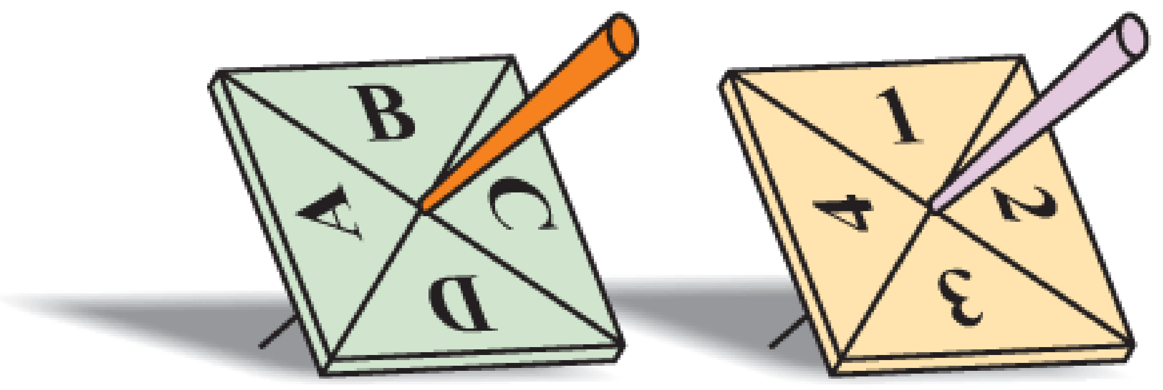
- a point on the grid
- a "branch" on the tree diagram.

EXERCISE 11C

1 List using set notation, the sample space for:
  - **a** twirling a square spinner labelled A, B, C, D
  - **b** spinning a wheel with sectors labelled with the numbers 1 to 8
  - **c** the sexes of a 2-child family.

2 One ticket is drawn from a box containing tickets labelled with the numbers 1 to 16 inclusive.
  - **a** Write down the sample space U.
  - **b** List the outcomes in the following events:
    - **i** $A =$ the ticket's number is a multiple of 4
    - **ii** $B =$ the ticket's number is a perfect square.
  - **c** Draw a Venn diagram to illustrate the sample space U and the events A and B.

3 Illustrate on a 2-dimensional grid the sample space for:
  - **a** rolling a die and tossing a coin simultaneously
  - **b** rolling two dice
  - **c** rolling a die and spinning a spinner with sides A, B, C, D
  - **d** twirling two square spinners, one labelled A, B, C, D and the other 1, 2, 3, 4.

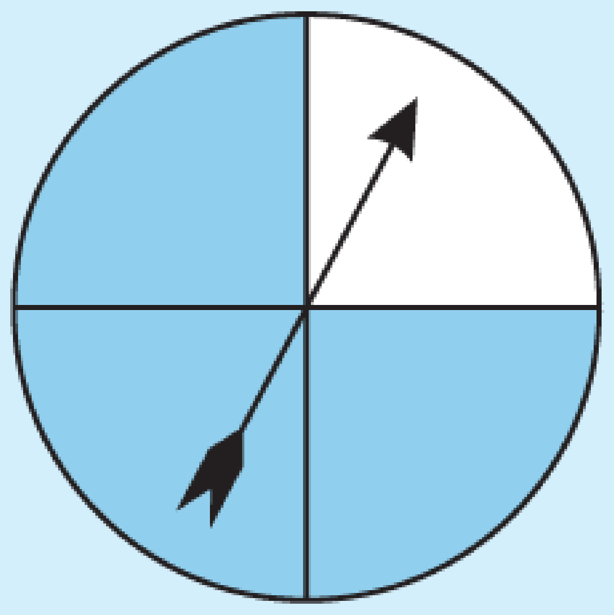


4 Illustrate on a tree diagram the sample space for:
  - **a** tossing a 5-cent and a 10-cent coin simultaneously
  - **b** tossing a coin and twirling an equilateral triangular spinner labelled A, B, C
  - **c** twirling two equilateral triangular spinners labelled 1, 2, 3, and X, Y, Z
  - **d** drawing two tickets from a hat containing a large number of pink, blue, and white tickets.

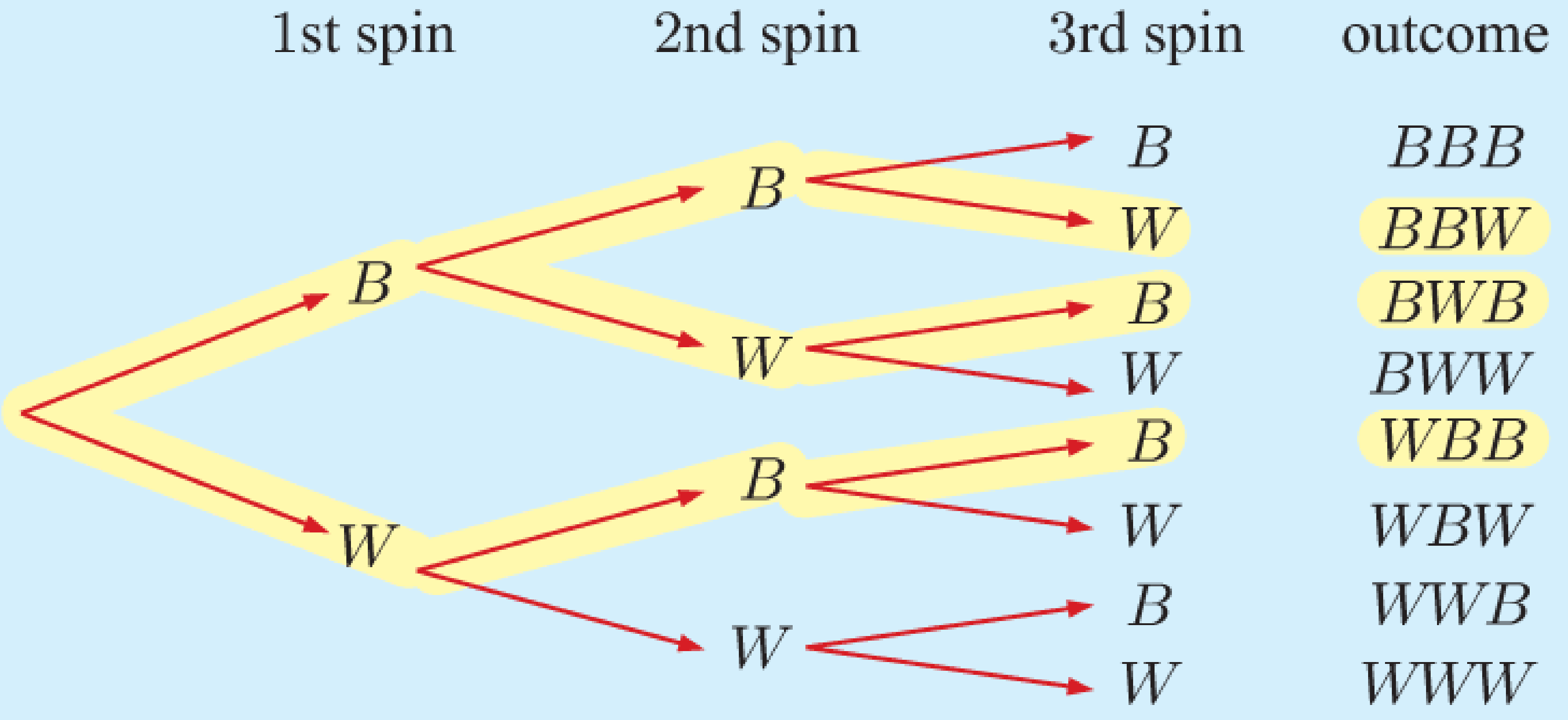
Example 5 **Self Tutor**

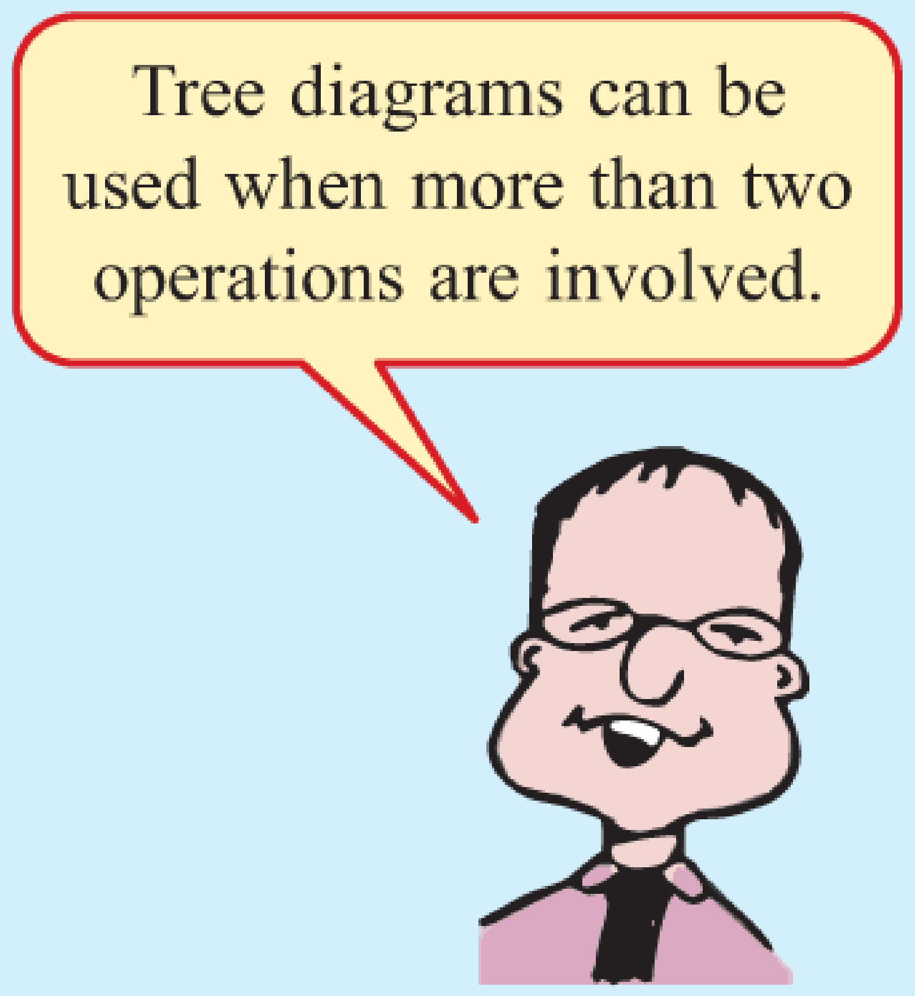
Use a tree diagram to illustrate the possible outcomes when this spinner is spun three times.

Highlight the outcomes corresponding to the event "obtaining blue twice".



Let B represent blue and W represent white.

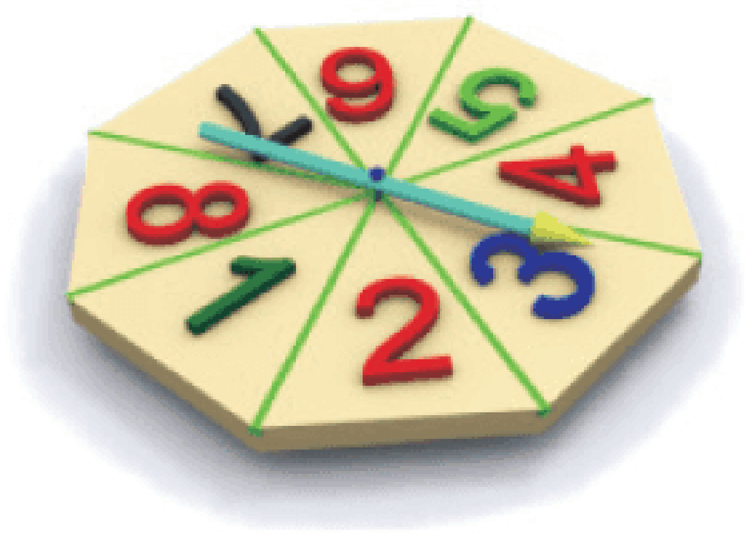



Tree diagrams can be used when more than two operations are involved.

5 From the whole numbers 1 to 7, Adam and Bill each select a number. Illustrate the sample space on a 2-dimensional grid. Circle the outcomes in the event "Adam and Bill's numbers are the same".

6 Suppose three coins are tossed simultaneously. Draw a tree diagram to illustrate the sample space. Highlight the outcomes corresponding to the event "getting at least 1 head".

D THEORETICAL PROBABILITY



The sample space when spinning the octagonal spinner shown is {1, 2, 3, 4, 5, 6, 7, 8}.

Since the spinner is symmetrical, we expect that each of the eight outcomes will be **equally likely** to occur. We say that the **theoretical probability** of any particular outcome occurring is 1 in 8, or $\frac{1}{8}$.

If a sample space has n outcomes which are **equally likely** to occur when the experiment is performed once, then each outcome has probability $\dfrac{1}{n}$ of occurring.

Consider the event of *spinning a prime number* with the spinner above. Of the 8 possible outcomes, the four outcomes 2, 3, 5, and 7 all correspond to this event. So, the probability of rolling a prime number is 4 in 8, or $\frac{4}{8}$.

When the outcomes of an experiment are equally likely, the probability that an event A occurs is:

$$\mathrm{P}(A) = \frac{\textbf{number of outcomes corresponding to } A}{\textbf{number of outcomes in the sample space}} = \frac{n(A)}{n(U)}$$

Example 6 **Self Tutor**

A ticket is *randomly selected* from a basket containing 3 green, 4 yellow, and 5 blue tickets. Determine the probability of getting:

a a green ticket
b a green or yellow ticket
c an orange ticket
d a green, yellow, or blue ticket.

There are $3 + 4 + 5 = 12$ tickets which could be selected with equal chance.

a $\text{P}(\text{G})$
$= \frac{3}{12}$
$= \frac{1}{4}$

b $\text{P}(\text{G or Y})$
$= \frac{3+4}{12}$
$= \frac{7}{12}$

c $\text{P}(\text{O})$
$= \frac{0}{12}$
$= 0$

d $\text{P}(\text{G, Y, or B})$
$= \frac{3+4+5}{12}$
$= 1$

Example 7 **Self Tutor**

An ordinary six-sided die is rolled once. Determine the chance of:

a getting a 6
b not getting a 6
c getting a 1 or 2
d not getting a 1 or 2.

The sample space is $\{1, 2, 3, 4, 5, 6\}$.

a $\text{P}(6)$
$= \frac{1}{6}$

b $\text{P}(\text{not a } 6)$
$= \text{P}(1, 2, 3, 4, \text{ or } 5)$
$= \frac{5}{6}$

c $\text{P}(1 \text{ or } 2)$
$= \frac{2}{6}$
$= \frac{1}{3}$

d $\text{P}(\text{not a } 1 \text{ or } 2)$
$= \text{P}(3, 4, 5, \text{ or } 6)$
$= \frac{4}{6}$
$= \frac{2}{3}$

In **Example 7**, notice that $\text{P}(6) + \text{P}(\text{not a } 6) = 1$
and that $\text{P}(1 \text{ or } 2) + \text{P}(\text{not a } 1 \text{ or } 2) = 1$.

This is no surprise as "6" and "not a 6" are complementary events. It is certain that exactly one of the events will occur, and impossible for both of them to occur at the same time.

Likewise, "1 or 2" and "not a 1 or 2" are complementary events.

In general, $\boldsymbol{\text{P}(A) + \text{P}(A') = 1}$.

EXERCISE 11D

1 A marble is randomly selected from a box containing 5 green, 3 red, and 7 blue marbles. Determine the probability that the marble is:

a red
b green
c blue
d not red
e neither green nor blue
f green or red.

2 A barrel contains tickets numbered from 1 to 200. If a ticket is selected at random, find the probability that the ticket number:

a is even
b is a multiple of 12
c contains a 2.

3 A giant spinner has 36 sectors labelled 1 to 36. Determine the probability that when it is spun, the arrow will land on a sector labelled with:

a a multiple of 4
b a number between 6 and 9 inclusive
c a number greater than 20
d a multiple of 13
e an odd number that is a multiple of 3
f a number containing a 1
g a multiple of both 4 and 6
h a multiple of 4 or 6, or both.

4 Find the probability that a randomly chosen person has his or her next birthday:

a on a Tuesday
b on a weekend
c in July
d in January or February
e in a month containing the letter "a".

5 **a** List the 8 possible 3-child families according to the gender of the children. For example, GGB means "*the first is a girl, the second is a girl, the third is a boy*".

b Assuming that each of these is equally likely to occur, determine the probability that a randomly selected 3-child family consists of:

i all boys
ii all girls
iii boy then girl then girl
iv two girls and a boy
v a girl for the eldest
vi at least one boy.

6 **a** List the 24 different orders in which four people A, B, C, and D may sit in a row.

b Determine the probability that when the four people sit at random in a row:

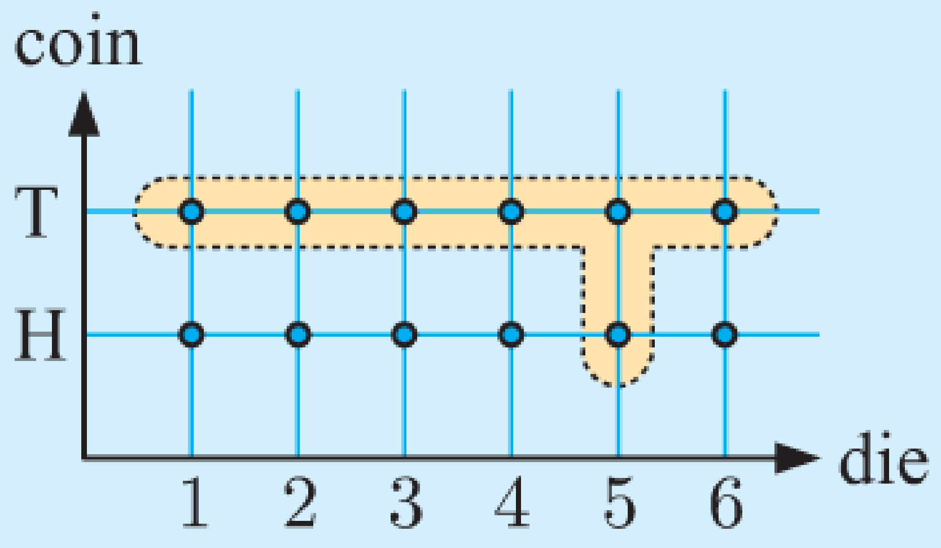
i A sits on one of the end seats
ii B sits on one of the two middle seats
iii A and B are seated together
iv A, B, and C are seated together, not necessarily in that order.

Example 8

Self Tutor

Use a 2-dimensional grid to illustrate the sample space for tossing a coin and rolling a die simultaneously. Hence determine the probability of:

a tossing a head
b tossing a tail and rolling a 5
c tossing a tail or rolling a 5.

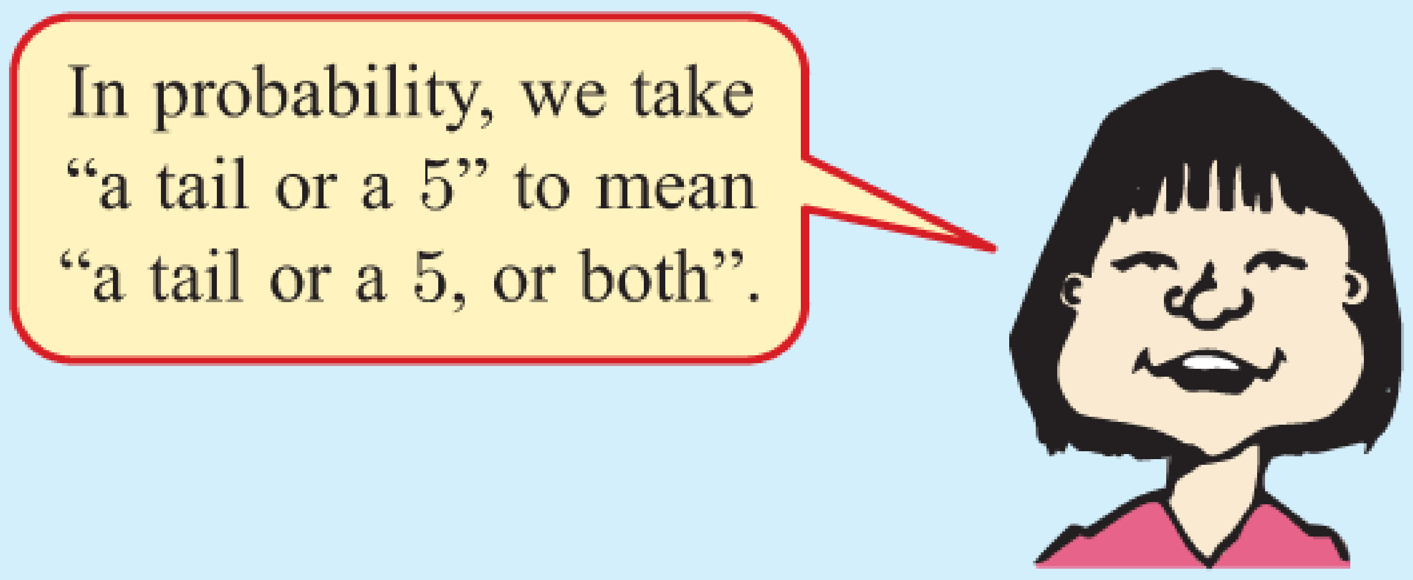


There are 12 outcomes in the sample space.

a P(head) $= \frac{6}{12} = \frac{1}{2}$

b P(tail and a 5) $= \frac{1}{12}$

c P(tail or a 5) $= \frac{7}{12}$ {the points in the shaded region}



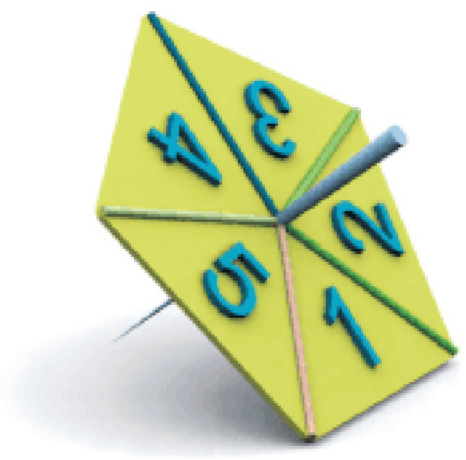
7 A 5-cent and a 20-cent coin are tossed simultaneously.

a Draw the grid of the sample space.

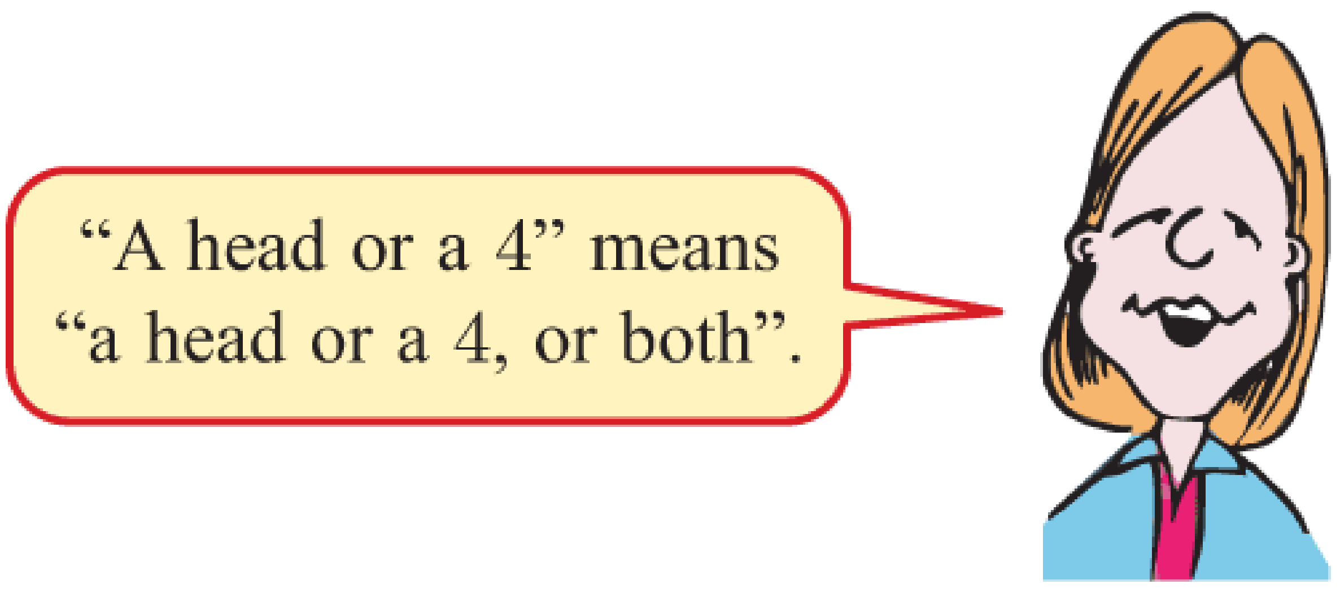
b Hence determine the probability of tossing:

i two heads
ii two tails
iii exactly one tail
iv at most one tail.

8 A coin and a pentagonal spinner with sectors 1, 2, 3, 4, and 5 are tossed and spun respectively.

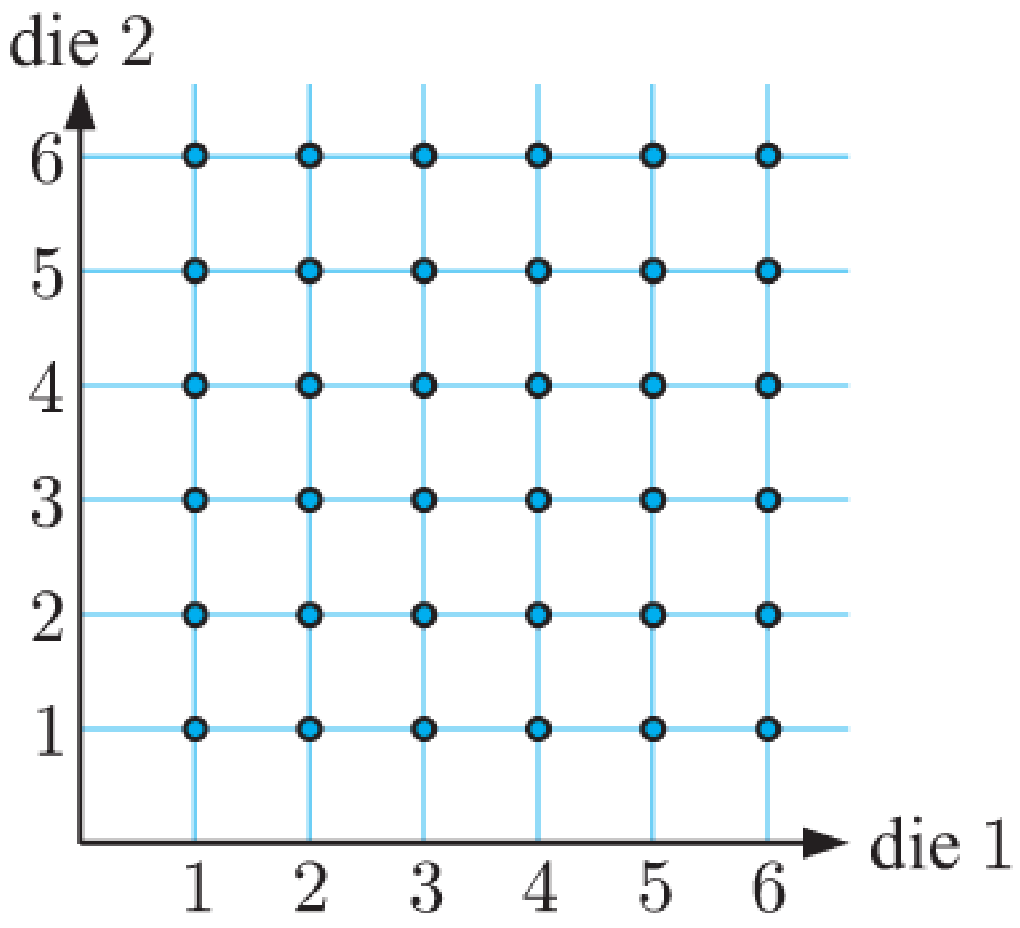


- **a** Draw a grid to illustrate the sample space of possible outcomes.
- **b** Use your grid to determine the chance of getting:
  - **i** a head and a 5
  - **ii** a tail and a prime number
  - **iii** an even number
  - **iv** a head or a 4.



9 The 36 different possible results from rolling two dice are illustrated on the 2-dimensional grid.
Use the grid to find the probability of rolling:

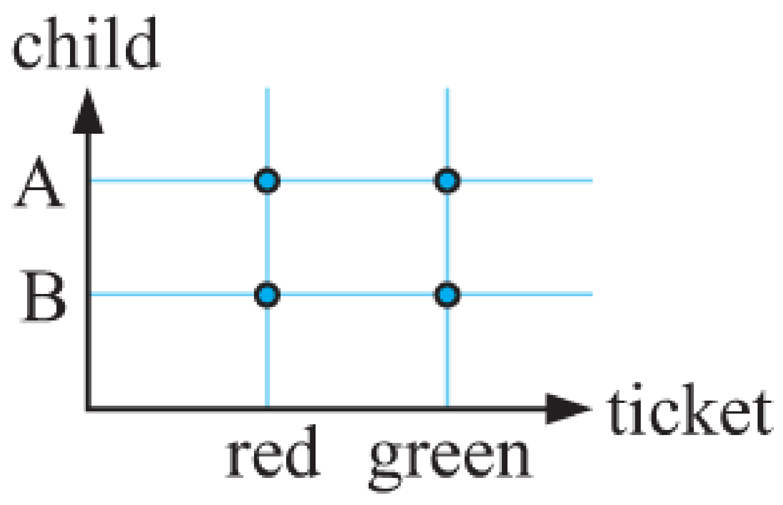
- **a** two 3s
- **b** a 5 and a 6
- **c** a 5 or a 6 (or both)
- **d** at least one 6
- **e** exactly one 6
- **f** no sixes.



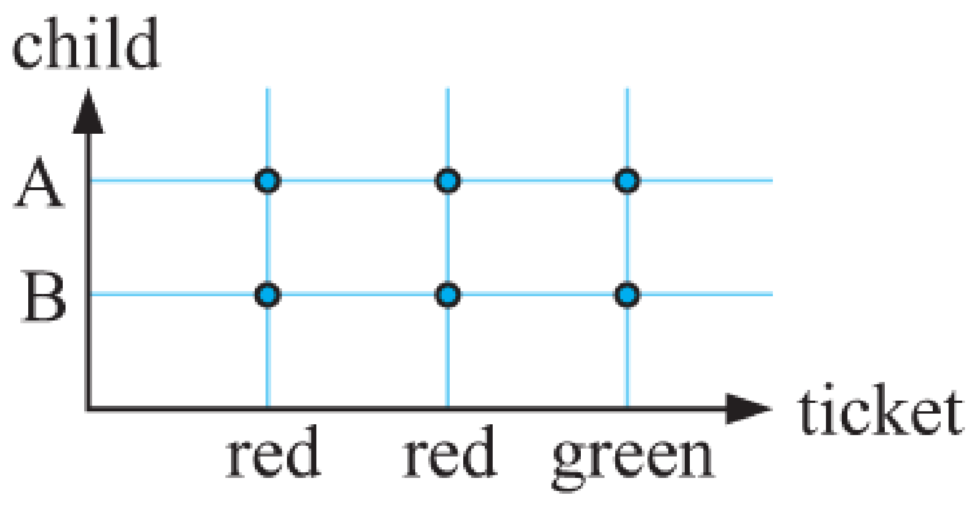
10 Two children A and B toss a coin to determine which of them will select a ticket from a bag. The bag contains two red tickets and one green ticket.

- **a** Which of these grids shows the sample space correctly? Discuss your answer.

A



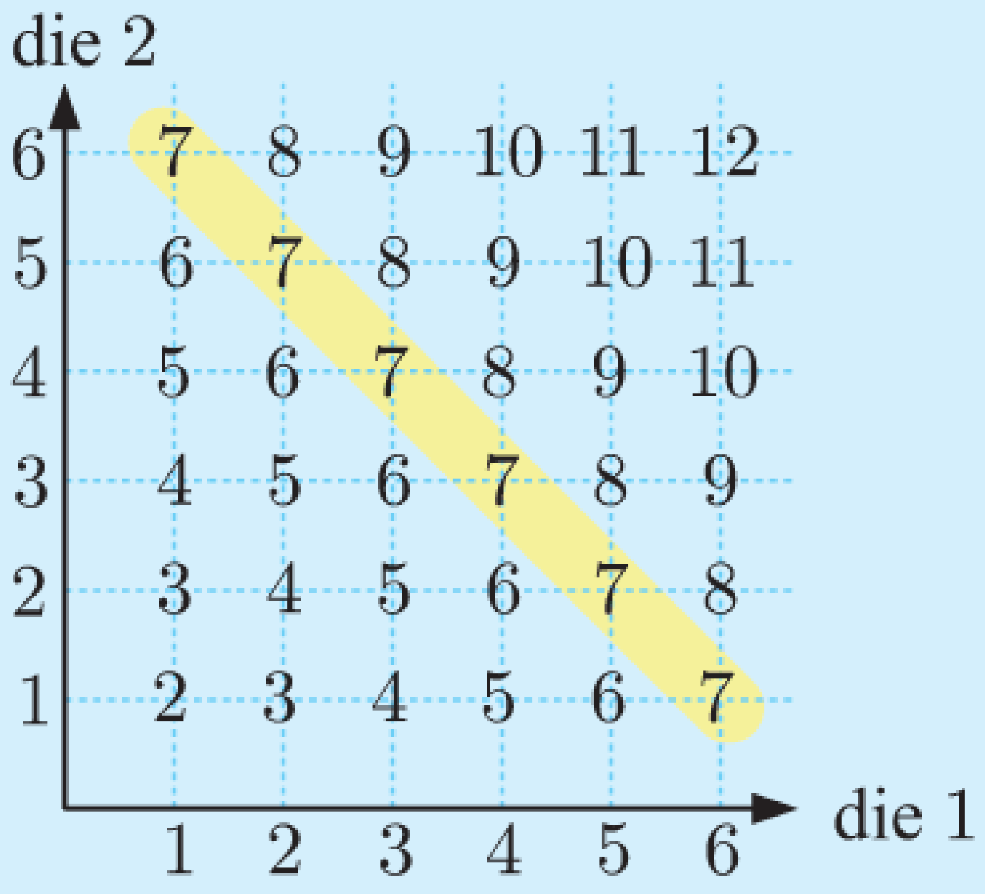
B



- **b** Use the appropriate grid to find the probability that child B will select a green ticket.

Example 9 **Self Tutor**

Display the possible results when two dice are rolled and the scores are added together.
Hence find the probability that the sum of the dice is 7.



die 2						
6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
	1	2	3	4	5	6

Of the 36 possible combinations of scores from the two dice, six have the sum 7.

$\therefore$ the probability $= \frac{6}{36} = \frac{1}{6}$

11
- **a** Display the possible results when two dice are rolled and the scores are added together.
- **b** Hence find the probability that the sum of the dice is:
  - **i** 11
  - **ii** 6
  - **iii** 8 or 9
  - **iv** less than 6
  - **v** greater than 8
  - **vi** no more than 8.

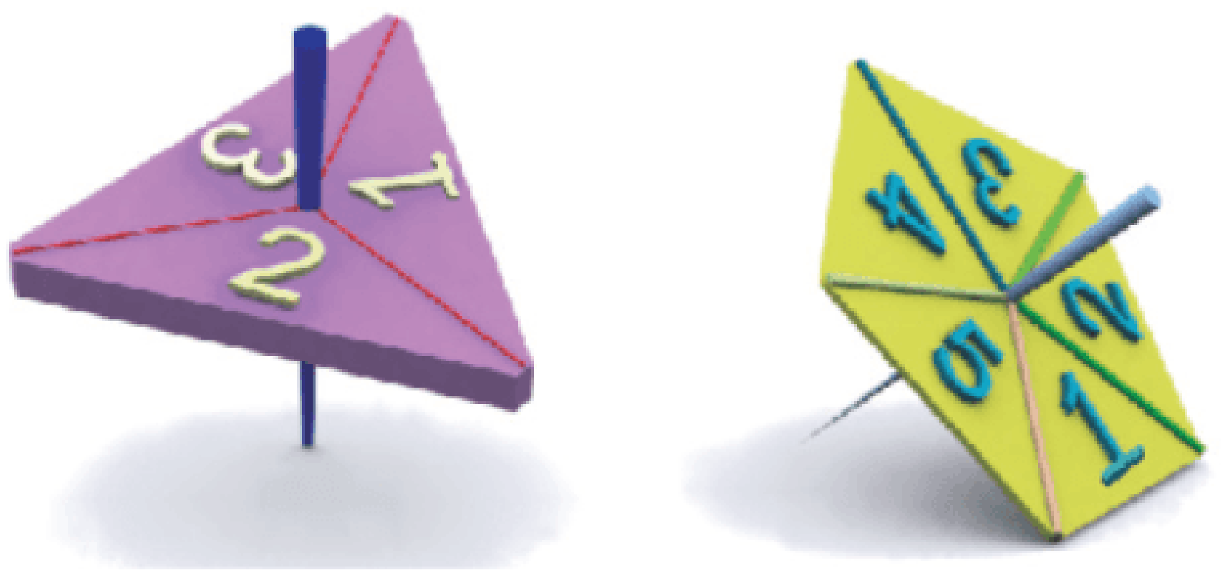
12 **a** Display the possible results when two dice are rolled and the difference between the numbers is found.

b Hence find the probability that the resulting value is:

i 0 **ii** 2 **iii** 1 or 2

iv more than 3 **v** less than 3.

13 The spinners alongside are spun, and the scores are multiplied together.



a Display the possible results.

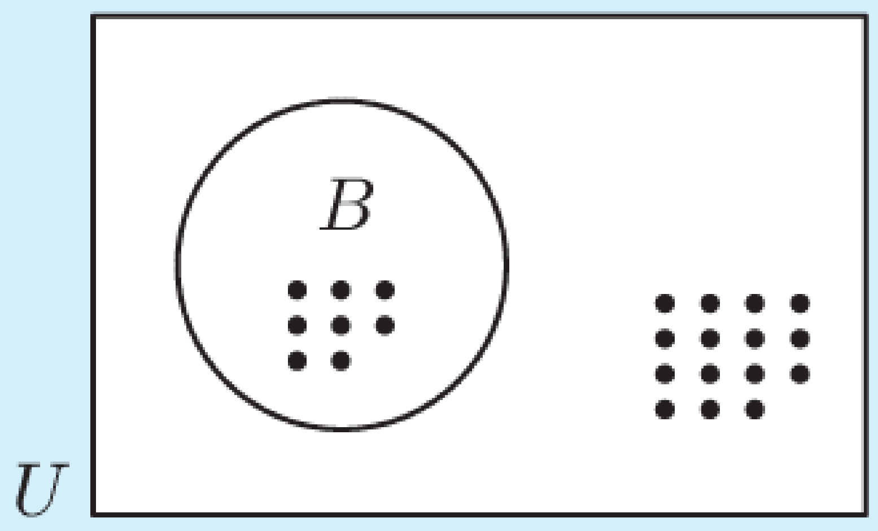
b Hence find the probability that the result is:

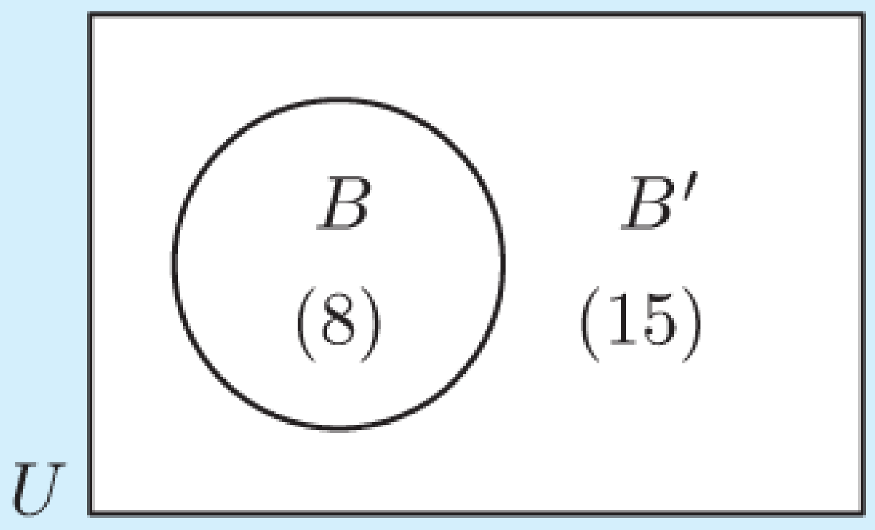
i 6 **ii** less than 5 **iii** odd.

Example 10 **Self Tutor**

In this Venn diagram, the universal set U is the children in a class. Each dot represents a student. The event B is that a student has blue eyes. Find the probability that a randomly selected child:

a has blue eyes **b** does not have blue eyes.





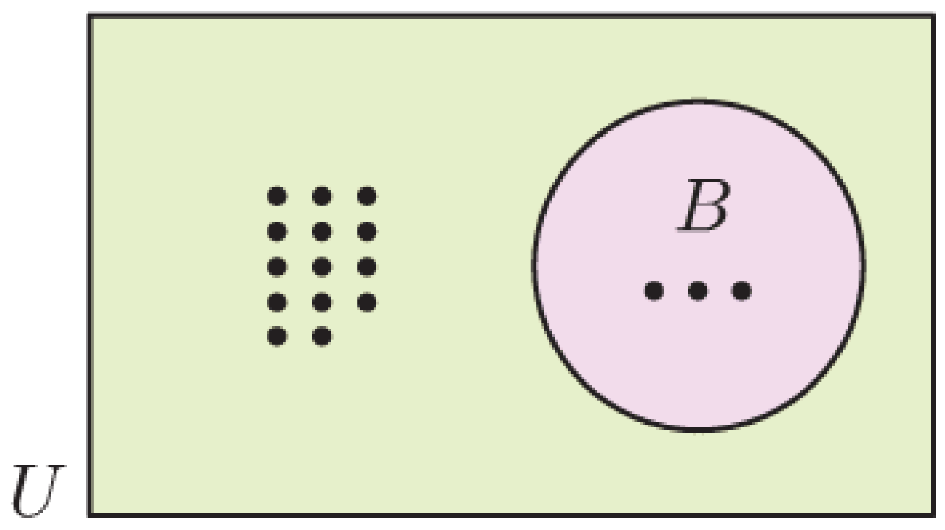
$n(U) = 23, \quad n(B) = 8$

a $\text{P(blue eyes)} = \dfrac{n(B)}{n(U)} = \frac{8}{23}$

b $\text{P(not blue eyes)} = \dfrac{n(B')}{n(U)} = \frac{15}{23}$

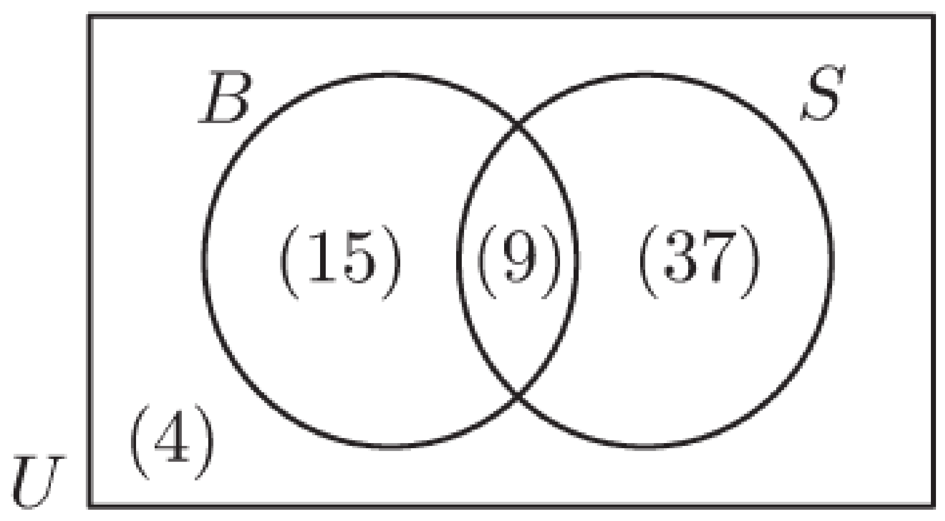
or $\text{P(not blue)} = 1 - \text{P(blue eyes)} = 1 - \frac{8}{23} = \frac{15}{23}$

14 In this Venn diagram, the universal set U is the sheep in a pen. Each dot represents a sheep. The event B is that a sheep has black wool.
Find the probability that a randomly selected sheep:



a has black wool **b** does not have black wool.

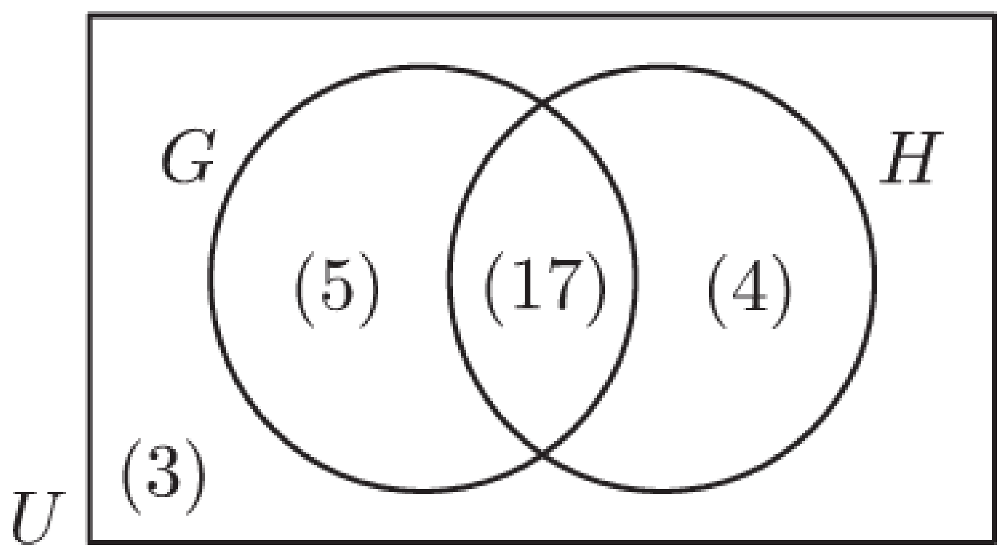
15 In a survey at an alpine resort, people were asked whether they liked skiing (S) or snowboarding (B). The results are shown in the Venn diagram. Find the probability that a randomly chosen person at the resort likes:



a both activities **b** neither activity

c exactly one activity.

16 This Venn diagram shows the number of students in a particular class who study Geography (G) and History (H). Find the probability that a randomly selected student in the class studies:



a both of these subjects

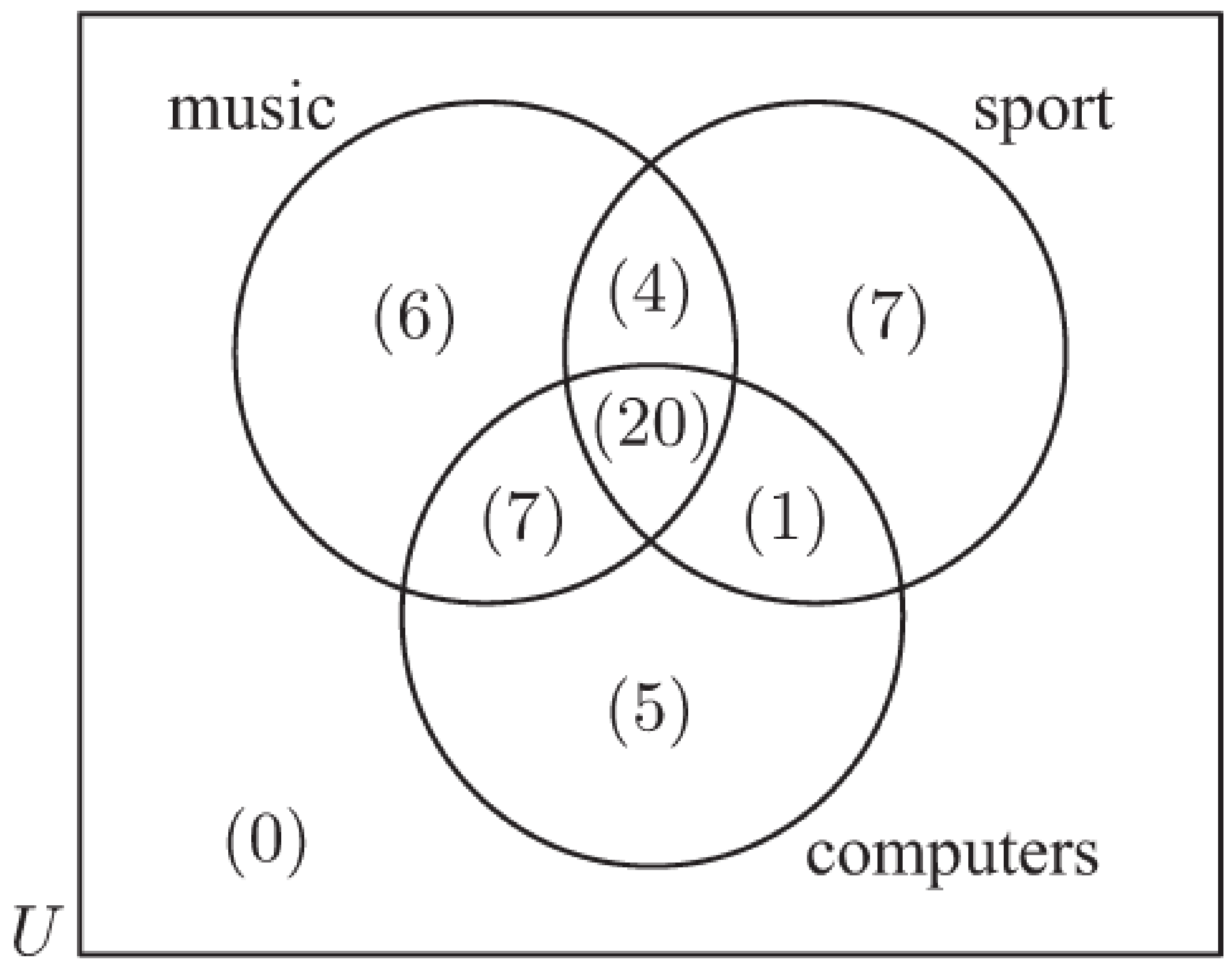
b at least one of these subjects

c only Geography.

17 A group of 50 employees were surveyed regarding their interest in music, sport, and computers. The number of employees interested in each area is shown in the Venn diagram.

If an employee is selected at random, determine the probability that they are:

- **a** interested in music
- **b** interested in music, sport, and computers
- **c** not interested in computers.



Example 11

Self Tutor

In a class of 30 students, 19 study Physics, 17 study Chemistry, and 15 study both of these subjects.

- **a** Display this information on a Venn diagram.
- **b** Hence determine the probability that a randomly selected student from this class studies:
  - **i** both subjects
  - **ii** at least one of the subjects
  - **iii** Physics but not Chemistry
  - **iv** exactly one of the subjects
  - **v** neither subject.

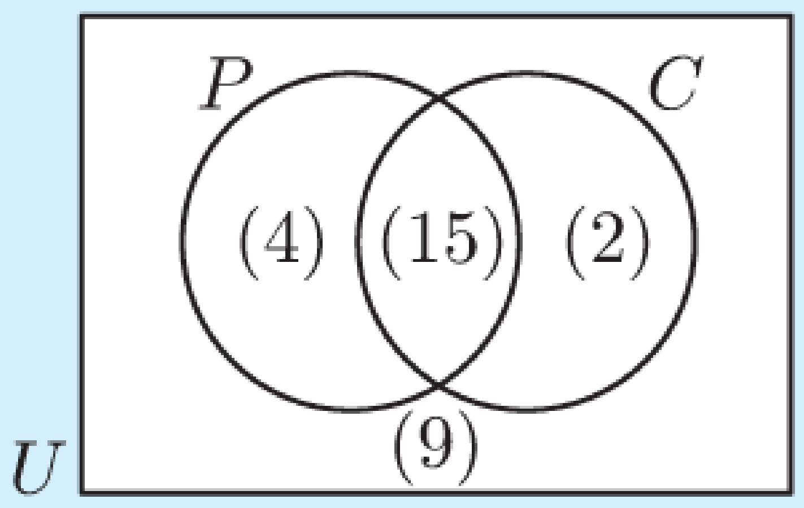
a Let P represent the event "the student studies Physics" and C represent the event "the student studies Chemistry".

$$n(P \cap C) = 15$$

$$\therefore \; n(P \cap C') = 19 - 15 = 4$$

$$\text{and} \quad n(P' \cap C) = 17 - 15 = 2$$

$$\therefore \; n(P' \cap C') = 30 - 15 - 4 - 2 = 9$$



b **i** P(studies both)

$$= \tfrac{15}{30} = \tfrac{1}{2}$$

ii P(studies at least one subject)

$$= \tfrac{4+15+2}{30} = \tfrac{7}{10}$$

iii P(P but not C)

$$= \tfrac{4}{30} = \tfrac{2}{15}$$

iv P(studies exactly one)

$$= \tfrac{4+2}{30} = \tfrac{1}{5}$$

v P(studies neither)

$$= \tfrac{9}{30} = \tfrac{3}{10}$$

18 50 married men were asked whether they gave their wife flowers or chocolates for her last birthday. 31 gave chocolates, 12 gave flowers, and 5 gave both chocolates and flowers.

- **a** Display this information on a Venn diagram.
- **b** If one of the married men was chosen at random, determine the probability that he gave his wife:
  - **i** chocolates or flowers
  - **ii** chocolates but not flowers
  - **iii** neither chocolates nor flowers.

19 In a class of 40 students, 19 play tennis, 20 play netball, and 8 play neither of these sports. A student is randomly chosen from the class. Determine the probability that the student:

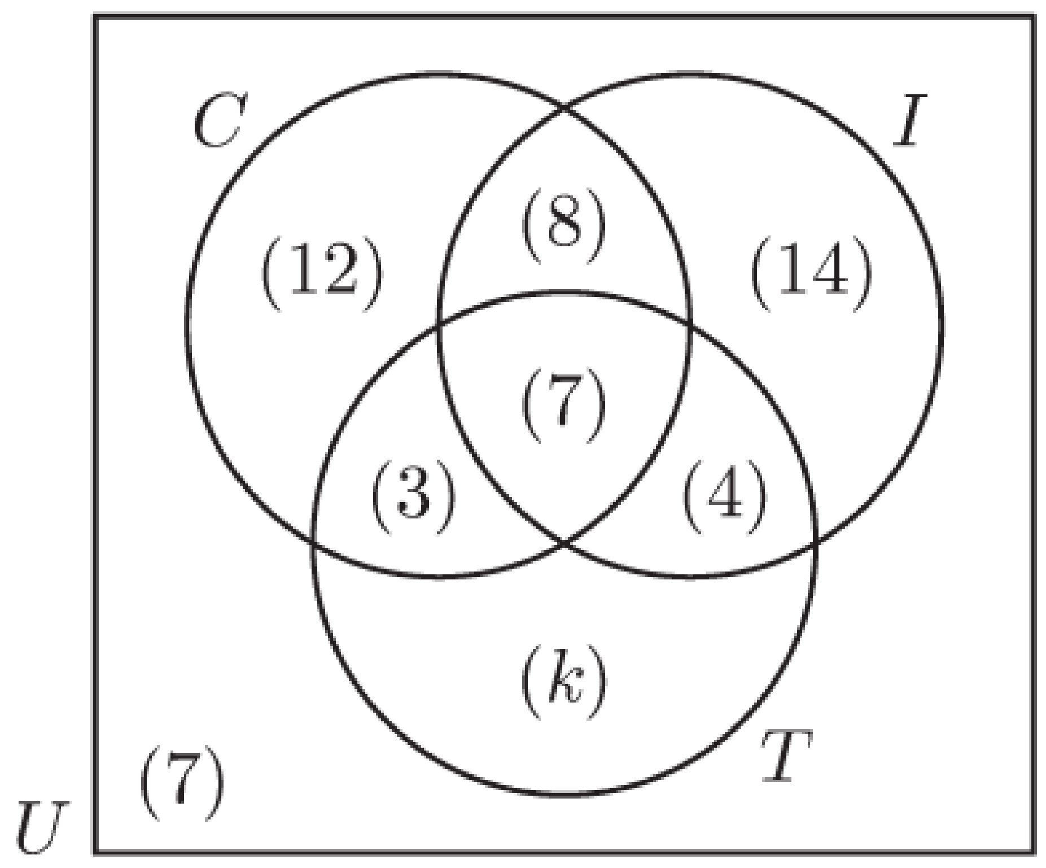
- **a** plays tennis
- **b** does not play netball
- **c** plays at least one of the sports
- **d** plays exactly one of the sports
- **e** plays netball but not tennis.

20 The medical records for a class of 30 children showed that 24 had previously had measles, 12 had previously had measles and mumps, and 26 had previously had at least one of measles or mumps. If one child from the class is selected at random, determine the probability that he or she has had:

- **a** mumps
- **b** mumps but not measles
- **c** neither mumps nor measles.

21 In this Venn diagram, U is the set of all 60 members of a club.

The members indicate their liking for Chinese (C), Italian (I), and Thai (T) food.



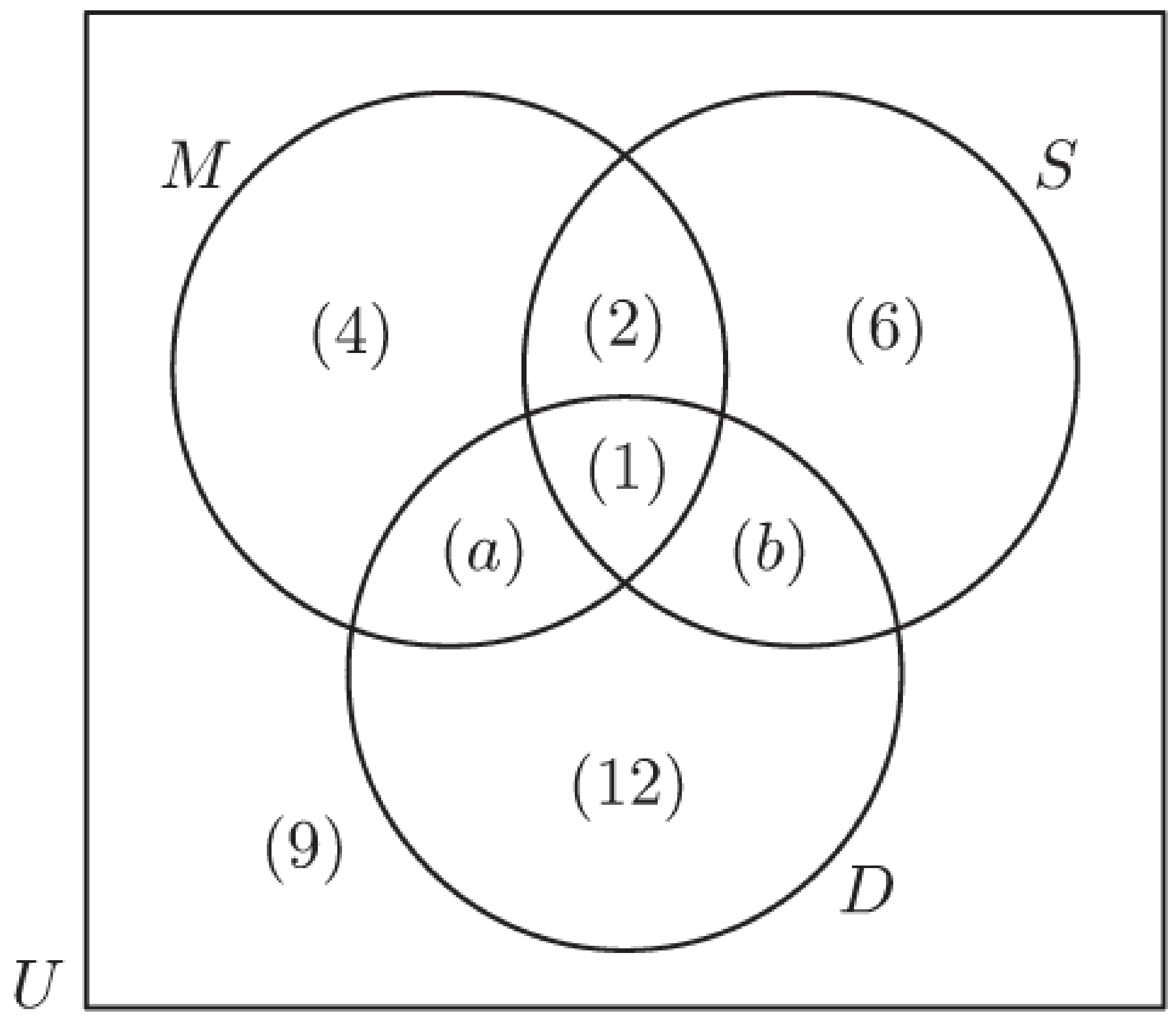
- **a** Find the value of k.
- **b** Find the probability that a randomly chosen member likes:
  - **i** only Italian
  - **ii** Italian and Thai
  - **iii** none of these foods
  - **iv** at least one of these foods
  - **v** all of these foods
  - **vi** Chinese and Italian, but not Thai
  - **vii** Thai or Italian
  - **viii** exactly one of these foods.

22 As a group bonding project, 50 delegates at a European conference were asked what languages they had conversations in at lunch time. The data collected is summarised alongside.

Languages	*Delegates*
English only	17
French only	7
Spanish only	12
English and French only	3
English and Spanish only	6
French and Spanish only	4
English, French, and Spanish	1

- **a** Construct a Venn diagram to display the information.
- **b** Find the probability that a randomly selected delegate had a conversation in:
  - **i** English
  - **ii** French
  - **iii** Spanish, but not in English
  - **iv** French, but not in Spanish
  - **v** French, and also one in English.

23 The Venn diagram opposite indicates the types of programs a group of 40 individuals watched on television last night. M represents movies, S represents sports, and D represents dramas.



- **a** Given that 10 people watched a movie last night, calculate a and b.
- **b** Find the probability that one of these individuals, selected at random, watched:
  - **i** sport
  - **ii** drama and sport
  - **iii** a movie but not sport
  - **iv** drama but not a movie
  - **v** drama or a movie.

DISCUSSION

Three children have been tossing a coin in the air and recording the outcomes. They have done this 10 times and have recorded 10 tails. Before the next toss they make these statements:

Jackson: "It's got to be a head next time!"

Sally: "No, it always has an equal chance of being a head or a tail. The coin cannot remember what the outcomes have been."

Amy: "Actually, I think it will probably be a tail again, because I think the coin must be biased. It might be weighted so it is more likely to give a tail."

Discuss the statements of each child. Who do you think is correct?

E MAKING PREDICTIONS USING PROBABILITY

In **Section A** we saw that if we perform an experiment a number of times, then the experimental probability of an event occurring is

$$\begin{aligned}\text{experimental probability} &= \text{relative frequency of event}\\ &= \frac{\text{number of times event occurs}}{\text{number of trials}}.\end{aligned}$$

So, we start with an experiment and use it to generate a probability. In the study of expectation, we go the other way.

Rearranging the equation, we obtain:

$$\text{number of times event occurs} = \text{experimental probability} \times \text{number of trials.}$$

However in this case, we use a theoretical probability to predict the results.

If there are n trials of an experiment, and an event has probability p of occurring in each of the trials, then the number of times we *expect* the event to occur is np.

DISCUSSION

In most cases, the expected value np will not be an integer. It will therefore be impossible to actually get the "expected" value. Is this a problem?

Example 12 **Self Tutor**

In his basketball career, Michael Jordan made 83.53% of shots from the free throw line. If he had played one more game and had 18 attempts from the free throw line, how many shots would you expect him to have made?

$n = 18$ throws

$p = \text{P(successfully makes free throw)} = 0.8353$

We would expect him to have made $np = 18 \times 0.8353 \approx 15$ shots.

EXERCISE 11E

1 A goalkeeper has probability $\frac{3}{10}$ of saving a penalty attempt. How many goals would he expect to save from 90 attempts?

2 The coach of a lacrosse team has calculated that Brayden scores on about 23% of his attempts at goal. If Brayden has 68 attempts to score this season, how many times would you expect him to score?

3 **a** If 2 coins are tossed, what is the chance that they both fall heads?

b If the 2 coins are tossed 200 times, on how many occasions would you expect them to both fall heads?

4 During the snow season there is a $\frac{3}{7}$ probability of snow falling on any particular day. If Udo skis for five weeks, on how many days could he expect to see snow falling?

5 If two dice are rolled simultaneously 180 times, on how many occasions would you expect to get a double?

6 In a pre-election poll, residents indicated their voting intentions. The number of voters that favoured each candidate A, B, and C are shown alongside.

A	B	C
165	87	48

a Estimate the probability that a randomly chosen voter in the electorate will vote for:

i A **ii** B **iii** C.

b If 7500 people vote in the election, how many do you expect to vote for:

i A **ii** B **iii** C?

F THE ADDITION LAW OF PROBABILITY

We now more carefully consider **compound events** where there is more than one event in our sample space. This may be because the experiment has more than one process, or because we are interested in more than one property of the outcome.

Suppose there are two events A and B in a sample space U. Following set notation:

- The event that both A **and** B occur is written $\boldsymbol{A \cap B}$, and read as "A intersection B".
- The event that A **or** B **or both** occur is written $\boldsymbol{A \cup B}$, and read as "A union B".

INVESTIGATION 3 THE ADDITION LAW OF PROBABILITY

In this Investigation we look for a formula connecting the probabilities for $P(A \cap B)$ and $P(A \cup B)$.

What to do:

1 Suppose $U = \{x \mid x \text{ is a positive integer less than } 100\}$.

Let $A = \{\text{multiples of } 7 \text{ in } U\}$ and $B = \{\text{multiples of } 5 \text{ in } U\}$.

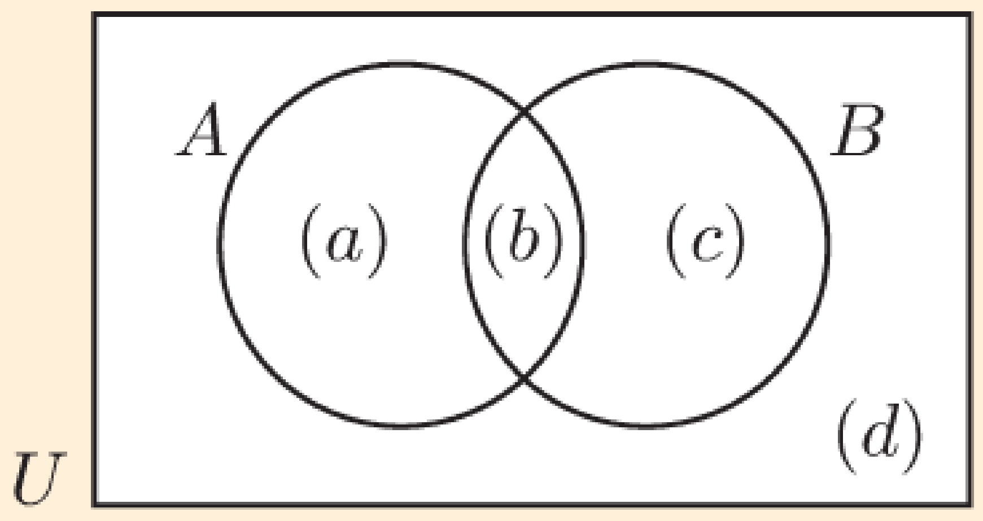
a How many elements are there in:

i A **ii** B **iii** $A \cap B$ **iv** $A \cup B$?

b Show that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

2 By comparing regions of the Venn diagram, verify that for all sets A and B in a universal set U:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$



3 By dividing both sides of the above equation by $n(U)$, establish the connection between $\text{P}(A \cup B)$ and $\text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$.

From the **Investigation** you should have discovered the **addition law of probability**:

For two events A and B, $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$

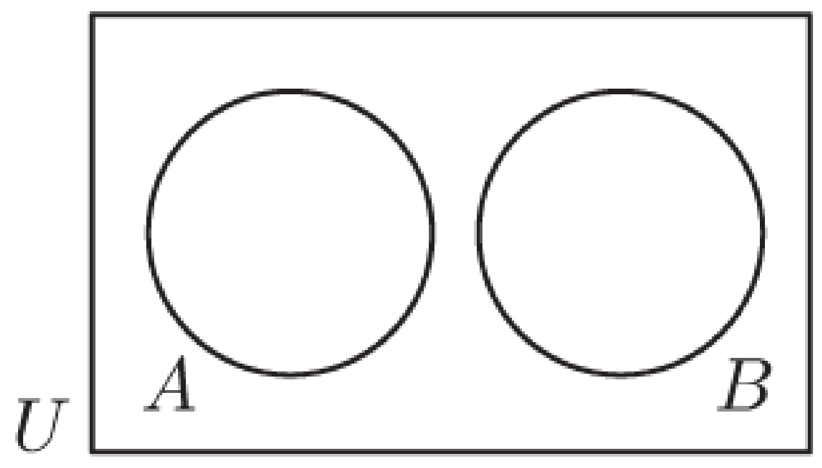
which means: P(**either** A **or** B **or** both) $= \text{P}(A) + \text{P}(B) -$ P(**both** A **and** B).

Example 13 **Self Tutor**

If $\text{P}(A) = 0.6$, $\text{P}(A \cup B) = 0.7$, and $\text{P}(A \cap B) = 0.3$, find $\text{P}(B)$.

$$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$$
$$\therefore\ 0.7 = 0.6 + \text{P}(B) - 0.3$$
$$\therefore\ \text{P}(B) = 0.4$$

If A and B are disjoint **mutually exclusive** events then $\text{P}(A \cap B) = 0$ and so the addition law becomes $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B)$.



Example 14 **Self Tutor**

30 students were given a History test. 7 students scored an A and 11 students scored a B.

A student is randomly selected. Let A be the event that the student scored an A, and B be the event that the student scored a B.

a Are A and B mutually exclusive?

b Find:

i $\text{P}(A)$ **ii** $\text{P}(B)$ **iii** $\text{P}(A \cap B)$ **iv** $\text{P}(A \cup B)$

a It is impossible for a student to score both an A and a B for the test.

$\therefore$ A and B are mutually exclusive.

b **i** $\text{P}(A) = \frac{7}{30}$

ii $\text{P}(B) = \frac{11}{30}$

iii $\text{P}(A \cap B) = 0$

{A and B are mutually exclusive}

iv $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B)$

$$= \frac{7}{30} + \frac{11}{30}$$
$$= \frac{3}{5}$$

EXERCISE 11F

1 If $P(A) = 0.2$, $P(B) = 0.4$, and $P(A \cap B) = 0.05$, find $P(A \cup B)$.

2 If $P(A) = 0.4$, $P(A \cup B) = 0.9$, and $P(A \cap B) = 0.1$, find $P(B)$.

3 If $P(X) = 0.6$, $P(Y) = 0.5$, and $P(X \cup Y) = 0.9$, find $P(X \cap Y)$.

4 Suppose $P(A) = 0.25$, $P(B) = 0.45$, and $P(A \cup B) = 0.7$.

- **a** Find $P(A \cap B)$.
- **b** What can you say about A and B?

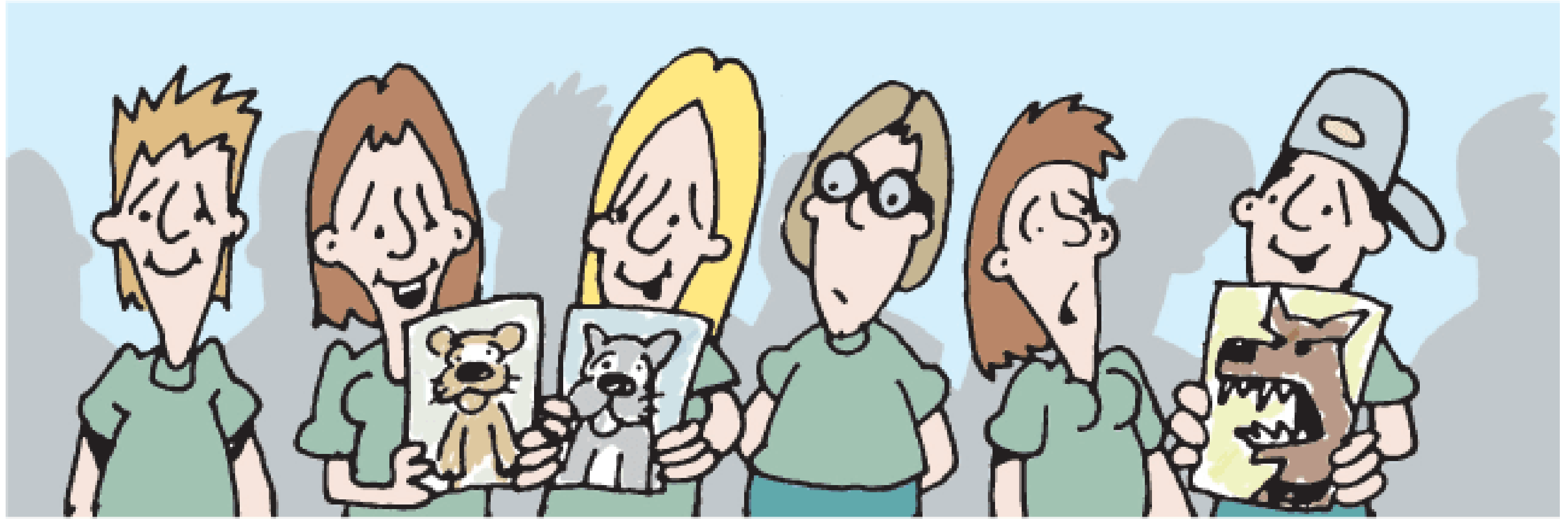
5 A and B are mutually exclusive events.
If $P(B) = 0.45$ and $P(A \cup B) = 0.8$, find $P(A)$.

6 Tickets numbered 1 to 15 are placed in a hat, and one ticket is chosen at random. Let A be the event that the number drawn is greater than 11, and B be the event that the number drawn is less than 8.

- **a** Are A and B mutually exclusive?
- **b** Find: **i** $P(A)$ **ii** $P(B)$ **iii** $P(A \cup B)$.

7 A class consists of 25 students.

11 students are fifteen years old (F).
12 students are sixteen years old (S).
8 students own a dog (D).
7 students own a cat (C).
4 students do not own any pets (N).



A student is chosen at random. If possible, find:

- **a** $P(F)$ **b** $P(S)$ **c** $P(D)$ **d** $P(C)$ **e** $P(N)$
- **f** $P(F \cup S)$ **g** $P(F \cup D)$ **h** $P(C \cup N)$ **i** $P(C \cup D)$ **j** $P(D \cup N)$

8 Suppose A and B are mutually exclusive, and that A' and B' are mutually exclusive. Find $P(A \cup B)$.

G INDEPENDENT EVENTS

Two events are **independent** if the occurrence of each event does not affect the occurrence of the other.

INVESTIGATION 4 INDEPENDENT EVENTS

WORKSHEET

In this Investigation we seek a rule for calculating $P(A \cap B)$ for two independent events A and B.

What to do:

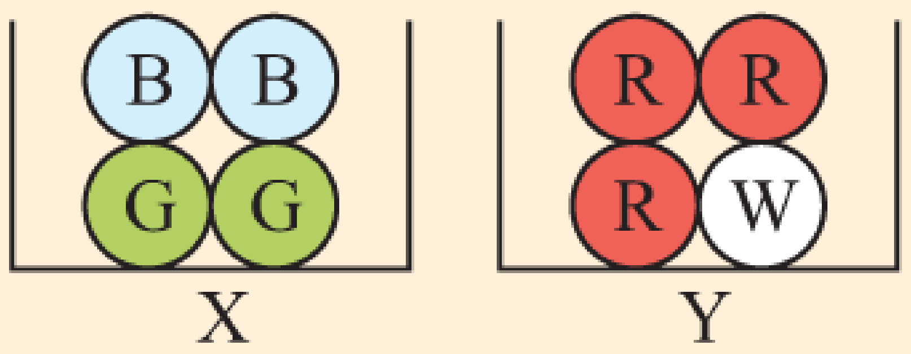
1 Suppose a coin is tossed and a die is rolled at the same time.

- **a** Does the outcome of the coin toss affect the outcome of the die roll, or vice versa?
- **b** Draw a 2-dimensional grid to show the possible outcomes of the experiment.

c Copy and complete this table of probabilities for different events A and B:

	A	B	$\mathrm{P}(A)$	$\mathrm{P}(B)$	$\mathrm{P}(A \cap B)$
i	head	4			
ii	head	odd number			
iii	tail	number greater than 1			
iv	tail	number less than 3			

2 Consider randomly selecting a ball from each of the boxes alongside.



a Does the outcome of the draw from either box affect the occurrence of the other?

b Draw a 2-dimensional grid to show the possible outcomes of the experiment.

c Copy and complete this table of probabilities for different events A and B:

	A	B	$\mathrm{P}(A)$	$\mathrm{P}(B)$	$\mathrm{P}(A \cap B)$
i	green from box X	red from box Y			
ii	green from box X	white from box Y			
iii	blue from box X	red from box Y			
iv	blue from box X	white from box Y			

3 For independent events A and B, what is the connection between $\mathrm{P}(A \cap B)$, $\mathrm{P}(A)$, and $\mathrm{P}(B)$?

From the **Investigation** you should have concluded that:

If A and B are independent events, then $\mathbf{P}(\boldsymbol{A} \cap \boldsymbol{B}) = \mathbf{P}(\boldsymbol{A}) \times \mathbf{P}(\boldsymbol{B})$.

This rule can be extended for any number of independent events.

For example: If A, B, and C are all independent events, then $\mathrm{P}(A \cap B \cap C) = \mathrm{P}(A) \times \mathrm{P}(B) \times \mathrm{P}(C)$.

Example 15 **Self Tutor**

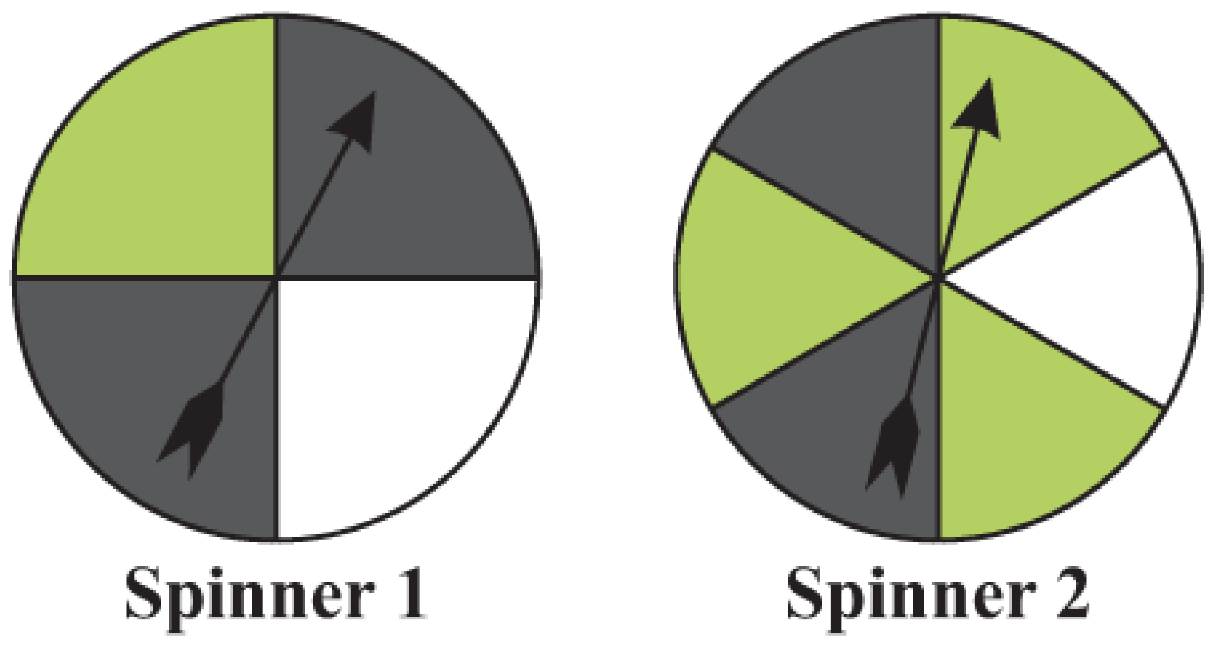
A coin is tossed and a die is rolled simultaneously. Determine the probability of getting a head and a 3, without using a grid.

$$\begin{aligned}\mathrm{P}(\text{a head} \cap \text{a } 3) &= \mathrm{P}(\mathrm{H}) \times \mathrm{P}(3) \qquad \{\text{events are independent}\}\\ &= \tfrac{1}{2} \times \tfrac{1}{6}\\ &= \tfrac{1}{12}\end{aligned}$$

EXERCISE 11G

1 Each of these spinners is spun once. Find the probability of spinning:

- **a** green with spinner 1 and white with spinner 2
- **b** black with both spinners.



2 A coin is tossed 3 times. Determine the probability of getting the following sequences of results:

- **a** head, head, head
- **b** tail, head, tail.

3 A school has two photocopiers. On any given day, machine A has an 8% chance of malfunctioning and machine B has a 12% chance of malfunctioning. Determine the probability that on any given day, both machines will:

- **a** malfunction
- **b** work effectively.

4 Two marksmen fire at a target simultaneously. Jiri hits the target 70% of the time and Benita hits it 80% of the time. Determine the probability that:

- **a** they both hit the target
- **b** they both miss the target
- **c** Jiri hits but Benita misses
- **d** Benita hits but Jiri misses.

5 An archer hits the bullseye on average 2 out of every 5 shots. If 3 arrows are fired at the target, determine the probability that the bullseye is hit:

- **a** every time
- **b** the first two times, but not on the third shot
- **c** on no occasion.

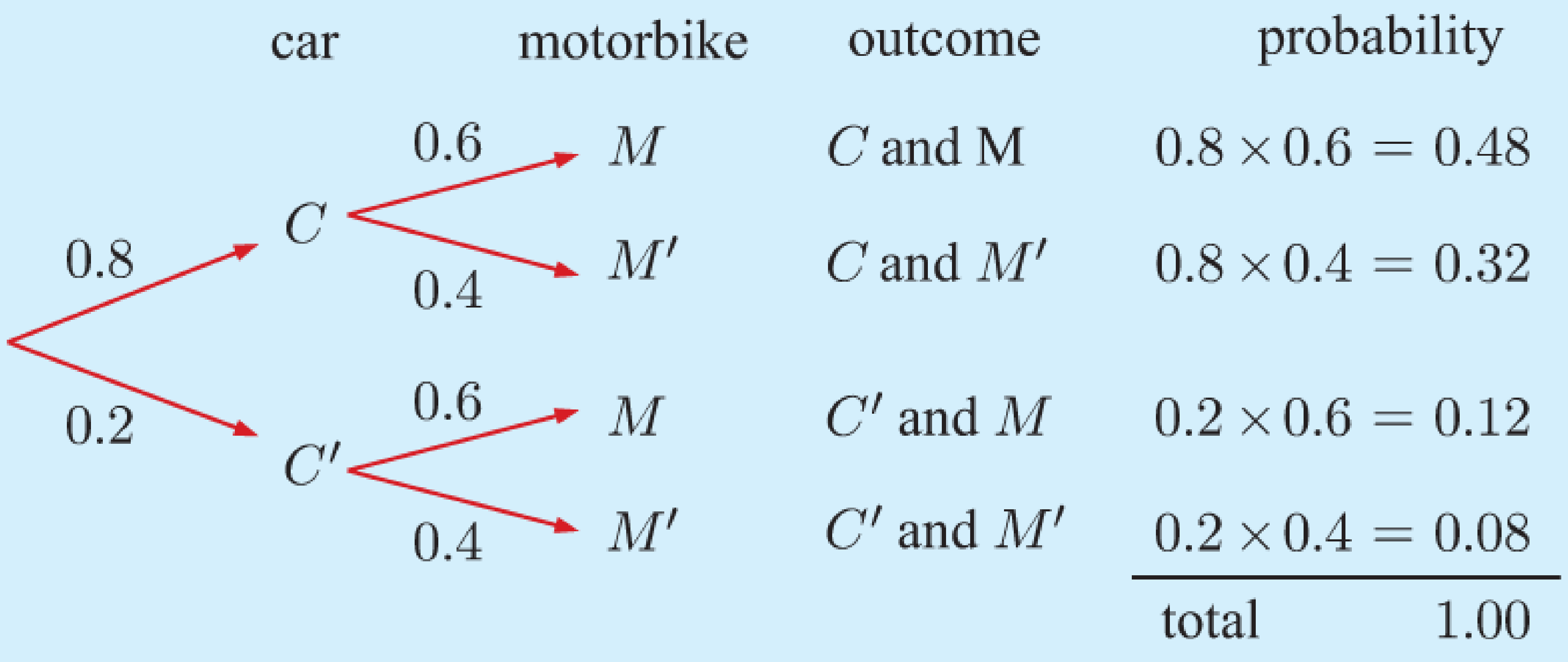
Example 16

Self Tutor

Carl is not having much luck lately. His car will only start 80% of the time and his motorbike will only start 60% of the time, independently of one another.

- **a** Draw a tree diagram to illustrate this situation.
- **b** Use the tree diagram to determine the chance that on the next attempt:
  - **i** both will start
  - **ii** Carl can only use his car.

a Let C be the event that Carl's car starts, and M be the event that his motorbike starts.



The probability of each outcome is obtained by **multiplying** the probabilities along its branch.

b **i** P(both start)
$= \text{P}(C \cap M)$
$= 0.8 \times 0.6$
$= 0.48$

ii P(car starts and motorbike does not)
$= \text{P}(C \cap M')$
$= 0.8 \times 0.4$
$= 0.32$

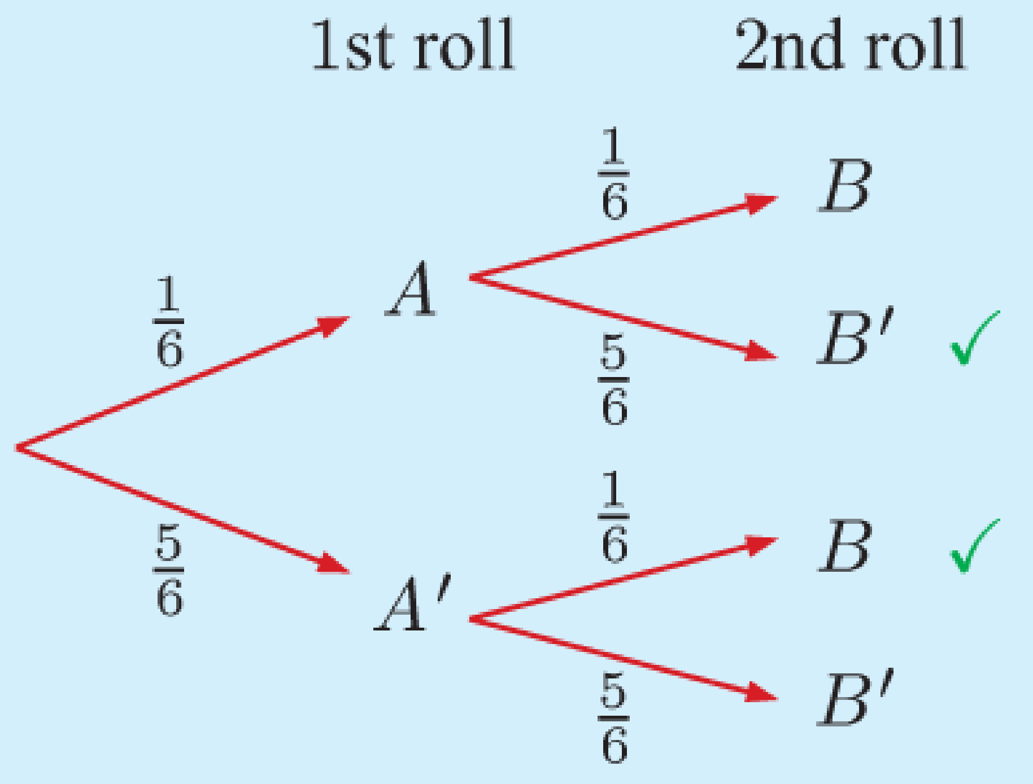
6 For a particular household, there is a 90% chance that at the end of the week the rubbish bin is full, and a 50% chance that the recycling bin is full, independently of one another.

- **a** Draw a tree diagram to illustrate this situation.
- **b** Find the probability that at the end of the week:
  - **i** both bins are full
  - **ii** the recycling bin is full but the rubbish bin is not.

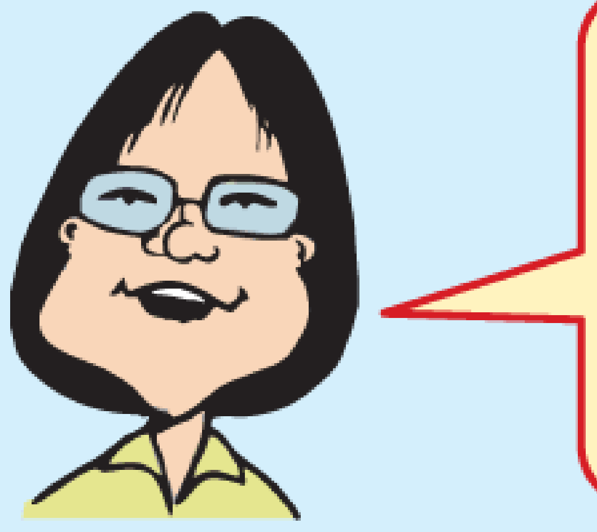
Example 17 **Self Tutor**

Liam rolls a six-sided die twice. Determine the probability that exactly 1 four is rolled.

Let A be the event that a four is rolled on the first roll, and B be the event that a four is rolled on the second roll.



$$\begin{aligned}&\text{P}(1\text{ four})\\ &= \text{P}(A \cap B') + \text{P}(A' \cap B)\\ &= \tfrac{1}{6} \times \tfrac{5}{6} + \tfrac{5}{6} \times \tfrac{1}{6} \quad \{\text{branches marked } \checkmark\}\\ &= \tfrac{5}{36} + \tfrac{5}{36}\\ &= \tfrac{10}{36}\\ &= \tfrac{5}{18}\end{aligned}$$



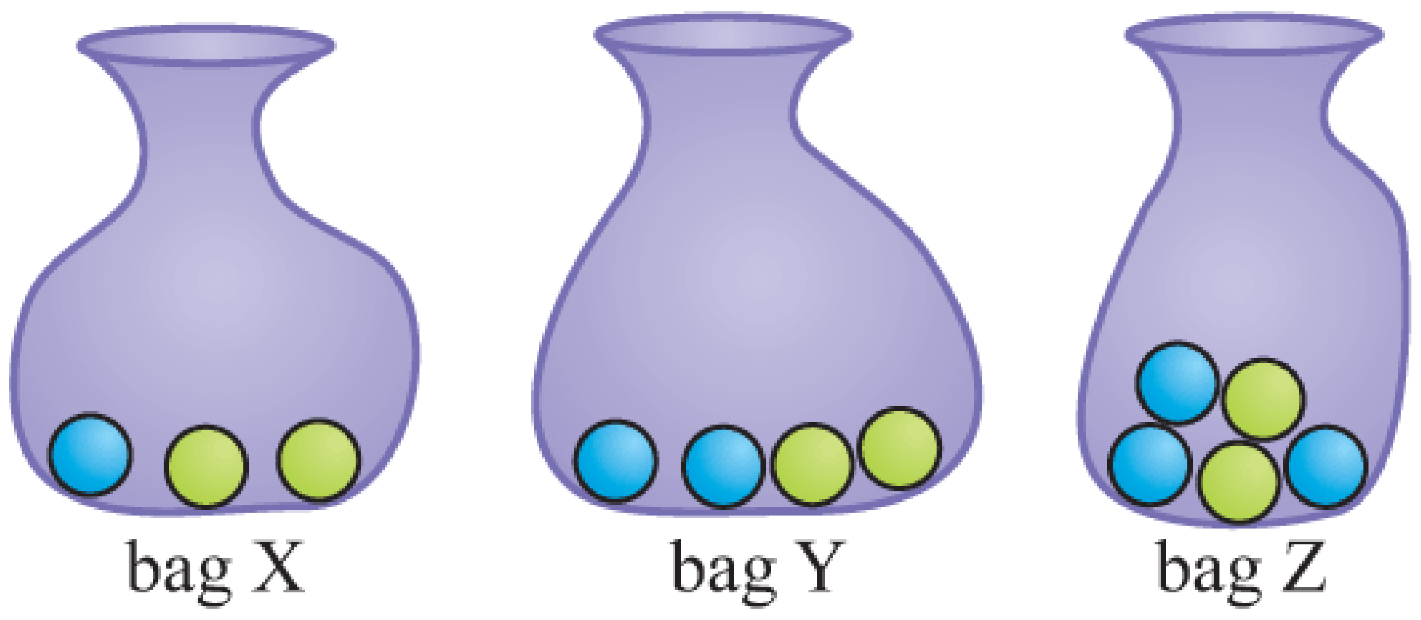
If more than one outcome corresponds to an event, **add** the probabilities of these outcomes.

7 Two baskets each contain 5 red apples and 2 green apples. Celia chooses an apple at random from each basket.

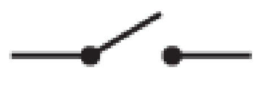
- **a** Draw a tree diagram to illustrate the possible outcomes.
- **b** Find the probability that Celia chooses:
  - **i** two red apples
  - **ii** one red and one green apple.

8 One ball is drawn from each of the bags shown.

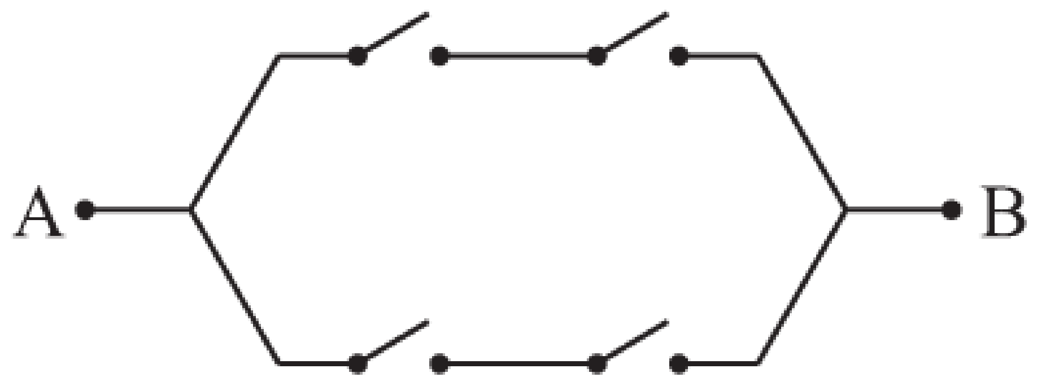
- **a** Draw a tree diagram to illustrate this situation.
- **b** Find the probability that:
  - **i** 3 blue balls are drawn
  - **ii** green balls are drawn from bags Y and Z
  - **iii** at least one blue ball is drawn.



9 The diagram shows a simple electrical network.

Each symbol  represents a switch.

All four switches operate independently, and the probability of each one of them being closed is p.



- **a** In terms of p, find the probability that the current flows from A to B.
- **b** Find the least value of p for which the probability of current flow is at least 0.5 .

10 Kane plays 3 matches in a darts challenge, alternating between Penny and Quentin as his opponents. Kane must win 2 matches in a row to win the challenge. He is allowed to choose which opponent he plays first. He knows that Penny is the better darts player.
Which strategy should Kane use to maximise his chances of winning the challenge? Justify your answer.

H DEPENDENT EVENTS

Suppose a hat contains 5 red and 3 blue tickets. One ticket is randomly chosen, its colour is noted, and it is then put aside and so *not* put back in the hat. A second ticket is then randomly selected. What is the chance that it is red?

If the first ticket was red, P(second is red) $= \frac{4}{7}$ — 4 reds remaining / 7 to choose from

If the first ticket was blue, P(second is red) $= \frac{5}{7}$ — 5 reds remaining / 7 to choose from

The probability of the second ticket being red *depends* on what colour the first ticket was. We therefore have **dependent events**.

> Two or more events are **dependent** if the occurrence of one of the events *does affect* the occurrence of the other events.
>
> Events are **dependent** if they are **not independent**.
>
> If A and B are dependent events then
>
> $$\text{P}(A \cap B) = \text{P}(A) \times \text{P}(B \text{ given that } A \text{ has occurred}).$$

In general, when an experiment involves sampling:

- **without replacement** we have dependent events
- **with replacement** we have independent events.

Not all scenarios we study involve sampling. However, they may still involve dependent events.

For example, the event *Pahal walks to school today* is dependent on the event *it will rain today*.

Example 18 **Self Tutor**

A box contains 4 red and 2 yellow tickets. Two tickets are randomly selected from the box one by one *without* replacement. Find the probability that:

a both are red **b** the first is red and the second is yellow.

a P(both red)

$=$ P(first selected is red $\cap$ second is red)

$=$ P(first selected is red) $\times$ P(second is red given that the first is red)

$= \frac{4}{6} \times \frac{3}{5}$

4 reds out of a total of 6 tickets

If a red is drawn first, 3 reds remain out of a total of 5.

$= \frac{2}{5}$

b P(first is red $\cap$ second is yellow)

$=$ P(first is red) $\times$ P(second is yellow given that the first is red)

$= \frac{4}{6} \times \frac{2}{5}$

4 reds out of a total of 6 tickets

If a red is drawn first, 2 yellows remain out of a total of 5.

$= \frac{4}{15}$

EXERCISE 11H

1 A box contains 7 red and 3 green balls. Two balls are drawn one after another from the box without replacement. Determine the probability that:

a both are red **b** the first is green and the second is red.

2 A bag contains 4 blue and 6 white tokens. Two tokens are drawn from the bag one after another, without replacement. Find the probability that:

a both are blue **b** the first is blue and the second is white.

Example 19 **Self Tutor**

A hat contains 20 tickets numbered 1, 2, 3, ..., 20. If three tickets are drawn from the hat without replacement, determine the probability that they are all prime numbers.

$\{2, 3, 5, 7, 11, 13, 17, 19\}$ are primes.

$\therefore$ 8 of the 20 numbers are primes.

$\therefore$ P(3 primes)

= P(1st drawn is prime $\cap$ 2nd is prime $\cap$ 3rd is prime)

$= \frac{8}{20}$ ← 8 primes out of 20 numbers

$\times \frac{7}{19}$ ← 7 primes out of 19 numbers after a successful first draw

$\times \frac{6}{18}$ ← 6 primes out of 18 numbers after two successful draws

≈ 0.0491

In each fraction, the numerator is the number of outcomes in the event. The denominator is the total number of possible outcomes.

3 A box contains 12 identically shaped chocolates of which 8 are strawberry creams. Three chocolates are selected simultaneously from the box. Determine the probability that:

a they are all strawberry creams

b none of them are strawberry creams.

Drawing three chocolates *simultaneously* implies there is no replacement.

4 A lottery has 100 tickets which are placed in a barrel. Three tickets are drawn at random from the barrel, without replacement, to decide 3 prizes. If John has 3 tickets in the lottery, determine his probability of winning:

a first prize **b** first and second prize **c** all 3 prizes **d** none of the prizes.

5 A hat contains 7 names of players in a tennis squad including the captain and the vice captain. If a team of three is chosen at random by drawing the names from the hat, determine the probability that it does *not* contain:

a the captain **b** the captain or the vice captain.

6 Two students are chosen at random from a group of two girls and five boys, all of different ages. Find the probability that the two students chosen will be:

a two boys **b** the eldest two students.

Example 20 **Self Tutor**

Two boxes each contain 6 petunia plants. Box A contains 2 plants with purple flowers and 4 plants with white flowers. Box B contains 5 plants with purple flowers and 1 plant with white flowers. A box is selected by tossing a coin, and one plant is removed at random from it.

Find the probability that the plant:

a was taken from box A and has white flowers

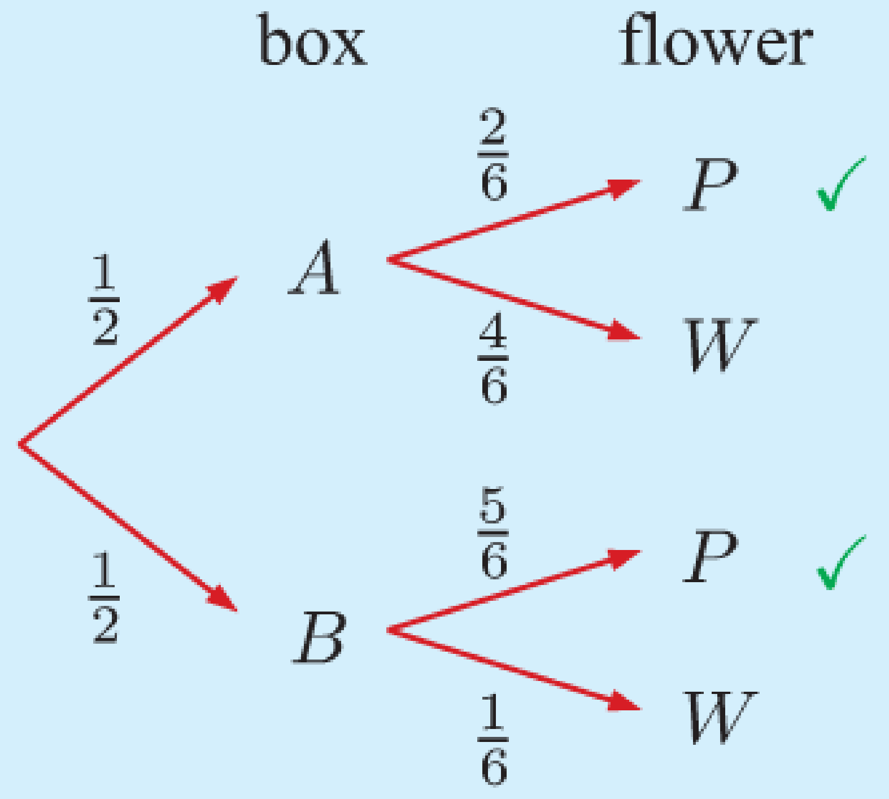
b has purple flowers.

Box A

P	W	W
W	P	W

Box B

P	P	P
W	P	P



a P(from box A ∩ white flowers)

$= \mathrm{P}(A \cap W)$

$= \frac{1}{2} \times \frac{4}{6}$

$= \frac{1}{3}$

b P(purple flowers)

$= \mathrm{P}(A \cap P) + \mathrm{P}(B \cap P)$

$= \frac{1}{2} \times \frac{2}{6} + \frac{1}{2} \times \frac{5}{6}$ {branches marked ✓}

$= \frac{7}{12}$

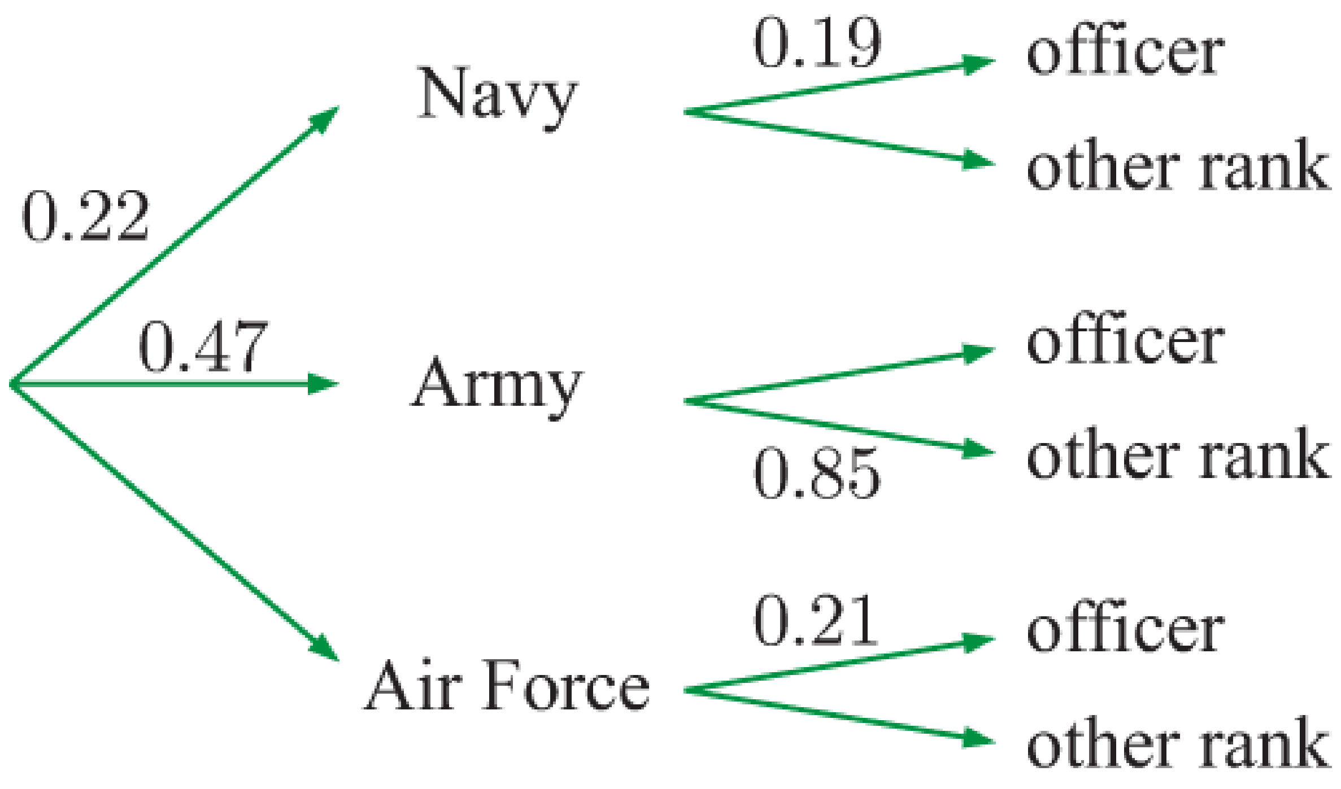
7 **a** Copy and complete this tree diagram about people in the armed forces.

b Find the probability that a member of the armed forces:

i is an officer

ii is not an officer in the navy

iii is not an army or air force officer.



8 Of the students in a class playing musical instruments, 60% are female. 20% of the females and 30% of the males play the violin.
Find the probability that a randomly selected student:

a is male and does not play the violin **b** plays the violin.

9 The probability of rain tomorrow is $\frac{1}{5}$. If it rains, Mudlark will start favourite in the horse race, with probability $\frac{1}{2}$ of winning. If it is fine, Mudlark only has a 1 in 20 chance of winning.

a Display the sample space of possible results for the horse race on a tree diagram.

b Hence determine the probability that Mudlark will win tomorrow.

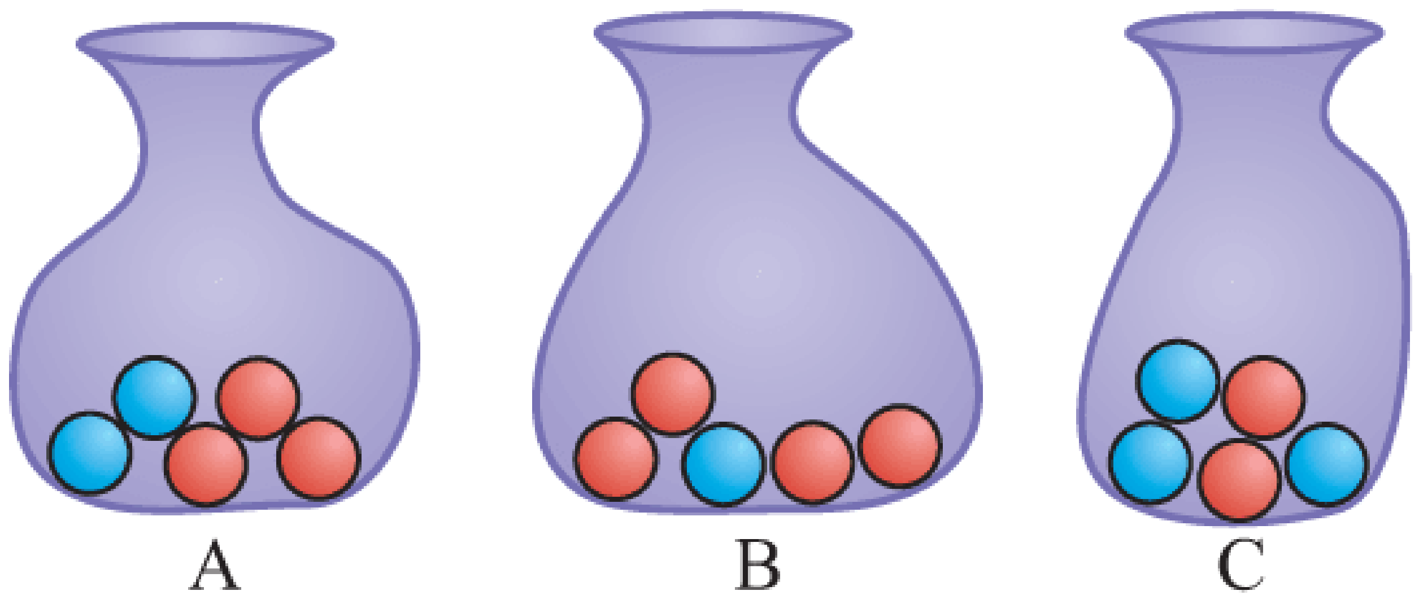
10 Machine A makes 40% of the bottles produced at a factory. Machine B makes the rest. Machine A spoils 5% of its product, while machine B spoils only 2%. Using an appropriate tree diagram, determine the probability that the next bottle inspected at this factory is spoiled.

11 The English Premier League consists of 20 teams. Tottenham is currently in 8th place on the table. It has 20% chance of winning and 50% chance of losing against any team placed above it. If a team is placed below it, Tottenham has a 50% chance of winning and a 30% chance of losing. Find the probability that Tottenham will draw its next game.

12 Three bags contain different numbers of blue and red marbles.

A bag is selected using a die which has three A faces, two B faces, and one C face. One marble is then randomly selected from the bag.

Determine the probability that the marble is:

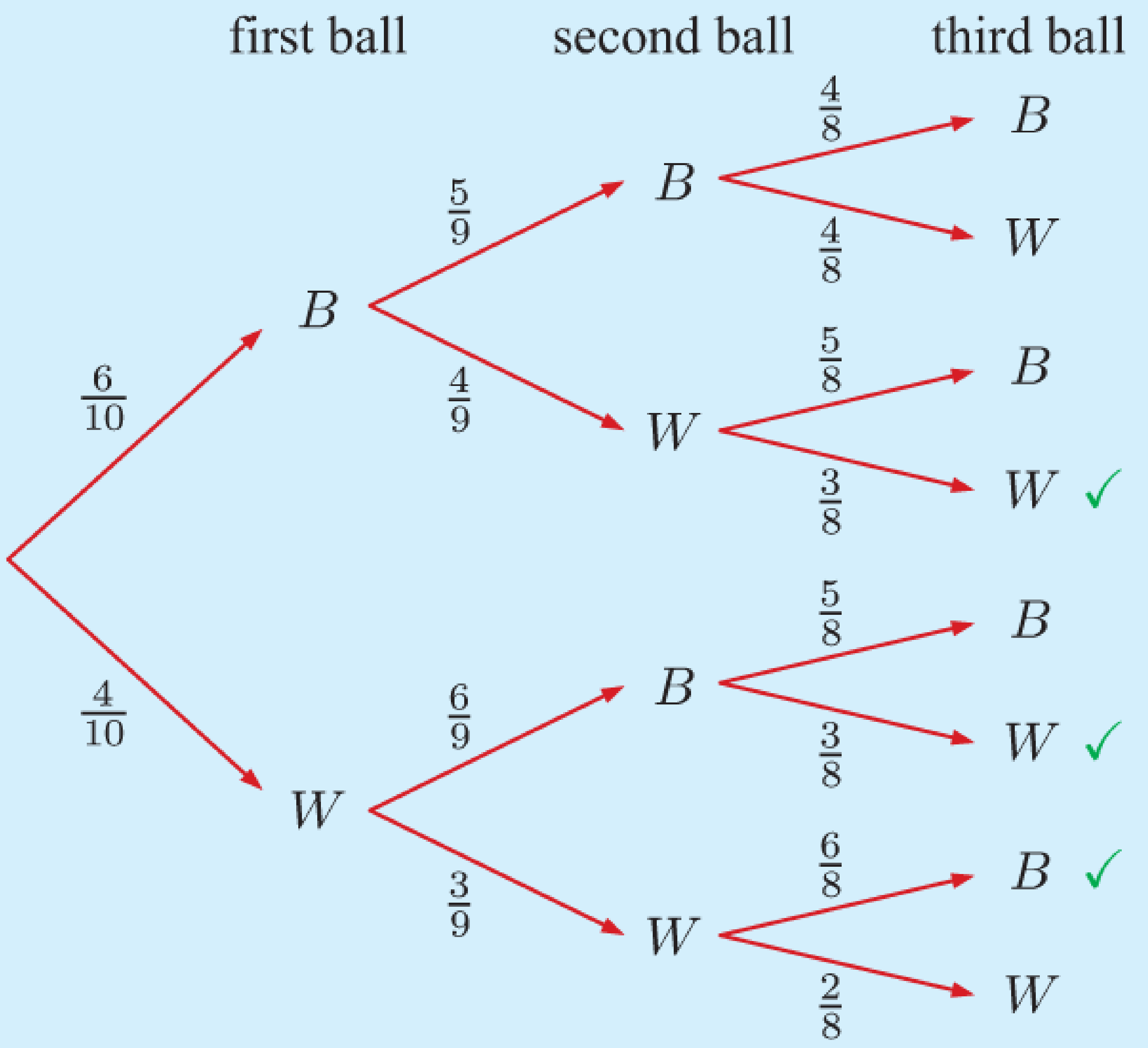


a blue **b** red.

Example 21 **Self Tutor**

A bag contains 10 balls. 6 balls are black and 4 balls are white. Three balls are drawn from the bag without replacement. Determine the probability that 1 black ball is drawn.

Let B represent drawing a black ball and W represent drawing a white ball.



$$\begin{aligned}&\text{P(1 black ball)}\\&= \text{P}(BWW) + \text{P}(WBW) + \text{P}(WWB)\\&= \left(\tfrac{6}{10}\right)\left(\tfrac{4}{9}\right)\left(\tfrac{3}{8}\right) + \left(\tfrac{4}{10}\right)\left(\tfrac{6}{9}\right)\left(\tfrac{3}{8}\right) + \left(\tfrac{4}{10}\right)\left(\tfrac{3}{9}\right)\left(\tfrac{6}{8}\right)\\&\qquad\qquad\qquad\qquad\{\text{branches marked ✓}\}\\&= \tfrac{1}{10} + \tfrac{1}{10} + \tfrac{1}{10}\\&= \tfrac{3}{10}\end{aligned}$$

13 In a class of 25 students, 11 students participate in extra-curricular activities. Suppose 3 students are randomly selected to be on the student representative council. Find the probability that at least two students selected for the council also participate in extra-curricular activities.

14 A standard deck of playing cards contains 52 cards. Four cards are drawn from a well-shuffled deck without replacement. Find the probability that:

a two red cards are drawn **b** at least one black card is drawn.

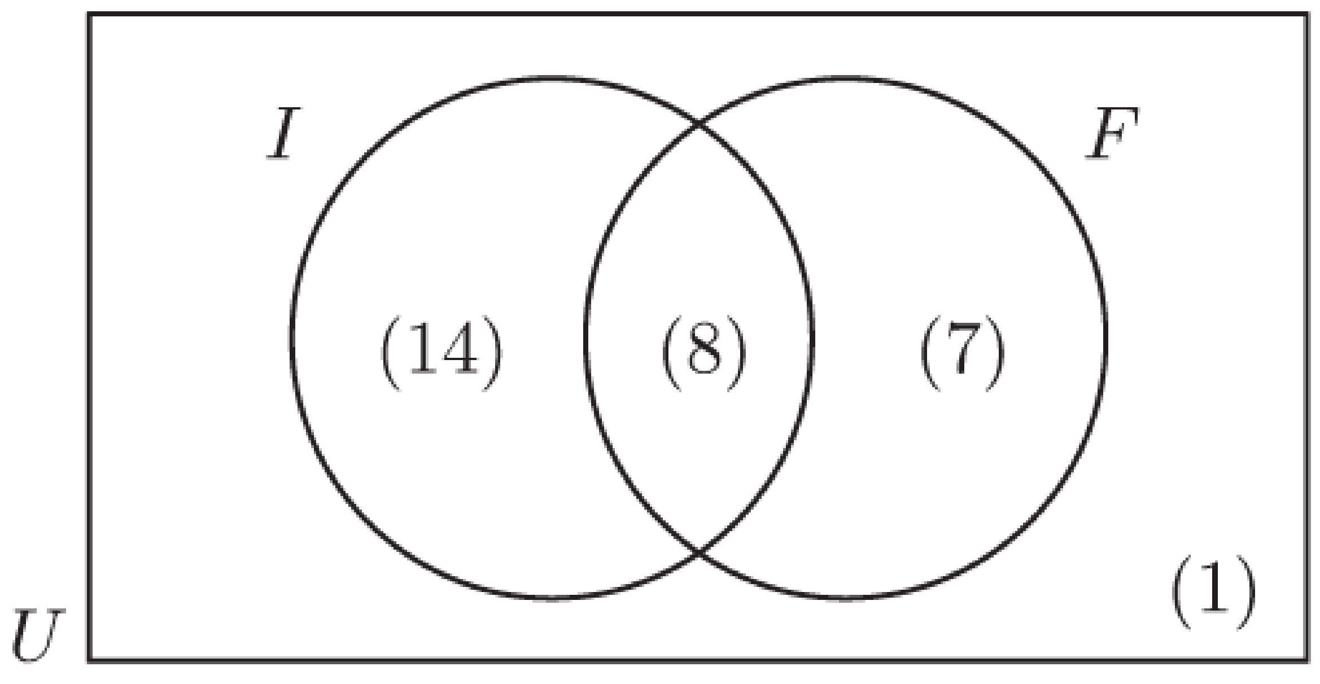
ACTIVITY 1 PÓLYA'S URN

Pólya's urn is a statistical model devised by the Hungarian mathematician **George Pólya** (1887 - 1985).

Click on the icon to obtain this Activity.

I CONDITIONAL PROBABILITY

This Venn diagram shows the numbers of students in a class who study Italian (I) and French (F).



Suppose a student is randomly selected from the class and it is found that the student studies French.

We can determine the probability that this student also studies Italian. We call this a **conditional probability** because it is the probability of I occurring on the *condition* that F has occurred.

$$\text{P}(I \text{ given that } F \text{ has occurred}) = \frac{8}{15} \begin{array}{l} \leftarrow \text{number of students who study Italian and French} \\ \leftarrow \text{number of students who study French} \end{array}$$

For events A and B, we use the notation "$A \mid B$" to represent the event "*A given that B has occurred*".

$$\mathbf{P}(\boldsymbol{A} \mid \boldsymbol{B}) = \frac{\boldsymbol{n}(\boldsymbol{A} \cap \boldsymbol{B})}{\boldsymbol{n}(\boldsymbol{B})}$$

If the outcomes in each of the events are equally likely, notice that:

$$\frac{n(A \cap B)}{n(B)} = \frac{\frac{n(A \cap B)}{n(U)}}{\frac{n(B)}{n(U)}} = \frac{\text{P}(A \cap B)}{\text{P}(B)}$$

This gives us the **conditional probability formula**:

$$\mathbf{P}(\boldsymbol{A} \mid \boldsymbol{B}) = \frac{\mathbf{P}(\boldsymbol{A} \cap \boldsymbol{B})}{\mathbf{P}(\boldsymbol{B})}$$

EXERCISE 11I

1 Find $\text{P}(A \mid B)$ if:

- **a** $\text{P}(A \cap B) = 0.1$ and $\text{P}(B) = 0.4$
- **b** $\text{P}(A) = 0.3$, $\text{P}(B) = 0.4$, and $\text{P}(A \cup B) = 0.5$
- **c** A and B are mutually exclusive.

2 The probability that it is cloudy on a particular day is 0.4. The probability that it is cloudy *and* rainy on a particular day is 0.2. Find the probability that it will be rainy on a day when it is cloudy.

3 In a group of 50 students, 40 study Mathematics, 32 study Physics, and each student studies at least one of these subjects.

- **a** Use a Venn diagram to find how many students study both subjects.
- **b** If a student from this group is randomly selected, find the probability that he or she:
  - **i** studies Mathematics but not Physics
  - **ii** studies Physics given that he or she studies Mathematics.

4 Out of 40 boys, 23 have dark hair, 18 have brown eyes, and 26 have dark hair, brown eyes, or both.

- **a** Draw a Venn diagram to display this information.
- **b** One of the boys is selected at random. Determine the probability that he has:
  - **i** dark hair and brown eyes
  - **ii** brown eyes given that he has dark hair.

5 50 hikers participated in an orienteering event during summer. 23 were sunburnt, 22 were bitten by ants, and 5 were both sunburnt and bitten by ants.

a Draw a Venn diagram to display this information.

b Determine the probability that a randomly selected hiker:

i avoided being bitten **ii** was bitten or sunburnt (or both)

iii was bitten given that he or she was sunburnt

iv was sunburnt given that he or she was not bitten.

Example 22 **Self Tutor**

In a town, 25% of the residents own a cat, 55% own a dog, and 30% do not own either animal.

a Draw a Venn diagram to describe the situation.

b Find the probability that a randomly selected resident:

i owns a cat given that they own a dog

ii does not own a dog given that they own a cat.

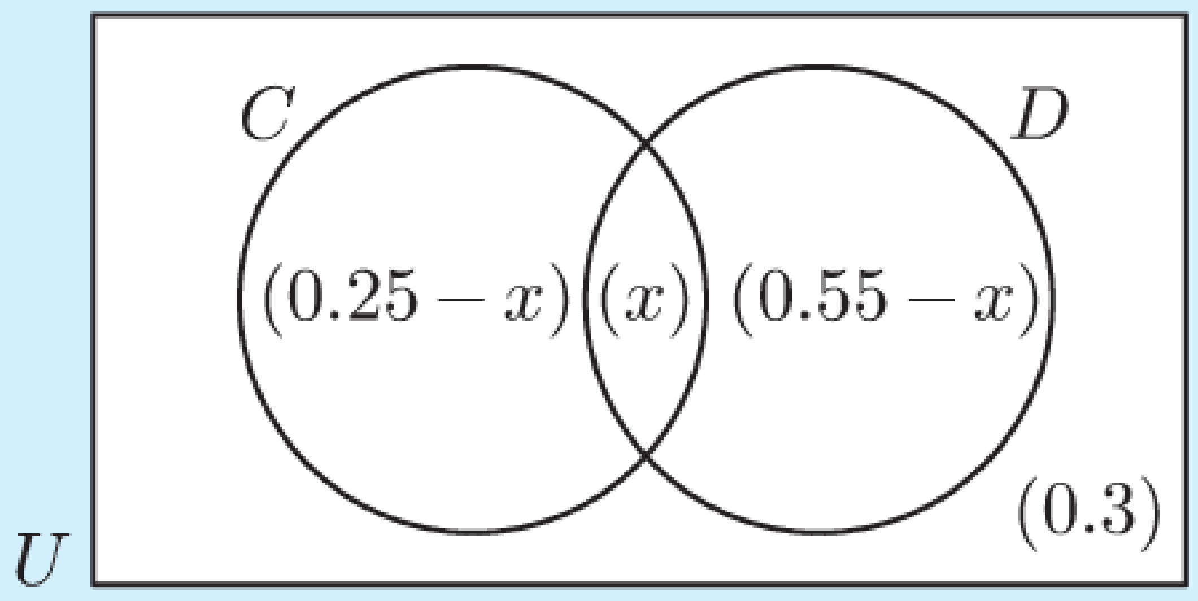
a Let C represent residents who own a cat and D represent residents who own a dog.

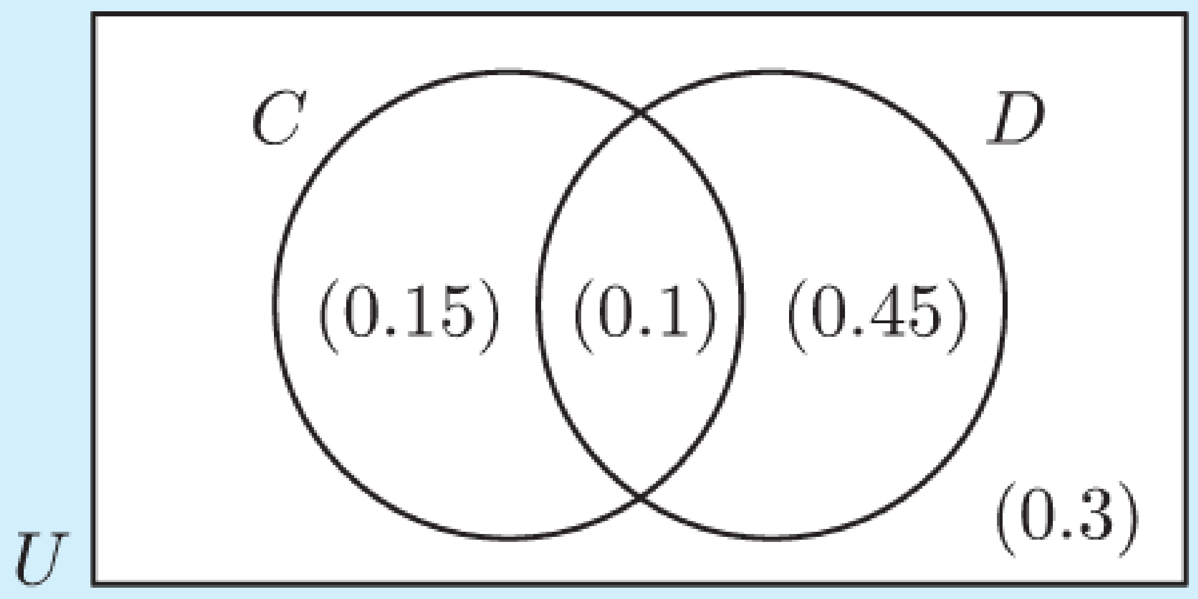
Let the proportion of residents in $C \cap D$ be x.

$\therefore$ the proportion in $C \cap D'$ is $0.25 - x$ and the proportion in $C' \cap D$ is $0.55 - x$.

The proportion in $C' \cap D'$ is 0.3.

$$\therefore \quad (0.25 - x) + x + (0.55 - x) = 0.7$$
$$\therefore \quad 0.8 - x = 0.7$$
$$\therefore \quad x = 0.1$$





b **i** $$\mathrm{P}(C \mid D) = \frac{\mathrm{P}(C \cap D)}{\mathrm{P}(D)} = \frac{0.1}{0.55} \approx 0.182$$

ii $$\mathrm{P}(D' \mid C) = \frac{\mathrm{P}(D' \cap C)}{\mathrm{P}(C)} = \frac{0.15}{0.25} = 0.6$$

6 400 families were surveyed. It was found that 90% had a TV set and 80% had a computer. Every family had at least one of these items. One of the families is randomly selected, and it is found that they have a computer. Find the probability that they also have a TV set.

7 In a certain town three newspapers are published. 20% of the population read A, 16% read B, 14% read C, 8% read A and B, 5% read A and C, 4% read B and C, and 2% read all 3 newspapers. A person is selected at random. Use a Venn diagram to help determine the probability that the person reads:

a none of the papers **b** at least one of the papers

c exactly one of the papers **d** A or B (or both)

e A, given that the person reads at least one paper

f C, given that the person reads either A or B or both.

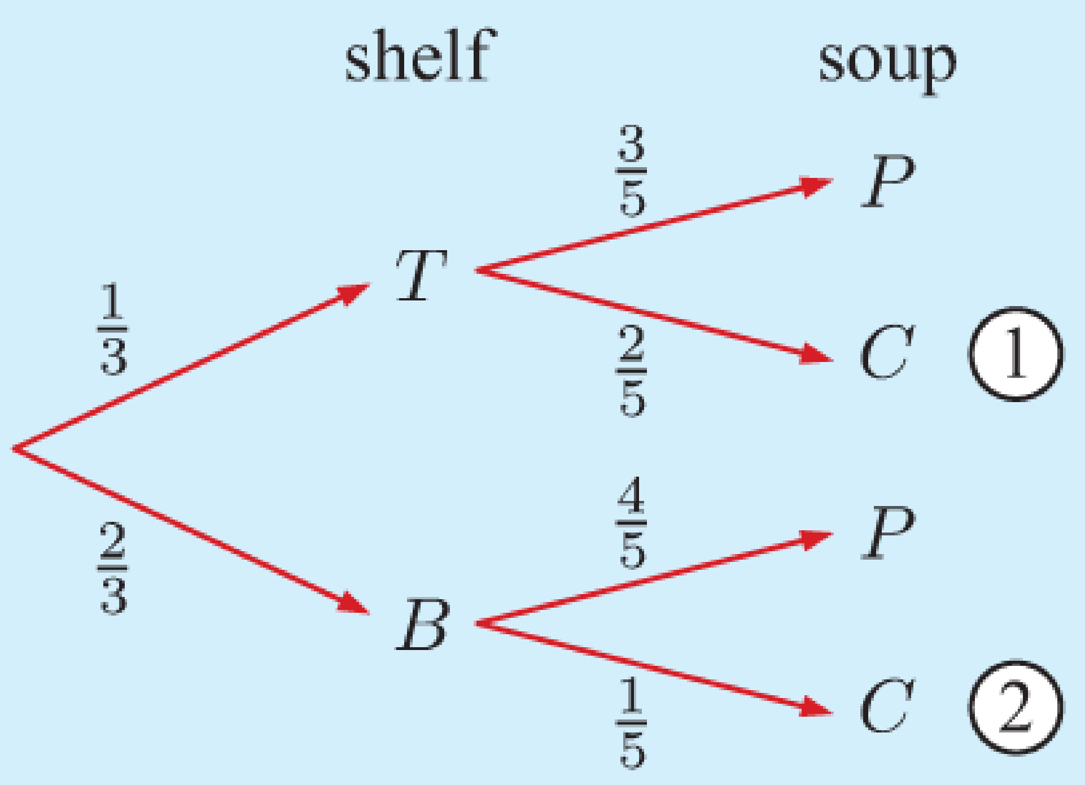
Example 23 Self Tutor

The top shelf in a cupboard contains 3 cans of pumpkin soup and 2 cans of chicken soup. The bottom shelf contains 4 cans of pumpkin soup and 1 can of chicken soup. Lukas is twice as likely to take a can from the bottom shelf as he is from the top shelf.

Suppose Lukas takes one can of soup without looking at the label. Find the probability that it:

a is chicken **b** was taken from the top shelf given that it is chicken.

Let T represent the top shelf, B represent the bottom shelf, P represent the pumpkin soup, and C represent the chicken soup.



a $\text{P}(C)$

$$= \underbrace{\text{P}(T \cap C)}_{\text{branch ①}} + \underbrace{\text{P}(B \cap C)}_{\text{branch ②}}$$

$$= \frac{1}{3} \times \frac{2}{5} + \frac{2}{3} \times \frac{1}{5}$$

$$= \frac{4}{15}$$

b $\text{P}(T \mid C)$

$$= \frac{\text{P}(T \cap C)}{\text{P}(C)}$$

$$= \frac{\frac{1}{3} \times \frac{2}{5}}{\frac{4}{15}} \quad \begin{array}{l} \longleftarrow \text{branch ①} \\ \longleftarrow \text{from } \textbf{a} \end{array}$$

$$= \frac{1}{2}$$

8 Urn A contains 2 red and 3 blue marbles, and urn B contains 4 red and 1 blue marble. Peter selects an urn by tossing a coin, and takes a marble from that urn.

a Determine the probability that the marble is red.

b Given that the marble is red, what is the probability that it came from urn B?

9 When Greta's mother goes shopping, the probability that she takes Greta with her is $\frac{2}{5}$. When Greta goes shopping with her mother she gets an ice cream 70% of the time. When Greta does not go shopping with her mother she gets an ice cream 30% of the time.
Determine the probability that:

a when Greta's mother goes shopping, she buys Greta an ice cream

b Greta went shopping with her mother, given that her mother buys her an ice cream.

10 On a given day, machine X has a 10% chance of malfunctioning and machine Y has a 7% chance of the same.

a Last Thursday *exactly* one of the machines malfunctioned. Find the probability that it was machine X.

b At least one of the machines malfunctioned today. Find the probability that machine Y malfunctioned.

11 Bags A and B each contain red and green balls. In bag B, the ratio of red balls to green balls is 1 : 4.
When a bag is randomly selected and a ball is randomly chosen from it, the probability that the ball is red is $\frac{1}{3}$.
Find the proportion of red balls in bag A.

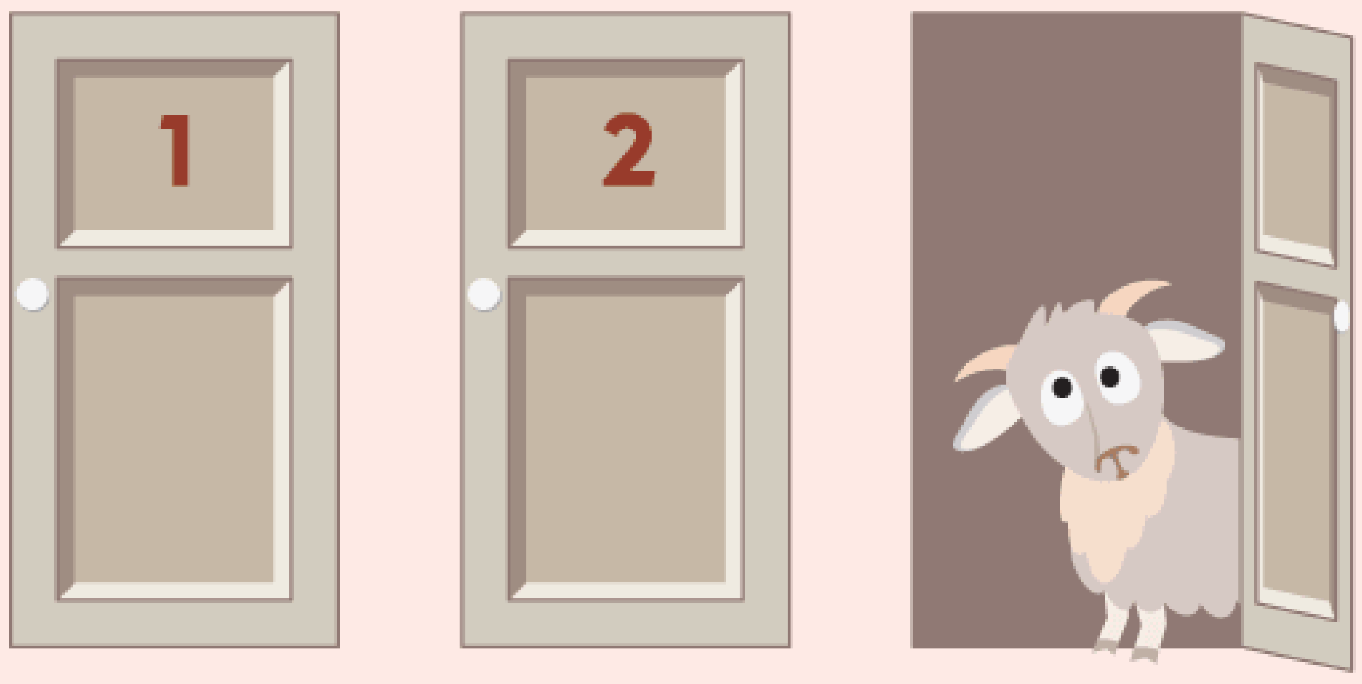
ACTIVITY 2 THE MONTY HALL PROBLEM

The Monty Hall problem is a mathematical paradox first posed by **Steve Selvin** to the *American Statistician* in 1975. It became famous after its publication in *Parade* magazine in 1990. The problem is named after the original host of the American television game show *Let's Make a Deal*, on which the problem is loosely based.

The problem as posed in *Parade* reads:

> *Suppose you're on a game show, and you're given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to switch your choice to door No. 2?" Is it to your advantage to switch your choice?*

What to do:



1. Draw a tree diagram to represent the problem. Let C represent the event that a contestant's choice is correct, and C' represent the event that the choice is incorrect.
2. Find the probability that:
   - **a** the contestant's first choice has the car
   - **b** the contestant's second choice has the car *given* they decide to change their guess.
3. Suppose this game is being played in front of a live studio audience. One of the audience members arrives late, so when they enter the room, they see two closed doors and the third (incorrect) door open. They do not know the contestant's original choice.
   - **a** If this audience member is asked to choose a door, what is the probability they will choose the one with the car?
   - **b** Explain why the contestant has an advantage over this audience member.

ACTIVITY 3 PENNEY'S GAME

Invented by **Walter Penney** in 1969, **Penney's Game** is a 2-player coin tossing game.

PENNEY'S GAME

Click on the icon to obtain this Activity.

J FORMAL DEFINITION OF INDEPENDENCE

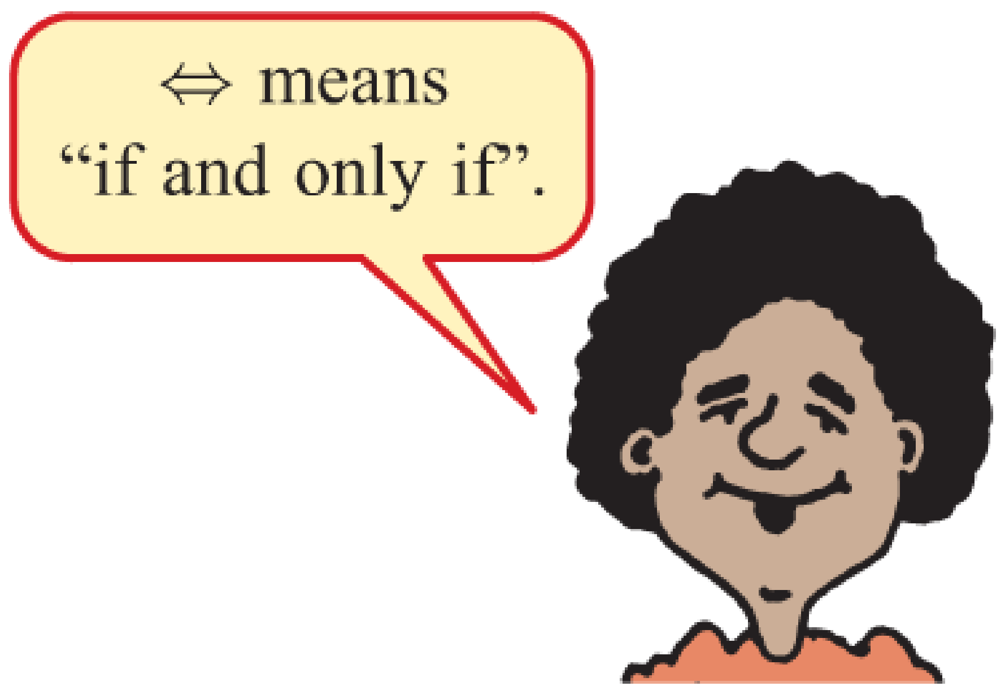
In **Section G** we saw that two events are **independent** if the occurrence of each event does not affect the probability that the other occurs. We can write this definition more formally using conditional probability notation:

> A and B are **independent events** if the occurrence of each one of them does not affect the probability that the other occurs.
>
> This means that $\mathrm{P}(A \mid B) = \mathrm{P}(A \mid B') = \mathrm{P}(A)$
>
> and that $\mathrm{P}(B \mid A) = \mathrm{P}(B \mid A') = \mathrm{P}(B)$.

Using $\text{P}(A \cap B) = \text{P}(A \mid B)\,\text{P}(B)$ we see that

A and B are **independent events** $\Leftrightarrow$ $\text{P}(A \cap B) = \text{P}(A)\,\text{P}(B)$

which is the result we saw earlier.



Example 24 **Self Tutor**

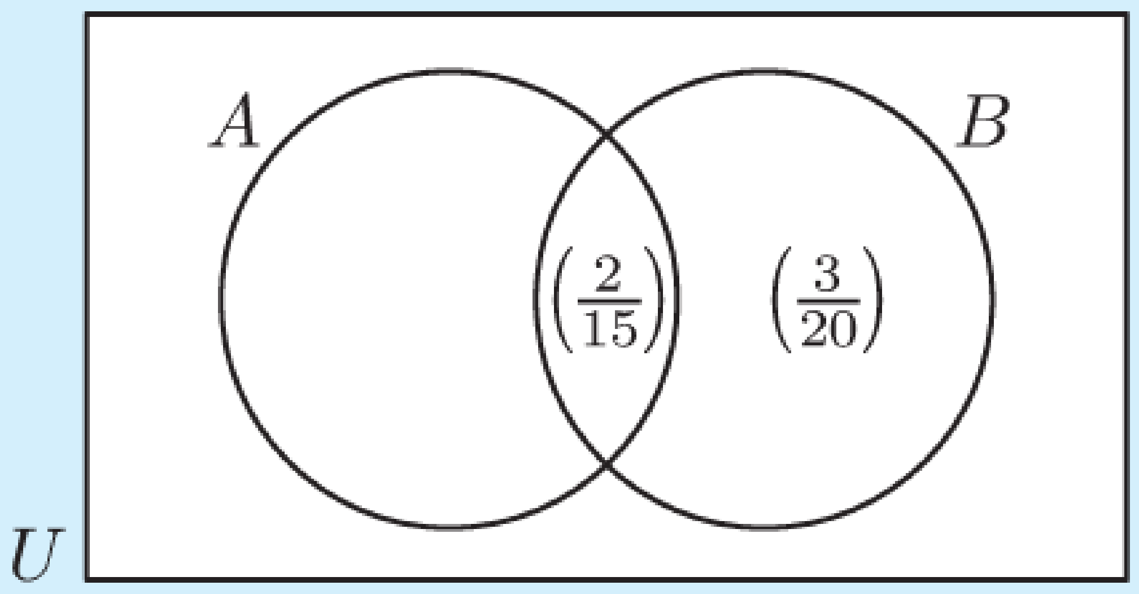
Suppose $\text{P}(A) = \frac{2}{5}$, $\text{P}(B \mid A) = \frac{1}{3}$, and $\text{P}(B \mid A') = \frac{1}{4}$.

a Find $\text{P}(B)$.

b Are A and B independent events? Justify your answer.

$$\begin{aligned}\text{P}(B \cap A) &= \text{P}(B \mid A)\,\text{P}(A) \\ &= \tfrac{1}{3} \times \tfrac{2}{5} \\ &= \tfrac{2}{15}\end{aligned} \qquad \begin{aligned}\text{P}(B \cap A') &= \text{P}(B \mid A')\,\text{P}(A') \\ &= \tfrac{1}{4} \times \tfrac{3}{5} \\ &= \tfrac{3}{20}\end{aligned}$$

$\therefore$ the Venn diagram is:



a $\text{P}(B) = \frac{2}{15} + \frac{3}{20} = \frac{17}{60}$

b $\text{P}(B) \neq \text{P}(B \mid A)$, so A and B are not independent events.

EXERCISE 11J

1 Suppose $\text{P}(R) = 0.4$, $\text{P}(S) = 0.5$, and $\text{P}(R \cup S) = 0.7$. Are R and S independent events? Justify your answer.

2 Suppose $\text{P}(A) = \frac{2}{5}$, $\text{P}(B) = \frac{1}{3}$, and $\text{P}(A \cup B) = \frac{1}{2}$.

- **a** Find:
  - **i** $\text{P}(A \cap B)$
  - **ii** $\text{P}(B \mid A)$
  - **iii** $\text{P}(A \mid B)$
- **b** Are A and B independent events? Justify your answer.

3 Suppose $\text{P}(X) = 0.5$, $\text{P}(Y) = 0.7$, and that X and Y are independent events. Determine the probability of the occurrence of:

- **a** both X and Y
- **b** X or Y or both
- **c** neither X nor Y
- **d** X but not Y
- **e** X given that Y occurs.

4 A and B are independent events. Prove that A' and B' are also independent events.

5 Suppose A and B are independent, mutually exclusive events, and that $\text{P}(A) = \frac{5}{7}$. Find $\text{P}(B)$.

6 Suppose $\mathrm{P}(A \cap B) = 0.1$ and $\mathrm{P}(A \cap B') = 0.4$. Given that A and B are independent, find $\mathrm{P}(A \cup B')$.

7 Suppose $\mathrm{P}(C) = \frac{9}{20}$, $\mathrm{P}(D \mid C) = \frac{1}{4}$, and $\mathrm{P}(D \mid C') = \frac{1}{5}$.

- **a** Find $\mathrm{P}(D)$.
- **b** Are C and D independent events? Justify your answer.

8 What can be deduced if $A \cap B$ and $A \cup B$ are independent events?

K BAYES' THEOREM

HISTORICAL NOTE

Bayes' theorem is named after **Reverend Thomas Bayes** (1701 - 1761) who was an English statistician, philosopher, and Presbyterian minister. Bayes first described a special case of the theorem in "*An Essay towards solving a Problem in the Doctrine of Chances*" which was posthumously published in 1764.

Despite being attributed to Thomas Bayes, most of the work on the theorem and its interpretation was actually done by French mathematician **Pierre-Simon Laplace**.

Today, Bayes' theorem lies at the heart of a field of statistics called **Bayesian statistics**.

Reverend Thomas Bayes

In **Section I**, we saw the conditional probability formula: $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ .... (1)

Notice that we can also write: $\mathrm{P}(B \mid A) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)}$

$$\therefore \quad \mathrm{P}(A \cap B) = \mathrm{P}(B \mid A)\,\mathrm{P}(A) \quad .... \ (2)$$

Now substituting (2) into (1) gives:

$$\boxed{\mathbf{P}(A \mid B) = \frac{\mathbf{P}(B \mid A)\,\mathbf{P}(A)}{\mathbf{P}(B)}}$$

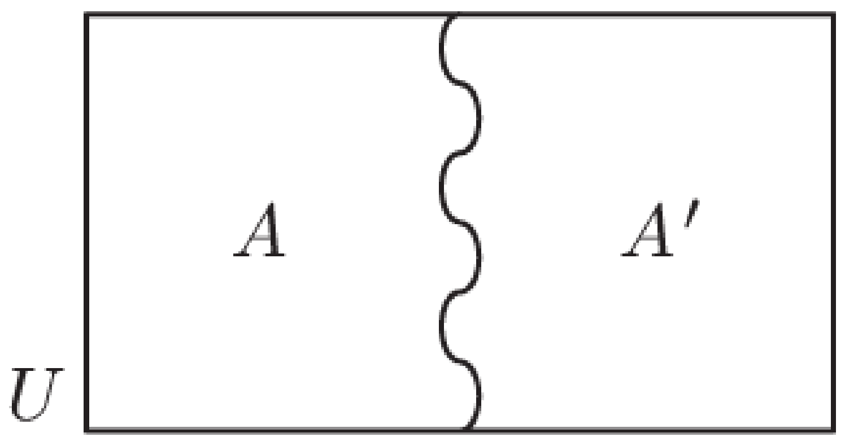
This result is known as **Bayes' theorem**.

PARTITIONS OF THE SAMPLE SPACE

For any event A and its complement A':

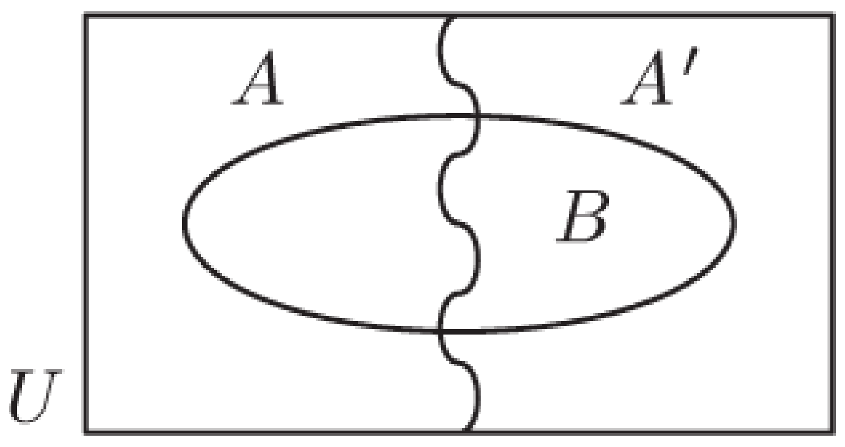
- A and A' are mutually exclusive, so $A \cap A' = \varnothing$
- $A \cup A' = U$, the sample space.

We say that A and A' **partition** the sample space, and we can represent this on a Venn diagram as shown.



For any other event B in the sample space U,

$$\begin{aligned} P(B) &= P(B \cap A) + P(B \cap A') \\ &= P(B \mid A)\,P(A) + P(B \mid A')\,P(A') \end{aligned}$$



So, Bayes' theorem can alternatively be written as:

$$\mathbf{P(A \mid B) = \frac{P(B \mid A)\,P(A)}{P(B \mid A)\,P(A) + P(B \mid A')\,P(A')}}$$

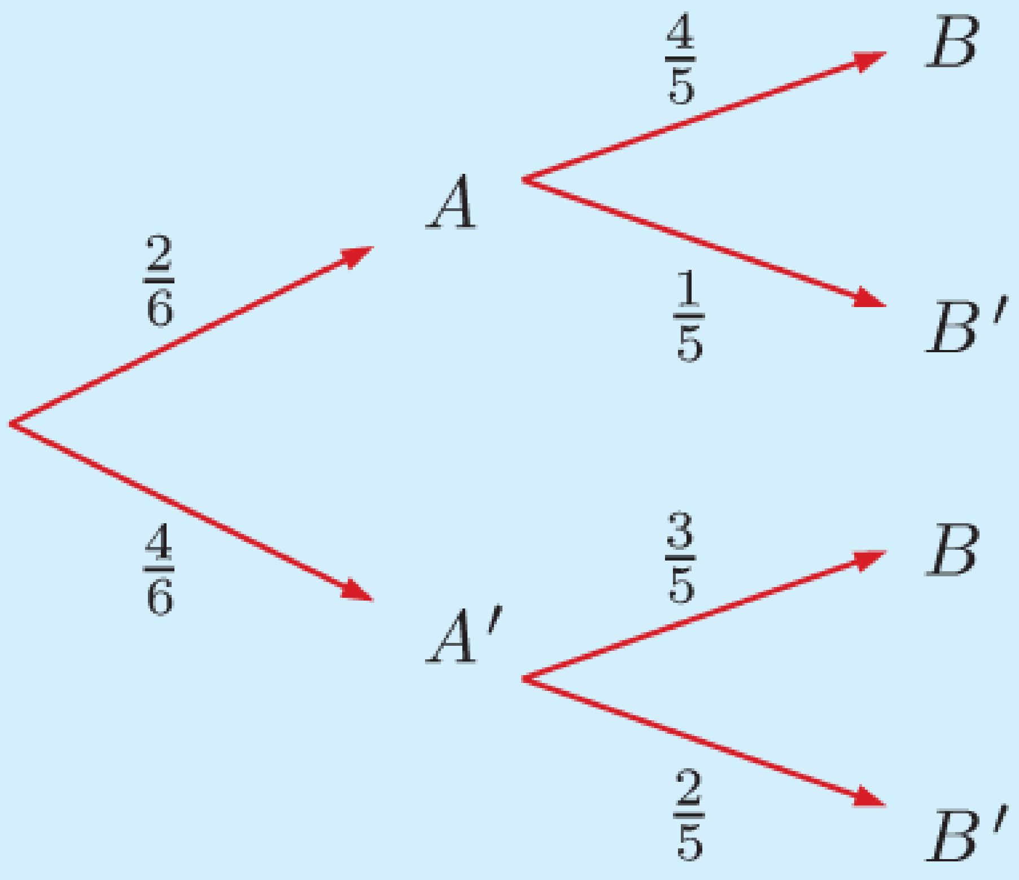
Example 25 **Self Tutor**

A can contains 4 blue and 2 green marbles. One marble is randomly drawn from the can without replacement, and its colour is noted. A second marble is then drawn. Find the probability that:

a the second marble is blue

b the first marble was green, given that the second marble is blue.

Let A be the event that the first marble is green
and B be the event that the second marble is blue.



a P(second marble is blue)

$= P(B)$

$= P(B \mid A)\,P(A) + P(B \mid A')\,P(A')$

$= \frac{4}{5} \times \frac{2}{6} + \frac{3}{5} \times \frac{4}{6}$

$= \frac{2}{3}$

b P(first was green | second is blue)

$= P(A \mid B)$

$= \dfrac{P(B \mid A)\,P(A)}{P(B)}$ {Bayes' theorem}

$= \dfrac{\frac{4}{5} \times \frac{2}{6}}{\frac{2}{3}}$ {using **a**}

$= \frac{2}{5}$

EXERCISE 11K

1 Coffee machines A and B produce coffee in identically shaped plastic cups. Machine A produces 65% of the coffee sold each day, and machine B produces the remainder. Machine A underfills a cup 4% of the time, while machine B underfills a cup 5% of the time.

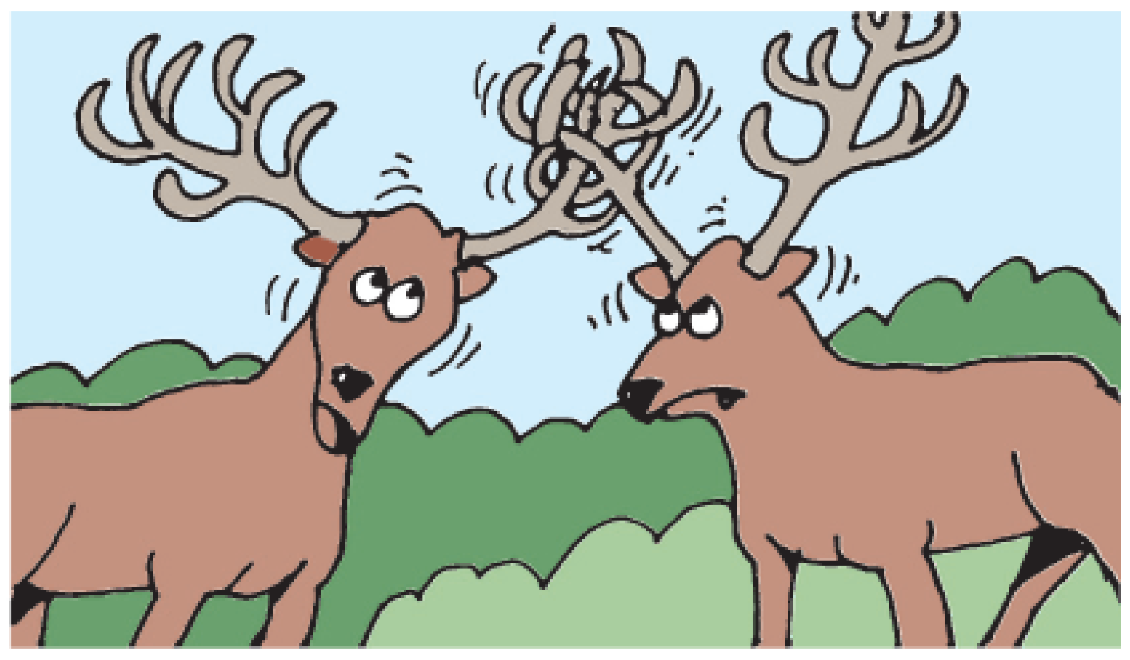
- **a** A cup of coffee is chosen at random. Find the probability that it is underfilled.
- **b** A cup of coffee is randomly chosen and is found to be underfilled. Find the probability that it came from machine A.

2 54% of the students at a university are female. 8% of the male students are colour-blind, and 2% of the female students are colour-blind.

- **a** A randomly chosen student is colour-blind. Find the probability that the student is male.
- **b** A randomly chosen student is not colour-blind. Find the probability that the student is female.

3 A marble is randomly chosen from a can containing 3 red and 5 blue marbles. It is replaced by two marbles of the other colour. Another marble is then randomly chosen from the can. Given that the marbles chosen have the same colour, what is the probability that they are both blue?

4 35% of the animals in a deer herd carry the TPC gene. 58% of these deer also carry the SD gene, while 23% of the deer without the TPC gene carry the SD gene. A deer is randomly chosen and is found to carry the SD gene. Find the probability that it does not carry the TPC gene.



5 A blood test has been designed to detect a particular form of cancer. The probability that the test correctly identifies someone with the cancer is 0.97, and the probability that the test correctly identifies someone without the cancer is 0.93. Approximately 0.1% of the general population are known to contract this cancer.

When a patient has a blood test, the test results are positive for the cancer. Find the probability that the patient actually has the cancer.

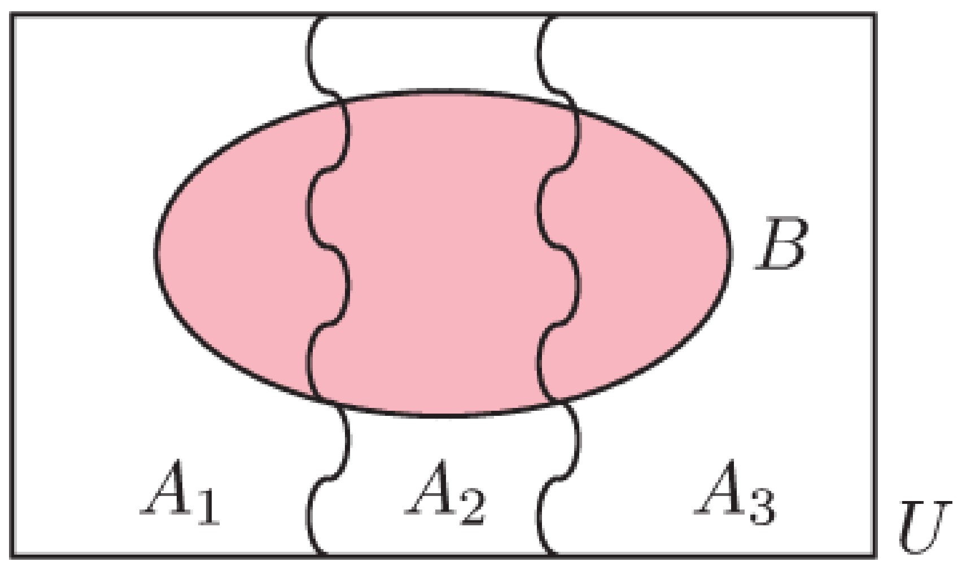
6 A man drives his car to work 80% of the time. The remainder of the time he rides his bicycle. When he rides his bicycle to work he is late 25% of the time. When he drives his car to work he is late 15% of the time. On a particular day, the man arrives early. Find the probability that he rode his bicycle to work that day.

7 The probabilities that Hiran's mother and father will be alive after ten years are 0.99 and 0.98 respectively. If only one of them is alive after ten years, find the probability that it will be his mother.

8 A manufacturer produces drink bottles using two machines. Machine A produces 60% of the bottles, and 3% of those it makes are defective. Machine B produces 40% of the bottles, and 5% of those it makes are defective. Find the probability that a defective bottle comes from:

a machine A **b** machine B.

9



A sample space U is partitioned into three by the mutually exclusive events A_1, A_2, and A_3.

The sample space also contains another event B.

a Show that $\mathrm{P}(B) = \mathrm{P}(B \mid A_1)\,\mathrm{P}(A_1) + \mathrm{P}(B \mid A_2)\,\mathrm{P}(A_2) + \mathrm{P}(B \mid A_3)\,\mathrm{P}(A_3)$.

b Hence show that Bayes' theorem for the case of three partitions is

$$\mathrm{P}(A_i \mid B) = \frac{\mathrm{P}(B \mid A_i)\,\mathrm{P}(A_i)}{\mathrm{P}(B)}, \quad i \in \{1, 2, 3\} \quad \text{where} \quad \mathrm{P}(B) = \sum_{j=1}^{3} \mathrm{P}(B \mid A_j)\,\mathrm{P}(A_j).$$

10 A newspaper printer has three presses A, B, and C which print 30%, 40%, and 30% of daily production respectively. Due to the age of the machines, the presses will produce streaks on their output 3%, 5%, and 7% of the time, respectively.

a Find the probability that a randomly chosen newspaper does not have streaks.

b If a randomly chosen newspaper does not have streaks, find the probability that it was printed by press A.

c If a randomly chosen newspaper has streaks, find the probability that it was printed by either press A or C.

11 12% of the over-60 population of Agento have lung cancer. Of those with lung cancer, 50% were heavy smokers, 40% were moderate smokers, and 10% were non-smokers. Of those without lung cancer, 5% were heavy smokers, 15% were moderate smokers, and 80% were non-smokers.

A member of the over-60 population of Agento is chosen at random.

- **a** Find the probability that the person was a heavy smoker.
- **b** Given the person was a moderate smoker, find the probability that the person has lung cancer.
- **c** Given the person was a non-smoker, find the probability that the person has lung cancer.

12 205 teachers and 52 headmasters attended an educational conference. Attendees were given t-shirts at the start of the conference.

Of the attending teachers, 39 received a small t-shirt, 120 received a medium t-shirt, and 46 received a large t-shirt. Of the attending headmasters, 16 received a small t-shirt, 25 received a medium t-shirt, and 11 received a large t-shirt.

- **a** A teacher lost their conference t-shirt. Find the probability that it is a large t-shirt.
- **b** A large t-shirt was left behind at a plenary talk. Find the probability that it belongs to a teacher.
- **c** Use Bayes' theorem to explain why your answers to **a** and **b** are different.

THEORY OF KNOWLEDGE

Modern probability theory began in 1653 when gambler Chevalier de Mere contacted mathematician **Blaise Pascal** with a problem on how to divide the stakes when a gambling game is interrupted during play. Pascal involved **Pierre de Fermat**, a lawyer and amateur mathematician, and together they solved the problem. In the process they laid the foundations upon which the laws of probability were formed.

Applications of probability are now found from quantum physics to medicine and industry.


Agner Krarup Erlang

The first research paper on **queueing theory** was published in 1909 by the Danish engineer **Agner Krarup Erlang** who worked for the Copenhagen Telephone Exchange. In the last hundred years this theory has become an integral part of the huge global telecommunications industry, but it is equally applicable to modelling car traffic or queues at your local supermarket.

Statistics and probability are used extensively to predict the behaviour of the global stock market. For example, American mathematician **Edward Oakley Thorp** developed and applied hedge fund techniques for the financial markets in the 1960s.

On the level of an individual investor, money is put into the stock market if there is a good probability that the value of the shares will increase. This investment has risk, however, as witnessed by historic stock market crashes like that of Wall Street in 1929 which triggered the Great Depression, the Black Monday crash of 1987, and the Global Financial Crisis of 2008 - 2009.

However, the question "What is probability?" is as much a philosophical question as it is mathematical.

In statistics, there are two main interpretations of probability:

- The **frequentist** interpretation considers probability as a measure of how *frequently* an event occurs if an experiment is repeated many times.
- The **Bayesian** interpretation considers probability to be a measure of the strength of one's prior beliefs of an event occurring. Such beliefs are often *subjective*.

1 Discuss the frequentist and Bayesian interpretations. Which interpretation makes the most sense to you?

2 How does a knowledge of probability theory affect decisions we make?

3 What roles should ethics play in the use of mathematics? You may wish to consider:

- What responsibility does a casino have to operate as a functioning business? What responsibility does it have to the welfare of habitual gamblers? How do these responsibilities affect the way casinos operate?
- By clever mathematical modelling of the global stock markets, you may be able to gain a market advantage. In your gain, does somebody else lose? What rules are in place to protect against financial corruption?
- Do rich countries adopt foreign policies and control trade in order that poor countries remain poor? How much power is associated with financial wealth?

REVIEW SET 11A

1 Kate recorded the number of emails she sent each day for 30 days. Find, to 2 decimal places, the experimental probability that tomorrow she will send:

a 5 emails **b** less than 3 emails.

Number of emails	*Frequency*
0	2
1	5
2	9
3	5
4	4
5	4
6	1

2 A coin is tossed and a square spinner labelled A, B, C, D is twirled.

a Draw a 2-dimensional grid to illustrate the sample space.

b Determine the probability of obtaining:

i a head and consonant **ii** a tail and C

iii a tail or a vowel (or both).

3 Explain what is meant by:

a independent events **b** mutually exclusive events.

4 The students A, B, and C have 10%, 20%, and 30% chance of independently solving a certain maths problem. If they all try independently of one another, what is the probability that at least one of them will solve the problem?

5 A and B are mutually exclusive events where $P(A) = x$ and $P(B') = 0.43$.

a Write $P(A \cup B)$ in terms of x. **b** Find x given that $P(A \cup B) = 0.73$.

6 On any one day, there is a 25% chance of rain and 36% chance that it will be windy.

- **a** Draw a tree diagram showing the probabilities of wind or rain on a particular day.
- **b** Hence determine the probability that on a particular day there will be:
  - **i** rain and wind
  - **ii** rain or wind (or both).
- **c** What assumption have you made in your answers?

7 Given $P(Y) = 0.35$ and $P(X \cup Y) = 0.8$, and that X and Y are mutually exclusive events, find:

- **a** $P(X \cap Y)$
- **b** $P(X)$
- **c** $P(X \text{ or } Y \text{ but not both})$.

8

- **a** Graph the sample space of all possible outcomes when a pair of dice is rolled.
- **b** Hence determine the probability of getting:
  - **i** a sum of 7 or 11
  - **ii** a sum of at least 8.

9 In a group of 40 students, 22 study Economics, 25 study Law, and 3 study neither of these subjects.

- **a** Draw a Venn diagram to display this information.
- **b** Determine the probability that a randomly chosen student studies:
  - **i** both Economics and Law
  - **ii** at least one of these subjects
  - **iii** Economics given that they study Law.

10 The probability that a tomato seed will germinate is 0.87. If a market gardener plants 5000 seeds, how many are expected to germinate?

11 A bag contains 3 red, 4 yellow, and 5 blue marbles. Two marbles are randomly selected from the bag with replacement. Find the probability that:

- **a** both are blue
- **b** they are the same colour
- **c** at least one is red
- **d** exactly one is yellow.

12 Of the people who apply for executive positions, those with a university degree have 0.33 chance of eventually being successful, while those without have only 0.17 chance. If 78% of all applicants for a particular executive position have a university degree, find the probability that the successful applicant does not have one.

13 A survey of 200 people included 90 females. It found that 60 people smoked, 40 of whom were male.

- **a** Use the given information to complete the two-way table.
- **b** A person is selected at random. Find the probability that this person is:
  - **i** a female non-smoker
  - **ii** a male given the person was a non-smoker.

	Female	*Male*	*Total*
Smoker			
Non-smoker			
Total			

- **c** If two people from the survey are selected at random, calculate the probability that:
  - **i** both of them are non-smoking females
  - **ii** one is a smoker and the other is a non-smoker.

14 The government of Australia is divided up into the Senate and the House of Representatives. As of 2018:

- Of the 76 Senators, 31 are part of the Coalition, 26 are part of the Opposition, and 19 are crossbenchers.
- Of the 150 members of parliament (MPs) in the House of Representatives, 73 are part of the Coalition, 69 are part of the Opposition, and 8 are crossbenchers.

Given that a randomly selected politician is part of the Opposition, find the probability that the politician is an MP.

15 For two events A and B, it is known that $\mathrm{P}(A) = \frac{2}{5}$, $\mathrm{P}(B) = \frac{3}{10}$, and $\mathrm{P}(B \mid A) = \frac{1}{2}$.

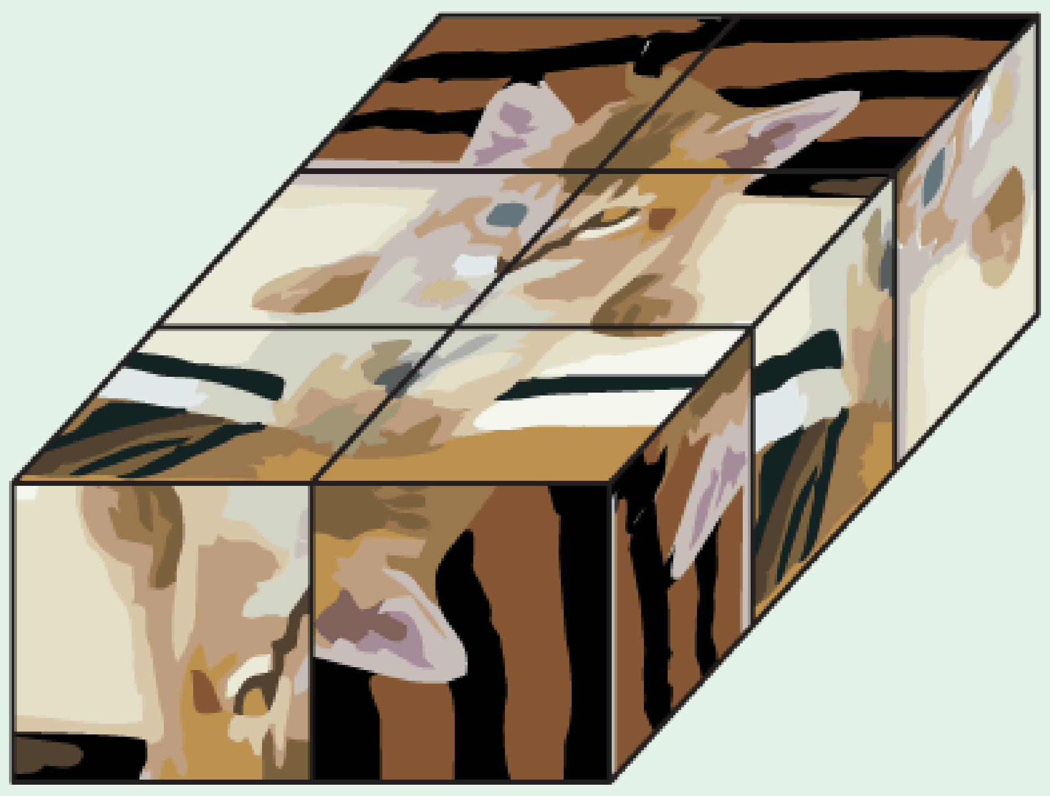
a Calculate $\mathrm{P}(A \cap B)$.

b Show that A and B are not independent.

c Calculate $\mathrm{P}(A \mid B)$.

16 A picture is divided into 6 squares of equal size. One of each of the squares is printed on each of the 6 faces of a cube. The cube is then copied until there are 6 identical cubes.

If the 6 cubes are "rolled", find the probability that the upmost faces can be assembled into the original image.



REVIEW SET 11B

1 T and M are events such that $n(U) = 30$, $n(T) = 10$, $n(M) = 17$, and $n((T \cup M)') = 5$.

a Draw a Venn diagram to display this information.

b Hence find:

i $\mathrm{P}(T \cap M)$ **ii** $\mathrm{P}((T \cap M) \mid M)$

2 A school photocopier has a 95% chance of working on any particular day. Find the probability that it will be working on at least one of the next two days.

3 Suppose A and B are independent events, $\mathrm{P}(A) = 0.4$, and $\mathrm{P}(B) = 0.7$.

a Calculate $\mathrm{P}(A \cap B)$. Hence explain why A and B cannot be mutually exclusive.

b Calculate $\mathrm{P}(A \cup B)$.

4 The probability that a particular salesman will leave his sunglasses behind in any store is $\frac{1}{5}$. Suppose the salesman visits two stores in succession and leaves his sunglasses behind in one of them. What is the probability that the salesman left his sunglasses in the first store?

5 A survey of 50 men and 50 women was conducted to see how many people prefer coffee or tea. It was found that 15 men and 24 women prefer tea.

Let C represent the people who prefer coffee and M represent the men.

a Represent C and M on a Venn diagram.

b Calculate $\mathrm{P}(M \mid C)$.

6 Niklas and Rolf play tennis with the winner being the first to win two sets. Niklas has a 40% chance of beating Rolf in any set.

a Draw a tree diagram showing the possible outcomes.
b Hence determine the probability that Niklas will win the match.

7 A and B are independent events where $\mathrm{P}(A) = 0.8$ and $\mathrm{P}(B) = 0.65$. Determine:

a $\mathrm{P}(A \cup B)$ **b** $\mathrm{P}(A \mid B)$ **c** $\mathrm{P}(A' \mid B')$ **d** $\mathrm{P}(B \mid A)$.

8 If I buy 4 tickets in a 500 ticket lottery and the prizes are drawn without replacement, determine the probability that I will win:

a the first 3 prizes **b** at least one of the first 3 prizes.

9 The students in a school are all invited to participate in a survey. 48% of the students at the school are males, of whom 16% will participate in the survey. 35% of the females will also participate in the survey. A student is randomly chosen from the school. Find the probability that the student:

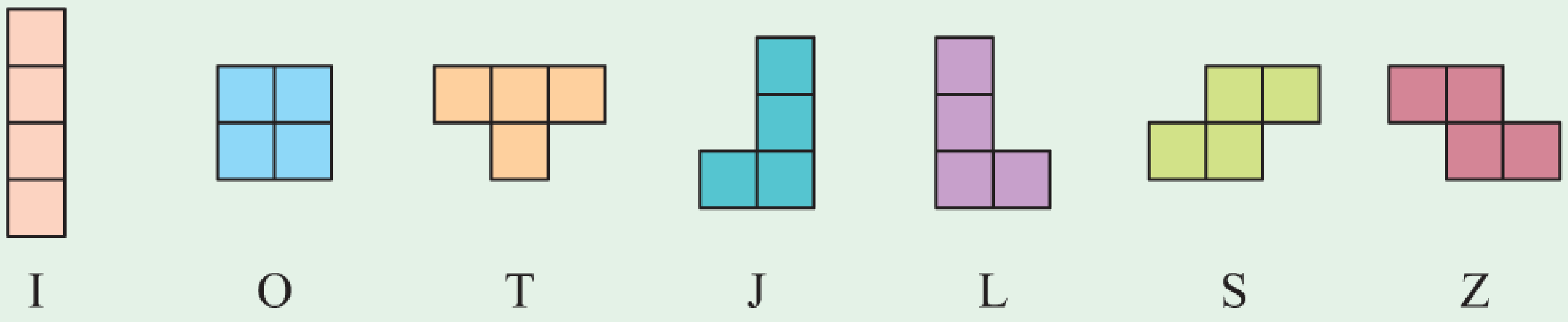
a will participate in the survey
b is female given that he or she will participate in the survey.

10 For the two events A and B, $\mathrm{P}(A) = \frac{3}{7}$ and $\mathrm{P}(B') = \frac{2}{3}$.

a Determine $\mathrm{P}(B)$.
b Calculate $\mathrm{P}(A \cup B)$ if A and B are:
  i mutually exclusive **ii** independent.

11 Suppose $\mathrm{P}(X' \mid Y) = \frac{2}{3}$, $\mathrm{P}(Y) = \frac{5}{6}$, and $X' \cap Y' = \varnothing$. Find $\mathrm{P}(X)$.

12 In a particular video game there are 7 distinct pieces called *tetrominos*, each of which represents a letter:



The pieces are given to the player in a random sequence.

a Find the probability of getting at least one "I" in a sequence of 7 pieces.
b Find the expected number of "O"s in a sequence of 100 pieces.

13 Jon goes cycling on three random mornings of each week. When he goes cycling he has eggs for breakfast 70% of the time. When he does not go cycling he has eggs for breakfast 25% of the time. Determine the probability that Jon:

a has eggs for breakfast
b goes cycling given that he has eggs for breakfast.

14 With each pregnancy, a particular woman will give birth to either a single baby or twins. There is a 15% chance of having twins during each pregnancy. Suppose that after 2 pregnancies she has given birth to 3 children. Find the probability that she had twins first.

15 The table alongside shows the number of balloons in a giant party pack.

	Red	*Yellow*	*Blue*
Large	12	5	9
Medium	15	8	10
Small	24	11	6

- **a** State the:
  - **i** total number of balloons in the pack
  - **ii** number of medium balloons in the pack.
- **b** One balloon is chosen at random from the pack. Find the probability that:
  - **i** the balloon is not yellow
  - **ii** the balloon is either medium or small.
- **c** Two balloons are selected at random from the pack. Find the probability that:
  - **i** both balloons are red
  - **ii** neither of the balloons are large
  - **iii** exactly one of the balloons is blue
  - **iv** at least one of the balloons is blue.
- **d** Three balloons are selected at random from the pack. Find the probability that:
  - **i** all three balloons are small and yellow
  - **ii** exactly two balloons are medium and red.

16 Answer the questions in the **Opening Problem** on page **248**.

Chapter 12

Sampling and data

Contents:

OPENING PROBLEM

A supermarket sells 1 kg bags of grapes.

Things to think about:

- **a** Would you expect every bag of grapes to weigh *exactly* 1 kg?
- **b** In what range of weights would you expect most of the bags of grapes to lie?
- **c** Adriana weighed 50 bags of grapes which were delivered to the supermarket in one day.
  - **i** Do you think this sample will be *representative* of all the bags of grapes sold by the supermarket? Explain your answer.
  - **ii** What type of graph should Adriana use to display her results?
  - **iii** What would you expect Adriana's graph to look like?

In statistics we collect information about a group of individuals, then analyse this information to draw conclusions about those individuals.

You should already be familiar with these words which are commonly used in statistics:

Data:	information about the characteristics of a group of individuals
Categorical variable:	describes a particular characteristic which can be divided into categories
Quantitative variable:	describes a characteristic which has a numerical value that can be counted or measured
Population:	an entire collection of individuals about which we want to draw conclusions
Census:	the collection of information from the **whole population**
Parameter:	a numerical quantity measuring some aspect of a population
Sample:	a group of individuals selected from a population
Survey:	the collection of information from a **sample**
Statistic:	a quantity calculated from data gathered from a sample, usually used to estimate a population parameter

STATISTICAL INVESTIGATIONS

A **statistical investigation** should include the following steps.

Step 1: **State the problem**
We need to decide what we are investigating and then describe it exactly. We also need to decide on **variables** we can measure to help us understand our topic of interest.

Step 2: **Choose a sample**
In general it is impossible to obtain data from every single individual in a population. We therefore need to choose a **sample** which is representative of the whole population.

Step 3: **Collect the data**
Having chosen the sample, we now need to decide exactly what we are going to ask the individuals, or how we are going to measure the variables. This is important to ensure the data we collect is actually going to be appropriate or useful for our specific investigation.

Step 4: **Organise and display the data**
How we display data will depend on its form. Our aim is to identify features of the data more easily.

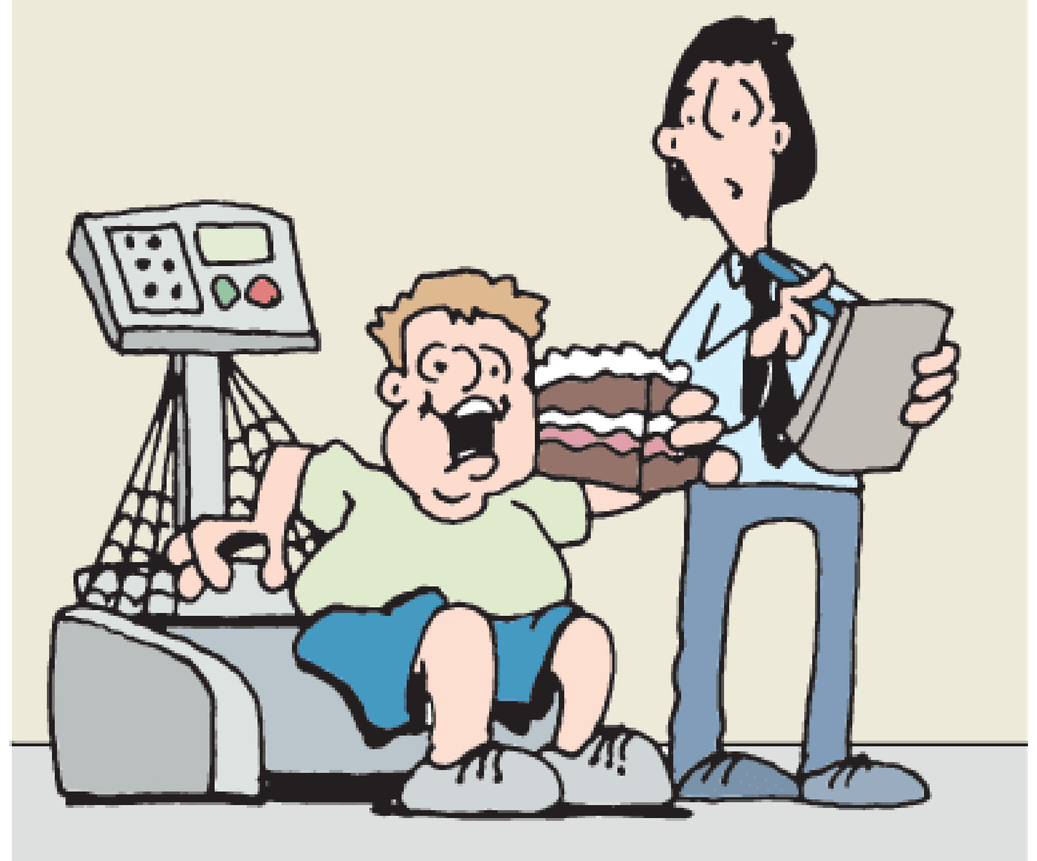
Step 5: **Calculate descriptive statistics**
Descriptive statistics help us to understand the **centre** and **spread** of the data.

Step 6: **Interpret statistics**
We need to describe what the data and its statistics tell us about the problem being investigated.

DESCRIBING A PROBLEM

Before we start collecting data, we need to consider what the **goals** of our investigation are. This will help us decide what **variables** we will need to measure in order to achieve these goals.

For example, if the aim of our investigation was to study the health of primary school children, we could consider variables such as *height*, *weight*, *daily nutritional intake*, *amount of exercise*, *amount of sleep*, *lung capacity*, *eye test results*, and *medical conditions*.

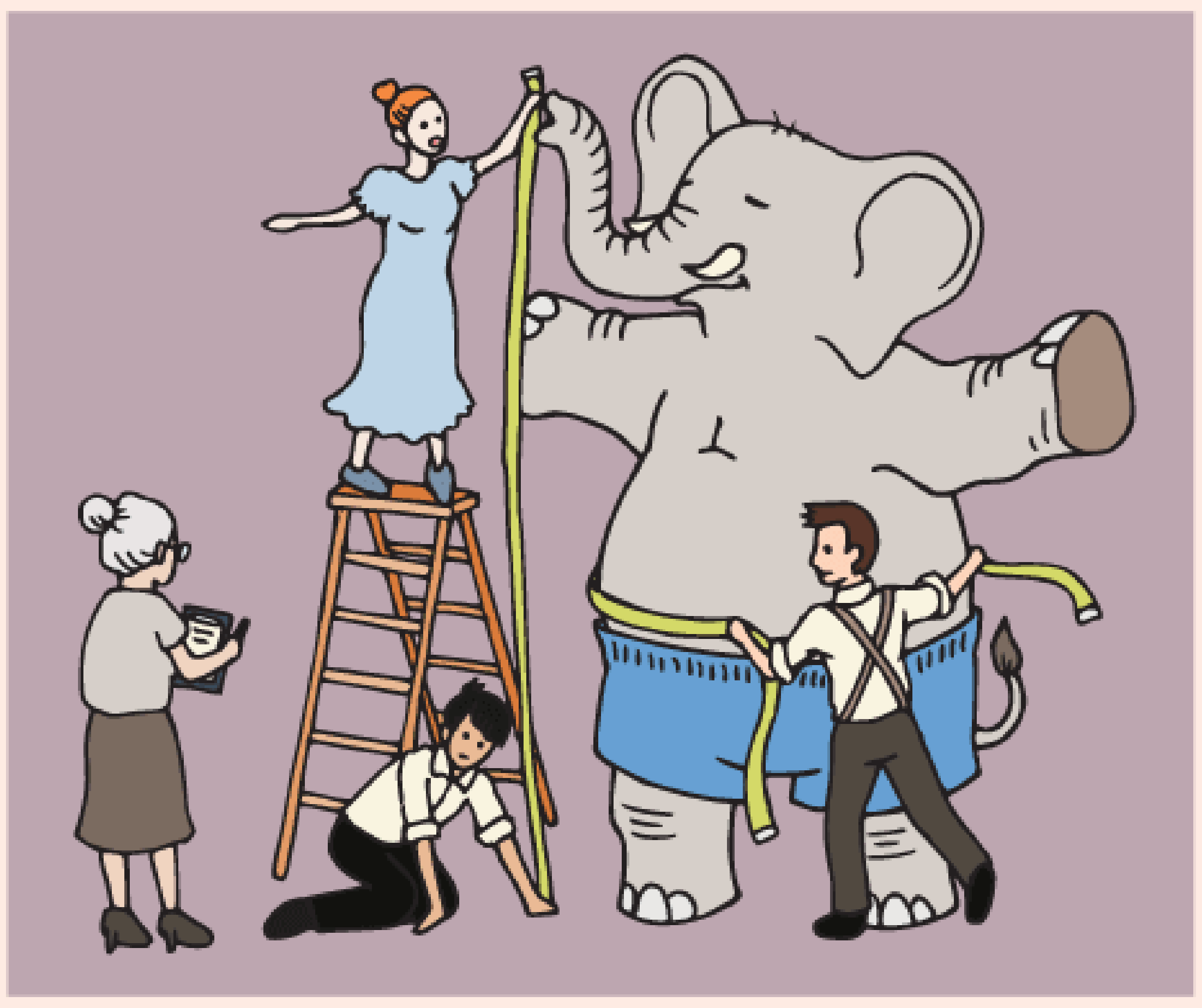


Factors that should be considered when selecting variables include:

- **relevance to the investigation**
  For example, if we wanted to focus on childhood obesity, the most relevant variables would be *weight*, *amount of exercise*, and *daily nutritional intake*.
- **limitations in measuring a variable**
  Some variables can be very difficult or impractical to measure, so they might not be suitable for our investigation.
  For example:
  - Counting every hair on a person's head within a reasonable amount of time is unrealistic.
  - We may not have the equipment to properly measure *lung capacity*.
  - A wooden ruler would not be suitable for measuring the distance around an athletics track.

DISCUSSION

1 **How big is an elephant?**
Before we can answer this question, we need to decide what we mean by "big", and what variable(s) we will use to measure it.



a Which of these variables would be suitable for measuring how big an elephant is?

- height
- volume
- length of trunk
- area of ears
- circumference of tail
- length
- mass
- girth
- size of feet

b Is it reasonable to use more than one variable to describe how "big" an elephant is?

c Can you think of other suitable variables for this investigation?

d What other things do we need to consider to specify *exactly* what we are investigating? For example, how might gender, age, subspecies, environment, and capacity status be important?

2 **How much sleep does a person get each night?**
Is this question sufficient to describe a statistical investigation? Discuss the question, work out what *you* want it to mean, then rewrite the question so it better describes the goals of your investigation.

3 **Should test results be used to measure academic performance?**
Test and examination results are often used to assess the academic performance of students. Do you think this is necessarily fair?

A ERRORS IN SAMPLING AND DATA COLLECTION

A **census** is the most accurate way to investigate a population of interest. However, in most situations it is impractical or impossible to obtain data from the entire population. Instead, we can conduct a **survey** of a well-chosen **sample** of the population.

When we collect data to estimate a characteristic of a population, our estimate will almost certainly be different from the actual characteristic of the population. This difference is referred to as **error**.

There are four main categories of error: **sampling error**, **coverage error**, **non-response error**, and **measurement error**.

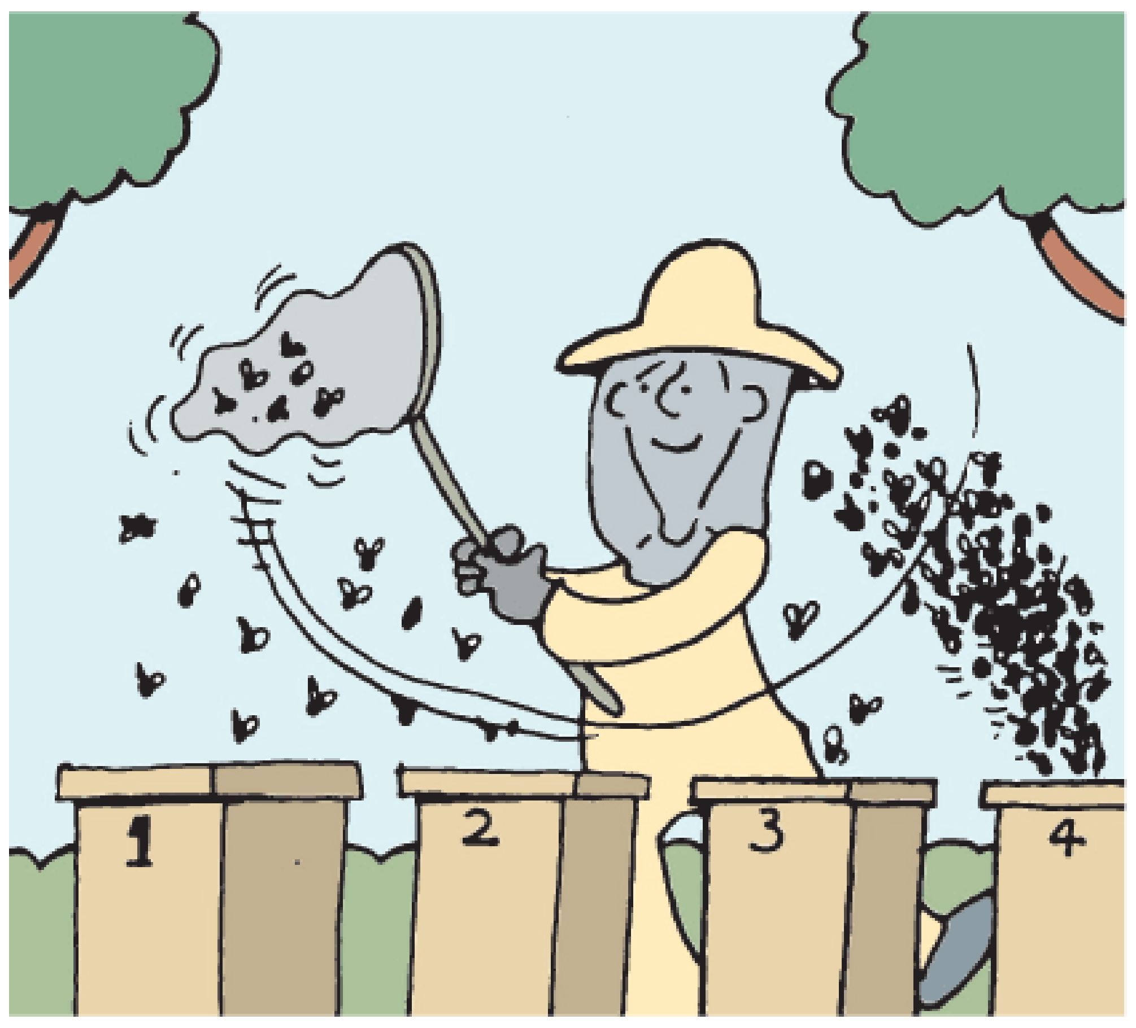
Sampling error occurs when a characteristic of a sample differs from that of the whole population. This error is random, and will occur even for samples which are well-chosen to avoid bias.

Coverage errors occur when a sample does not truly reflect the population we are trying to find information about.

To avoid coverage errors, samples should be **sufficiently large** and **unbiased**.

For example, suppose you are interested in the health of bees on a particular island.

- If you only collect data from 10 bees, you will not get a reliable idea of the health of all bees on the island.
- If you only collect data from one particular bee hive, the sample may not be **representative** of all of the bees on the island. For example, the hive you pick may be stressed and preparing to swarm, whereas its neighbouring hives may be healthy. The sample would therefore be a **biased sample**, and would be unreliable for forming conclusions about the whole population.



Non-response errors occur when a large number of people selected for a survey choose not to respond to it. For example:

- An online survey is less likely to be completed by elderly people who are unfamiliar with technology. This means that elderly people will be under-represented in the survey.
- In surveys on customer satisfaction, people are more likely to respond if they are dissatisfied.

Measurement error refers to inaccuracies in measurement at the data collection stage. For example:

- When we record a person's height to the nearest centimetre, the recorded height is slightly different from the person's *exact* height.
- If the questions in a survey are not well worded, they may be misunderstood and produce answers which are not relevant to the question. Survey construction is discussed more thoroughly in **Section C**.

EXERCISE 12A

1 A new drug called Cobrasyl has been developed for the treatment of high blood pressure in humans. A derivative of cobra venom, it is able to reduce blood pressure to an acceptable level. Before its release, a research team treated 7 high blood pressure patients with the drug, and in 5 cases it reduced their blood pressure to an acceptable level.

Do you think this sample can be used to draw reliable conclusions about the drug's effectiveness for all patients? Explain your answer.

2 50 people in a Toronto shopping mall were surveyed. It was found that 20 of them had been to an ice hockey game in the past year. From this survey, it was concluded that "40% of people living in Canada have been to an ice hockey game in the past year".

Give *two* reasons why this conclusion is unreliable.

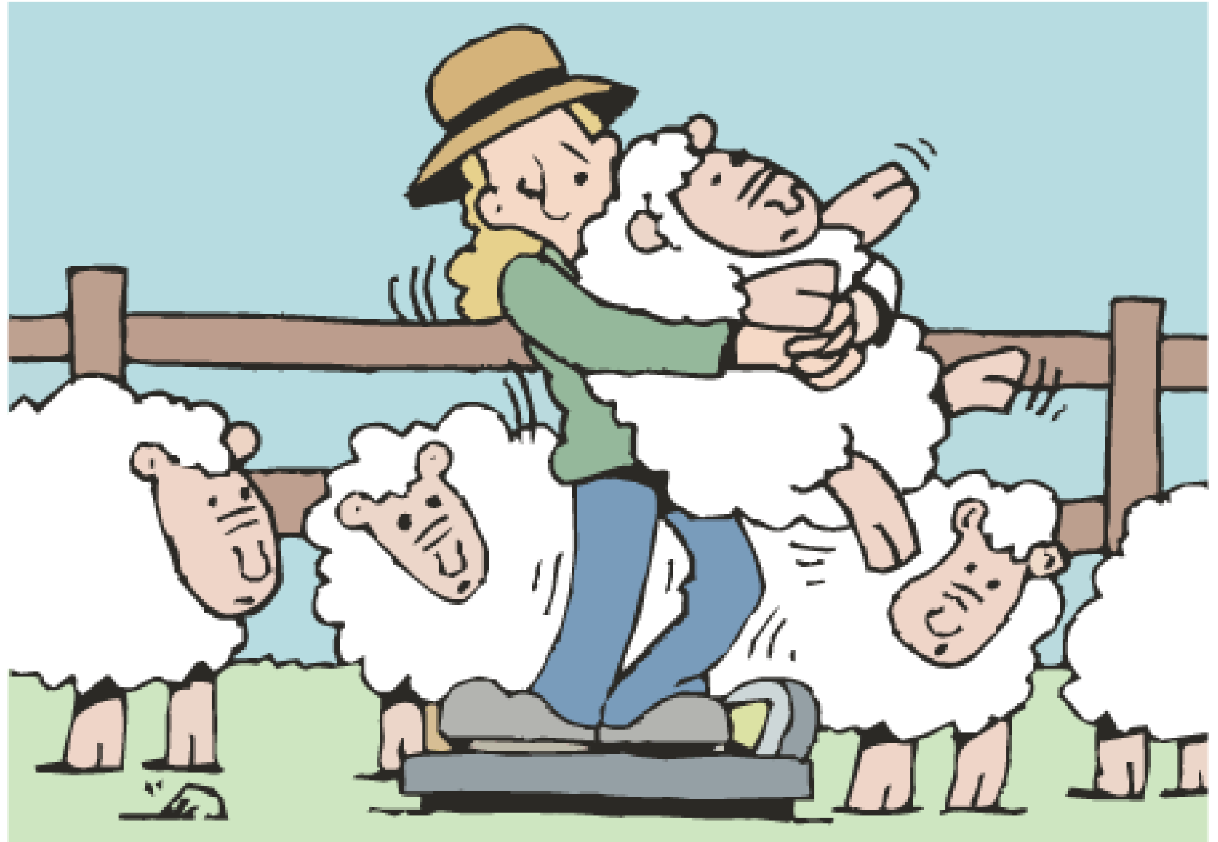
3 A polling agency is employed to investigate the voting intention of residents in a particular electorate. From the data collected, they want to predict the election result for that electorate in the next election. Explain why each of the following situations may produce a biased sample:

- **a** A random selection of people in the local large shopping complex is surveyed between 1 pm and 3 pm on a weekday.
- **b** The members of the local golf club are surveyed.
- **c** A random sample of people at the local train station between 7 am and 9 am are surveyed.
- **d** A door to door visit is undertaken, surveying every voter in a particular street.

4 Jennifer wants to estimate the average weight of the 2000 sheep on her farm. She selects a sample of 10 sheep, and weighs them.

Explain why this approach may produce a:

a coverage error **b** measurement error.



5 Jack owns 800 apple trees. To determine how many apples the trees are producing, he instructs his four sons to each count the apples from 200 trees.

- **a** Explain why there will be no sampling error in this process.
- **b** Two of the sons only count the apples on the tree itself, whilst the other two sons also count the apples on the ground beneath the tree. What type of error is this?

6 A survey company is interested in whether people feel overworked at their jobs. They mail out a survey to 5000 workers, and ask the workers to mail back the survey.

- **a** Explain why this survey may produce a significant non-response error.
- **b** What would be the advantages and disadvantages of conducting the survey online instead of by mail?

7 A national sporting organisation has over 300 000 members. Every member is invited to complete an online survey regarding the management structure of the organisation. Only 16% of the members responded.

- **a** Do you think the non-response error in this situation is likely to produce a biased sample? Explain your answer.
- **b** Does such a high non-response error necessarily invalidate findings from the survey? Discuss your answer.

DISCUSSION

- Why do you think companies offer incentives for people to complete their surveys?
- Which of the following incentives for completing a survey would be more effective?
    - A chance to win a prize as shown alongside.
    - A guaranteed discount or promotional code for the participant to use on their next purchase.
- Is it ethical to offer monetary compensation for completing a survey?

COMPLETE OUR SURVEY
FOR YOUR CHANCE TO
WIN
A $50 GIFT CARD

B SAMPLING METHODS

In general, the best way to avoid bias when selecting a sample is to make sure the sample is **randomly selected**. This means that each member of the population has the same chance of being selected in the sample.

We will look at five sampling methods:

- **simple random sampling**
- **stratified sampling**
- **systematic sampling**
- **quota sampling**
- **convenience sampling**

SIMPLE RANDOM SAMPLING

Suppose 3 students are to be sampled from a class of 30 students. The names of all students in the class are placed in a barrel, and 3 names are drawn from the barrel.

Notice that:

- Each student has the same chance $(\frac{1}{10})$ of being selected.
- Each set of 3 students is just as likely to be selected as any other. For example, the selection {Bruce, Jane, Sean} is just as likely to occur as {Jane, Peter, Vanessa}.

This type of sampling is called **simple random sampling**.

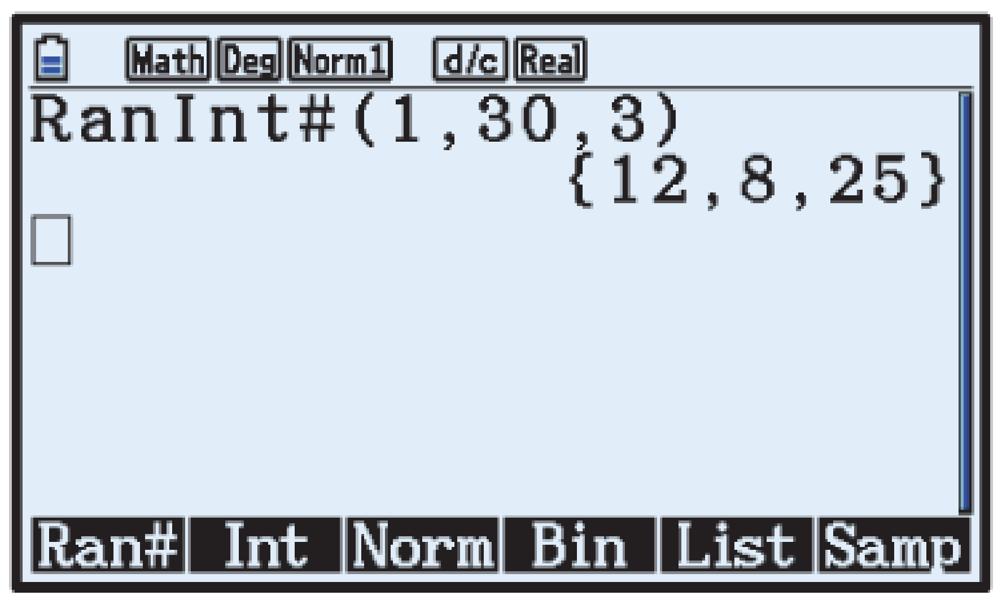
For a **simple random sample** of size n from a population:

- Each member of the population has the same chance of being selected in the sample.
- Each set of n members of the population has the same chance of being selected as any other set of n members.

Instead of drawing names from a barrel, it is usually more practical to number the members of the population, and use a random number generator to select the sample.

You can use your calculator to generate random numbers. In this case, the 8th, 12th, and 25th students would be selected for the sample.

GRAPHICS CALCULATOR INSTRUCTIONS

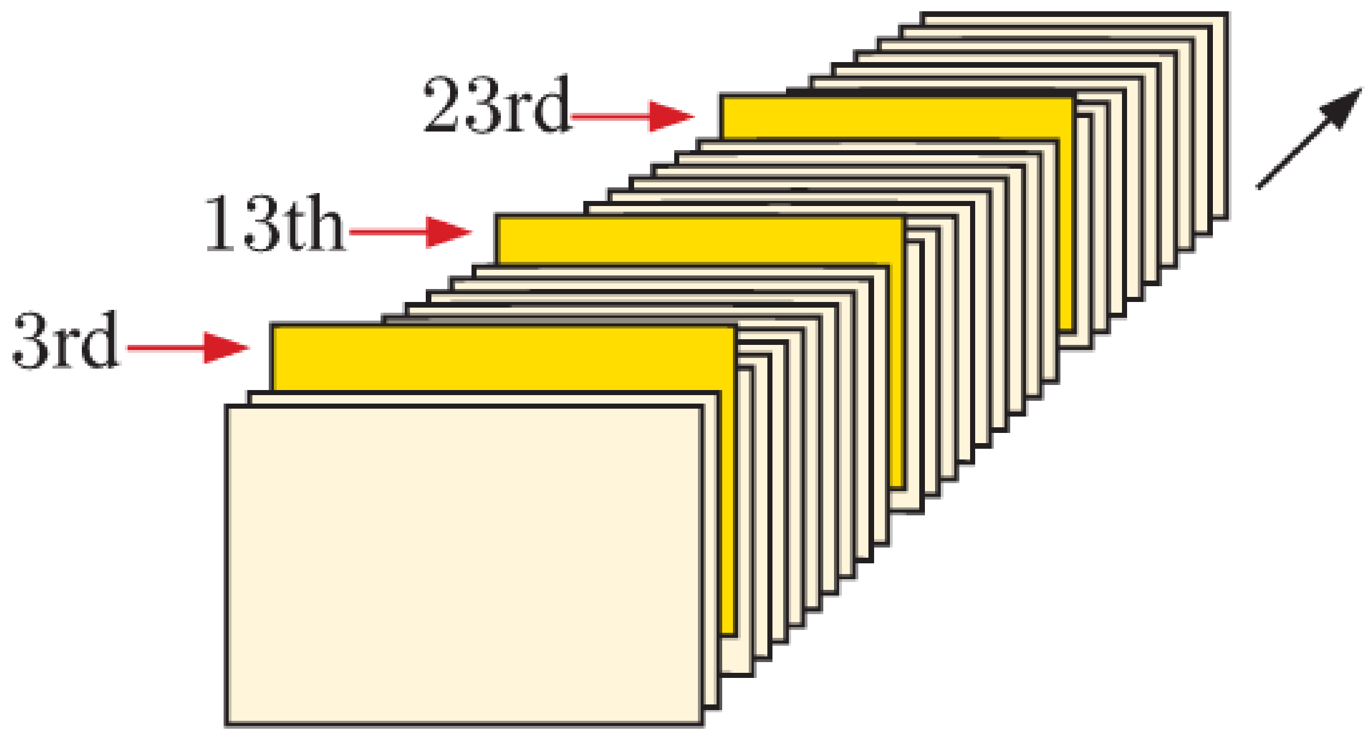


```
RanInt#(1,30,3)
           {12,8,25}
```

SYSTEMATIC SAMPLING

In **systematic sampling**, the sample is created by selecting members of the population at regular intervals.

For example, an accountancy firm may wish to sample the files of $\frac{1}{10}$th of their clients. They choose a starting file from 1 to 10 (for example, 3), and then select every 10th file after that. So, they would select the 3rd file, then the 13th, 23rd, 33rd, and so on.



Systematic sampling is useful when not all members of the population are available for sampling at the same time. An example of this is the sampling of cars which pass through a particular intersection during the day.

Example 1 **Self Tutor**

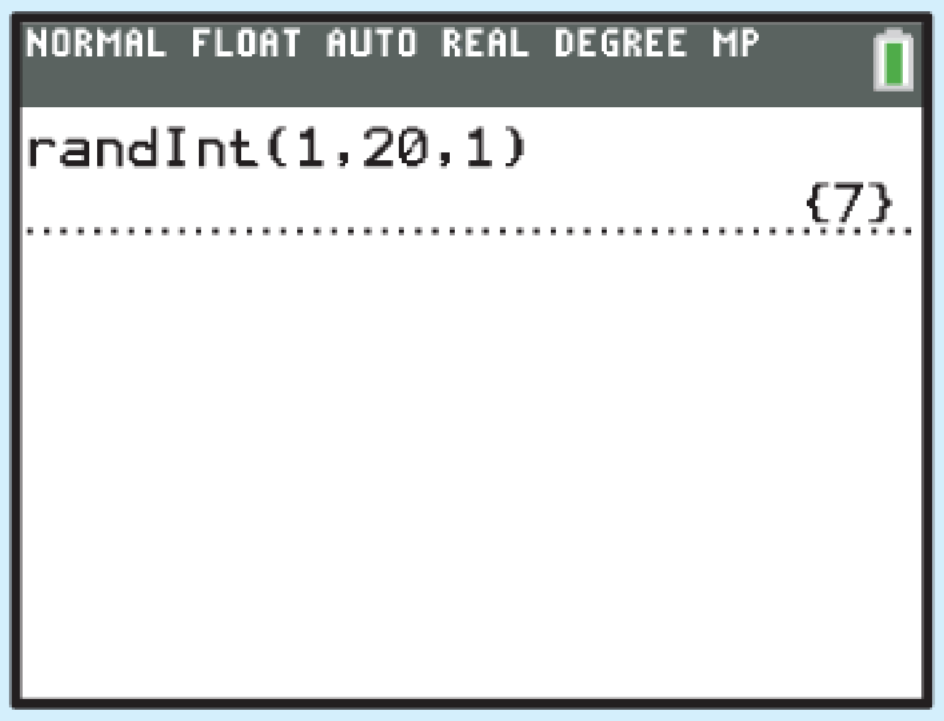
Management of a large city store wishes to find out whether potential customers like the look of a new product. They decide to sample 5% of the customers using a systematic sample. Show how this sample would be selected.

$5\% = \frac{5}{100} = \frac{1}{20}$

So, every 20th customer will be sampled.

A starting customer is selected from 1 to 20. In this case it is customer 7.

So, the store would select the 7th customer, then the 27th, 47th, 67th, and so on.



```
NORMAL FLOAT AUTO REAL DEGREE MP
randInt(1,20,1)
                                   {7}
```

CONVENIENCE SAMPLING

In many situations, people are chosen simply because they are easier to select or more likely to respond.

For example, consider a researcher conducting a survey regarding environmental issues. The researcher decides to stand in a pedestrian mall and ask people walking past. It is easiest for the researcher to ask people who are:

- walking closest to them
- walking slowly
- not already in a conversation or using their phone.

These types of samples are known as **convenience samples** because they are convenient for the experimenter.

DISCUSSION

Do you think convenience samples will often be biased?

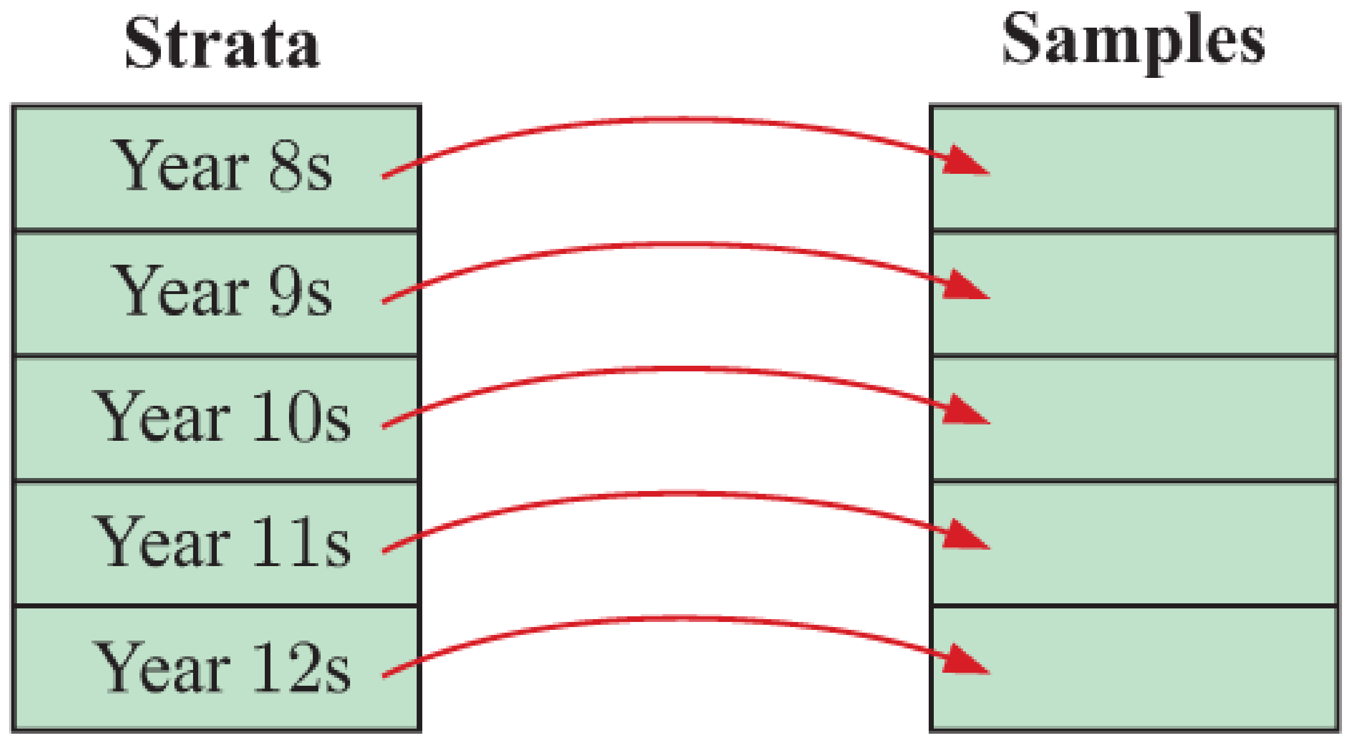
Discuss any possible bias if the researcher in the mall was studying:

- mobile internet usage
- personal relationships
- social media
- mental health issues.

STRATIFIED SAMPLING AND QUOTA SAMPLING

Stratified sampling and **quota sampling** are useful when the population can be divided into subgroups, and you want to make sure each subgroup is represented fairly in the sample.

For example, a school may want to know the opinions of its students on which charities it should support in the school fun run. To make sure each year level is represented fairly, the number of students sampled from each year level should be proportional to the fraction of the total number of students that year level represents.



Example 2 **Self Tutor**

In our school there are 137 students in Year 8, 152 in Year 9, 174 in Year 10, 168 in Year 11, and 121 in Year 12. A sample of 50 students is needed. How many should be randomly selected from each year?

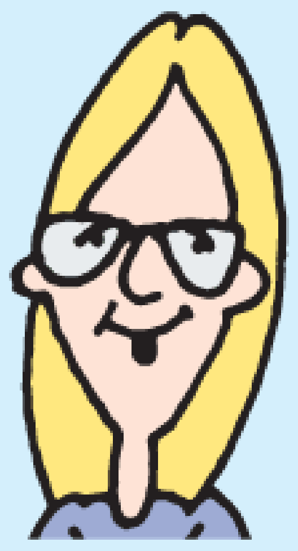
Total number of students in the school $= 137 + 152 + 174 + 168 + 121 = 752$

For the sample, we want:

$$\begin{aligned}
\text{number of Year 8 students} &= \tfrac{137}{752} \times 50 \approx 9 \\
\text{number of Year 9 students} &= \tfrac{152}{752} \times 50 \approx 10 \\
\text{number of Year 10 students} &= \tfrac{174}{752} \times 50 \approx 12 \\
\text{number of Year 11 students} &= \tfrac{168}{752} \times 50 \approx 11 \\
\text{number of Year 12 students} &= \tfrac{121}{752} \times 50 \approx 8
\end{aligned}$$

We should select 9 students from Year 8, 10 from Year 9, 12 from Year 10, 11 from Year 11, and 8 from Year 12.

Year 8 students represent $\frac{137}{752}$ of the school, so they should also represent $\frac{137}{752}$ of the sample.



Ideally, we would want the individuals from each strata to be randomly selected to minimise bias. If this can be done, the sample is a **stratified sample**. Otherwise, if the individuals are specifically selected by the experimenter (such as in a convenience sample) then the sample is a **quota sample**.

EXERCISE 12B

1 Use your calculator to select a random sample of:

- **a** 6 different numbers between 5 and 25 inclusive
- **b** 10 different numbers between 1 and 25 inclusive
- **c** 6 different numbers between 1 and 45 inclusive
- **d** 5 different numbers between 100 and 499 inclusive.

You may need to generate additional random numbers if a number appears more than once.

2 A chocolate factory produces 80 000 blocks of chocolate per day. Today, the factory operator wants to sample 2% of the blocks for quality testing. He uses a systematic sample, starting from the 17th block.

- **a** List the first five blocks to be sampled.
- **b** Find the total size of the sample.

3 Click on the icon to obtain a printable calendar for 2019 showing the weeks of the year. Each day is numbered.

CALENDAR

Using a random number generator, choose a sample from the calendar of:

- **a** five different dates
- **b** a complete week starting with a Monday
- **c** a month
- **d** three different months
- **e** three consecutive months
- **f** four different Wednesdays.

Explain your method of selection in each case.

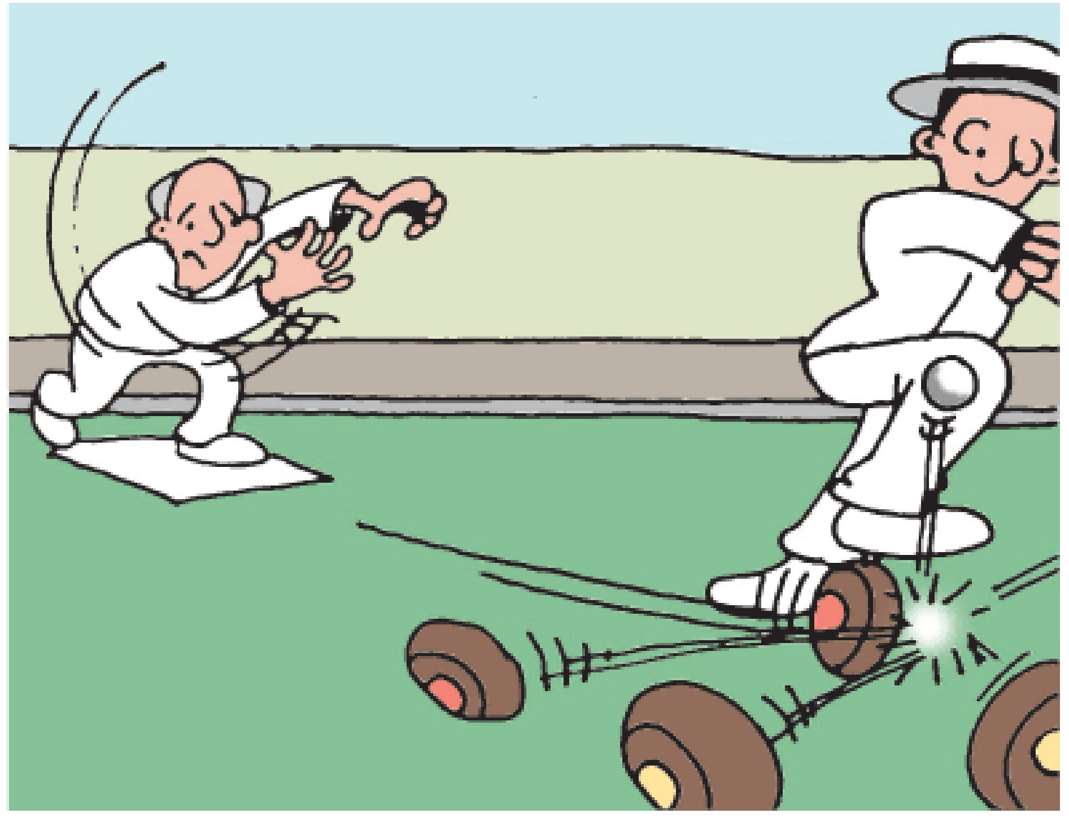
January	February	March	April	May
1 Tu (1) Wk 1	1 Fr (32)	1 Fr (60)	1 Mo (91)	1 We (121)
2 We (2)	2 Sa (33)	2 Sa (61)	2 Tu (92) Wk 14	2 Th (122)
3 Th (3)	3 Su (34)	3 Su (62)	3 We (93)	3 Fr (123)
4 Fr (4)	4 Mo (35)	4 Mo (63)	4 Th (94)	4 Sa (124)
5 Sa (5)	5 Tu (36) Wk 6	5 Tu (64) Wk 10	5 Fr (95)	5 Su (125)
6 Su (6)	6 We (37)	6 We (65)	6 Sa (96)	6 Mo (126)
7 Mo (7)	7 Th (38)	7 Th (66)	7 Su (97)	7 Tu (127) Wk 19
8 Tu (8) Wk 2	8 Fr (39)	8 Fr (67)	8 Mo (98)	8 We (128)
9 We (9)	9 Sa (40)	9 Sa (68)	9 Tu (99) Wk 15	9 Th (129)
…	…	…	…	…

4 An annual dog show averages 3540 visitors. The catering manager is conducting a survey to investigate the proportion of visitors who will spend more than €20 on food and drinks at the show. He decides to survey the first 40 people through the gate.

- **a** Identify the sampling method used.
- **b** Discuss any problems with the sampling method.
- **c** Suggest a better sampling method that includes a suitable sample size and which better represents the population.

5 A library manager is interested in the number of people using the library each day. She decides to perform a count every 28th day for one year, starting next Monday.

- **a** What type of sampling method is this?
- **b** How many days will be in her sample?
- **c** Explain why the sample may be biased.



6 A sporting club wants to ask its members some questions about the clubhouse. The club has 80 tennis members, 60 lawn bowls members, and 20 croquet members.

- **a** How many members does the club have in total?
- **b** The club decides to use a sample of 40. How many members of each sport should be sampled?

7 A large retail store has 10 departmental managers, 24 supervisors, 65 senior sales staff, 98 junior sales staff, and 28 shelf packers. The company director wishes to interview a sample of 30 staff to obtain their view of operating procedures. How many of each group should be selected for the sample?

8 Mona wants to gauge the opinions of her peers on the design of the school's yearbook. She uses her own home room class as her sample.

- **a** Explain why Mona's sample is a convenience sample.
- **b** In what ways will Mona's sample be biased?
- **c** Suggest a more appropriate sampling method that Mona should use.

9 Lucian is a school counsellor. He wants to raise awareness of student cyber-bullying with the students' parents. Lucian therefore wants to find out whether students at the school have discussed the issue with their parents.

- **a** Why might it be impractical for Lucian to use a simple random sample or systematic sample?
- **b** Lucian wants to make sure that each gender is appropriately represented in his sample. Should he use a stratified sample or quota sample?

10 The 200 students in Years 11 and 12 at a high school were asked whether or not they had ever smoked a cigarette. The replies received were:

nnnny nnnyn ynnnn yynyy ynyny ynnyn nyynn yynyn ynynn nyynn
ynnyn yynyy nnyyy yyyyy nnnyy nnnnn nnyny yynny nnnyy ynyyn
nynnn ynyyn nnyny ynyyy ynnnn yyyyn yynnn nynyn yyyny ynnyy
nynnn yynny nyynn yynyn ynynn nyyyn ynnyy nyyny nnyny ynnnn

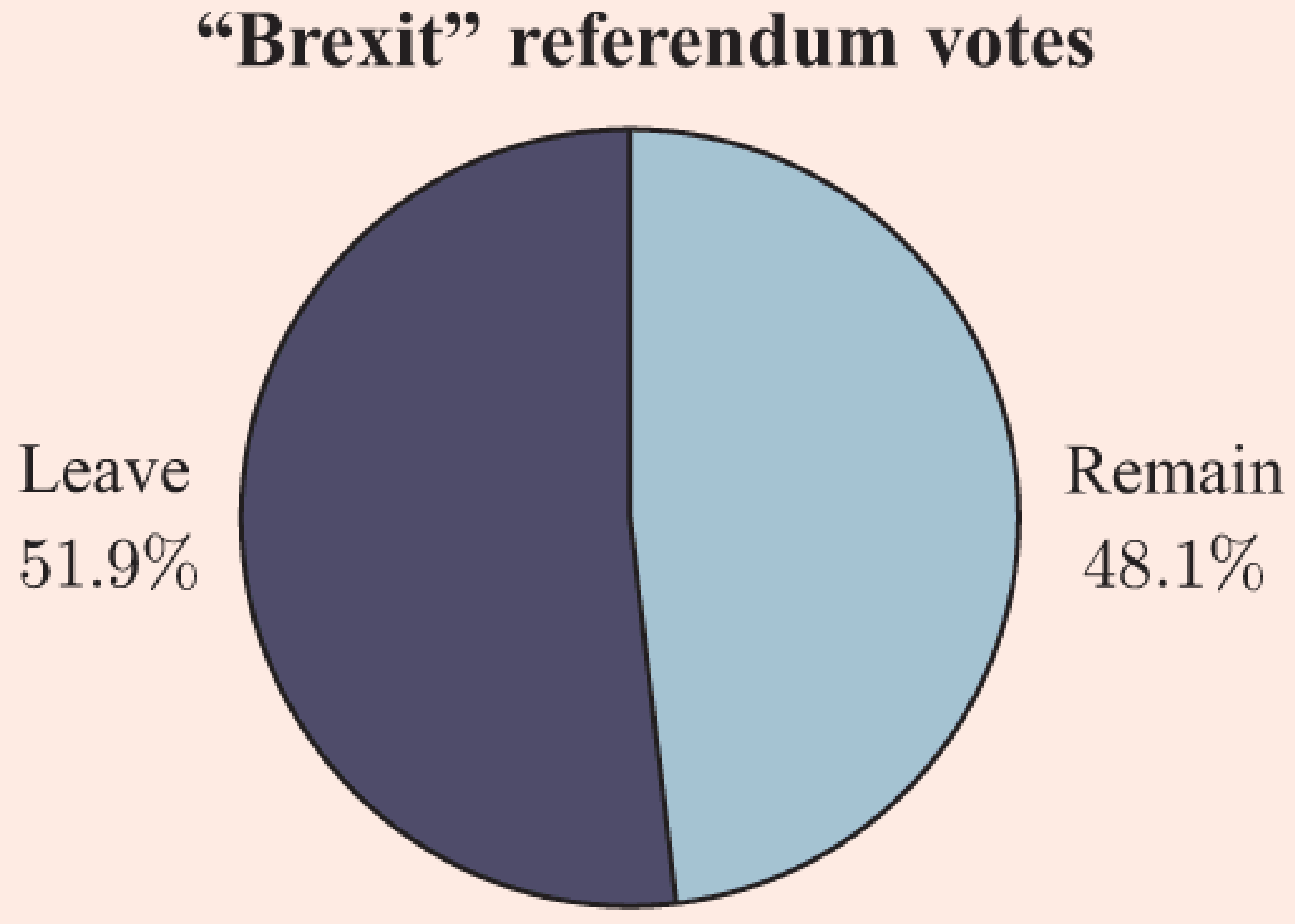
- **a** Why is this considered to be a census?
- **b** Find the actual proportion of all students who said they had smoked.
- **c** Discuss the validity and usefulness of the following sampling methods which could have been used to estimate the proportion in **b**:
  - **i** sampling the first five replies
  - **ii** sampling the first ten replies
  - **iii** sampling every second reply
  - **iv** sampling the fourth member of every group of five
  - **v** randomly selecting 30 numbers from 1 to 200 and choosing the response corresponding to that number
  - **vi** sampling 20% of Year 11 students and 20% of Year 12 students.

- **d** Are any of the methods in **c** examples of simple random sampling, systematic sampling, stratified sampling, or quota sampling?

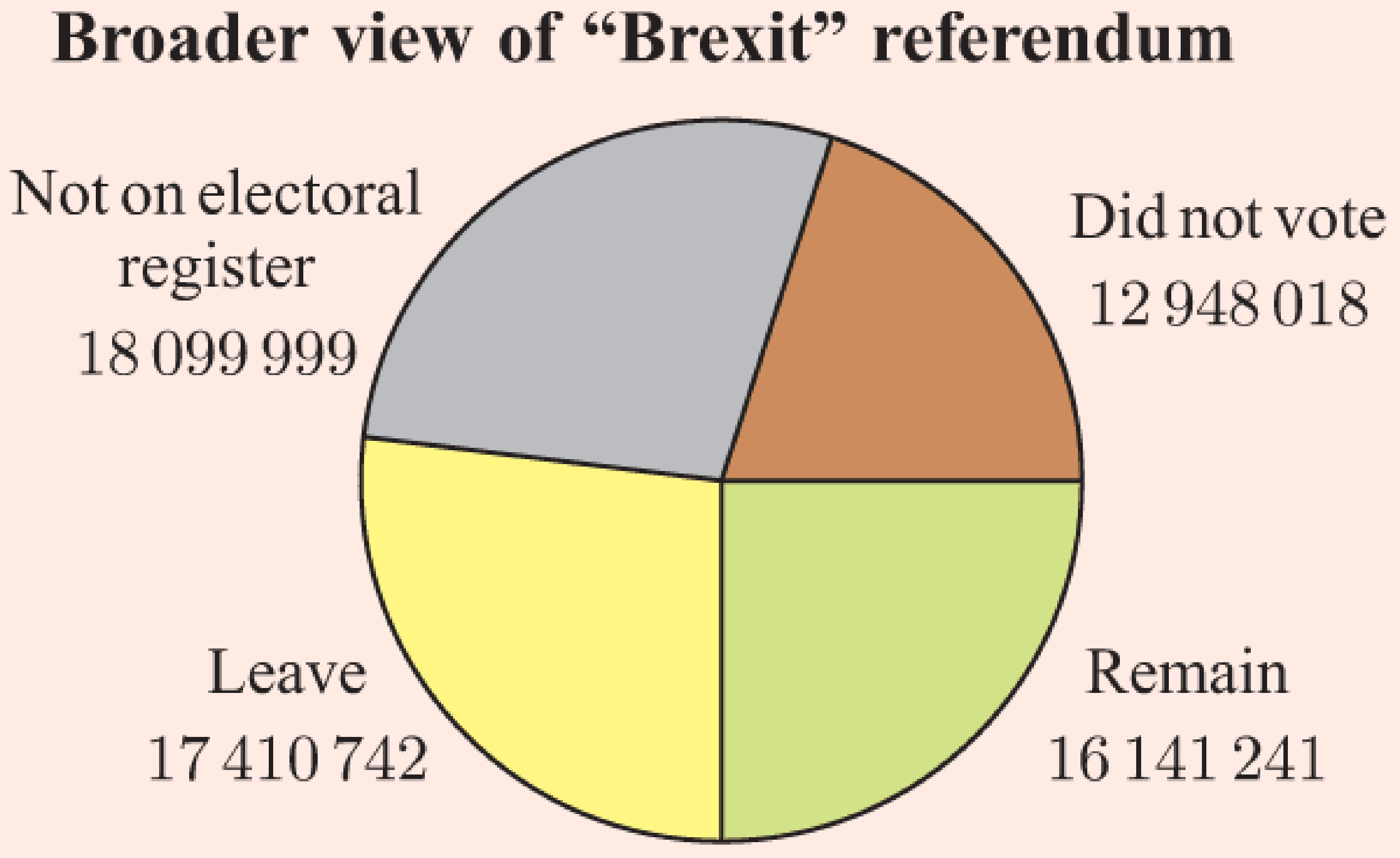
DISCUSSION

The so-called "Brexit" referendum of 2016 to determine whether the United Kingdom would remain part of the European Union is one of the most controversial democratic referendums in recent history.

1. Was the referendum a census or a sample?
2. What sampling errors may have been present? In what ways might the sample have been biased?
3. 33 551 983 votes were counted in the referendum, and it was decided by a simple majority of 51.9% to 48.1% that the United Kingdom would leave. This is shown in the first pie chart.



"Brexit" referendum votes



Broader view of "Brexit" referendum

In the second pie chart we take a broader view to include those who did not vote and those not on the electoral register.

a Do you think that the United Kingdom leaving the European Union can be considered "the will of the people"?

b Do you think it is a good idea to have a non-compulsory referendum which can be carried with only a simple majority?

THEORY OF KNOWLEDGE

Clinical trials are commonly used in medical research to test the effectiveness of new treatments for conditions or diseases. They require randomly sampling patients who have the condition or disease in question.

A clinical trial usually involves dividing the sampled individuals into 2 groups:

- A **control group** that serves as the baseline for comparison. This group is usually given either a placebo or the best currently available treatment.
- A **treatment group** that receives the new treatment to be tested.

Ideally, the two groups should be as similar as possible so that differences between the results for the two groups more accurately reflect the differences between treatments, rather than the differences between individuals.

1 Are clinical trials necessarily practical for studying treatments of extremely rare diseases?

2 How should groups be sampled?

As clinical trials are experiments involving people, the consideration of **ethics** is particularly important. In 1975, the **Declaration of Helsinki**[1] was written by the World Medical Association to provide guidelines on human experimentation.

Key points of the Declaration include:

- Patients must be made fully aware of all possible risks before they give their consent to participate.
- Patients must never be given a treatment that is known to be inferior.
- Patients are allowed to withdraw from the study at any time.

3 Are these ethical guidelines applicable to *any* experiment or survey that involves people?

4 Are there the same ethical concerns about experiments involving:

a animals **b** non-living things?

In June 2014, Facebook published a study on the effects of omitting certain words from posts in a user's "News feed" on their moods and emotions[2]. This study was heavily criticised for its lack of ethical consideration. The only consent to use people's data was obtained within Facebook's general Terms and Conditions document, which must be agreed to upon making an account.

5 Do you think this study was an acceptable use of users' data?

6 Do you think using the general Terms and Conditions document to justify "informed consent" is *fair*?

7 Are ethics more important than research?

[1] www.wma.net/policy/current-policies/

[2] Adam D. I. Kramer, Jamie E. Guillory, and Jeffrey T. Hancock. "Experimental evidence of massive-scale emotional contagion through social networks". In: *Proceedings of the National Academy of Sciences* 111.24 (2014), pp. 8788 - 8790. ISSN: 0027-8424. DOI: 10.1073/pnas.1320040111. eprint: http://www.pnas.org/content/111/24/8788.full.pdf. URL: http://www.pnas.org/content/111/24/8788.

C WRITING SURVEYS

Surveys are one of the most common and simple methods for collecting data from a sample. Surveys usually consist of a series of questions which can be asked in a written **questionnaire** or an oral **interview**.

When writing survey questions we need to be very careful about how they are **worded**. Poorly worded questions can lead to misinterpretation and yield unintended or inaccurate answers. Such answers would be regarded as **measurement errors**.

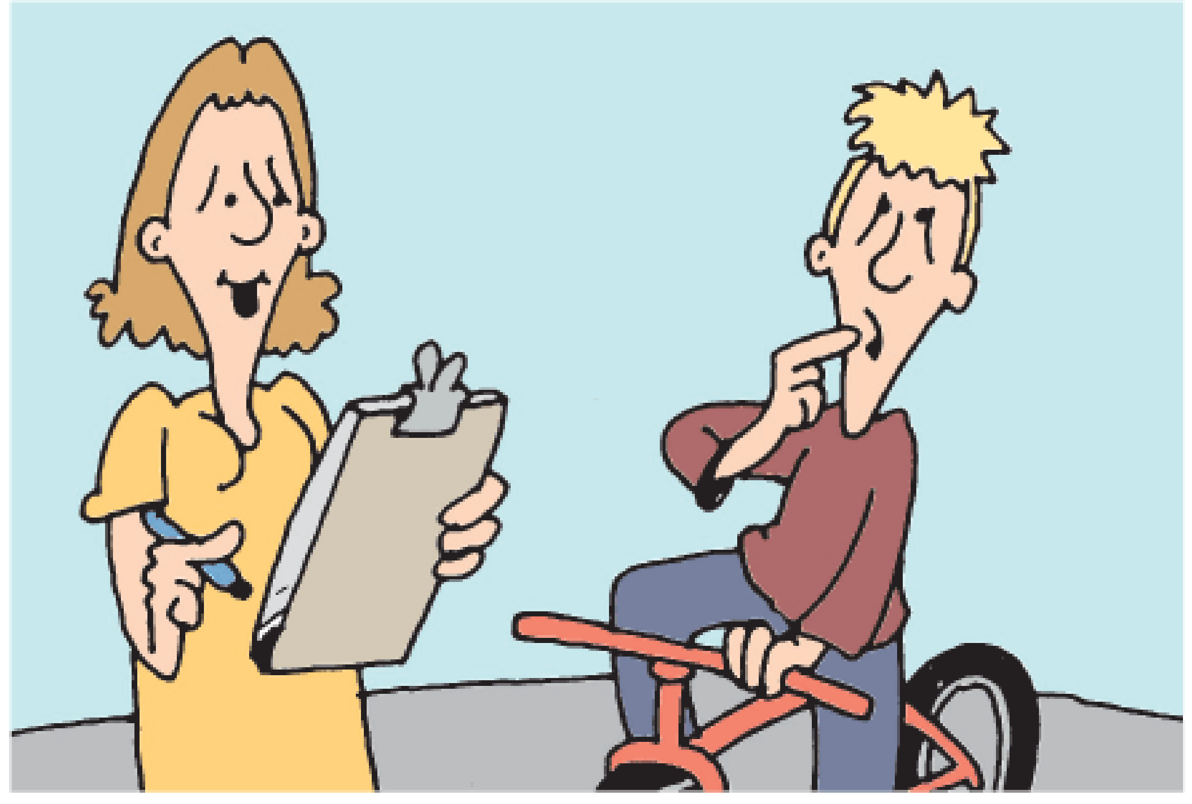
GENERAL GUIDELINES FOR QUESTION WRITING

- **Keep questions simple and clear**
  Simple wording will help respondents understand what the question is asking, and the context in which it is being asked. You should only ask for **one** response per question.
  Questions containing double negatives, such as "Do you disagree with not vaccinating children?", should be avoided as they often confuse the respondent.

- **Ask specific questions instead of general questions**
  General questions are easier to misinterpret than specific questions.
  Asking a series of specific questions gives more information and a deeper understanding of the respondent's opinion than asking one general question.
  For example, instead of asking "Do you support Party A?", we can ask a series of questions:
  - "Do you support the environmental policies that Party A introduced?"
  - "Why did you choose Party A over Party B in the last election?"
  - "Will you vote for Party A in the next election?"

- **Choose between structured versus unstructured questions**
  Questions without any restrictions on the answer required are called **unstructured questions**. Unstructured questions are useful for exploratory surveys where the purpose of the investigation is to gauge opinions.
  Questions with a set of answers for the respondent to choose from are called **structured questions**. The choices presented in structured questions can prompt respondents to remember things that they would have otherwise forgotten. An "Other" category which allows the respondent to input their own answer may be useful.

- **Keep the tone of questions neutral**
  Questions containing opinions or emotive language are **biased** and can lead the respondent to answer in a certain way.
  For example, the question "Do you support the dangerous practice of cycling without a helmet?" invites the respondent to answer "no", since the question contains the judgement that riding without a helmet is dangerous.



- **Be aware of personal questions**

  Questions that ask for sensitive information or opinions are called **personal questions**.
  For example:
  - "How much do you weigh?"
  - "What is your religious affiliation?"
  - "What is your gender?"
  - "What is your sexual orientation?"

  Such questions can make the respondent uncomfortable. They may not answer truthfully, or they may not answer at all.

  If you *need* to ask a personal question, justify why the information is needed. Respondents are more likely to divulge personal information if they understand why it is needed.

EXERCISE 12C

1 Consider the question "Is your shirt red, blue, yellow, or white?"

- **a** State one problem with posing this as a structured question.
- **b** Rewrite the question as an unstructured question.
- **c** The colour of a person's shirt can be *subjective* because it usually depends on individual interpretation. Explain how this might be a problem.

2 Consider the question "Do you have any allergies?"

- **a** List ways in which the question can be interpreted. Include any possible misinterpretations.
- **b** Rewrite the question so it is more specific.

3 Consider the question "Do you have any pets?"

- **a** List ways in which the question can be interpreted. Include any possible misinterpretations.
- **b** Rewrite the question so it is more specific.

4 The government has released a new proposal to move funding from education to health. A journalist wants to understand the public's feelings about this proposal. She asks 100 people the question "Do you support the Government's proposed cuts to education?".

- **a** Explain why this survey may produce a measurement error.
- **b** How could the question be worded so the public's feelings about the proposal would be more accurately measured?

5 Consider the question "Where do you live?"

- **a** Explain why this question is likely to have a high non-response error.
- **b** List ways in which the question could be improved.

6 For each of the following survey questions:

  - **i** Identify any problems with how the question is worded.
  - **ii** Rewrite the question to fix these problems.

- **a** Have you or have you not been immunised against the infectious meningococcal disease?
- **b** Do you believe that climate change is a major issue or do you think that it is a topic thrown around by politicians to gain support?
- **c** Considering the fact that "fair trade cocoa" ensures that cocoa farmers are paid a minimum wage and helps prevent child labour, do you agree that fair trade certified chocolate should be more expensive than uncertified chocolate?

DISCUSSION

If a question provides choices for answers which use a rating scale, is it better to have an even or odd number of ratings?

ACTIVITY

Lily wants to survey students at her school about the school canteen.

What to do:

1. List various characteristics about the canteen that Lily might be interested in.
2. Decide which characteristics would be most relevant if:
   - **a** a new canteen was being constructed
   - **b** there was a new manager in charge of the canteen.
3. What sampling method(s) do you think would be most suitable for Lily? Explain your answer.
4. Write a complete survey for Lily, including questions about each of the characteristics listed in **1**. Consider carefully your choices of unstructured and structured questions.

RESEARCH PRECISE QUESTIONING

Precise questioning (PQ) is a question writing framework aimed at obtaining clear answers and communicating more efficiently, particularly in business settings. It was developed at Stanford University during the mid-1990s.

The seven main categories of questions in PQ are:

(1) Go/No go
(2) Clarification
(3) Assumptions
(4) Basic critical question
(5) Causes
(6) Effects
(7) Action

What to do:

1. Research PQ and explain what each category of questions is meant to achieve. Give an example of each type of question in context.
2. Which categories of questions might be applicable when writing a questionnaire?
3. Do you think this framework could apply to statistical investigations? Discuss your answer.

D TYPES OF DATA

In previous years you should have seen how variables can be described either as **categorical** or **numerical**.

CATEGORICAL VARIABLES

A **categorical variable** describes a particular quality or characteristic.

The data is divided into **categories**, and the information collected is called **categorical data**.

Some examples of categorical data are:

- *computer operating system*:

  The categories could be Windows, macOS, or Linux.

- *gender*:

  The categories are male and female.

QUANTITATIVE OR NUMERICAL VARIABLES

A **quantitative variable** has a numerical value. The information collected is called **numerical data**.

Quantitative variables can either be **discrete** or **continuous**.

A **discrete quantitative variable** or just **discrete variable** takes exact number values. It is usually a result of **counting**.

Some examples of discrete variables are:

- *the number of apricots on a tree*:

  The variable could take the values 0, 1, 2, 3, ... up to 1000 or more.

- *the number of players in a game of tennis*:

  The variable could take the values 2 or 4.

A **continuous quantitative variable** or just **continuous variable** can take any numerical value within a certain range. It is usually a result of **measuring**.

Some examples of continuous variables are:

- *the times taken to run a* 100 *m race*:

  The variable would likely be between 9.5 and 25 seconds.

- *the distance of each hit in baseball*:

  The variable could take values from 0 m to 100 m.

Example 3

Self Tutor

Classify each variable as categorical, discrete, or continuous:

a the number of heads when 3 coins are tossed

b the brand of toothpaste used by the students in a class

c the heights of a group of 15 year old children.

a We count the number of heads. The result could be 0, 1, 2, or 3. It is a discrete variable.

b The variable describes the brands of toothpaste. It is a categorical variable.

c We measure the height of each child. The data can take any value between certain limits, though when measured we round off the data to an accuracy determined by the measuring device. It is a continuous variable.

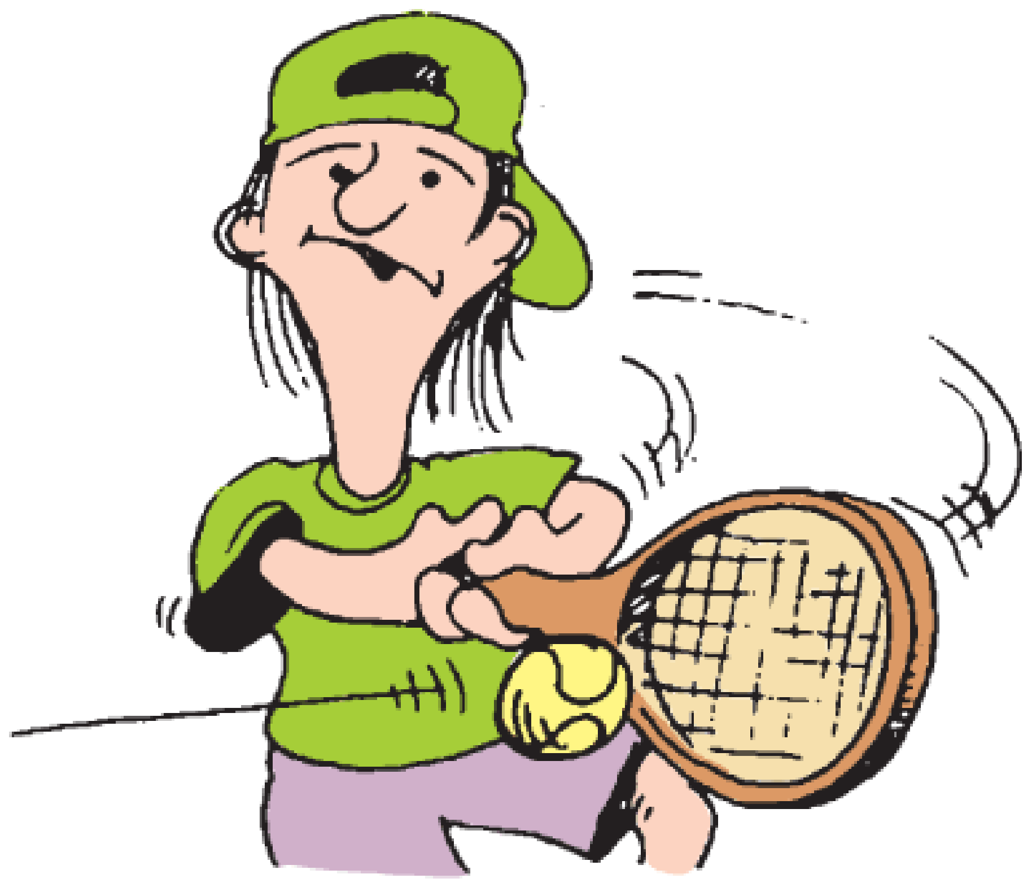
EXERCISE 12D

1 Classify each variable as categorical, discrete, or continuous.
If the variable is categorical, list some possible categories.
If the variable is quantitative, suggest possible values or a range of values the variable may take.

- **a** The number of brothers a person has.
- **b** The colours of lollies in a packet.
- **c** The time children spend brushing their teeth each day.
- **d** The heights of the trees in a garden.
- **e** The brand of car a person drives.
- **f** The number of petrol pumps at a service station.
- **g** The most popular holiday destinations.
- **h** The scores out of 10 in a diving competition.
- **i** The amount of water a person drinks each day.
- **j** The number of hours spent per week at work.
- **k** The average temperatures of various cities.
- **l** The items students ate for breakfast before coming to school.
- **m** The number of televisions in each house.

2 Consider the following statistics for a tennis player:

Name: Vance McFarland
Age: 28
Height: 191 cm
Country: Ireland
Tournament wins: 14
Average serving speed: 185 km h^{-1}
Ranking: 6
Career prize money: £3 720 000



Classify each variable as categorical, discrete, or continuous.

E SIMPLE DISCRETE DATA

ORGANISING DISCRETE DATA

One of the simplest ways to organise data is using a **tally and frequency table** or just **frequency table**.

For example, consider the data set:

1 3 1 2 4   2 4 1 5 3   1 3 2 2 4
1 3 4 1 2   3 2 4 1 3   2 1 2 5 2

A **tally** is used to count the number of 1s, 2s, 3s, and so on. As we read the data from left to right, we place a vertical stroke in the tally column. We use 卌 to represent 5 occurrences.

The **frequency** column summarises the number of occurrences of each particular data value.

The **relative frequency** of a data value is the frequency divided by the total number of recorded values. It indicates the proportion of results which take that value.

Value	*Tally*	*Frequency* (f)	*Relative frequency*
1	𝍸 |||	8	≈ 0.267
2	𝍸 ||||	9	0.3
3	𝍸 |	6	0.2
4	𝍸	5	≈ 0.167
5	||	2	≈ 0.0667
	Total	30	

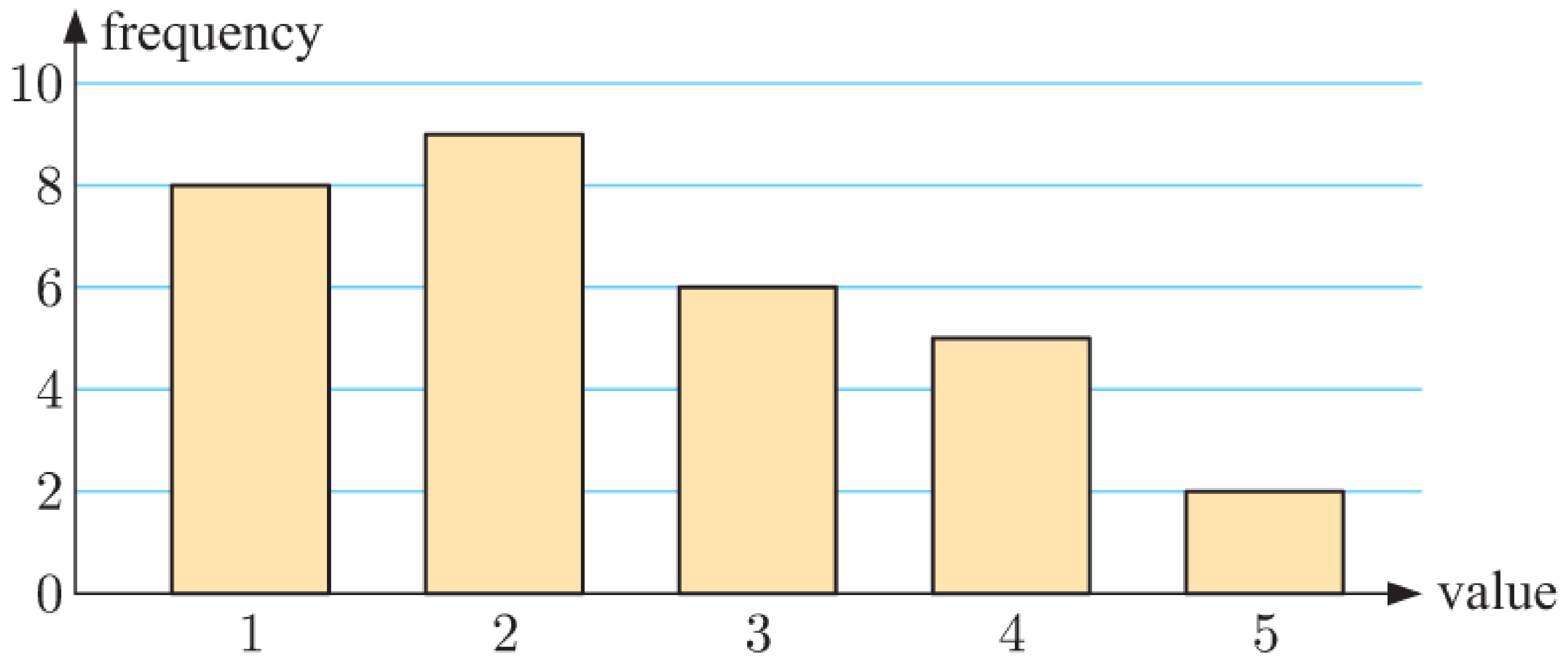
A tally column is not essential for a frequency table, but is useful in the counting process.

DISPLAYING DISCRETE DATA

Discrete data is displayed using a **column graph**. For this type of graph:

- The possible data values are placed on the horizontal axis.
- The frequency of data values is read from the vertical axis.
- The column widths are equal and the column height represents the frequency of the data value.
- There are gaps between columns to indicate the data is discrete.

A column graph for the data set on the previous page is shown alongside.

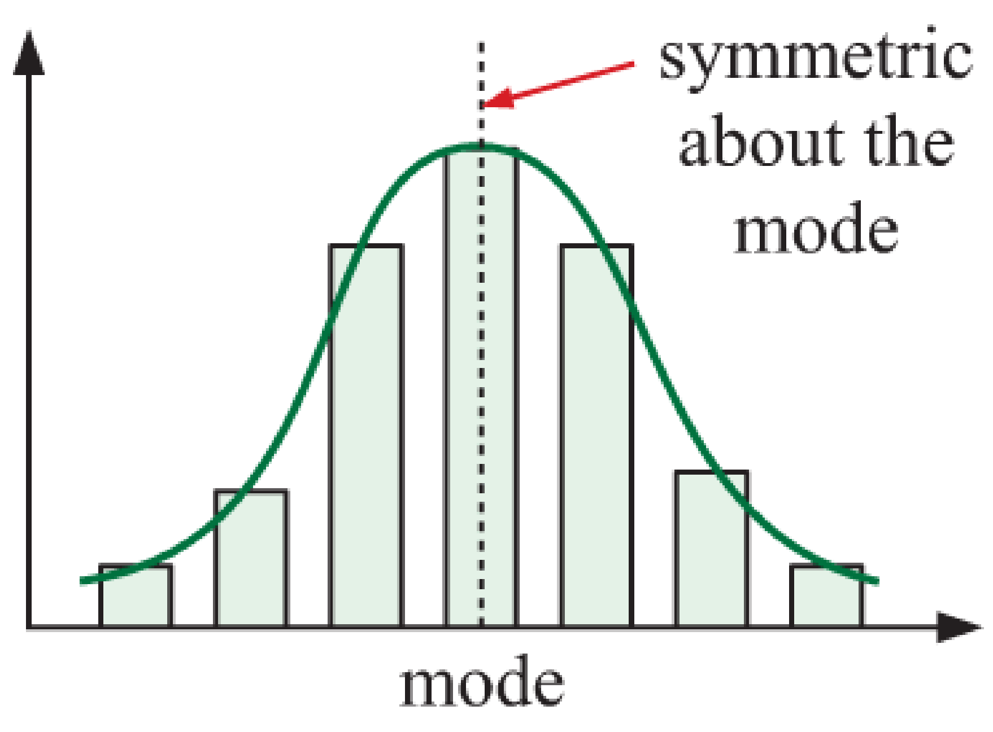


The **mode** of a data set is the most frequently occurring value. On a column graph, the mode will have the highest column. In this case the mode is 2.

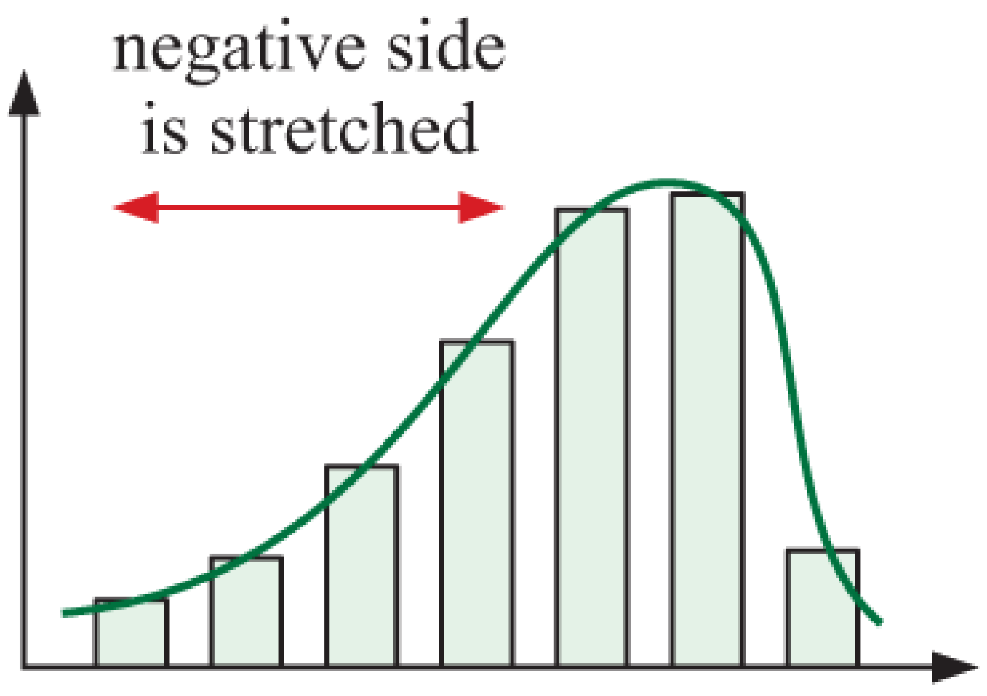
DESCRIBING THE DISTRIBUTION OF A DATA SET

A column graph allows us to quickly observe the **distribution** or **shape** of the data set. We can describe the distribution as:

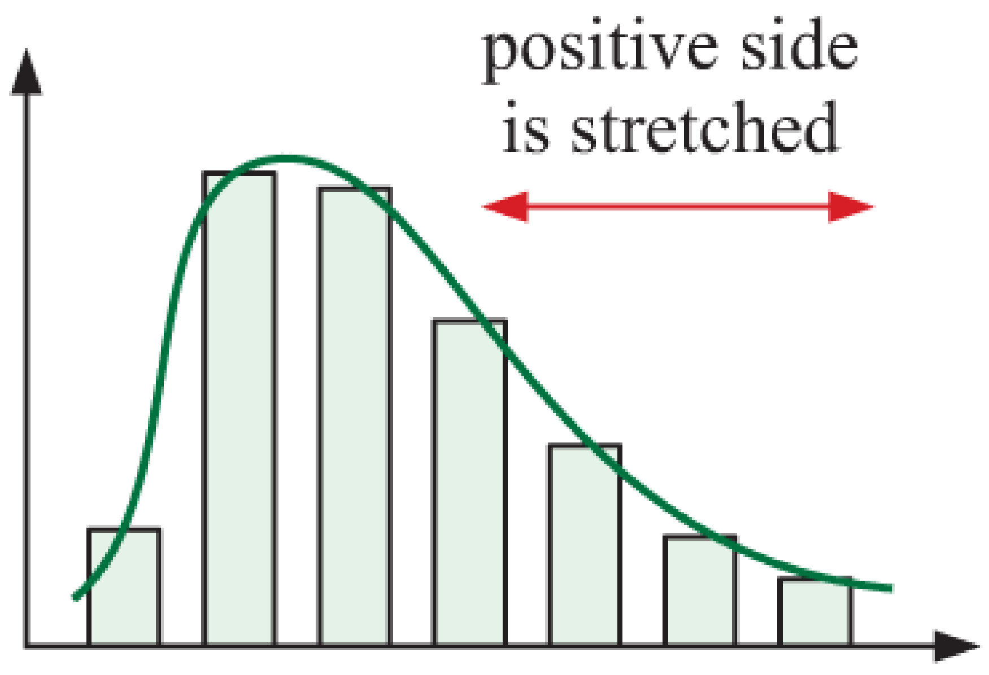
- **Symmetric**



- **Negatively skewed**



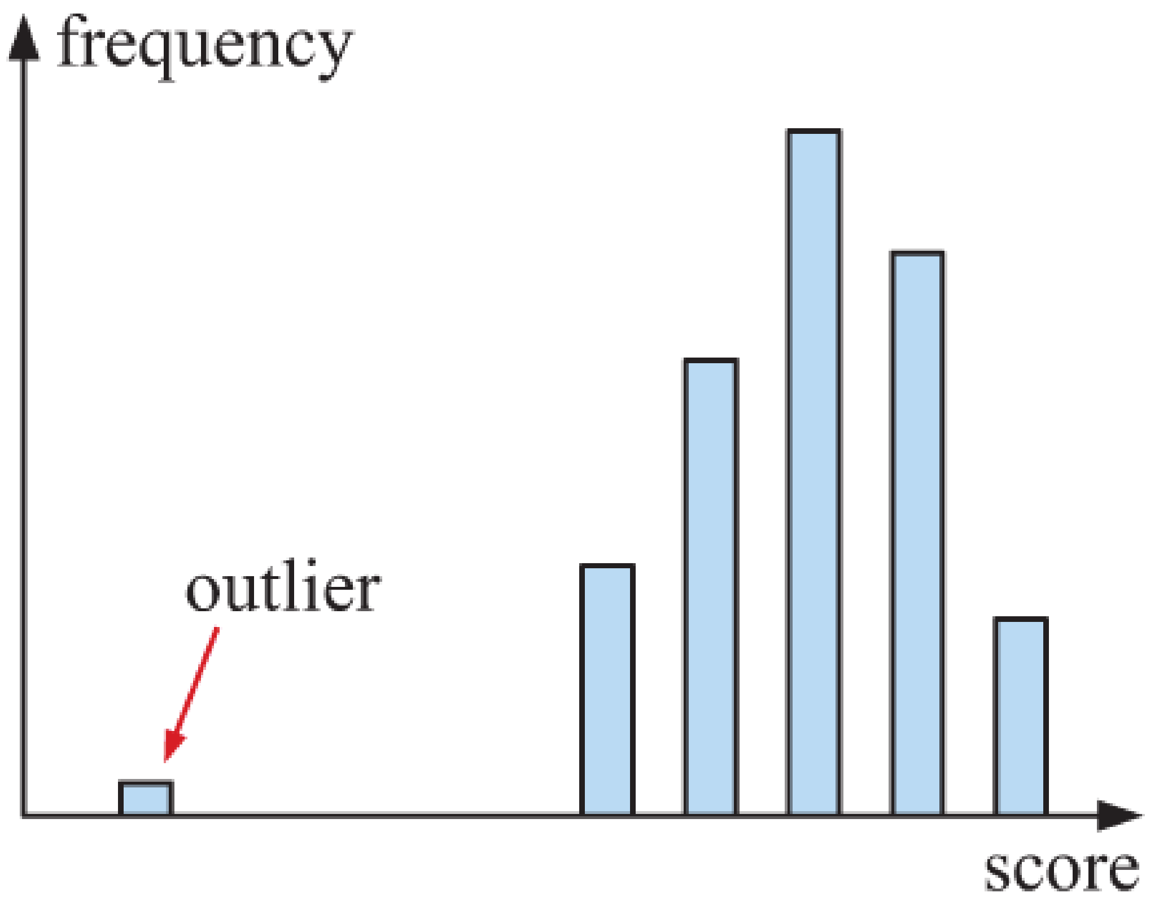
- **Positively skewed**



Outliers are data values that are either much larger or much smaller than the general body of data.

Outliers appear separated from the body of data on a column graph.

If an outlier is a genuine piece of data, it should be retained for analysis. However, if it is found to be the result of an error in the data collection process, it should be removed from the data.



Example 4 **Self Tutor**

30 children attended a library holiday programme. Their year levels at school were:

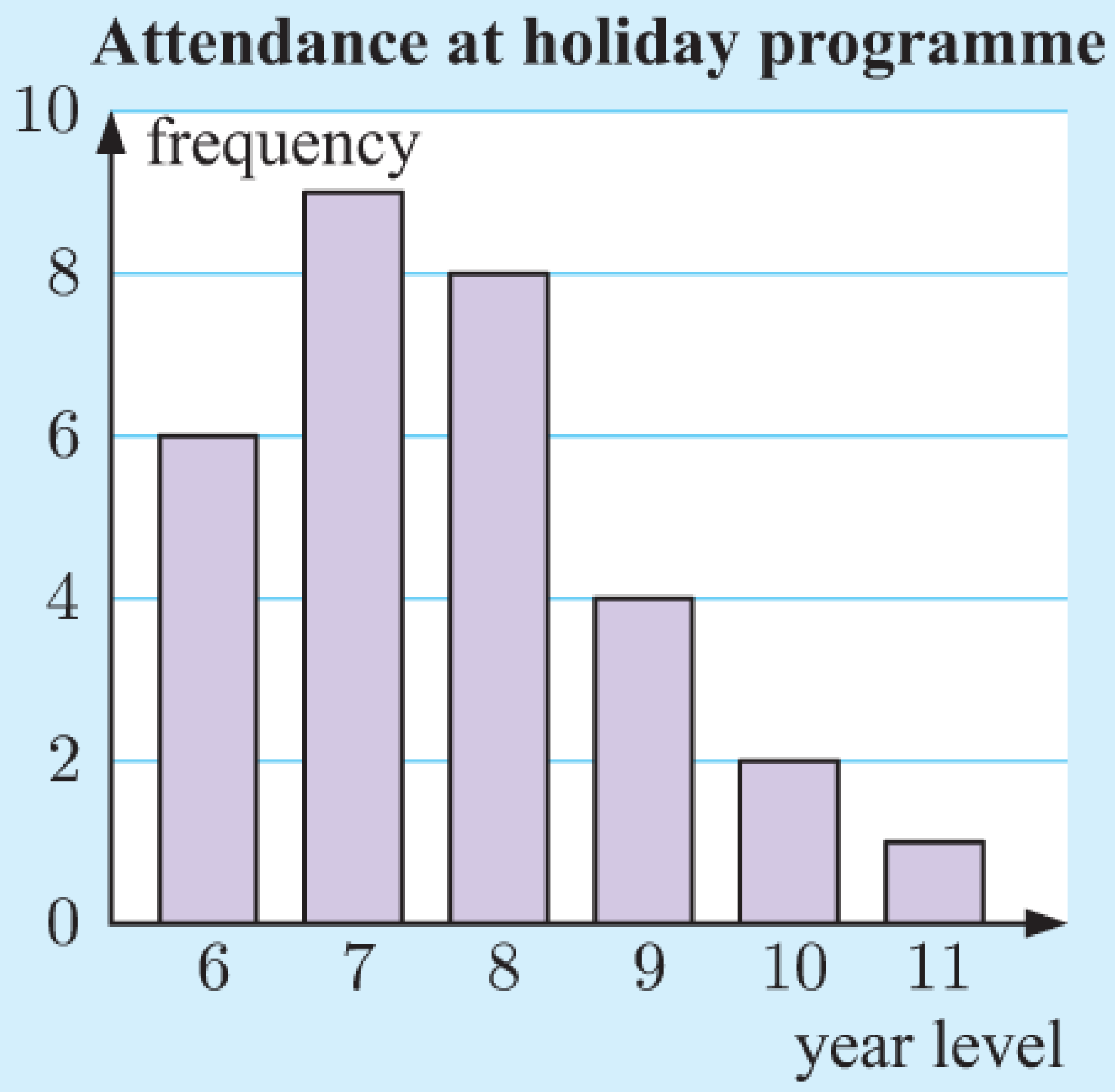
8 7 6 7 7 7 9 7 7 11 8 10 8 8 9
10 7 7 8 8 8 8 7 6 6 6 6 9 6 9

a Record this information in a frequency table. Include a column for relative frequency.
b Construct a column graph to display the data.
c What is the modal year level of the children?
d Describe the shape of the distribution. Are there any outliers?
e What percentage of the children were in Year 8 or below?
f What percentage of the children were above Year 9?

a

Year level	*Tally*	*Frequency*	*Relative frequency*
6	𝍸 |	6	0.2
7	𝍸 ||||	9	0.3
8	𝍸 |||	8	≈ 0.267
9	||||	4	≈ 0.133
10	||	2	≈ 0.067
11	|	1	≈ 0.033
	Total	30	

b


Attendance at holiday programme

c The modal year level is Year 7.

d The distribution of children's year levels is positively skewed. There are no outliers.

e $\dfrac{6+9+8}{30} \times 100\% \approx 76.7\%$ of the children were in Year 8 or below.

or the sum of the relative frequencies is
$0.2 + 0.3 + 0.267 = 0.767$
$\therefore$ 76.7% were in Year 8 or below.

f $\dfrac{2+1}{30} \times 100\% = 10\%$ of the children were above Year 9.

or $0.067 + 0.033 = 0.1$ $\therefore$ 10% were above Year 9.

Due to rounding, the relative frequencies will not always appear to add to *exactly* 1.

EXERCISE 12E

1 In the last football season, the Flames scored the following numbers of goals in each game:

2 0 1 4 0 1 2 1 1 0 3 1
3 0 1 1 6 2 1 3 1 2 0 2

- **a** What is the variable being considered here?
- **b** Explain why the data is discrete.
- **c** Construct a frequency table to organise the data. Include a column for relative frequency.
- **d** Draw a column graph to display the data.
- **e** What is the modal score for the team?
- **f** Describe the distribution of the data. Are there any outliers?
- **g** In what percentage of games did the Flames fail to score?

2 Prince Edward High School prides itself on the behaviour of its students. However, from time to time they misbehave and as a result are placed on detention. The studious school master records the number of students on detention each week throughout the year:

0 2 1 5 0 1 4 2 3 1 4 3 0 2 9 2 1 5 0 3
6 4 2 1 5 1 0 2 1 4 3 1 2 0 4 3 2 1 2 3

- **a** Construct a column graph to display the data.
- **b** What is the modal number of students on detention in a week?
- **c** Describe the distribution of the data, including the presence of outliers.
- **d** In what percentage of weeks were more than 4 students on detention?

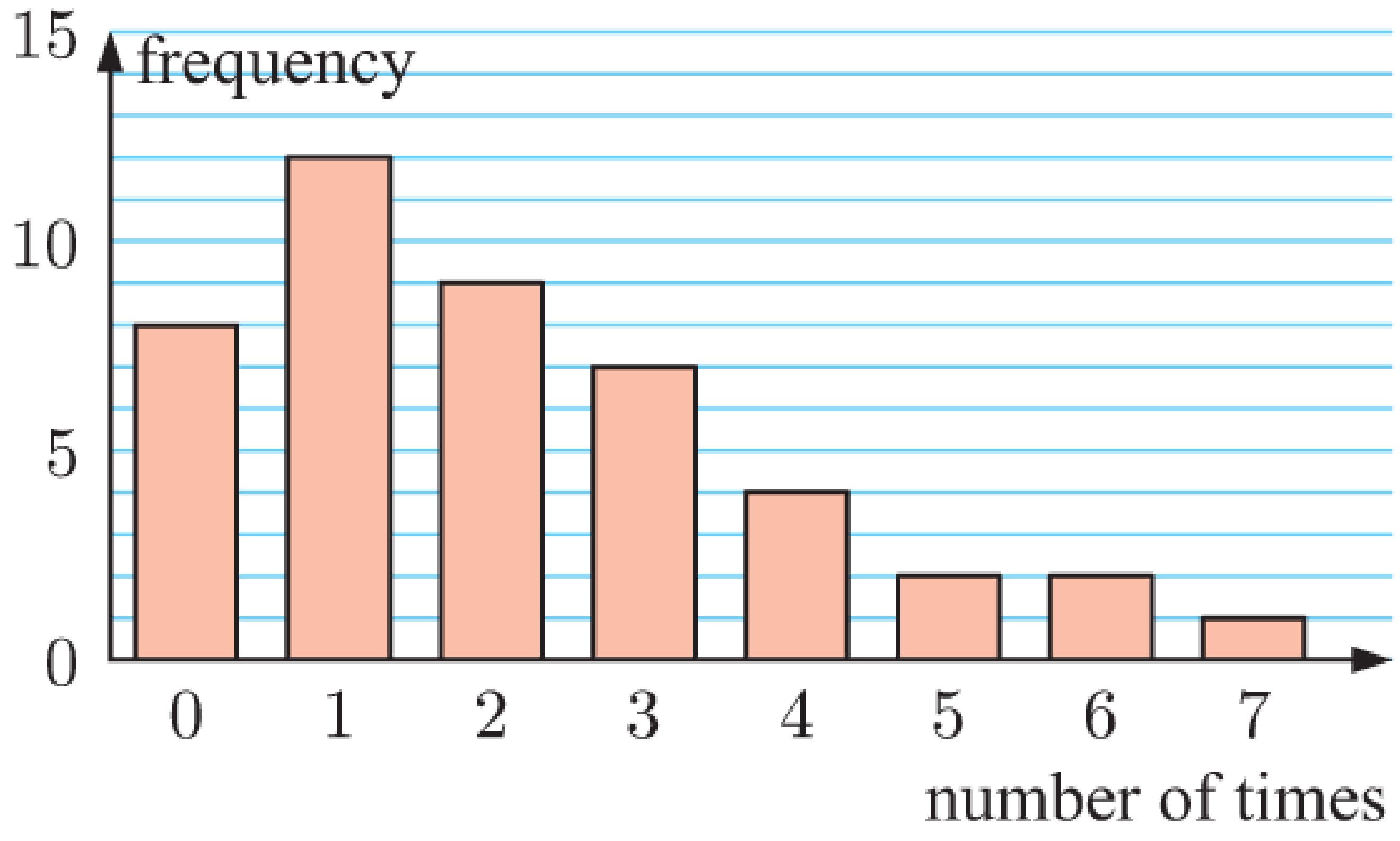
3 Each time Joan visits the cinema, she records the number of previews for other films which are shown before the feature. She has obtained these results:

2 4 3 1 3 2 3 4 2 5 2 3 4 3 6 5 4
3 3 6 3 4 6 4 3 1 4 2 5 4 3 5 4 5

- **a** Construct a frequency table to organise the data.
- **b** Draw a column graph to display the data.
- **c** Find the mode of the data.
- **d** Describe the distribution of the data. Are there any outliers?
- **e** On what percentage of occasions were at least 3 previews shown?

4 A random sample of people were asked "How many times did you eat out last week?" A column graph was used to display the results.

- **a** How many people were surveyed?
- **b** Find the mode of the data.
- **c** How many people surveyed did not eat out at all last week?
- **d** What percentage of people surveyed ate out more than three times last week?
- **e** Describe the distribution of the data.



F GROUPED DISCRETE DATA

A local kindergarten is concerned about the number of vehicles passing by between 8:45 am and 9:00 am. Over 30 consecutive weekdays they recorded data:

27, 30, 17, 13, 46, 23, 40, 28, 38, 24, 23, 22, 18, 29, 16,
35, 24, 18, 24, 44, 32, 52, 31, 39, 32, 9, 41, 38, 24, 32

In situations like this there are many different data values with very low frequencies. This makes it difficult to study the distribution of the data. It is more meaningful to **group** the data into **class intervals** and then compare the frequencies of the classes.

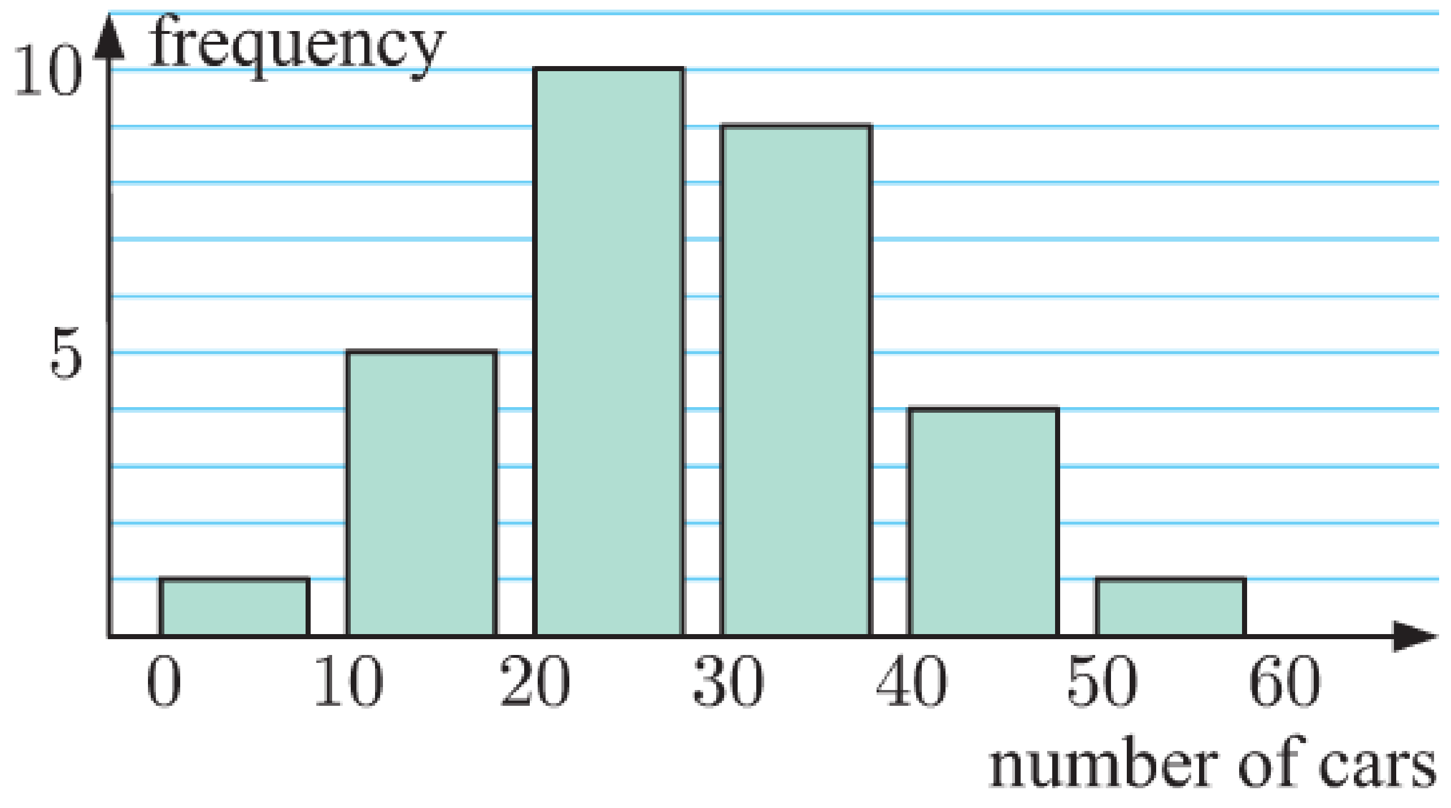
For the data given we use class intervals of width 10. The frequency table for the grouped data is shown alongside.

The **modal class**, or class with the highest frequency, is from 20 to 29 cars.

Number of cars	*Tally*	*Frequency*
0 to 9	|	1
10 to 19	卌	5
20 to 29	卌 卌	10
30 to 39	卌 ||||	9
40 to 49	||||	4
50 to 59	|	1
	Total	30

We construct **column graphs** for grouped discrete data in the same way as for simple data.

Vehicles passing kindergarten between 8:45 am and 9:00 am



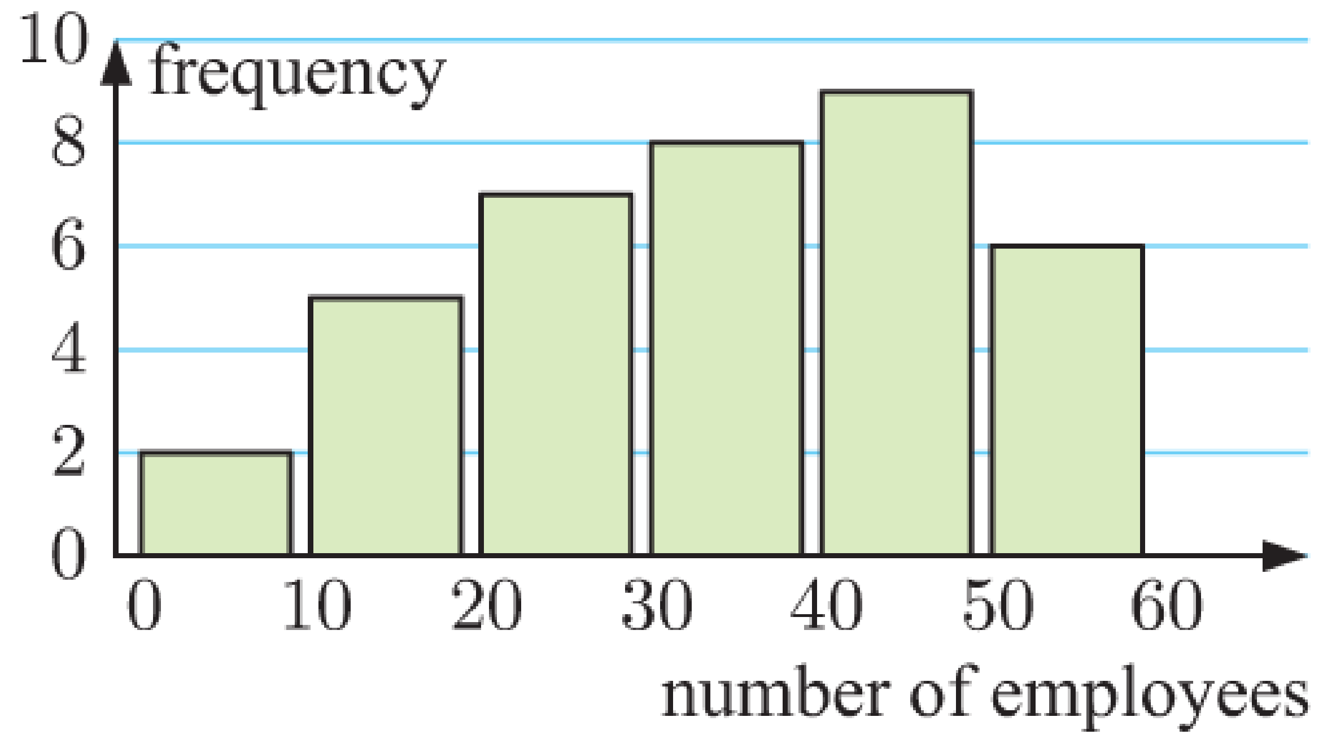
DISCUSSION

- If we are given a set of raw data, how can we efficiently find the lowest and highest data values?
- If the data values are grouped in classes on a frequency table or column graph, do we still know what the lowest and highest values are?

EXERCISE 12F

1 A selection of businesses were asked how many employees they had. The results are displayed on this column graph.

Number of employees



a How many businesses were surveyed?

b Find the modal class.

c Describe the distribution of the data.

d What percentage of businesses surveyed had less than 30 employees?

e Can you determine the highest number of employees a business had?

2 Arthur catches the train to school from a suburban train station. Over the course of 30 days he counts the number of people waiting at the station when the train arrives.

17	25	32	19	45	30	22	15	38	8
21	29	37	25	42	35	19	31	26	7
22	11	27	44	24	22	32	18	40	29

- **a** Construct a tally and frequency table for this data using class intervals 0 - 9, 10 - 19, ..., 40 - 49.
- **b** On how many days were there less than 10 people at the station?
- **c** On what percentage of days were there at least 30 people at the station?
- **d** Draw a column graph to display the data.
- **e** Find the modal class of the data.

3 A city council is interested in the number of houses in each street of a suburb, because it intends to place collection bins for unwanted clothing. The data they find is:

42	15	20	6	34	19	8	5	11	38	56	23	24	24
35	47	22	36	39	18	14	44	25	6	34	35	28	12
27	32	36	34	30	40	32	12	17	6	37	32		

- **a** Construct a frequency table for this data using class intervals 0 - 9, 10 - 19, ..., 50 - 59.
- **b** Hence draw a column graph to display the data.
- **c** Write down the modal class.
- **d** What percentage of the streets contain at least 20 houses?

G CONTINUOUS DATA

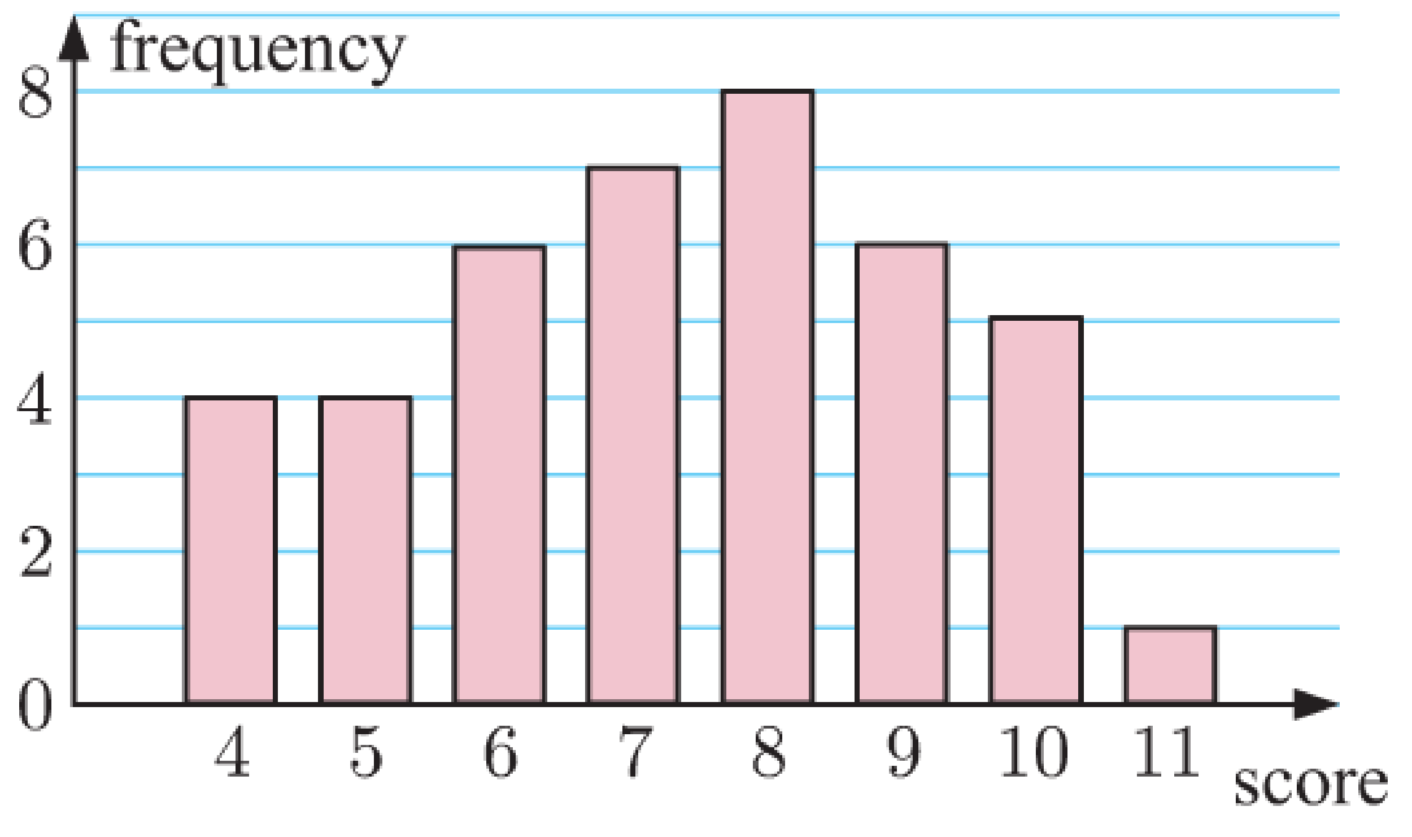
When we measure data that is **continuous**, we cannot write down an exact value. Instead we write down an approximation which is only as accurate as the measuring device.

Since no two data values will be *exactly* the same, it does not make sense to talk about the frequency of a particular value. Instead we group the data into **class intervals** of **equal width**. We can then talk about the frequency of each class interval.

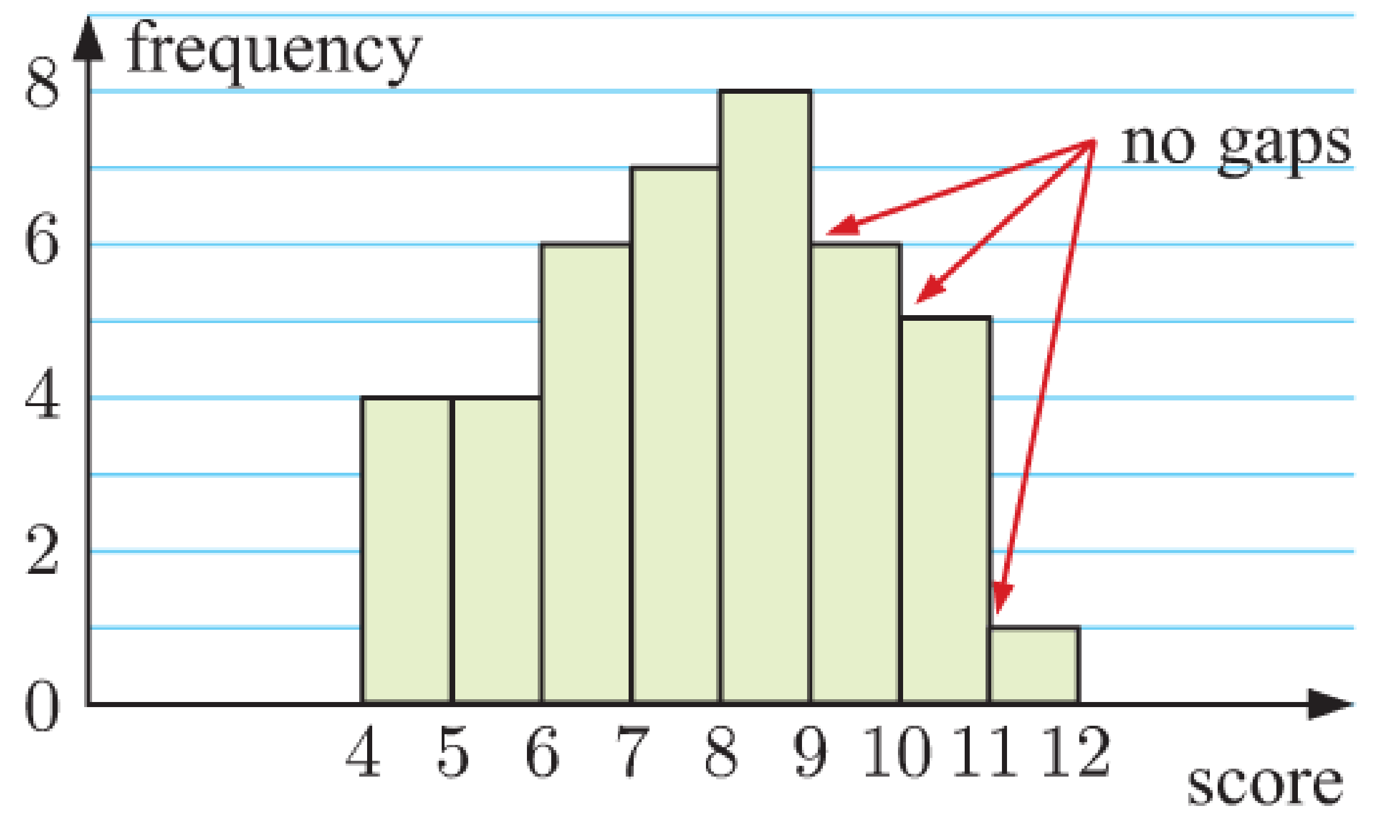
A special type of graph called a **frequency histogram** or just **histogram** is used to display continuous data. This is similar to a column graph, but the "columns" are joined together and the values at the edges of each column indicate the boundaries of that class interval.

The **modal class**, or class of values that appears most often, is easy to identify from a frequency histogram.

Column graph



Frequency histogram



INVESTIGATION CHOOSING CLASS INTERVALS

When dividing data values into intervals, the choice of how many intervals to use is important. It affects not only the width of each class interval, but how much detail of the distribution is seen on the histogram.

What to do:

1 Click on the icon to access a demonstration which draws histograms of data sets with different sizes and distributions.

DEMO

- **a** Select the symmetrical data set with $n = 1000$ data values. Use the slider to vary the number of intervals used in the histogram.
  - **i** Comment on what happens to the *shape* of the histogram.
  - **ii** Are there features of the data that can only be seen when there are many class intervals?
  - **iii** When there are many class intervals, is the frequency axis necessarily useful?
- **b** Repeat your investigation in **a** for other values of n. Try $n = 100$, $10\,000$, and $100\,000$. Record your observations.
- **c** For each value $n = 100$, 1000, $10\,000$, and $100\,000$, try using $\approx \sqrt{n}$ class intervals. Discuss whether you think this is an appropriate number.

2 Experiment with the other distribution types. If the distribution is not symmetric, do you need more or fewer class intervals?

Example 5

Self Tutor

A sample of 20 juvenile lobsters was randomly selected from a tank containing several hundred. The length of each lobster is recorded in cm alongside.

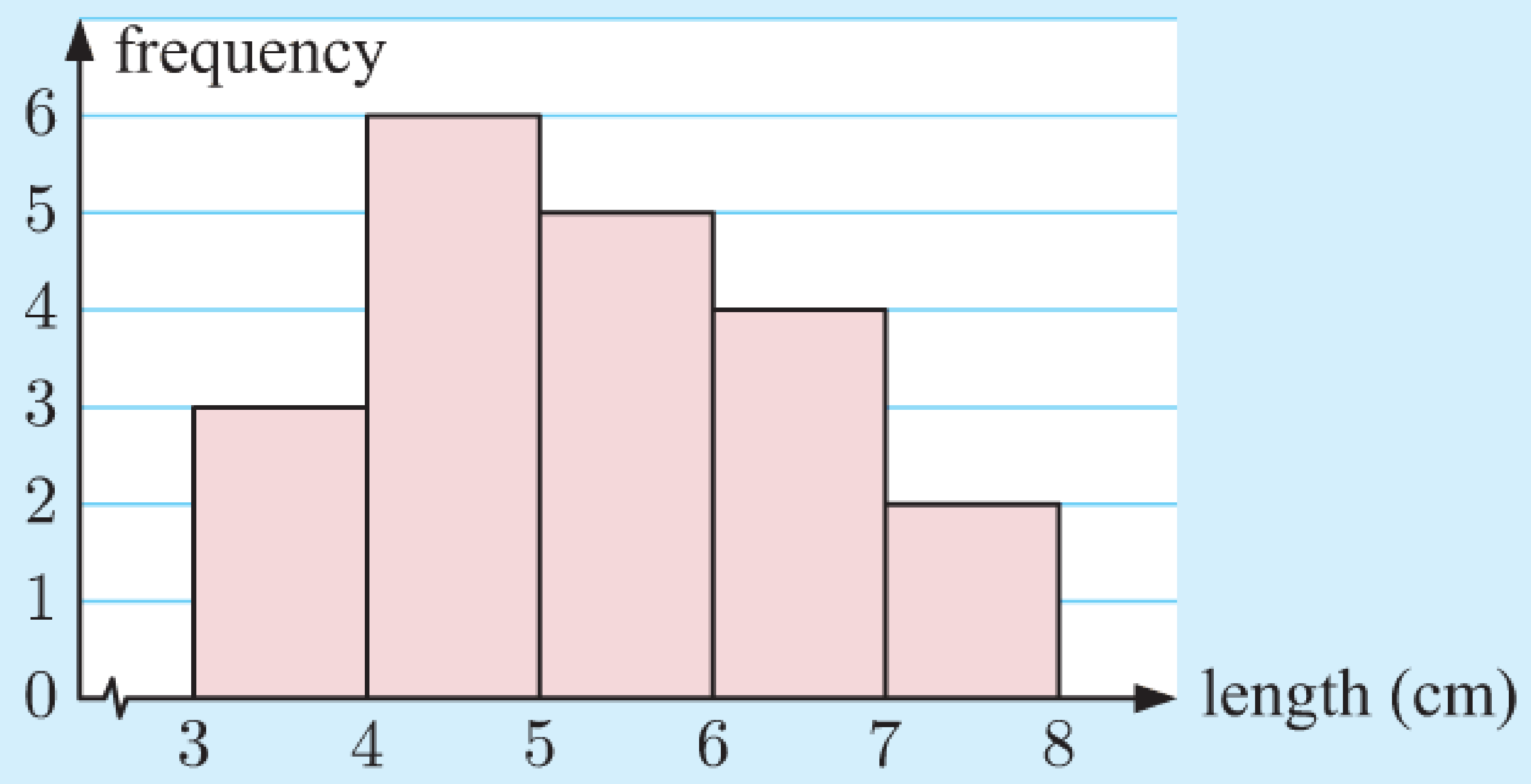
4.9	5.6	7.2	6.7	3.1
4.6	6.0	5.0	3.7	7.3
6.0	5.4	4.2	6.6	4.7
5.8	4.4	3.6	4.2	5.4

- **a** Organise the data using a frequency table, and hence graph the data.
- **b** State the modal class and explain what this means.
- **c** Describe the distribution of the data.

a The variable *length of a lobster* is continuous, even though lengths have been rounded to the nearest mm. The shortest length is 3.1 cm and the longest is 7.3 cm, so we will use class intervals of width 1 cm.

Length (l cm)	*Frequency*
$3 \leqslant l < 4$	3
$4 \leqslant l < 5$	6
$5 \leqslant l < 6$	5
$6 \leqslant l < 7$	4
$7 \leqslant l < 8$	2

Lengths of lobsters



b The modal class $4 \leqslant l < 5$ occurs most frequently. More lobsters have lengths in this interval than in any other interval.

c The distribution is positively skewed with no outliers.

EXERCISE 12G

1 A frequency table for the heights of a volleyball squad is given alongside.

Height (H cm)	*Frequency*
$170 \leqslant H < 175$	1
$175 \leqslant H < 180$	8
$180 \leqslant H < 185$	9
$185 \leqslant H < 190$	11
$190 \leqslant H < 195$	9
$195 \leqslant H < 200$	3
$200 \leqslant H < 205$	3

- **a** Explain why *height* is a continuous variable.
- **b** Construct a frequency histogram for the data. Carefully mark and label the axes, and include a heading for the graph.
- **c** What is the modal class? Explain what this means.
- **d** Describe the distribution of the data.

2 A school has conducted a survey of 60 students to investigate the time it takes for them to travel to school. The following data gives their travel times to the nearest minute.

12	15	16	8	10	17	25	34	42	18	24	18	45	33	38
45	40	3	20	12	10	10	27	16	37	45	15	16	26	32
35	8	14	18	15	27	19	32	6	12	14	20	10	16	14
28	31	21	25	8	32	46	14	15	20	18	8	10	25	22

- **a** Is travel time a discrete or continuous variable?
- **b** Construct a frequency table for the data using class intervals $0 \leqslant t < 10$, $10 \leqslant t < 20$, ..., $40 \leqslant t < 50$.
- **c** Hence draw a histogram to display the data.
- **d** Describe the distribution of the data.
- **e** What is the modal travelling time?

3 A group of 25 junior athletes participated in a javelin competition. They achieved the following distances in metres:

17.6	25.7	21.3	30.9	13.0	31.6	22.3	28.3	7.4
38.4	19.1	24.0	40.0	16.2	42.9	31.9	28.1	41.8
13.6	27.4	33.7	9.2	23.3	39.8	25.1		

- **a** Choose suitable class intervals to group the data.
- **b** Organise the data in a frequency table.
- **c** Draw a frequency histogram to display the data.
- **d** Find the modal class.
- **e** What percentage of athletes threw the javelin 30 m or further?

4 A horticulturalist takes a random sample of six month old seedlings from a nursery and measures their heights. The results are shown in the table.

Height (h mm)	*Frequency*
$300 \leqslant h < 325$	12
$325 \leqslant h < 350$	18
$350 \leqslant h < 375$	42
$375 \leqslant h < 400$	28
$400 \leqslant h < 425$	14
$425 \leqslant h < 450$	6

- **a** Display the data on a frequency histogram.
- **b** How many of the seedlings are 400 mm or higher?
- **c** What percentage of the seedlings are between 350 mm and 400 mm high?
- **d** In total there are 1462 seedlings in the nursery. Estimate the number of seedlings which measure:
  - **i** less than 400 mm
  - **ii** between 375 and 425 mm.

5 The weights, in grams, of 50 laboratory rats are given below.

261	133	173	295	265	142	140	271	185	251
166	100	292	107	201	234	239	159	153	263
195	151	156	117	144	189	234	171	233	182
165	122	281	149	152	289	168	260	256	156
239	203	101	268	241	217	254	240	214	221

- **a** Choose suitable class intervals to group the data.
- **b** Organise the data in a frequency table.
- **c** Draw a frequency histogram to display the data.
- **d** What percentage of the rats weigh less than 200 grams?

REVIEW SET 12A

1 Andrew is interested in the cultural background of the students at his school. He puts together a survey which he hands out to students in his Italian class.

- **a** Explain why Andrew's sample may be biased.
- **b** Suggest an alternative sampling method that Andrew can use so that his results will be more representative of his population of interest.

2 A golf club has 1800 members with ages shown alongside. A member survey is to be undertaken to determine the proportion of members who are in favour of changes to dress regulations.

- **a** Explain why the golf club would not question all members on the proposed changes to dress regulations.
- **b** If a sample size of 350 is used, how many of each age group will be surveyed?

Age range	*Members*
under 18	257
18 - 39	421
40 - 54	632
55 - 70	356
over 70	134

3 Classify each variable as categorical, discrete, or continuous:

- **a** the number of pages in a daily newspaper
- **b** the maximum daily temperature in a city
- **c** the manufacturer of a television
- **d** a person's favourite flavour of ice cream
- **e** the position taken by a player on a lacrosse field
- **f** the time it takes to run one kilometre
- **g** the length of a person's feet
- **h** a person's shoe size
- **i** the cost of a bicycle.

4 On a Saturday night, a team of police officers set up a drug and alcohol testing station to test drivers leaving the centre of town on a major road.

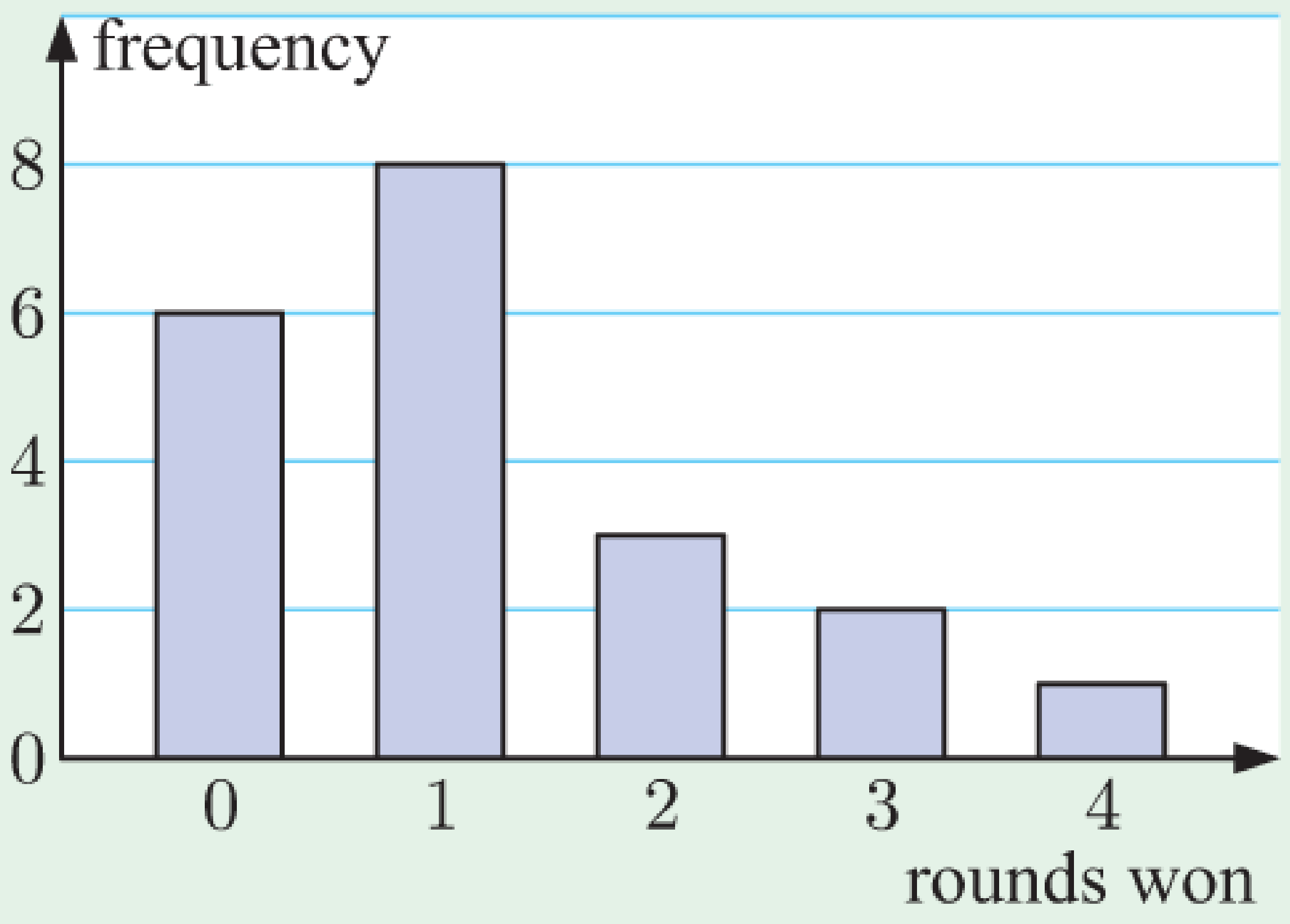
- **a** What type of sampling method is this?
- **b** Do you think the sample will be biased? If so, do you think it is *sensible* for it to be biased? Explain your answer.

5 Consider the question "Are you healthy?"

- **a** List ways in which the question can be interpreted. Include any possible misinterpretations.
- **b** Rewrite the question so it is more specific.

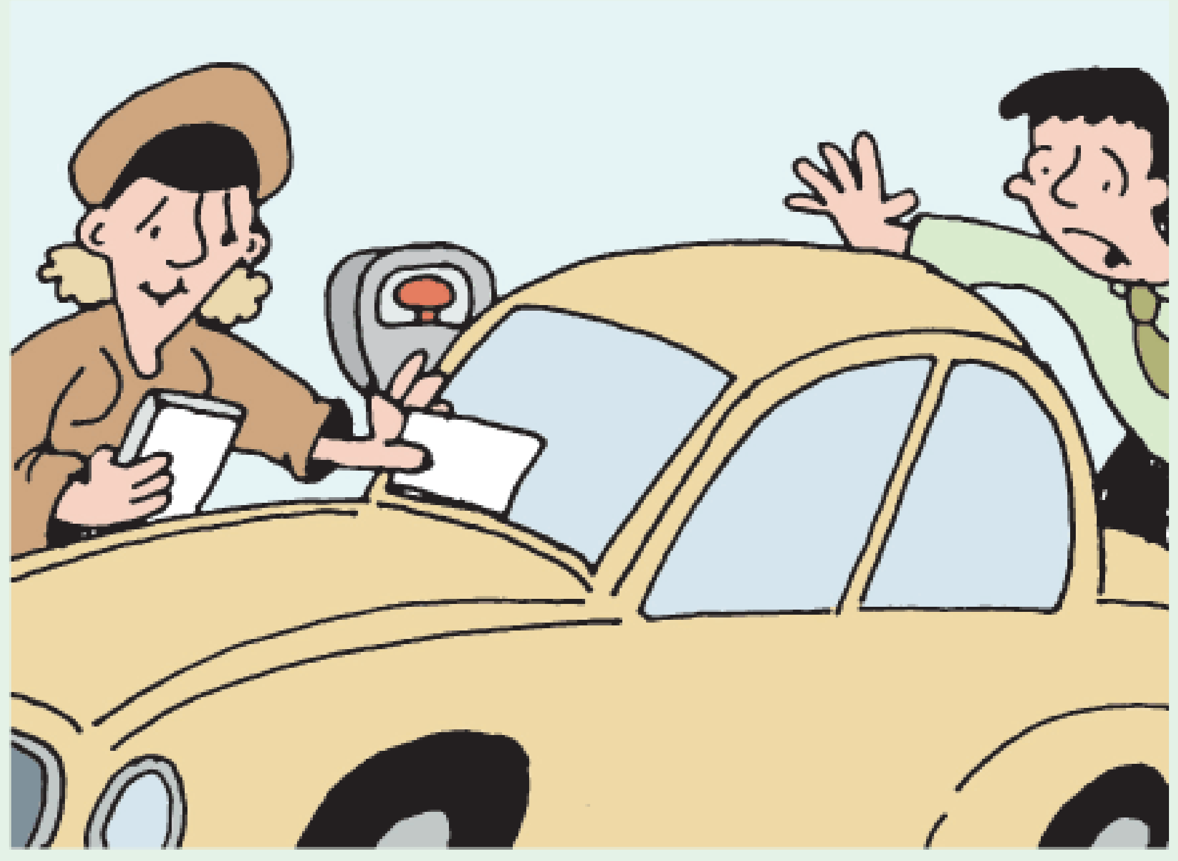
6 This column graph shows the number of rounds a player wins in a judo tournament.

- **a** Is the *number of rounds won* a discrete or continuous variable?
- **b** State the modal number of rounds won.
- **c** Describe the distribution of the data.



7 A parking inspector recorded the number of parking tickets she issued each day for four weeks. Her results are shown below:

2	4	2	3	5	0	3
3	2	4	6	3	3	3
4	1	3	3	4	5	2
3	1	3	1	2	5	4



- **a** Construct a tally and frequency table to organise the data.
- **b** Draw a column graph to display the data.
- **c** Describe the distribution of the data. Are there any outliers?

8 The data supplied below is the diameter (in cm) of a number of bacteria colonies as measured by a microbiologist 12 hours after seeding.

0.4	2.1	3.4	3.9	4.7	3.7	0.8	3.6	4.1	4.9	2.5	3.1	1.5	2.6	4.0
1.3	3.5	0.9	1.5	4.2	3.5	2.1	3.0	1.7	3.6	2.8	3.7	2.8	3.2	3.3

- **a** Is the *diameter of bacteria colonies* a discrete or continuous variable?
- **b** Organise the data into 5 class intervals of equal width.
- **c** Draw a histogram to display the data.
- **d** State the modal class.
- **e** Describe the distribution.

REVIEW SET 12B

1 Classify the following variables as discrete or continuous:

- **a** the number of pages in a book
- **b** the distance travelled by hikers in one day
- **c** the attendance figures for a music festival.

2 A sales promoter decides to visit 10 houses in a street and offer special discounts on a new window treatment. The street has 100 houses numbered from 1 to 100. The sales promoter selects a random number between 1 and 10 inclusive and calls on the house with that street number. After this the promoter calls on every tenth house.

- **a** What sampling technique is used by the sales promoter?
- **b** Explain why every house in the street has an equal chance of being visited.
- **c** Explain why this is not a simple random sample.

3 Petra emailed a questionnaire to her teacher colleagues about general student behaviour in their classes.

- **a** Explain why Petra's questionnaire may produce a high non-response error.
- **b** Of the 20 teachers who were emailed the questionnaire, 10 responded. Petra decides to use these 10 responses as her sample. Explain why Petra is likely to encounter a coverage error.

4 Rewrite the question "How did you learn about our services?" as a structured question. Include the options that respondents should choose from.

5 Consider the question "Were you a naughty child?"

- **a** Identify any problems with how the question is worded.
- **b** Rewrite the question to address these problems.

6 The winning margins in 100 rugby games were recorded as follows:

Margin (points)	1 - 10	11 - 20	21 - 30	31 - 40	41 - 50
Frequency	13	35	27	18	7

Draw a column graph to present this information.

7 The Dzungarian or Przewalski's horse is an endangered species native to the Mongolian steppes. The adult horses in an established breeding program are weighed in kg. The results are:

274	298	302	316	296	279	325
303	286	318	286	325	306	303
261	315	326	293	281		

- **a** Explain why the *mass of a horse*, m kg, is a continuous quantitative variable.
- **b** Organise the data in a frequency table using the intervals $260 \leqslant m < 270$, $270 \leqslant m < 280$, ..., $320 \leqslant m < 330$.
- **c** Identify the modal class.
- **d** Draw a histogram to display the data.
- **e** Hence describe the distribution of the data.

8 The data below are the lengths, in metres, of yachts competing in a sailing race.

14.7	14.1	21.6	16.2	15.7	12.8	10.1	13.9	14.4	13.0
11.7	14.6	17.2	13.4	12.1	11.3	13.1	21.6	23.5	16.4
14.4	15.8	12.6	19.7	18.0	16.2	27.4	21.9	14.4	12.4

- **a** Is the data discrete or continuous?
- **b** Organise the data using a frequency table.
- **c** Draw an appropriate graph to display the data.
- **d** Describe the distribution of the data.

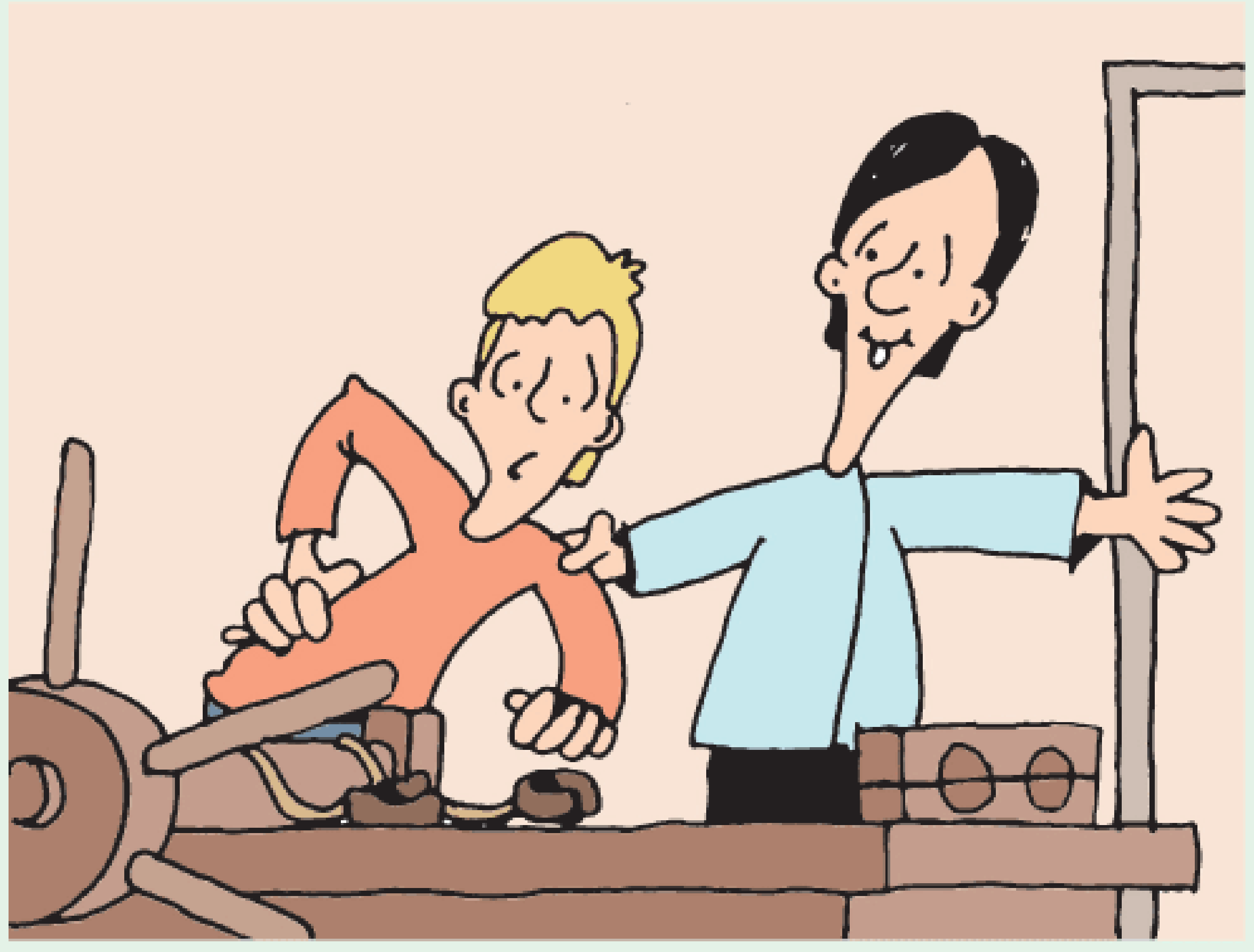
Chapter 13

Statistics

Contents:

OPENING PROBLEM

Nick believes he has devised a series of stretches which can help relieve back pain. He invites people with back pain to perform the stretches for several weeks.



The participants rate their level of back pain on a scale of 1 to 10 (10 being the greatest) before and after the experiment:

Before:	7	9	5	6	9	7	10	6	8	9
	8	7	9	8	10	4	8	6	7	8
After:	4	7	6	3	8	5	9	4	5	8
	7	6	5	4	7	2	5	3	4	6

Things to think about:

- **a** What statistics can we calculate to measure the *centre* of each data set?
- **b** How can we use a graph to make a visual comparison between the data sets?
- **c** Do you believe that Nick's stretching exercises reduce back pain? Explain your answer.

In the previous Chapter, we looked at how data can be collected, organised, and displayed. By looking at appropriate graphs, we can get an idea of a data set's **distribution**.

We can get a better understanding of a data set if we can locate its **middle** or **centre**, and measure its **spread** or dispersion. Knowing one of these without the other is often of little use.

However, whatever statistics we calculate, it is essential to view and interpret them in the context of what we are studying.

A MEASURING THE CENTRE OF DATA

There are three statistics that are used to measure the **centre** of a data set. These are the **mode**, the **mean**, and the **median**.

THE MODE

In the previous Chapter we saw that:

- For discrete data, the **mode** is the most frequently occurring value in the data set.
- For continuous data, we cannot talk about a mode in this way because no two data values will be *exactly* equal. Instead we talk about a **modal class**, which is the class or group that has the highest frequency.

If a data set has two values which both occur most frequently, we say it is **bimodal**.

If a data set has three or more values which all occur most frequently, the mode is not an appropriate measure of centre to use.

THE MEAN

The **mean** of a data set is the statistical name for its arithmetic average.

For the data set $\{x_1, x_2, x_3, ..., x_n\}$,

$$\textbf{mean} = \frac{\textbf{sum of all data values}}{\textbf{the number of data values}} = \frac{x_1 + x_2 + x_3 + ... + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

We use $\overline{x}$ to represent the mean of a **sample**, and μ to represent the mean of a **population**.

In many cases we do not have data from all of the members of a population, so the exact value of μ is unknown. Instead we collect data from a sample of the population, and use the mean of the sample $\overline{x}$ as an approximation for μ.

μ is the Greek letter "mu" which we pronounce as "mew".

THE MEDIAN

The **median** is the *middle value* of an ordered data set.

An ordered data set is obtained by listing the data from the smallest to the largest value.

The median splits the data in halves. Half of the data values are less than or equal to the median, and half are greater than or equal to it.

For example, if the median mark for a test is 73% then you know that half the class scored less than or equal to 73% and half scored greater than or equal to 73%.

For an **odd number** of data values, the median is one of the original data values.

For an **even number** of data values, the median is the average of the two middle values, and hence may not be in the original data set.

If there are n data values listed in order from smallest to largest, the median is the $\left(\frac{n+1}{2}\right)$th data value.

For example:

DEMO

If $n = 13$, $\frac{n+1}{2} = 7$, so the median is the 7th ordered data value.

If $n = 14$, $\frac{n+1}{2} = 7.5$, so the median is the average of the 7th and 8th ordered data values.

Example 1 **Self Tutor**

The numbers of faulty products returned to an electrical goods store each day over a 21 day period are:

3 4 4 9 8 8 6 4 7 9 1 3 5 3 5 9 8 6 3 7 1

a For this data set, find:

i the mean **ii** the median **iii** the mode.

b On the 22nd day there were 9 faulty products returned. How does this affect the measures of the centre?

a **i** $$\text{mean} = \frac{3+4+4+ .... +3+7+1}{21}$$

← sum of all the data values

← 21 data values

$$= \tfrac{113}{21}$$

$$\approx 5.38 \text{ faulty products}$$

ii As $n = 21$, $\dfrac{n+1}{2} = 11$

The ordered data set is: ~~1 1 3 3 3 3 4 4 4 5~~ $\boxed{5}$ ~~6 6 7 7 8 8 8 9 9 9~~

↑ 11th value

$\therefore$ median $= 5$ faulty products

iii 3 is the data value which occurs most often, so the mode is 3 faulty products.

b We expect the mean to increase since the new data value is greater than the old mean.

In fact, the new mean $= \dfrac{113+9}{22} = \dfrac{122}{22} \approx 5.55$ faulty products.

Since $n = 22$, $\dfrac{n+1}{2} = 11.5$

The new ordered data set is:

~~1 1 3 3 3 3 4 4 4 5~~ $\underbrace{\boxed{5\ \ 6}}_{\text{two middle data values}}$ ~~6 7 7 8 8 8 9 9 9 9~~

$\therefore$ the new median $= \dfrac{5+6}{2} = 5.5$ faulty products.

The new data set has two modes which are 3 and 9 faulty products.

You can use your **graphics calculator** or the **statistics package** to find measures of centre.

GRAPHICS CALCULATOR INSTRUCTIONS

STATISTICS PACKAGE

EXERCISE 13A

1 Phil kept a record of the number of cups of coffee he drank each day for 15 days:

2, 3, 1, 1, 0, 0, 4, 3, 0, 1, 2, 3, 2, 1, 4

Without using technology, find the **a** mode **b** median **c** mean of the data.

2 For each data set, find the: **i** mean **ii** median **iii** mode.

a 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 6, 7, 7, 8, 8, 8, 9, 9

b 10, 12, 12, 15, 15, 16, 16, 17, 18, 18, 18, 18, 19, 20, 21

c 22.4, 24.6, 21.8, 26.4, 24.9, 25.0, 23.5, 26.1, 25.3, 29.5, 23.5

Check your answers using technology.

3 The sum of 7 scores is 63. What is their mean?

4 The scores obtained by two ten-pin bowlers over a 10 game series are:

Gordon:	160,	175,	142,	137,	151,	144,	169,	182,	175,	155
Ruth:	157,	181,	164,	142,	195,	188,	150,	147,	168,	148

Who had the higher mean score?

5 Consider the two data sets:

Data set A: 3, 4, 4, 5, 6, 6, 7, 7, 7, 8, 8, 9, 10
Data set B: 3, 4, 4, 5, 6, 6, 7, 7, 7, 8, 8, 9, 15

a Find the mean of both data set A and data set B.
b Find the median of both data set A and data set B.
c Comment on your answers to **a** and **b**.

6 An Indian dessert shop keeps a record of how many motichoor ladoo and malai jamun they sell each day for a month:

Motichoor ladoo

62	76	55	65	49	78	71	82
79	47	60	72	58	82	76	67
50	61	70	85	77	69	48	74
63	56	81	75	63	74	54	

Malai jamun

37	52	71	59	63	47	56	68
43	67	38	73	54	55	61	49
50	48	53	39	45	60	46	51
38	57	41	72	50	44	76	

a Find the:
  i mean number of motichoor ladoo and malai jamun sold
  ii median number of motichoor ladoo and malai jamun sold.
b Which item was more popular? Explain your answer.

7 A bus and tram travel the same route many times during the day. The drivers counted the number of passengers on each trip one day, as listed below.

Bus

30	43	40	53	70	50	63
41	38	21	28	23	43	48
20	26	35	48	41	33	

Tram

58	68	43	45	70	79
38	23	30	22	63	73
25	35	60	53		

a Use technology to calculate the mean and median number of passengers for both the *Bus* and *Tram* data.
b Which method of transport do you think is more popular? Explain your answer.

8 A basketball team scored 43, 55, 41, and 37 points in their first four matches.

a Find the mean number of points scored for these four matches.
b What score does the team need to shoot in their next match to maintain the same mean score?
c The team scores only 25 points in the fifth match.
  i Will this increase or decrease their overall mean score? Explain your answer.
  ii Find the mean number of points scored for the five matches.

Example 2 ◀) **Self Tutor**

If 6 people have a mean mass of 53.7 kg, find their total mass.

$$\frac{\text{sum of masses}}{6} = 53.7 \text{ kg}$$

$$\therefore \quad \text{sum of masses} = 53.7 \times 6$$

$$\therefore \quad \text{the total mass} = 322.2 \text{ kg}$$

9 This year, the mean monthly sales for a clothing store have been €15 467. Calculate the total sales for the store for the year.

10 Given $\overline{x} = 11.6$ and $n = 10$, calculate $\sum_{i=1}^{10} x_i$.

11 Towards the end of a season, a netballer had played 14 matches and scored an average of 16.5 goals per game. In the final two matches of the season she scored 21 goals and 24 goals. Find the netballer's average for the whole season.

12 Find x if 5, 9, 11, 12, 13, 14, 17, and x have a mean of 12.

13 Find a if 3, 0, a, a, 4, a, 6, a, and 3 have a mean of 4.

14 Over the entire assessment period, Aruna averaged 35 out of a possible 40 marks for her Mathematics tests. However, when checking her files, she could only find 7 of the 8 tests. For these she scored 29, 36, 32, 38, 35, 34, and 39. How many marks out of 40 did she score for the eighth test?

15 A sample of 10 measurements has a mean of 15.7, and a sample of 20 measurements has a mean of 14.3. Find the mean of all 30 measurements.

16 The mean and median of a set of 9 measurements are both 12. Seven of the measurements are 7, 9, 11, 13, 14, 17, and 19. Find the other two measurements.

INVESTIGATION 1 EFFECTS OF OUTLIERS

We have seen that an **outlier** or **extreme value** is a value which is much greater than, or much less than, the other values.

Your task is to examine the effect of an outlier on the three measures of centre.

What to do:

1 Consider the set of data: 4, 5, 6, 6, 6, 7, 7, 8, 9, 10. Calculate:

a the mean **b** the mode **c** the median.

2 Suppose we introduce the extreme value 100 to the data, so the data set is now: 4, 5, 6, 6, 6, 7, 7, 8, 9, 10, 100. Calculate:

a the mean **b** the mode **c** the median.

3 Comment on the effect that the extreme value has on:

a the mean **b** the mode **c** the median.

4 Which of the three measures of centre is most affected by the inclusion of an outlier?

5 Discuss situations with your class when it would *not* be appropriate to use a particular measure of centre of a data set.

B CHOOSING THE APPROPRIATE MEASURE

The mean, mode, and median can all be used to indicate the centre of a set of numbers. The most appropriate measure will depend upon the type of data under consideration. When selecting which one to use for a given set of data, you should keep the following properties in mind.

Statistic	*Properties*
Mode	• gives the most usual value • only takes common values into account • not affected by extreme values
Mean	• commonly used and easy to understand • takes all values into account • affected by extreme values
Median	• gives the halfway point of the data • only takes middle values into account • not affected by extreme values

For example:

- A shoe store is investigating the sizes of shoes sold over one month. The mean shoe size is not useful to know, since it probably will not be an actual shoe size. However, the mode shows at a glance which size the store most commonly has to restock.
- On a particular day a computer shop makes sales of $900, $1250, $1000, $1700, $1140, $1100, $1495, $1250, $1090, and $1075. In this case the mode is meaningless, the median is $1120, and the mean is $1200. The mean is the best measure of centre as the salesman can use it to predict average profit.
- When looking at real estate prices, the mean is distorted by the few sales of very expensive houses. For a typical house buyer, the median will best indicate the price they should expect to pay in a particular area.

EXERCISE 13B

1 The selling prices of the last 10 houses sold in a certain district were as follows:

$346 400, $327 600, $411 000, $392 500, $456 400,
$332 400, $348 000, $329 500, $331 400, $362 500

a Calculate the mean and median selling prices. Comment on your results.

b Which measure would you use if you were:

i a vendor wanting to sell your house

ii looking to buy a house in the district?

2 The annual salaries of ten office workers are:

$33 000, $56 000, $33 000, $48 000, $34 000,
$33 000, $33 000, $48 000, $33 000, $42 000

a Find the mode, mean, and median salaries of this group.

b Explain why the mode is an unsatisfactory measure of the centre in this case.

c Is the median a satisfactory measure of the centre of this data set?

3 The following raw data is the daily rainfall, to the nearest millimetre, for a month:

3, 1, 0, 0, 0, 0, 0, 2, 0, 0, 3, 0, 0, 0, 7, 1, 1, 0, 3, 8, 0, 0, 0, 42, 21, 3, 0, 3, 1, 0, 0

- **a** Use technology to find the mean, median, and mode of the data.
- **b** Explain why the median is not the most suitable measure of centre for this set of data.
- **c** Explain why the mode is not the most suitable measure of centre for this set of data.
- **d** Identify the outliers in this data set.
- **e** The outliers are genuine pieces of data and not the result of recording errors. Should they be removed before calculating statistics?

4 Esmé runs a day-tour business in Amsterdam. She wants to offer a "family package" that includes the charges for two adults and their children. To investigate the number of children she should include in the package, she asks 30 randomly selected customers with children how many children they have. Their responses are:

2 2 2 3 4 1 1 2 1 1 1 2 2 3 4
1 4 4 2 3 1 1 1 2 1 1 2 2 3 2

- **a** Calculate the mean, median, and modal number of children per family.
- **b** Is the mode a useful statistic in this case?
- **c** Suggest how many children Esmé should include in the package, giving reasons for your answer.

THEORY OF KNOWLEDGE

We have seen that the mean, the median, and mode are all statistics that give an *indication* of a data set's centre. The actual things that they measure are quite different!

- The mode is the value with the highest frequency. It is a measure of centre in terms of *frequency*.
- The median divides the data into halves. It is a measure of centre in terms of *proportion*.
- The mean is the arithmetic average. It can be thought of as the "balancing point" of the data set's distribution.

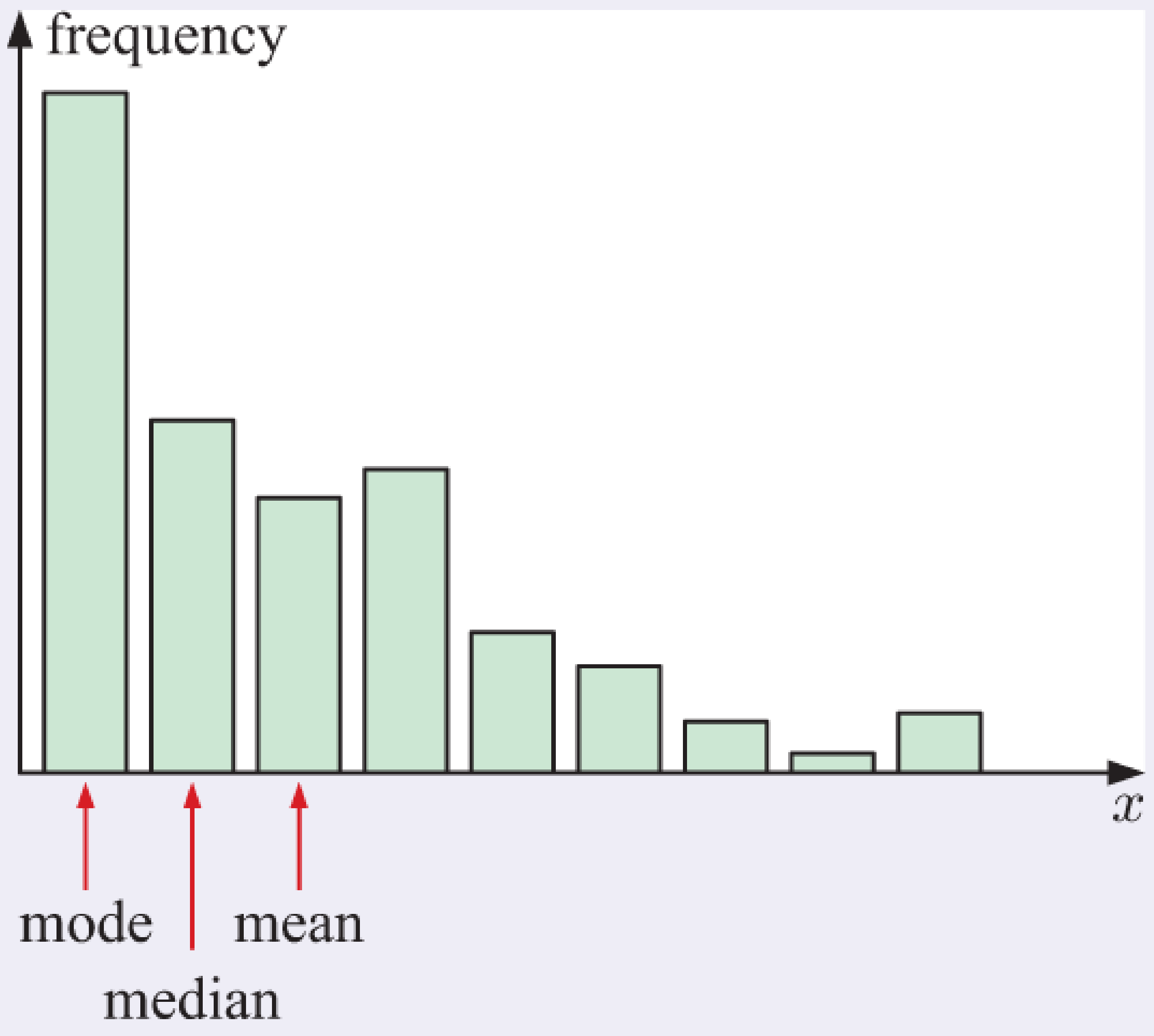
Other less commonly used measures for a data set $\{x_1, x_2, ..., x_n\}$ include the:

- **geometric mean** $= \sqrt[n]{x_1 \times x_2 \times .... \times x_n}$
- **mid-range value** $= \dfrac{\text{maximum} + \text{minimum}}{2}$.

We have seen that the most appropriate measure of centre will depend on what we are investigating. In a way, we change how the "centre" of a data set is defined to suit the purpose of our investigation.

1 When we have data that is heavily skewed, the mode will be on the far left or far right on its column graph.

- **a** Does the mode give an accurate indication of a data set's centre in these cases?
- **b** Is the relationship between mode and the centre of a data set purely coincidental?



2 For what kinds of data sets would the geometric mean and the mid-range value be useful?

3 How would *you* define the “centre” of a data set?

4 What makes a measure of centre objectively “better” than another measure?

5 Is there a *canonical* measure of centre, which means a measure of centre that is “better” than any other in all cases?

C USING FREQUENCY TABLES

We have already seen how to organise data into a **frequency table** like the one alongside.

The mode of the data is found directly from the *Frequency* column.

Value	*Frequency*
3	1
4	1
5	3
6	7
7	15
8	8
9	5

mode →

THE MEAN

Adding a “Product” column to the table helps to add the data values.

For example, the value 7 occurs 15 times, and these add to $15 \times 7 = 105$.

Value (x)	*Frequency* (f)	*Product* (xf)
3	1	$3 \times 1 = 3$
4	1	$4 \times 1 = 4$
5	3	$5 \times 3 = 15$
6	7	$6 \times 7 = 42$
7	15	$7 \times 15 = 105$
8	8	$8 \times 8 = 64$
9	5	$9 \times 5 = 45$
Total	$\sum f = 40$	$\sum xf = 278$

Since the mean $= \dfrac{\text{sum of all data values}}{\text{the number of data values}}$, we find

$$\overline{x} = \frac{x_1 f_1 + x_2 f_2 + x_3 f_3 + \dots + x_k f_k}{f_1 + f_2 + f_3 + \dots + f_k}$$

where k is the number of *different* values in the data.

$$\therefore \quad \overline{x} = \frac{\sum\limits_{j=1}^{k} x_j f_j}{\sum\limits_{j=1}^{k} f_j}$$

which we often abbreviate as $\dfrac{\sum xf}{\sum f}$.

In this case the mean $= \dfrac{278}{40} = 6.95$.

THE MEDIAN

Since $\frac{n+1}{2} = \frac{41}{2} = 20.5$, the median is the average of the 20th and 21st ordered data values.

In the table, the blue numbers show the accumulated frequency values, or **cumulative frequency**.

We can see that the 20th and 21st ordered data values are both 7s.

$\therefore$ the median $= \frac{7+7}{2} = 7$

Value	*Frequency*	*Cumulative frequency*
3	1	1 ← one number is 3
4	1	2 ← two numbers are 4 or less
5	3	5 ← five numbers are 5 or less
6	7	12 ← 12 numbers are 6 or less
7	15	27 ← 27 numbers are 7 or less
8	8	35 ← 35 numbers are 8 or less
9	5	40 ← all numbers are 9 or less
Total	40	

Example 3

Self Tutor

The table below shows the number of aces served by a sample of tennis players in their first sets of a tournament.

Number of aces	1	2	3	4	5	6
Frequency	4	11	18	13	7	2

Determine the: **a** mean **b** median **c** mode for this data.

Number of aces (x)	*Frequency* (f)	*Product* (xf)	*Cumulative frequency*
1	4	4	4
2	11	22	15
3	18	54	33
4	13	52	46
5	7	35	53
6	2	12	55
Total	$\sum f = 55$	$\sum xf = 179$	

a $\overline{x} = \frac{\sum xf}{\sum f}$

$= \frac{179}{55}$

≈ 3.25 aces

In this case $\frac{\sum xf}{\sum f}$ is short for $\frac{\sum_{j=1}^{6} x_j f_j}{\sum_{j=1}^{6} f_j}$.

b There are 55 data values, so $n = 55$. $\frac{n+1}{2} = 28$, so the median is the 28th ordered data value. From the cumulative frequency column, the 16th to 33rd ordered data values are 3 aces.

$\therefore$ the 28th ordered data value is 3 aces.

$\therefore$ the median is 3 aces.

c Looking down the frequency column, the highest frequency is 18. This corresponds to 3 aces, so the mode is 3 aces.

EXERCISE 13C

1 The table alongside shows the number of people in cars on a road.
Calculate the:

a mode **b** median **c** mean.

Check your answers using your graphics calculator.

GRAPHICS CALCULATOR INSTRUCTIONS

Number of people	*Frequency*
1	13
2	8
3	4
4	5
Total	30

2 The frequency table alongside shows the number of phone calls made in a day by 50 fifteen-year-olds.

a For this data set, find the:

i mean **ii** median **iii** mode.

b Construct a column graph for the data and show the position of the mean, median, and mode on the horizontal axis.

c Describe the distribution of the data.

d Why is the mean larger than the median?

e Which measure of centre would be the most suitable for this data set?

Number of phone calls	*Frequency*
0	5
1	8
2	13
3	8
4	6
5	3
6	3
7	2
8	1
11	1

3 Families at a school in Manchester were surveyed, and the number of children in each family was recorded. The results of the survey are shown alongside.

a Calculate the:

i mean **ii** mode **iii** median.

b The average British family has 2.075 children. How does this school compare to the national average?

c Describe the skewness of the data.

d How has the skewness of the data affected the measures of the centre of the data set?

Number of children	*Frequency*
1	5
2	28
3	15
4	8
5	2
6	1
Total	59

4 The column graph shows the weekly pocket money for a class of children.

a Construct a frequency table from the graph.

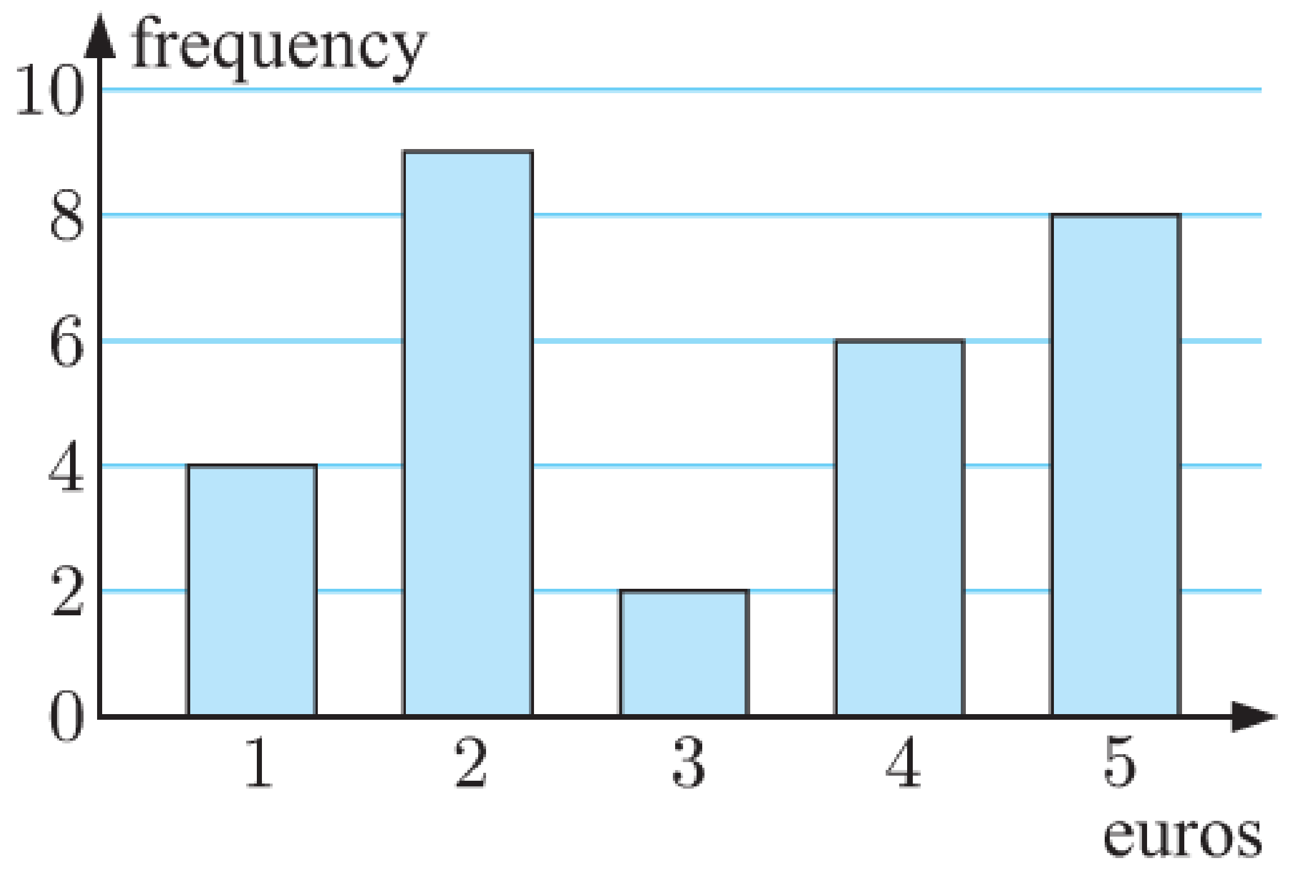
b Determine the total number of children in the class.

c Find the:

i mean **ii** median **iii** mode of the data.

d Which of the measures of centre can be found easily using the graph only?

Column graph of pocket money



5 Out of 31 measurements, 15 are below 10 cm and 12 are above 11 cm. Find the median if the other 4 measurements are 10.1 cm, 10.4 cm, 10.7 cm, and 10.9 cm.

6 In an office of 20 people there are only 4 salary levels paid:

$100 000 (1 person), $84 000 (3 people),
$70 000 (6 people), $56 000 (10 people)

a Calculate:

i the median salary **ii** the modal salary

iii the mean salary.

b Which measure of central tendency might be used by the boss who is against a pay rise for the other employees?

7 The table shows the test scores for a class of students. A pass is a score of 5 or more.

Score	2	3	4	5	6	7	8
Frequency	0	2	3	5	x	4	1

a Given that the mean score was 5.45, find x.

b Find the percentage of students who passed.

D GROUPED DATA

When information has been gathered in groups or classes, we use the **midpoint** or **mid-interval value** to represent all data values within each interval.

We are assuming that the data values within each class are evenly distributed throughout that interval. The mean calculated is an **approximation** of the actual value, and we cannot do better than this without knowing each individual data value.

INVESTIGATION 2 MID-INTERVAL VALUES

When mid-interval values are used to represent all data values within each interval, what effect will this have on estimating the mean of the grouped data?

This table summarises the marks out of 50 received by students in a Physics examination. The exact results for each student have been lost.

Marks	*Frequency*
0 - 9	2
10 - 19	31
20 - 29	73
30 - 39	85
40 - 49	28

What to do:

1 Suppose that all of the students scored the lowest possible result in their class interval, so 2 students scored 0, 31 students scored 10, and so on.
Calculate the mean of these results, and hence complete:
"The mean Physics examination mark must be *at least* ......"

2 Now suppose that all of the students scored the highest possible result in their class interval. Calculate the mean of these results, and hence complete:
"The mean Physics examination mark must be *at most* ......"

3 We now have two extreme values between which the actual mean must lie.
Now suppose that all of the students scored the mid-interval value in their class interval. We assume that 2 students scored 4.5, 31 students scored 14.5, and so on.

a Calculate the mean of these results.

b How does this result compare with lower and upper limits found in **1** and **2**?

c Copy and complete: "The mean Physics examination mark was approximately ......"

4 Discuss with your class how accurate you think an estimate of the mean using mid-interval values will be. How is this accuracy affected by the number and width of the class intervals?

Example 4 **Self Tutor**

The table below shows the ages of bus drivers. Estimate the mean age, to the nearest year.

Age (years)	21 - 25	26 - 30	31 - 35	36 - 40	41 - 45	46 - 50	51 - 55
Frequency	11	14	32	27	29	17	7

Age (years)	*Frequency* (f)	*Midpoint* (x)	xf
21 - 25	11	23	253
26 - 30	14	28	392
31 - 35	32	33	1056
36 - 40	27	38	1026
41 - 45	29	43	1247
46 - 50	17	48	816
51 - 55	7	53	371
Total	$\sum f = 137$		$\sum xf = 5161$

$$\overline{x} = \frac{\sum xf}{\sum f}$$

$$= \frac{5161}{137}$$

$$\approx 37.7$$

$\therefore$ the mean age of the drivers is about 38 years.

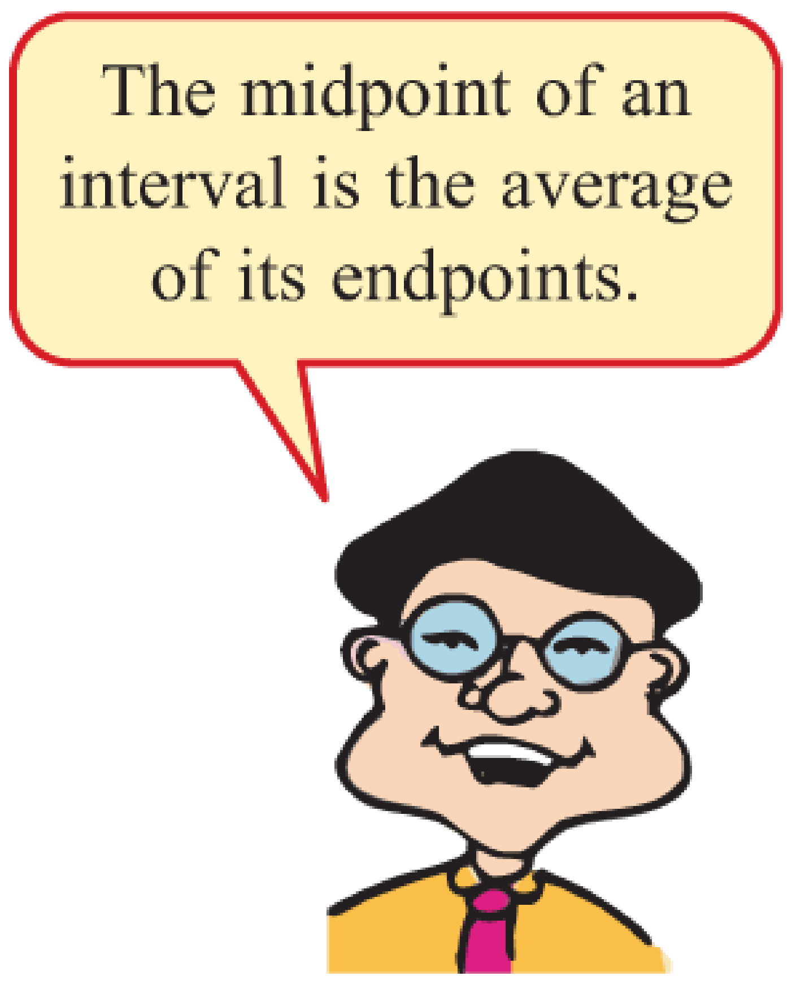
EXERCISE 13D

1 Simone recorded the lengths of her phone calls for one week. The results are shown in the table alongside.

a How many phone calls did she make during the week?

b Estimate the mean length of the calls.

Time (t min)	*Frequency*
$0 \leqslant t < 10$	17
$10 \leqslant t < 20$	10
$20 \leqslant t < 30$	9
$30 \leqslant t < 40$	4



2 50 students sat a Mathematics test. Estimate the mean score given these results:

Score	0 - 9	10 - 19	20 - 29	30 - 39	40 - 49
Frequency	2	5	7	27	9


GRAPHICS CALCULATOR INSTRUCTIONS

Check your answers using your calculator.

3 The table shows the petrol sales in one day by a number of city service stations.

a How many service stations were involved in the survey?

b Estimate the total amount of petrol sold for the day by the service stations.

c Estimate the mean amount of petrol sold for the day.

d Find the modal class for this distribution. Explain your answer.

Amount of petrol (P L)	*Frequency*
$2000 < P \leqslant 3000$	4
$3000 < P \leqslant 4000$	4
$4000 < P \leqslant 5000$	9
$5000 < P \leqslant 6000$	14
$6000 < P \leqslant 7000$	23
$7000 < P \leqslant 8000$	16

4 The data below shows the runs scored by Jeff over an entire cricket season.

17	5	22	13	6	0	15	20
14	7	28	36	13	28	9	18
2	23	12	27	5	22	3	0
32	8	13	25	9			

- **a** Organise the data into the groups 0 - 9, 10 - 19, 20 - 29, 30 - 39.
- **b** Use your grouped data to estimate the mean number of runs scored.
- **c** Use the raw data to find the exact mean number of runs scored. How accurate was your estimate in **b**?

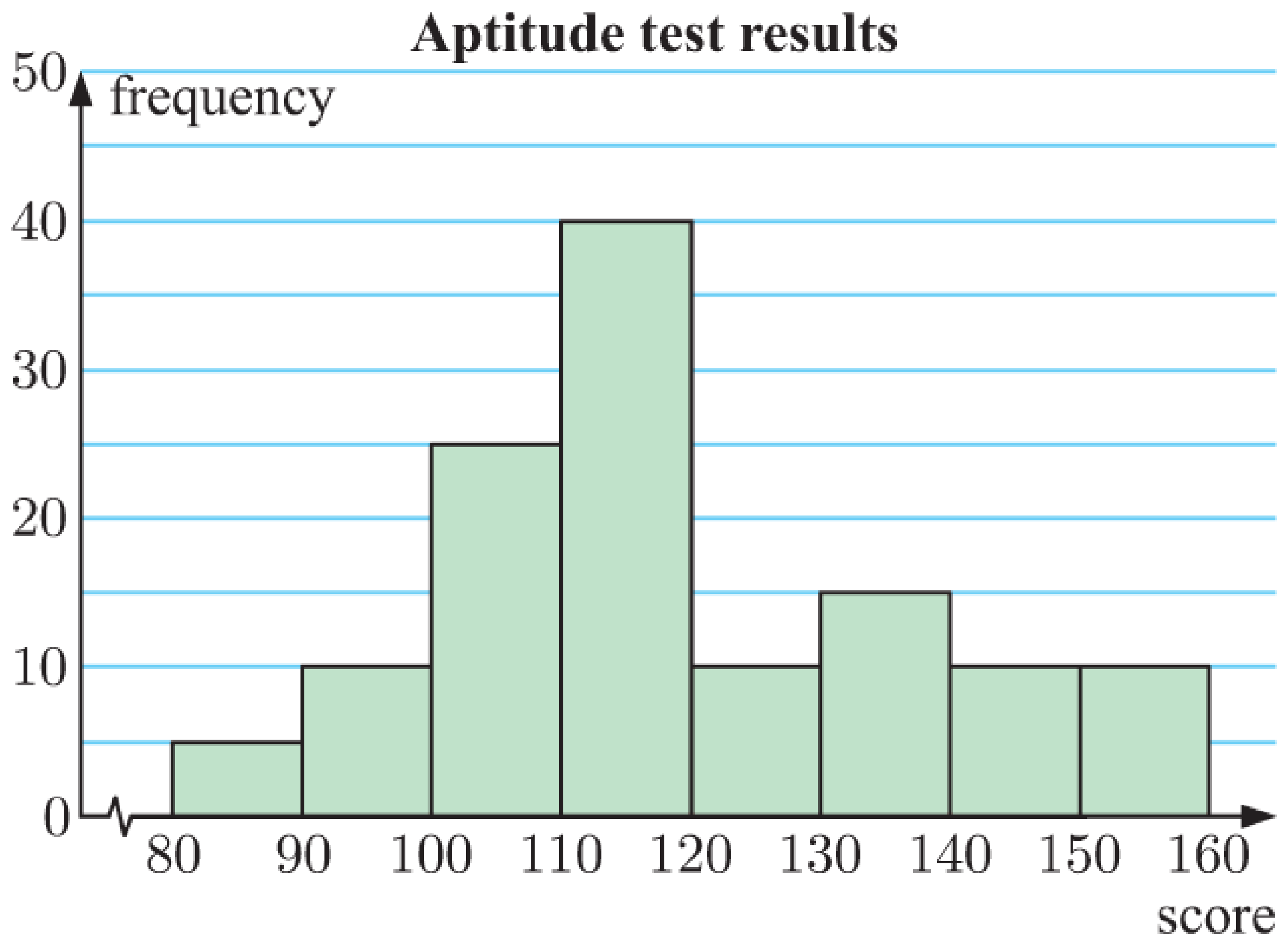
5 The manager of a bank decides to investigate the time customers wait to be served. The results for 300 customers are shown in the table alongside.

- **a** Determine the value of p.
- **b** Estimate the mean waiting time.
- **c** What percentage of customers waited for at least 5 minutes?

Waiting time (t min)	*Frequency*
$0 \leqslant t < 1$	p
$1 \leqslant t < 2$	42
$2 \leqslant t < 3$	50
$3 \leqslant t < 4$	78
$4 \leqslant t < 5$	60
$5 \leqslant t < 6$	30
$6 \leqslant t < 7$	16

6 This frequency histogram illustrates the results of an aptitude test given to a group of people seeking positions in a company.

- **a** How many people took the test?
- **b** Estimate the mean score for the test.
- **c** What fraction of the people scored less than 100 for the test?
- **d** What percentage of the people scored more than 130 for the test?



E MEASURING THE SPREAD OF DATA

Consider the following statements:

- The mean height of 20 boys in a Year 12 class was found to be 175 cm.
- A carpenter used a machine to cut 20 planks of length 175 cm.

Even though the means of the two data sets are the same, there will clearly be a greater *variation* in the heights of boys than in the lengths of the planks.

Commonly used statistics that measure the spread of a data set are:

- the **range**
- the **interquartile range**
- the **variance**
- the **standard deviation**.

We will look at variance and standard deviation later in the Chapter.

THE RANGE

The **range** is the difference between the **maximum** data value and the **minimum** data value.

range = maximum − minimum

As a statistic for discussing the spread of a data set, the range is not considered to be particularly reliable. This is because it only uses two data values. It may be influenced by extreme values or outliers.

However, the range is useful for purposes such as choosing class intervals.

Example 5 **Self Tutor**

The weight, in kilograms, of the pumpkins in Herb's crop are:
2.3, 3.1, 2.7, 4.1, 2.9, 4.0, 3.3, 3.7, 3.4, 5.1, 4.3, 2.9, 4.2
Find the range of the data.

$$\begin{aligned}\text{Range} &= \text{maximum} - \text{minimum}\\ &= 5.1 - 2.3\\ &= 2.8 \text{ kg}\end{aligned}$$

THE INTERQUARTILE RANGE

The median divides the ordered data set into two halves, and these halves are divided in half again by the **quartiles**.

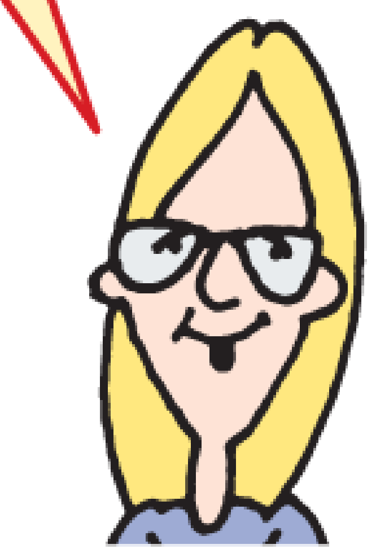
The middle value of the *lower* half is called the **lower quartile** (Q_1).

The middle value of the *upper* half is called the **upper quartile** (Q_3).

The **interquartile range** (**IQR**) is the range of the middle half of the data.

interquartile range = upper quartile − lower quartile
IQR = $\mathbf{Q_3 - Q_1}$

The median is sometimes referred to as Q_2 because it is the 2nd quartile.



Example 6 **Self Tutor**

For the data set 5 5 7 3 8 2 3 4 6 5 7 6 4, find:

a the median **b** Q_1 and Q_3 **c** the interquartile range.

The ordered data set is: 2 3 3 4 4 5 5 5 6 6 7 7 8 (13 data values)

a Since $n = 13$, $\dfrac{n+1}{2} = 7$ $\therefore$ the median is the 7th data value.

~~2 3 3 4 4 5~~ 5 ~~5 6 6 7 7 8~~

$\therefore$ median = 5

b Since the median is a data value we now ignore it and split the remaining data into two:

$$\overbrace{2\ \ 3\ \ 3\ \ 4\ \ 4\ \ 5}^{\text{lower half}} \qquad \overbrace{5\ \ 6\ \ 6\ \ 7\ \ 7\ \ 8}^{\text{upper half}}$$

Q_1 = median of lower half $= \dfrac{3+4}{2} = 3.5$

Q_3 = median of upper half $= \dfrac{6+7}{2} = 6.5$

c IQR $= Q_3 - Q_1 = 6.5 - 3.5 = 3$

Example 7 **Self Tutor**

For the data set 12 24 17 10 16 29 22 18 32 20, find:

a the median **b** Q_1 and Q_3 **c** the interquartile range.

The ordered data set is: 10 12 16 17 18 20 22 24 29 32 (10 data values)

a Since $n = 10$, $\dfrac{n+1}{2} = 5.5$ $\therefore$ the median is the average of the 5th and 6th data values.

~~10 12 16 17~~ 18 20 ~~22 24 29 32~~

$$\therefore \quad \text{median} = \frac{\text{5th value} + \text{6th value}}{2} = \frac{18+20}{2} = 19$$

b We have an even number of data values, so we include all data values when we split the data set into two:

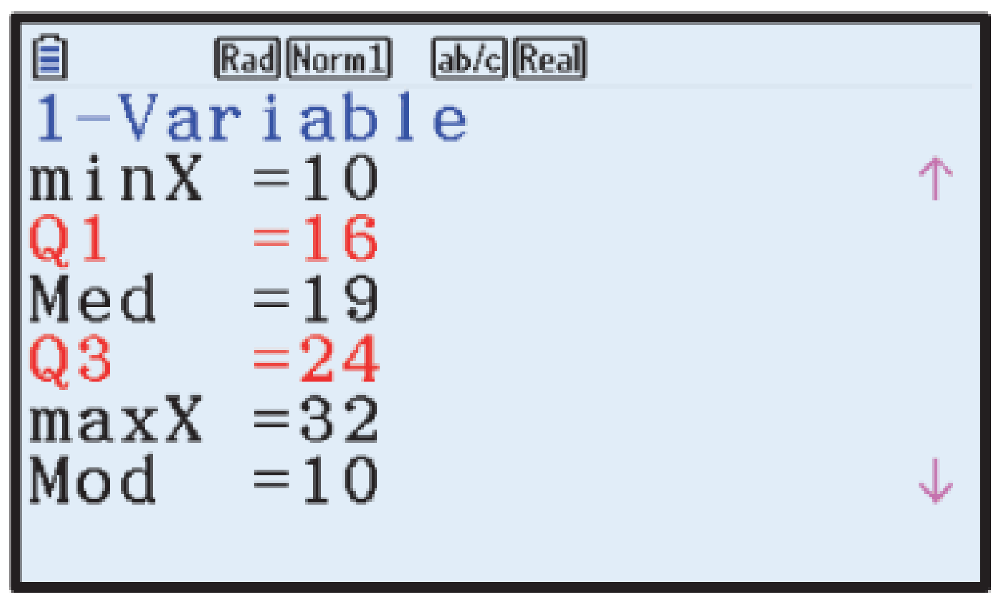
$$\overbrace{10\ \ 12\ \ 16\ \ 17\ \ 18}^{\text{lower half}}\ \ \overbrace{20\ \ 22\ \ 24\ \ 29\ \ 32}^{\text{upper half}}$$

Q_1 = median of lower half = 16

Q_3 = median of upper half = 24

c $\text{IQR} = Q_3 - Q_1 = 24 - 16 = 8$

Technology can be used to help calculate the interquartile range. Your graphics calculator gives the values of Q_1 and Q_3, from which the interquartile range is found using $\text{IQR} = Q_3 - Q_1$.



```
1-Variable
minX =10
Q1   =16
Med  =19
Q3   =24
maxX =32
Mod  =10
```

GRAPHICS CALCULATOR INSTRUCTIONS

STATISTICS PACKAGE

EXERCISE 13E

1 For each of the following data sets, make sure the data is ordered and then find:

- **i** the median
- **ii** the lower and upper quartiles
- **iii** the range
- **iv** the interquartile range.

a 5, 6, 9, 10, 11, 13, 15, 16, 18, 20, 21

b 7, 7, 10, 13, 14, 15, 18, 19, 21, 21, 23, 24, 24, 26

c 21, 24, 19, 32, 15, 43, 38, 29

d 32, 45, 26, 28, 52, 57, 41, 69, 33, 20

Check your answers using your graphics calculator.

2 Jane and Ashley's monthly telephone bills are shown below:

Jane: \$35, \$47, \$29, \$38, \$29, \$34, \$42, \$29, \$36, \$40, \$36, \$31

Ashley: \$19, \$24, \$26, \$19, \$23, \$40, \$35, \$59, \$32, \$42, \$26, \$24

a Find the mean and median for each data set.

b Find the range and interquartile range for each data set.

c Which person generally pays more for their telephone bills?

d Which person has the greater variability in their telephone bills?

3 **a** Find the range and interquartile range for the data set:

9	12	7	15	14	22	18	11	20	15
20	10	13	67	25	18	11	7	14	19

b Identify the outlier in the data set.

c Recalculate the range and interquartile range with the outlier removed.

d Which measure of spread is more affected by the outlier?

4 Derrick and Gareth recorded the number of minutes they slept each night for 15 nights:

Derrick: 420, 435, 440, 415, 380, 400, 430, 450, 210, 445, 425, 445, 450, 420, 425
Gareth: 360, 420, 460, 430, 480, 340, 450, 490, 500, 460, 330, 470, 340, 480, 370

a Calculate the range and interquartile range for each data set.

b Which person's data has the lower:

i range **ii** interquartile range?

c Which measure of spread is more appropriate for determining who is generally the more consistent sleeper? Explain your answer.

5 a, b, c, d, e, f, g, h, i, j, k, l, and m are 13 data values which have been arranged in *ascending* order.

a Which variable represents the median?

b Write down an expression for:

i the range **ii** the interquartile range.

6 A data set has the following known measures of centre and spread:

Measure	median	mode	range	interquartile range
Value	9	7	13	6

Find the new value of each of these measures if every member of the data set is:

a increased by 2 **b** doubled.

DISCUSSION

Consider the data set:

5, 7, 7, 8, 9, 11, 11, 12, 14, 14, 15

1 Calculate Q_1, Q_3, and the IQR:

- by hand
- using your graphics calculator
- using a spreadsheet.

Do you get the same answers in all 3 cases?

2 If there is an odd number of data values, some statistical packages calculate quartiles by *including* the median in each half of the data.

a Check to see whether your spreadsheet calculates quartiles this way.

b Does this method necessarily change the *interpretation* of the calculated values?

c Are statistical packages that do this necessarily "wrong"?

F BOX AND WHISKER DIAGRAMS

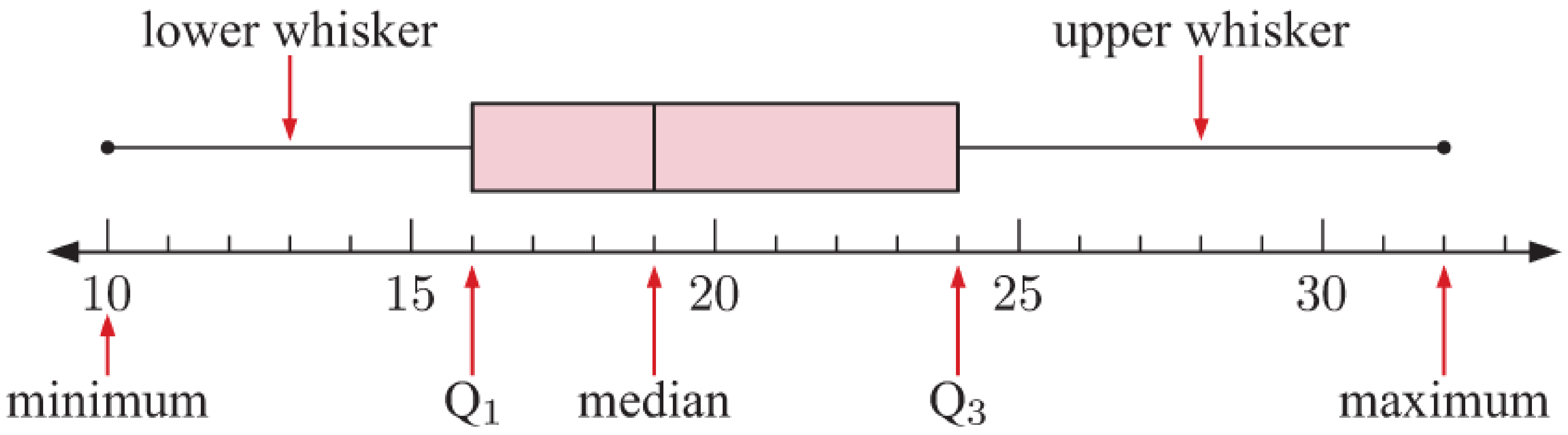
A **box and whisker diagram** or simply **box plot** is a visual display of some of the descriptive statistics of a data set. It shows:

- the minimum value
- the lower quartile (Q_1)
- the median (Q_2)
- the upper quartile (Q_3)
- the maximum value

These five numbers form the **five-number summary** of the data set.

For the data set in **Example 7** on page **330**, the five-number summary and box plot are:

minimum $= 10$
$Q_1 = 16$
median $= 19$
$Q_3 = 24$
maximum $= 32$

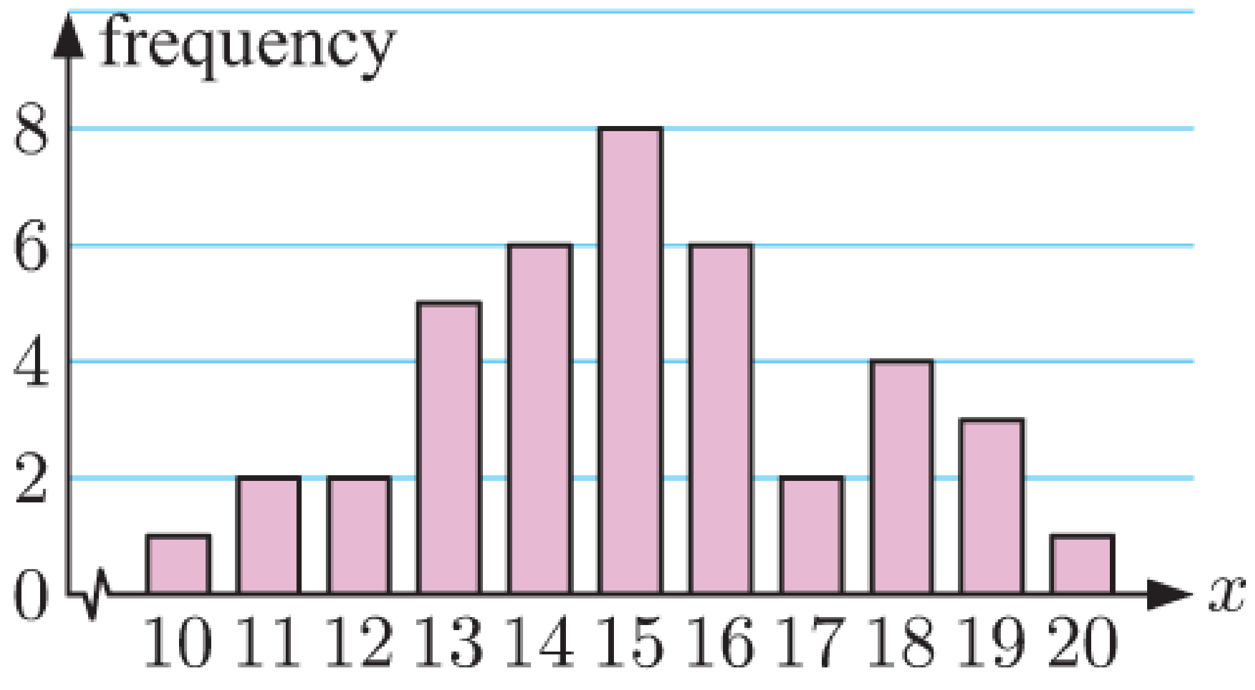


You should notice that:

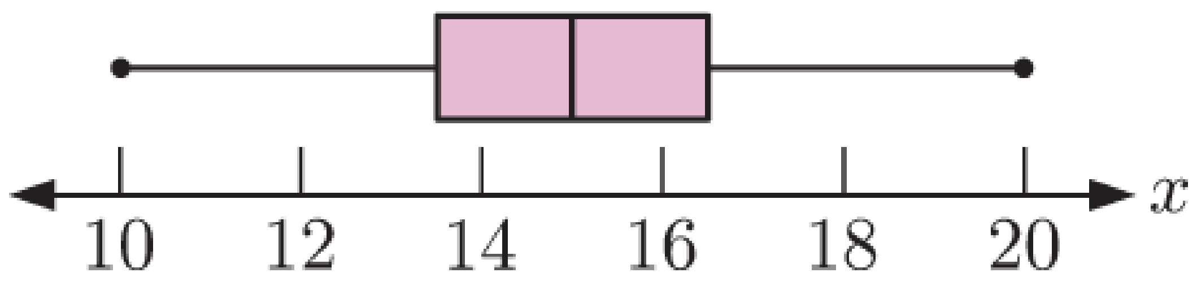
- The rectangular box represents the "middle" half of the data set.
- The lower whisker represents the 25% of the data with smallest values.
- The upper whisker represents the 25% of the data with greatest values.

INTERPRETING A BOX PLOT

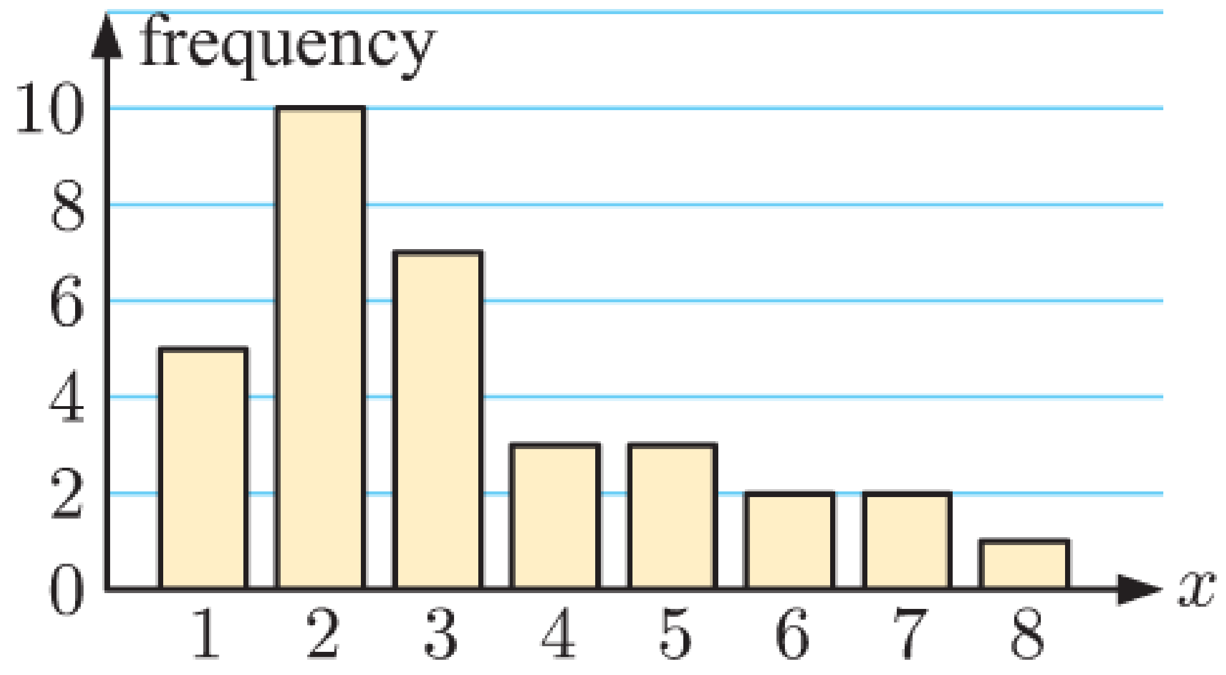
A set of data with a **symmetric distribution** will have a symmetric box plot.



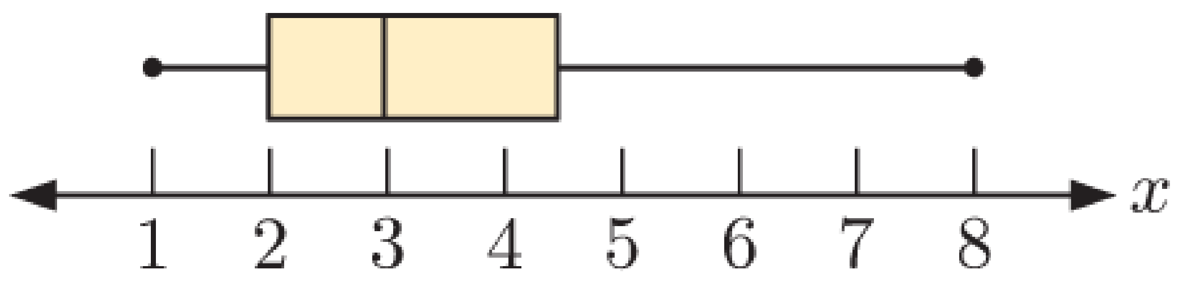
The whiskers of the box plot are the same length and the median line is in the centre of the box.



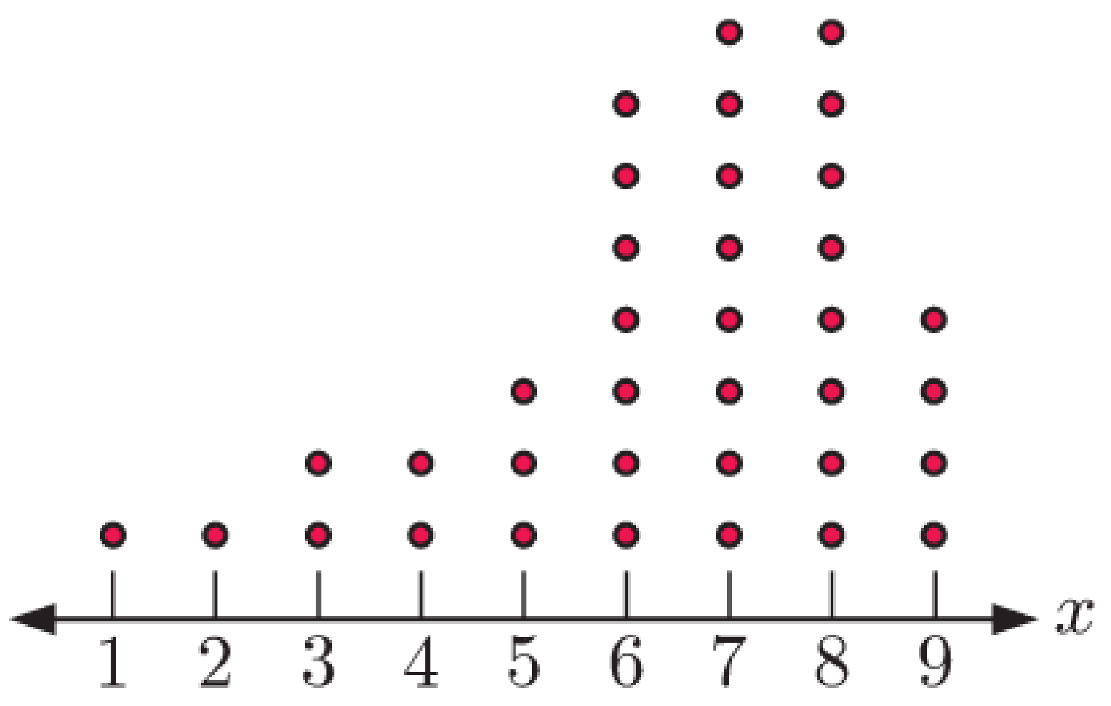
A set of data which is **positively skewed** will have a positively skewed box plot.



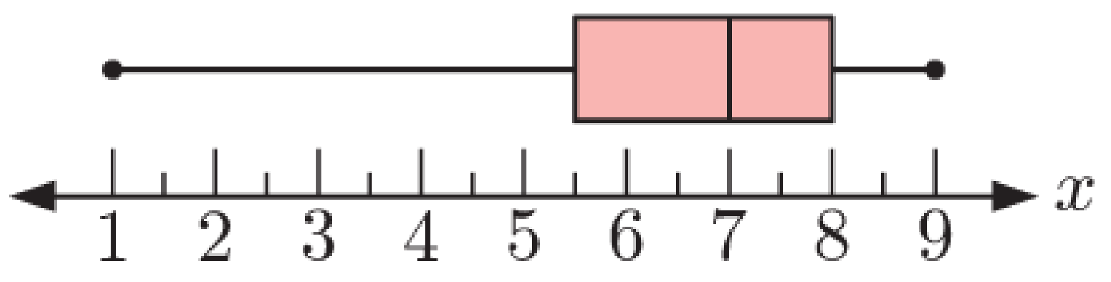
The upper whisker is longer than the lower whisker and the median line is closer to the left hand side of the box.



A set of data which is **negatively skewed** will have a negatively skewed box plot.



The lower whisker is longer than the upper whisker and the median line is closer to the right hand side of the box.



Example 8

Self Tutor

Consider the data set: 8 2 3 9 6 5 3 2 2 6 2 5 4 5 5 6

a Find the five-number summary for this data.

b Draw a box plot for the data.

c Find the:

i range **ii** interquartile range.

d Find the percentage of data values which are less than 3.

STATISTICS PACKAGE

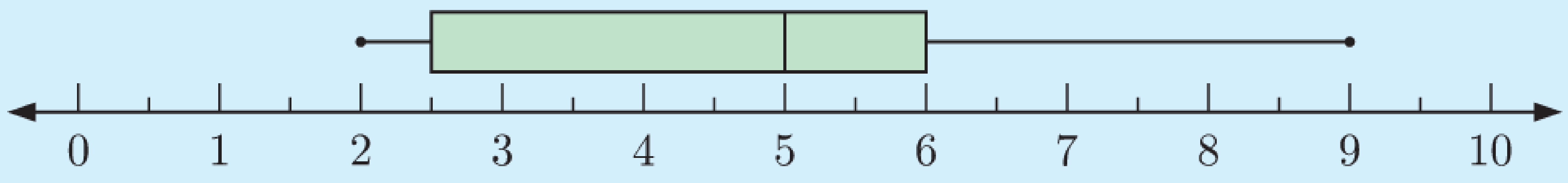
a The ordered data set is:

2 2 2 2 3 3 4 5 5 5 5 6 6 6 8 9 {16 data values}

$Q_1 = 2.5$ median $= 5$ $Q_3 = 6$

The five-number summary is: $\begin{cases} \text{minimum} = 2 & Q_1 = 2.5 \\ \text{median} = 5 & Q_3 = 6 \\ \text{maximum} = 9 & \end{cases}$

b

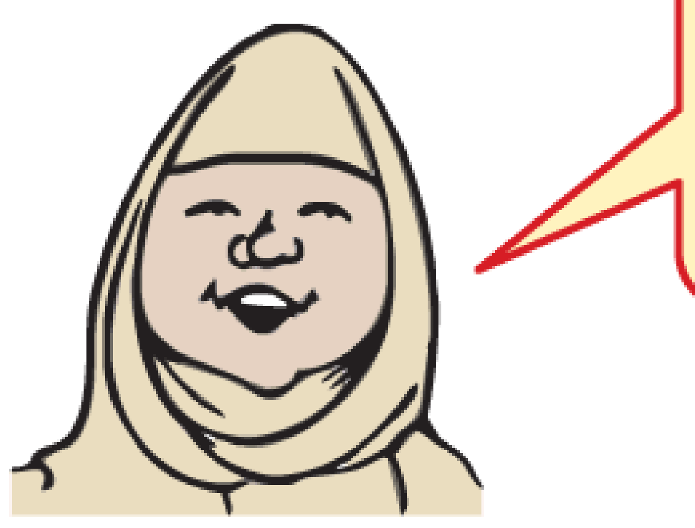


c **i** range = maximum − minimum
= 9 − 2
= 7

ii IQR = $Q_3 - Q_1$
= 6 − 2.5
= 3.5

d Using the ordered data set in **a**, 4 out of 16 data values are less than 3.

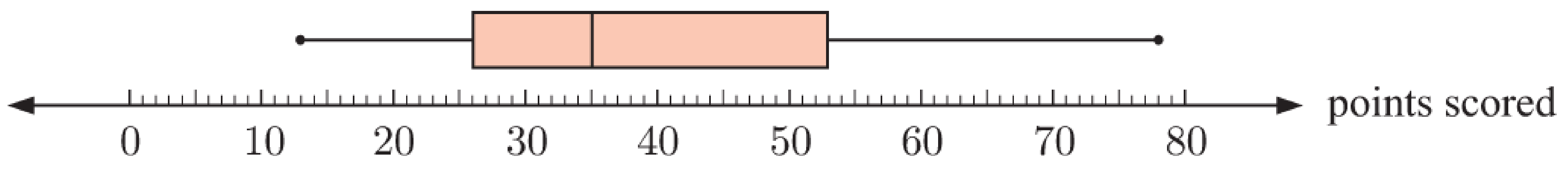
$\therefore$ 25% of the data values are less than 3.



Part **d** can be seen from the original data set. We cannot read it straight from the box plot because the box plot does not tell us that all of the data values are integers.

EXERCISE 13F

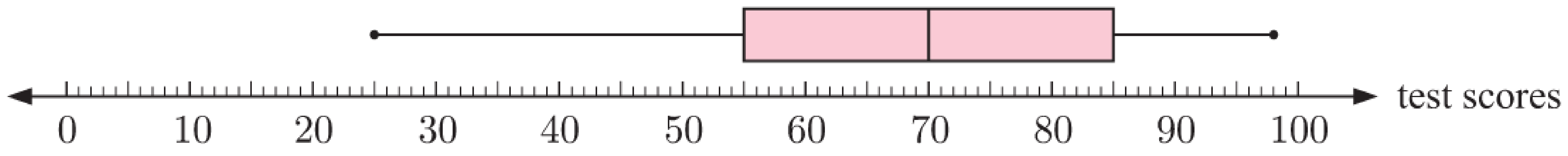
1 The box plot below summarises the points scored by a basketball team.



State the:

a median **b** maximum value **c** minimum value
d upper quartile **e** lower quartile **f** range
g interquartile range.

2 The box plot below summarises the class results for a test out of 100 marks.



a Copy and complete the following statements about the test results:
- **i** The highest mark scored for the test was ...... and the lowest mark was ......
- **ii** Half of the class scored a mark greater than or equal to ......
- **iii** The top 25% of the class scored at least ...... marks.
- **iv** The middle half of the class had scores between ...... and ......

b Find the range and interquartile range of the data set.

3 For the following data sets:
- **i** Construct a five-number summary.
- **ii** Draw a box plot.
- **iii** Find the range and interquartile range.

STATISTICS PACKAGE

GRAPHICS CALCULATOR INSTRUCTIONS

a 3, 4, 5, 5, 5, 6, 6, 6, 7, 7, 8, 8, 9, 10
b 3, 7, 0, 1, 4, 6, 8, 8, 8, 9, 7, 5, 6, 8, 7, 8, 8, 2, 9
c 23, 44, 31, 33, 26, 17, 30, 35, 47, 31, 51, 47, 20, 31, 28, 49, 26, 49

4 Enid counts the number of beans in 33 pods. Her results are:

5, 8, 10, 4, 2, 12, 6, 5, 7, 7, 5, 5, 5, 13, 9, 3, 4, 4, 7, 8, 9, 5, 5, 4, 3, 6, 6, 6, 6, 9, 8, 7, 6

a Find the median, lower quartile, and upper quartile of the data set.
b Find the interquartile range of the data set. **c** Draw a box plot of the data set.

5 Ranji counts the number of bolts in several boxes and tabulates the data as follows:

Number of bolts	33	34	35	36	37	38	39	40
Frequency	1	5	7	13	12	8	0	1

a Draw a box plot of the data set. **b** Find the range and the IQR.

GAME

Click on the icon to play a card game about box plots.

CARD GAME

G OUTLIERS

We have seen that **outliers** are extraordinary data that are separated from the main body of the data.

However, we have so far identified outliers rather informally by looking at the data directly, or at a column graph of the data.

A commonly used test to identify outliers involves the calculation of upper and lower boundaries:

- **upper boundary = upper quartile + 1.5 × IQR**
  Any data larger than the upper boundary is an outlier.
- **lower boundary = lower quartile − 1.5 × IQR**
  Any data smaller than the lower boundary is an outlier.

Outliers are marked with an asterisk on a box plot. There may be more than one outlier at either end.

Each whisker extends to the last value that is not an outlier.

Example 9 **Self Tutor**

Test the following data for outliers. Hence construct a box plot for the data.

3, 7, 8, 8, 5, 9, 10, 12, 14, 7, 1, 3, 8, 16, 8, 6, 9, 10, 13, 7

The ordered data set is:

1 3 3 5 6 7 7 7 8 8 8 8 9 9 10 10 12 13 14 16 $\{n = 20\}$

min = 1 $Q_1 = 6.5$ median = 8 $Q_3 = 10$ max = 16

$\text{IQR} = Q_3 - Q_1 = 3.5$

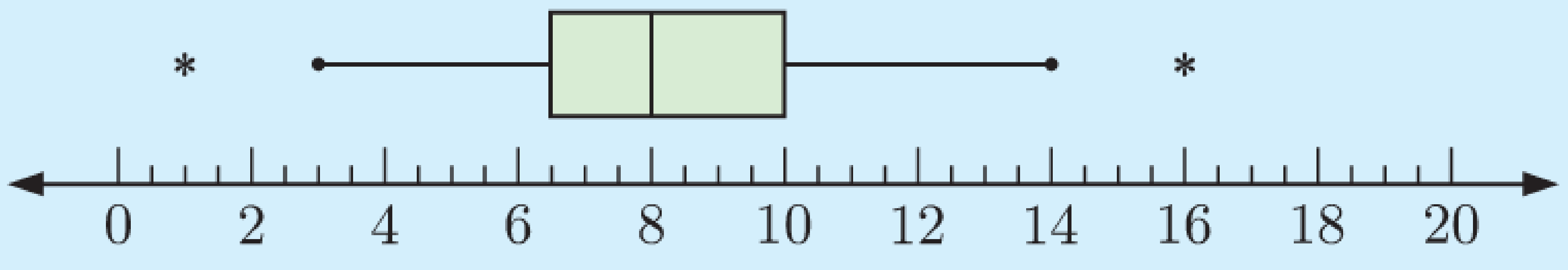
Test for outliers: upper boundary and lower boundary

$= \text{upper quartile} + 1.5 \times \text{IQR}$ $= \text{lower quartile} - 1.5 \times \text{IQR}$

$= 10 + 1.5 \times 3.5$ $= 6.5 - 1.5 \times 3.5$

$= 15.25$ $= 1.25$

16 is above the upper boundary, so it is an outlier.
1 is below the lower boundary, so it is an outlier.



Each whisker is drawn to the last value that is not an outlier.

EXERCISE 13G

1 A data set has lower quartile = 31.5, median = 37, and upper quartile = 43.5.

a Calculate the interquartile range for this data set.

b Calculate the boundaries that identify outliers.

c The smallest values of the data set are 13 and 20. The largest values are 52 and 55. Which of these are outliers?

d Draw a box plot of the data set.

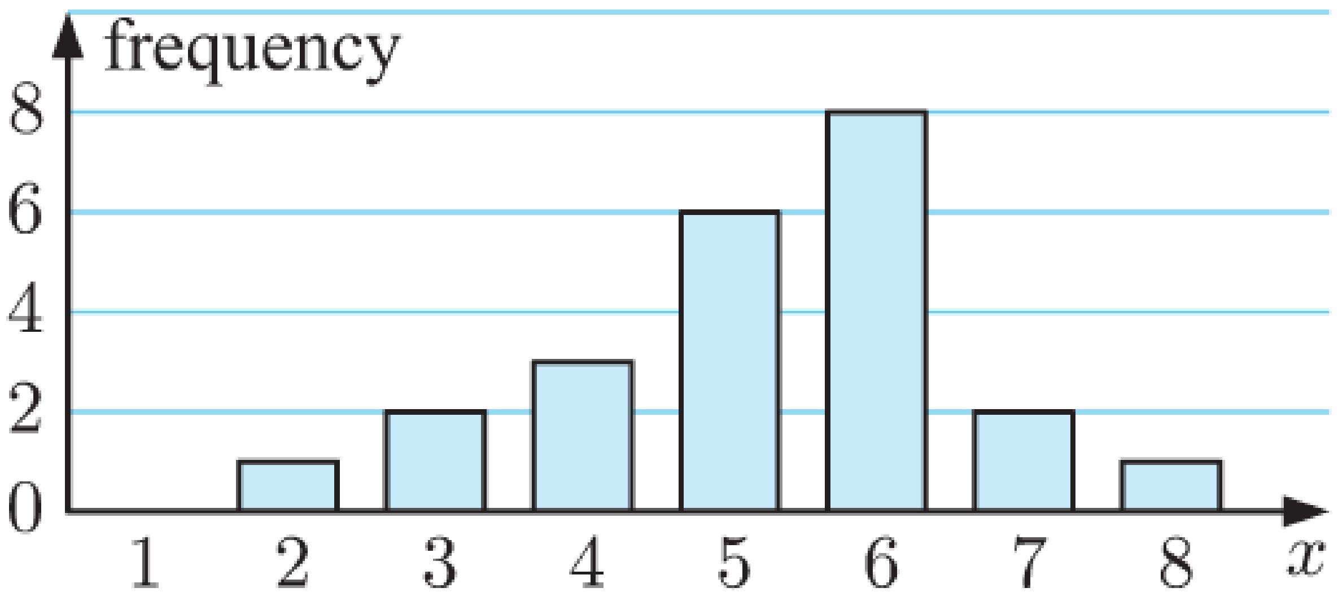
2 James goes bird watching for 25 days. The number of birds he sees each day are:

12, 5, 13, 16, 8, 10, 12, 18, 9, 11, 14, 14,
22, 9, 10, 7, 9, 11, 13, 7, 10, 6, 13, 3, 8

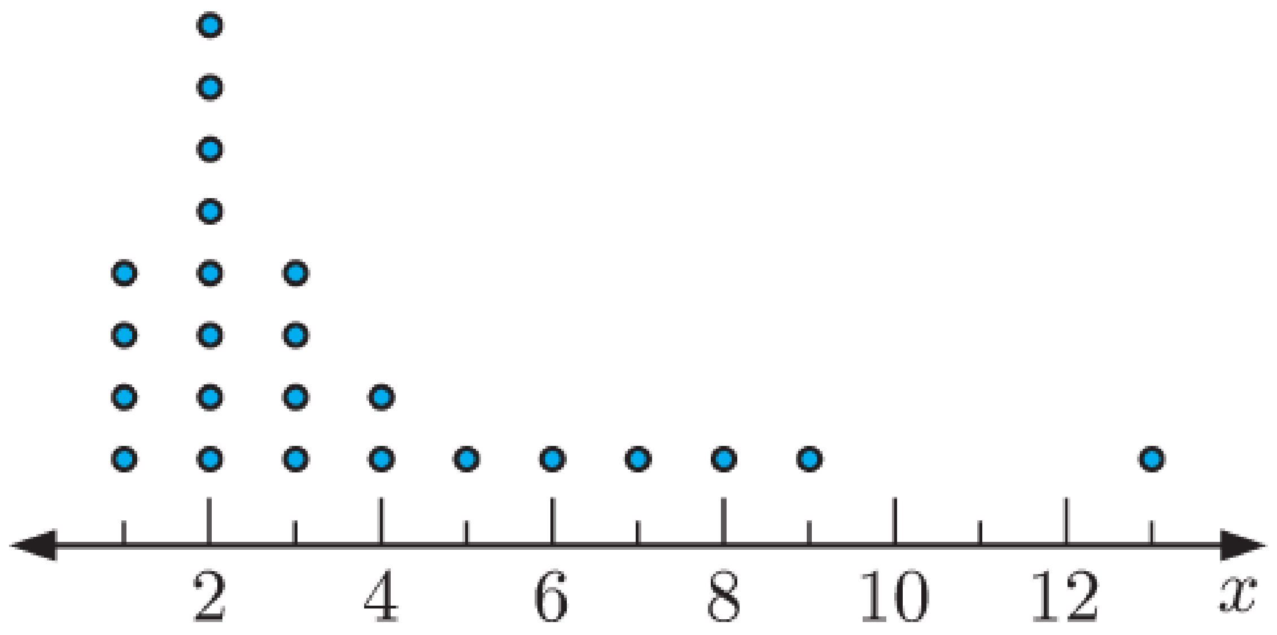
- **a** Find the median, lower quartile, and upper quartile of the data set.
- **b** Find the interquartile range of the data set.
- **c** Find the lower and upper boundaries, and hence identify any outliers.
- **d** Draw a box plot of the data set.

3 Match each graph with its box plot:

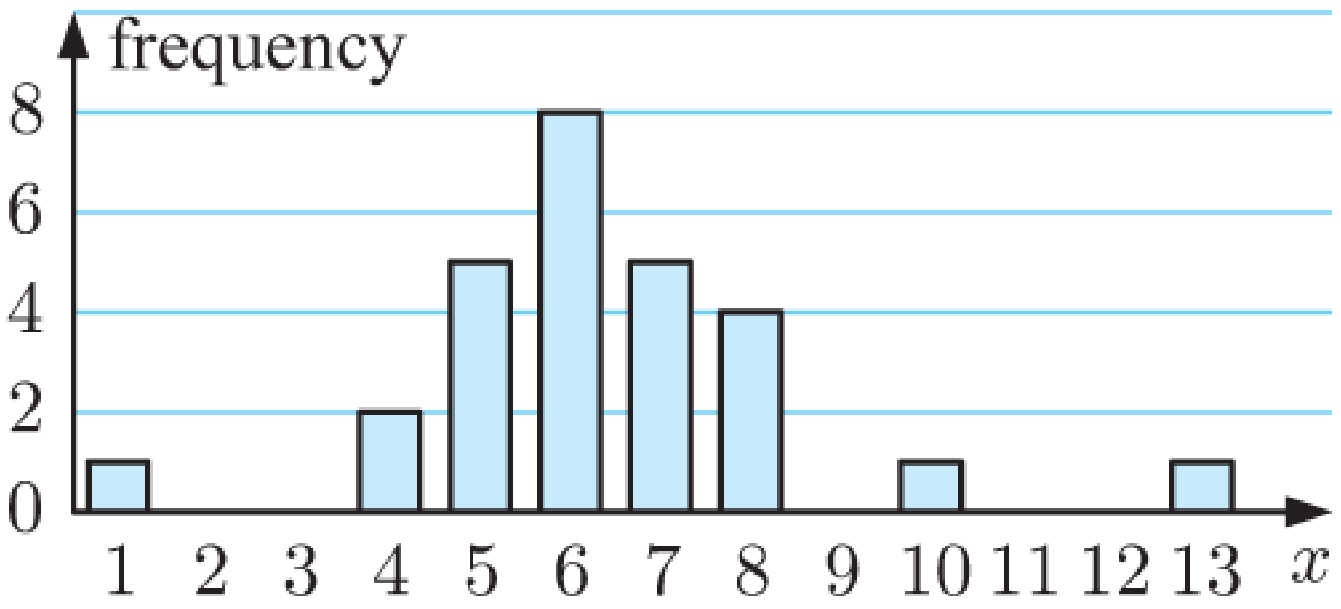
a



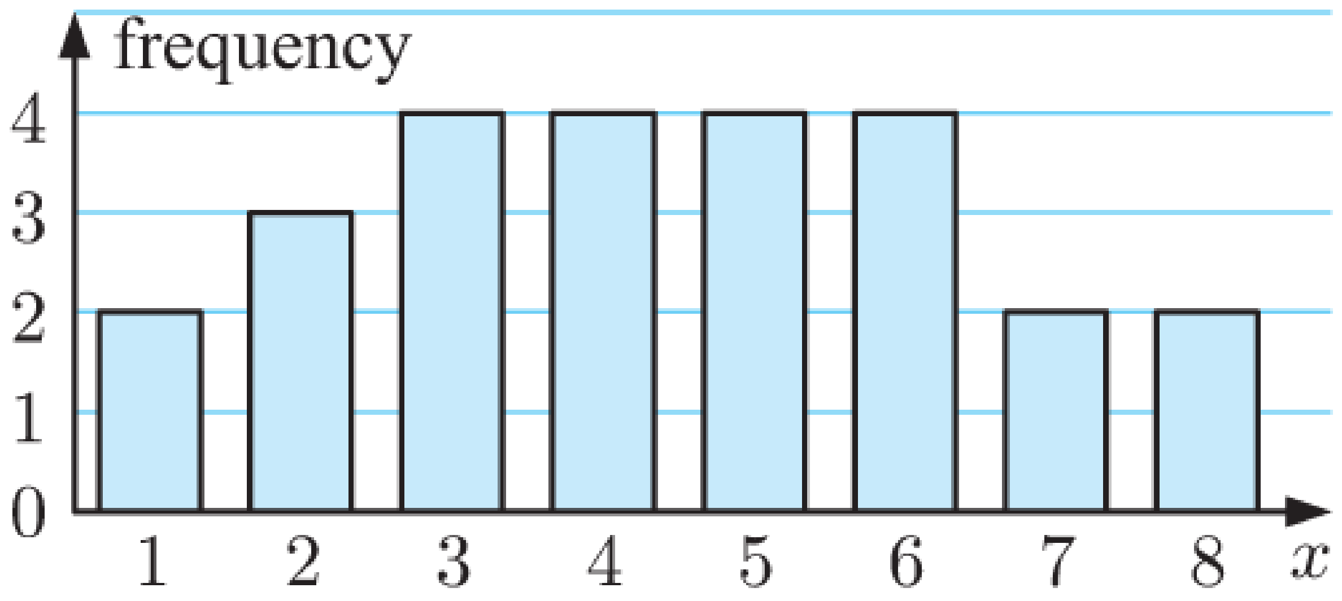
b



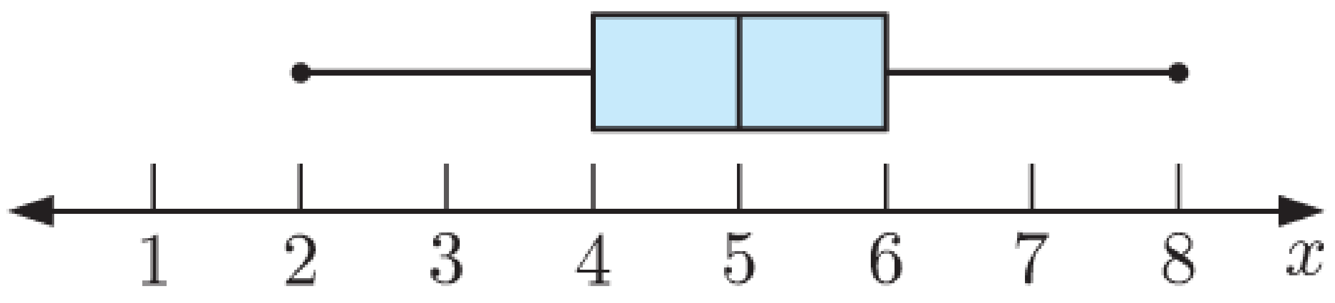
c



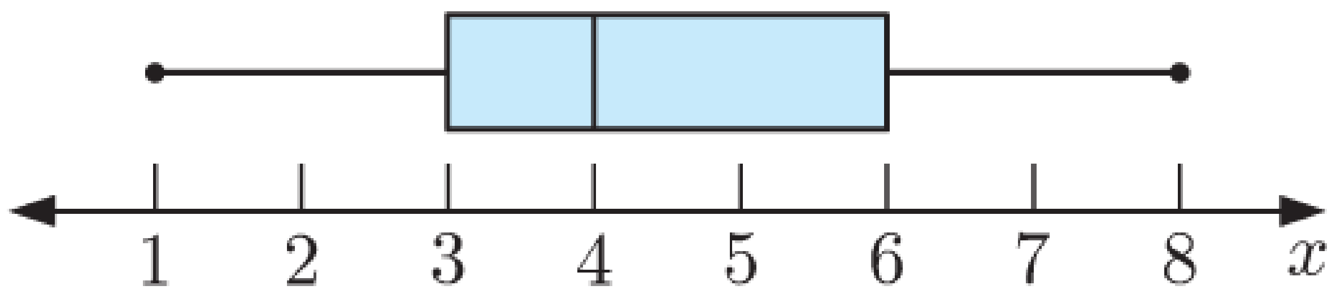
d



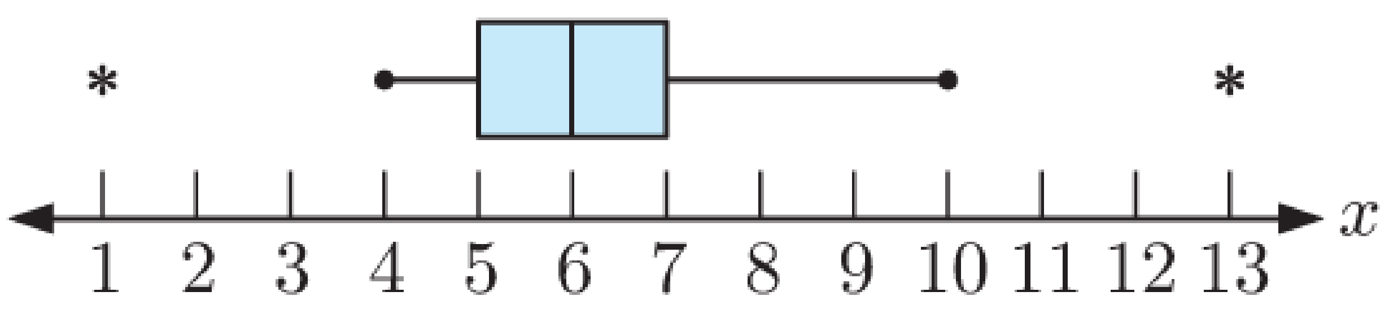
A



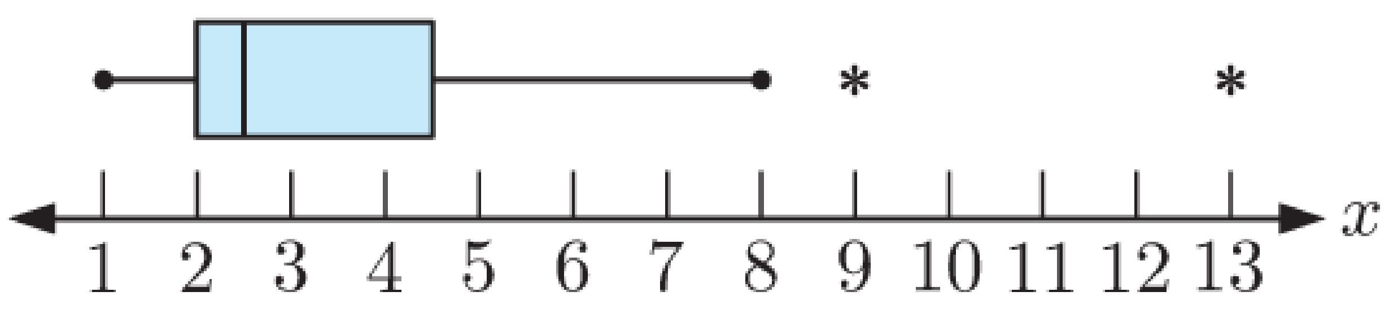
B



C



D



4 The data below shows the number of properties sold by a real estate agent each week in 2018:

2	2	1	3	2	2	2	2	1	4	1	5	1
1	1	2	2	2	3	1	7	2	2	2	0	2
2	4	4	3	3	1	0	2	4	1	2	1	3
0	2	3	1	2	1	3	4	2	2	2	1	3

- **a** Draw a column graph to display the data.
- **b** From the column graph, does the data appear to have any outliers?
- **c** Calculate the upper and lower boundaries to test for outliers and hence check your answer to **b**.
- **d** Construct a box plot for the data.

H PARALLEL BOX AND WHISKER DIAGRAMS

A **parallel box and whisker diagram** or **parallel box plot** enables us to make a *visual comparison* of the distributions of two data sets. We can easily compare descriptive statistics such as their median, range, and interquartile range.

Example 10 **Self Tutor**

A hospital trialling a new anaesthetic has collected data on how long the new and old drugs take before the patient becomes unconscious. They wish to know which drug acts faster and which is more predictable.

Old drug times (s): 8, 12, 9, 8, 16, 10, 14, 7, 5, 21,
13, 10, 8, 10, 11, 8, 11, 9, 11, 14

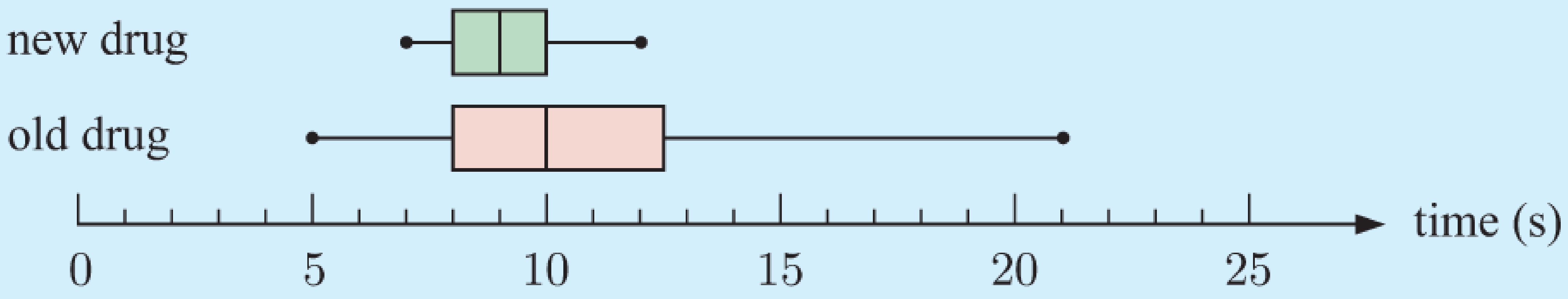
New drug times (s): 8, 12, 7, 8, 12, 11, 9, 8, 10, 8,
10, 9, 12, 8, 8, 7, 10, 7, 9, 9

Draw a parallel box plot for the data sets and use it to compare the two drugs.

The five-number summaries are:

For the old drug: min $= 5$, $Q_1 = 8$, median $= 10$, $Q_3 = 12.5$, max $= 21$

For the new drug: min $= 7$, $Q_1 = 8$, median $= 9$, $Q_3 = 10$, max $= 12$



Using the median, 50% of the time the new drug takes 9 seconds or less, compared with 10 seconds for the old drug. So, the new drug is generally a little quicker.

Comparing the spreads:

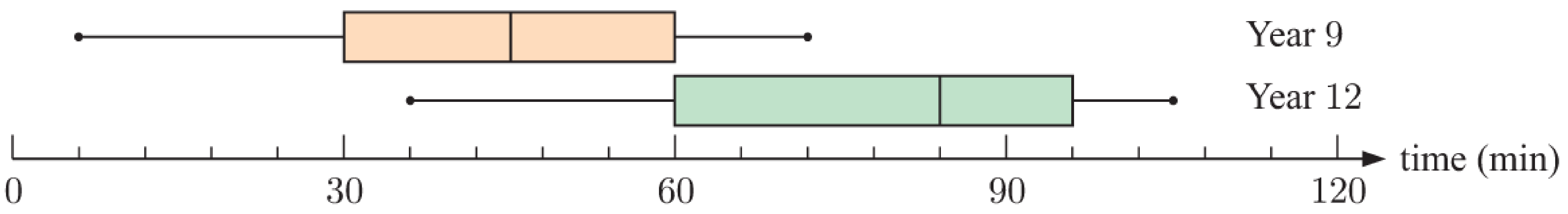
$$\begin{aligned}\text{range for old drug} &= 21 - 5\\ &= 16\end{aligned} \qquad \begin{aligned}\text{range for new drug} &= 12 - 7\\ &= 5\end{aligned}$$

$$\begin{aligned}\text{IQR for old drug} &= Q_3 - Q_1\\ &= 12.5 - 8\\ &= 4.5\end{aligned} \qquad \begin{aligned}\text{IQR for new drug} &= Q_3 - Q_1\\ &= 10 - 8\\ &= 2\end{aligned}$$

The new drug times are less "spread out" than the old drug times, so the new drug is more predictable.

EXERCISE 13H

1 The following parallel box plots compare the times students in Years 9 and 12 spend on homework.



a Copy and complete:

Statistic	*Year 9*	*Year 12*
minimum		
Q_1		
median		
Q_3		
maximum		

b For each group, determine the:

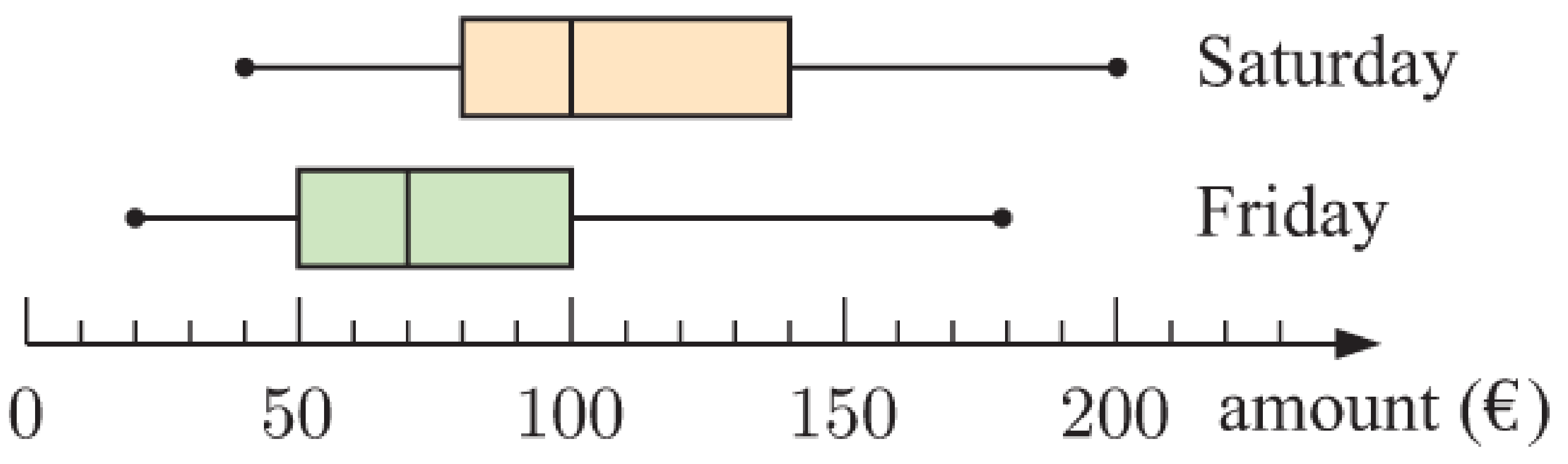
i range **ii** interquartile range.

c Determine whether the following statements are true or false, or if there is not enough information to tell:

i On average, Year 12 students spend about twice as much time on homework as Year 9 students.

ii Over 25% of Year 9 students spend less time on homework than all Year 12 students.

2 The amounts of money withdrawn from an ATM were recorded on a Friday and on a Saturday. The results are displayed on the parallel box plot shown.

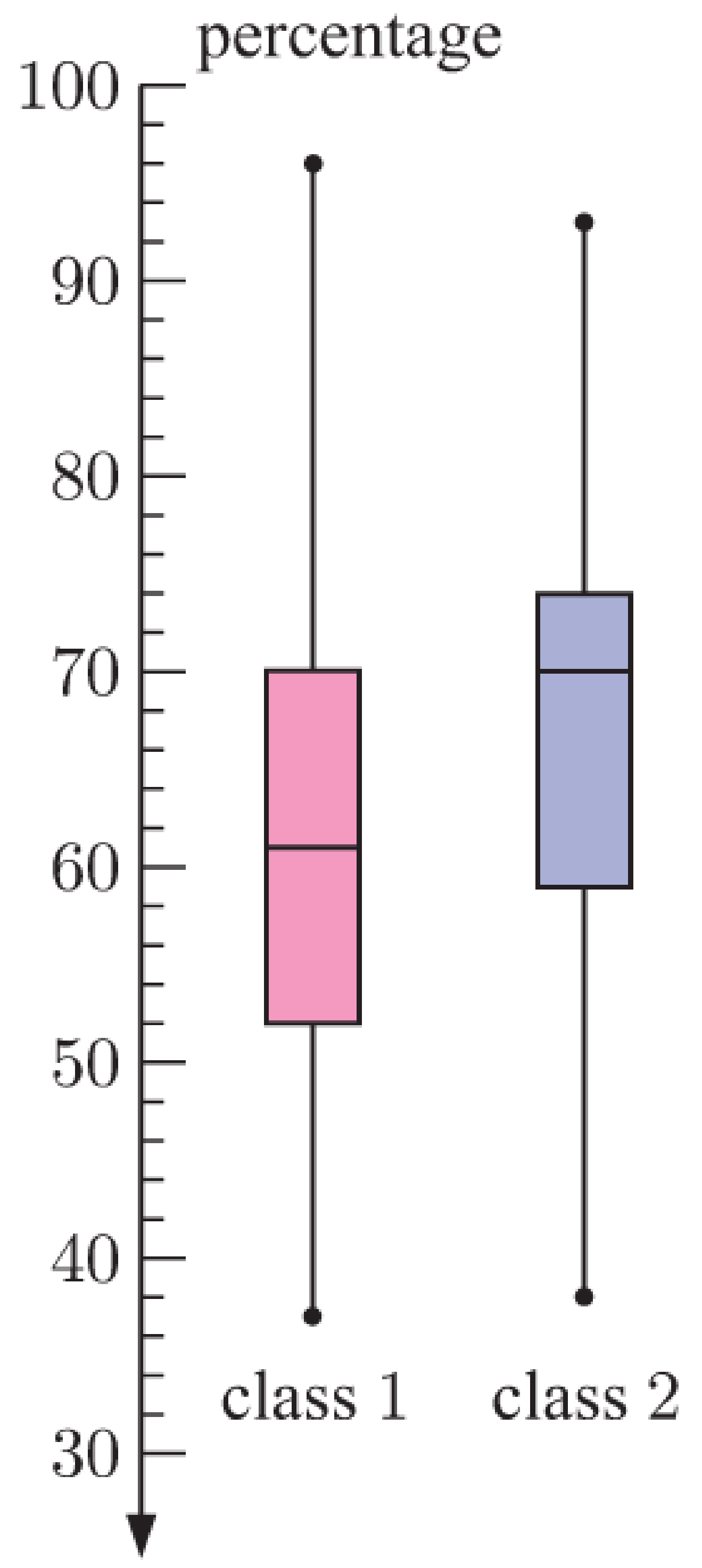


a Find the five-number summary for each data set.

b For each data set, determine the

i range **ii** interquartile range.

3 After the final examination, the results of two classes studying the same subject were compiled in this parallel box plot.



a In which class was:

i the highest mark **ii** the lowest mark

iii there a larger spread of marks?

b Find the interquartile range of class 1.

c Find the range of class 2.

d Students who scored at least 70% received an achievement award. Find the percentage of students who received an award in:

i class 1 **ii** class 2.

e Describe the distribution of marks in:

i class 1 **ii** class 2.

f Copy and complete:

The students in class ...... generally scored higher marks.

The marks in class ...... were more varied.

4 The data below are the durations, in minutes, of Kirsten and Erika's last 25 phone calls.

Kirsten:	1.7	2.0	3.9	3.4	0.9	1.4	2.5	1.1	5.1	4.2	1.5	2.6	0.8
	4.0	1.5	1.0	2.9	3.2	2.5	0.8	1.8	3.1	6.9	2.3	1.2	
Erika:	2.0	4.8	1.2	7.5	3.2	5.7	3.9	0.2	2.7	6.8	3.4	5.2	3.2
	7.2	1.7	11.5	4.0	2.4	3.7	4.2	10.7	3.0	2.0	0.9	5.7	

a Find the five-number summary for each data set.

b Display the data in a parallel box plot.

c Compare and comment on the distributions of the data.

5 Emil and Aaron play in the same handball team and are fierce but friendly rivals when it comes to scoring. During a season, the numbers of goals they scored in each match were:

Emil:	1	6	2	0	3	4	1	4	2	3	0	3	2	4	3	4	3	3
	3	4	2	4	3	2	3	3	0	5	3	5	3	2	4	3	4	3
Aaron:	7	2	4	8	1	3	4	2	3	0	5	3	5	2	3	1	2	0
	4	3	4	0	3	3	0	2	5	1	1	2	2	5	1	4	0	1

a Is the variable discrete or continuous?

b Enter the data into a graphics calculator or statistics package.

c Produce a column graph for each data set.

d Describe the shape of each distribution.

e Compare the measures of the centre of each distribution.

f Compare the spreads of each distribution.

g Draw a parallel box plot for the data.

h What conclusions can be drawn from the data?

6 A manufacturer of light globes claims that their new design has a 20% longer life than those they are presently selling. Forty of each globe are randomly selected and tested. Here are the results to the nearest hour:

Old type:	103	96	113	111	126	100	122	110	84	117	103	113	104	104
	111	87	90	121	99	114	105	121	93	109	87	118	75	111
	87	127	117	131	115	116	82	130	113	95	108	112		
New type:	146	131	132	160	128	119	133	117	139	123	109	129	109	131
	191	117	132	107	141	136	146	142	123	144	145	125	164	125
	133	124	153	129	118	130	134	151	145	131	133	135		

a Is the variable discrete or continuous?

b Enter the data into a graphics calculator or statistics package. Compare the measures of centre and spread.

c Draw a parallel box plot.

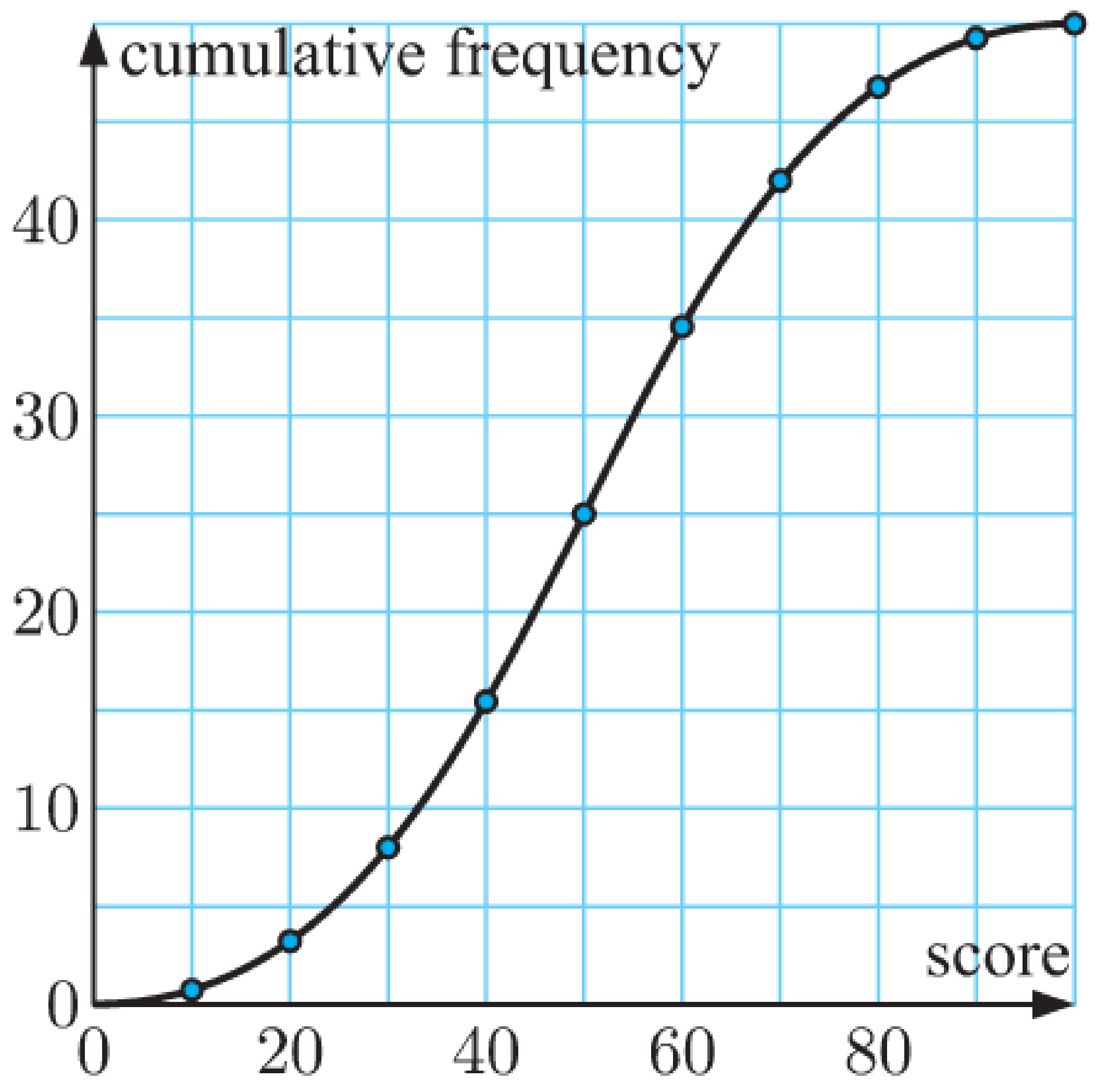
d Describe the shape of each distribution.

e What conclusions, if any, can be drawn from the data?

I CUMULATIVE FREQUENCY GRAPHS

If we want to know the number or proportion of scores that lie above or below a particular value, we add a **cumulative frequency** column to a **frequency table**, and use a graph called a **cumulative frequency graph** to represent the data.

The cumulative frequencies are plotted and the points joined by a smooth curve. This differs from an ogive or cumulative frequency polygon where neighbouring points are joined by straight lines.



PERCENTILES

A **percentile** is the score below which a certain percentage of the data lies.

For example:

- the 85th percentile is the score below which 85% of the data lies.
- If your score in a test is the 95th percentile, then 95% of the class have scored less than you.

Notice that:

- the **lower quartile** (Q_1) is the 25th percentile
- the **median** (Q_2) is the 50th percentile
- the **upper quartile** (Q_3) is the 75th percentile.

A cumulative frequency graph provides a convenient way to find percentiles.

Example 11 **Self Tutor**

The data shows the results of the women's marathon at the 2008 Olympics, for all competitors who finished the race.

a Add a cumulative frequency column to the table.

b Represent the data on a cumulative frequency graph.

c Use your graph to estimate the:

  i median finishing time

  ii number of competitors who finished in less than 155 minutes

  iii percentage of competitors who took more than 159 minutes to finish

  iv time taken by a competitor who finished in the top 20% of runners completing the marathon.

Time (t min)	*Frequency*
$146 \leqslant t < 148$	8
$148 \leqslant t < 150$	3
$150 \leqslant t < 152$	9
$152 \leqslant t < 154$	11
$154 \leqslant t < 156$	12
$156 \leqslant t < 158$	7
$158 \leqslant t < 160$	5
$160 \leqslant t < 168$	8
$168 \leqslant t < 176$	6

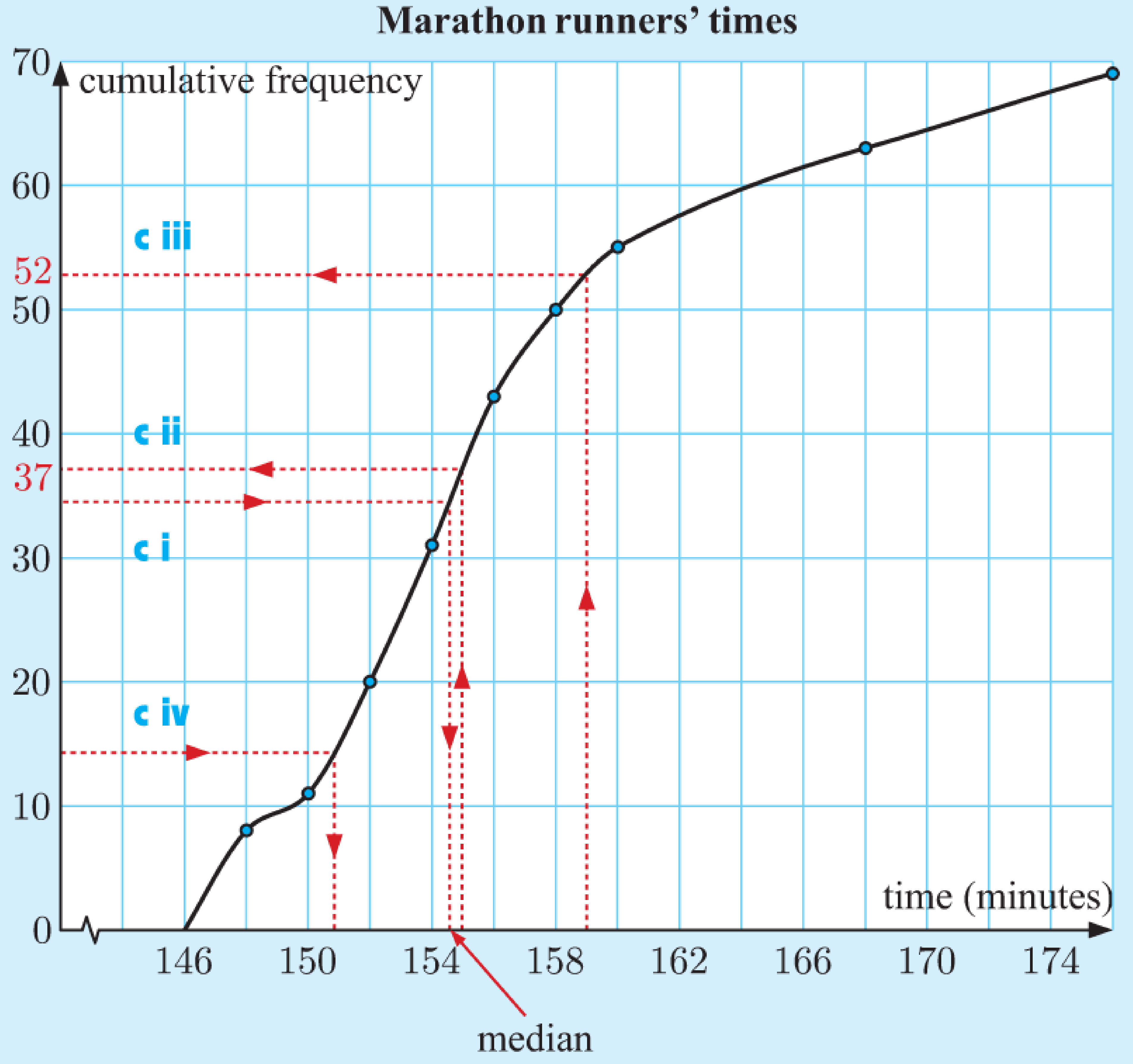
a

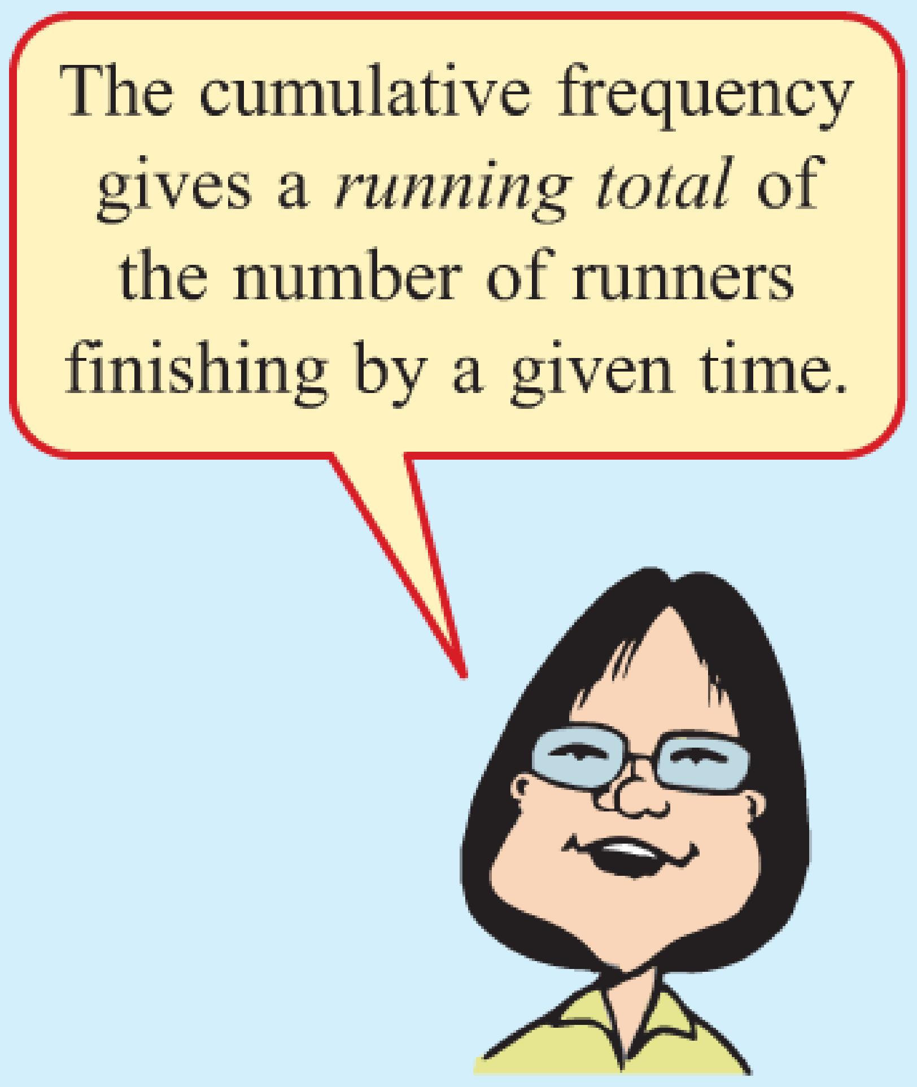
Time (t min)	*Frequency*	*Cumulative frequency*
$146 \leqslant t < 148$	8	8
$148 \leqslant t < 150$	3	11
$150 \leqslant t < 152$	9	20
$152 \leqslant t < 154$	11	31
$154 \leqslant t < 156$	12	43
$156 \leqslant t < 158$	7	50
$158 \leqslant t < 160$	5	55
$160 \leqslant t < 168$	8	63
$168 \leqslant t < 176$	6	69

$8 + 3 = 11$ competitors completed the marathon in less than 150 minutes.

50 competitors completed the marathon in less than 158 minutes.

b


Marathon runners' times


The cumulative frequency gives a *running total* of the number of runners finishing by a given time.

c **i** The median is the 50th percentile. As 50% of 69 is 34.5, we start with the cumulative frequency 34.5 and find the corresponding time.

The median ≈ 154.5 min.

ii Approximately 37 competitors took less than 155 min to complete the race.

iii $69 - 52 = 17$ competitors took more than 159 min.

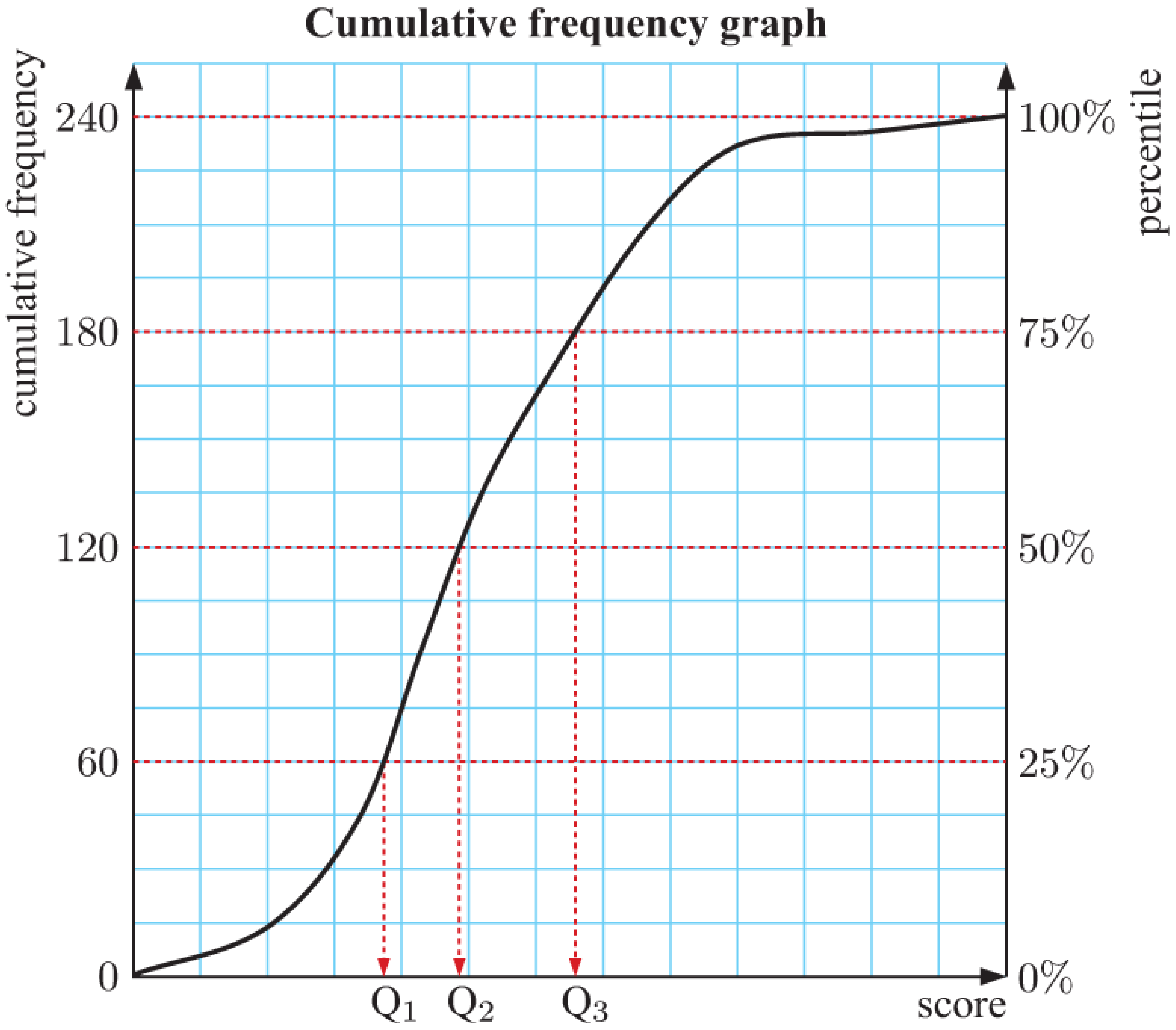
$\therefore \ \frac{17}{69} \approx 24.6\%$ took more than 159 min.

iv As 20% of 69 is 13.8, we start with the cumulative frequency 14 and find the corresponding time.

The top 20% of competitors took less than 151 min.

Another way to calculate percentiles is to add a separate scale to the cumulative frequency graph.

For example, on the graph alongside, the cumulative frequency is read from the axis on the left side, and each value corresponds to a percentile on the right side.



EXERCISE 13I

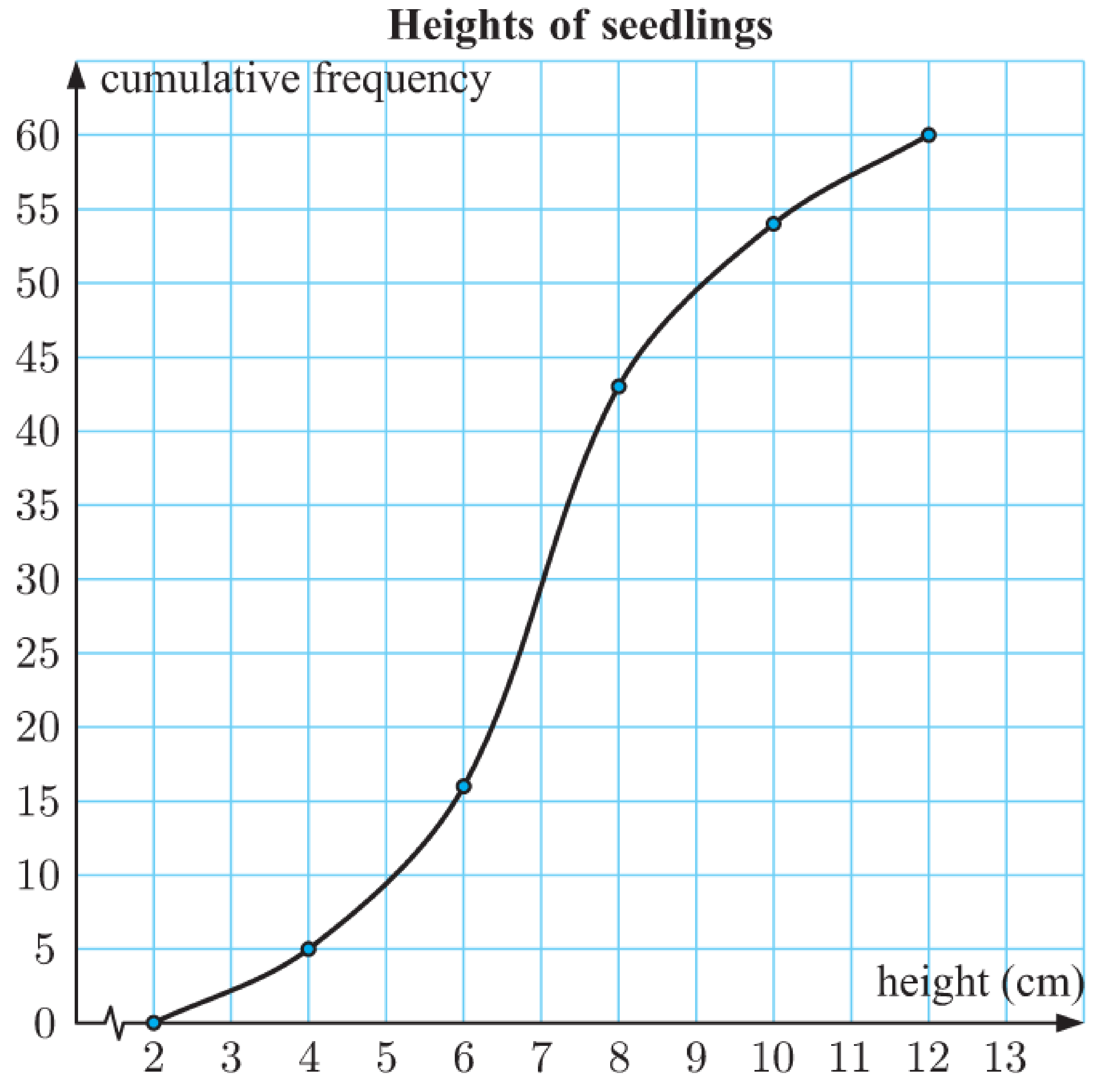
1 The examination scores of a group of students are shown in the table.

Score (x)	*Frequency*
$10 \leqslant x < 20$	2
$20 \leqslant x < 30$	5
$30 \leqslant x < 40$	7
$40 \leqslant x < 50$	21
$50 \leqslant x < 60$	36
$60 \leqslant x < 70$	40
$70 \leqslant x < 80$	27
$80 \leqslant x < 90$	9
$90 \leqslant x < 100$	3

- **a** Draw a cumulative frequency graph for the data.
- **b** Find the median examination mark.
- **c** How many students scored 65 marks or less?
- **d** How many students scored at least 50 but less than 70 marks?
- **e** If the pass mark was 45, how many students failed?
- **f** If the top 16% of students were awarded credits, what was the credit mark?

2 A botanist has measured the heights of 60 seedlings and has presented her findings on this cumulative frequency graph.

- **a** How many seedlings have heights of 5 cm or less?
- **b** What percentage of seedlings are taller than 8 cm?
- **c** Find the median height.
- **d** Find the interquartile range for the heights.
- **e** Find the 90th percentile for the data and explain what this value represents.



3 The following table summarises the age groups of car drivers involved in accidents in a city for a given year.

a Draw a cumulative frequency graph for the data.

b Estimate the median age of the drivers involved in accidents.

c Estimate the percentage of drivers involved in accidents who had an age of 23 or less.

d Estimate the probability that a driver involved in an accident is aged:

i 27 years or less **ii** 27 years.

Age (x years)	*Number of accidents*
$16 \leqslant x < 20$	59
$20 \leqslant x < 25$	82
$25 \leqslant x < 30$	43
$30 \leqslant x < 35$	21
$35 \leqslant x < 40$	19
$40 \leqslant x < 50$	11
$50 \leqslant x < 60$	24
$60 \leqslant x < 80$	41

4 The following data are the lengths of 30 trout caught in a lake during a fishing competition. The measurements were rounded *down* to the next centimetre.

31 38 34 40 24 33 30 36 38 32 35 32 36 27 35
40 34 37 44 38 36 34 33 31 38 35 36 33 33 28

a Construct a cumulative frequency table for trout lengths, x cm, using the intervals $24 \leqslant x < 27$, $27 \leqslant x < 30$, and so on.

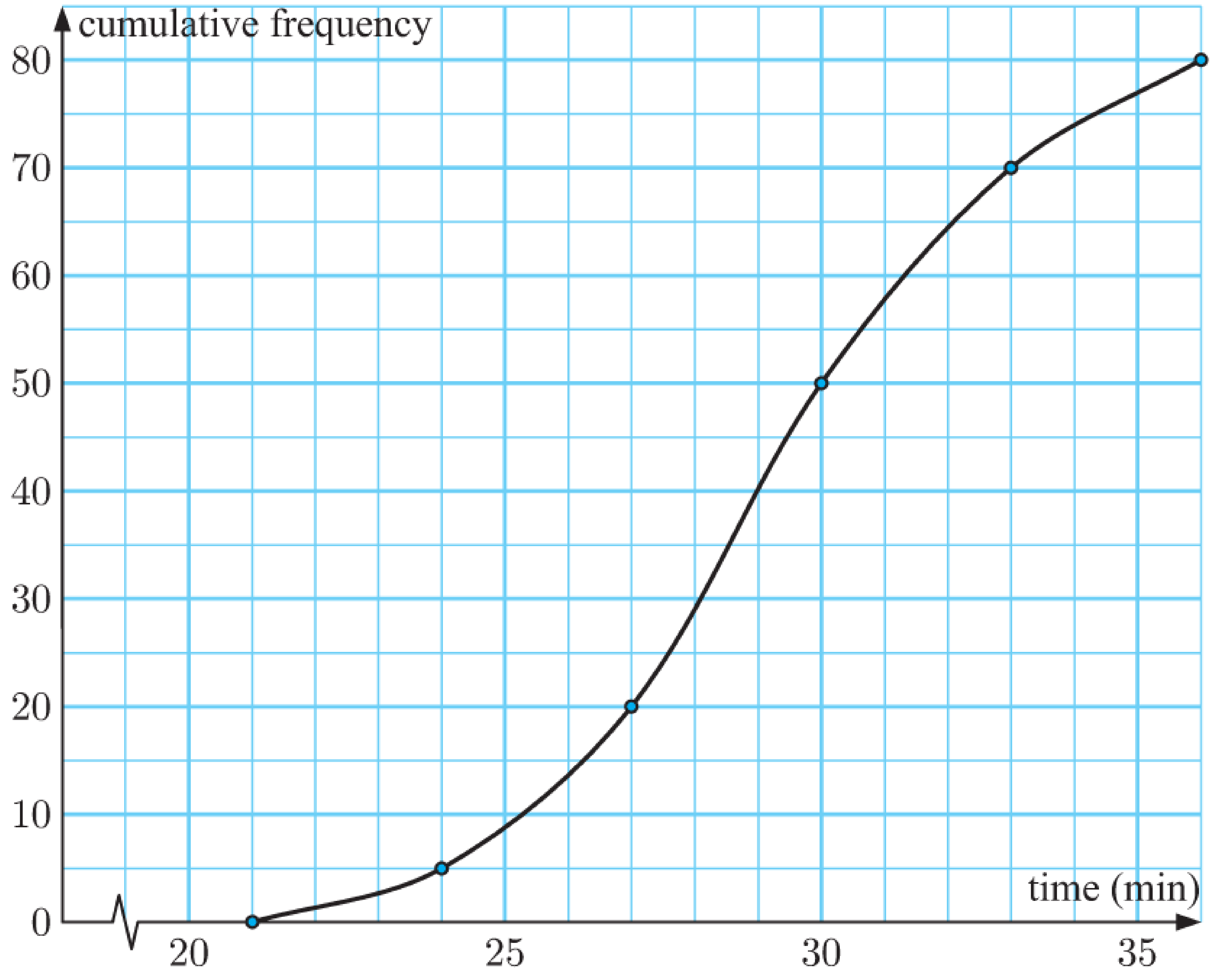
b Draw a cumulative frequency graph for the data.

c Hence estimate the median length.

d Use the original data to find its median and compare your answer with **c**.

5 The following cumulative frequency graph displays the performances of 80 competitors in a cross-country race.

Cross-country race times



a Find the lower quartile.

b Find the median.

c Find the upper quartile.

d Find the IQR.

e Estimate the 40th percentile.

f Use the cumulative frequency curve to complete the following table:

Time (t min)	$21 \leqslant t < 24$	$24 \leqslant t < 27$	$27 \leqslant t < 30$	$30 \leqslant t < 33$	$33 \leqslant t < 36$
Number of competitors					

6 The table shows the lifetimes of a sample of electric light globes.

Life (l hours)	*Number of globes*
$0 \leqslant l < 500$	5
$500 \leqslant l < 1000$	17
$1000 \leqslant l < 2000$	46
$2000 \leqslant l < 3000$	79
$3000 \leqslant l < 4000$	27
$4000 \leqslant l < 5000$	4

- **a** Draw a cumulative frequency graph for the data.
- **b** Estimate the median life of a globe.
- **c** Estimate the percentage of globes which had a life of 2700 hours or less.
- **d** Estimate the number of globes which had a life between 1500 and 2500 hours.

7 The following frequency distribution was obtained by asking 50 randomly selected people to measure the length of their feet. Their answers are given to the nearest centimetre.

Foot length (cm)	20	21	22	23	24	25	26	27	28	29	30
Frequency	1	1	0	3	5	13	17	7	2	0	1

- **a** Between what limits are lengths rounded to 20 cm?
- **b** Rewrite the frequency table to show the data in the class intervals you have just described.
- **c** Hence draw a cumulative frequency graph for the data.
- **d** Estimate:
  - **i** the median foot length
  - **ii** the number of people with foot length 26 cm or more.

J VARIANCE AND STANDARD DEVIATION

The problem with using the range and the IQR as measures of spread or dispersion is that both of them only use two values in their calculation. As a result, some data sets can have their spread characteristics hidden when only the range or IQR are quoted.

So we need to consider alternative measures of spread which take into account all data values of a data set. We therefore turn to the **variance** and **standard deviation**.

POPULATION VARIANCE AND STANDARD DEVIATION

The **population variance** of a data set $\{x_1, x_2, x_3, ..., x_n\}$ is

$$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}$$

where μ is the population mean
and n is the number of data values.

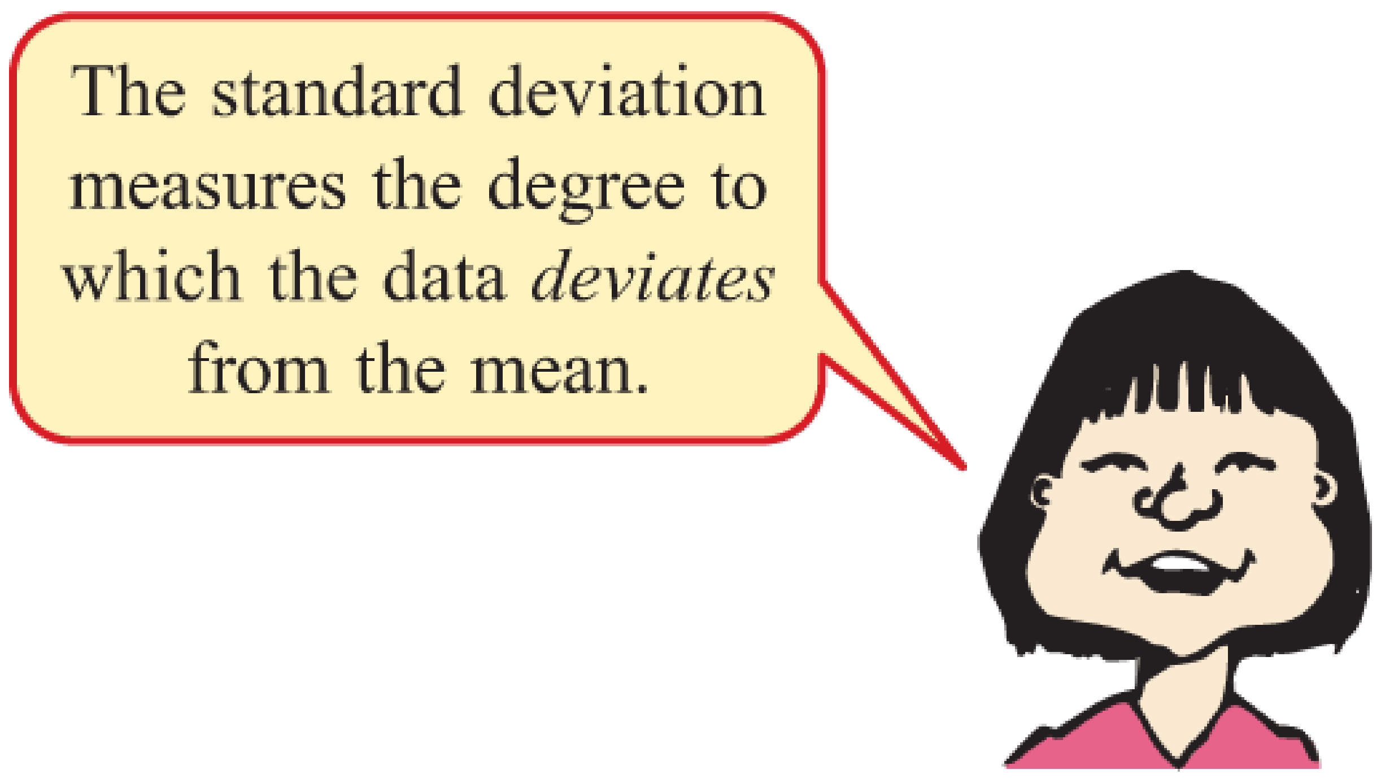
The variance is the average of the squares of the distances from the mean.

We observe that if the data values x_i are situated close together around the mean μ, then the values $(x_i - \mu)^2$ will be small, and so the variance will be small.

The **standard deviation** is the square root of the variance.

The **population standard deviation** of a data set $\{x_1, x_2, x_3, ..., x_n\}$ is

$$\boldsymbol{\sigma = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{n}}}.$$



The square root in the standard deviation is used to correct the units. For example, if x_i is the weight of a student in kg, the variance σ^2 would be in kg^2, and σ would be in kg.

The standard deviation is a **non-resistant** measure of spread. This is due to its dependence on the mean and because extreme data values will give large values for $(x_i - \mu)^2$. It is only a useful measure if the distribution is approximately symmetrical.

The IQR and percentiles are more appropriate tools for measuring spread if the distribution is considerably skewed.

Example 12 **Self Tutor**

Find the population variance and standard deviation for the data set: 3 12 8 15 7

The mean $\mu = \dfrac{3+12+8+15+7}{5} = 9$

The population variance $\sigma^2 = \dfrac{\sum(x-\mu)^2}{n}$

$= \dfrac{86}{5}$

$= 17.2$

The population standard deviation $\sigma = \sqrt{17.2}$

≈ 4.15

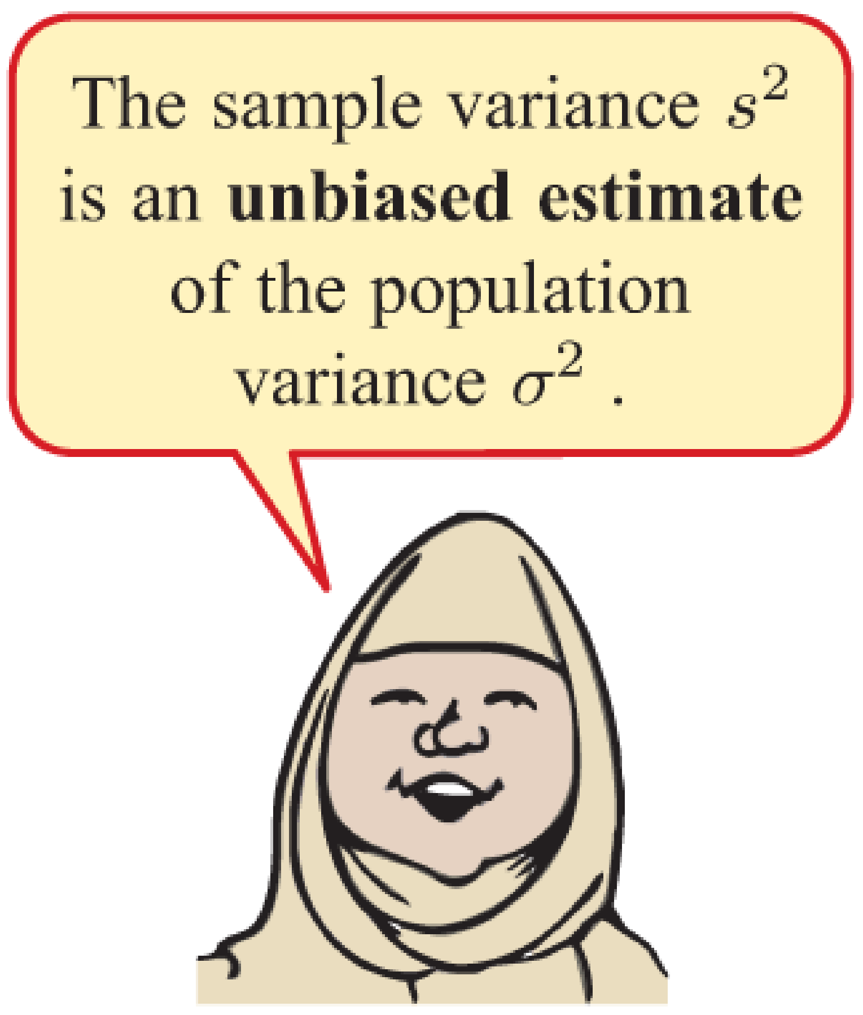
x	$x-\mu$	$(x-\mu)^2$
3	-6	36
12	3	9
8	-1	1
15	6	36
7	-2	4
	Total	86

SAMPLE VARIANCE AND STANDARD DEVIATION

If we are only given a *sample* of data from a larger population, we calculate a statistic called the **sample variance**. This statistic is used to *estimate* the variance of the population.

For a sample of n data values $\{x_1, x_2, x_3, ..., x_n\}$ with sample mean $\overline{x}$:

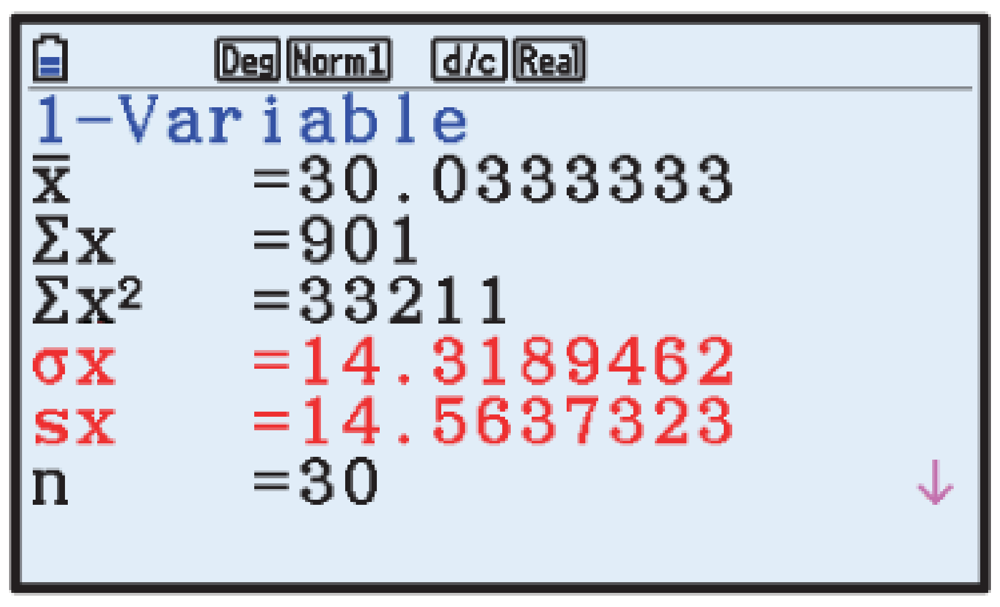
- The **sample variance** is $\boldsymbol{s^2 = \dfrac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}}$.
- The **sample standard deviation** is $\boldsymbol{s = \sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}}}$.



CALCULATING VARIANCE AND STANDARD DEVIATION

We commonly use technology to find the standard deviation. If we are given the whole population we use σ_x, and if we are given a sample we use s_x. We can then square these values to find the population and sample variance, $\sigma_x{}^2$ and $s_x{}^2$ respectively.

GRAPHICS CALCULATOR INSTRUCTIONS



```
1-Variable
x̄     =30.0333333
Σx    =901
Σx²   =33211
σx    =14.3189462
sx    =14.5637323
n     =30
```

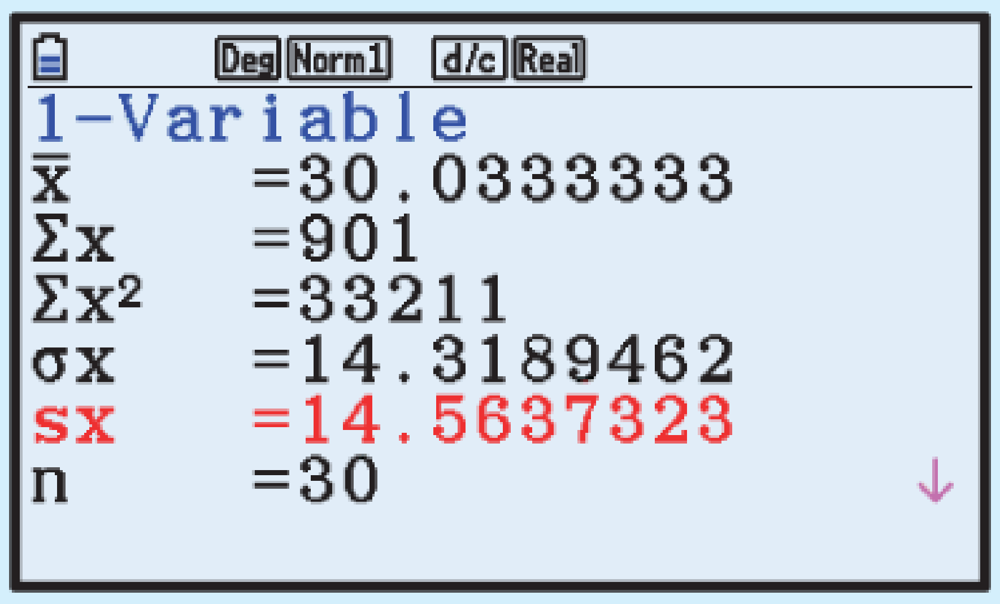
Example 13 **Self Tutor**

Kylie is interested in the ages of spectators at a rugby match. She selects a sample of 30 spectators. Their ages are shown below:

17	24	30	10	42	48	37	19	28	53	29	40	11	21	9
43	22	59	46	52	31	13	7	26	32	47	22	15	26	42

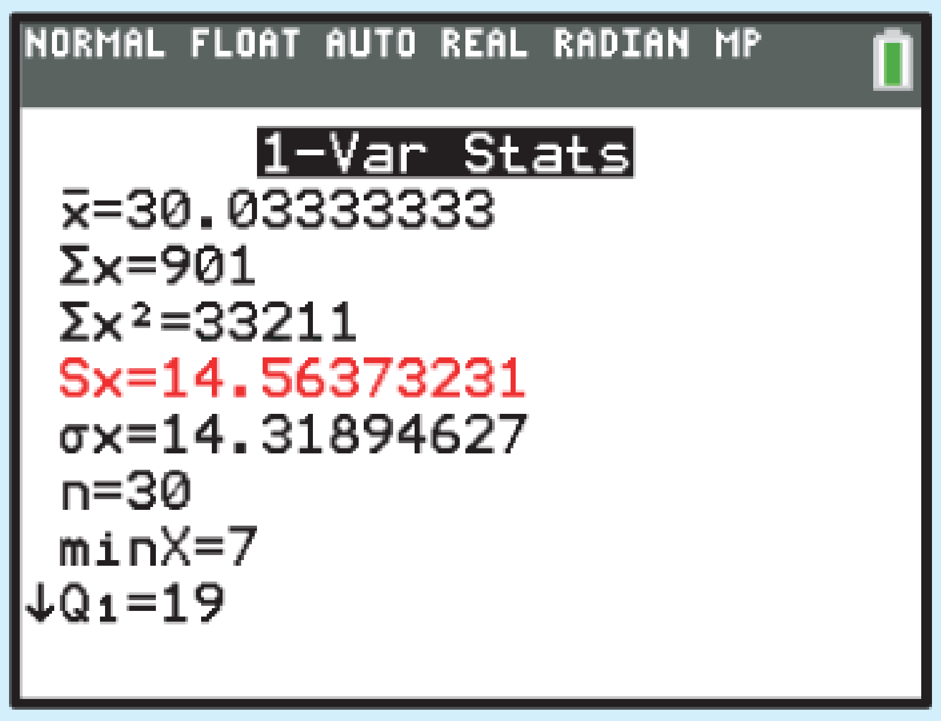
Use technology to find the sample standard deviation.

Casio fx-CG50



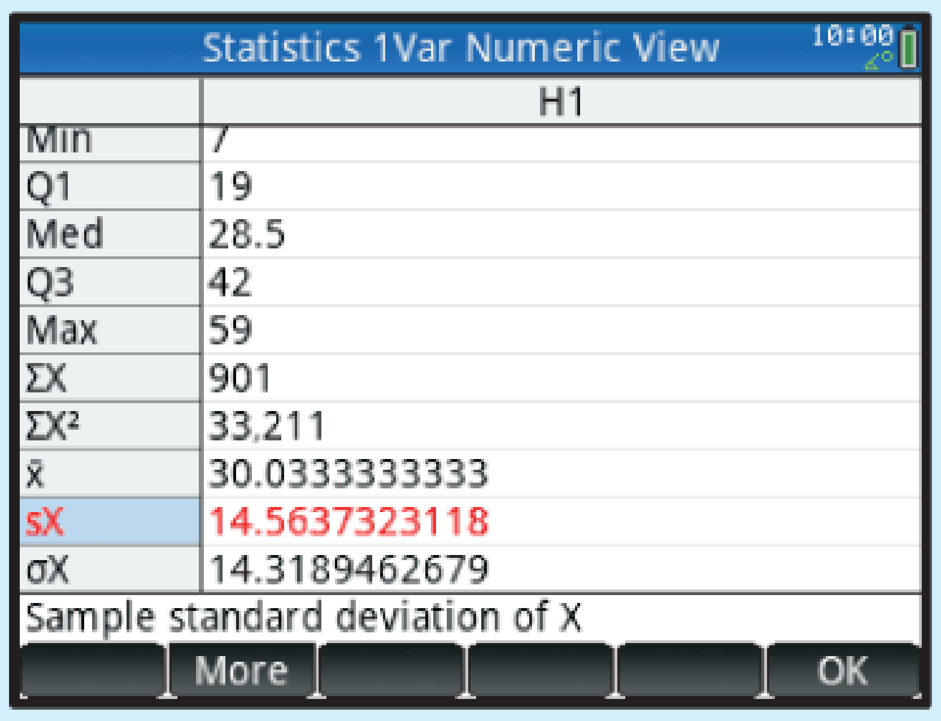
```
1-Variable
x̄     =30.0333333
Σx    =901
Σx²   =33211
σx    =14.3189462
sx    =14.5637323
n     =30
```

TI-84 Plus CE



```
1-Var Stats
x̄=30.03333333
Σx=901
Σx²=33211
Sx=14.56373231
σx=14.31894627
n=30
minX=7
↓Q₁=19
```

HP Prime



	H1
Min	7
Q1	19
Med	28.5
Q3	42
Max	59
ΣX	901
ΣX²	33,211
x̄	30.0333333333
sX	14.5637323118
σX	14.3189462679

Sample standard deviation of X

The sample standard deviation $s \approx 14.6$ years.

The reason there are two formulae for standard deviation is that if we have data which is sampled from a large population, the sample standard deviation s provides a better estimate for the actual population standard deviation σ than if we use the formula for σ on the sample.

However, in the **Mathematics: Analysis and Approaches HL** course you are expected to calculate all standard deviations as though they were populations. For this reason, two answers are given for some questions in the following Exercise.

EXERCISE 13J

1 Consider the following data sets:

Data set A: 10 7 5 8 10
Data set B: 4 12 11 14 1 6

- **a** Show that each data set has mean 8.
- **b** Which data set appears to have the greater spread? Explain your answer.
- **c** Find the population variance and standard deviation of each data set. Use technology to check your answers.

2 Skye recorded the number of pets owned by each student in her class.

0	2	3	1	2	4	0	0	1	5	2	3	6
2	3	1	1	0	4	1	1	0	2	1	2	0

- **a** Describe the population in this case.
- **b** Use technology to find the population standard deviation of the data.
- **c** Find the population variance of the data.

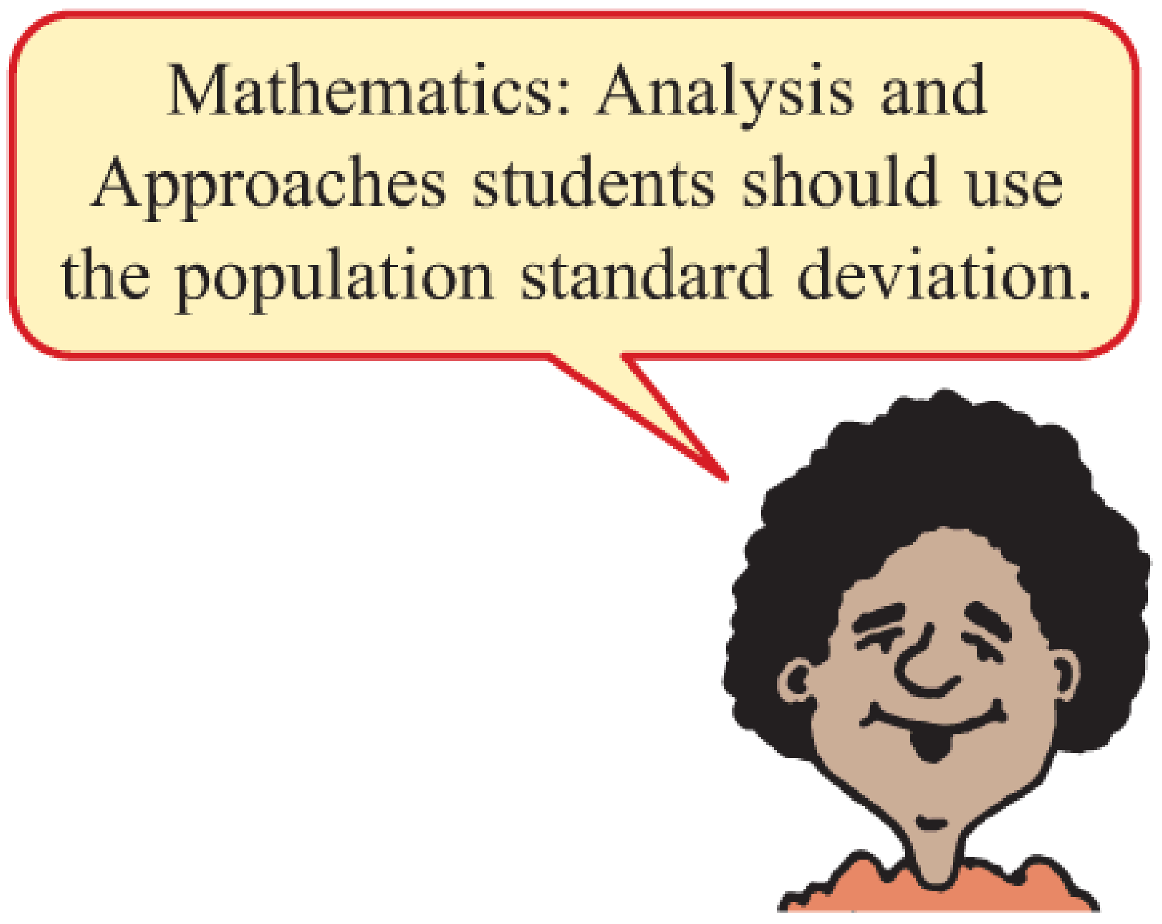
3 The ages of members of an Olympic water polo team are: 22, 25, 23, 28, 29, 21, 20, 26.

- **a** Calculate the mean and population standard deviation for this group.
- **b** The same team members are chosen to play in the next Olympic Games 4 years later. Calculate the mean and population standard deviation of their ages at the next Olympic Games.
- **c** Comment on your results in general terms.

4 A hospital selected a sample of 20 patients and asked them how many glasses of water they had consumed that day. The results were:

5	2	1	0	4	1	0	2	7	4
8	2	7	6	1	2	3	8	0	2

Find the standard deviation of the data.



5 Danny and Jennifer recorded how many hours they spent on homework each day for 14 days.

Danny: $3\frac{1}{2}$, $3\frac{1}{2}$, 4, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 3, $1\frac{1}{2}$, 3, 4, $2\frac{1}{2}$, 4, 4, 3

Jennifer: $2\frac{1}{2}$, 1, $2\frac{1}{2}$, 2, 2, $2\frac{1}{2}$, $1\frac{1}{2}$, 2, 2, $2\frac{1}{2}$, 2, 2, 2, $1\frac{1}{2}$

- **a** Calculate the mean number of hours each person spent on homework.
- **b** Which person generally studies for longer?
- **c** Calculate the standard deviation for each data set.
- **d** Which person studies more consistently?

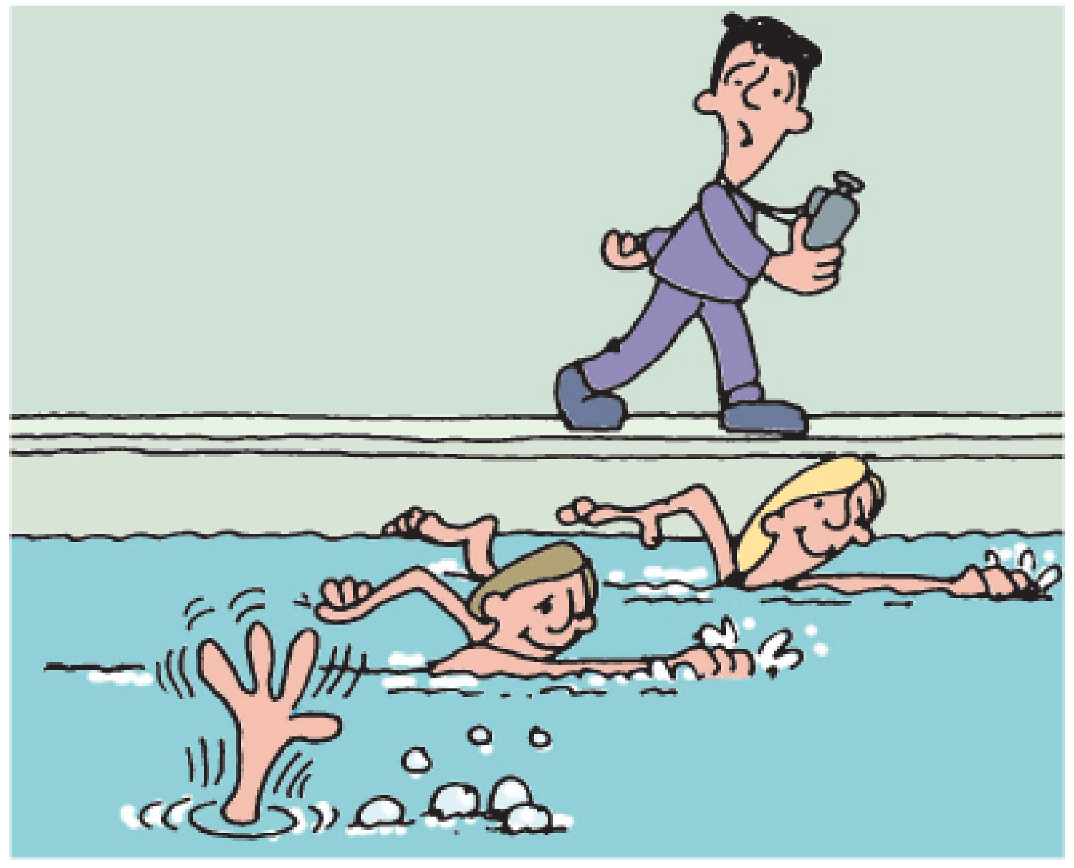
6 Tyson wants to compare the swimming speeds of boys and girls at his school. He randomly selects 10 boys and 10 girls, and records the time, in seconds, each person takes to swim two laps of the 25 m school pool.

Boys: 32.2, 26.4, 35.6, 30.8, 28.5, 40.2, 27.3, 38.9, 29.0, 31.3

Girls: 36.2, 33.5, 28.1, 39.8, 31.6, 35.7, 37.3, 36.0, 39.7, 29.8

- **a** Copy and complete the table:

	Boys	*Girls*
Mean $\overline{x}$		
Median		
Standard deviation		
Range		



- **b** Which group:
  - **i** generally swims faster
  - **ii** has the greater spread of swimming speeds?
- **c** How could Tyson improve the reliability of his findings?

7 Two baseball coaches compare the number of runs scored by their teams in their last ten games:

Rockets	0	10	1	9	11	0	8	5	6	7
Bullets	4	3	4	1	4	11	7	6	12	5

a Show that the two teams have the same mean and range of runs scored.

b Which team's performance do you suspect is more variable? Check your answer by finding the standard deviation for each data set.

c Does the range or the standard deviation give a better indication of variability?

8 The number of visitors to a museum and an art gallery each day during December are shown.

Museum:	1108	1019	850	1243	1100	923	964	847	918	820	781
	963	814	881	742	911	1101	952	864	943	1087	1132
	906	1050	0	826	986	1040	1127	1084	981		
Art gallery:	1258	1107	1179	1302	1236	1386	1287	1313	1269	1332	1094
	1153	1275	1168	1086	1276	1342	1153	1227	1305	1187	1249
	1300	1156	1074	1168	1299	1257	1134	1259	1366		

a For each data set, calculate the:

i mean **ii** standard deviation.

b Which place had the greater spread of visitor numbers?

c **i** Identify the outlier in the *Museum* data.

ii Give a reason why this outlier may have occurred.

iii Do you think it is reasonable to remove the outlier when comparing the numbers of visitors to these places? Explain your answer.

iv Recalculate the mean and standard deviation with the outlier removed.

v Discuss the effect of the outlier on the standard deviation.

9 Anna and Apple want to compare the distributions of weights from two different populations A and B, so they take samples from each population. Apple uses the sample standard deviation formula, and finds that $s_A > s_B$. Anna uses the formula for the population standard deviation on the samples. Will Anna find that $\sigma_A > \sigma_B$ for the samples? Explain your answer.

10 A set of 8 integers $\{1, 3, 5, 7, 4, 5, p, q\}$ has mean 5 and population variance 5.25. Find p and q given that $p < q$.

11 A set of 10 integers $\{3, 9, 5, 5, 6, 4, a, 6, b, 8\}$ has mean 6 and population variance 3.2. Find a and b given that $a > b$.

12 **a** Prove that $\sum_{i=1}^{n}(x_i - \mu)^2 = \sum_{i=1}^{n}\left({x_i}^2\right) - n\mu^2$.

b The data set $\{x_1, x_2, ..., x_{25}\}$ has $\sum_{i=1}^{25} {x_i}^2 = 2568.25$ and population standard deviation 5.2.

Find the mean of the data set.

13 Find the population standard deviation of this data set.

Value	*Frequency*
3	1
4	3
5	11
6	5

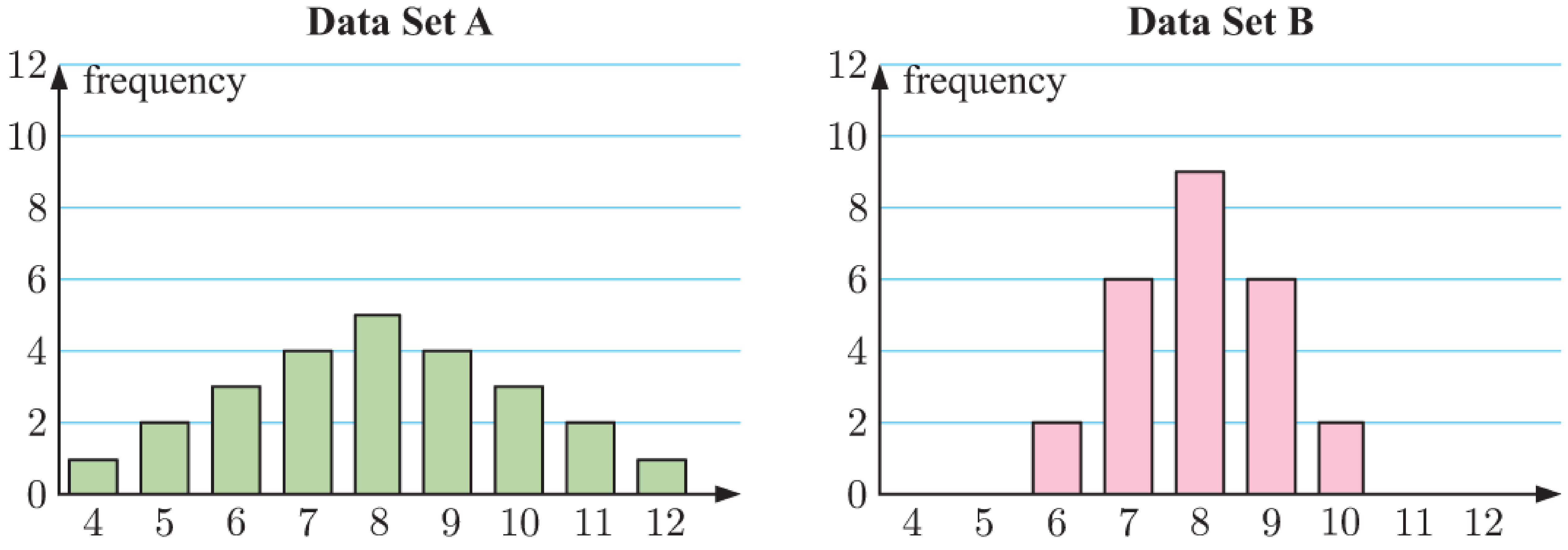
GRAPHICS CALCULATOR INSTRUCTIONS

14 The table shows the ages of squash players at the Junior National Squash Championship.

Age	11	12	13	14	15	16	17	18
Frequency	2	1	4	5	6	4	2	1

Find the mean and population standard deviation of the ages.

15 The column graphs show two distributions:



a By looking at the graphs, which distribution appears to have wider spread?

b Find the mean of each data set.

c Find the population standard deviation for each data set. Comment on your answers.

d The other measures of spread for the two data sets are given in the table.

Data set	*Range*	*IQR*
A	8	3
B	4	2

In what way does the standard deviation give a better description of how the data is distributed?

16 The table alongside shows the results obtained by female and male students in a test out of 20 marks.

a Looking at the table:

  i Which group appears to have scored better in the test?

  ii Which group appears to have a greater spread of scores?

Justify your answers.

b Calculate the mean and population standard deviation for each group.

Score	*Females*	*Males*
12	0	1
13	0	0
14	0	2
15	0	3
16	2	4
17	6	2
18	5	0
19	1	1
20	1	0

17 Brianna and Jess are conducting a survey of their class.
Brianna asked every student (including Jess and herself) how many children there are in their family.
Jess asked every student (including Brianna and herself) how many siblings or step-siblings they have.
How will their results compare in terms of mean and standard deviation? Explain your answer.

Example 14 **Self Tutor**

The table alongside summarises the examination scores for 80 randomly selected students. Estimate the standard deviation for the data.

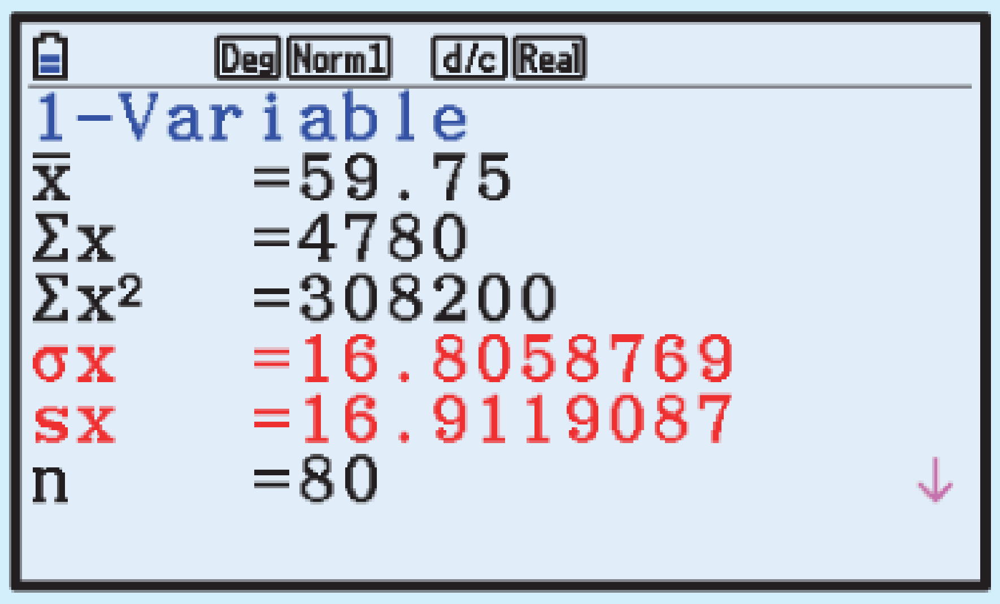
Mark	*Frequency*
0 - 9	1
10 - 19	1
20 - 29	2
30 - 39	4
40 - 49	11

Mark	*Frequency*
50 - 59	16
60 - 69	24
70 - 79	13
80 - 89	6
90 - 99	2

Class interval	*Mid-interval value*	*Frequency*
0 - 9	4.5	1
10 - 19	14.5	1
20 - 29	24.5	2
30 - 39	34.5	4
40 - 49	44.5	11
50 - 59	54.5	16
60 - 69	64.5	24
70 - 79	74.5	13
80 - 89	84.5	6
90 - 99	94.5	2

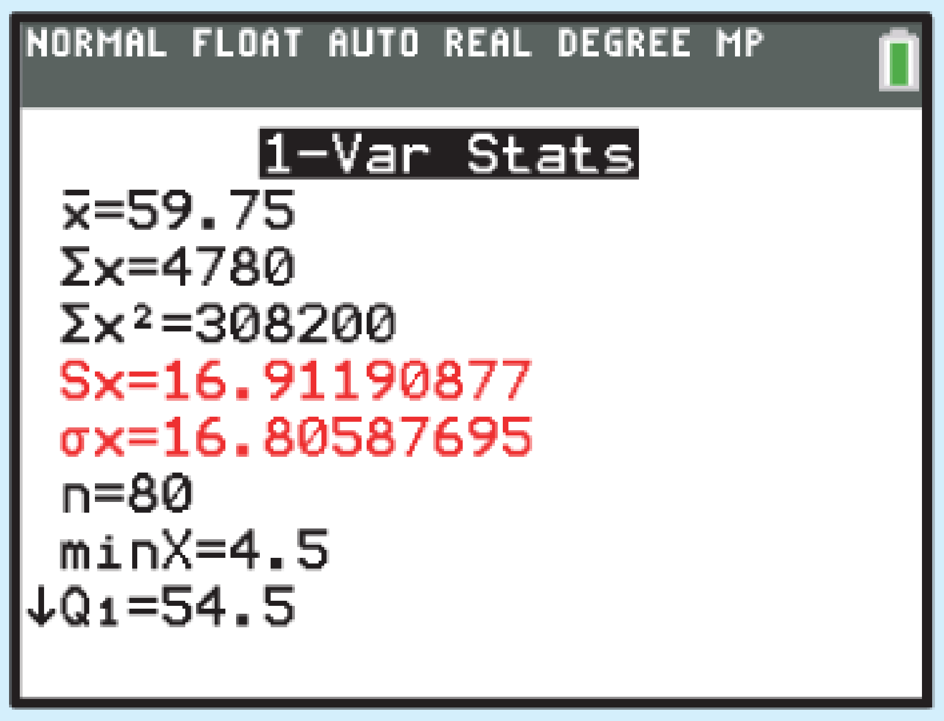
For continuous data or data grouped in classes, use the mid-interval value to represent all data in that interval.

Casio fx-CG50



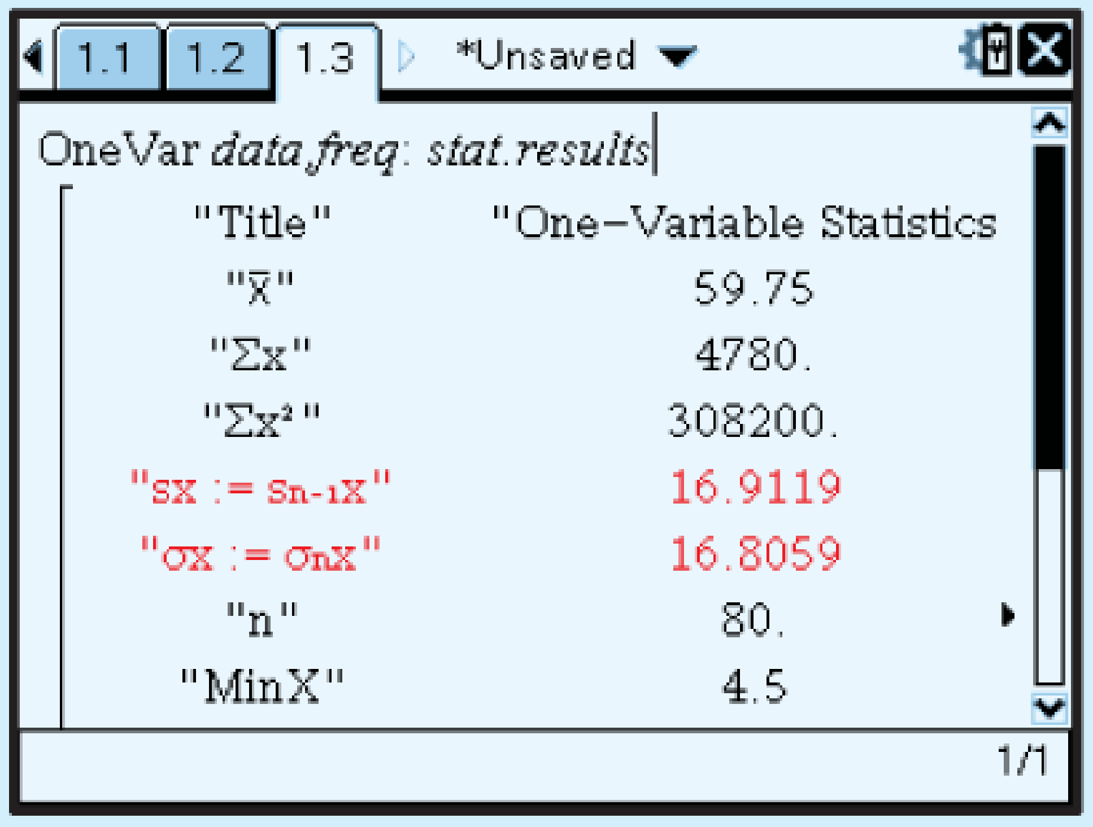
```
1-Variable
x̄     =59.75
Σx    =4780
Σx²   =308200
σx    =16.8058769
sx    =16.9119087
n     =80
```

TI-84 Plus CE



```
NORMAL FLOAT AUTO REAL DEGREE MP
       1-Var Stats
 x̄=59.75
 Σx=4780
 Σx²=308200
 Sx=16.91190877
 σx=16.80587695
 n=80
 minX=4.5
↓Q₁=54.5
```

TI-*n*spire



```
OneVar data,freq: stat.results
   "Title"          "One-Variable Statistics"
   "x̄"              59.75
   "Σx"             4780.
   "Σx²"            308200.
   "sx := sn-1x"    16.9119
   "σx := σnx"      16.8059
   "n"              80.
   "MinX"           4.5
```

The sample standard deviation ≈ 16.9.

The population standard deviation ≈ 16.8.

18 The lengths of 30 randomly selected 12-day old babies were measured and the following data obtained:

For the given data, estimate the:

a mean **b** standard deviation.

Length (L cm)	*Frequency*
$40 \leqslant L < 42$	1
$42 \leqslant L < 44$	1
$44 \leqslant L < 46$	3
$46 \leqslant L < 48$	7
$48 \leqslant L < 50$	11
$50 \leqslant L < 52$	5
$52 \leqslant L < 54$	2

19 A traffic survey revealed that the following numbers of vehicles passed through a suburban intersection in 5 minute intervals during the day.

Vehicles	1 - 5	6 - 10	11 - 15	16 - 20	21 - 25	26 - 30	31 - 35	36 - 40
Frequency	4	16	22	28	14	9	5	2

For the given data, estimate the:

a mean **b** standard deviation.

20 The weekly wages of 200 randomly selected steel workers are given alongside:

For the given data, estimate the:

a mean **b** standard deviation.

Wage ($*W*)	*Number of workers*
$720 \leqslant W < 740$	17
$740 \leqslant W < 760$	38
$760 \leqslant W < 780$	47
$780 \leqslant W < 800$	57
$800 \leqslant W < 820$	18
$820 \leqslant W < 840$	10
$840 \leqslant W < 860$	10
$860 \leqslant W < 880$	3

21 The hours worked last week by 40 employees of a local clothing factory were as follows:

38	40	46	32	41	39	44	38	40	42	38	40	43	41
47	36	38	39	34	40	48	30	49	40	40	43	45	36
35	39	42	44	48	36	38	42	46	38	39	40		

a Calculate the mean and standard deviation for this data.

b Now group the data into classes 30 - 33, 34 - 37, and so on. Calculate the mean and standard deviation using these groups. Examine any differences between the two sets of answers.

INVESTIGATION 3 TRANSFORMING DATA

In this Investigation we will explore the effects of transforming a data set on its mean and standard deviation.

We will use this data set as a basis:

4	2	3	3	5	2	9	7	3	5
2	1	5	3	6	6	3	3	6	7

What to do:

1 Calculate the mean and population standard deviation for the data set.

2 **a** Suppose we add 5 to each data value. Calculate the mean and population standard deviation for the new data set.

b What do you expect to happen to the mean and standard deviation if k is added to each value in a data set?

c Check your answer by:

i adding 11 to each data value **ii** subtracting 3 from each data value.

3 **a** Suppose we multiply each value in the original data set by 4. Calculate the mean and population standard deviation for the new data set.

b What do you expect to happen to the mean and standard deviation if each value in a data set is multiplied by a?

c Check your answer by:
  i multiplying each value by 9   **ii** dividing each value by 4.

4 Suppose a data set $\{x_i\}$ has mean μ and standard deviation σ. Write down the mean and standard deviation for the data set:
  a $\{ax_i\}$ **b** $\{x_i + k\}$ **c** $\{ax_i + k\}$

INVESTIGATION 4 ESTIMATING THE VARIANCE AND STANDARD DEVIATION OF A POPULATION

In this Investigation we consider the accuracy of using a sample to make inferences about a whole population. This will help you to see why statisticians have a subtly different formula for the standard deviation of a sample.

The Year 12 students at a school were asked to record how many minutes they spent travelling to school. The results were collected in a survey the following morning.

There are a total of 150 Year 12 students at the school, and these are split into 6 classes.

What to do:

1 Click on the icon to obtain a spreadsheet containing all of the responses to the survey.

SPREADSHEET

  a Use the frequency table in the spreadsheet to draw a histogram for the data. Describe this distribution.
  b The summary statistics in the spreadsheet are calculated using all of the survey responses, and hence are the *true* population values. Find the true population variance.

2 10 students were randomly selected from each class to form 6 samples. Their responses to the survey are shown below:

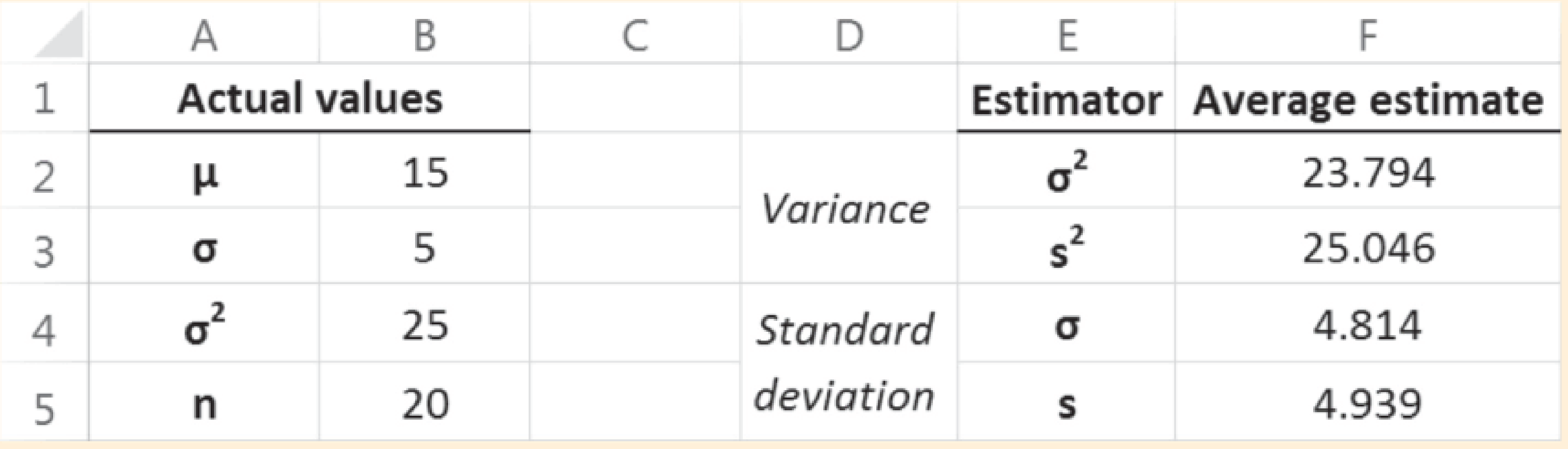
Sample 1:	10	14	16	9	16	15	15	21	9	21
Sample 2:	11	9	11	16	16	13	10	12	21	16
Sample 3:	12	10	14	7	13	11	21	20	15	9
Sample 4:	20	19	19	19	13	19	22	15	10	19
Sample 5:	19	13	23	11	17	4	14	21	13	11
Sample 6:	19	11	16	6	8	13	10	22	20	11

  a Calculate the *sample* statistics s and s^2 for each sample.
  b Calculate the *population* statistics σ and σ^2 for each sample.
  c Which set of estimates from **a** and **b** are generally closer to the true population variance and standard deviation?
  d Does your answer to **c** explain why we have different variance and standard deviation formulae for a sample as opposed to a population?

3 To see which set of estimators (population or sample) are better at estimating the true population variance and standard deviation, we will consider a simulation based on the survey responses from the school.

Click on the icon to obtain a spreadsheet with 1000 simulations of the survey results. The values s, s^2, σ, and σ^2 are calculated for each simulated sample. The average values for each estimator are shown in the table on the sheet labelled "Summary".

SPREADSHEET



	A	B	C	D	E	F
1	Actual values				Estimator	Average estimate
2	μ	15		*Variance*	σ^2	23.794
3	σ	5			s^2	25.046
4	σ^2	25		*Standard deviation*	σ	4.814
5	n	20			s	4.939

Based on the calculations in the spreadsheet, which set of estimates (population or sample) are generally closer to the true values? Does your conclusion agree with your answer to **2 c**?

4 Change the values for μ and σ in the spreadsheet. This will now effectively simulate the results for a different distribution, perhaps the travel times for the students at a different school. Does your choice of μ or σ affect your conclusion regarding the choice of estimators?

5 Why is it important to have accurate estimates of the variance and standard deviation of a population?

REVIEW SET 13A

1 For each of the following data sets, find the: **i** mean **ii** median.

a 0, 2, 3, 3, 4, 5, 5, 6, 6, 7, 7, 8 **b** 2.9, 3.1, 3.7, 3.8, 3.9, 3.9, 4.0, 4.5, 4.7, 5.4

2 Katie loves cats. She visits every house in her street to find out how many cats live there. The responses are given below:

Number of cats	0	1	2	3	4	5
Frequency	36	9	11	5	1	1

a Draw a graph to display this data.

b Describe the distribution.

c Find the:

i mode **ii** mean **iii** median.

d Which of the measures of centre is most appropriate for this data? Explain your answer.

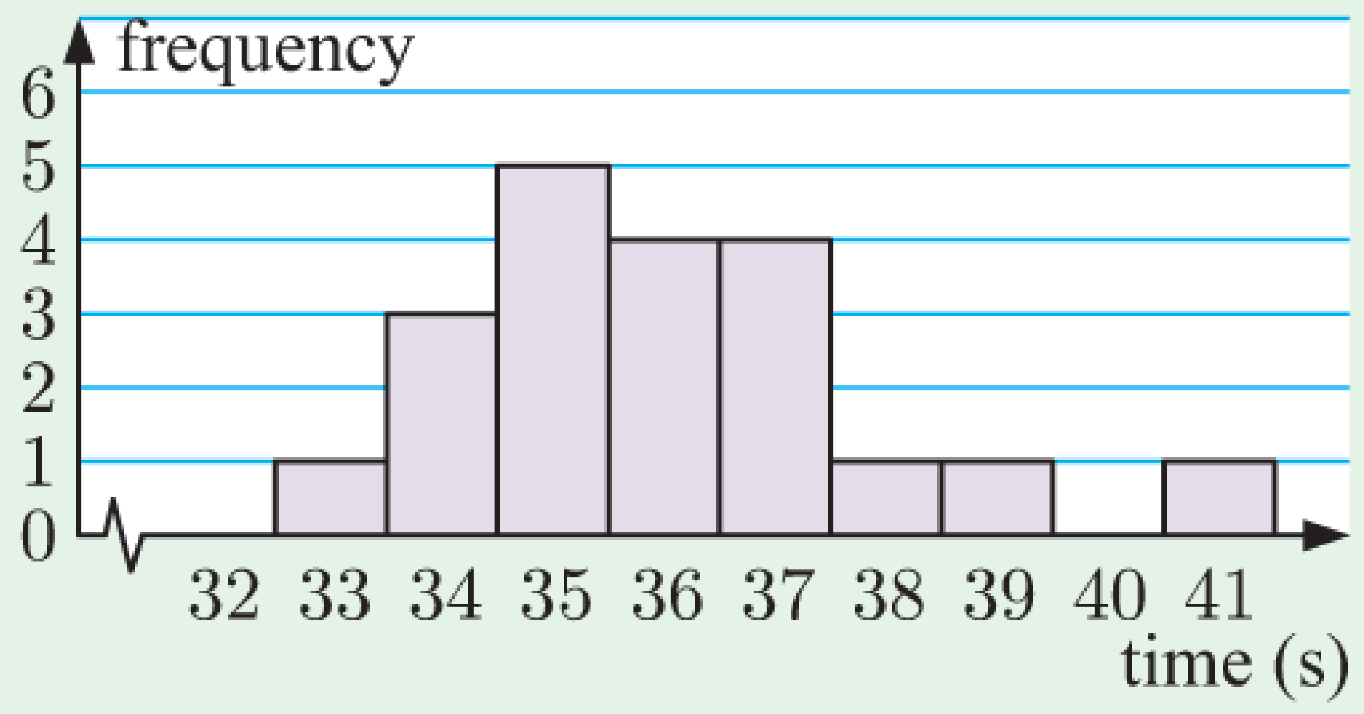
3 The histograms alongside show the times for the 50 metre freestyle recorded by members of a swimming squad.

a Copy and complete:

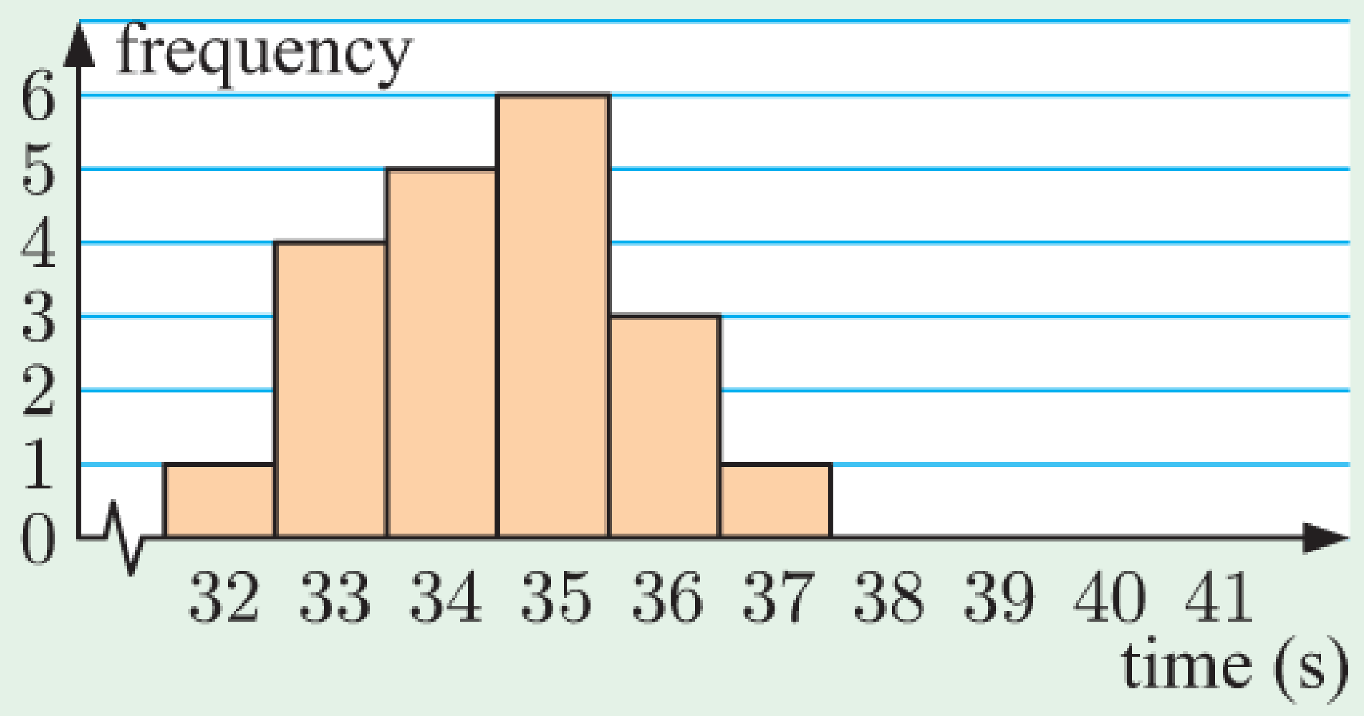
Distribution	*Girls*	*Boys*
median		
mean		
modal class		

b Discuss the distributions of times for the boys and girls. What conclusion can you make?

Girls



Boys



4 The data set $4, 6, 9, a, 3, b$ has a mean and mode of 6. Find the values of a and b given that $a > b$.

5 Consider the data set: $k - 2,\ k,\ k + 3,\ k + 3$.

- **a** Show that the mean of the data set is equal to $k + 1$.
- **b** Suppose each number in the data set is increased by 2. Find the new mean of the data set in terms of k.

6 The winning margins in 100 basketball games were recorded. The results are summarised alongside.

- **a** Explain why you cannot calculate the mean winning margin from the table exactly.
- **b** Estimate the mean winning margin.

Margin (points)	*Frequency*
1 - 10	13
11 - 20	35
21 - 30	27
31 - 40	18
41 - 50	7

7 Consider this data set:

19, 7, 22, 15, 14, 10, 8, 28, 14, 18, 31, 13, 18, 19, 11, 3, 15, 16, 19, 14

- **a** Find the five-number summary for the data.
- **b** Find the range and IQR.
- **c** Draw a box plot of the data set.

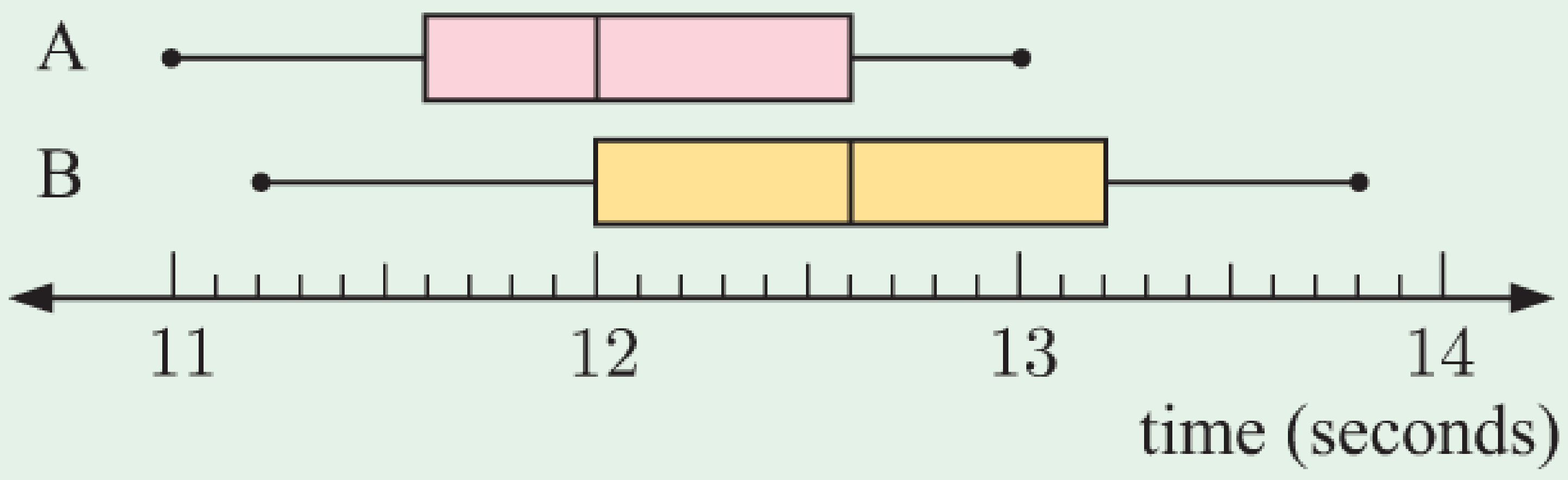
8 Katja's golf scores for her last 20 rounds were:

90	106	84	103	112	100	105	81	104	98
107	95	104	108	99	101	106	102	98	101

For this data set, find the:

- **a** median
- **b** interquartile range
- **c** mean
- **d** standard deviation.

9 The parallel box plot alongside shows the 100 metre sprint times for the members of two athletics squads.

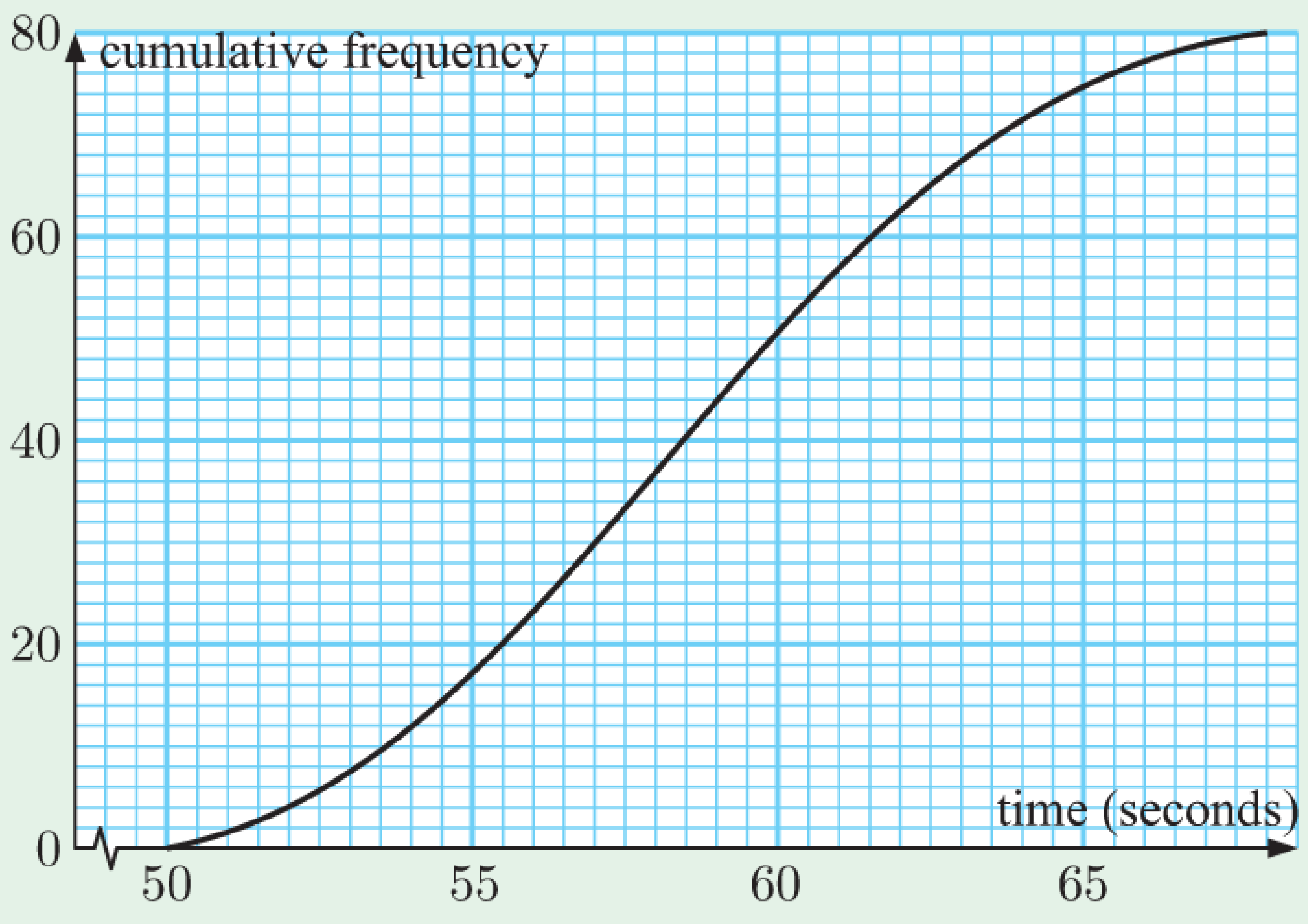


- **a** Determine the five-number summaries for both A and B.
- **b** For each group, calculate the range and interquartile range.
- **c** Copy and complete:
  - **i** The members of squad ...... generally ran faster because ......
  - **ii** The times in squad ...... are more varied because ......

10 80 senior students ran 400 metres in a Physical Education program. Their times were recorded and the results were used to produce the cumulative frequency graph shown.

Estimate:

- **a** the median
- **b** the interquartile range
- **c** the time corresponding to the top 10% of runners.



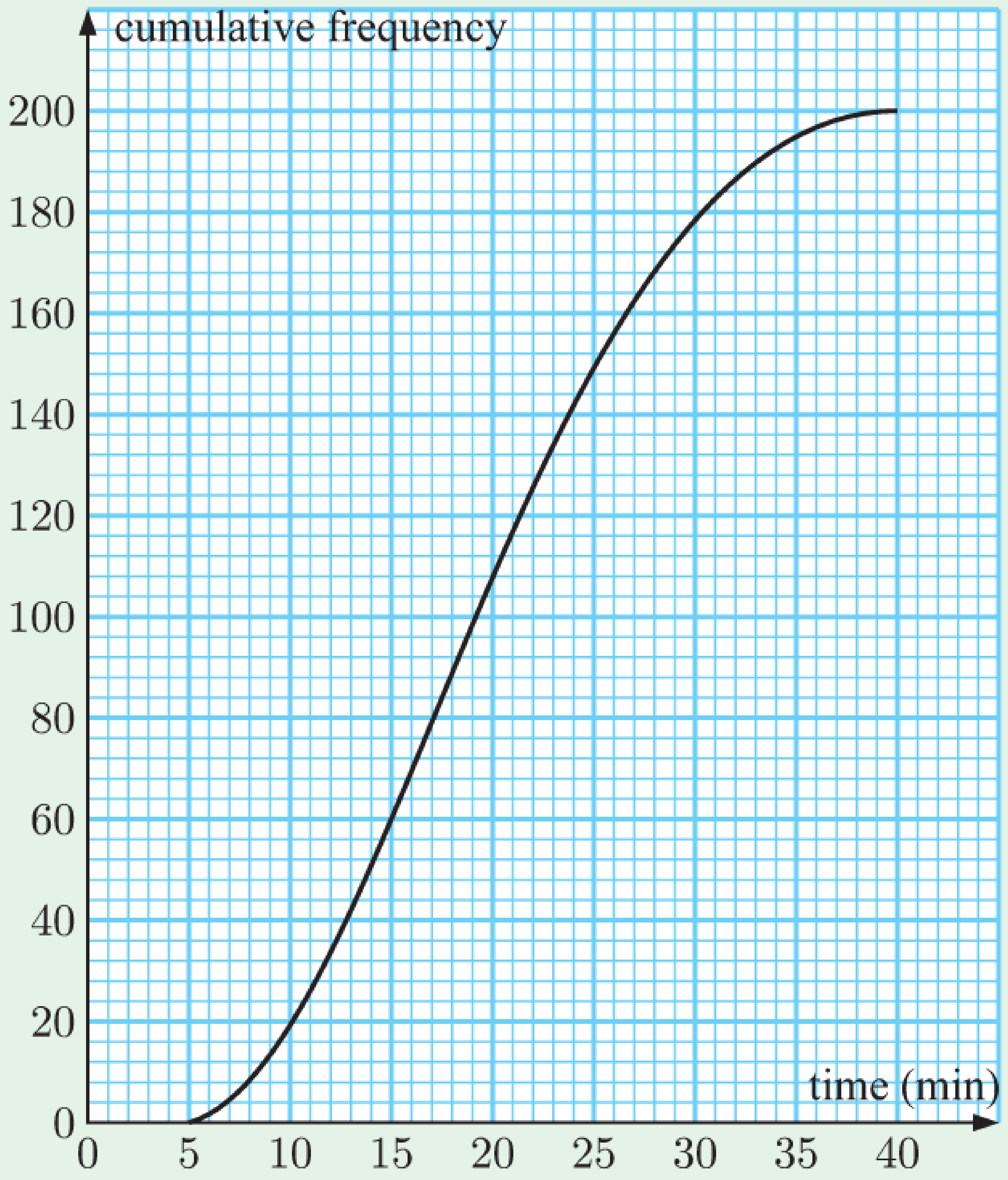
11 This cumulative frequency curve shows the times taken for 200 students to travel to school by bus.

a Estimate how many of the students spent between 10 and 20 minutes travelling to school.

b 30% of the students spent more than m minutes travelling to school. Estimate the value of m.

c Use the cumulative frequency curve to complete the following table:

Time (t min)	*Frequency*
$5 \leqslant t < 10$	
$10 \leqslant t < 15$	
$\vdots$	
$35 \leqslant t < 40$	



12 Find the population variance and standard deviation for each data set:

a 117, 129, 105, 124, 123, 128, 131, 124, 123, 125, 108

b 6.1, 5.6, 7.2, 8.3, 6.6, 8.4, 7.7, 6.2

13 The table alongside shows the number of matches in a sample of boxes.

Number	47	48	49	50	51	52
Frequency	21	29	35	42	18	31

a Find the mean and standard deviation for this data.

b Does this result justify a claim that the average number of matches per box is 50?

14 The number of litres of petrol purchased by a random sample of motor vehicle drivers is shown alongside. For the given data, estimate the:

a mean **b** standard deviation.

Litres (L)	*Number of vehicles*
$15 \leqslant L < 20$	5
$20 \leqslant L < 25$	13
$25 \leqslant L < 30$	17
$30 \leqslant L < 35$	29
$35 \leqslant L < 40$	27
$40 \leqslant L < 45$	18
$45 \leqslant L < 50$	7

15 Pratik is a quality control officer for a biscuit company. He needs to check that 250 g of biscuits go into each packet, but realises that the weight in each packet will vary slightly.

a Would you expect the standard deviation for the whole population to be the same for one day as it is for one week? Explain your answer.

b If a sample of 100 packets is measured each day, what measure would be used to check:

  i that an average of 250 g of biscuits goes into each packet

  ii the variability of the mass going into each packet?

c Explain the significance of a low standard deviation in this case.

REVIEW SET 13B

1 Heike is preparing for an athletics carnival. She records her times in seconds for the 100 m sprint each day for 4 weeks.

Week 1:	16.4	15.2	16.3	16.3	17.1	15.5	14.9
Week 2:	14.9	15.7	15.1	15.1	14.7	14.7	15.3
Week 3:	14.3	14.2	14.6	14.6	14.3	14.3	14.4
Week 4:	14.0	14.0	13.9	14.0	14.1	13.8	14.2

a Calculate Heike's mean and median time for each week.

b Do you think Heike's times have improved over the 4 week period? Explain your answer.

2 A die was rolled 50 times.
The results are shown in the table alongside.
Find the:

a mode **b** mean **c** median.

Number	*Frequency*
1	10
2	7
3	8
4	5
5	12
6	8

3 The data in the table alongside has mean 5.7.

a Find the value of x.

b Find the median of the distribution.

Value	2	5	x	$x+6$
Frequency	3	2	4	1

4 A set of 14 data is: 6, 8, 7, 7, 5, 7, 6, 8, 6, 9, 6, 7, p, q.
The mean and mode of the set are both 7.
Find p and q.

5 The table alongside shows the number of patrons visiting an art gallery on various days.
Estimate the mean number of patrons per day.

Number of patrons	*Frequency*
250 - 299	14
300 - 349	34
350 - 399	68
400 - 449	72
450 - 499	54
500 - 549	23
550 - 599	7

6 Draw a box and whisker diagram for the following data:

11, 12, 12, 13, 14, 14, 15, 15, 15, 16, 17, 17, 18.

7 Consider the data set: 120, 118, 132, 127, 135, 116, 122, 93, 128.

a Find the standard deviation for the data.

b Find the upper and lower quartiles of the data set.

c Are there any outliers in the data set?

d Draw a box plot to display the data.

8 The number of peanuts in a jar varies slightly from jar to jar. Samples of 30 jars were taken for each of two brands X and Y, and the number of peanuts in each jar was recorded.

Brand X						*Brand Y*					
871	885	878	882	889	885	909	906	913	891	898	901
916	913	886	905	907	898	894	894	928	893	924	892
874	904	901	894	897	899	927	907	901	900	907	913
908	901	898	894	895	895	921	904	903	896	901	895
910	904	896	893	903	888	917	903	910	903	909	904

a Copy and complete the table alongside.

b Display the data on a parallel box plot.

c Comment on which brand:

  i has more peanuts per jar

  ii has a more consistent number of peanuts per jar.

	Brand X	*Brand Y*
min		
Q_1		
median		
Q_3		
max		
IQR		

9 Consider the frequency table alongside:

a Find the values of p and m.

b Hence find the mode, median, and range of the data.

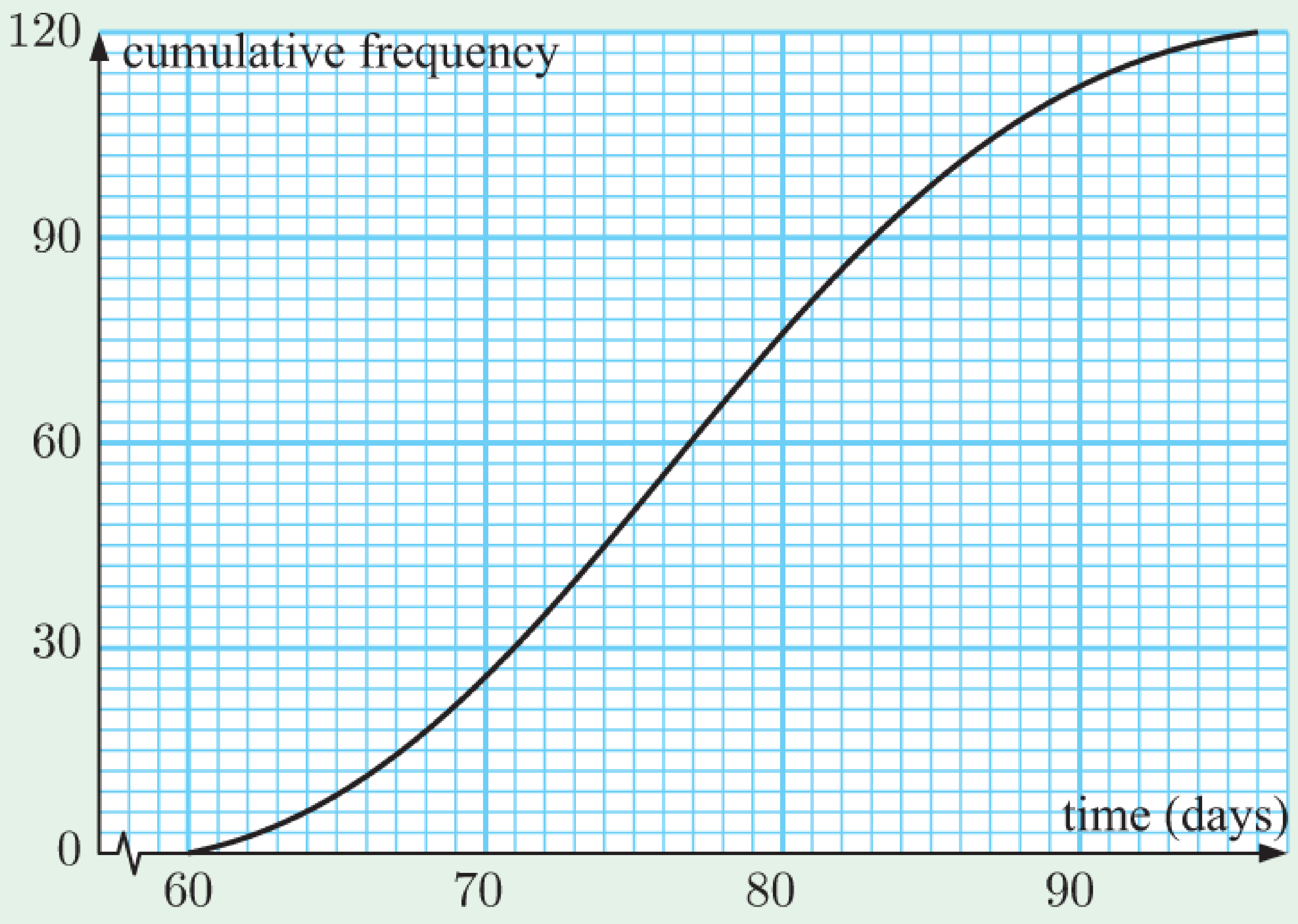
c Given that $\sum_{i=1}^{5} x_i f_i = 254$, write the mean $\overline{x}$ as a fraction.

Score	*Frequency*	*Cumulative frequency*
6	2	2
7	4	m
8	7	13
9	p	25
10	5	30

10 120 people caught whooping cough in an outbreak. The times for them to recover were recorded, and the results were used to produce the cumulative frequency graph shown. Estimate:

a the median

b the interquartile range.



11 Consider the data in the table below:

Scores (x)	$0 \leqslant x < 10$	$10 \leqslant x < 20$	$20 \leqslant x < 30$	$30 \leqslant x < 40$	$40 \leqslant x < 50$
Frequency	1	13	27	17	2

a Construct a cumulative frequency graph for the data.

b Estimate the:

  i median **ii** interquartile range **iii** mean **iv** standard deviation.

12 To test the difficulty level of a new computer game, a company measures the time taken for a group of players to complete the game. Their results are displayed in the table alongside.

Completion time (t min)	*Number of players*
$0 \leqslant t < 30$	1
$30 \leqslant t < 60$	4
$60 \leqslant t < 90$	12
$90 \leqslant t < 120$	18
$120 \leqslant t < 150$	7
$150 \leqslant t < 180$	2

- **a** How many players were surveyed?
- **b** Write down the modal class.
- **c** Draw a cumulative frequency graph for the data.
- **d** The game is considered too easy if either the mean or median completion time is below 90 minutes.
  - **i** Estimate the median completion time using your cumulative frequency graph.
  - **ii** Estimate the mean completion time.
  - **iii** Hence comment on whether the game is too easy.
- **e** Complete this sentence:
  The middle 50% of players completed the game in times between ...... and ...... minutes.

13 A random sample of weekly supermarket bills was recorded in the table alongside.
For the given data, estimate the:

a mean **b** standard deviation.

Bill (€b)	*Number of families*
$140 \leqslant b < 160$	27
$160 \leqslant b < 180$	32
$180 \leqslant b < 200$	48
$200 \leqslant b < 220$	25
$220 \leqslant b < 240$	37
$240 \leqslant b < 260$	21
$260 \leqslant b < 280$	18
$280 \leqslant b < 300$	7

14 Friends Kevin and Felicity each selected a sample of 20 crossword puzzles. The times they took, in minutes, to complete each puzzle were:

Kevin

37	53	47	33	39
49	37	48	32	36
39	42	34	29	52
48	33	56	39	41

Felicity

33	36	41	26	52
38	49	57	39	44
48	25	34	27	53
38	34	35	50	31

- **a** Find the mean of each data set.
- **b** Find the standard deviation for each person.
- **c** Who generally solves crossword puzzles faster?
- **d** Who is more consistent in their time taken to solve the puzzles?

15 A data set has $s^2 = 4.1$ and $\sigma^2 = 3.69$. How many data values does the data set have?

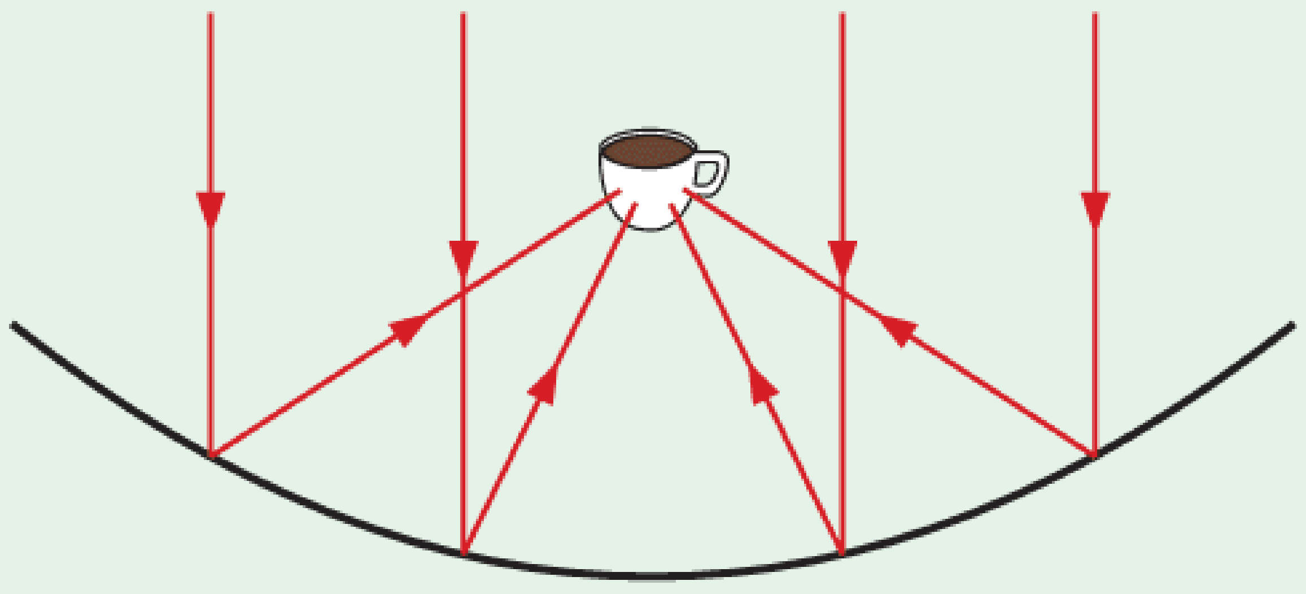
Chapter 14

Quadratic functions

Contents:

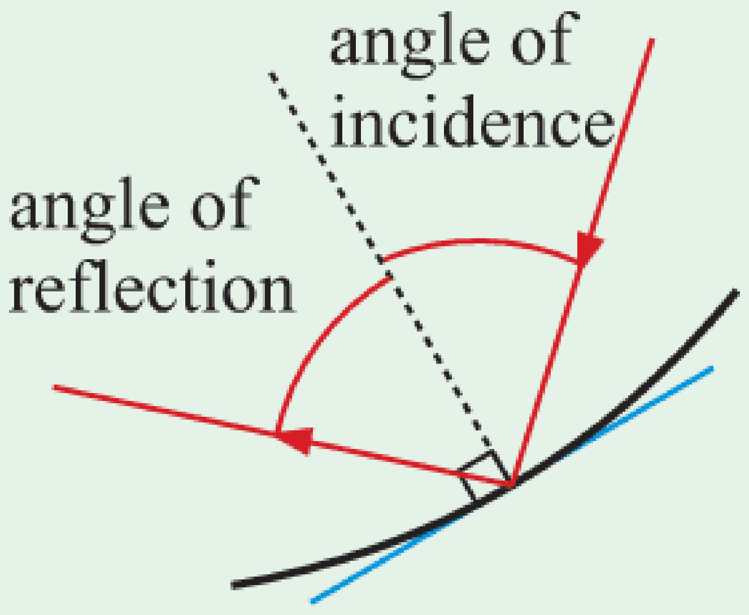
OPENING PROBLEM

Energy-conscious Misha wants to use solar energy to heat his cup of coffee. He has decided to build a reflecting surface to focus the sun's light on the cup.



He understands that the sun's rays will arrive parallel, and that each ray will bounce off the surface according to the law of reflection:

angle of incidence = angle of reflection



Things to think about:

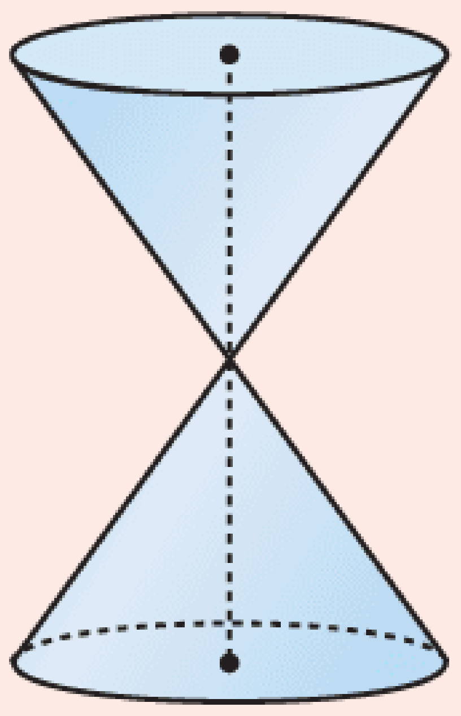
a What *shape* should the surface have?

b Can we write a *formula* which defines the shape of the surface?

In this Chapter we will study **quadratic functions** and investigate their graphs which are called **parabolas**. There are many examples of parabolas in everyday life, including water fountains, bridges, and radio telescopes.

We will see how the curve Misha needs in the **Opening Problem** is actually a parabola, and how the **Opening Problem** relates to the geometric definition of a parabola.

ACTIVITY 1 CONIC SECTIONS



A cone is *right-circular* if its apex is directly above the centre of the base.

Suppose we have two right-circular cones, and we place one upside-down on the first. Now suppose the cones are infinitely tall.

We call the resulting shape a **double inverted right-circular cone**.

When a double inverted right-circular cone is cut by a plane, 7 possible intersections may result, called **conic sections**:

- a point
- a line
- a line-pair
- a circle
- an ellipse
- a parabola
- a hyperbola

Click on the icon to explore the conic sections.

DEMO

You should observe how the parabola results when cutting the cone parallel to its slant edge.

A QUADRATIC FUNCTIONS

A **quadratic function** is a relationship between two variables x and y which can be written in the form $y = ax^2 + bx + c$ where a, b, c are constants, $a \neq 0$.

For any value of x, the corresponding value of y can be found by substitution.

Example 1 **Self Tutor**

Determine whether the given point satisfies the quadratic function:

a $y = 3x^2 + 2x$ (2, 16)

b $y = -x^2 - 2x + 1$ (−3, 1)

a When $x = 2$,

$$\begin{aligned} y &= 3(2)^2 + 2(2) \\ &= 12 + 4 \\ &= 16 \end{aligned}$$

$\therefore$ (2, 16) satisfies the function $y = 3x^2 + 2x$.

b When $x = -3$,

$$\begin{aligned} y &= -(-3)^2 - 2(-3) + 1 \\ &= -9 + 6 + 1 \\ &= -2 \end{aligned}$$

$\therefore$ (−3, 1) does not satisfy the function $y = -x^2 - 2x + 1$.

When we substitute a value for y into a quadratic function, we are left with a quadratic equation. Solving the quadratic equation gives us the values of x corresponding to that y-value. There may be 0, 1, or 2 solutions.

Example 2 **Self Tutor**

If $y = x^2 - 2x + 3$, find the value(s) of x when:

a $y = 2$

b $y = 18$.

a If $y = 2$ then

$$\begin{aligned} x^2 - 2x + 3 &= 2 \\ \therefore\ x^2 - 2x + 1 &= 0 \\ \therefore\ (x-1)^2 &= 0 \\ \therefore\ x &= 1 \end{aligned}$$

b If $y = 18$ then

$$\begin{aligned} x^2 - 2x + 3 &= 18 \\ \therefore\ x^2 - 2x - 15 &= 0 \\ \therefore\ (x-5)(x+3) &= 0 \\ \therefore\ x &= -3 \text{ or } 5 \end{aligned}$$

EXERCISE 14A

1 Copy and complete each table of values:

a $y = x^2 - 3x + 1$

x	−2	−1	0	1	2
y					

b $y = x^2 + 2x - 5$

x	−2	−1	0	1	2
y					

c $y = 2x^2 - x + 3$

x	−4	−2	0	2	4
y					

d $y = -3x^2 + 2x + 4$

x	−4	−2	0	2	4
y					

2 Determine whether the given point satisfies the quadratic function:

a $y = 2x^2 + 5$ $(0,\ 4)$

b $y = x^2 - 3x + 2$ $(2,\ 0)$

c $y = -x^2 + 2x - 5$ $(-1,\ -8)$

d $y = -2x^2 - x + 6$ $(3,\ -15)$

e $y = 3x^2 - 4x + 10$ $(2,\ 10)$

f $y = -\frac{1}{2}x^2 + 4x - 1$ $(2,\ 5)$

3 For each of the following quadratic functions, find the value(s) of x for the given value of y:

a $y = x^2 + 3x + 6$ when $y = 4$

b $y = x^2 - 4x + 7$ when $y = 3$

c $y = x^2 - 6x + 1$ when $y = -4$

d $y = 2x^2 + 5x + 1$ when $y = 4$

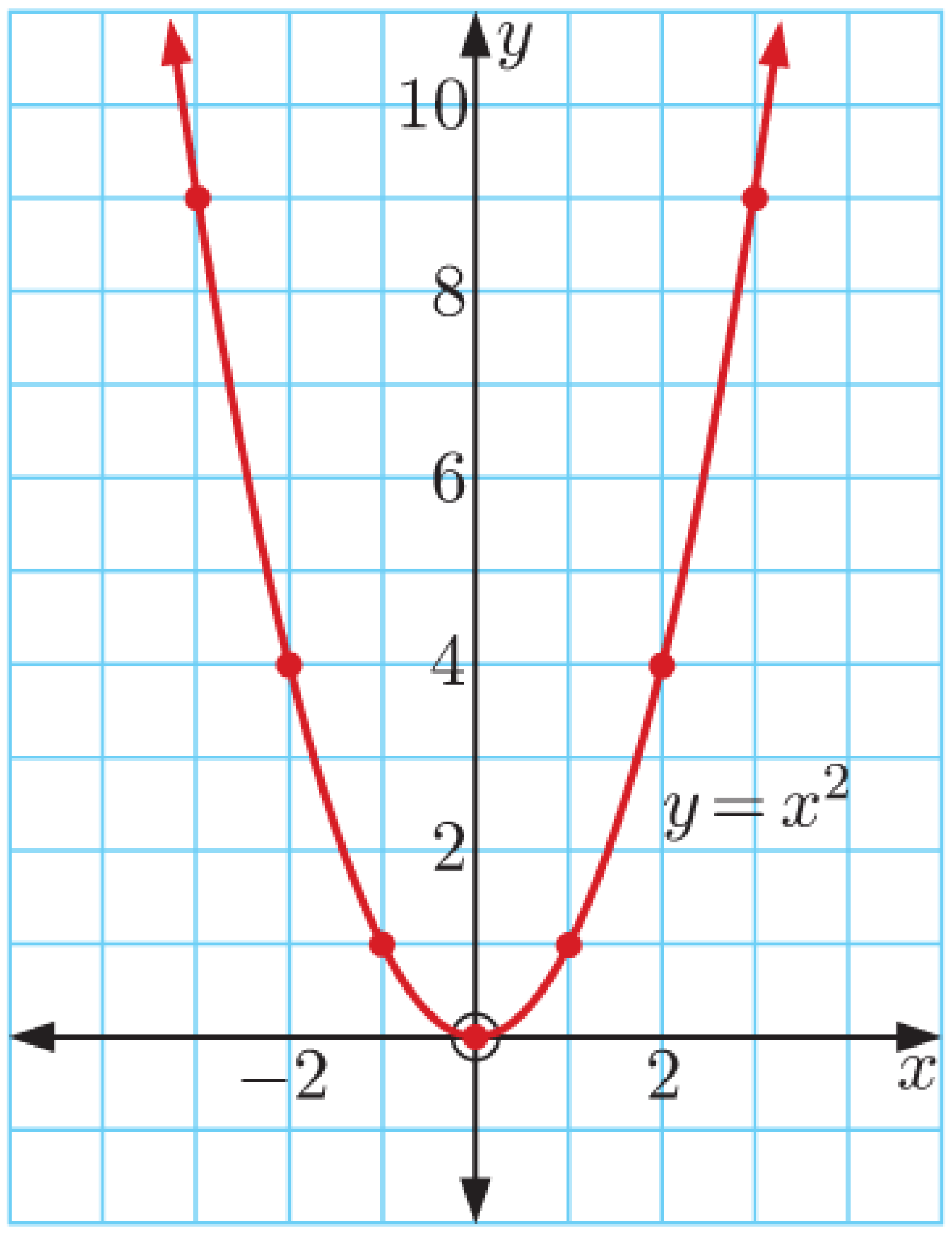
e $y = \frac{1}{2}x^2 + \frac{5}{2}x - 2$ when $y = 1$

f $y = -\frac{1}{2}x^2 + 2x - 1$ when $y = 2$

B GRAPHS OF QUADRATIC FUNCTIONS

The simplest quadratic function is $y = x^2$. Its graph can be drawn from a table of values.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9



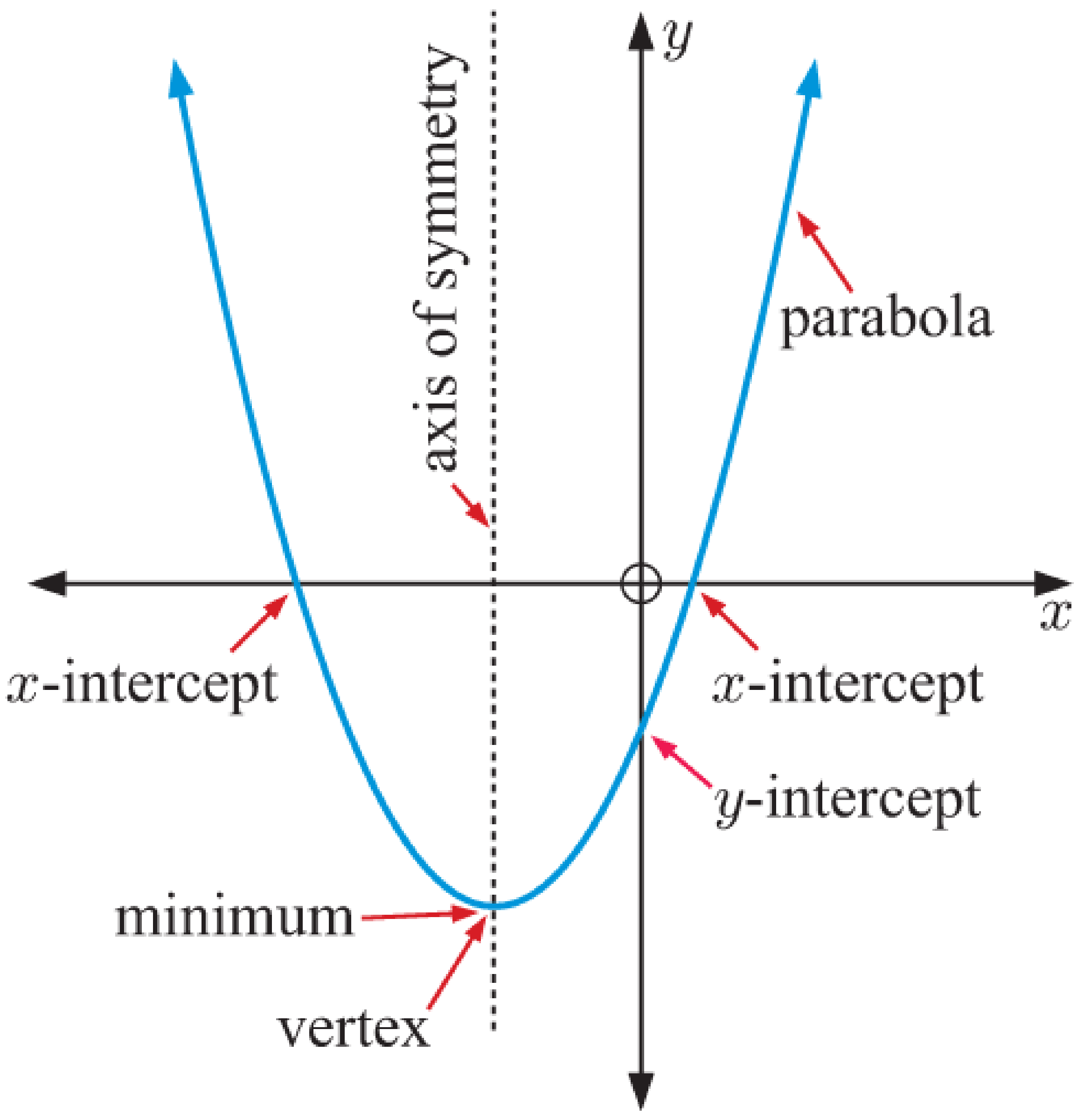
The graph of a quadratic function is called a **parabola**.

The point where the graph "turns" is called the **vertex**.

If the graph opens upwards, the vertex is the **minimum** or **minimum turning point**, and the graph is **concave upwards**.

If the graph opens downwards, the vertex is the **maximum** or **maximum turning point**, and the graph is **concave downwards**.

The vertical line that passes through the vertex is called the **axis of symmetry**. Every parabola is symmetrical about its axis of symmetry.



The value of y where the graph crosses the y-axis is the ***y*-intercept**.

The values of x (if they exist) where the graph crosses the x-axis are called the ***x*-intercepts**. They correspond to the **roots** of the quadratic equation $ax^2 + bx + c = 0$.

INVESTIGATION 1 GRAPHING $y = a(x-p)(x-q)$

In this Investigation we consider the properties of the graph of a quadratic stated in factored form. It is best done using a **graphing package** or **graphics calculator**.

What to do:

GRAPHING PACKAGE

1 **a** Use technology to help you to sketch:
$y = (x-1)(x-3)$, $y = 2(x-1)(x-3)$, $y = -(x-1)(x-3)$,
$y = -3(x-1)(x-3)$, and $y = -\frac{1}{2}(x-1)(x-3)$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of a in $y = a(x-1)(x-3)$?

2 **a** Use technology to help you to sketch:
$y = 2(x-1)(x-4)$, $y = 2(x-3)(x-5)$, $y = 2(x+1)(x-2)$,
$y = 2x(x+5)$, and $y = 2(x+2)(x+4)$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of p and q in $y = 2(x-p)(x-q)$?

3 **a** Use technology to help you to sketch:
$y = 2(x-1)^2$, $y = 2(x-3)^2$, $y = 2(x+2)^2$, and $y = 2x^2$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of p in $y = 2(x-p)^2$?

4 Copy and complete:

- If a quadratic has the form $y = a(x-p)(x-q)$ then it ...... the x-axis at ......
- If a quadratic has the form $y = a(x-p)^2$ then it ...... the x-axis at ......

INVESTIGATION 2 GRAPHING $y = a(x-h)^2 + k$

In this Investigation we consider the properties of the graph of a quadratic stated in completed square form. It is best done using a **graphing package** or **graphics calculator**.

What to do:

GRAPHING PACKAGE

1 **a** Use technology to help you to sketch:
$y = (x-3)^2 + 2$, $y = 2(x-3)^2 + 2$, $y = -2(x-3)^2 + 2$,
$y = -(x-3)^2 + 2$, and $y = -\frac{1}{3}(x-3)^2 + 2$

b Find the coordinates of the vertex for each function in **a**.

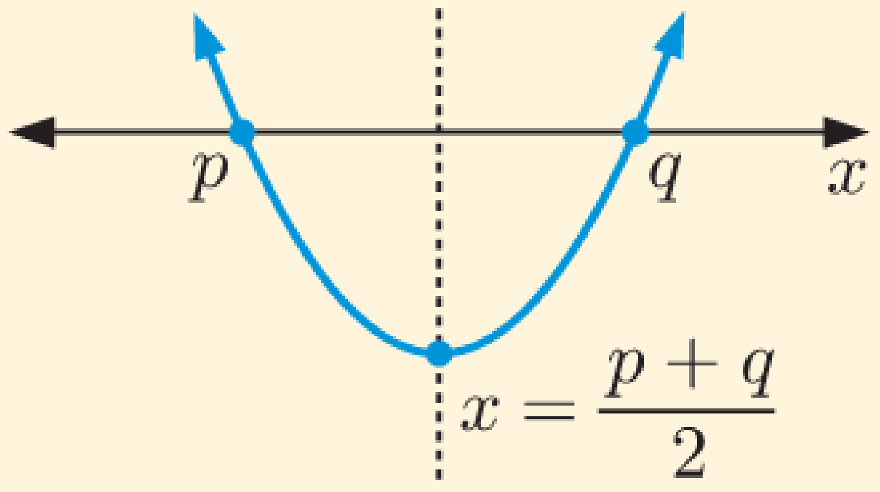
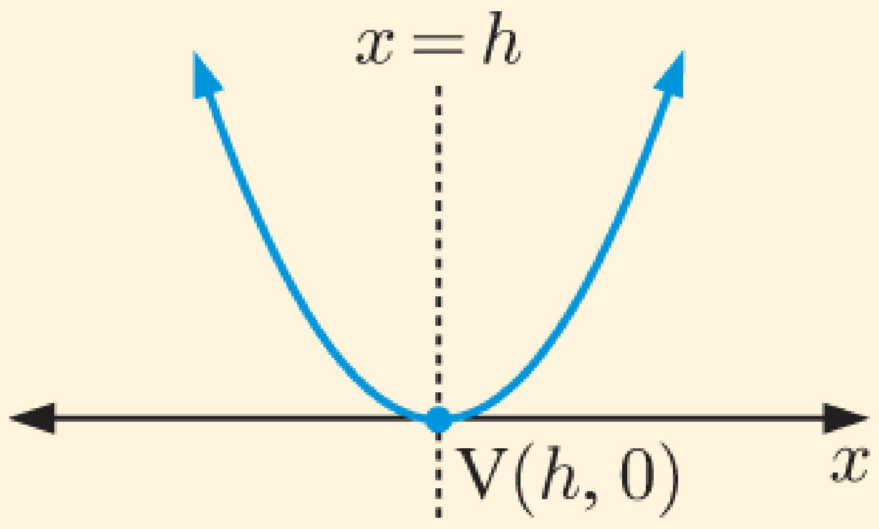
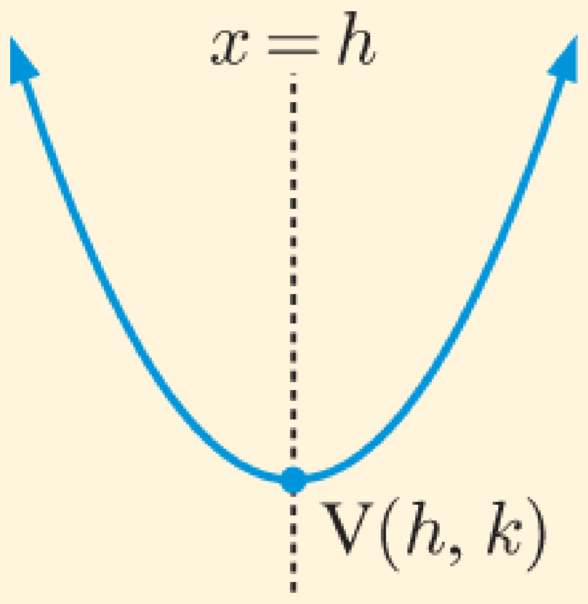
c What is the geometrical significance of a in $y = a(x-3)^2 + 2$?

2 **a** Use technology to help you to sketch:
$y = 2(x-1)^2 + 3$, $y = 2(x-2)^2 + 4$, $y = 2(x-3)^2 + 1$,
$y = 2(x+1)^2 + 4$, $y = 2(x+2)^2 - 5$, and $y = 2(x+3)^2 - 2$

b Find the coordinates of the vertex for each function in **a**.

c What is the geometrical significance of h and k in $y = 2(x-h)^2 + k$?

3 Copy and complete:
If a quadratic has the form $y = a(x-h)^2 + k$ then its vertex has coordinates ......

Quadratic form, $a \neq 0$	*Graph*	*Facts*
$y = a(x-p)(x-q)$ where $p,\ q \in \mathbb{R}$		• x-intercepts are p and q • axis of symmetry is $x = \frac{p+q}{2}$ • vertex has x-coordinate $\frac{p+q}{2}$
$y = a(x-h)^2$ where $h \in \mathbb{R}$		• touches x-axis at h • axis of symmetry is $x = h$ • vertex is $(h,\ 0)$
$y = a(x-h)^2 + k$ where $h,\ k \in \mathbb{R}$		• axis of symmetry is $x = h$ • vertex is $(h,\ k)$

You should have found that a, the coefficient of x^2, controls the width of the graph and whether it opens upwards or downwards.

For a quadratic function $y = ax^2 + bx + c$, $a \neq 0$:

- $a > 0$ produces the shape  called concave up.

  $a < 0$ produces the shape  called concave down.
- If $-1 < a < 1$, $a \neq 0$ the graph is wider than $y = x^2$.
  If $a < -1$ or $a > 1$ the graph is narrower than $y = x^2$.

Example 3 **Self Tutor**

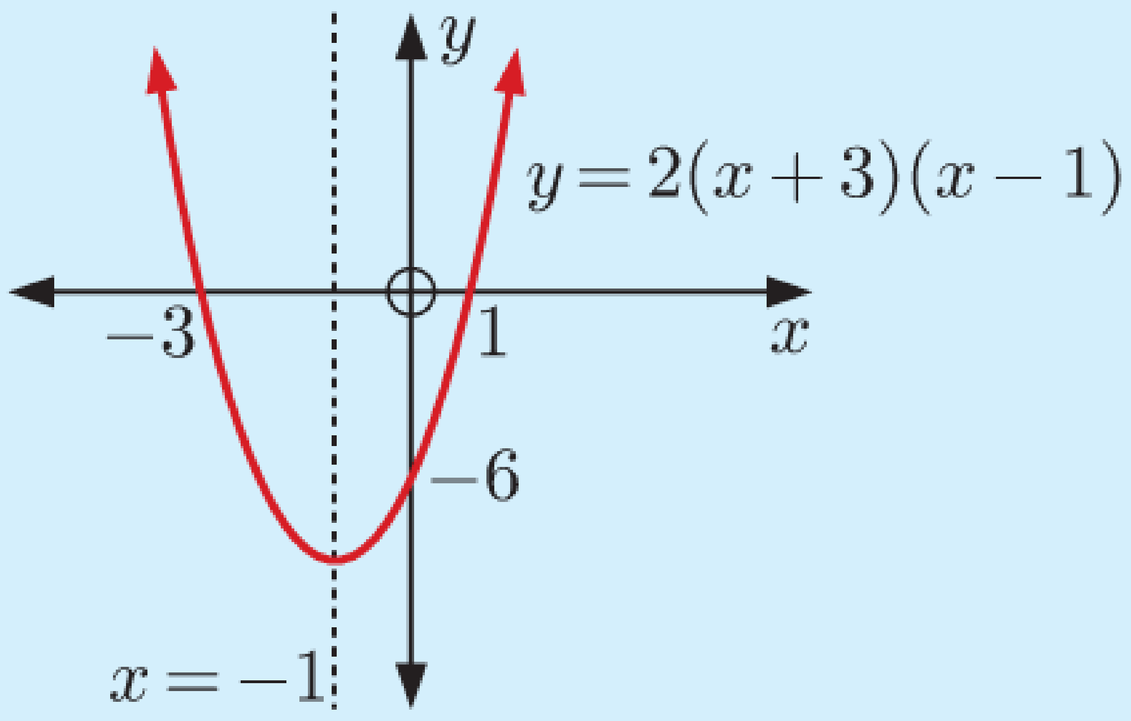
Sketch the graph using axes intercepts, and state the equation of the axis of symmetry:

a $y = 2(x+3)(x-1)$ **b** $y = -2(x-1)(x-2)$ **c** $y = \frac{1}{2}(x+2)^2$

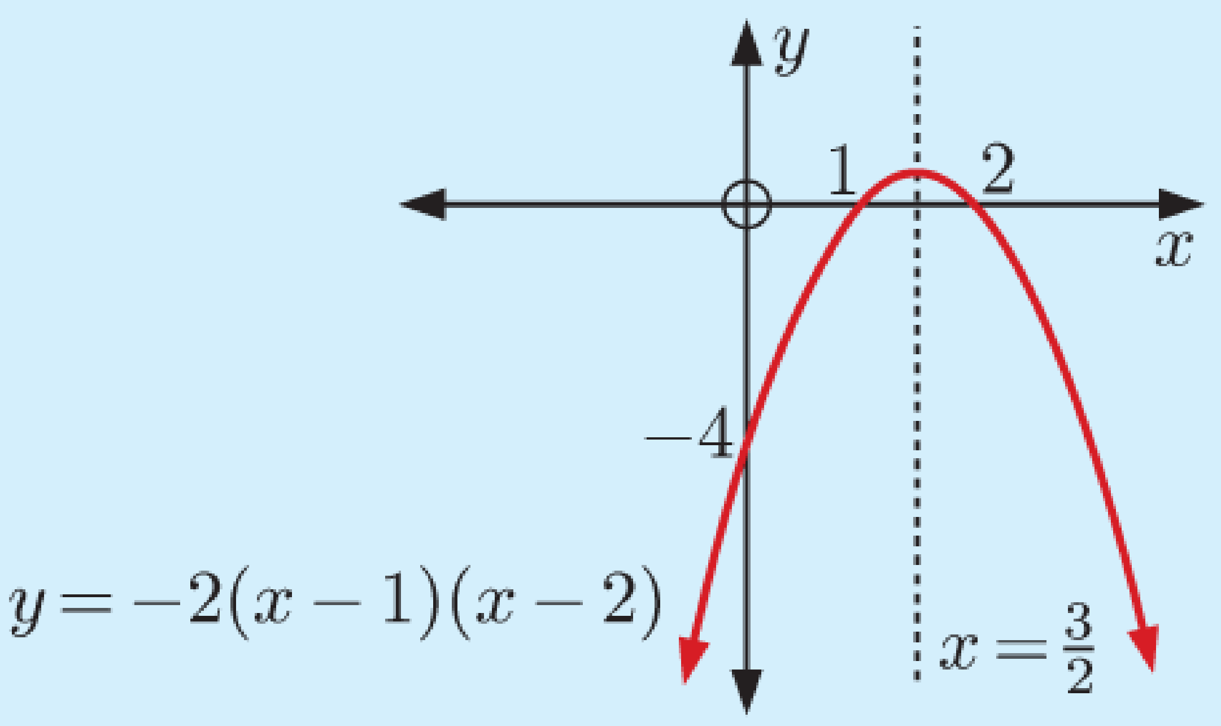
a $y = 2(x+3)(x-1)$
has x-intercepts -3, 1

When $x = 0$, $y = 2(3)(-1)$
$= -6$

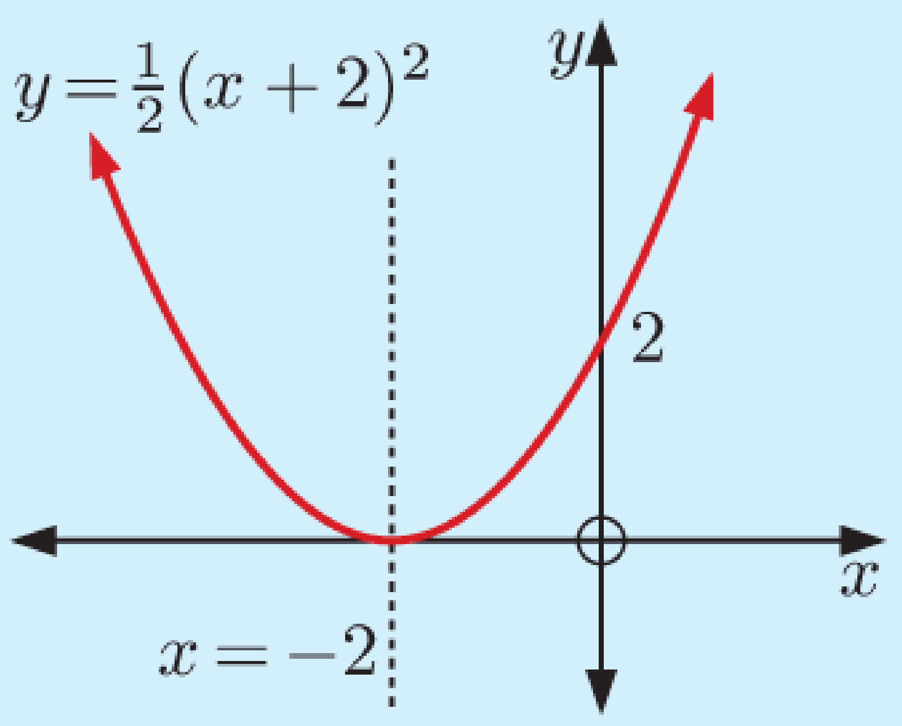
$\therefore$ y-intercept is -6



b $y = -2(x-1)(x-2)$
has x-intercepts 1, 2
When $x = 0$, $y = -2(-1)(-2)$
$= -4$
$\therefore$ y-intercept is -4



c $y = \frac{1}{2}(x+2)^2$
touches x-axis at -2
When $x = 0$, $y = \frac{1}{2}(2)^2$
$= 2$
$\therefore$ y-intercept is 2

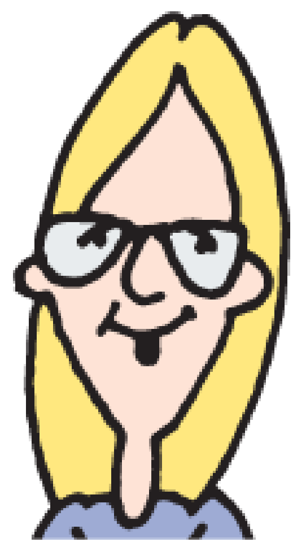


EXERCISE 14B.1

1 Sketch the graph using axes intercepts, and state the equation of the axis of symmetry:

- **a** $y = (x-4)(x+2)$
- **b** $y = -(x-4)(x+2)$
- **c** $y = 2(x+3)(x+5)$
- **d** $y = -3(x+1)(x+5)$
- **e** $y = 2(x+3)^2$
- **f** $y = -\frac{1}{4}(x+2)^2$

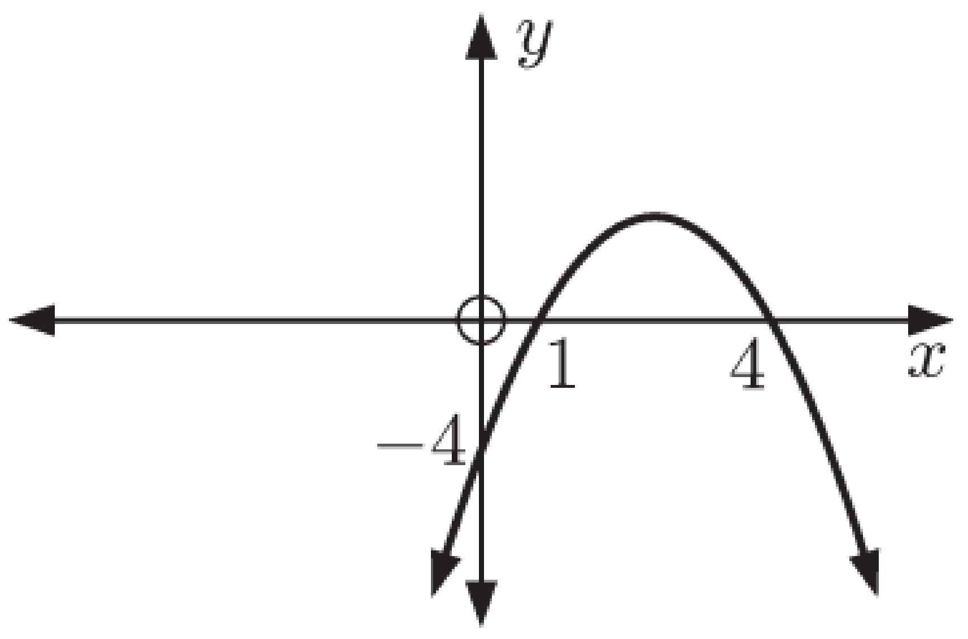
The axis of symmetry is midway between the x-intercepts.



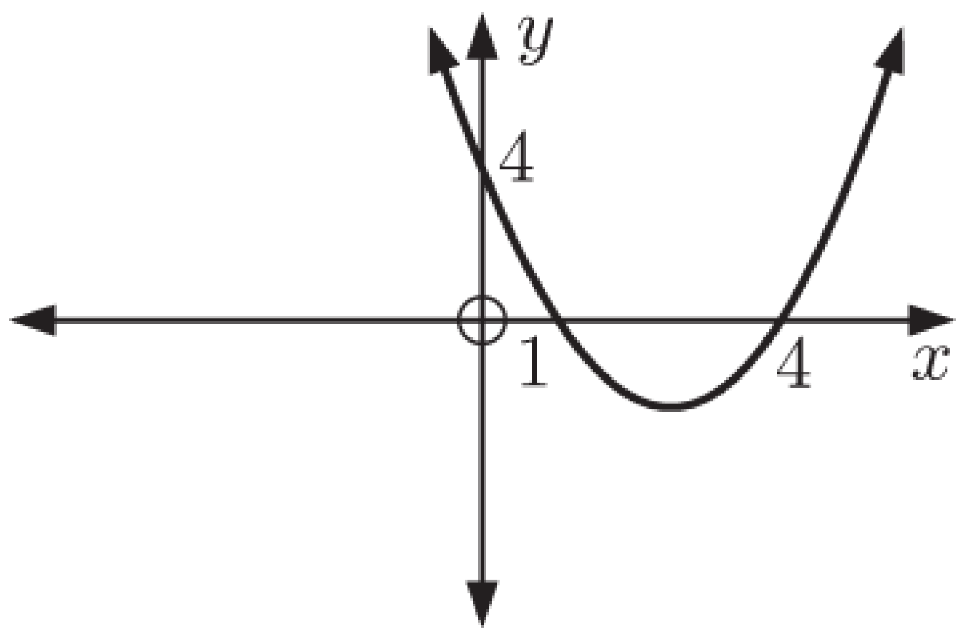
2 Match each quadratic function with its corresponding graph.

- **a** $y = 2(x-1)(x-4)$
- **b** $y = -(x+1)(x-4)$
- **c** $y = (x-1)(x-4)$
- **d** $y = (x+1)(x-4)$
- **e** $y = 2(x+4)(x-1)$
- **f** $y = -3(x+4)(x-1)$
- **g** $y = 2(x+1)(x+4)$
- **h** $y = -(x-1)(x-4)$
- **i** $y = -3(x-1)(x-4)$

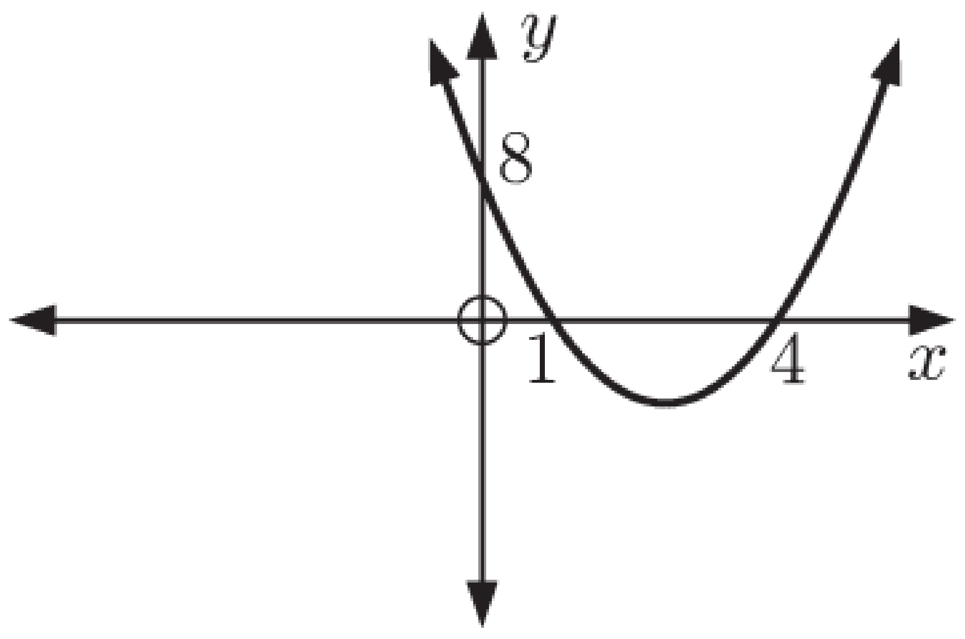
A



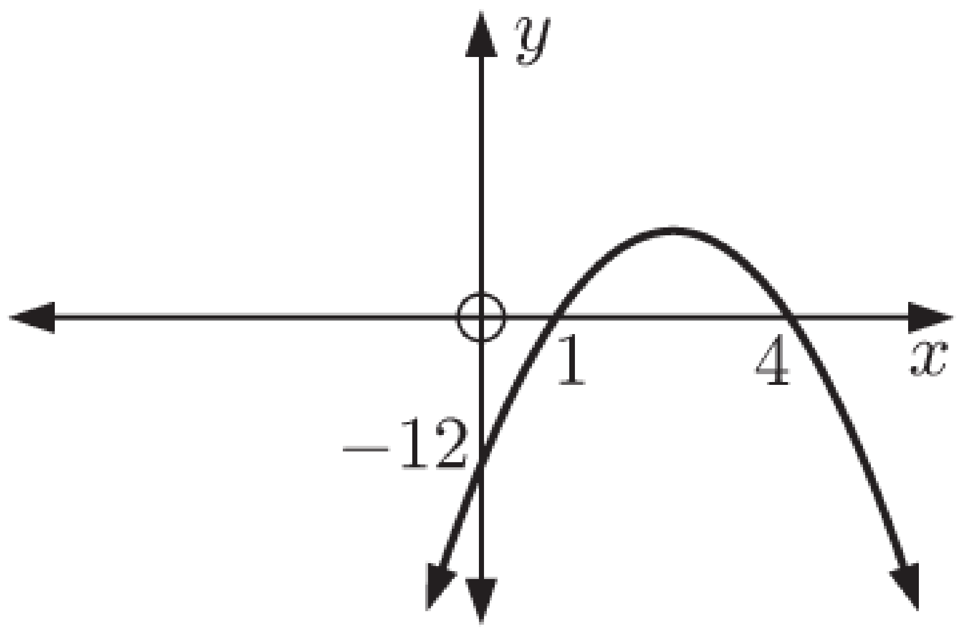
B



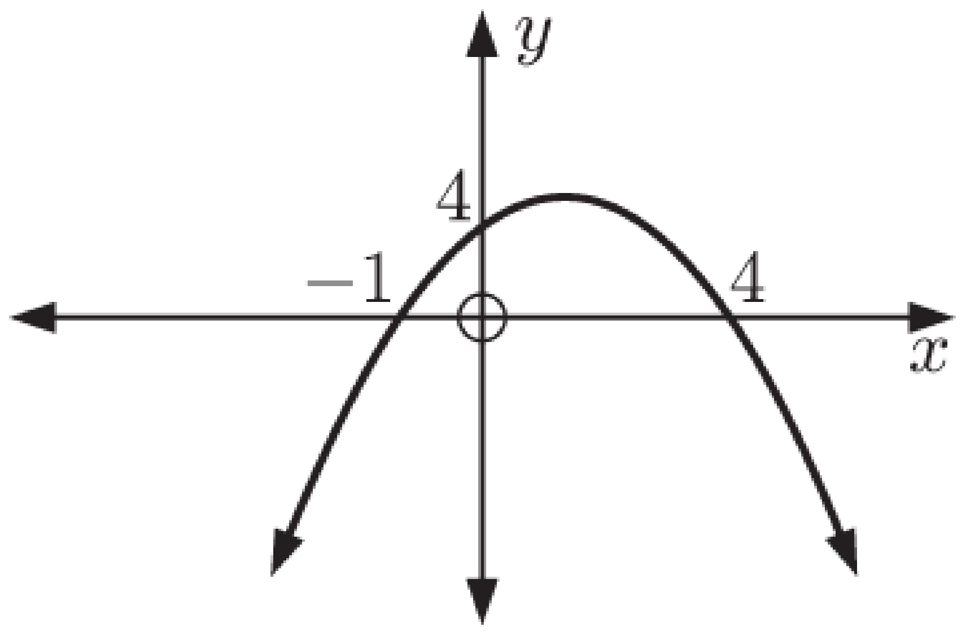
C



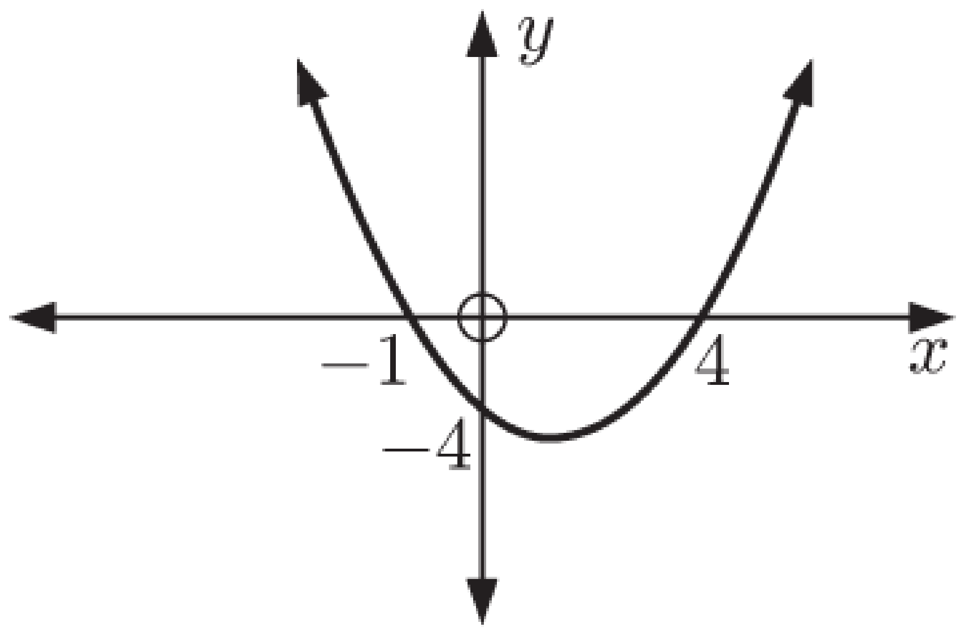
D



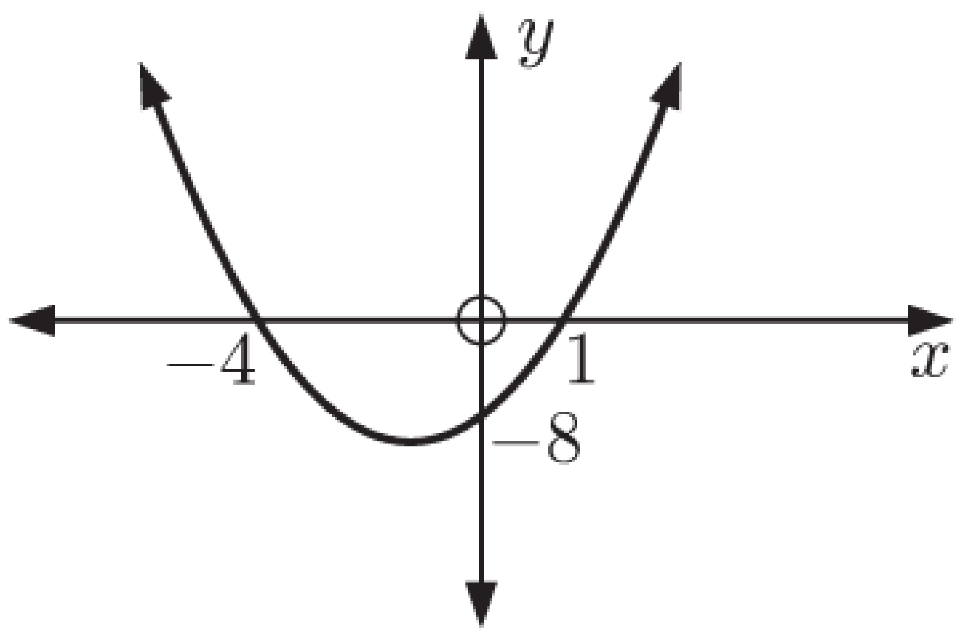
E



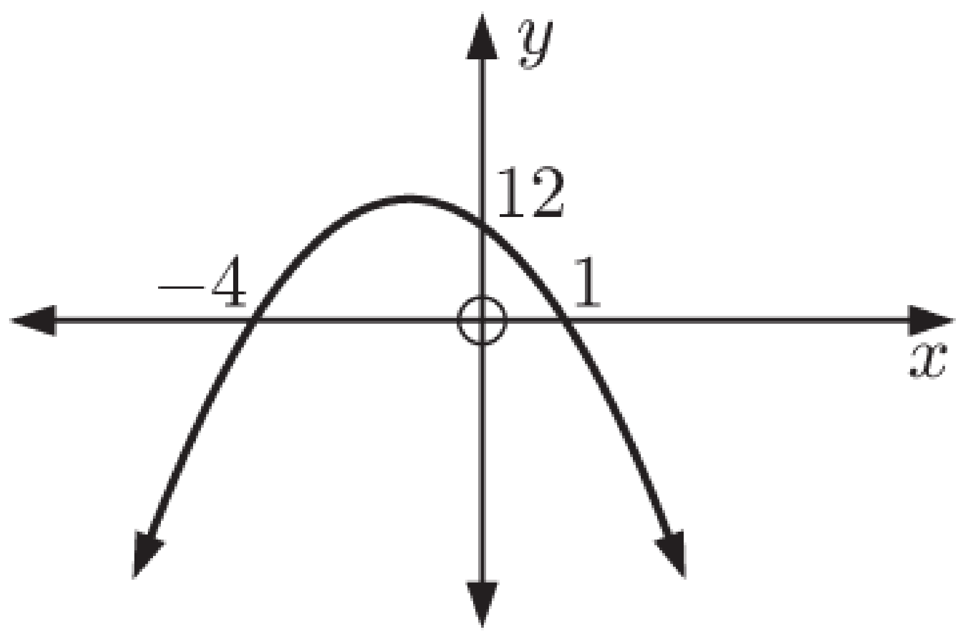
F



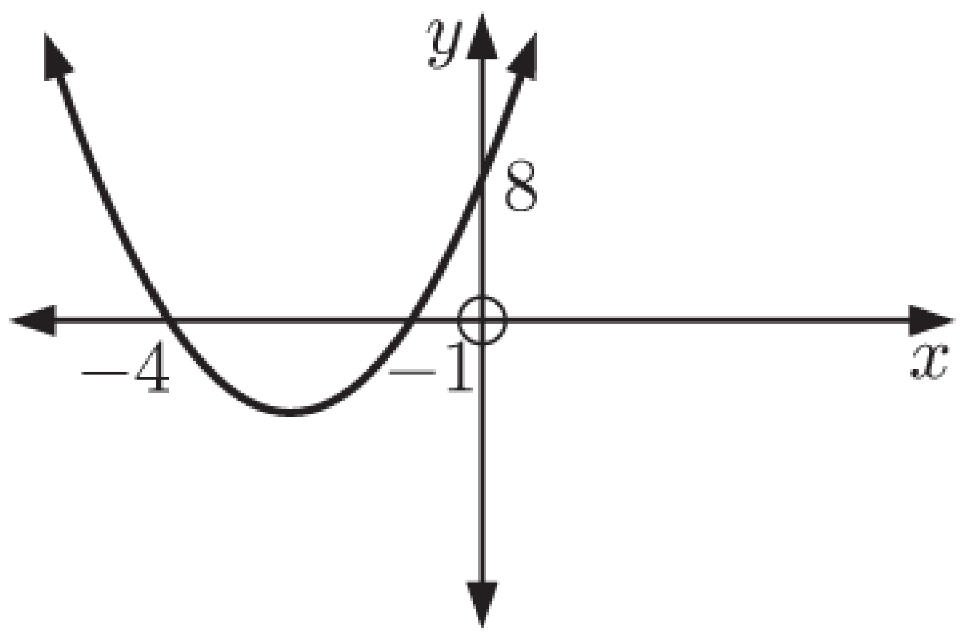
G



H



I



Example 4 Self Tutor

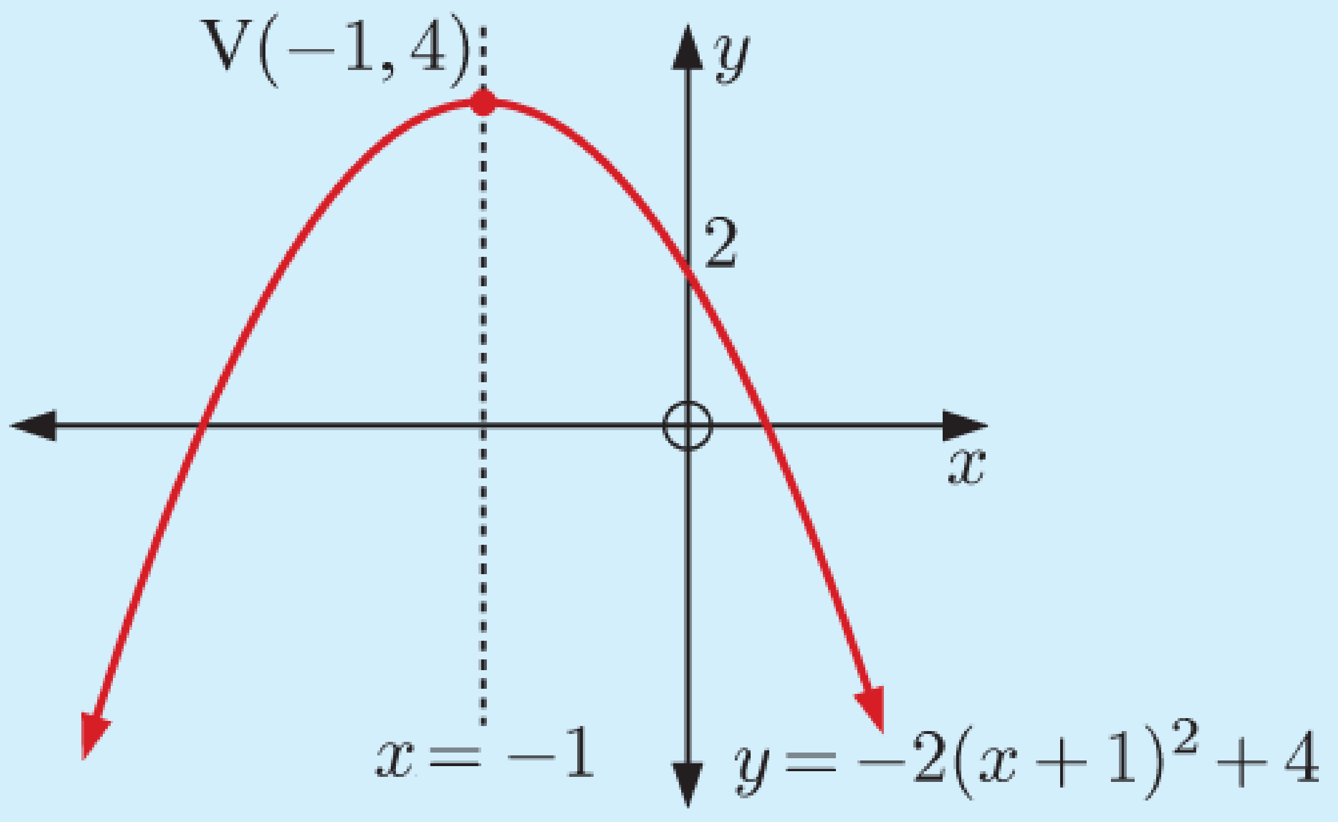
Use the vertex, axis of symmetry, and y-intercept to graph $y = -2(x+1)^2 + 4$.

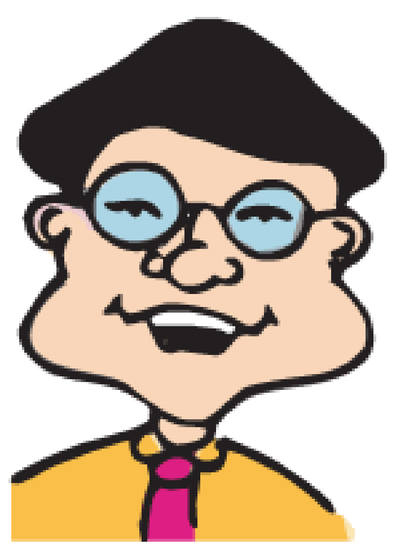
The axis of symmetry is $x = -1$.

The vertex is $(-1, 4)$.

When $x = 0$, $y = -2(1)^2 + 4$
$= 2$

$a < 0$ so the shape is 





$y = a(x-h)^2 + k$ is called **completed square form**.

3 Use the vertex, axis of symmetry, and y-intercept to graph:

a $y = (x-1)^2 + 3$ **b** $y = (x+3)^2 - 4$ **c** $y = -(x+4)^2 + 2$

d $y = 2(x+2)^2 + 1$ **e** $y = -2(x-1)^2 - 3$ **f** $y = \frac{1}{3}(x+6)^2 - 1$

g $y = \frac{1}{2}(x-3)^2 + 2$ **h** $y = -\frac{1}{3}(x-1)^2 + 4$ **i** $y = -\frac{1}{10}(x+2)^2 - 3$

SKETCHING GRAPHS BY "COMPLETING THE SQUARE"

If we wish to graph a quadratic given in general form $y = ax^2 + bx + c$, one approach is to use "**completing the square**" to convert it to the completed square form $y = a(x-h)^2 + k$. We can then read off the coordinates of the vertex (h, k).

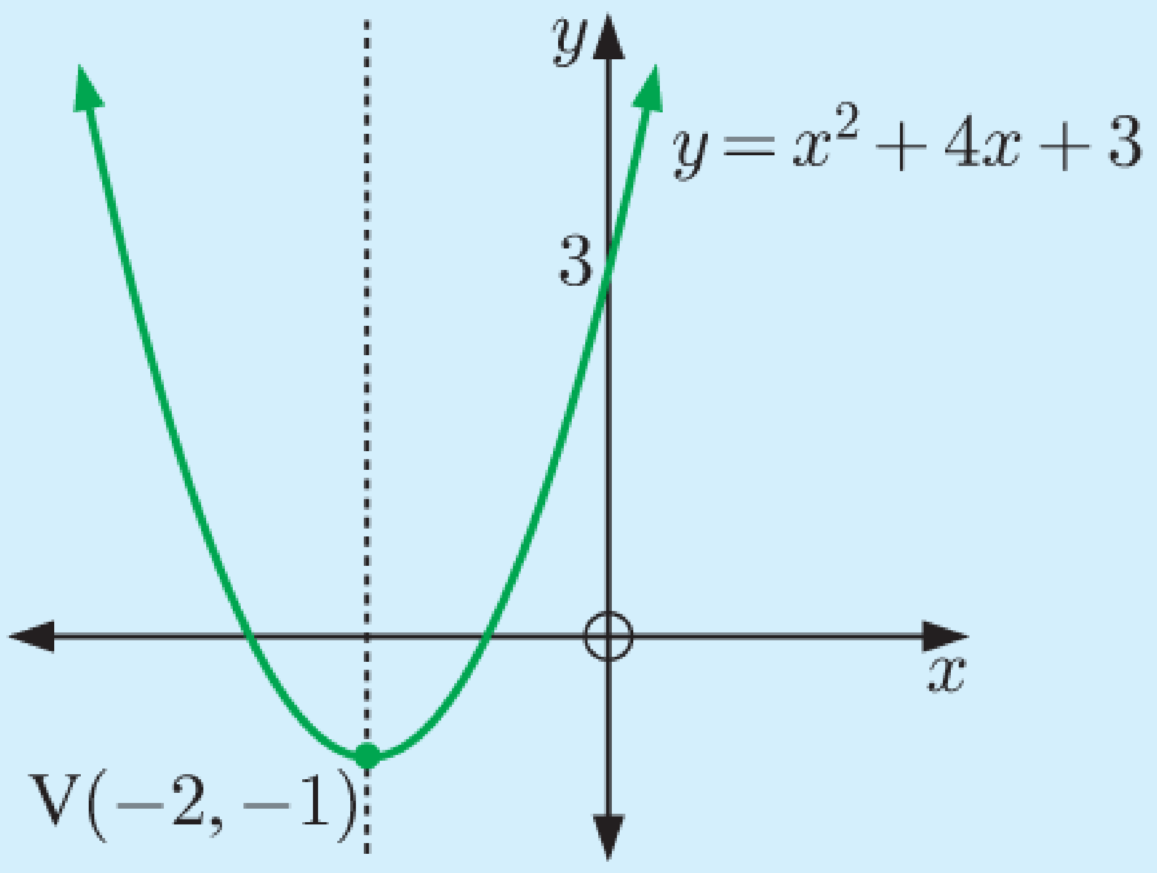
Example 5 Self Tutor

Write $y = x^2 + 4x + 3$ in the form $y = (x-h)^2 + k$ by "completing the square".
Hence sketch $y = x^2 + 4x + 3$, stating the coordinates of the vertex.

$y = x^2 + 4x + 3$
$\therefore \quad = x^2 + 4x + 2^2 + 3 - 2^2$
$\therefore \quad y = (x+2)^2 - 1$

So, the axis of symmetry is $x = -2$
and the vertex is $(-2, -1)$.

When $x = 0$, $y = 3$
$\therefore$ the y-intercept is 3.



EXERCISE 14B.2

1 Write the following quadratics in the form $y = (x-h)^2 + k$ by "completing the square".
Hence sketch each function, stating the coordinates of the vertex.

a $y = x^2 - 2x + 3$ **b** $y = x^2 + 4x - 2$ **c** $y = x^2 - 4x$

d $y = x^2 + 3x$ **e** $y = x^2 + 5x - 2$ **f** $y = x^2 - 3x + 2$

g $y = x^2 - 6x + 5$ **h** $y = x^2 + 8x - 2$ **i** $y = x^2 - 5x + 1$

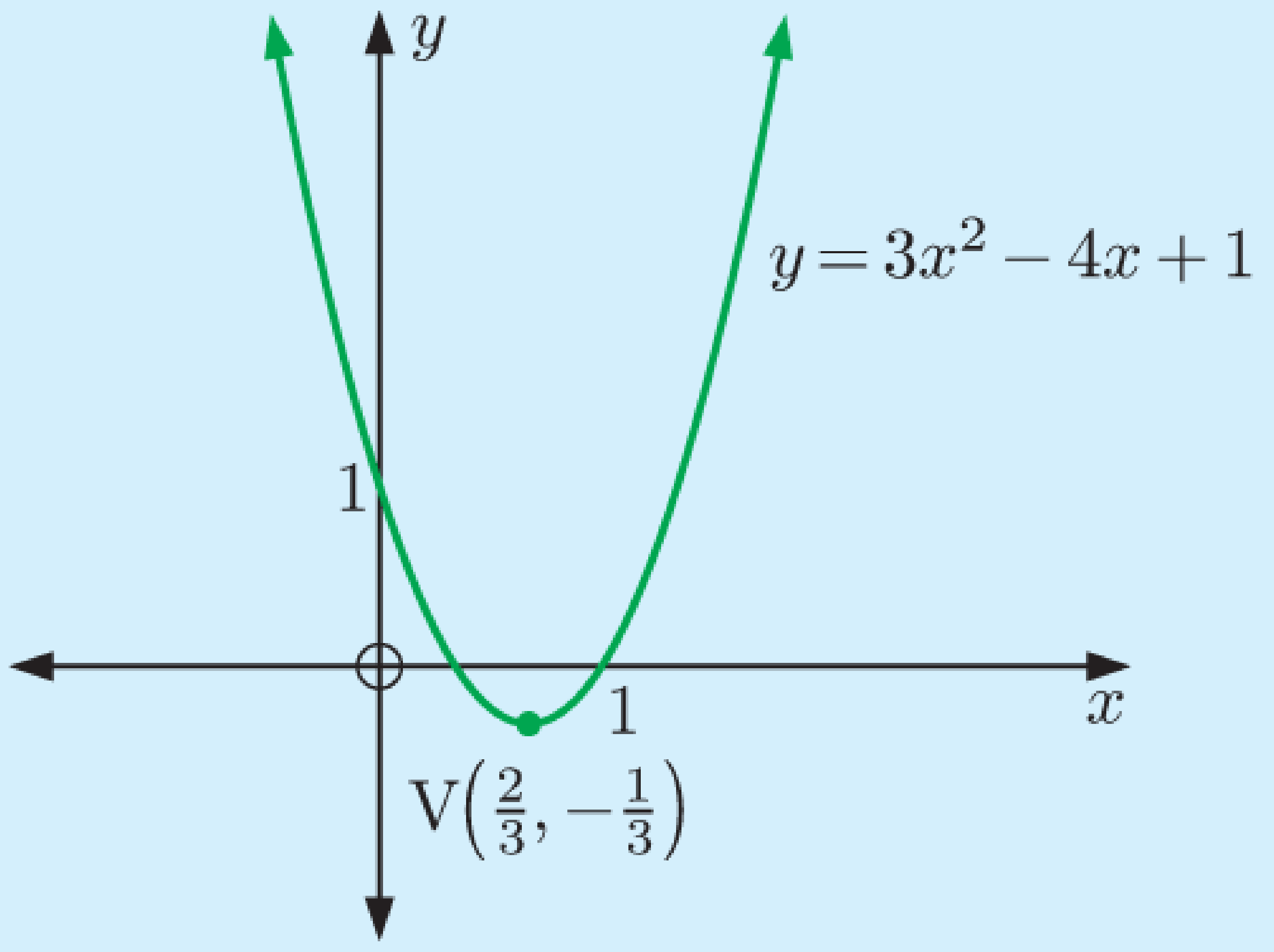
Example 6 **Self Tutor**

a Convert $y = 3x^2 - 4x + 1$ to the completed square form $y = a(x-h)^2 + k$.

b Hence write down the coordinates of the vertex, and sketch the quadratic.

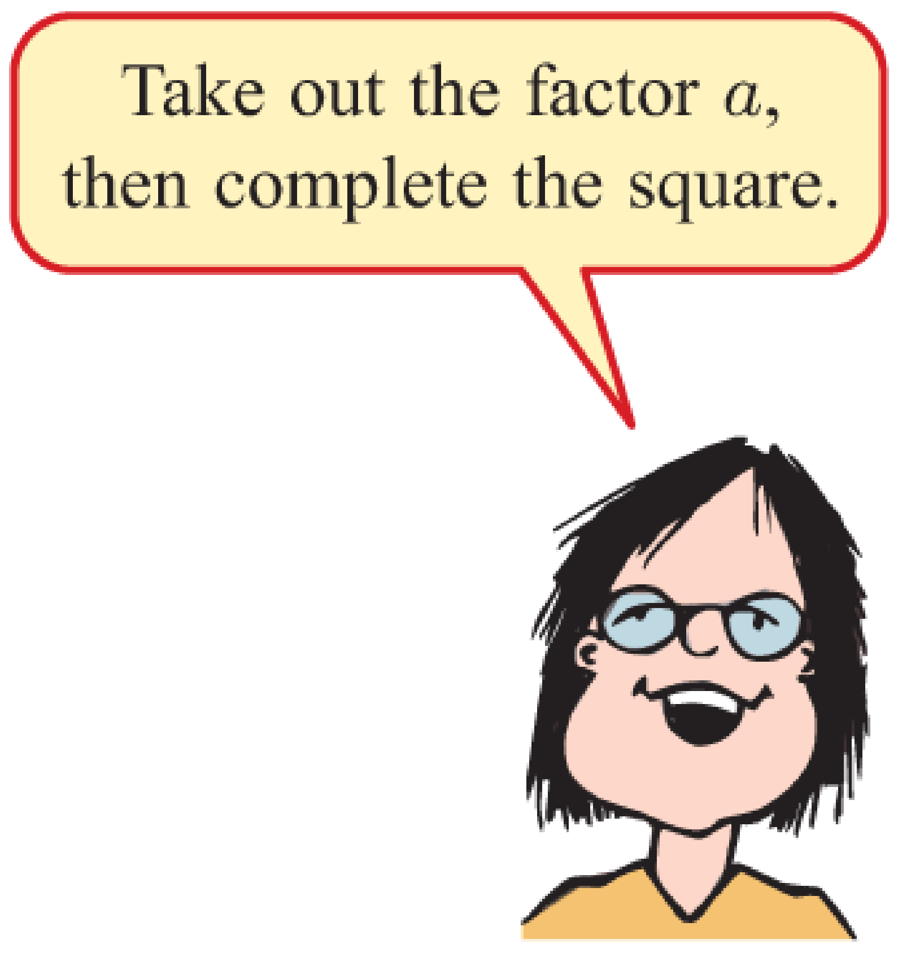
a
$$\begin{aligned} y &= 3x^2 - 4x + 1 \\ &= 3[x^2 - \tfrac{4}{3}x + \tfrac{1}{3}] \\ &= 3[x^2 - 2(\tfrac{2}{3})x + (\tfrac{2}{3})^2 + \tfrac{1}{3} - (\tfrac{2}{3})^2] \\ &= 3[(x - \tfrac{2}{3})^2 + \tfrac{3}{9} - \tfrac{4}{9}] \\ &= 3[(x - \tfrac{2}{3})^2 - \tfrac{1}{9}] \\ &= 3(x - \tfrac{2}{3})^2 - \tfrac{1}{3} \end{aligned}$$

b The vertex is $(\tfrac{2}{3}, -\tfrac{1}{3})$
and the y-intercept is 1.



2 For each of the following quadratics:

- **i** Write the quadratic in the completed square form $y = a(x-h)^2 + k$.
- **ii** State the coordinates of the vertex.
- **iii** Find the y-intercept.
- **iv** Sketch the graph of the quadratic.



Take out the factor a, then complete the square.

a $y = 2x^2 + 4x + 5$
b $y = 2x^2 - 8x + 3$
c $y = 2x^2 - 6x + 1$
d $y = 3x^2 - 6x + 5$
e $y = -x^2 + 4x + 2$
f $y = -2x^2 - 5x + 3$
g $y = -\frac{1}{3}x^2 + 2x - 3$
h $y = \frac{1}{2}x^2 + 3x - 4$

SKETCHING QUADRATICS IN THE GENERAL FORM $y = ax^2 + bx + c$

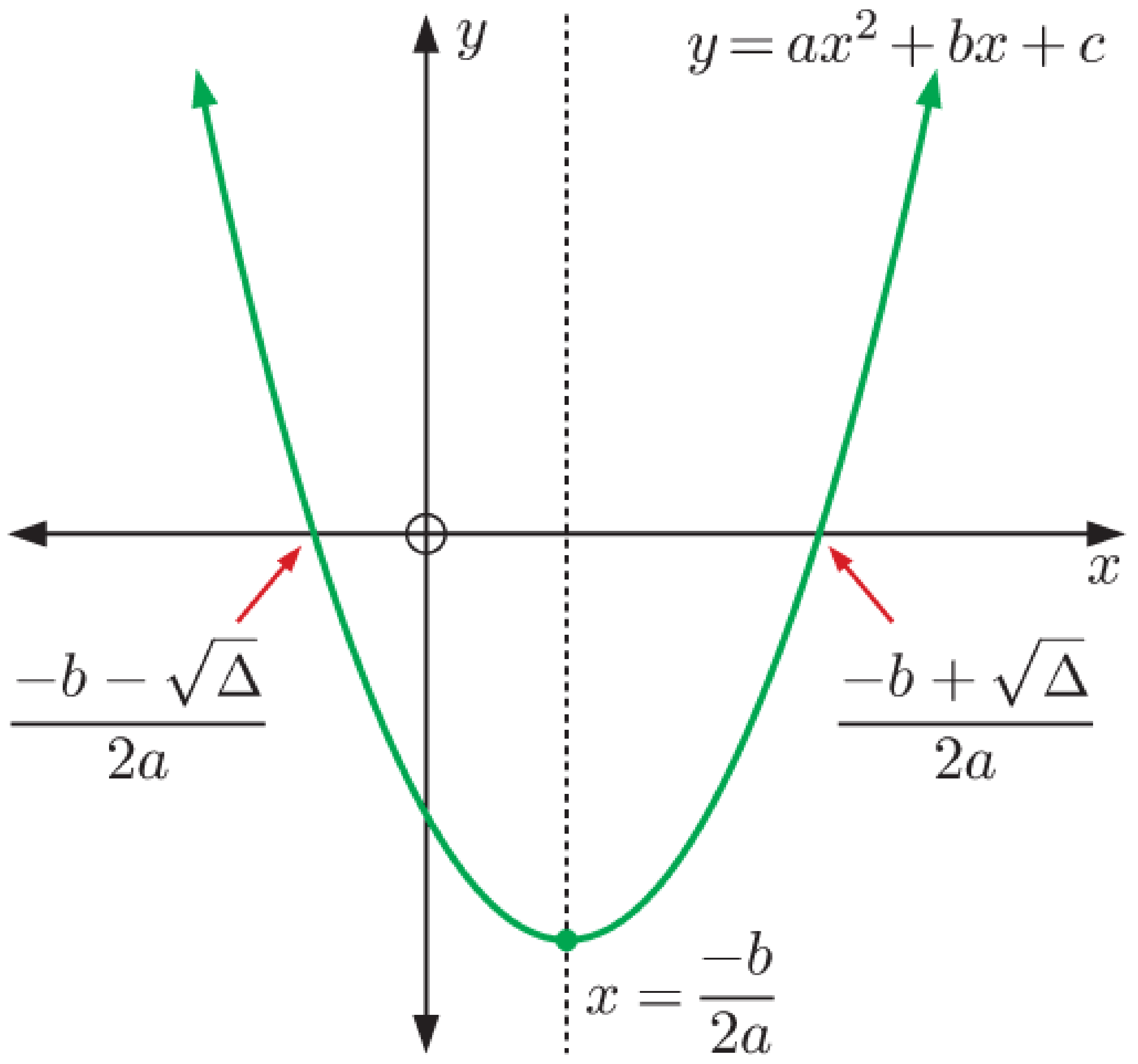
We now consider a method of graphing quadratics of the form $y = ax^2 + bx + c$ directly, without having to first convert them to a different form.

We know that the quadratic equation $ax^2 + bx + c = 0$ has solutions $\dfrac{-b \pm \sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac$.

If $\Delta \geqslant 0$, these are the x-intercepts of the graph of the quadratic function $y = ax^2 + bx + c$.

The average of the values is $\dfrac{-b}{2a}$, so we conclude that:

- the axis of symmetry is $x = \dfrac{-b}{2a}$
- the vertex of the quadratic has x-coordinate $\dfrac{-b}{2a}$.



To graph a quadratic of the form $y = ax^2 + bx + c$, we:

- Find the axis of symmetry $x = \dfrac{-b}{2a}$.
- Substitute this value to find the y-coordinate of the vertex.
- State the y-intercept c.
- Find the x-intercepts by solving $ax^2 + bx + c = 0$, either by factorisation or using the quadratic formula.
- Graph the quadratic using the information you have found.

Example 7 **Self Tutor**

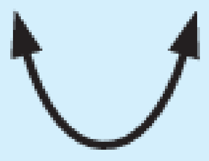
Consider the quadratic $y = 2x^2 + 8x - 10$.

a Find the axis of symmetry.

b Find the coordinates of the vertex.

c Find the axes intercepts.

d Hence sketch the quadratic.

$y = 2x^2 + 8x - 10$ has $a = 2$, $b = 8$, and $c = -10$. Since $a > 0$, the shape is 

a $\dfrac{-b}{2a} = \dfrac{-8}{2(2)} = -2$

The axis of symmetry is $x = -2$.

b When $x = -2$,

$$y = 2(-2)^2 + 8(-2) - 10$$
$$= -18$$

The vertex is $(-2, -18)$.

c The y-intercept is -10.

When $y = 0$, $2x^2 + 8x - 10 = 0$

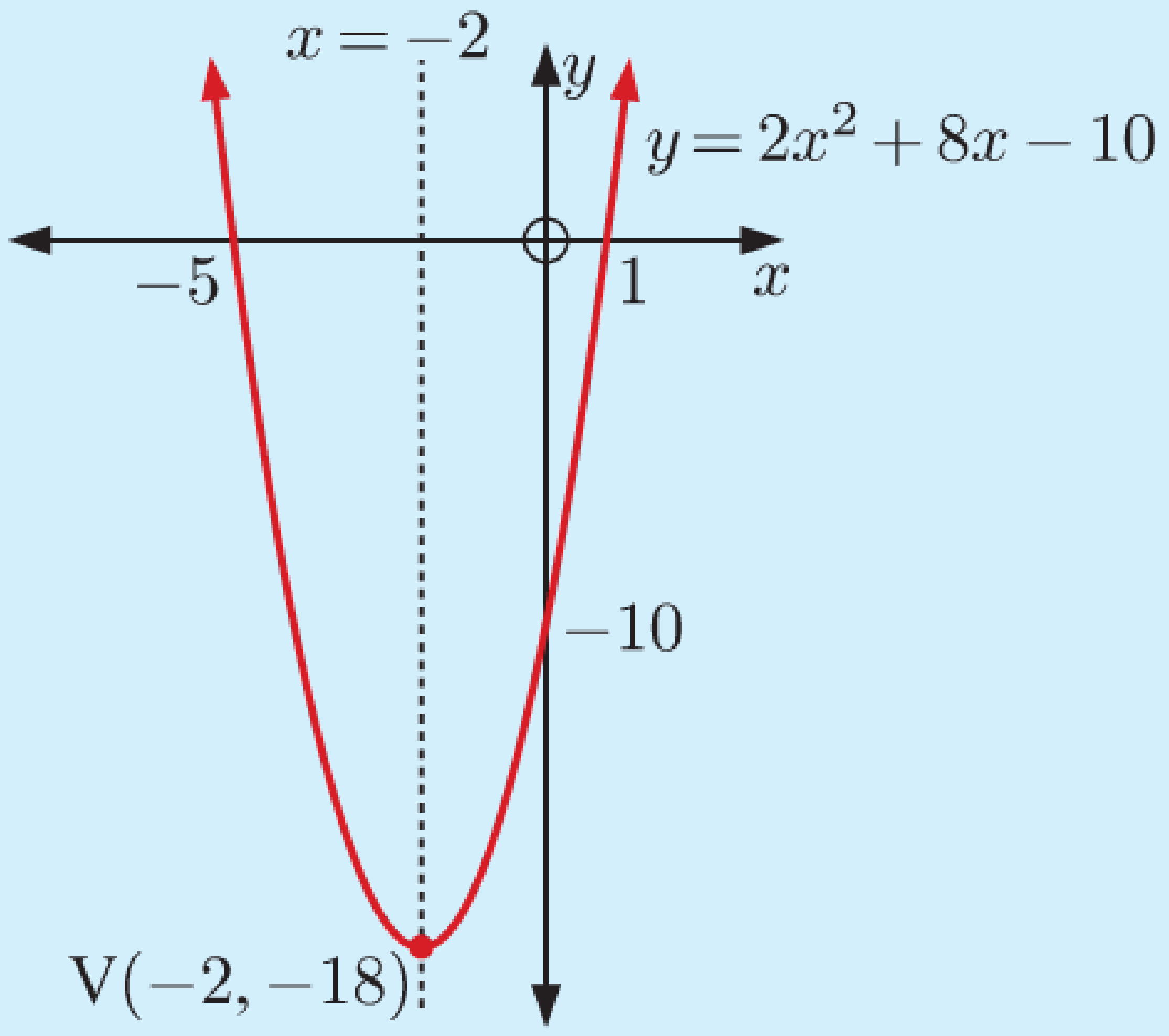
$\therefore\ 2(x^2 + 4x - 5) = 0$

$\therefore\ 2(x + 5)(x - 1) = 0$

$\therefore\ x = -5$ or 1

$\therefore$ the x-intercepts are -5 and 1.

d



EXERCISE 14B.3

1 For each of the following quadratics:

i Locate the turning point or vertex.

ii State whether the vertex is a minimum turning point or a maximum turning point.

a $y = x^2 - 4x + 2$

b $y = x^2 + 2x - 3$

c $y = 2x^2 + 4$

d $y = -3x^2 + 1$

e $y = 2x^2 + 8x - 7$

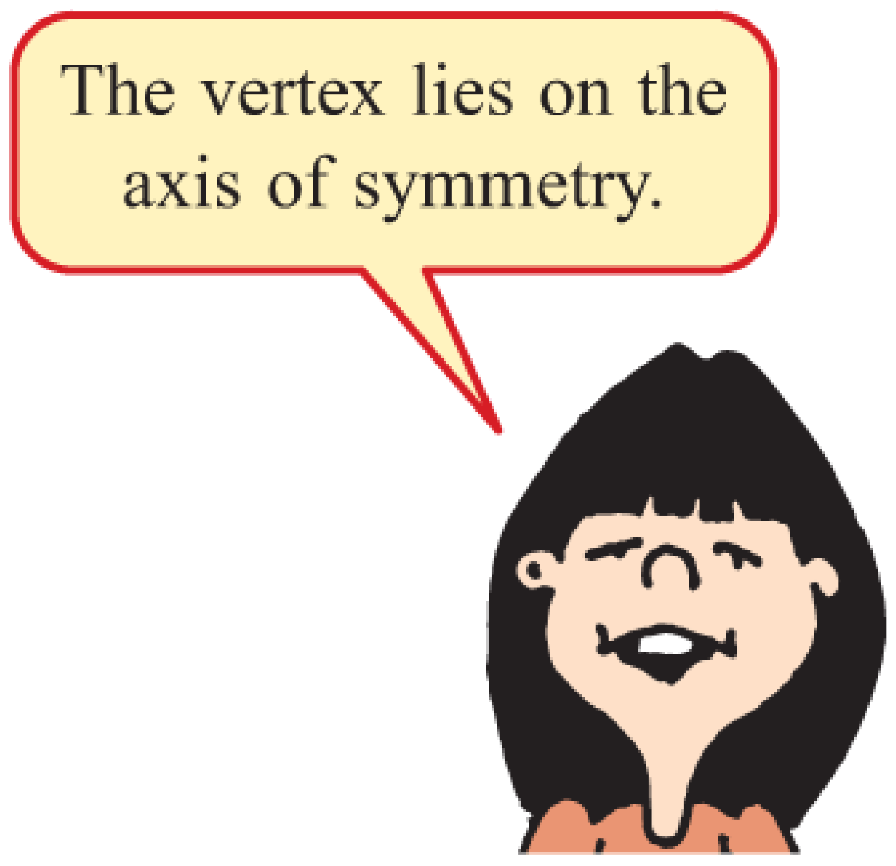
f $y = -x^2 - 4x - 9$

g $y = 2x^2 + 6x - 1$

h $y = 2x^2 - 10x + 3$

i $y = -\frac{1}{2}x^2 + x - 5$

j $y = \frac{1}{4}x^2 - 7x + 6$



2 For each of the following quadratics:

i State the axis of symmetry. **ii** Find the coordinates of the vertex.
iii Find the axes intercepts. **iv** Hence sketch the quadratic.

a $y = x^2 - 8x + 7$
b $y = -x^2 - 6x - 8$
c $y = 6x - x^2$
d $y = -x^2 + 3x - 2$
e $y = 2x^2 + 4x - 24$
f $y = -3x^2 + 4x - 1$
g $y = 2x^2 - 5x + 2$
h $y = 4x^2 - 8x - 5$
i $y = -\frac{1}{4}x^2 + 2x - 3$

3 For the quadratic function $y = ax^2 + bx + c$, suppose a and c remain constant, but b is allowed to vary. As b varies, the vertex of the quadratic changes. Show that the path formed by the vertex is itself a quadratic function, and that the vertex of this quadratic function always lies on the y-axis.

DYNAMIC GEOMETRY PACKAGE

ACTIVITY 2

Click on the icon to run a card game for quadratic functions.

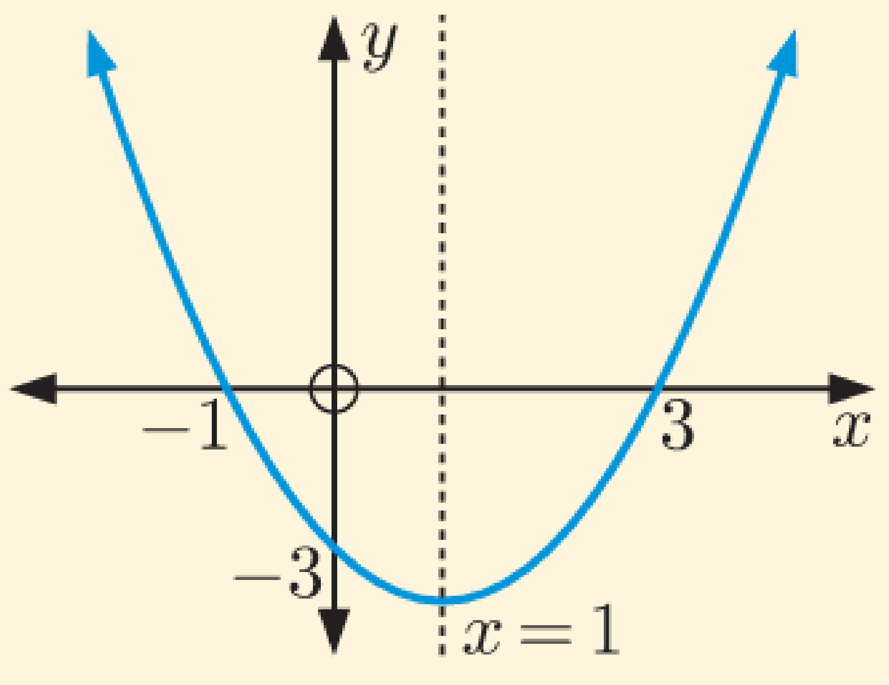
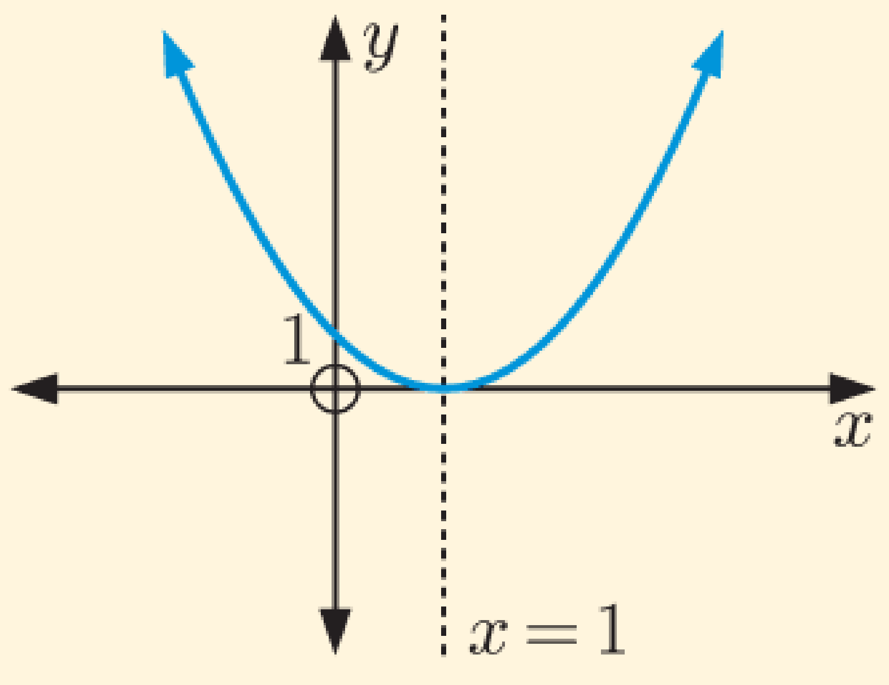
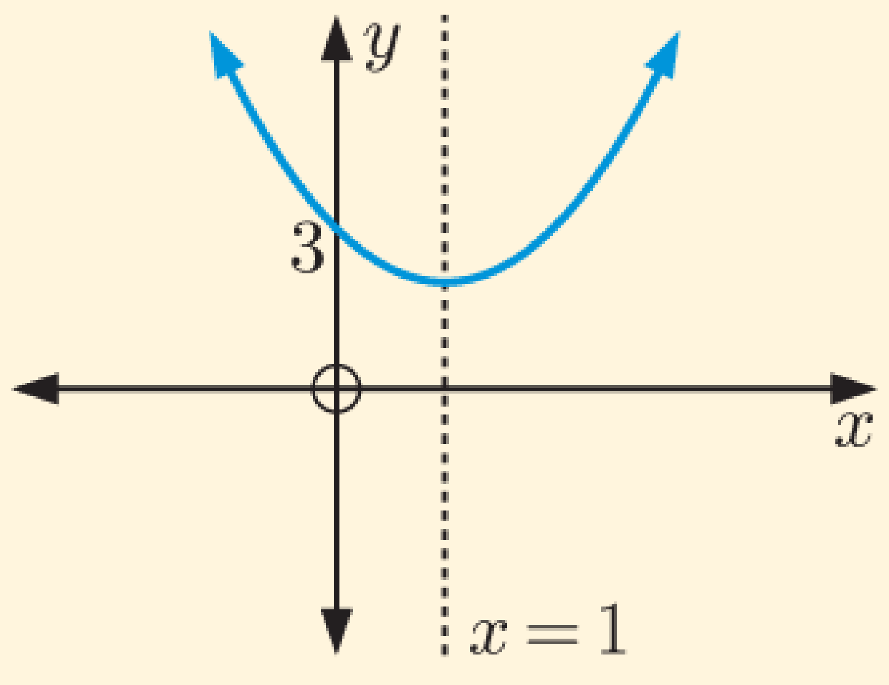
CARD GAME

C USING THE DISCRIMINANT

The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $\Delta = b^2 - 4ac$.

We have used Δ to determine the number of real roots of the equation. If they exist, these roots correspond to zeros of the quadratic $y = ax^2 + bx + c$. Δ therefore tells us about the relationship between the graph of a quadratic function and the x-axis.

The graphs of $y = x^2 - 2x - 3$, $y = x^2 - 2x + 1$, and $y = x^2 - 2x + 3$ all have the same axis of symmetry, $x = 1$.

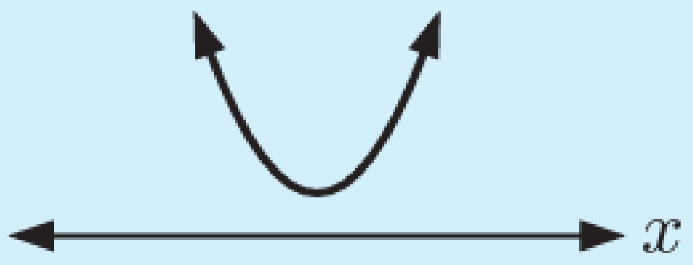
$y = x^2 - 2x - 3$	$y = x^2 - 2x + 1$	$y = x^2 - 2x + 3$
		
$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(-3)$ $= 16$ cuts the x-axis twice	$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(1)$ $= 0$ touches the x-axis	$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(3)$ $= -8$ does not cut the x-axis

For a quadratic function $y = ax^2 + bx + c$, we consider the discriminant $\Delta = b^2 - 4ac$.

- If $\Delta > 0$, the graph cuts the x-axis twice.
- If $\Delta = 0$, the graph *touches* the x-axis.
- If $\Delta < 0$, the graph does not cut the x-axis.

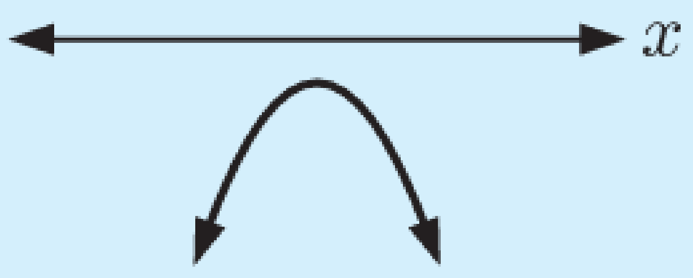
POSITIVE DEFINITE AND NEGATIVE DEFINITE QUADRATICS

Positive definite quadratics are quadratics which are positive for all values of x. So, $ax^2+bx+c>0$ for all $x \in \mathbb{R}$.



A quadratic is **positive definite** if and only if $a>0$ and $\Delta<0$.

Negative definite quadratics are quadratics which are negative for all values of x. So, $ax^2+bx+c<0$ for all $x \in \mathbb{R}$.



A quadratic is **negative definite** if and only if $a<0$ and $\Delta<0$.

Example 8

Self Tutor

Use the discriminant to determine the relationship between the graph of each function and the x-axis:

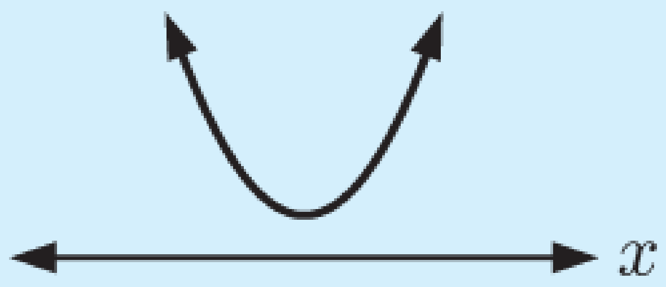
a $y=x^2+3x+4$

b $y=-2x^2+5x+1$

a $a=1, \ b=3, \ c=4$

$$\begin{aligned}\therefore \quad \Delta &= b^2-4ac\\ &= 9-4(1)(4)\\ &= -7\end{aligned}$$

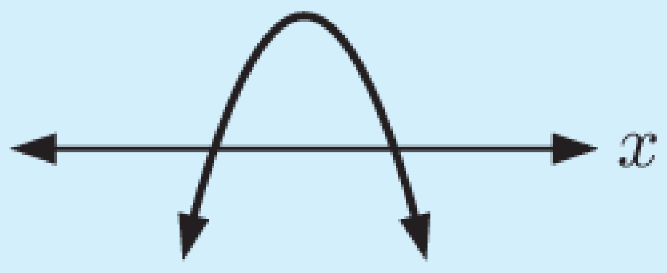
Since $\Delta<0$, the graph does not cut the x-axis.
Since $a>0$, the graph is concave up.
The graph is positive definite.



b $a=-2, \ b=5, \ c=1$

$$\begin{aligned}\therefore \quad \Delta &= b^2-4ac\\ &= 25-4(-2)(1)\\ &= 33\end{aligned}$$

Since $\Delta>0$, the graph cuts the x-axis twice.
Since $a<0$, the graph is concave down.
The quadratic is neither positive definite nor negative definite.



EXERCISE 14C

1 Use the discriminant to determine the relationship between the graph of each function and the x-axis:

a $y=x^2+x-2$
b $y=x^2-4x+1$
c $y=-x^2-3$
d $y=x^2+7x-2$
e $y=x^2+8x+16$
f $y=-2x^2+3x+1$
g $y=6x^2+5x-4$
h $y=-x^2+x+6$
i $y=9x^2+6x+1$

2 Consider the graph of $y=2x^2-5x+1$.

a Describe the shape of the graph.
b Use the discriminant to show that the graph cuts the x-axis twice.
c Find the x-intercepts, rounding your answers to 2 decimal places.
d State the y-intercept.
e Hence sketch the function.

3 Consider the graph of $y = -x^2 + 4x - 7$.

- **a** Use the discriminant to show that the graph does not cut the x-axis.
- **b** Is the graph positive definite or negative definite? Explain your answer.
- **c** Find the vertex and y-intercept.
- **d** Hence sketch the function.

4 Show that:

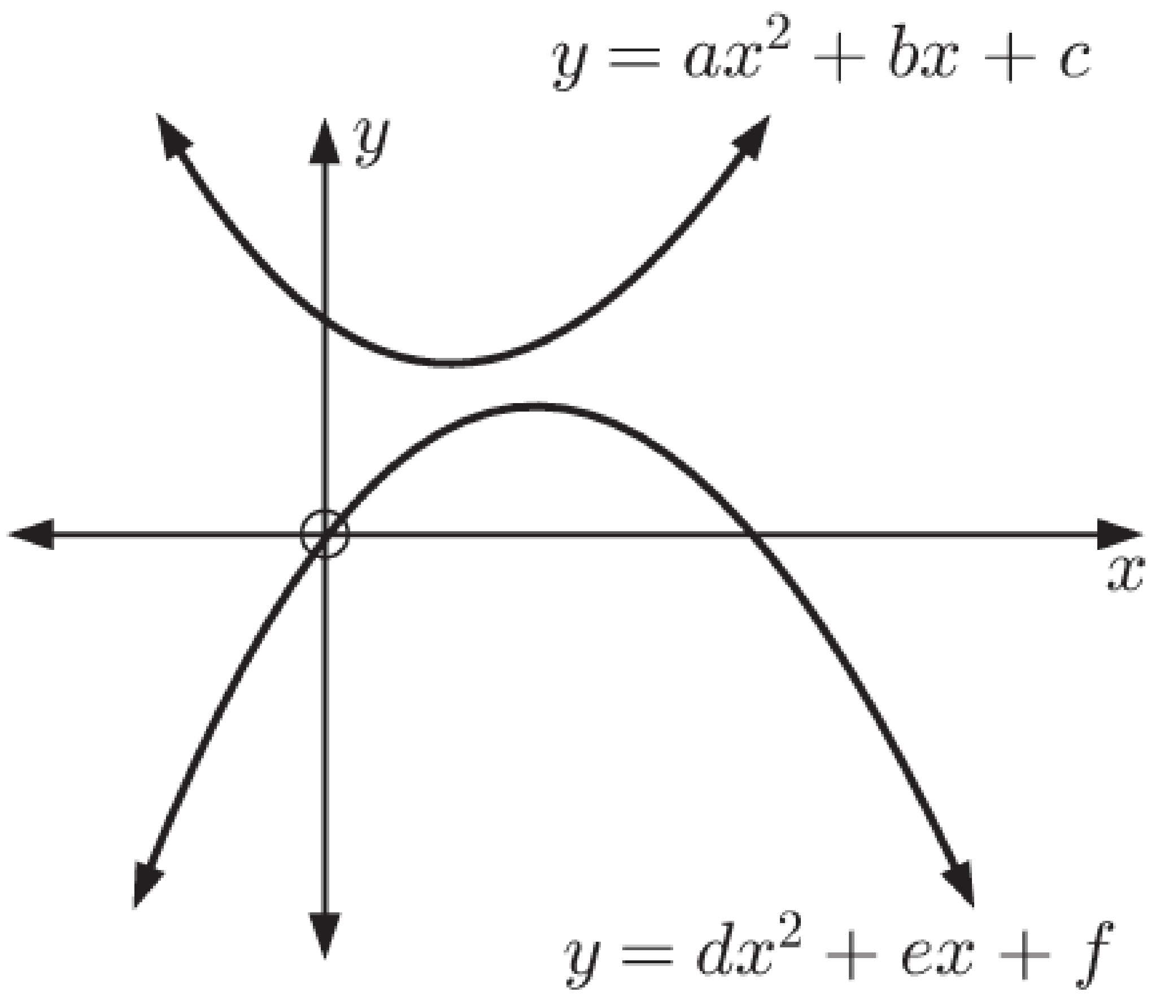
- **a** $2x^2 - 4x + 7$ is positive definite
- **b** $-2x^2 + 3x - 4$ is negative definite
- **c** $x^2 - 3x + 6 > 0$ for all x
- **d** $4x - x^2 - 6 < 0$ for all x.

5 Consider the graphs illustrated.

Let $y = ax^2 + bx + c$ have discriminant Δ_1, and $y = dx^2 + ex + f$ have discriminant Δ_2.

Copy and complete the following table by indicating whether each constant is positive, negative, or zero:

Constant	a	b	c	d	e	f	Δ_1	Δ_2
Sign								



Example 9 **Self Tutor**

Find the value(s) of k for which the function $y = x^2 - 6x + k$:

- **a** cuts the x-axis twice
- **b** touches the x-axis
- **c** misses the x-axis.

$a = 1, \quad b = -6, \quad c = k$

$$\begin{aligned} \therefore \quad \Delta &= b^2 - 4ac \\ &= (-6)^2 - 4(1)(k) \\ &= 36 - 4k \end{aligned}$$

a The graph cuts the x-axis twice if $\Delta > 0$.

$$\therefore \quad 36 - 4k > 0$$
$$\therefore \quad 4k < 36$$
$$\therefore \quad k < 9$$

b The graph touches the x-axis twice if $\Delta = 0$.

$$\therefore \quad 36 - 4k = 0$$
$$\therefore \quad k = 9$$

c The graph does not cut the x-axis if $\Delta < 0$.

$$\therefore \quad 36 - 4k < 0$$
$$\therefore \quad 4k > 36$$
$$\therefore \quad k > 9$$

6 For each quadratic function, find the value(s) of k for which the function:

- **i** cuts the x-axis twice
- **ii** touches the x-axis
- **iii** misses the x-axis.

- **a** $y = x^2 + 3x + k$
- **b** $y = kx^2 - 4x + 1$
- **c** $y = (k+1)x^2 - 2kx + (k-4)$

7 Explain why $3x^2 + kx - 1$ is never positive definite for any value of k.

8 Find the value of k such that $y = \frac{1}{2}x^2 + (k-2)x + k^2 + 4$ is *not* positive definite. What relationship does the graph have with the x-axis in this case?

9 b_1, c_1, b_2, and c_2 are real, non-zero numbers such that $b_1 b_2 = 2(c_1 + c_2)$. Show that at least one of the quadratics $y = x^2 + b_1 x + c_1$, and $y = x^2 + b_2 x + c_2$ cuts the x-axis twice.

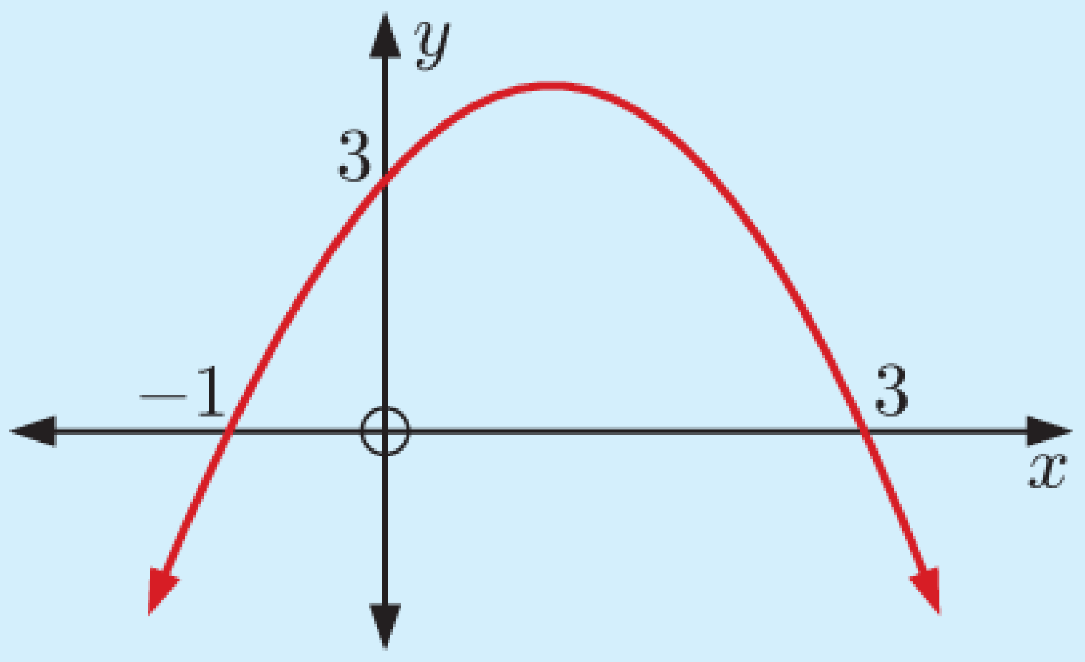
D FINDING A QUADRATIC FROM ITS GRAPH

If we are given sufficient information on or about a graph, we can determine the quadratic in whatever form is required.

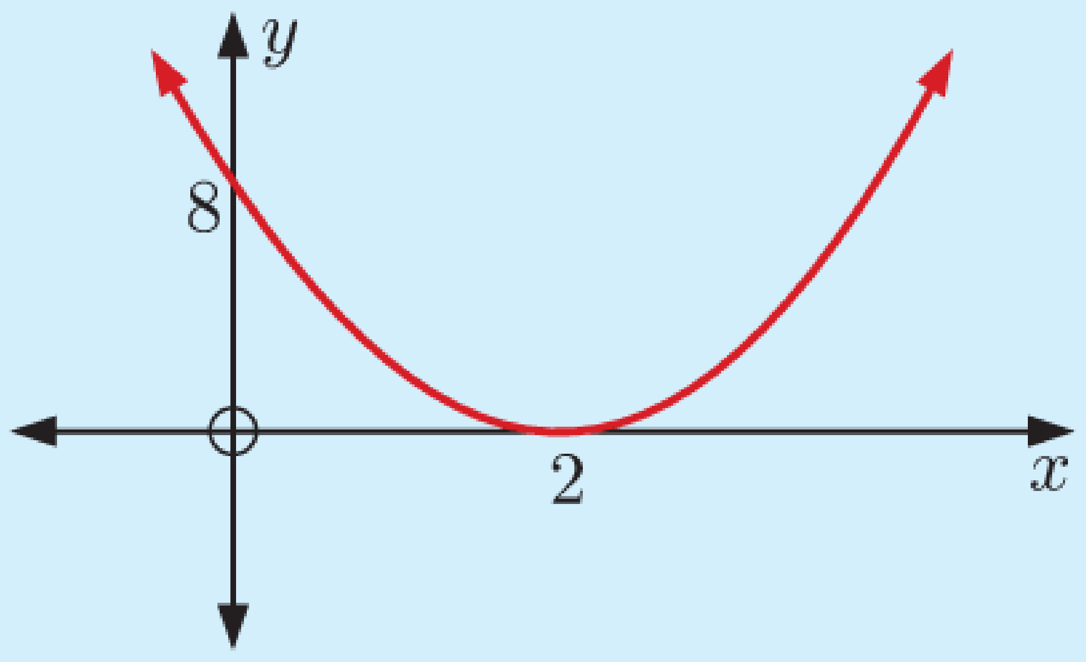
Example 10 Self Tutor

Find the equation of the quadratic with graph:

a



b

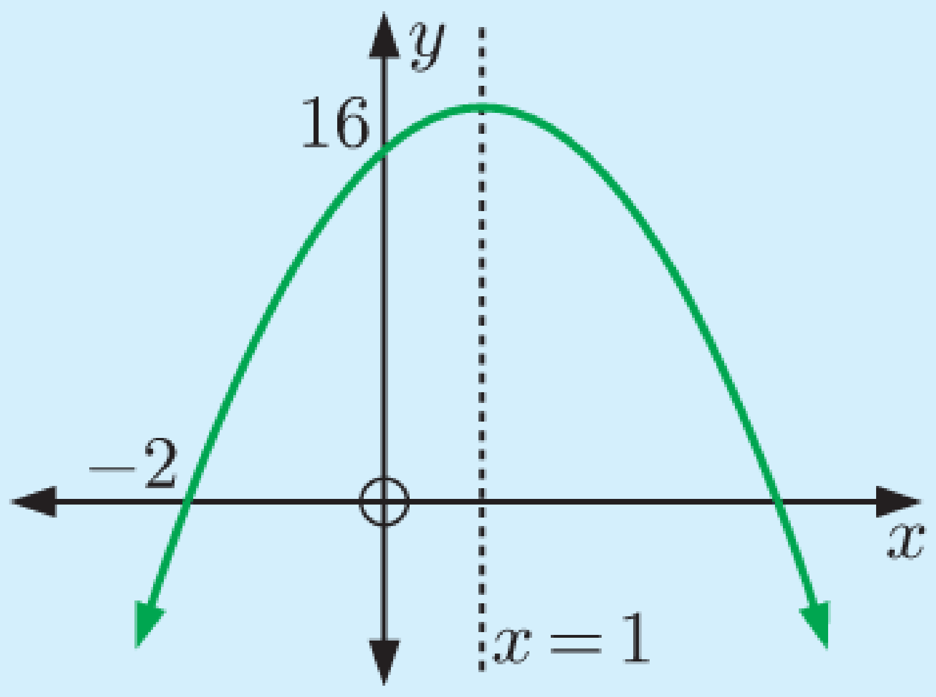


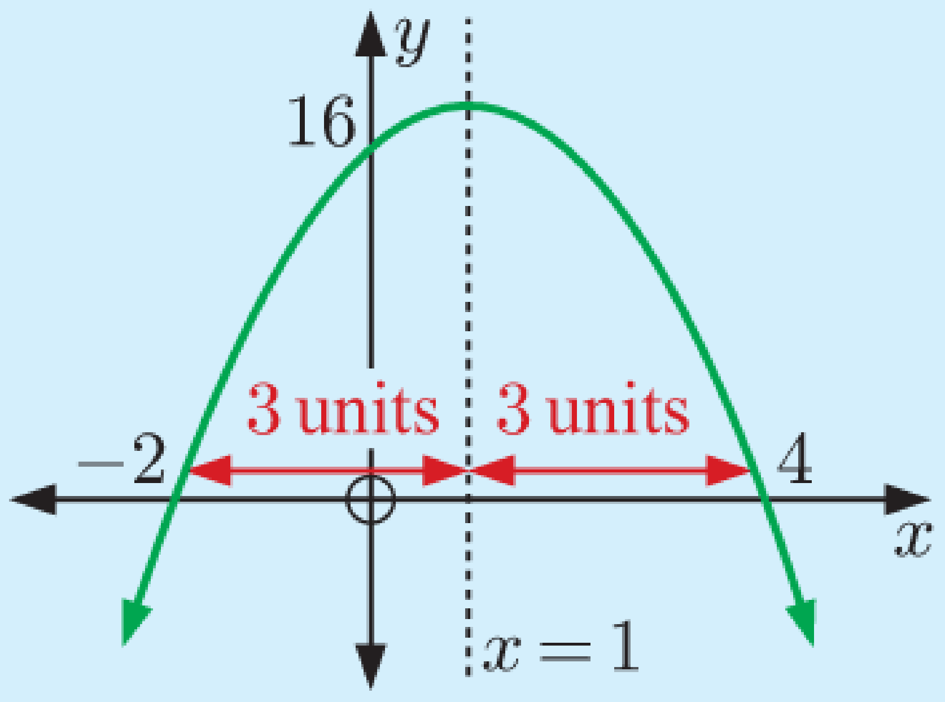
a Since the x-intercepts are -1 and 3,
$y = a(x+1)(x-3)$.
The graph is concave down, so $a < 0$.
When $x = 0$, $y = 3$
$\therefore\ 3 = a(1)(-3)$
$\therefore\ a = -1$
The quadratic is $y = -(x+1)(x-3)$.

b The graph touches the x-axis at $x = 2$,
so $y = a(x-2)^2$.
The graph is concave up, so $a > 0$.
When $x = 0$, $y = 8$
$\therefore\ 8 = a(-2)^2$
$\therefore\ a = 2$
The quadratic is $y = 2(x-2)^2$.

Example 11 Self Tutor

Find the equation of the quadratic with graph:





The axis of symmetry $x = 1$ lies midway between the x-intercepts.

$\therefore$ the other x-intercept is 4.

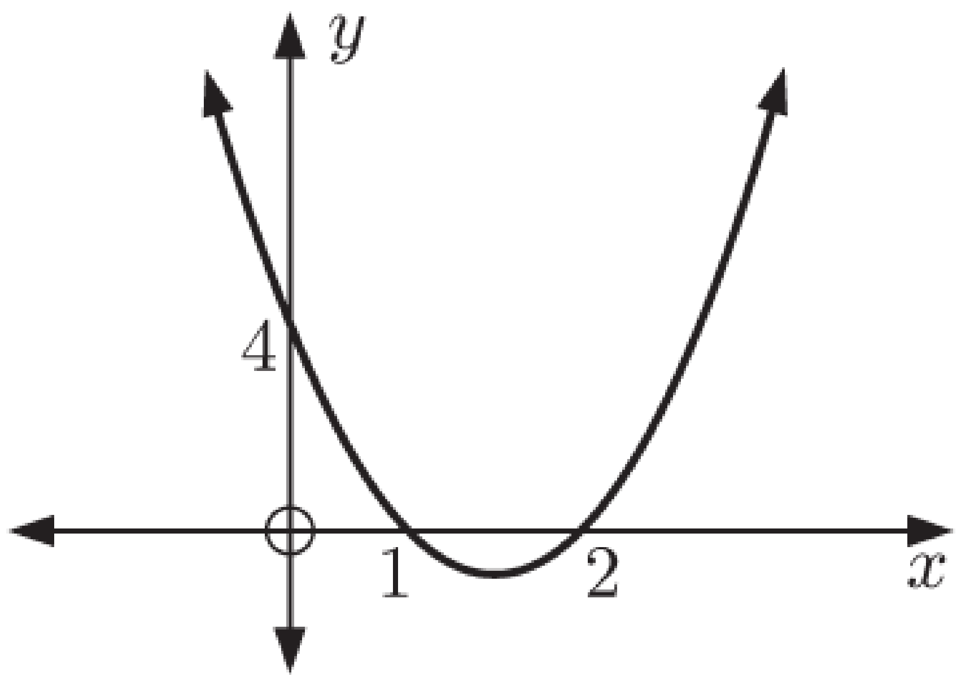
$\therefore$ the quadratic has the form
$y = a(x+2)(x-4)$ where $a < 0$
But when $x = 0$, $y = 16$
$\therefore\ 16 = a(2)(-4)$
$\therefore\ a = -2$

The quadratic is $y = -2(x+2)(x-4)$.

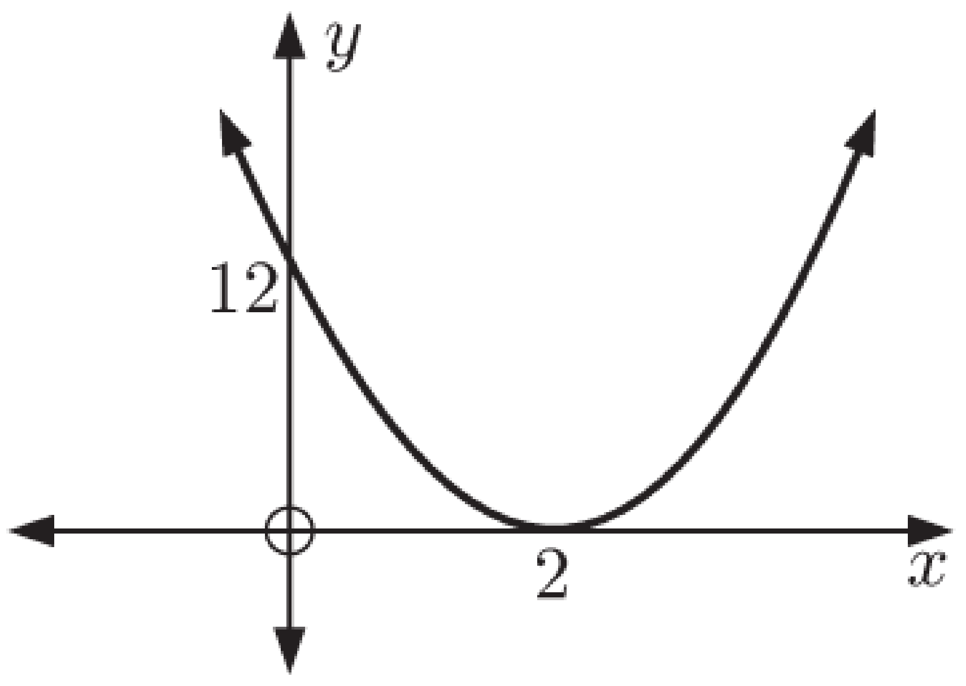
EXERCISE 14D

1 Find the equation of the quadratic with graph:

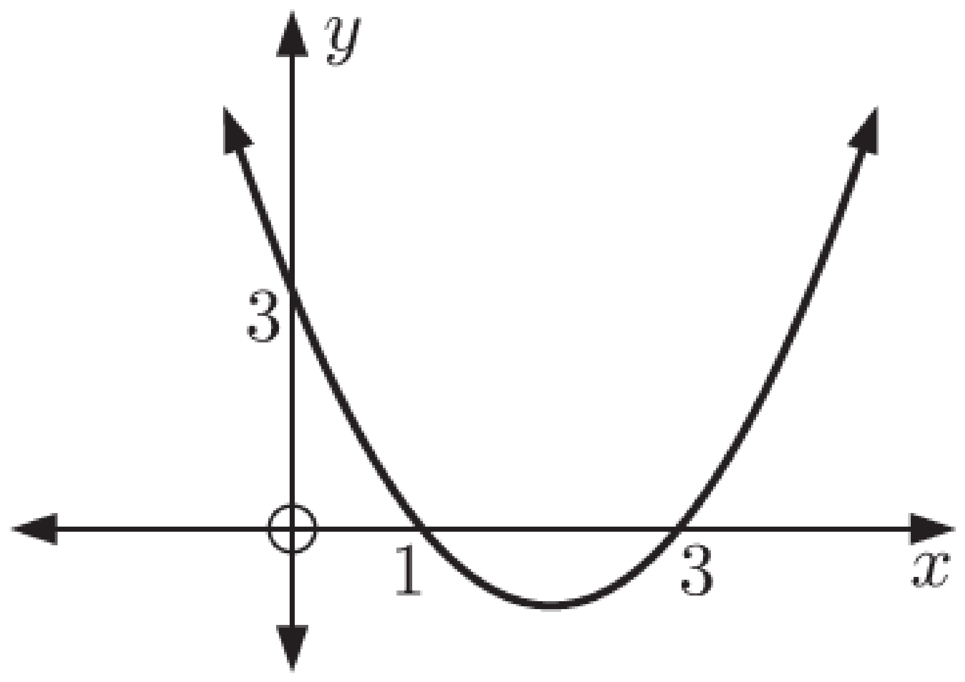
a



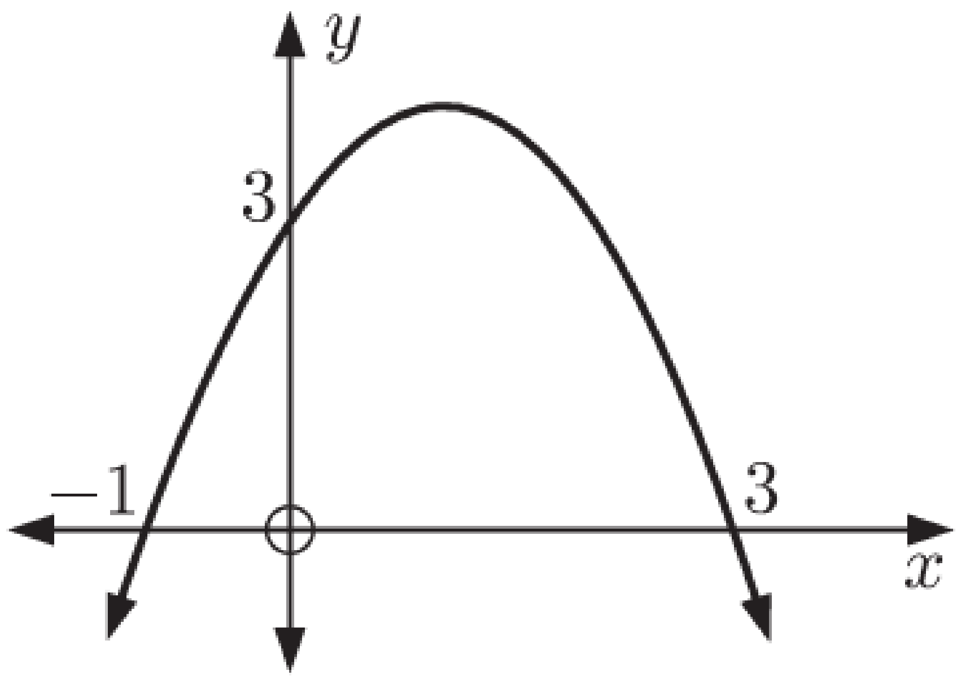
b



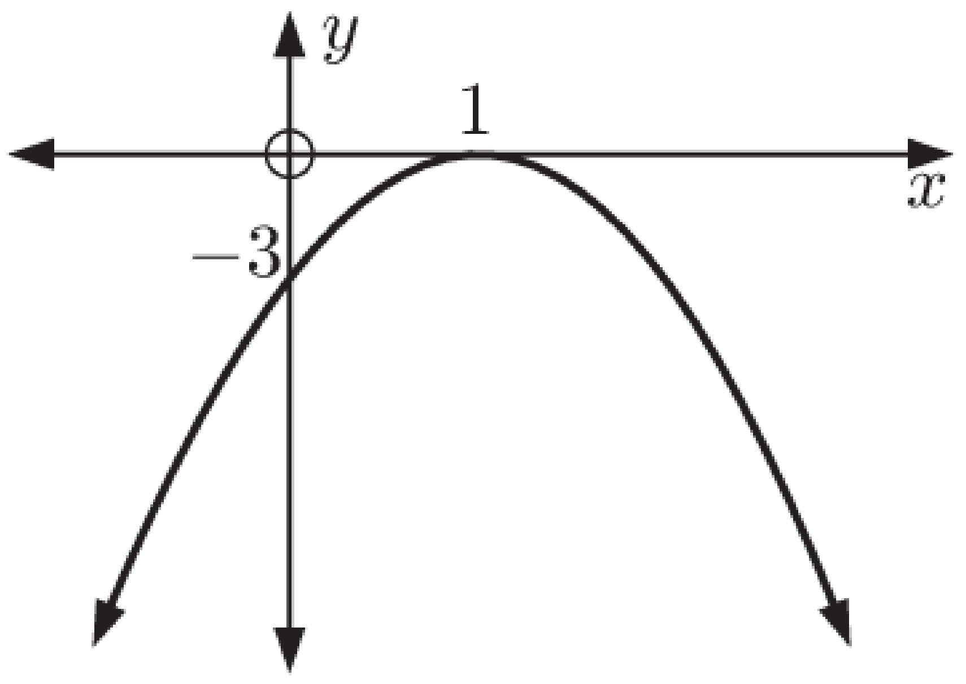
c



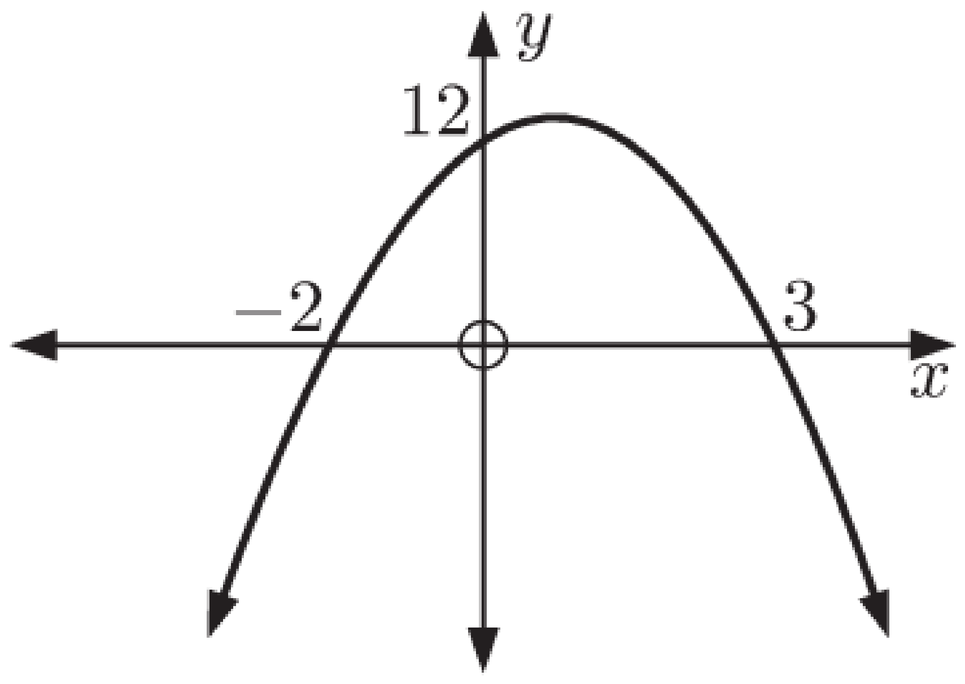
d



e

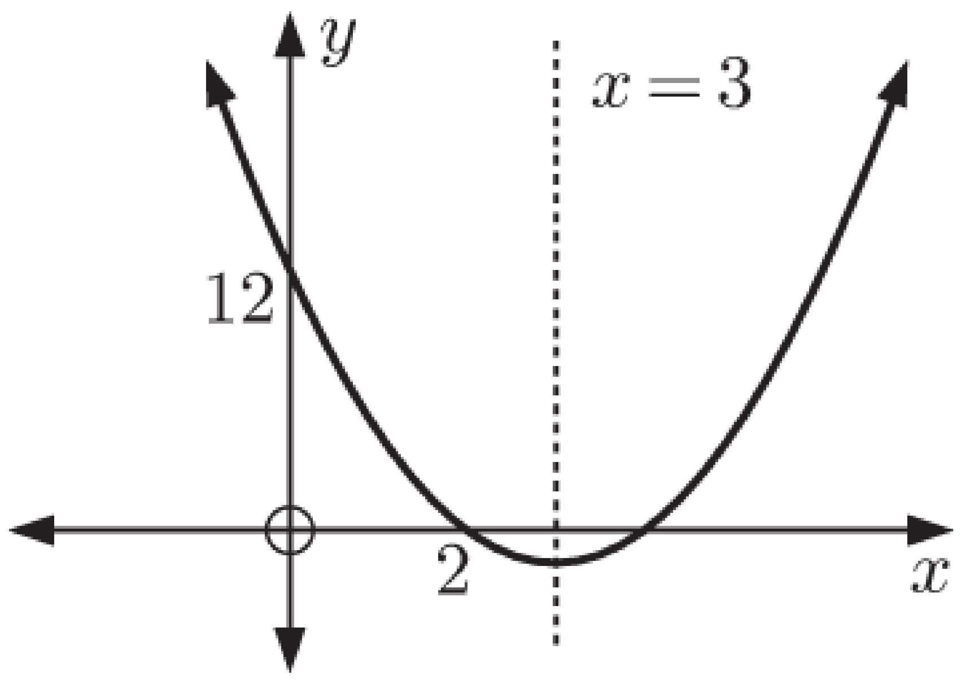


f

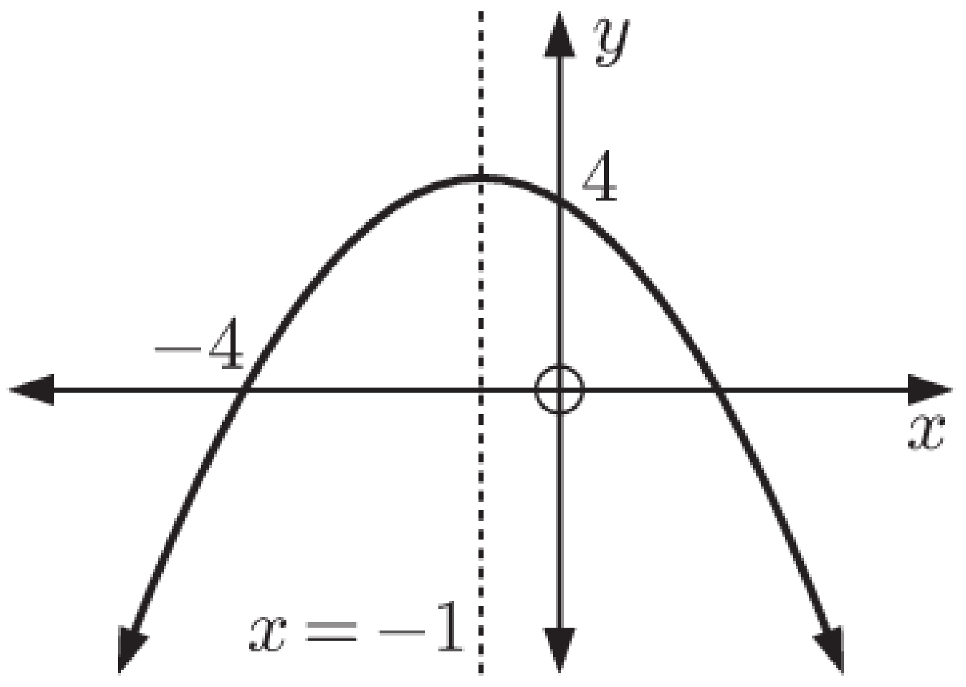


2 Find the equation of the quadratic with graph:

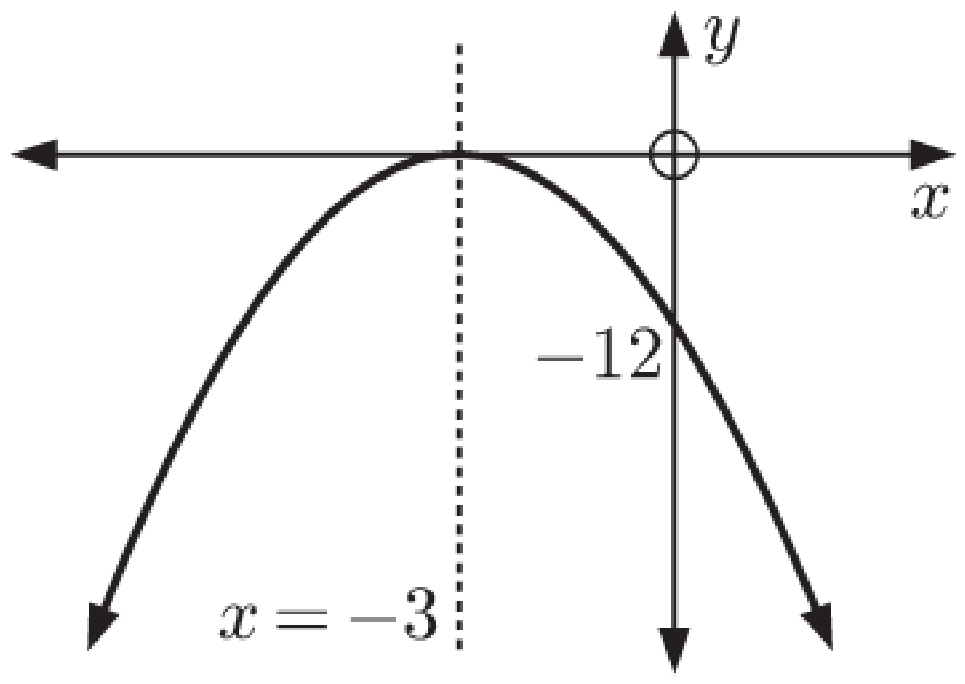
a



b



c



Example 12

Self Tutor

Find the equation of the quadratic whose graph cuts the x-axis at 4 and -3, and which passes through the point $(2, -20)$. Give your answer in the form $y = ax^2 + bx + c$.

Since the x-intercepts are 4 and -3, the quadratic has the form $y = a(x-4)(x+3)$, $a \neq 0$.

When $x = 2$, $y = -20$

$$\therefore\ -20 = a(2-4)(2+3)$$
$$\therefore\ -20 = a(-2)(5)$$
$$\therefore\ a = 2$$

The quadratic is
$$\begin{aligned} y &= 2(x-4)(x+3) \\ &= 2(x^2 - x - 12) \\ &= 2x^2 - 2x - 24 \end{aligned}$$

3 Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph:

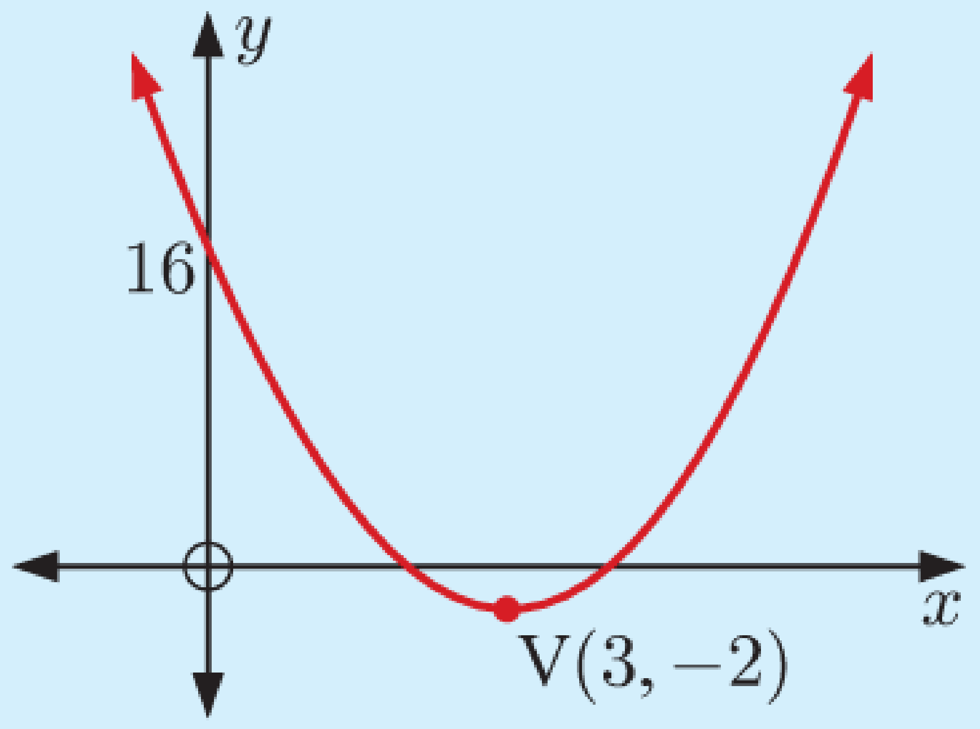
- **a** cuts the x-axis at 5 and 1, and passes through $(2, -9)$
- **b** cuts the x-axis at 2 and $-\frac{1}{2}$, and passes through $(3, -14)$
- **c** touches the x-axis at 3 and passes through $(-2, -25)$
- **d** touches the x-axis at -2 and passes through $(-1, 4)$
- **e** cuts the x-axis at 3, passes through $(5, 12)$, and has axis of symmetry $x = 2$
- **f** cuts the x-axis at 5, passes through $(2, 5)$, and has axis of symmetry $x = 1$.

Example 13

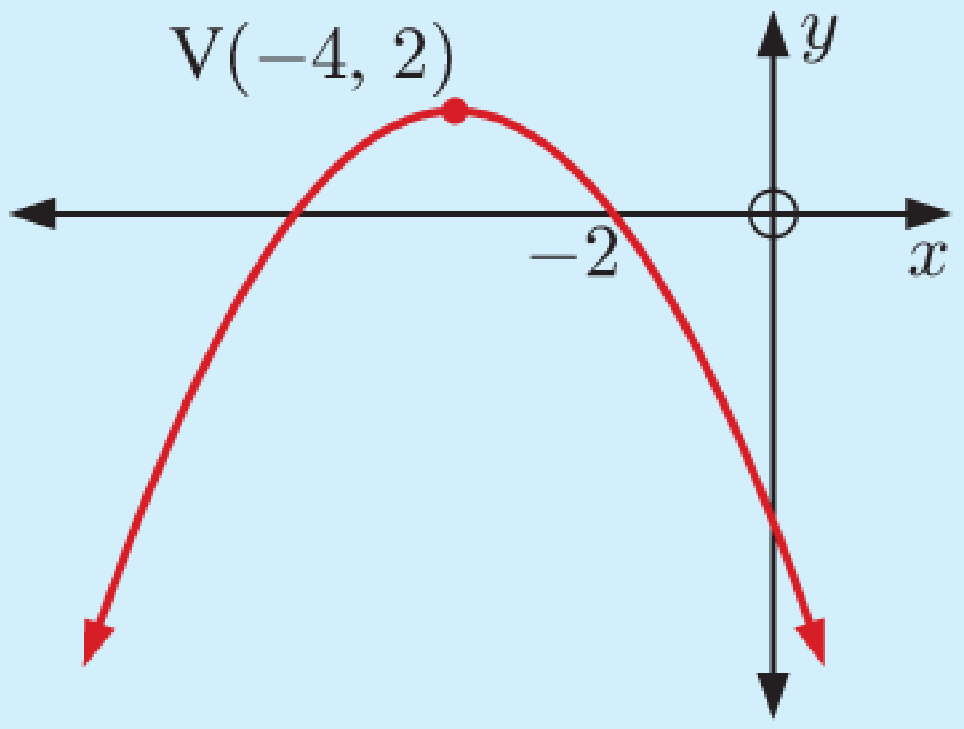
Self Tutor

Find the equation of the quadratic with graph:

a



b



a Since the vertex is $(3, -2)$, the quadratic has the form $y = a(x-3)^2 - 2$ where $a > 0$.

When $x = 0$, $y = 16$

$\therefore\ 16 = a(-3)^2 - 2$

$\therefore\ 16 = 9a - 2$

$\therefore\ 18 = 9a$

$\therefore\ a = 2$

The quadratic is $y = 2(x-3)^2 - 2$.

b Since the vertex is $(-4, 2)$, the quadratic has the form $y = a(x+4)^2 + 2$ where $a < 0$.

When $x = -2$, $y = 0$

$\therefore\ 0 = a(2)^2 + 2$

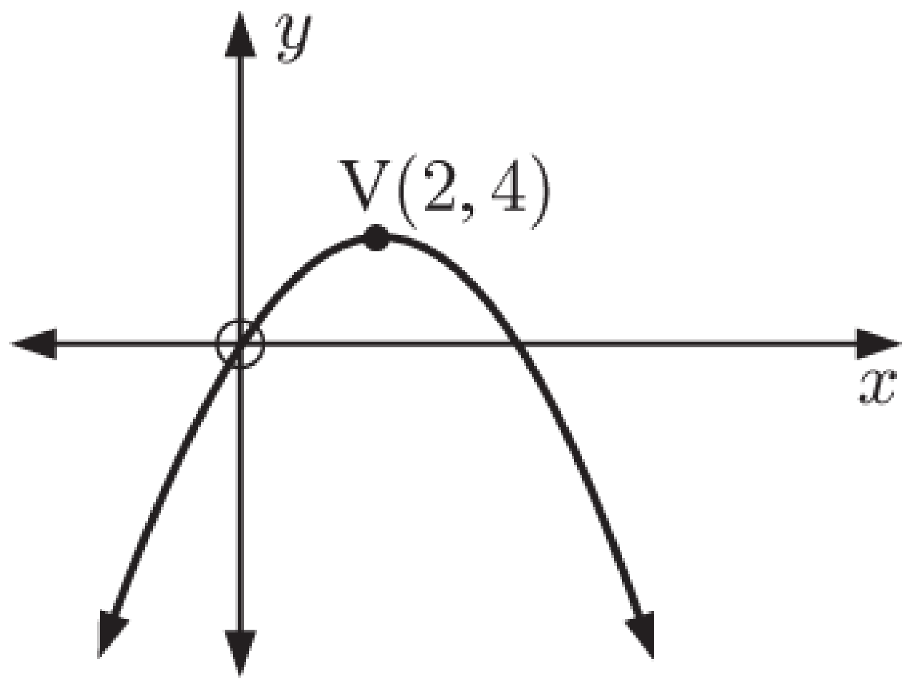
$\therefore\ 4a = -2$

$\therefore\ a = -\frac{1}{2}$

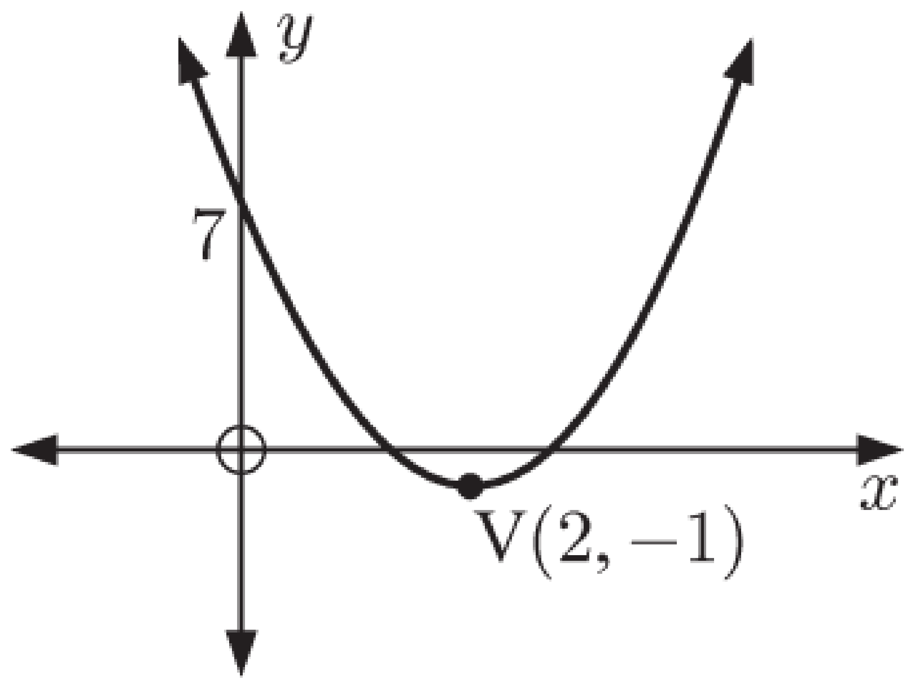
The quadratic is $y = -\frac{1}{2}(x+4)^2 + 2$.

4 If V is the vertex, find the equation of the quadratic with graph:

a



b



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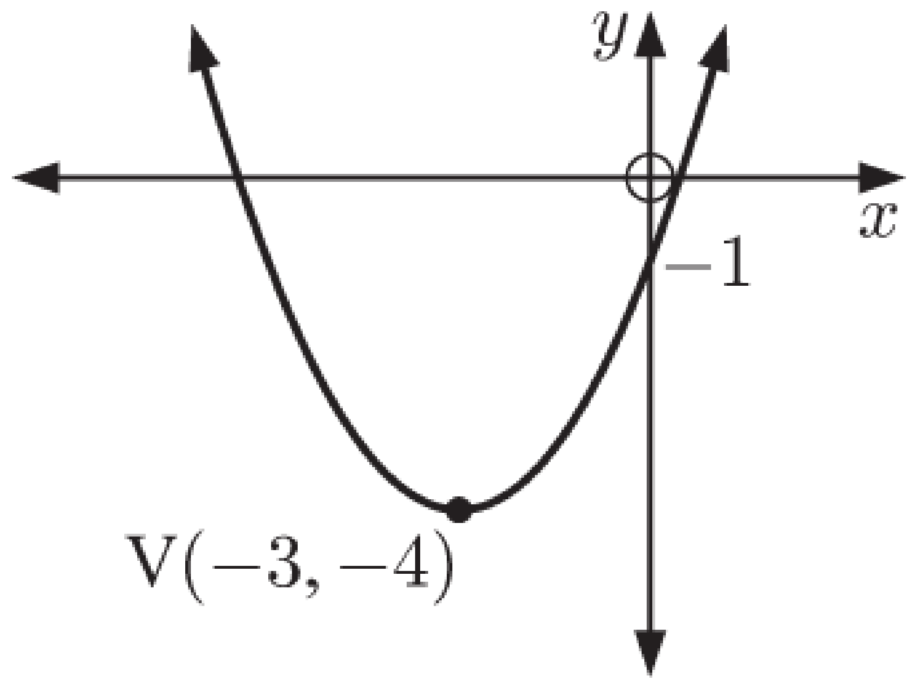

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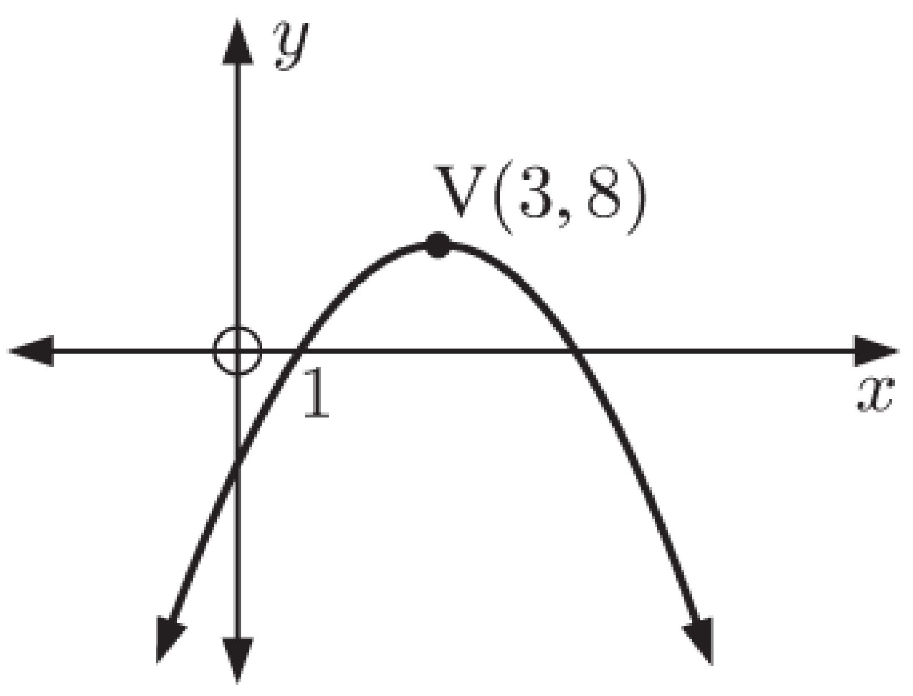

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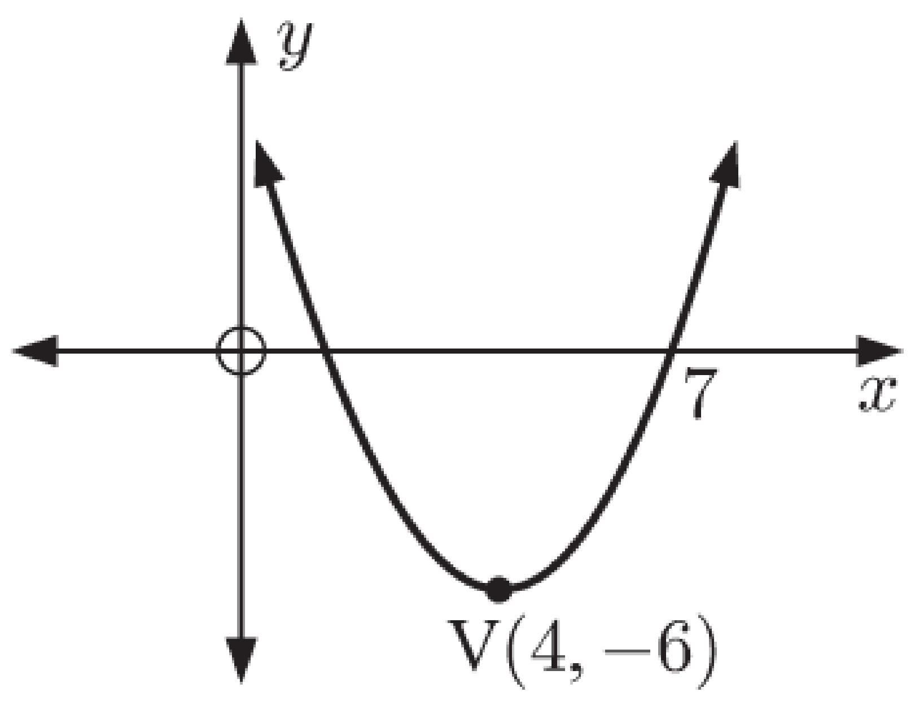

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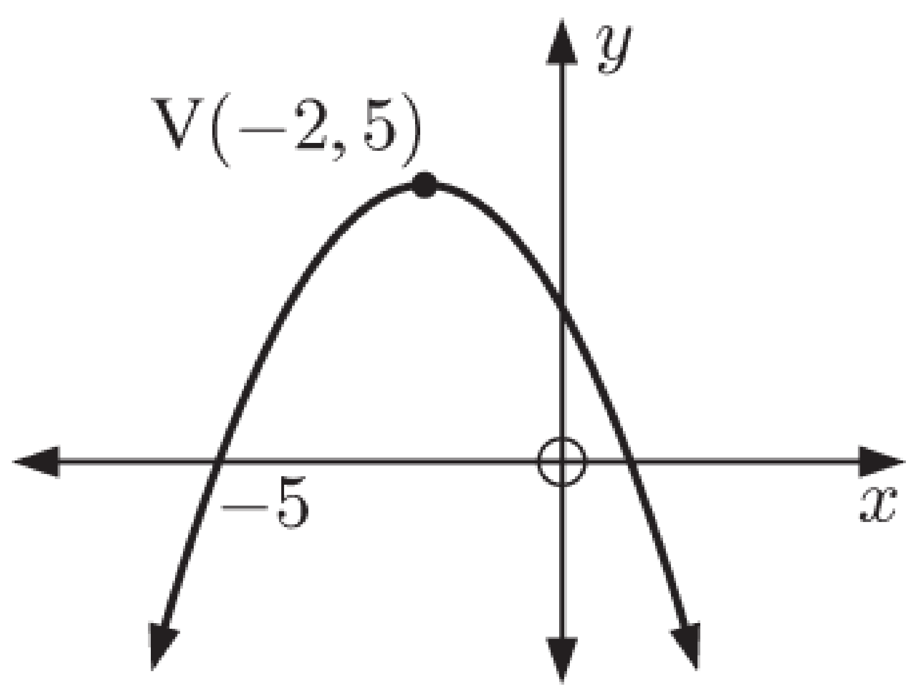
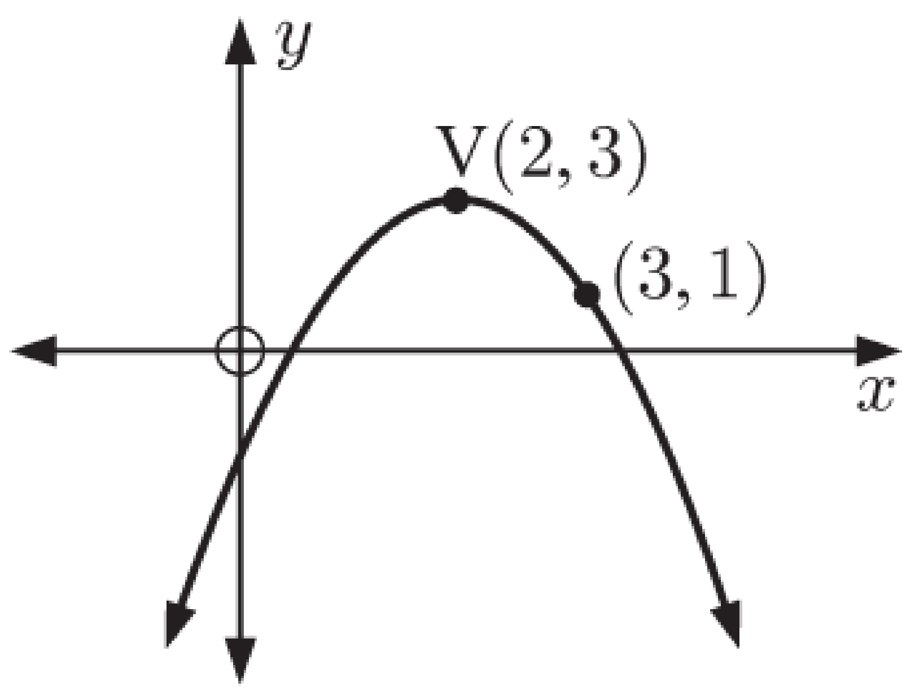
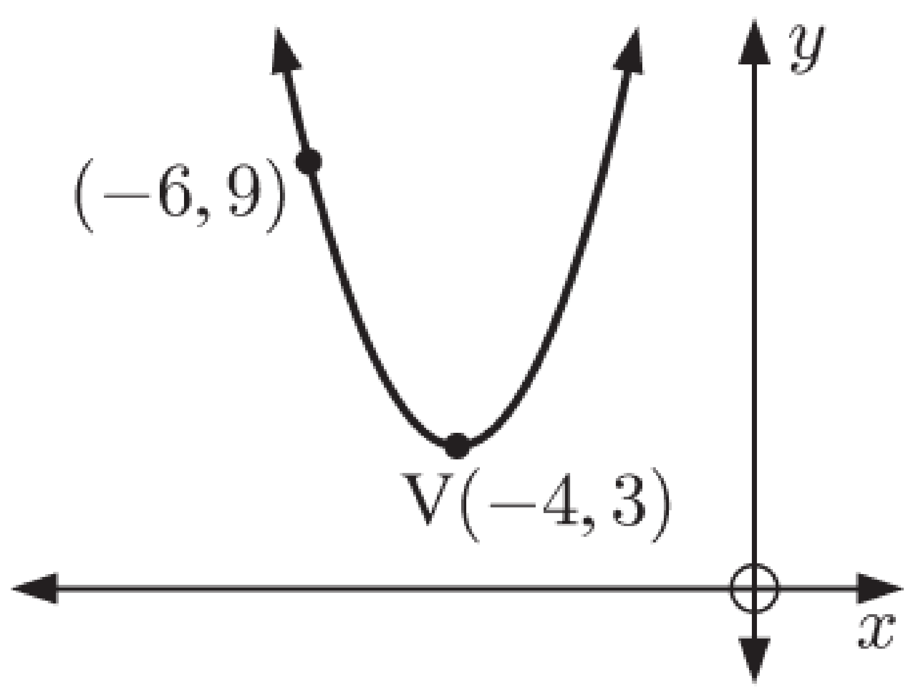


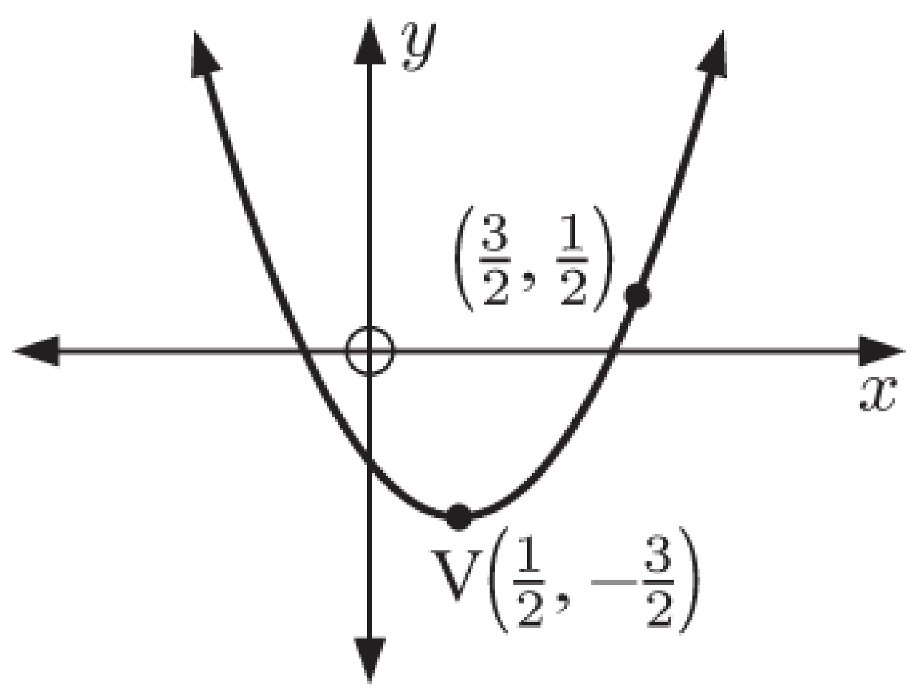
g



h



i



5 A quadratic has vertex $(2, -5)$, and passes through the point $(-1, 13)$. Find the value of the quadratic when $x = 4$.

INVESTIGATION 3 FINDING QUADRATICS

For the quadratic $y = 2x^2 + 3x + 7$ we can construct a table of values for $x = 0, 1, 2, 3, 4, 5$.

x	0	1	2	3	4	5
y	7	12	21	34	51	72

We turn this table into a **difference table** by adding two further rows:

- the row Δ_1 gives the differences between successive y-values
- the row Δ_2 gives the differences between successive Δ_1-values.

x	0	1	2	3	4	5
y	7	12	21	34	51	72
Δ_1		5	9	$\boxed{13}$	17	$\boxed{21}$
Δ_2		$\boxed{4}$	4	4	4	

9 − 5 34 − 21 72 − 51

What to do:

1 Construct difference tables for $x = 0, 1, 2, 3, 4, 5$ for each of the following quadratics:

a $y = x^2 + 4x + 3$

b $y = 3x^2 - 4x$

c $y = 5x - x^2$

d $y = 4x^2 - 5x + 2$

2 What do you notice about the Δ_2 row for each quadratic in **1**?

3 Consider the general quadratic $y = ax^2 + bx + c$, $a \neq 0$.

a Copy and complete the following difference table:

x	0	1	2	3	4	5
y	ⓒ	$a + b + c$	$4a + 2b + c$	......	......	......
Δ_1		○	......	......	......	......
Δ_2		○	......	......	......	

b Comment on the Δ_2 row.

c What can the circled numbers be used for?

4 Use your observations in **3** to determine, if possible, the quadratics with the following tables of values:

a

x	0	1	2	3	4
y	6	5	8	15	26

b

x	0	1	2	3	4
y	8	10	18	32	52

c

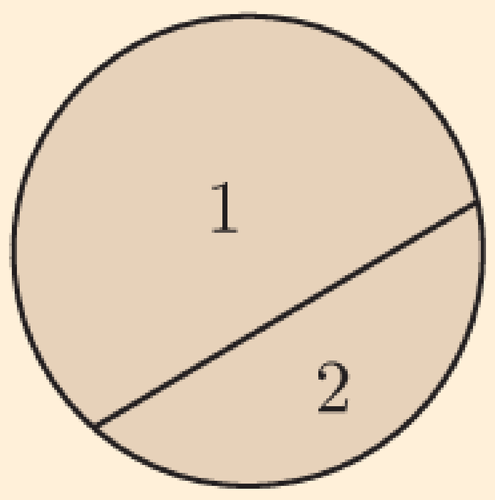
x	0	1	2	3	4
y	1	2	−1	−8	−19

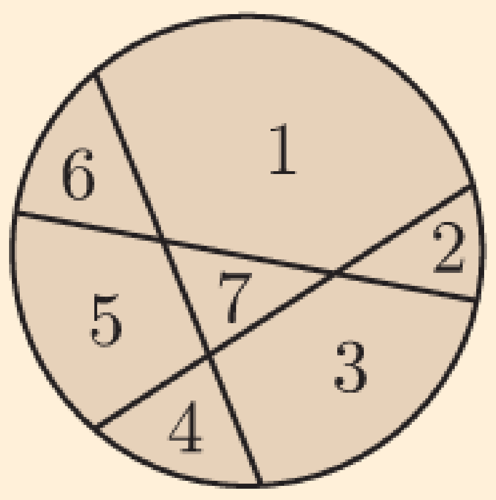
d

x	0	1	2	3	4
y	5	3	−1	−7	−15

5 We wish to determine the **maximum** number of pieces into which a pizza can be cut using n straight cuts across it. For example:

- for $n = 1$ we can make 2 pieces
- for $n = 3$ we can make 7 pieces.





a Copy and complete:

Number of cuts, n	0	1	2	3	4	5
Maximum number of pieces, P_n						

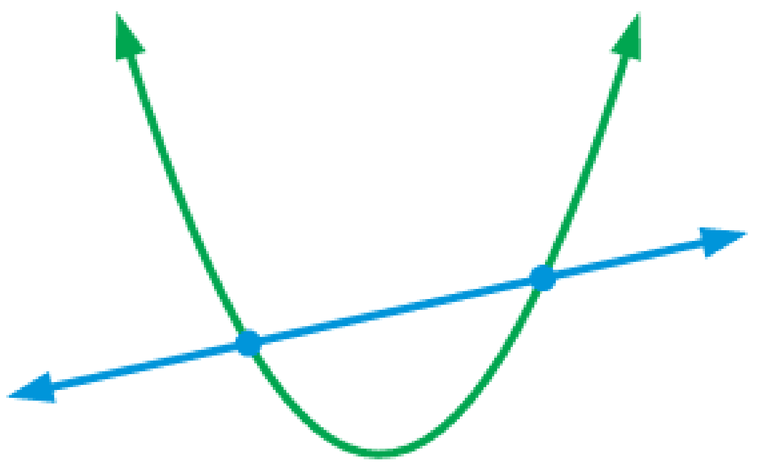
b Complete the Δ_1 and Δ_2 rows. Hence determine a quadratic formula for P_n.

c For a huge pizza with 12 cuts across it, find the maximum number of pieces which can result.

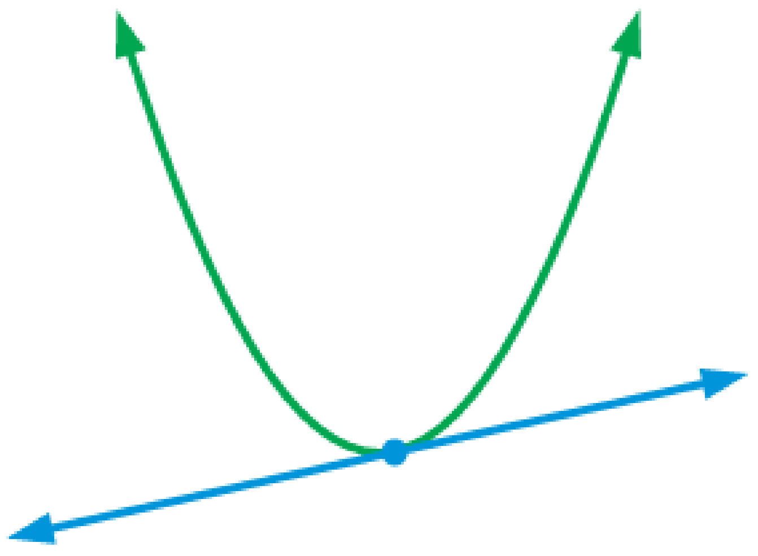
E THE INTERSECTION OF GRAPHS

Consider the graphs of a quadratic function and a linear function on the same set of axes.

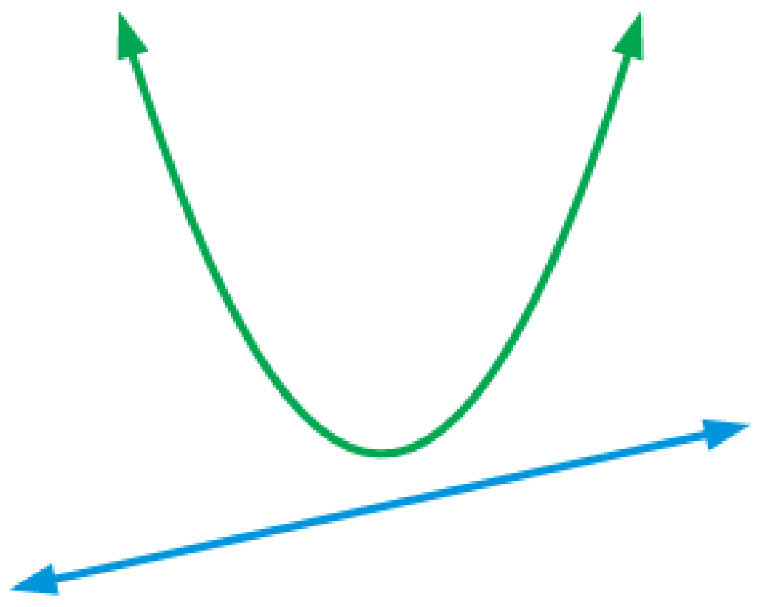
There are three possible scenarios for intersection:



cutting
(2 points of intersection)



touching
(1 point of intersection)



missing
(no points of intersection)

If the line *touches* the curve, we say that the line is a **tangent** to the curve.

The x-coordinates of any intersection points of the graphs can be found by solving the two equations **simultaneously**.

Example 14 Self Tutor

Find the coordinates of the point(s) of intersection of the graphs with equations $y = x^2 - x - 18$ and $y = x - 3$.

$y = x^2 - x - 18$ meets $y = x - 3$ where

$$x^2 - x - 18 = x - 3$$
$$\therefore\ x^2 - 2x - 15 = 0 \qquad \{\text{RHS} = 0\}$$
$$\therefore\ (x-5)(x+3) = 0 \qquad \{\text{factorising}\}$$
$$\therefore\ x = 5 \text{ or } -3$$

Substituting into $y = x - 3$, when $x = 5$, $y = 2$ and when $x = -3$, $y = -6$.

$\therefore$ the graphs meet at $(5,\ 2)$ and $(-3,\ -6)$.

Graphing each side of an inequality helps us to illustrate its solutions. Any points where the graphs intersect will lie at the endpoints of the interval(s) in the solution.

Example 15 **Self Tutor**

Consider the curves $y = x^2 + 5x + 6$ and $y = 2x^2 + 2x - 4$.

a Solve for x: $x^2 + 5x + 6 = 2x^2 + 2x - 4$.

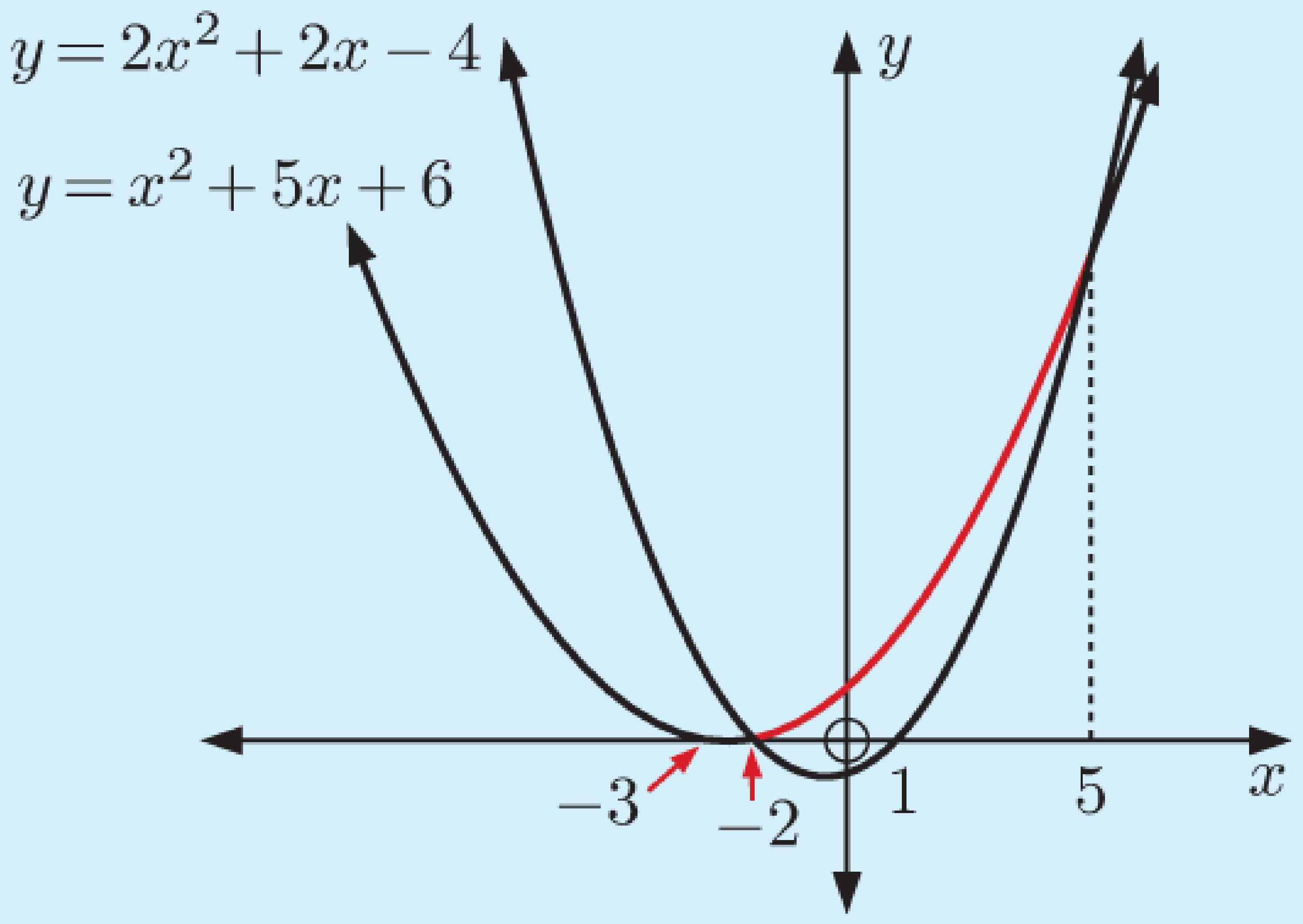
b Graph the two curves on the same set of axes.

c Hence solve for x: $x^2 + 5x + 6 > 2x^2 + 2x - 4$.

a
$$\begin{aligned} x^2 + 5x + 6 &= 2x^2 + 2x - 4 \\ \therefore\ x^2 - 3x - 10 &= 0 \\ \therefore\ (x+2)(x-5) &= 0 \\ \therefore\ x &= -2 \text{ or } 5 \end{aligned}$$

b $y = x^2 + 5x + 6$
$= (x+2)(x+3)$ has zeros -2 and -3.

$y = 2x^2 + 2x - 4$
$= 2(x^2 + x - 2)$
$= 2(x+2)(x-1)$ has zeros -2 and 1.



c If $x^2 + 5x + 6 > 2x^2 + 2x - 4$, the graph of $y = x^2 + 5x + 6$ is above the graph of $y = 2x^2 + 2x - 4$.
This occurs when $-2 < x < 5$.

EXERCISE 14E

1 Find the coordinates of the point(s) of intersection of:

a $y = x^2 - 2x + 8$ and $y = x + 6$

b $y = -x^2 + 3x + 9$ and $y = 2x - 3$

c $y = x^2 - 4x + 3$ and $y = 2x - 6$

d $y = -x^2 + 4x - 7$ and $y = 5x - 4$

2 Use technology to find the coordinates of the points of intersection of the graphs with equations:

a $y = x^2 - 3x + 7$ and $y = x + 5$

b $y = x^2 - 5x + 2$ and $y = x - 7$

c $y = -x^2 - 2x + 4$ and $y = x + 8$

d $y = -x^2 + 4x - 2$ and $y = 5x - 6$.

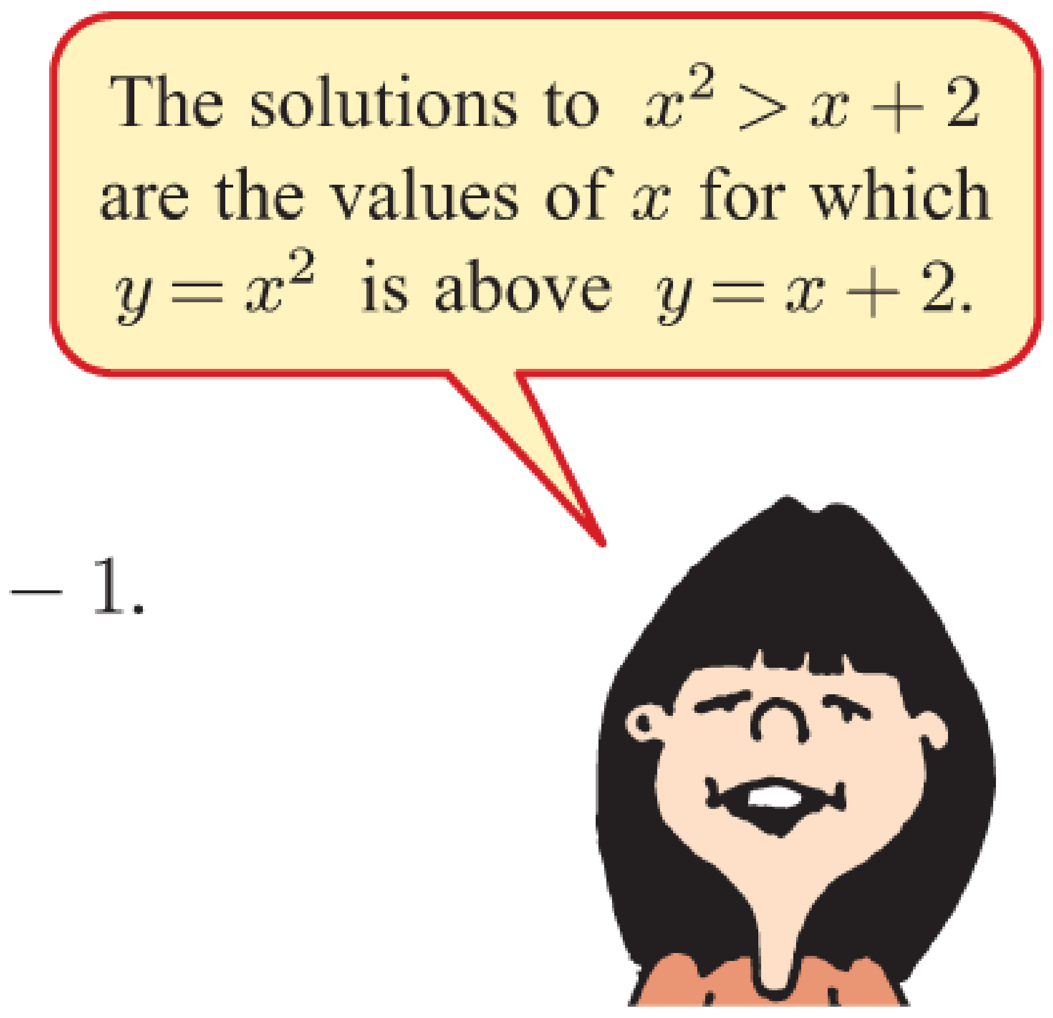
GRAPHING PACKAGE

3 Consider the graphs with equations $y = x^2$ and $y = x + 2$.

a Find the points where the graphs intersect.

b Plot the graphs on the same set of axes.

c Hence solve for x: $x^2 > x + 2$.



4 Consider the graphs with equations $y = x^2 + 2x - 3$ and $y = x - 1$.

a Find the points where the graphs intersect.

b Plot the graphs on the same set of axes.

c Hence solve for x: $x^2 + 2x - 3 > x - 1$.

5 Consider the curves $y = 2x^2 - x + 3$ and $y = 2 + x + x^2$.

- **a** Find the points where the curves intersect.
- **b** Plot the curves on the same set of axes.
- **c** Hence solve for x: $2x^2 - x + 3 > 2 + x + x^2$.

6 Consider the graphs with equations $y = \frac{4}{x}$ and $y = x + 3$.

- **a** Solve $\frac{4}{x} = x + 3$ using algebra.
- **b** Use technology to plot the graphs on the same set of axes.
- **c** Hence solve for x: $\frac{4}{x} > x + 3$.

Example 16

Self Tutor

$y = 2x + k$ is a tangent to $y = 2x^2 - 3x + 4$. Find k.

$y = 2x + k$ meets $y = 2x^2 - 3x + 4$ where

$$2x^2 - 3x + 4 = 2x + k$$
$$\therefore\ 2x^2 - 5x + (4 - k) = 0$$

Since the graphs touch, this quadratic has $\Delta = 0$

$$\therefore\ (-5)^2 - 4(2)(4-k) = 0$$
$$\therefore\ 25 - 8(4-k) = 0$$
$$\therefore\ 25 - 32 + 8k = 0$$
$$\therefore\ 8k = 7$$
$$\therefore\ k = \tfrac{7}{8}$$

A *tangent* is a line which *touches* the curve.

7 For what value of c is the line $y = 3x + c$ a tangent to the parabola with equation $y = x^2 - 5x + 7$?

8 Find the values of m for which the lines $y = mx - 2$ are tangents to the curve with equation $y = x^2 - 4x + 2$.

9 Find the gradients of the lines with y-intercept 1 that are tangents to the curve $y = 3x^2 + 5x + 4$.

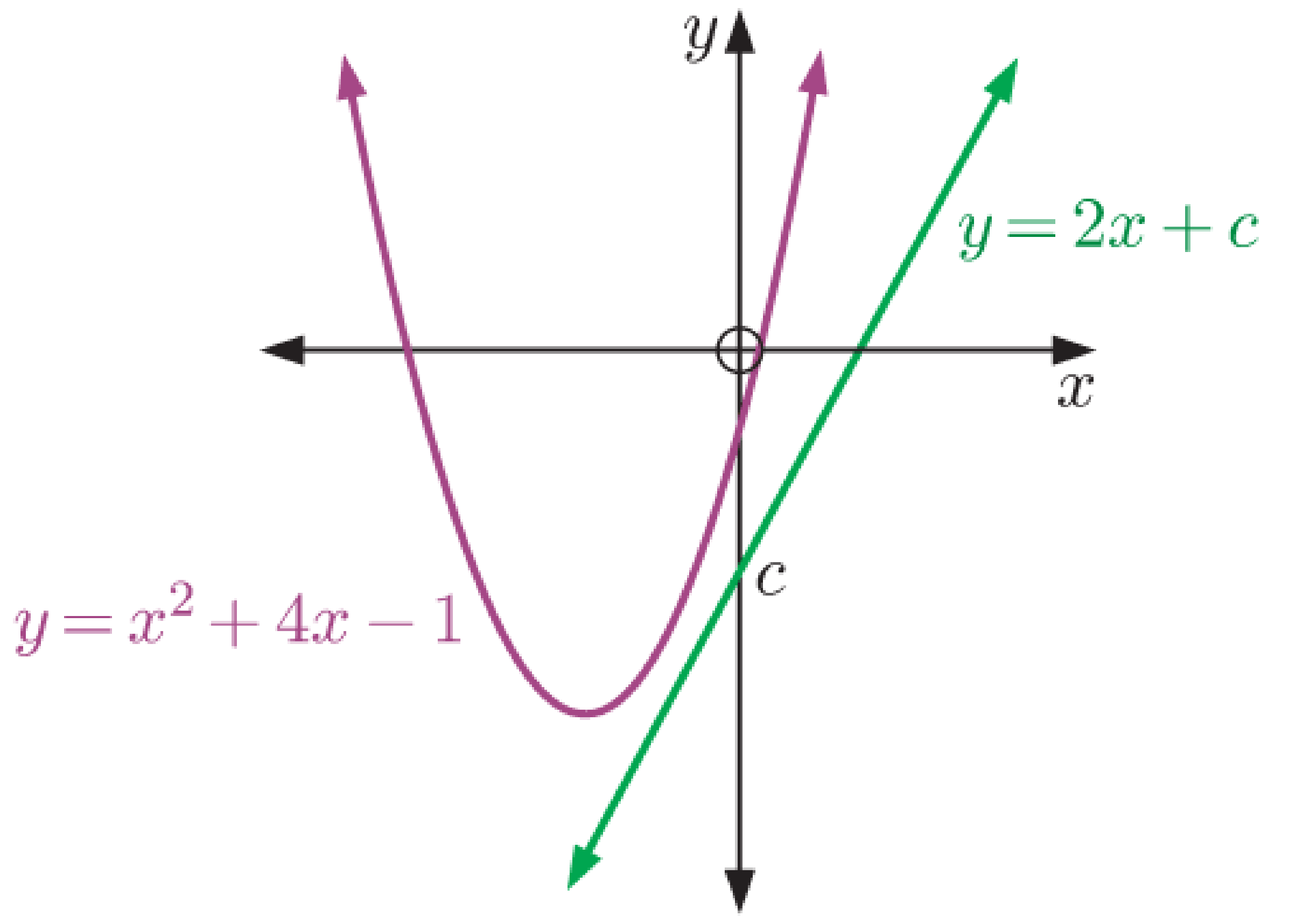
10 **a** For what values of c do the lines $y = x + c$ never meet the parabola with equation $y = 2x^2 - 3x - 7$?

- **b** Choose one of the values of c found in part **a**. Illustrate with a sketch that these graphs never meet.

11 Prove that two quadratic functions can intersect at most twice.

12 Consider the curve $y = x^2 + 4x - 1$ and the line $y = 2x + c$. Find the values of c for which the line:

- **a** meets the curve twice
- **b** is a tangent to the curve
- **c** does not meet the curve.



DYNAMIC GEOMETRY PACKAGE

13 Show that any linear function passing through $(0,\ 3)$ will meet the curve $y = 2x^2 - x - 2$ twice.

14 The graphs of $y = (x-2)^2$ and $y = -x^2 + bx + c$ touch when $x = 3$.
Find the values of b and c.

15 Consider the quadratic function $y = x^2$. The point $P(a,\ a^2)$ lies on the function.

- **a** The line $y = m(x-a) + c$ also passes through P. What, if anything, can you say about the values of m and c?
- **b** Suppose $y = m(x-a) + a^2$ touches $y = x^2$ at P. Find the value of m.

F PROBLEM SOLVING WITH QUADRATICS

Some real-world problems can be solved using a quadratic equation.

Any answer we obtain must be checked to see if it is reasonable. For example:

- if we are finding a length then it must be positive and we reject any negative solutions
- if we are finding “how many people are present” then the answer must be a positive integer.

We employ the following general problem solving method:

Step 1: If the information is given in words, translate it into algebra using a variable such as x. Be sure to define what x represents, and include units if appropriate. Write down the resulting equation.

Step 2: Solve the equation by a suitable method.

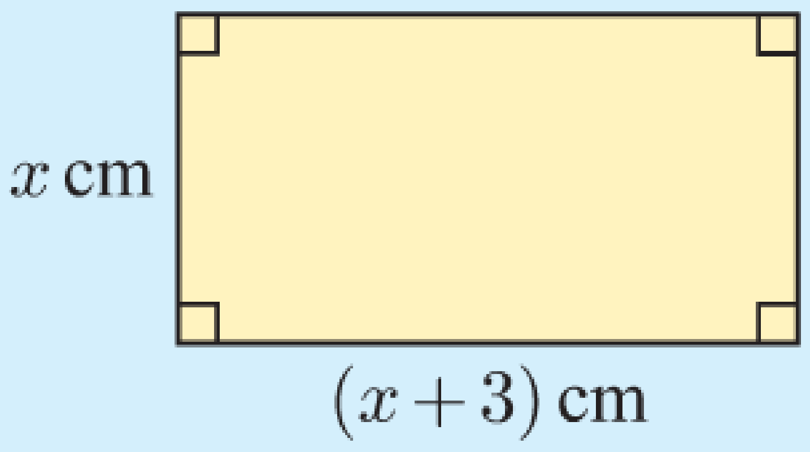
Step 3: Examine the solutions carefully to see if they are acceptable.

Step 4: Give your answer in a sentence, making sure you answer the question.

Example 17 **Self Tutor**

A rectangle has length 3 cm longer than its width, and its area is 42 cm^2. Find the width of the rectangle.

If the width is x cm then the length is $(x+3)$ cm.



$$\therefore\ x(x+3) = 42 \quad \{\text{equating areas}\}$$

$$\therefore\ x^2 + 3x - 42 = 0$$

$$\therefore\ x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-42)}}{2}$$

$$\therefore\ x = \frac{-3 \pm \sqrt{177}}{2}$$

$$\therefore\ x \approx -8.15 \quad \text{or} \quad 5.15$$

We reject the negative solution as lengths are positive.

The width is about 5.15 cm.

EXERCISE 14F

1 Two integers differ by 12, and the sum of their squares is 74. Find the integers.

2 The sum of a number and its reciprocal is $\frac{26}{5}$. Find the number.

3 The sum of a natural number and its square is 210. Find the number.

4 The product of two consecutive even numbers is 360. Find the numbers.

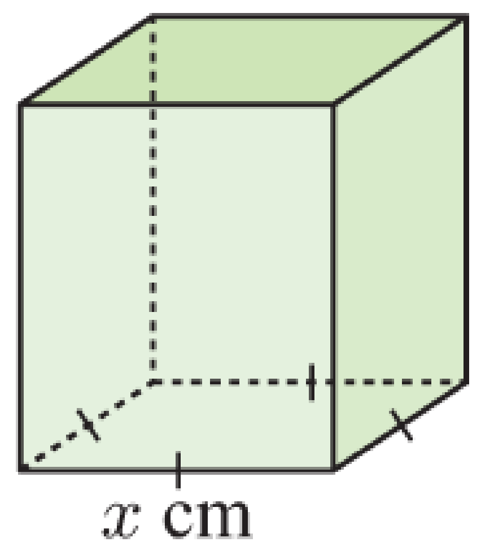
5 The product of two consecutive odd numbers is 255. Find the numbers.

6 The number of diagonals of an n-sided polygon is given by the formula $D = \frac{n}{2}(n-3)$.
A polygon has 90 diagonals. How many sides does it have?

7 The length of a rectangle is 4 cm longer than its width. The rectangle has area 26 cm^2. Find its width.

8 A rectangular box has a square base. Its height is 1 cm longer than its base side length. The total surface area of the box is 240 cm^2.
Suppose the sides of the base are x cm long.

- **a** Show that the total surface area is given by $A = 6x^2 + 4x$ cm^2.
- **b** Find the dimensions of the box.



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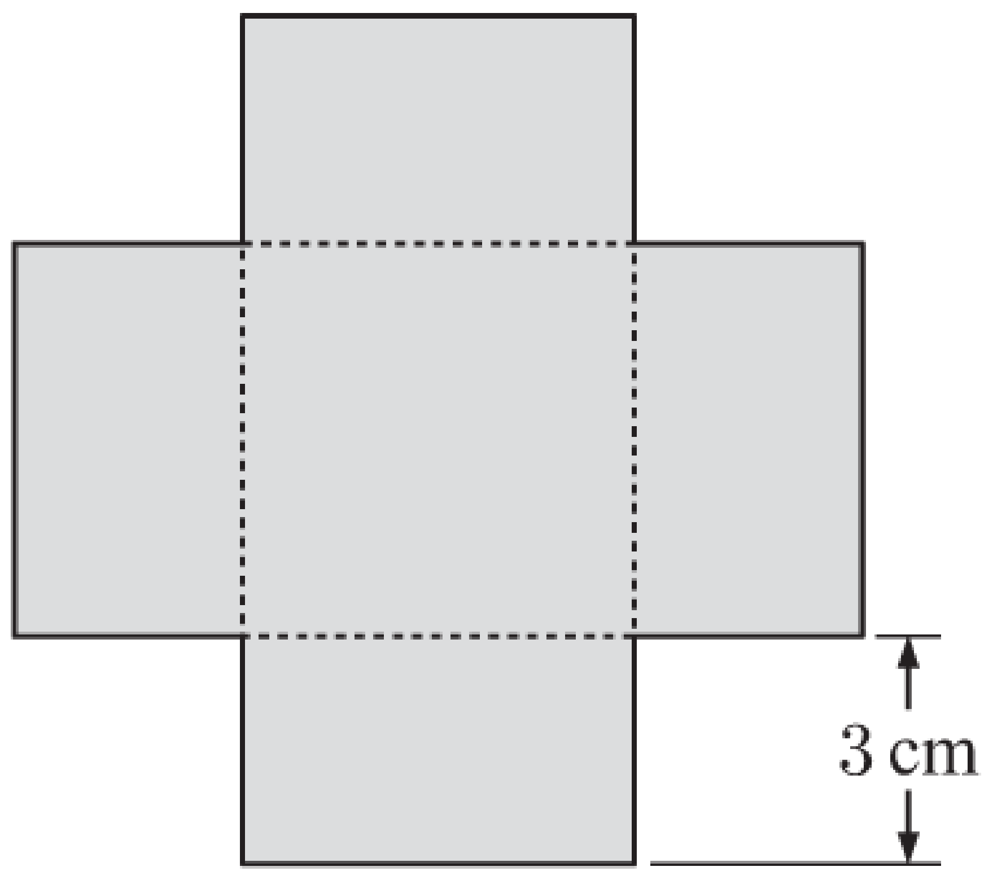


An open box can hold 80 cm^3. It is made from a square piece of tinplate with 3 cm squares cut from each of its 4 corners. Find the dimensions of the original piece of tinplate.

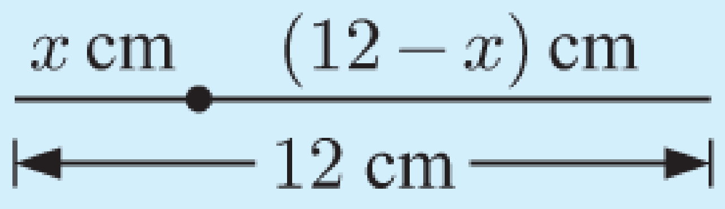
Example 18 Self Tutor

Is it possible to bend a 12 cm length of wire to form the perpendicular sides of a right angled triangle with area 20 cm^2 ?

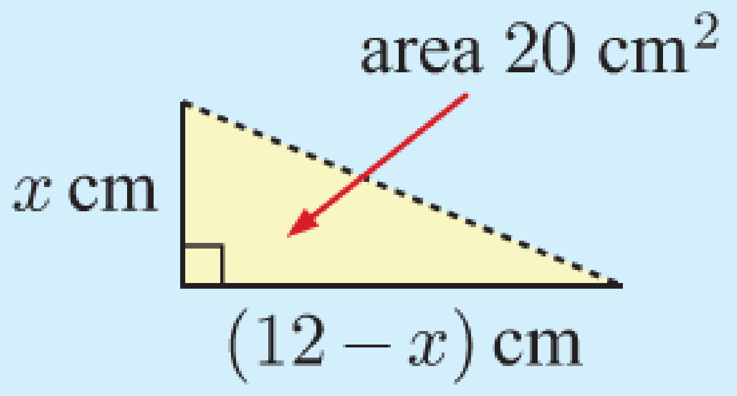
Suppose the wire is bent x cm from one end.

The area $A = \frac{1}{2}x(12-x)$

$$\therefore \quad \tfrac{1}{2}x(12-x) = 20$$
$$\therefore \quad x(12-x) = 40$$
$$\therefore \quad 12x - x^2 - 40 = 0$$
$$\therefore \quad x^2 - 12x + 40 = 0$$



becomes



Now $\Delta = (-12)^2 - 4(1)(40)$
$= -16$ which is < 0

There are no real solutions, indicating that this situation is **impossible**.

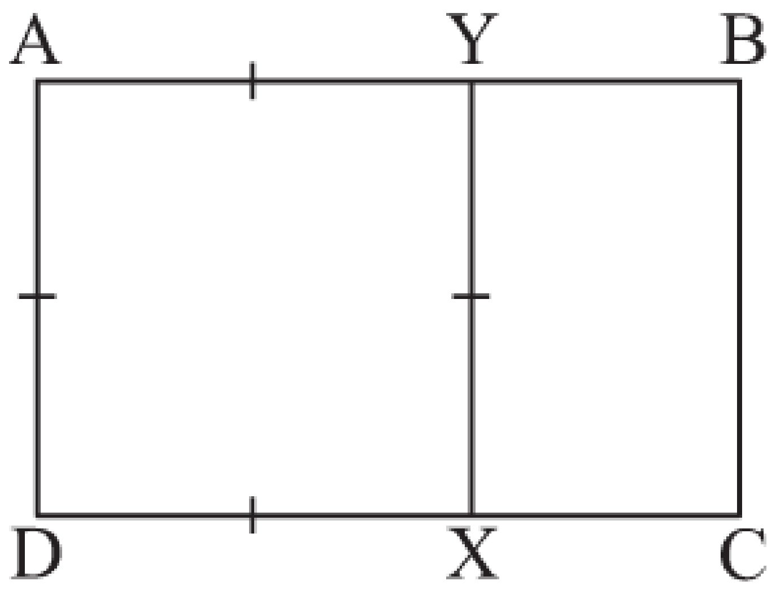
10 Is it possible to bend a 20 cm length of wire into a rectangle with area 30 cm^2 ?

11 The rectangle ABCD is divided into a square and a smaller rectangle by [XY] which is parallel to its shorter sides.
The smaller rectangle BCXY is *similar* to the original rectangle, so rectangle ABCD is a **golden rectangle**.

The ratio $\frac{\text{AB}}{\text{AD}}$ is called the **golden ratio**.

Show that the golden ratio is $\frac{1+\sqrt{5}}{2}$.

Hint: Let AB $= x$ units and AD $= 1$ unit.

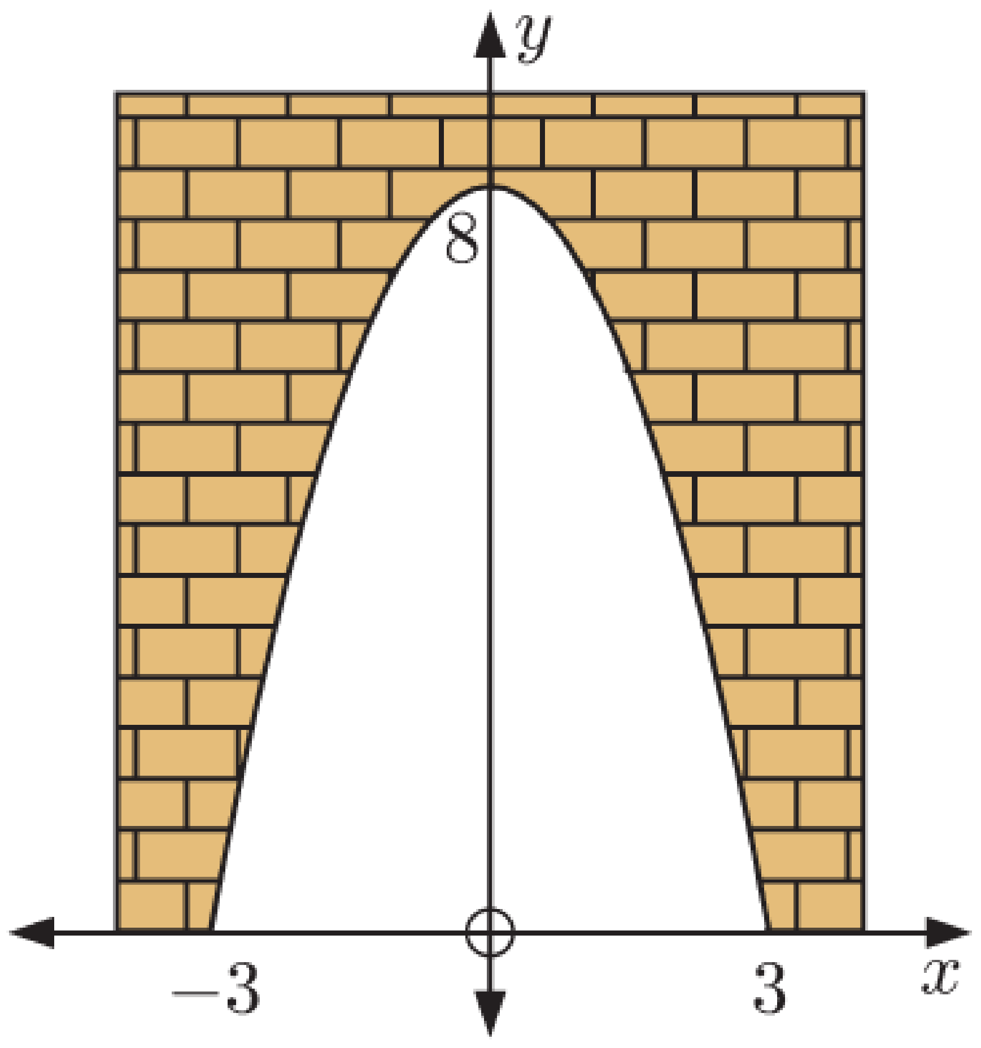


12 Two trains travel along a 160 km track each day. The express travels 10 km h^{-1} faster and takes 30 minutes less time than the normal train. Find the speed of the express.

13 A group of elderly citizens chartered a bus for \$160. Unfortunately, 8 of them fell ill and had to miss the trip. As a consequence, the other citizens had to pay an extra \$1 each. How many elderly citizens went on the trip?

14 A truck carrying a wide load needs to pass through the parabolic tunnel shown. The units are metres.
The truck is 5 m high and 4 m wide.

- **a** Find the quadratic function which describes the shape of the tunnel.
- **b** Determine whether the truck will fit.



15 A stone is thrown into the air from the top of a cliff 60 m above sea level. The stone reaches a maximum height of 80 m above sea level after 2 seconds.

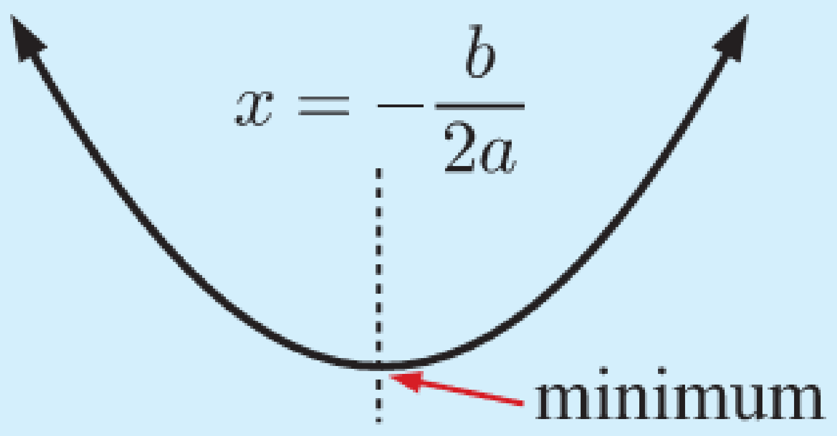
- **a** Find the quadratic function which describes the stone's height above sea level.
- **b** Find the stone's height above sea level after 3 seconds.
- **c** How long will it take for the stone to hit the water?

G OPTIMISATION WITH QUADRATICS

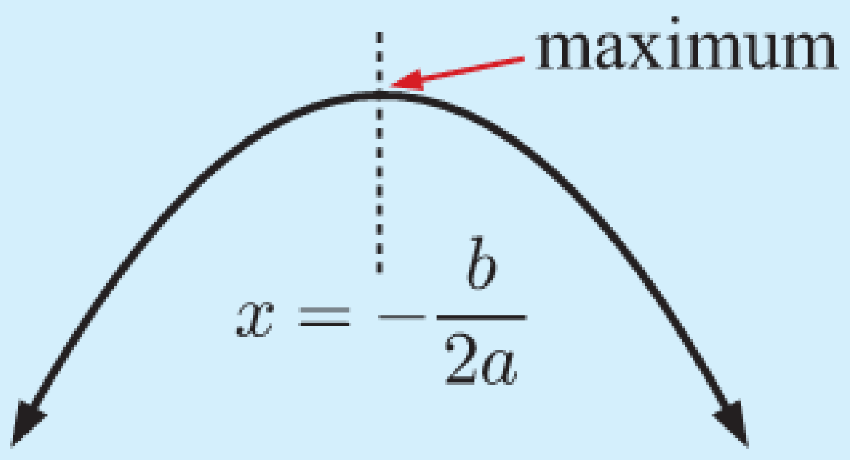
The process of finding a maximum or minimum value is called **optimisation**.

For the quadratic $y = ax^2 + bx + c$, we have seen that the vertex has x-coordinate $-\frac{b}{2a}$.

- If $a > 0$, the **minimum** value of y occurs at $x = -\frac{b}{2a}$.



- If $a < 0$, the **maximum** value of y occurs at $x = -\frac{b}{2a}$.

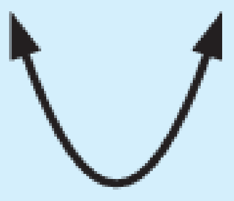


Example 19 **Self Tutor**

Find the maximum or minimum value of the quadratic, and the corresponding value of x:

a $y = x^2 + x - 3$ **b** $y = 3 + 3x - 2x^2$

a $y = x^2 + x - 3$ has $a = 1$, $b = 1$, and $c = -3$.

Since $a > 0$, the shape is 

The minimum value occurs when $x = \frac{-b}{2a} = -\frac{1}{2}$

and $y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 3 = -3\frac{1}{4}$

So, the minimum value of y is $-3\frac{1}{4}$, occurring when $x = -\frac{1}{2}$.

b $y = -2x^2 + 3x + 3$ has $a = -2$, $b = 3$, and $c = 3$.

Since $a < 0$, the shape is 

The maximum value occurs when $x = \frac{-b}{2a} = \frac{-3}{-4} = \frac{3}{4}$

and $y = -2(\frac{3}{4})^2 + 3(\frac{3}{4}) + 3 = 4\frac{1}{8}$

So, the maximum value of y is $4\frac{1}{8}$, occurring when $x = \frac{3}{4}$.

EXERCISE 14G

1 Find the maximum or minimum value for each quadratic, and the corresponding value of x:

a $y = x^2 - 2x$ **b** $y = 7 - 2x - x^2$ **c** $y = 8 + 2x - 3x^2$

d $y = 2x^2 + x - 1$ **e** $y = 4x^2 - x + 5$ **f** $y = 7x - 2x^2$

2 The profit in manufacturing x refrigerators per day, is given by $P = -3x^2 + 240x - 800$ euros.

a How many refrigerators should be made each day to maximise the total profit?

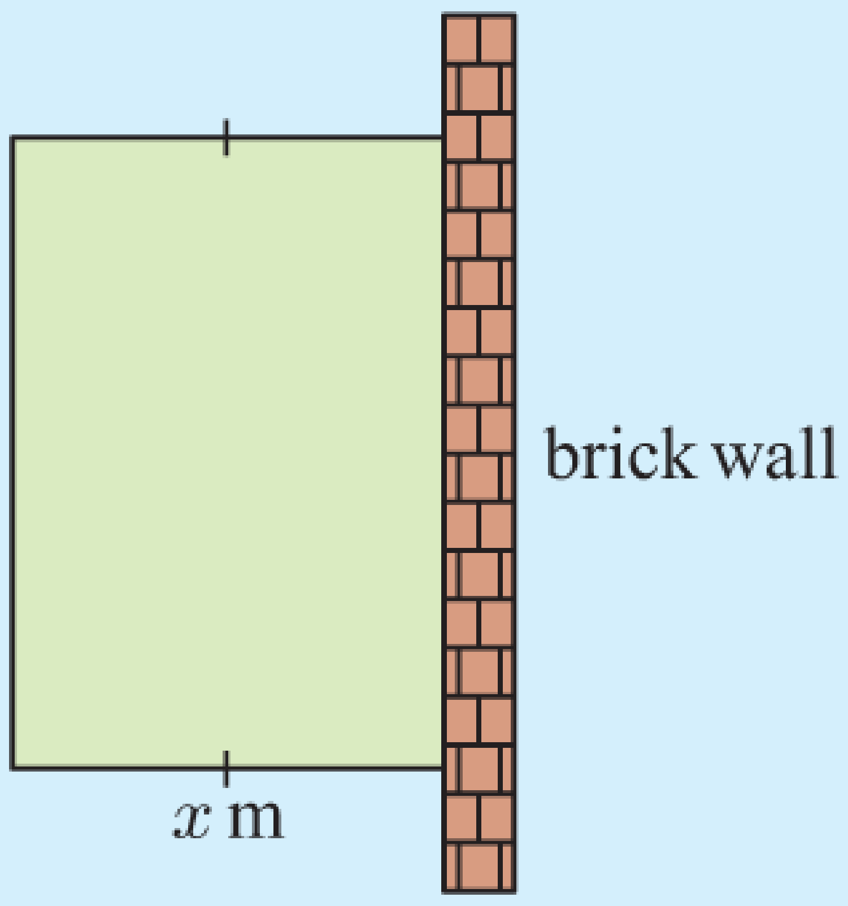
b What is the maximum profit?

Example 20 **Self Tutor**

A gardener has 40 m of fencing to enclose a rectangular garden plot, where one side is an existing brick wall. Suppose the two new equal sides are x m long.

a Show that the area enclosed is given by $A = x(40 - 2x)$ m^2.

b Find the dimensions of the garden of maximum area.



a Side [XY] has length $(40 - 2x)$ m.

Now, area = length × width

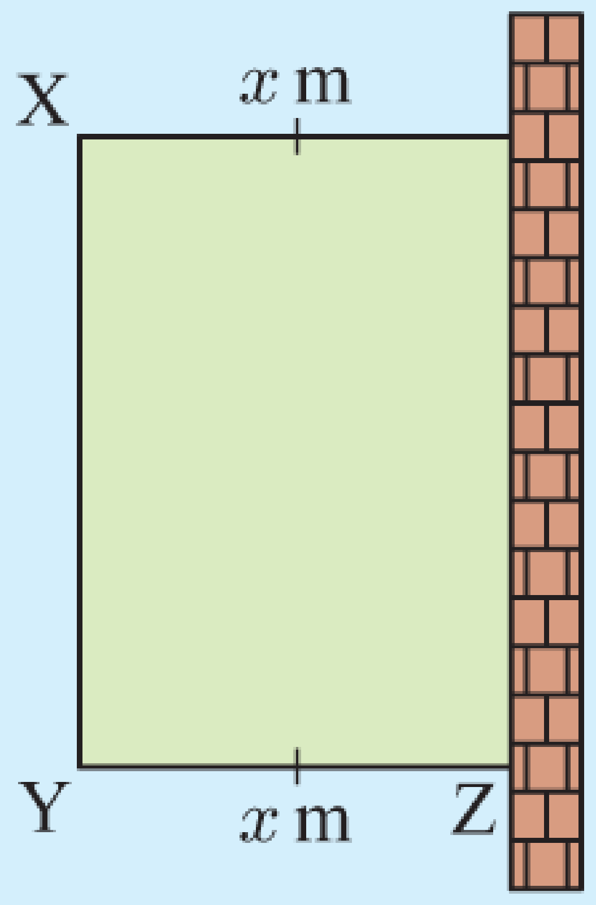
$\therefore$ $A = x(40 - 2x)$ m^2

b $A = 0$ when $x = 0$ or 20.

The vertex of the function lies midway between these values, so $x = 10$.

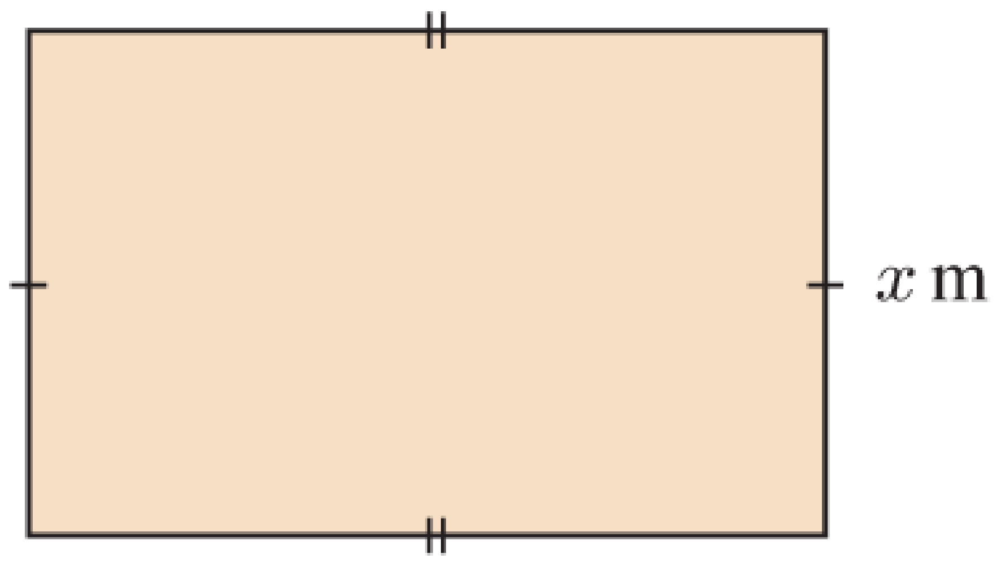
Since $a < 0$, the shape is 

$\therefore$ the area is maximised when YZ = 10 m and XY = 20 m.



3 A rectangular plot is enclosed by 200 m of fencing and has an area of A square metres. Show that:

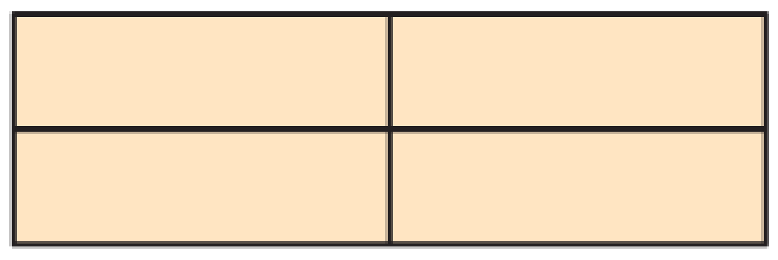
- **a** $A = 100x - x^2$ where x m is the length of one of its sides
- **b** the area is maximised if the rectangle is a square.



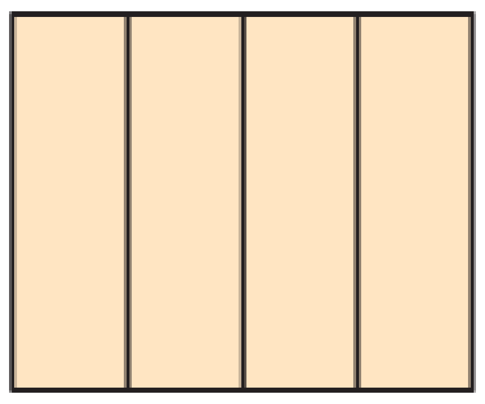
4 Three sides of a rectangular paddock are to be fenced, the fourth side being an existing straight water drain. If 1000 m of fencing is available, what dimensions should be used for the paddock to maximise its area?

5 500 m of fencing is available to make 4 rectangular pens of identical shape. Find the dimensions that maximise the area of each pen, if the plan is:

a



b



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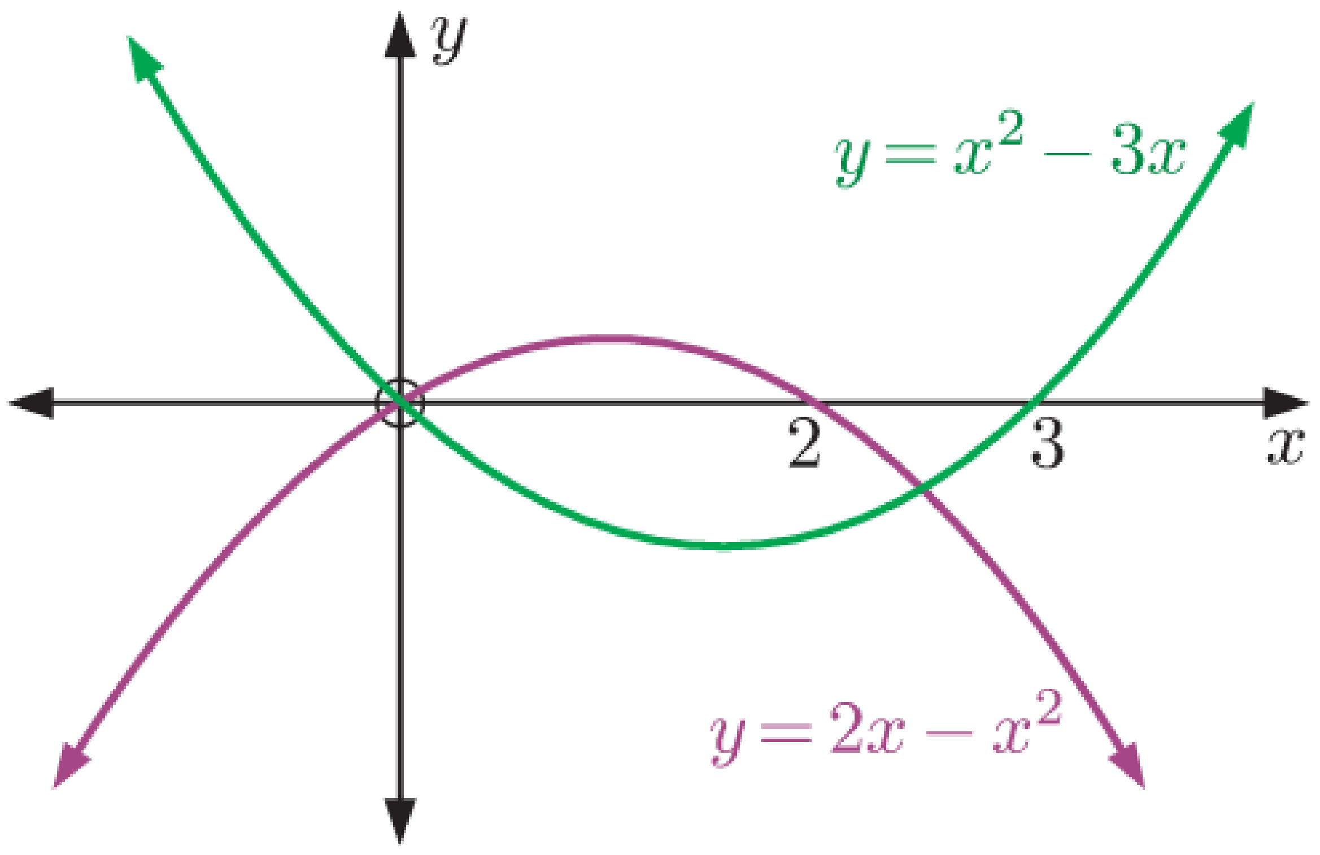
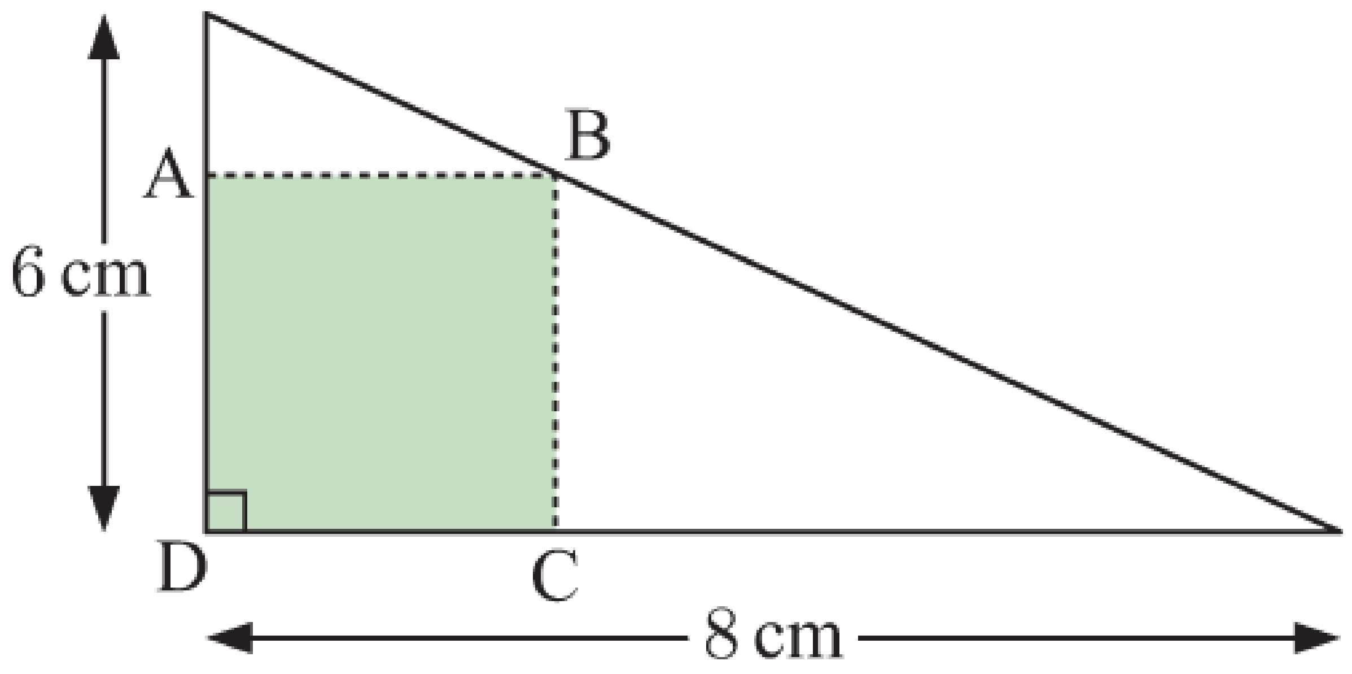


The graphs of $y = x^2 - 3x$ and $y = 2x - x^2$ are illustrated.

- **a** Show that the graphs meet where $x = 0$ and $x = 2\frac{1}{2}$.
- **b** Find the maximum vertical separation between the curves for $0 \leqslant x \leqslant 2\frac{1}{2}$.

7 Infinitely many rectangles may be inscribed within the right angled triangle shown alongside. One of them is illustrated.

- **a** Let AB $= x$ cm and BC $= y$ cm.
  Use similar triangles to find y in terms of x.
- **b** Find the dimensions of rectangle ABCD of maximum area.



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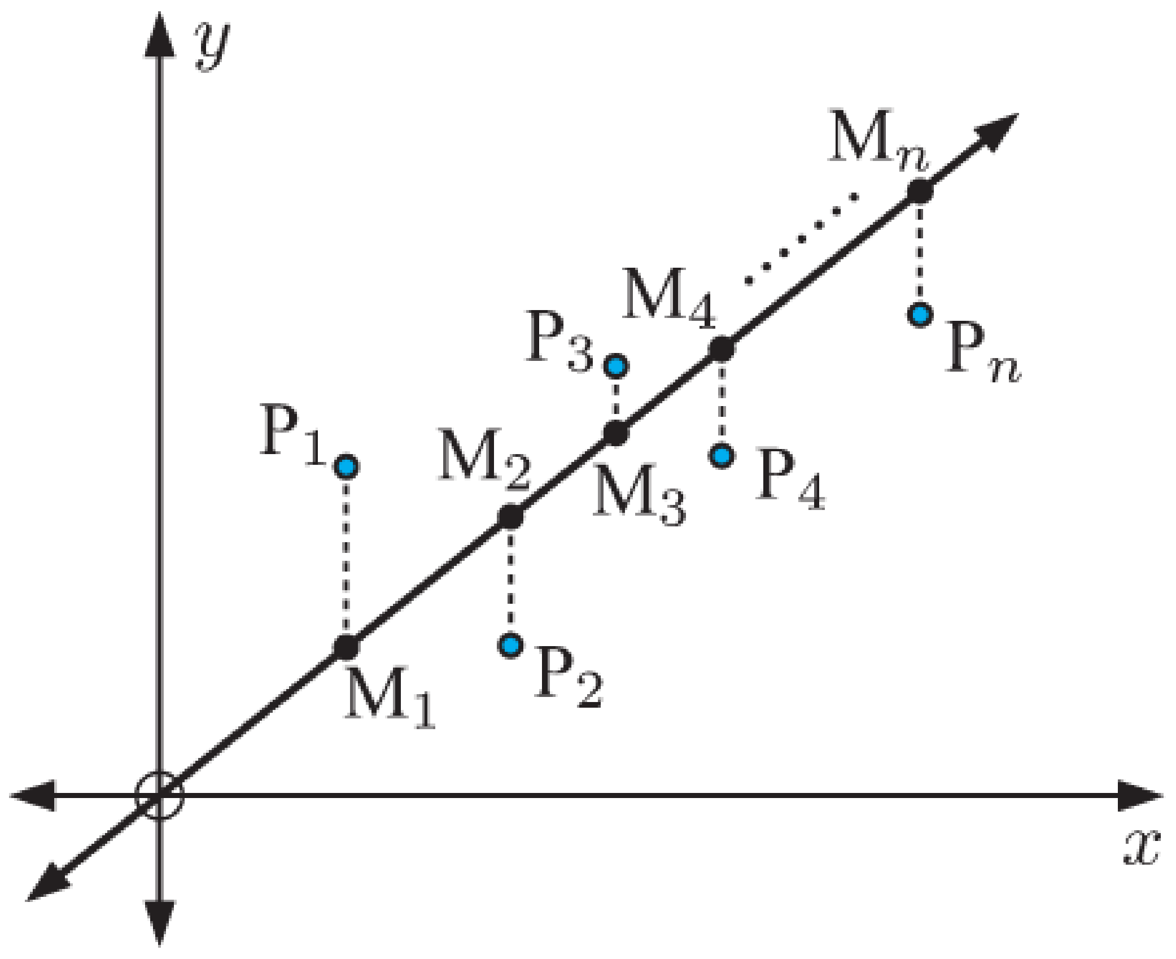


$P_1(a_1, b_1)$, $P_2(a_2, b_2)$, $P_3(a_3, b_3)$, ..., $P_n(a_n, b_n)$ are a set of data points which are approximately linear through the origin O(0, 0).

To find the equation of the "line of best fit" through the origin, we minimise

$(P_1M_1)^2 + (P_2M_2)^2 + (P_3M_3)^2 + ... + (P_nM_n)^2$ where M_i lies on the line and has x-coordinate a_i.

Find the gradient m of the "line of best fit" in terms of a_i and b_i, $i = 1, 2, 3, 4, ..., n$.

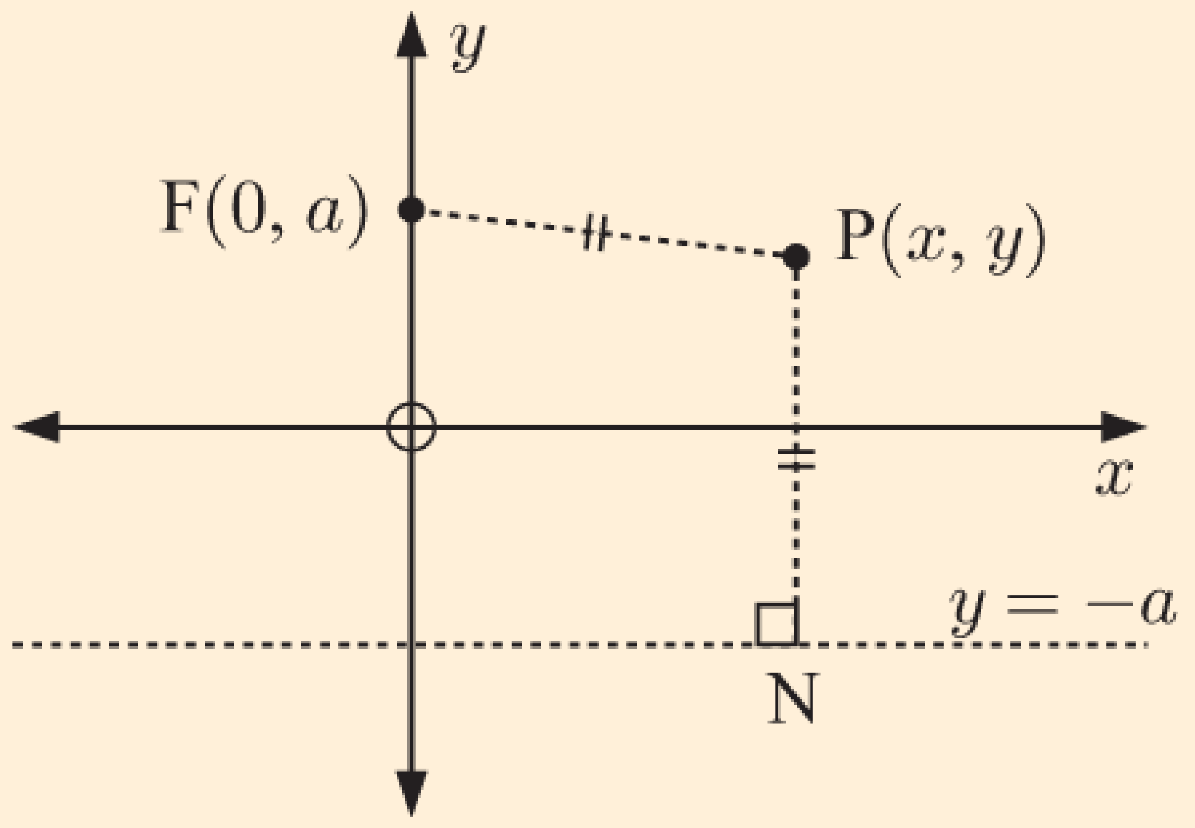
9 Assuming a and b are real constants, expand $y = (x - a - b)(x - a + b)(x + a - b)(x + a + b)$ and hence determine the least value of y.

10 By considering the function $y = (a_1x - b_1)^2 + (a_2x - b_2)^2$, use quadratic theory to prove the **Cauchy-Schwarz inequality** $|\,a_1b_1 + a_2b_2\,| \leqslant \sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}$.

INVESTIGATION 4 THE GEOMETRIC DEFINITION OF A PARABOLA

A **parabola** is defined as the locus of all points which are equidistant from a fixed point called the **focus** and a fixed line called the **directrix**.

Suppose the focus is F$(0, a)$ and the directrix is the horizontal line $y = -a$. The parabola is the set of all points P such that FP = NP where N is the closest point on the directrix to P.

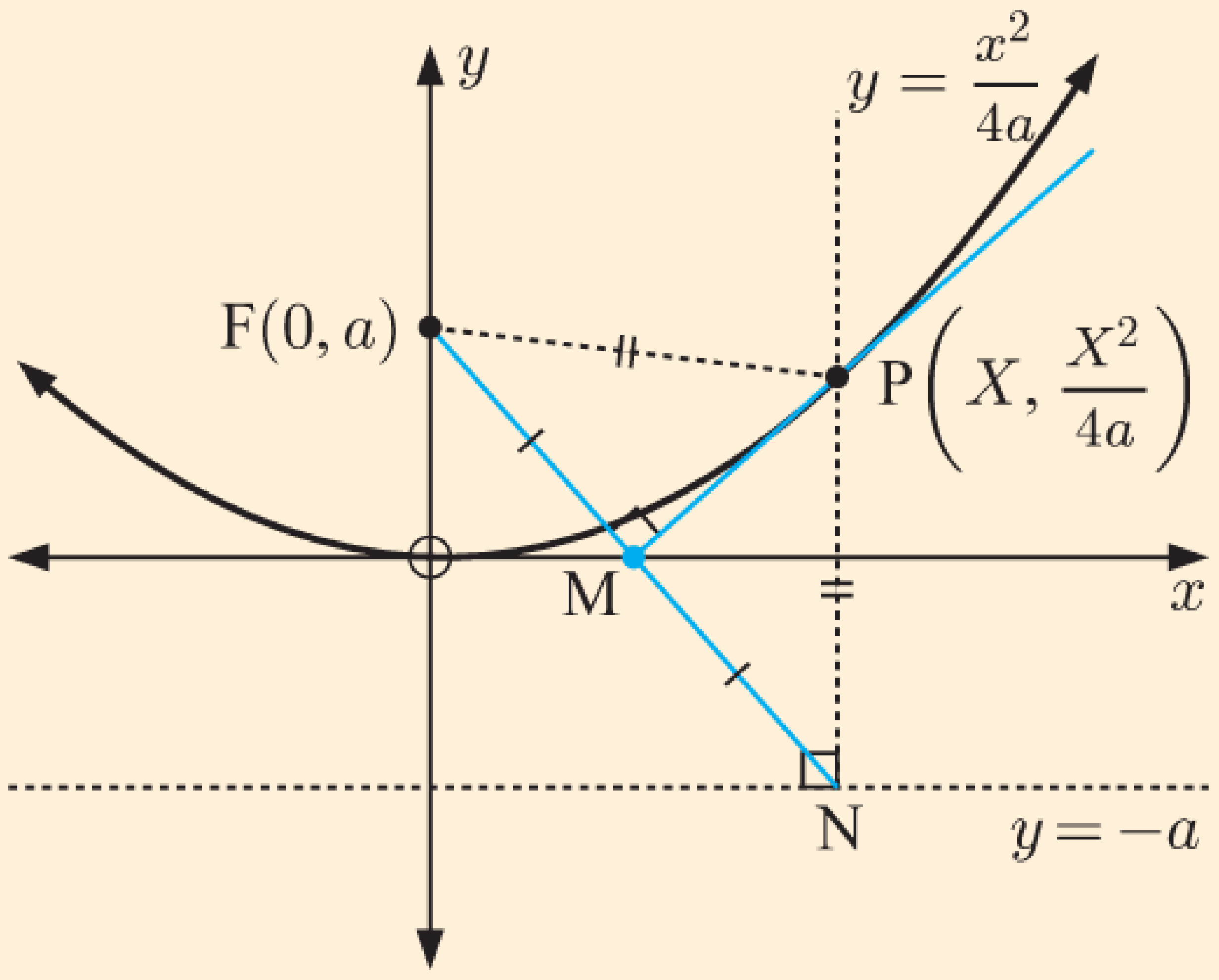


What to do:

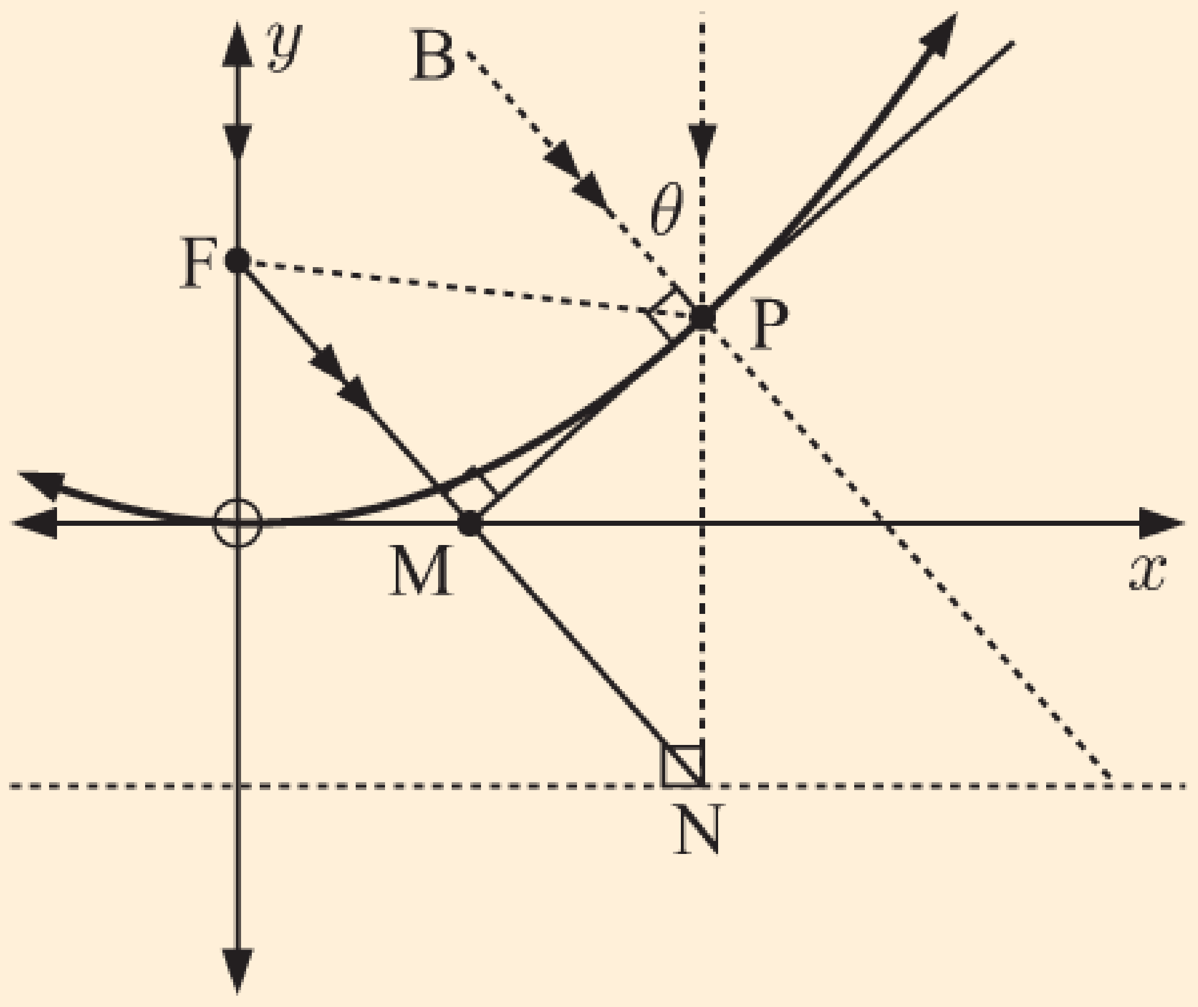
1. Suggest why it is convenient to let the focus be at $(0, a)$ and the directrix be the line $y = -a$.
2. Use the circular-linear graph paper provided to graph the parabola which has focus F$(0, 2)$ and directrix $y = -2$.
3. Using the definition above:
   - **a** Write down the coordinates of N.
   - **b** Write expressions for FP and NP.
   - **c** Show that the parabola has the equation $y = \dfrac{x^2}{4a}$.

PRINTABLE GRAPH PAPER

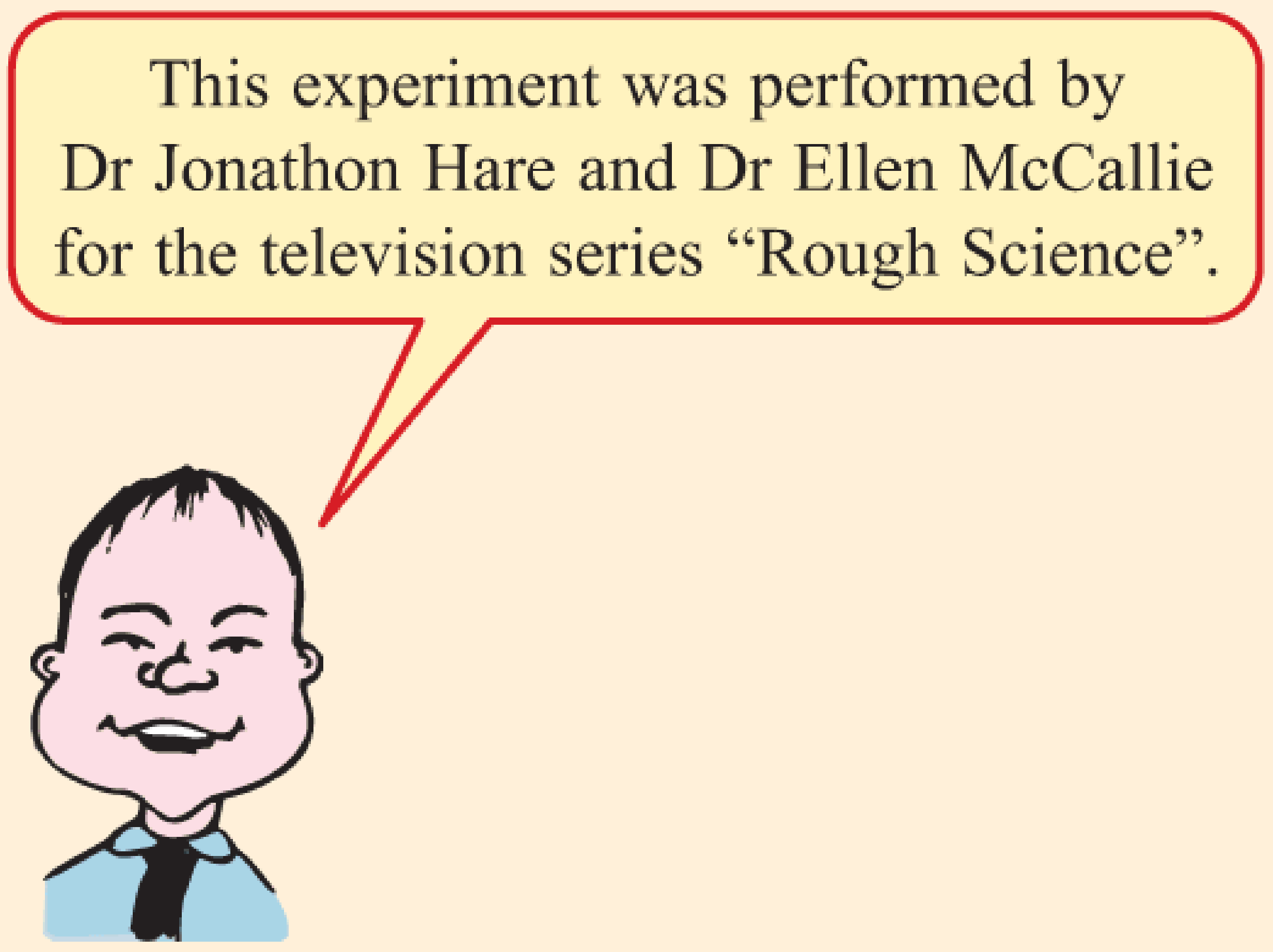
4. Consider a point P$\left(X, \dfrac{X^2}{4a}\right)$ on the parabola $y = \dfrac{x^2}{4a}$ with focus F$(0, a)$ and directrix $y = -a$.
   - **a** Let N be the closest point on the directrix to P, and M be the midpoint of [FN].
     - **i** Show that [MP] has equation $y = \dfrac{X}{2a}\left(x - \dfrac{X}{2}\right)$.
     - **ii** Hence prove that (MP) is a tangent to the parabola.



   - **b** Let B lie on the normal to the parabola. Suppose a ray of light shines vertically down onto the parabola with angle of incidence θ as shown.
     - **i** Explain why $\widehat{\text{MNP}}$ must equal θ.
     - **ii** Hence explain why $\widehat{\text{MFP}}$ must equal θ.
     - **iii** Hence explain why $\widehat{\text{FPB}}$ must equal θ.
     - **iv** Hence explain why any vertical ray of light shining down onto a parabolic mirror will be reflected to the focus of the parabola F.



     - **v** Explain what shape Misha needs in the **Opening Problem** and where he needs to place his cup.



H QUADRATIC INEQUALITIES

A **quadratic inequality** can be written in either the form $ax^2 + bx + c \geqslant 0$ or $ax^2 + bx + c > 0$ where $a \neq 0$.

We have seen that the solutions to a quadratic equation are the x-intercepts of the corresponding quadratic function.

In a similar way, the solutions to a quadratic *inequality* are the values of x for which the corresponding function has a particular *sign*.

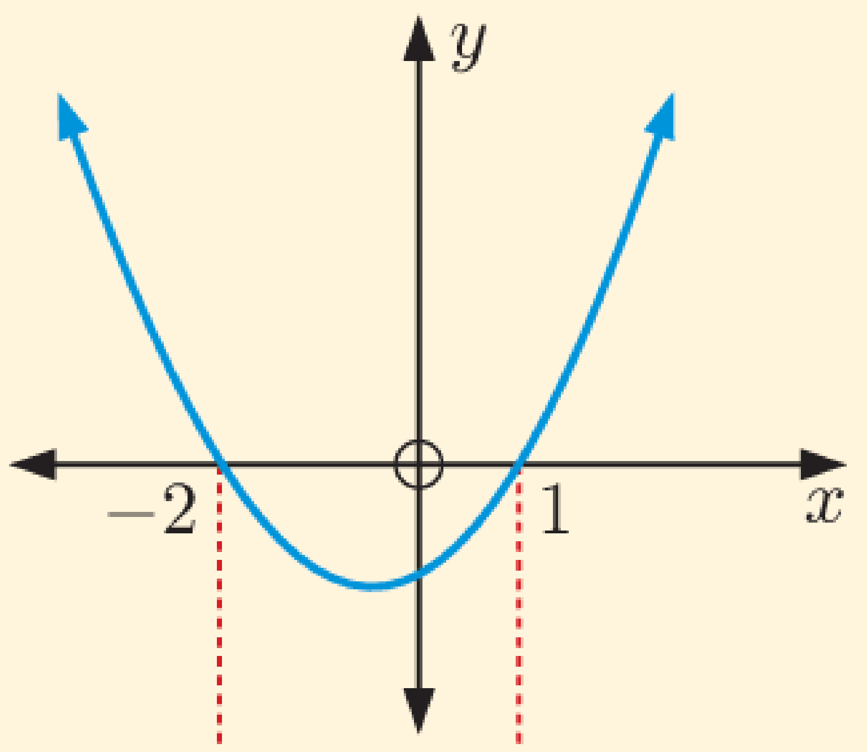
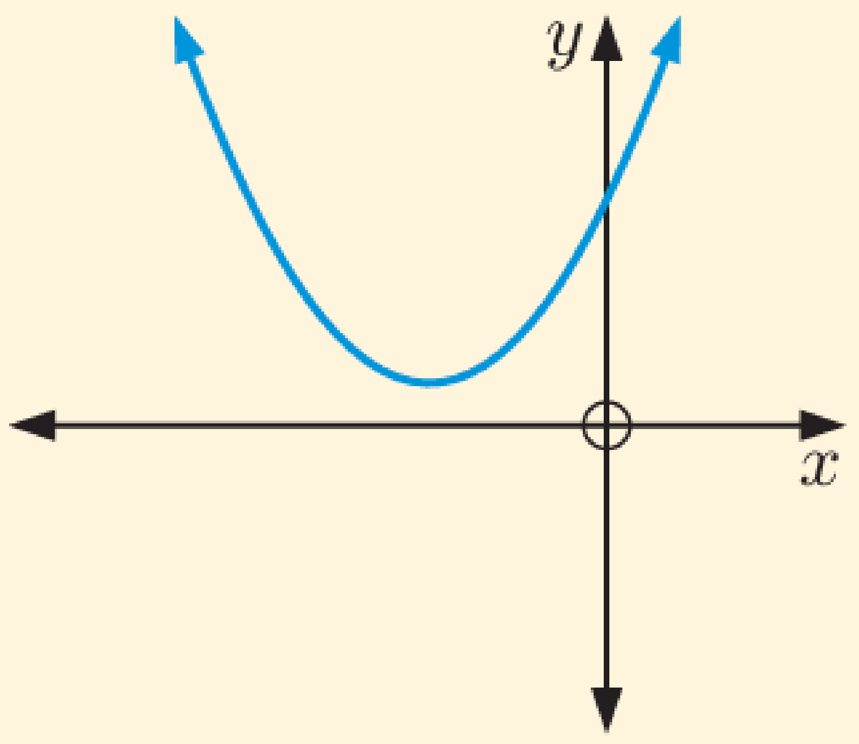
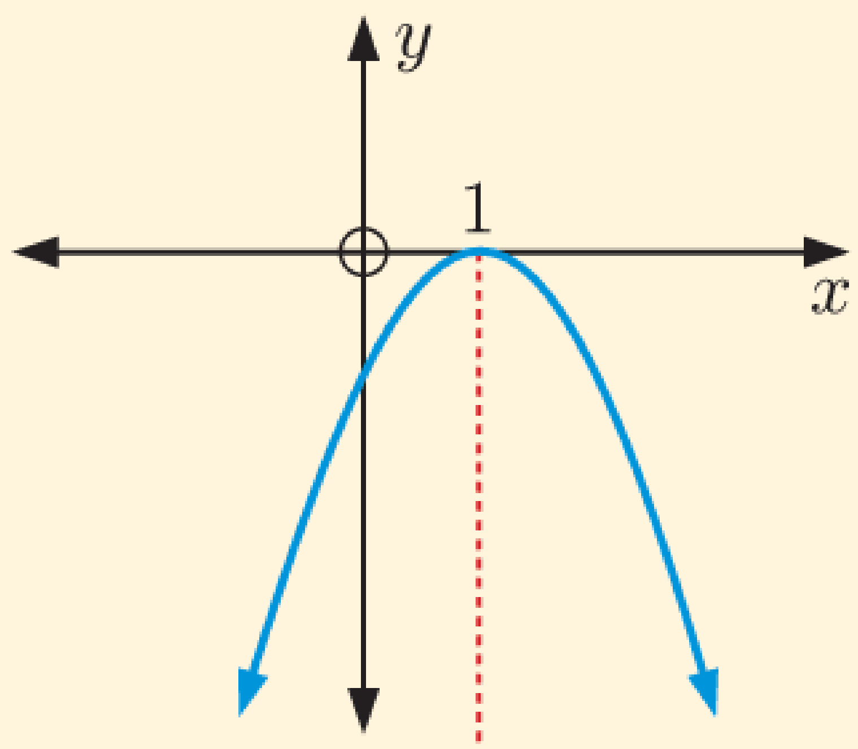
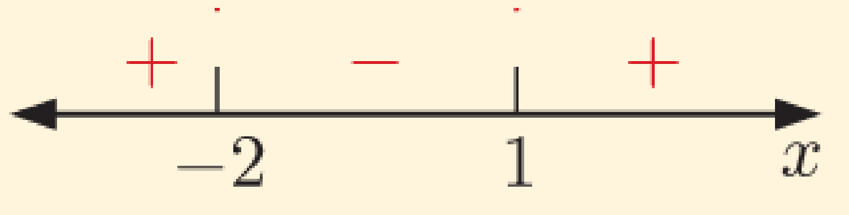
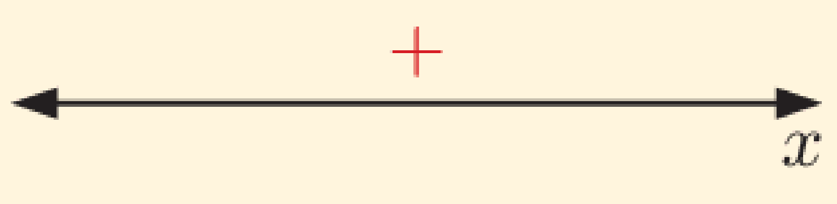
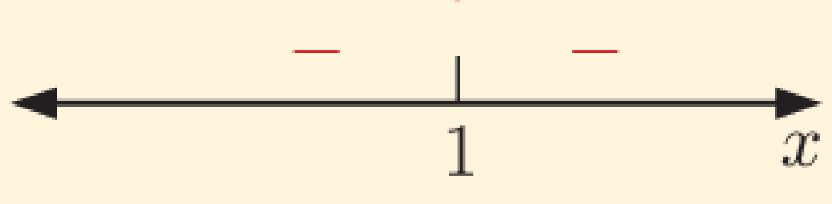
SIGN DIAGRAMS

A **sign diagram** is a number line which indicates the values of x for which a function is negative, zero, positive, or undefined.

A sign diagram consists of:

- a **horizontal line** which represents the x-axis
- **positive** $(+)$ and **negative** $(-)$ signs indicating where the graph is **above** and **below** the x-axis respectively
- the **zeros** of the function, which are the x-intercepts of its graph.

Consider the three functions below:

Function	$y = (x+2)(x-1)$	$y = (x+3)^2 + 2$	$y = -2(x-1)^2$
Graph			
Sign diagram			

You should notice that:

- A sign change occurs about a zero of the function for single linear factors such as $(x+2)$ and $(x-1)$. This indicates **cutting** of the x-axis.
- No sign change occurs about a zero of the function for squared linear factors such as $(x-1)^2$. This indicates **touching** of the x-axis.

In general:

- when a linear factor has an **odd power** there is a change of sign about that zero
- when a linear factor has an **even power** there is no sign change about that zero.

Example 21

Self Tutor

Draw a sign diagram for:

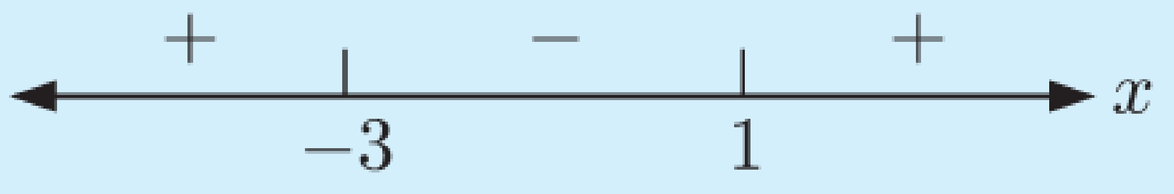
a x^2+2x-3

b $-4(x-3)^2$

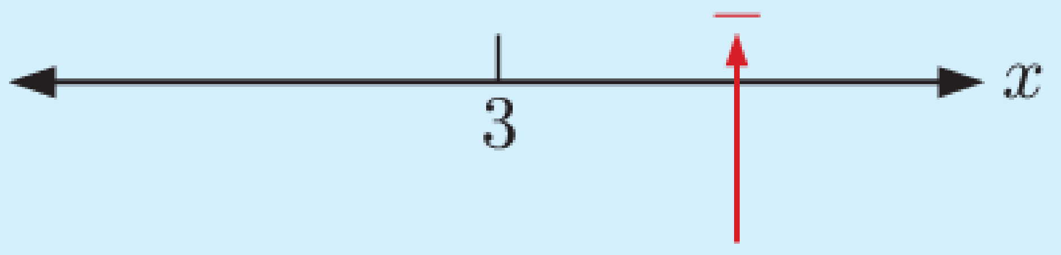
a $x^2+2x-3=(x+3)(x-1)$ which has zeros -3 and 1.

When $x=2$ we have $(5)(1)>0$, so we put a $+$ sign here.

As the factors are single, the signs alternate.

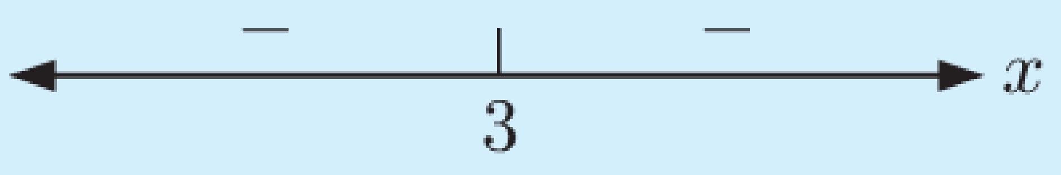


b $-4(x-3)^2$ has zero 3.



When $x=4$ we have $-4(1)^2<0$, so we put a $-$ sign here.

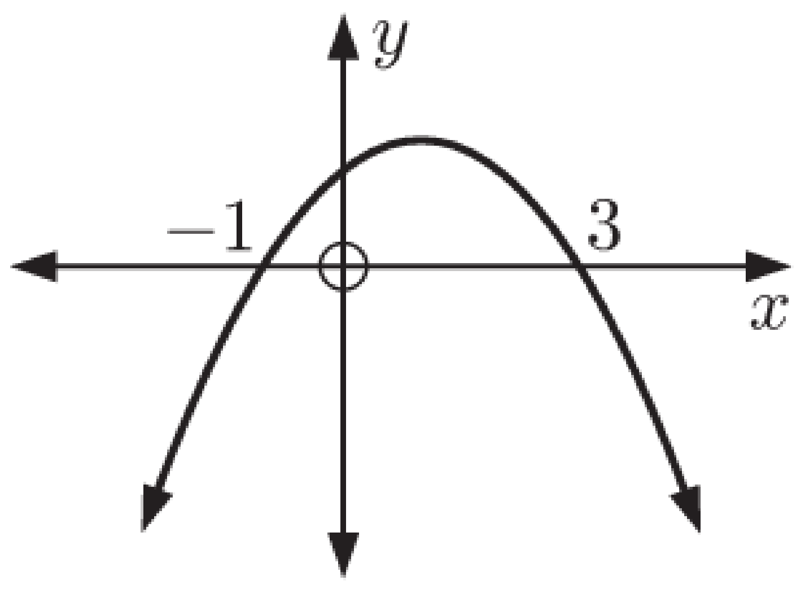
As the factor is squared, the signs do not change.



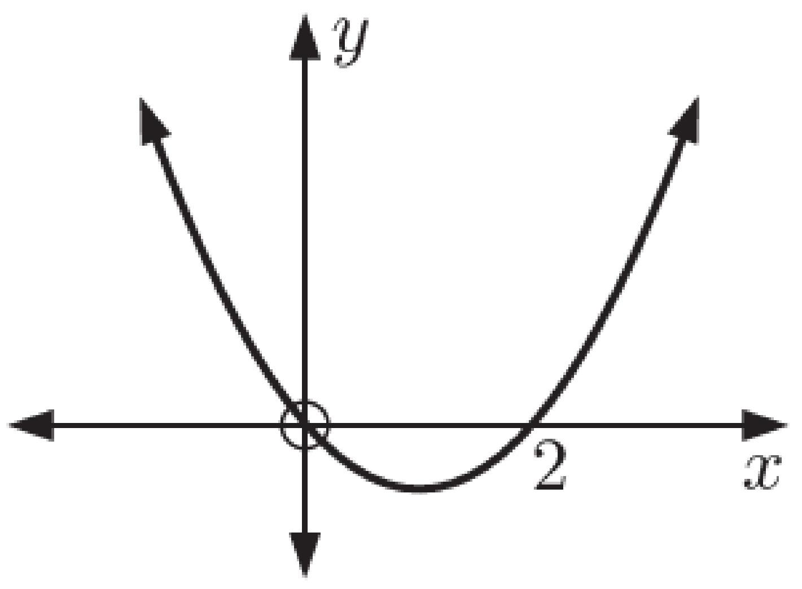
EXERCISE 14H.1

1 Draw a sign diagram for each graph:

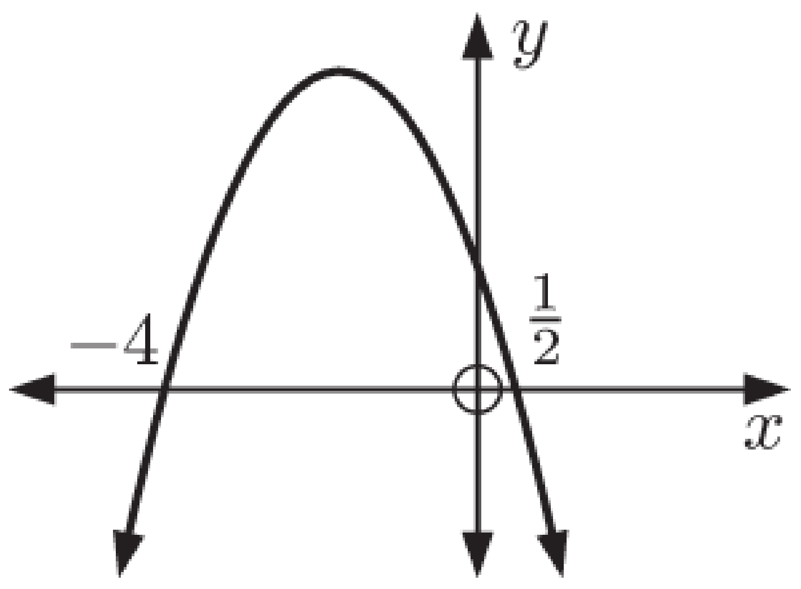
a



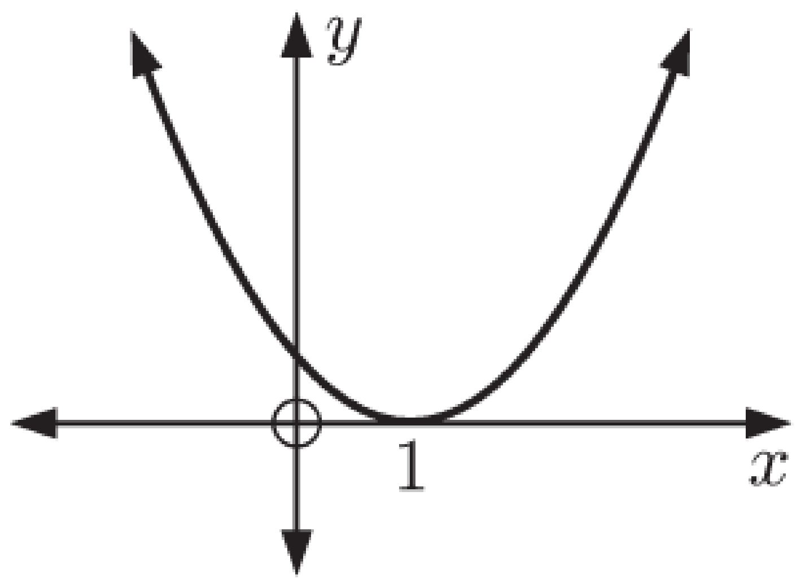
b



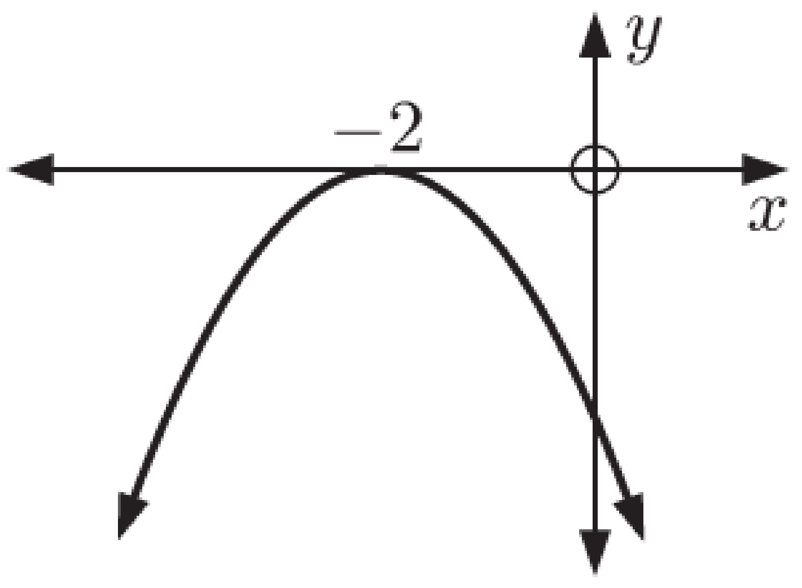
c



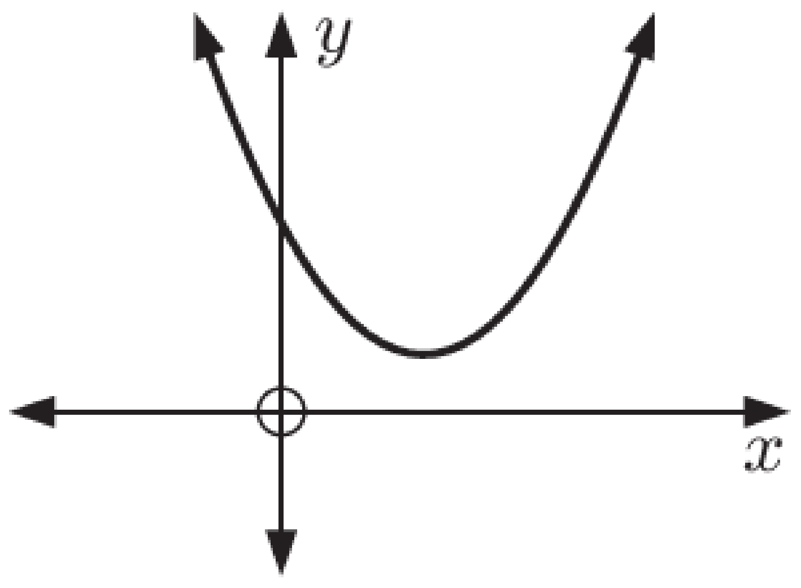
d



e



f



2 Draw a sign diagram for:

a $(x+4)(x-2)$
b $(x+1)(x-5)$
c $x(x-3)$
d $x(x+2)$
e $(2x+1)(x-4)$
f $-(x+1)(x-3)$
g $-(3x-2)(x+1)$
h $(2x-1)(3-x)$
i $(5-x)(1-2x)$

3 Draw a sign diagram for:

a $(x+2)^2$ **b** $(x-3)^2$ **c** $-(x-4)^2$

d $2(x+1)^2$ **e** $-3(x+4)^2$ **f** $-\frac{1}{2}(2x+5)^2$

4 Draw a sign diagram for:

a x^2-9 **b** $4-x^2$ **c** $5x-x^2$

d x^2-3x+2 **e** $2-8x^2$ **f** $6x^2+x-2$

g $6-16x-6x^2$ **h** $-2x^2+9x+5$ **i** $-15x^2-x+2$

5 Draw a sign diagram for:

a $x^2+10x+25$ **b** x^2-2x+1 **c** $-x^2+4x-4$

d $4x^2-4x+1$ **e** $-x^2-6x-9$ **f** $-4x^2+12x-9$

QUADRATIC INEQUALITIES

To solve quadratic inequalities we use the following procedure:

- Make the RHS zero by shifting all terms to the LHS.
- Fully factorise the LHS.
- Draw a sign diagram for the LHS.
- Determine the values required from the sign diagram.

Example 22 **Self Tutor**

Solve for x:

a $3x^2+5x \geqslant 2$ **b** $x^2+9<6x$

a

$$3x^2+5x \geqslant 2$$
$$\therefore\ 3x^2+5x-2 \geqslant 0$$
$$\therefore\ (3x-1)(x+2) \geqslant 0$$

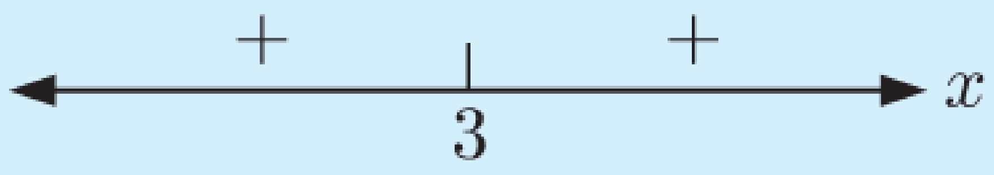
Sign diagram of LHS is

$x \leqslant -2$ or $x \geqslant \frac{1}{3}$

b

$$x^2+9<6x$$
$$\therefore\ x^2-6x+9<0$$
$$\therefore\ (x-3)^2<0$$

Sign diagram of LHS is



So, the inequality is not true for any real x.

EXERCISE 14H.2

1 Solve for x:

a $(x-2)(x+5) \leqslant 0$ **b** $(2-x)(x+3) \geqslant 0$ **c** $(x-1)^2<0$

d $(x+5)^2 \geqslant 0$ **e** $(2x+1)(3-x)>0$ **f** $(x-4)(2x+3)<0$

2 Solve for x:

a $x^2-x \geqslant 0$ **b** $3x^2+2x<0$ **c** $x^2+4x+4>0$

d $x^2+2x-15 \leqslant 0$ **e** $x^2-4x-12>0$ **f** $3x^2+9x-12<0$

3 Solve for x:

a $x^2 \geqslant 3x$ **b** $x^2 < 4$ **c** $2x^2 \geqslant 4$

d $x^2 - 21 \leqslant 4x$ **e** $x^2 + 30 > 11x$ **f** $x + 42 < x^2$

g $2x^2 \geqslant x + 3$ **h** $4x^2 - 4x + 1 < 0$ **i** $6x^2 + 7x < 3$

j $3x^2 > 8(x + 2)$ **k** $2x^2 - 4x + 2 > 0$ **l** $6x^2 + 1 \leqslant 5x$

m $1 + 5x < 6x^2$ **n** $12x^2 \geqslant 5x + 2$ **o** $2x^2 + 9 > 9x$

Example 23 **Self Tutor**

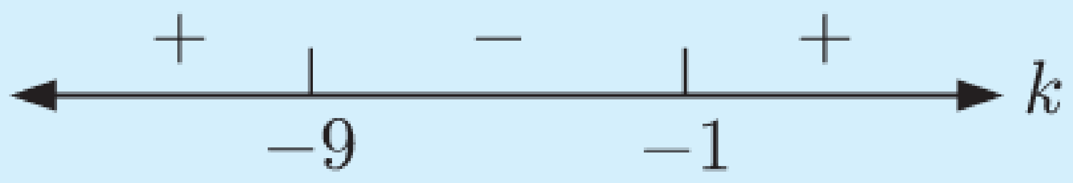
Find the value(s) of k for which the function $y = kx^2 + (k + 3)x - 1$:

a cuts the x-axis twice **b** touches the x-axis **c** misses the x-axis.

$a = k,\ \ b = k + 3,\ \ c = -1$

$$\begin{aligned} \therefore\ \ \Delta &= b^2 - 4ac \\ &= (k + 3)^2 - 4(k)(-1) \\ &= k^2 + 6k + 9 + 4k \\ &= k^2 + 10k + 9 \\ &= (k + 9)(k + 1) \end{aligned}$$

So, Δ has sign diagram:



a The graph cuts the x-axis twice if $\Delta > 0$

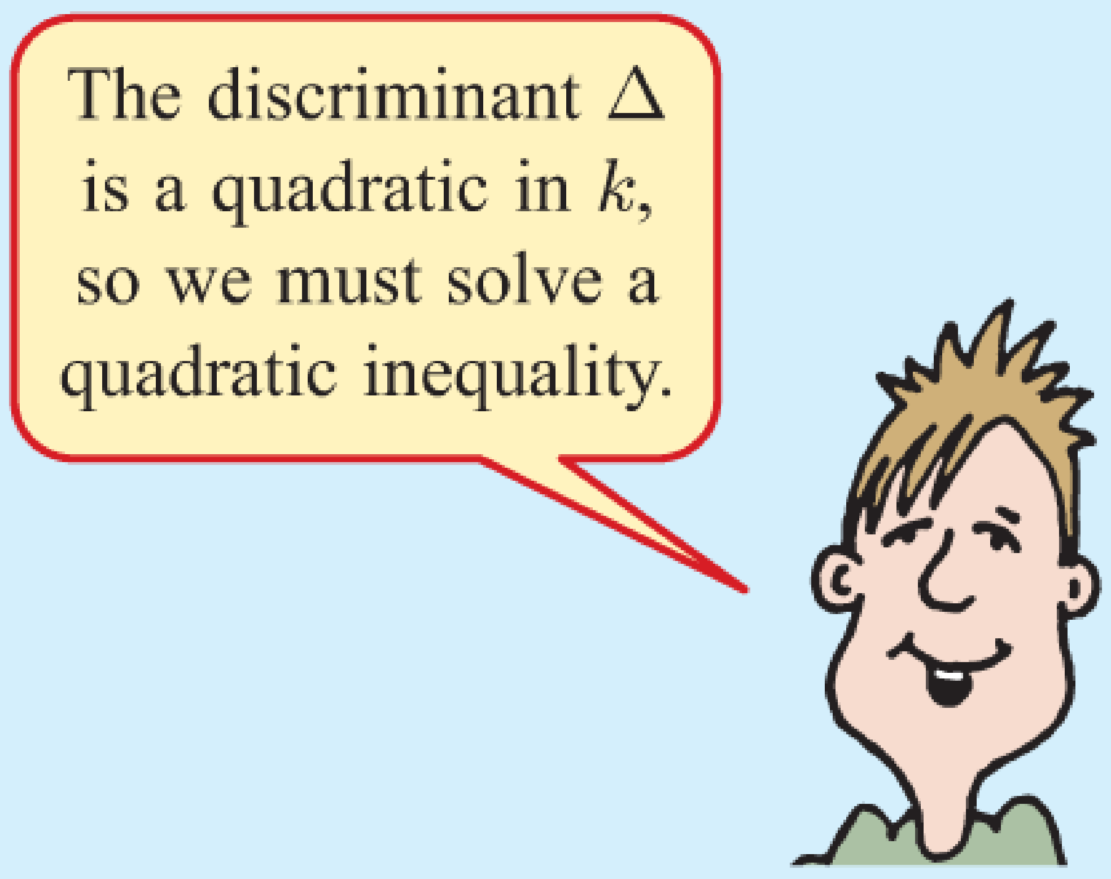
$\therefore\ \ k < -9$ or $k > -1,\ \ k \neq 0.$

b The graph touches the x-axis if $\Delta = 0$

$\therefore\ \ k = -9$ or $k = -1.$

c The graph misses the x-axis if $\Delta < 0$

$\therefore\ \ -9 < k < -1.$



4 For each quadratic function, find the values of k for which the function:

i cuts the x-axis twice **ii** touches the x-axis **iii** misses the x-axis.

a $y = 2x^2 + kx - k$ **b** $y = kx^2 - 2x + k$ **c** $y = x^2 + (k + 2)x + 4$

5 For each quadratic equation, find the values of k for which the equation has:

i two real roots **ii** a repeated real root **iii** no real roots.

a $2x^2 + (k - 2)x + 2 = 0$ **b** $x^2 + (3k - 1)x + (2k + 10) = 0$

c $(k + 1)x^2 + kx + k = 0$

6 For what values of m is $y = (m - 2)x^2 + 6x + 3m$:

a positive definite **b** negative definite?

7 Consider the curve $y = -x^2 + 3x - 6$ and the line $y = mx - 2$. Find the values of m for which the line:

a meets the curve twice **b** is a tangent to the curve

c does not meet the curve.

DYNAMIC GEOMETRY PACKAGE

8 For what values of a do the curves $y = ax^2 + 2x + 1$ and $y = -x^2 + ax - 1$:

a meet twice **b** touch **c** never meet?

REVIEW SET 14A

1 Use the vertex, axis of symmetry, and y-intercept to graph:

a $y = (x-2)^2 - 4$ **b** $y = -\frac{1}{2}(x+4)^2 + 6$

2 Find the points of intersection of $y = x^2 - 3x$ and $y = 3x^2 - 5x - 24$.

3 For what values of k does the graph of $y = -2x^2 + 5x + k$ *not* cut the x-axis?

4 Find the values of m for which $2x^2 - 3x + m = 0$ has:

a a repeated root **b** two distinct real roots **c** no real roots.

5 The sum of a number and its reciprocal is $2\frac{1}{30}$. Find the number.

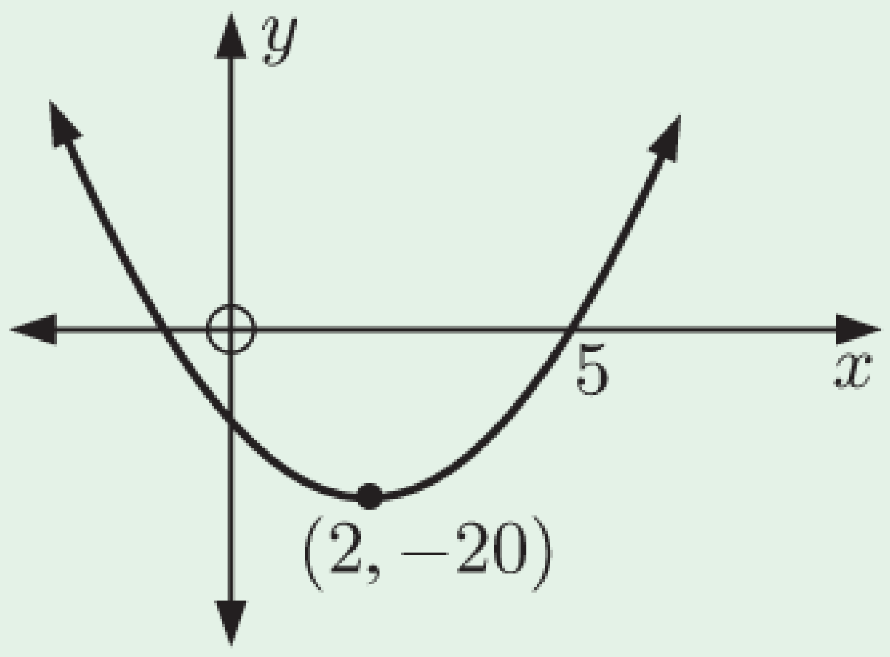
6 Show that no line with a y-intercept of 10 will ever be tangential to the curve with equation $y = 3x^2 + 7x - 2$.

7 **a** Write the quadratic $y = 2x^2 + 6x - 3$ in the form $y = a(x-h)^2 + k$.

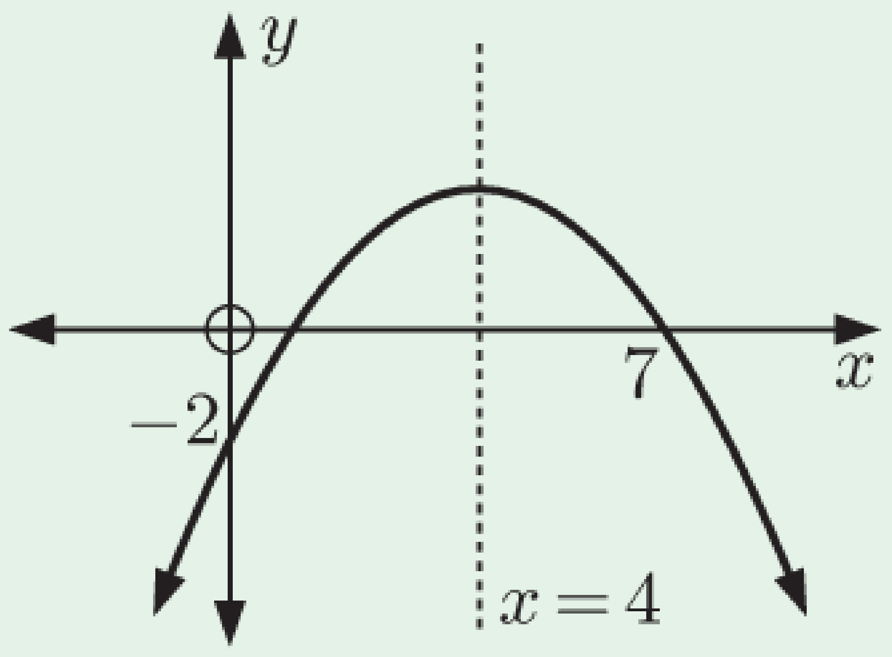
b Hence sketch the graph of the quadratic.

8 Find the equation of the quadratic with graph:

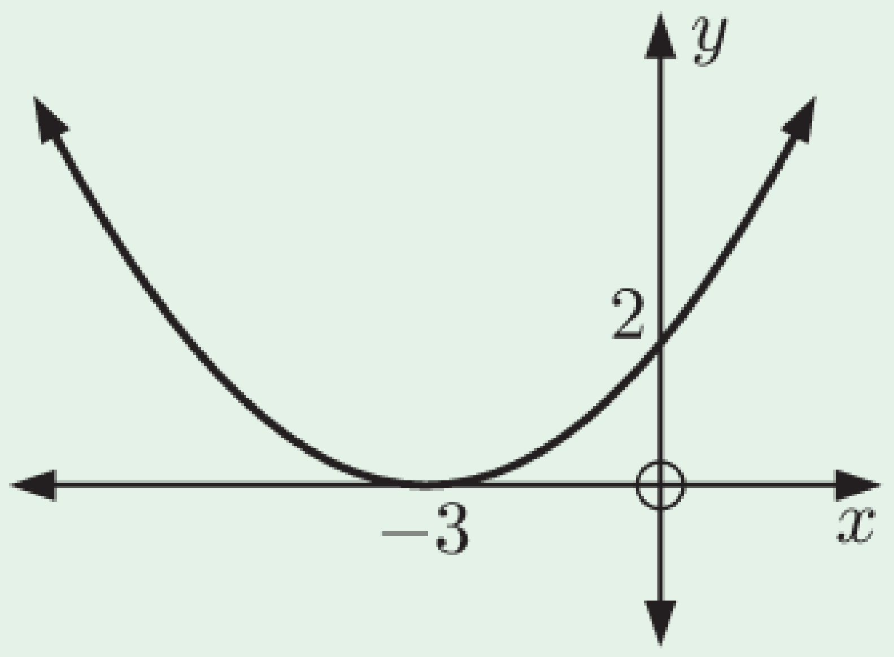
a



b



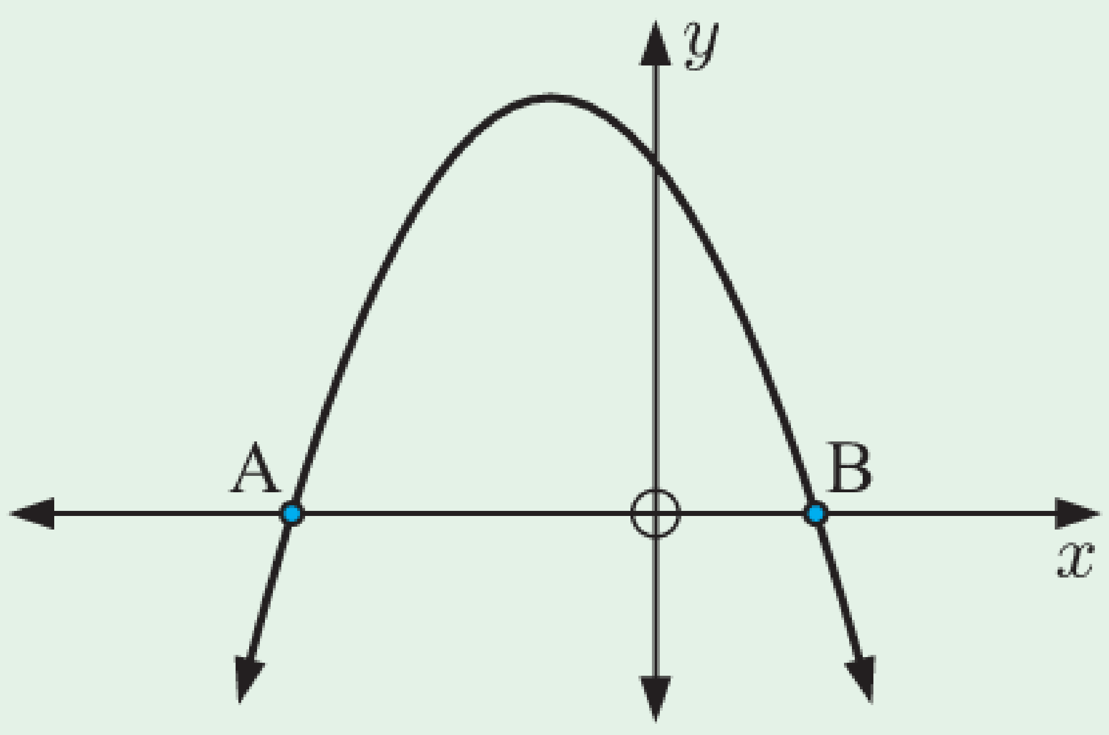
c



9 Draw the graph of $y = -x^2 + 2x$.

10 Find the y-intercept of the line with gradient -3 which is a tangent to the parabola $y = 2x^2 - 5x + 1$.

11 The graph shows the parabola $y = a(x+m)(x+n)$ where $m > n$.



a State the sign of:

i the discriminant Δ **ii** a.

b Find, in terms of m and n, the:

i coordinates of the x-intercepts A and B

ii equation of the axis of symmetry.

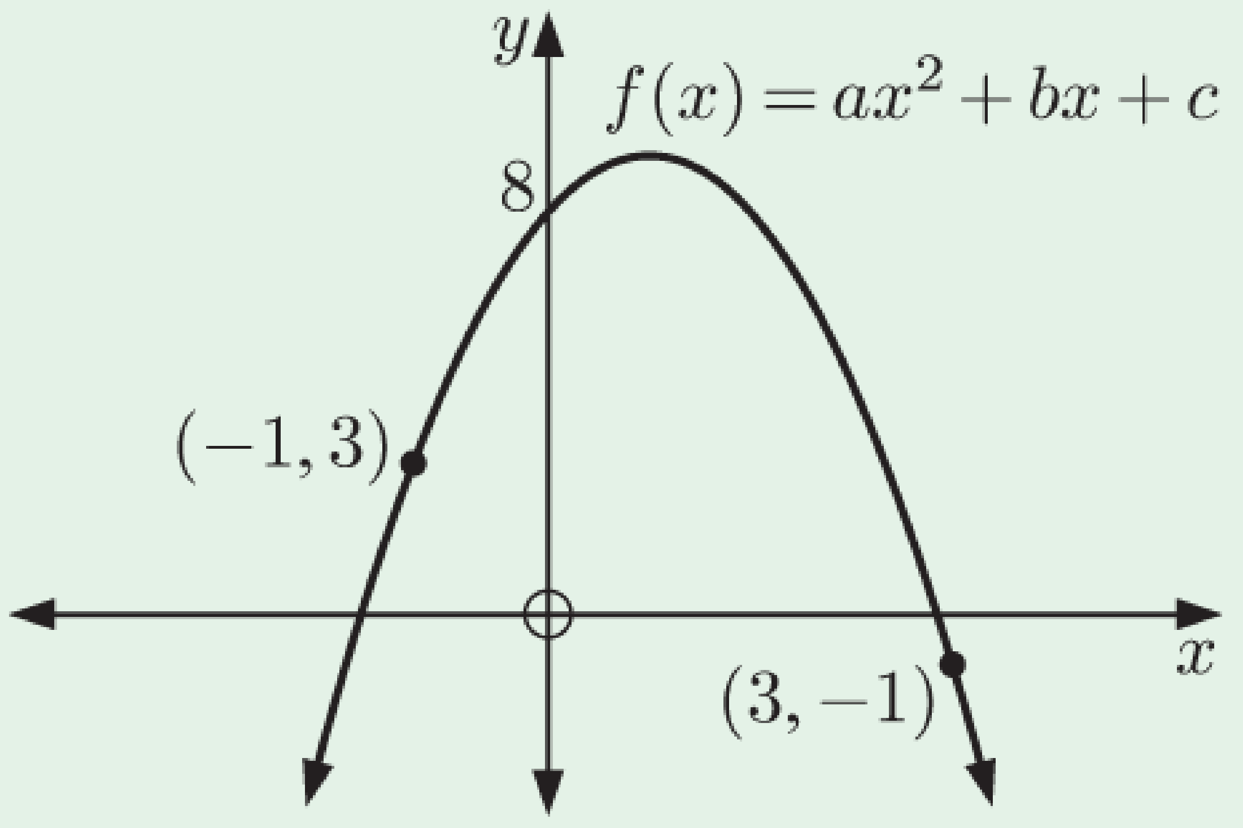
12 For a quadratic function $y = ax^2 + bx + c$, suppose the constants a, b, and c are consecutive terms of a geometric sequence. Show that the function does not cut the x-axis.

13 Find the quadratic function which cuts the x-axis at 3 and -2 and which has y-intercept 24. Give your answer in the form $y = ax^2 + bx + c$.

14 Find the value of k for which the x-intercepts of $y = 3x^2 + 2kx + k - 1$ are closest together.

15 Consider the function $y = ax^2 + bx + c$ shown.

- **a** State the value of c.
- **b** Use the other information to write two equations involving a and b.
- **c** Find a and b, and hence state the equation of the quadratic.



16 For what values of m are the lines $y = mx - 10$ tangents to the parabola $y = 3x^2 + 7x + 2$?

17 When Annie hits a softball, the height of the ball above the ground after t seconds is given by $h = -4.9t^2 + 19.6t + 1.4$ metres. Find the maximum height reached by the ball.

18 Draw a sign diagram for:

- **a** $(3x + 2)(4 - x)$
- **b** $-x^2 + 3x + 18$

19 Solve for x:

- **a** $(3 - x)(x + 2) < 0$
- **b** $x^2 - 4x - 5 \leqslant 0$
- **c** $2x^2 + x > 10$

20 Find the values of k for which the function $f(x) = x^2 + kx + (3k - 4)$:

- **a** cuts the x-axis twice
- **b** touches the x-axis
- **c** misses the x-axis.

REVIEW SET 14B

1 Consider the quadratic $y = \frac{1}{2}(x - 2)^2 - 4$.

- **a** State the equation of the axis of symmetry.
- **b** Find the coordinates of the vertex.
- **c** Find the y-intercept.
- **d** Sketch the function.

2 Consider the quadratic $y = -3x^2 + 8x + 7$. Find the equation of the axis of symmetry, and the coordinates of the vertex.

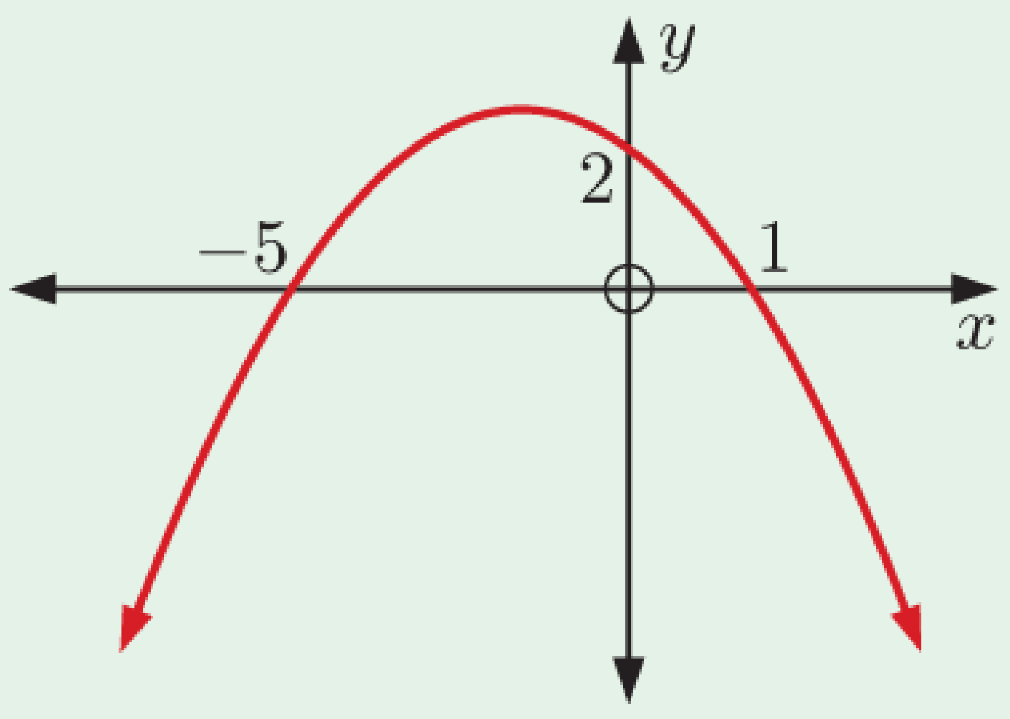
3 Use the discriminant only to find the relationship between the graph and the x-axis for:

- **a** $y = 2x^2 + 3x - 7$
- **b** $y = -3x^2 - 7x + 4$

4 Find the equation of the quadratic with vertex $(2, 25)$ and y-intercept 1.

5

- **a** Find the equation of the quadratic illustrated.
- **b** Hence find its vertex and axis of symmetry.



6 Consider the quadratic $y = 2x^2 + 4x - 1$.

a State the axis of symmetry.

b Find the coordinates of the vertex.

c Find the axes intercepts.

d Hence sketch the function.

7 Find, in the form $y = ax^2 + bx + c$, the quadratic function whose graph:

a touches the x-axis at 3 and passes through $(2, 2)$

b has x-intercepts 3 and -2, and y-intercept 3

c passes through $(-1, -9)$, $(1, 5)$, and $(2, 15)$

d has vertex $(3, 15)$ and passes through the point $(1, 7)$.

8 **a** For what values of c do the lines with equations $y = 3x + c$ intersect the parabola $y = x^2 + x - 5$ in two distinct points?

b Choose one such value of c and find the points of intersection in this case.

9 Find the maximum or minimum value of each quadratic, and the corresponding value of x:

a $y = 3x^2 + 4x + 7$

b $y = -2x^2 - 5x + 2$

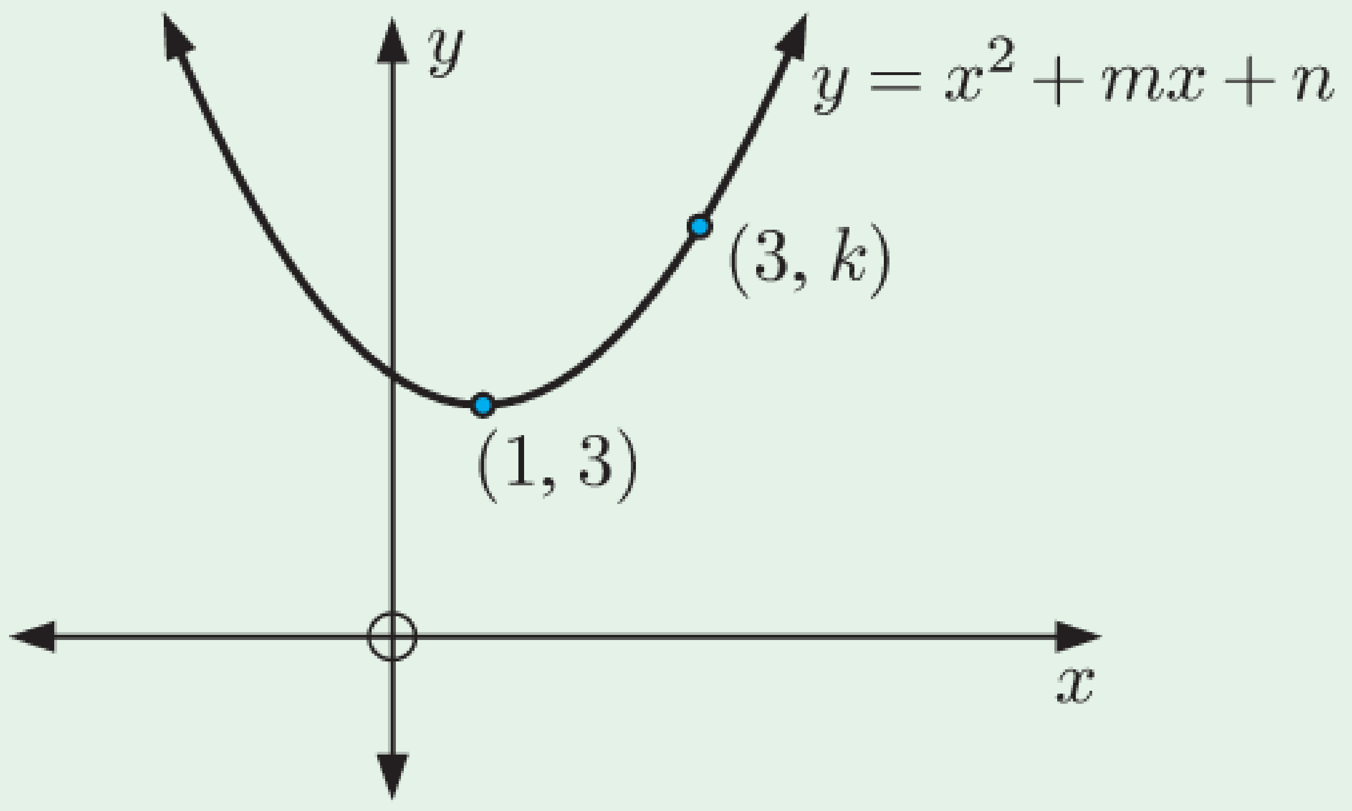
10 The graph of a quadratic function cuts the x-axis at -2 and 3, and passes through $(-3, 18)$.

a Find the equation of the function in the form $y = ax^2 + bx + c$.

b Write down the y-intercept of the function.

c Find the coordinates of the vertex.

11



Consider the graph of $y = x^2 + mx + n$.

a Determine the values of m and n.

b Hence find the value of k.

12 An open square-based box has capacity 120 mL. It is made from a square piece of tinplate with 4 cm squares cut from each of its corners. Find the dimensions of the original piece of tinplate.

13 Consider $y = -x^2 - 3x + 4$ and $y = x^2 + 5x + 4$.

a Solve for x: $-x^2 - 3x + 4 = x^2 + 5x + 4$.

b Sketch the curves on the same set of axes.

c Hence solve for x: $x^2 + 5x + 4 > -x^2 - 3x + 4$.

14 For each of the following quadratics:

i Write the quadratic in completed square form.

ii Write the quadratic in factored form.

iii Sketch the graph of the quadratic, identifying its axes intercepts, vertex, and axis of symmetry.

a $y = x^2 + 4x + 3$

b $y = x^2 + 2x - 3$

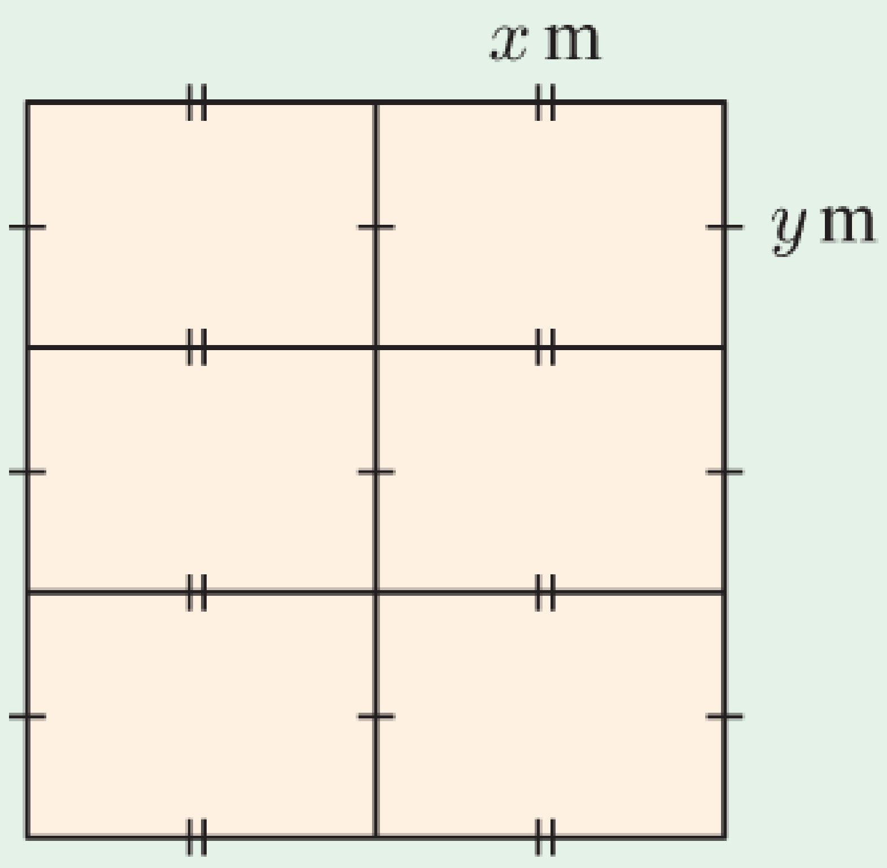
c $y = 2x^2 - 8x - 10$

d $y = -x^2 + 6x + 7$

15 Two different quadratic functions of the form $y = 9x^2 - kx + 4$ both *touch* the x-axis.

- **a** Find the two values of k.
- **b** Find the point of intersection of the two quadratic functions.

16 600 m of fencing is used to construct 6 rectangular animal pens as shown.



- **a** Show that the area A of each pen is
  $A = x\left(\frac{600 - 8x}{9}\right)$ m^2.
- **b** Find the dimensions of each pen so that it has the maximum possible area.
- **c** What is the area of each pen in this case?

17 A retailer sells sunglasses for \$45, and has 50 customers per day. From market research, the retailer discovers that for every \$1.50 increase in the price of the sunglasses, he will lose a customer per day.

Let \$$x$ be the price increase of the sunglasses.

- **a** Show that the revenue collected by the retailer each day is
  $R = (45 + x)\left(50 - \frac{x}{1.5}\right)$ dollars.
- **b** Find the price the retailer should set for his sunglasses in order to maximise his daily revenue. How much revenue is made per day at this price?

18 Draw a sign diagram for:

- **a** $x^2 - 3x - 10$
- **b** $-(x + 3)^2$

19 Solve for x:

- **a** $4x^2 - 3x < 0$
- **b** $2x^2 - 3x - 5 \geqslant 0$
- **c** $\frac{11}{3}x \leqslant 2x^2 + 1$

20 Find the values of m for which the function $y = mx^2 + 5x + (m + 12)$:

- **a** cuts the x-axis twice
- **b** touches the x-axis
- **c** misses the x-axis.

Chapter 15

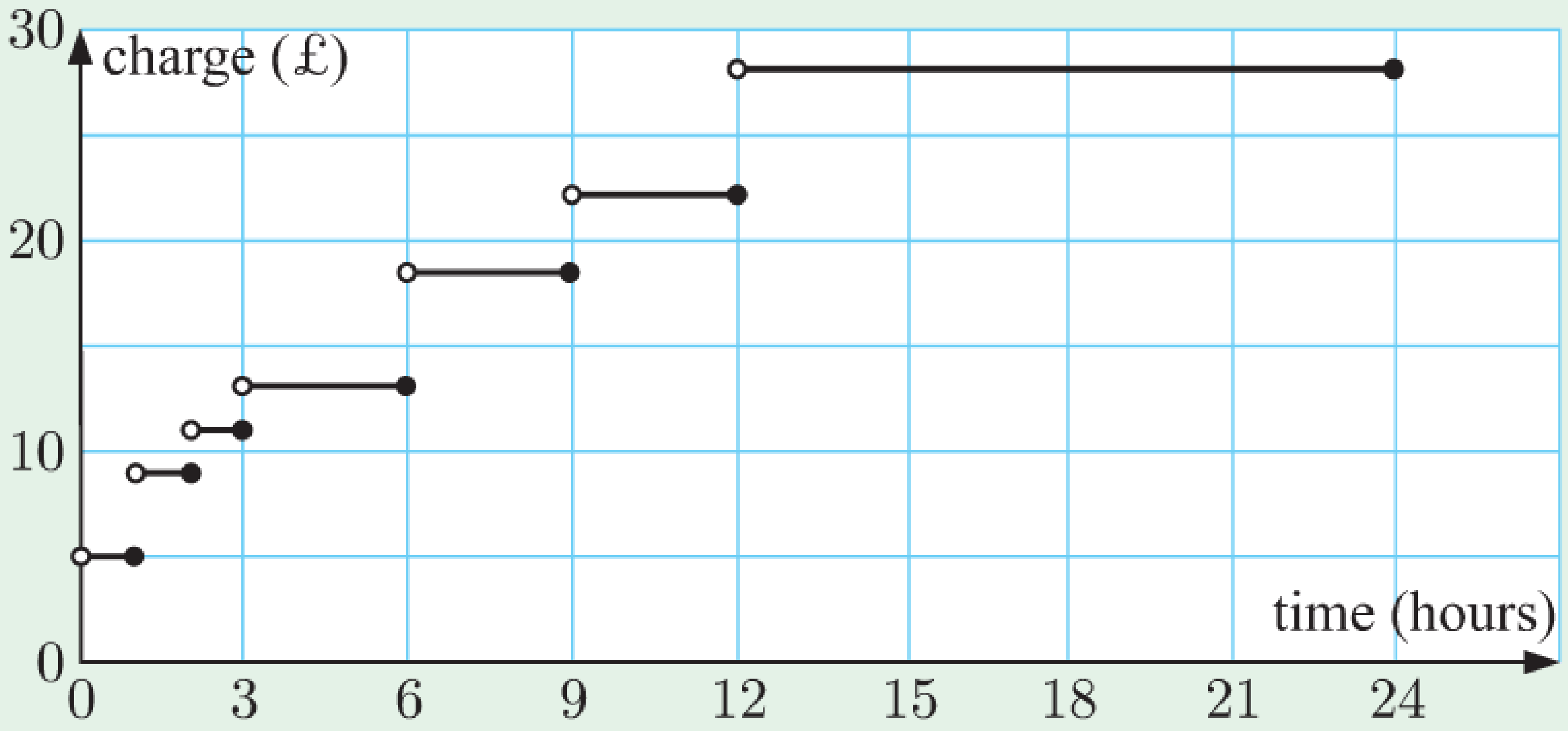
Functions

Contents:

OPENING PROBLEM

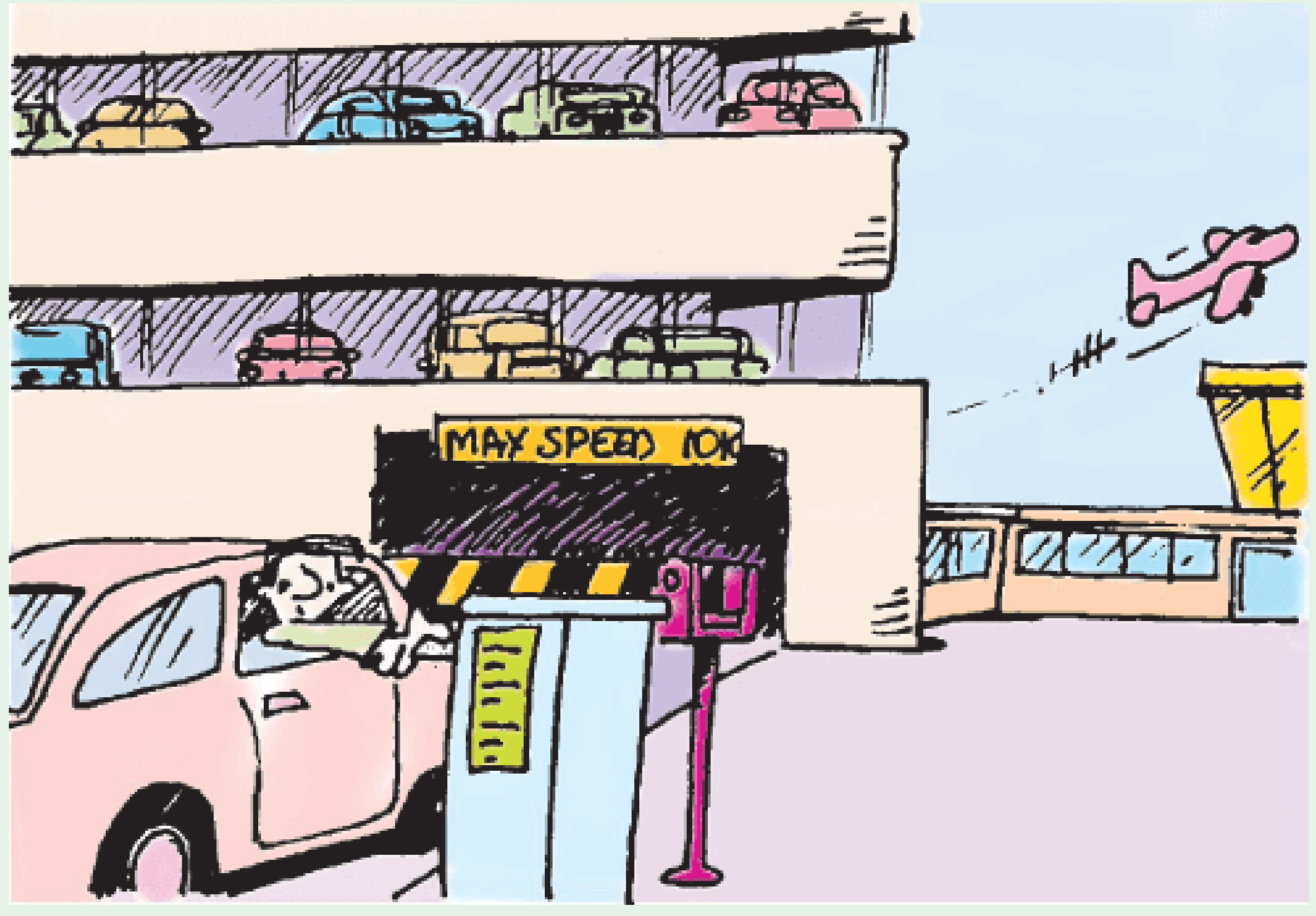
The charges for parking a car in a short-term car park at an airport are shown in the table and graph below. The total charge is *dependent* on the length of time t the car is parked.

Car park charges	
Time t (hours)	*Charge*
$0 < t \leqslant 1$	£5.00
$1 < t \leqslant 2$	£9.00
$2 < t \leqslant 3$	£11.00
$3 < t \leqslant 6$	£13.00
$6 < t \leqslant 9$	£18.00
$9 < t \leqslant 12$	£22.00
$12 < t \leqslant 24$	£28.00



Things to think about:

- **a** What values of *time* are illustrated in the graph?
- **b** What are the possible charges?
- **c** What feature of the graph ensures that there is only one charge for any given time?



In the course so far, we have studied several different relationships between variables. In particular, for two variables x and y:

- A **linear function** is a relationship which can be expressed in the form $y = ax + b$ where a, b are constants, $a \neq 0$.
- A **quadratic function** is a relationship which can be expressed in the form $y = ax^2 + bx + c$ where a, b, c are constants, $a \neq 0$.

In the **Opening Problem** we see another type of relationship, between the two variables *time* and *charge*. We call this a **piecewise function** because its graph has several sections.

In this Chapter we explore what it really means for the relationship between two variables to be called a **function**. We will then explore properties of functions which will help us work with and understand them.

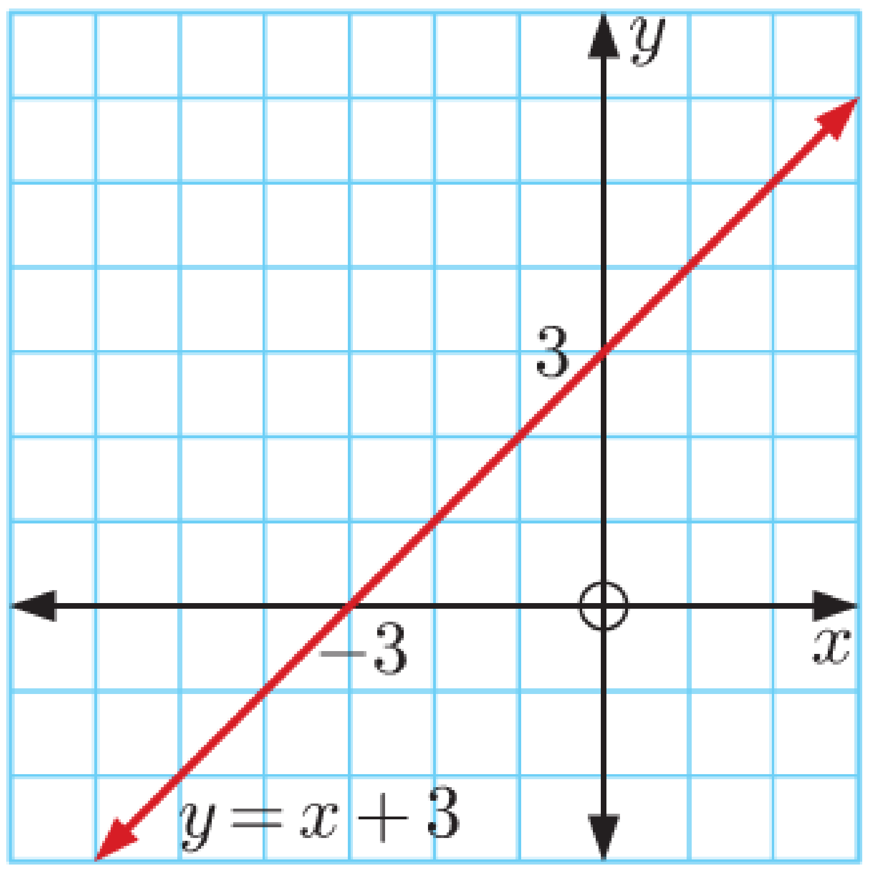
A RELATIONS AND FUNCTIONS

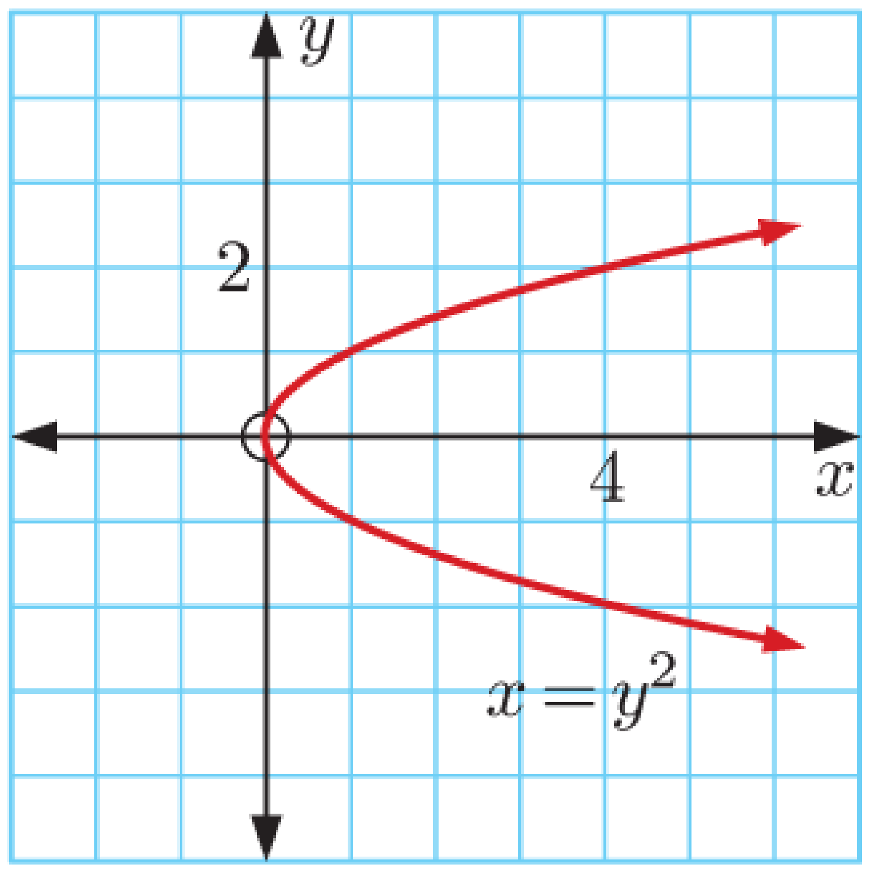
A **relation** between variables x and y is any set of points in the (x, y) plane.

We say that the points *connect* the two variables.

A relation is often expressed in the form of an **equation** connecting the **variables** x and y.

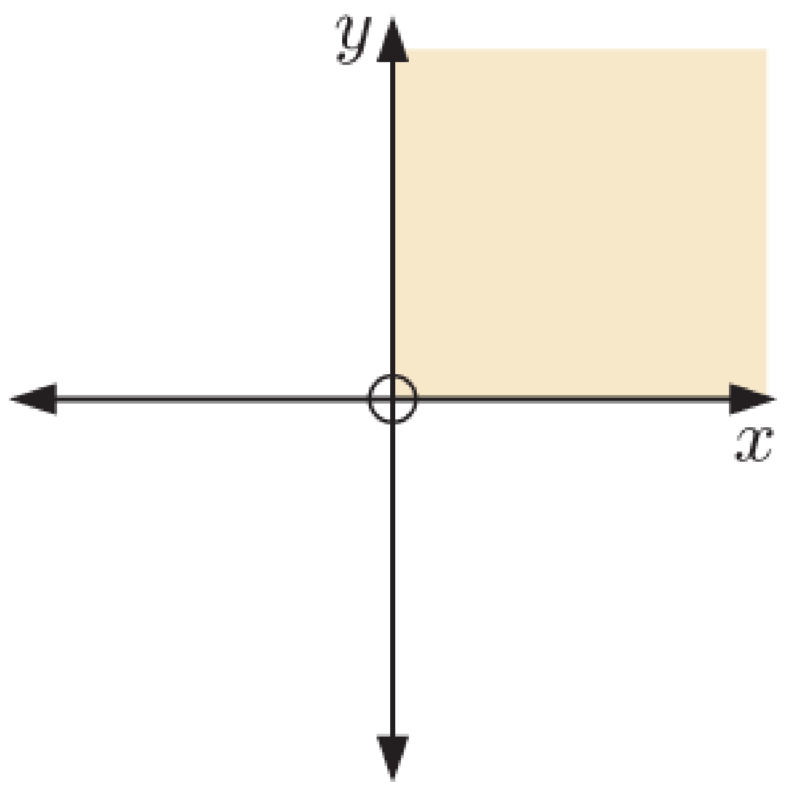
For example, $y = x + 3$ and $x = y^2$ are the equations of two relations. Each equation generates a set of ordered pairs, which we can graph:





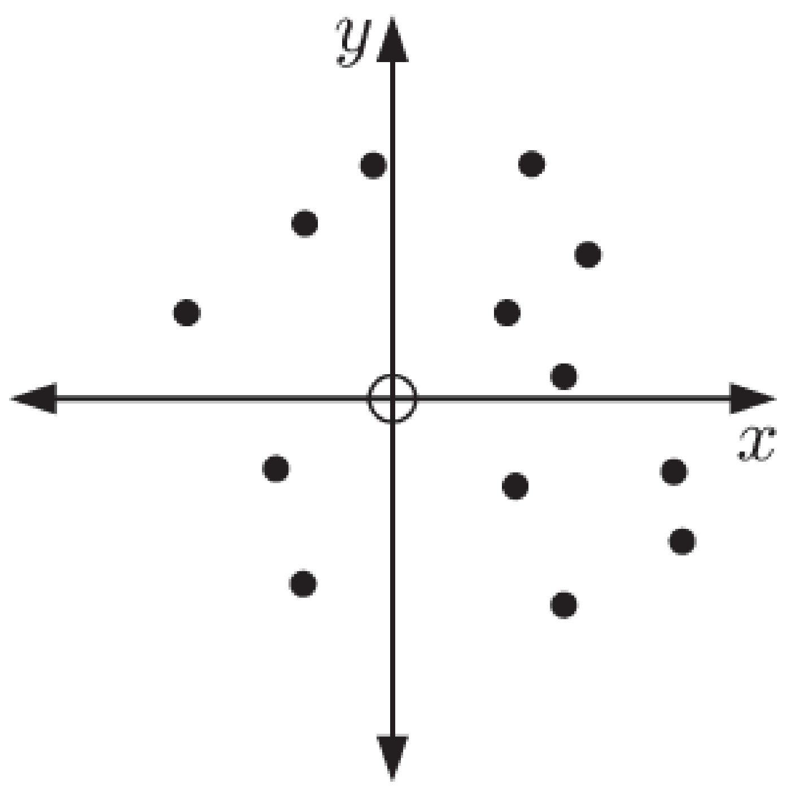
However, not all relations can be defined by an equation. Below are two examples:

(1)



The set of all points in the first quadrant is the relation $x > 0$, $y > 0$.

(2)



These 13 points form a relation. It can be described as a finite set of points, but not by an equation.

FUNCTIONS

A **function** is a relation in which no two different ordered pairs have the same x-coordinate or first component.

We can see from this definition that a function is a special type of relation.

Every function is a relation, but not every relation is a function.

ALGEBRAIC TEST FOR FUNCTIONS

Suppose a relation is given as an equation. If the substitution of any value for x results in at most one value of y, then the relation is a function.

For example:

- $y = 3x - 1$ is a function, since for any value of x there is only one corresponding value of y
- $x = y^2$ is not a function, since if $x = 4$ then $y = \pm 2$.

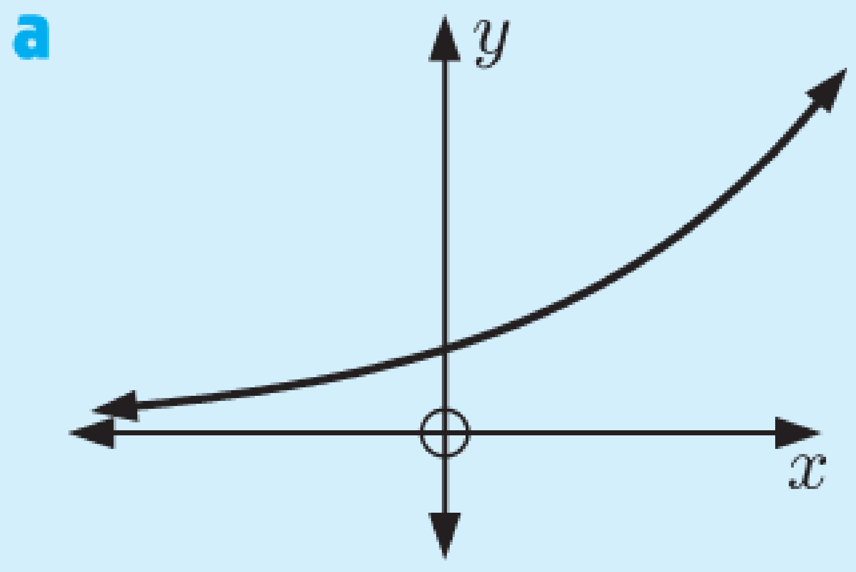
GEOMETRIC TEST OR VERTICAL LINE TEST FOR FUNCTIONS

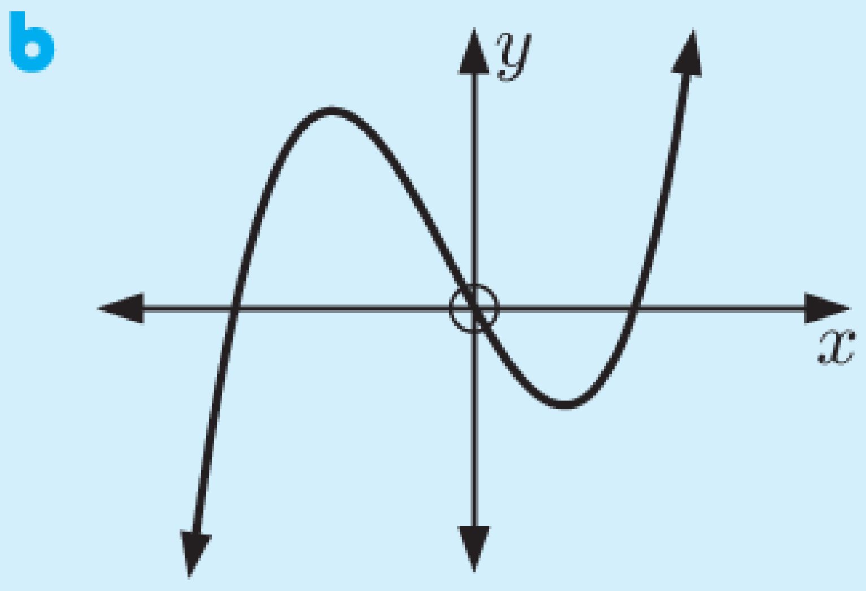
Suppose we draw all possible vertical lines on the graph of a relation.

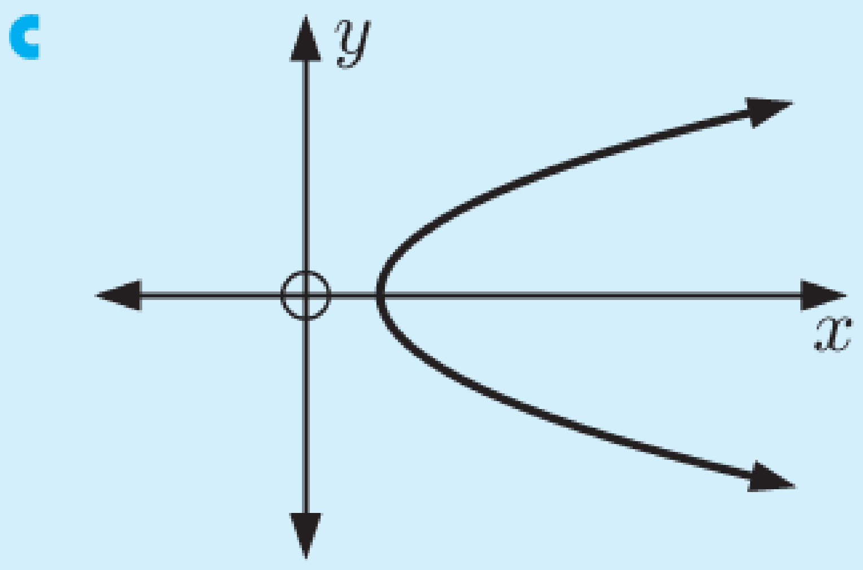
- If each line cuts the graph at most once, then the relation is a function.
- If at least one line cuts the graph more than once, then the relation is not a function.

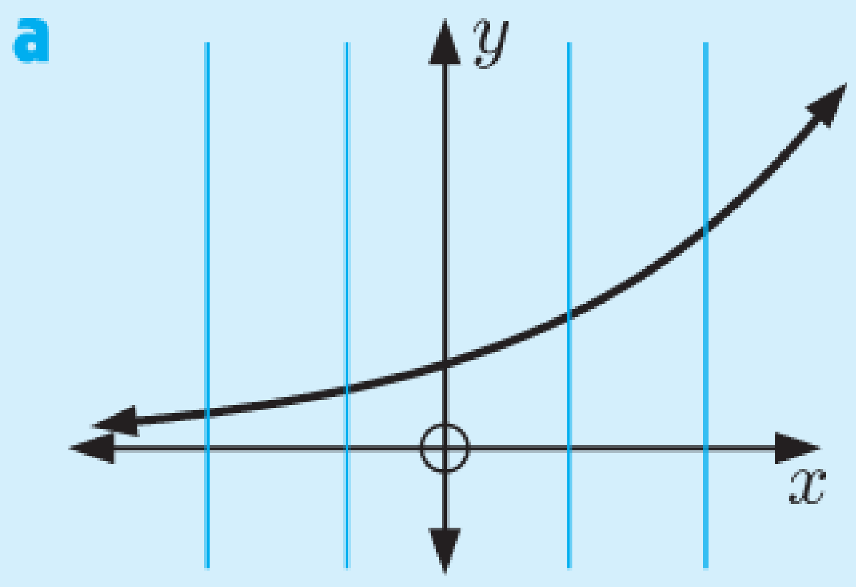
Example 1 **Self Tutor**

Which of the following relations are functions?

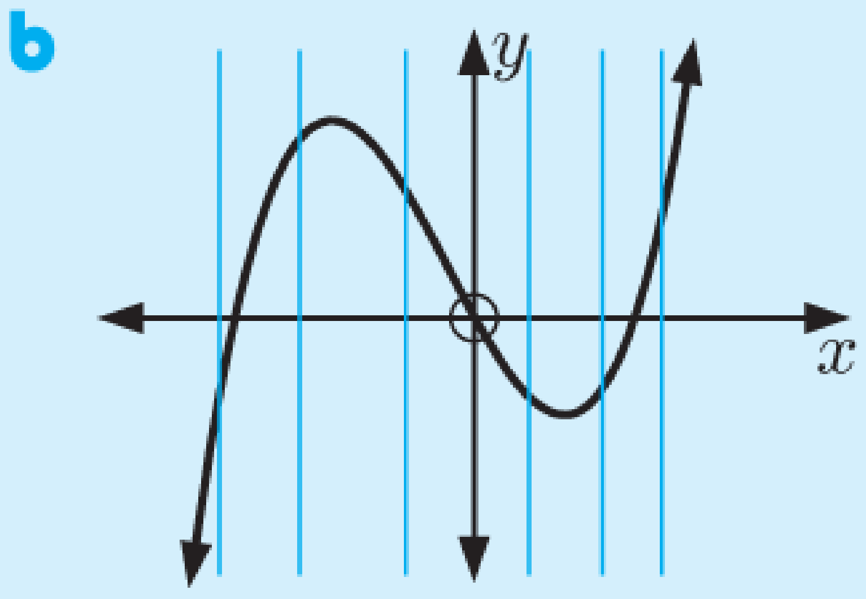




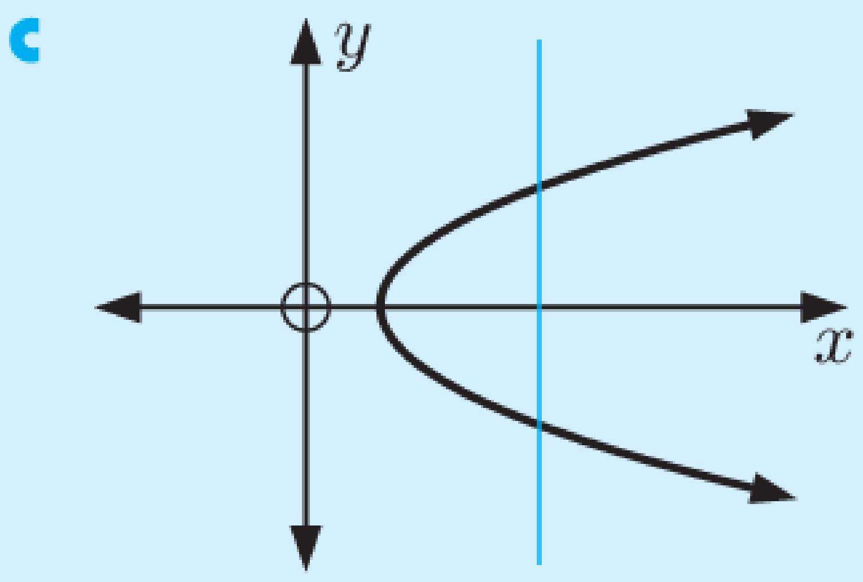




a function



a function



not a function

DEMO

GRAPHICAL NOTE

- If a graph contains a small **open circle** such as —o— , this point is **not included**.
- If a graph contains a small **filled-in circle** such as ——● , this point is **included**.
- If a graph contains an **arrowhead** at an end such as ——►, then the graph continues indefinitely in that general direction, or the shape may repeat as it has done previously.

EXERCISE 15A

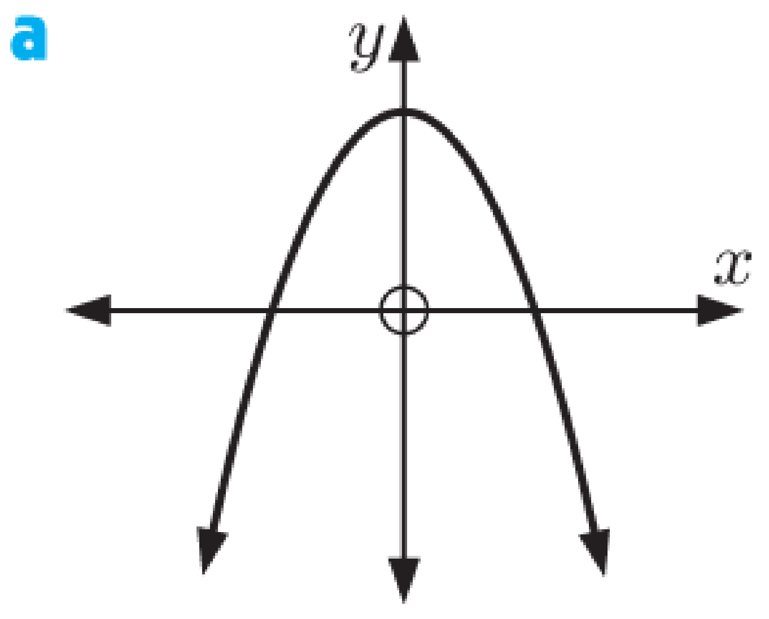
1 Which of the following sets of ordered pairs are functions? Explain your answers.

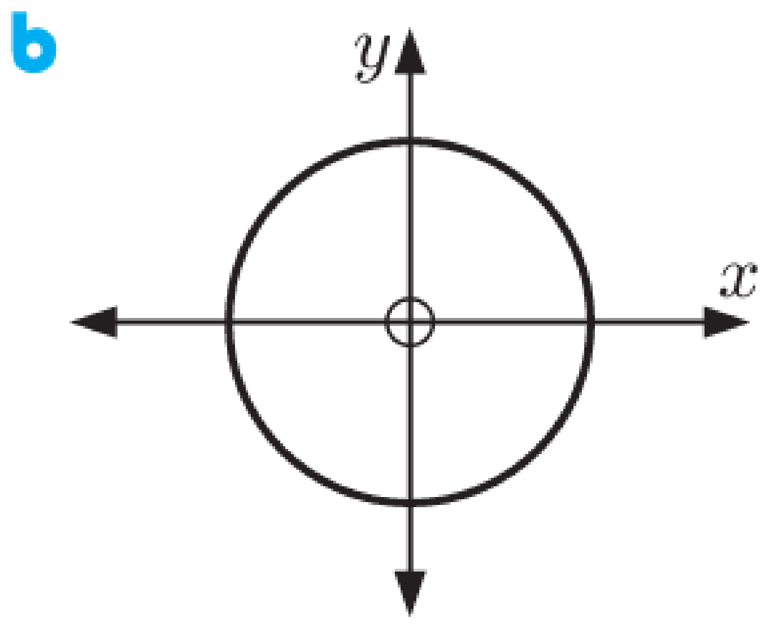
a $\{(1, 3), (2, 3), (3, 1), (4, 2)\}$ **b** $\{(2, 1), (3, 1), (-1, 2), (2, 0)\}$

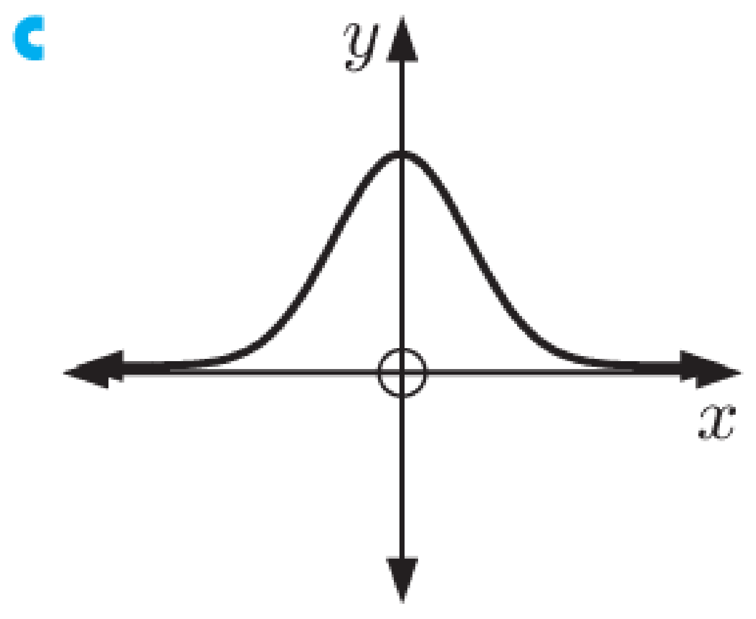
2 Use algebraic methods to decide whether these relations are functions. Explain your answers.

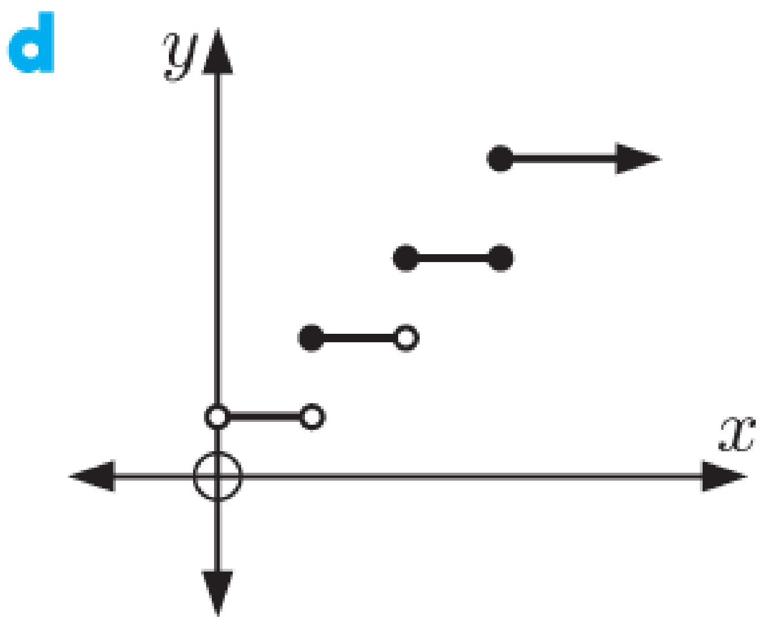
a $y = x^2 - 9$ **b** $x + y = 9$ **c** $x^2 + y^2 = 9$

3 Use the vertical line test to determine which of the following relations are functions:









4 The managers of a new amusement park are discussing the schedule of ticket prices. Maurice suggests the table alongside. Explain why this relation between *age* and *cost* is not a function, and discuss the problems that this will cause.

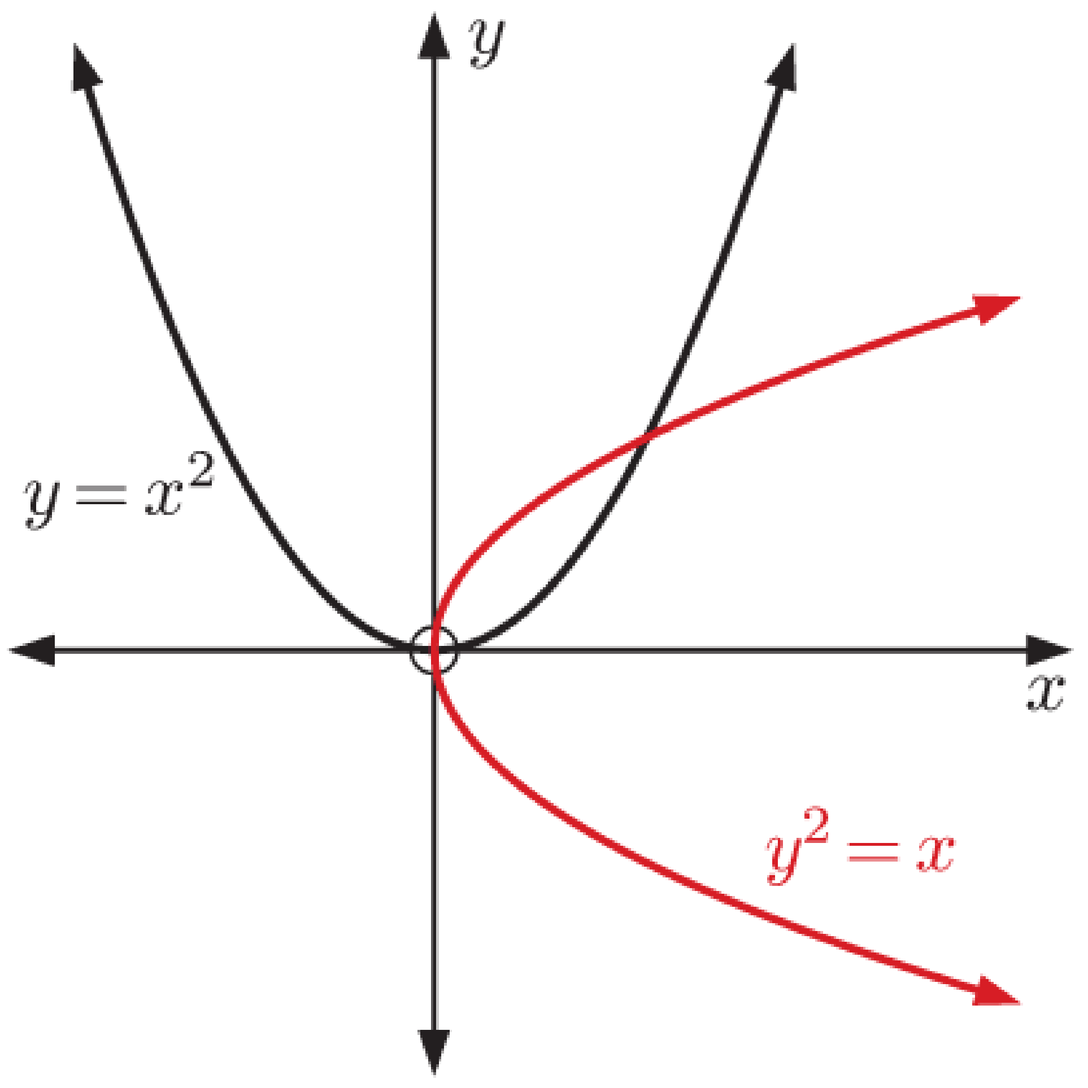
Age	*Cost*
0 - 2 years (infants)	\$0
2 - 16 years (children)	\$20
16+ years (adults)	\$30

5 Is it possible for a function to have more than one y-intercept? Explain your answer.

6 Is the graph of a straight line always a function? Give evidence to support your answer.

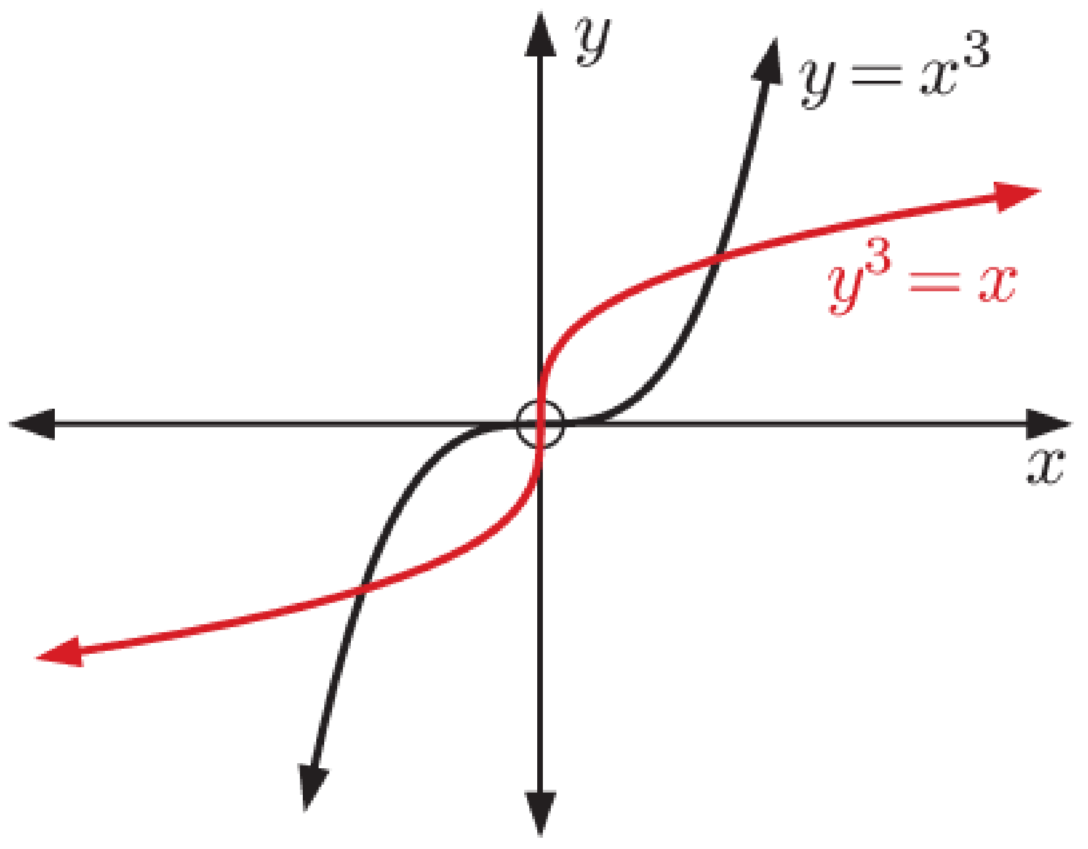
7 The graph alongside shows the curves $y = x^2$ and $y^2 = x$.

- **a** Discuss the similarities and differences between the curves, including whether each curve is a function. You may also consider what transformation(s) map one curve onto the other.
- **b** Using $y^2 = x$, we can write $y = \pm\sqrt{x}$.
  - **i** What part of the graph of $y^2 = x$ corresponds to $y = \sqrt{x}$?
  - **ii** Is $y = \sqrt{x}$ a function? Explain your answer.



8 The graph alongside shows the curves $y = x^3$ and $y^3 = x$.

- **a** Explain why both of these curves are functions.
- **b** For the curve $y^3 = x$, write y as a function of x.



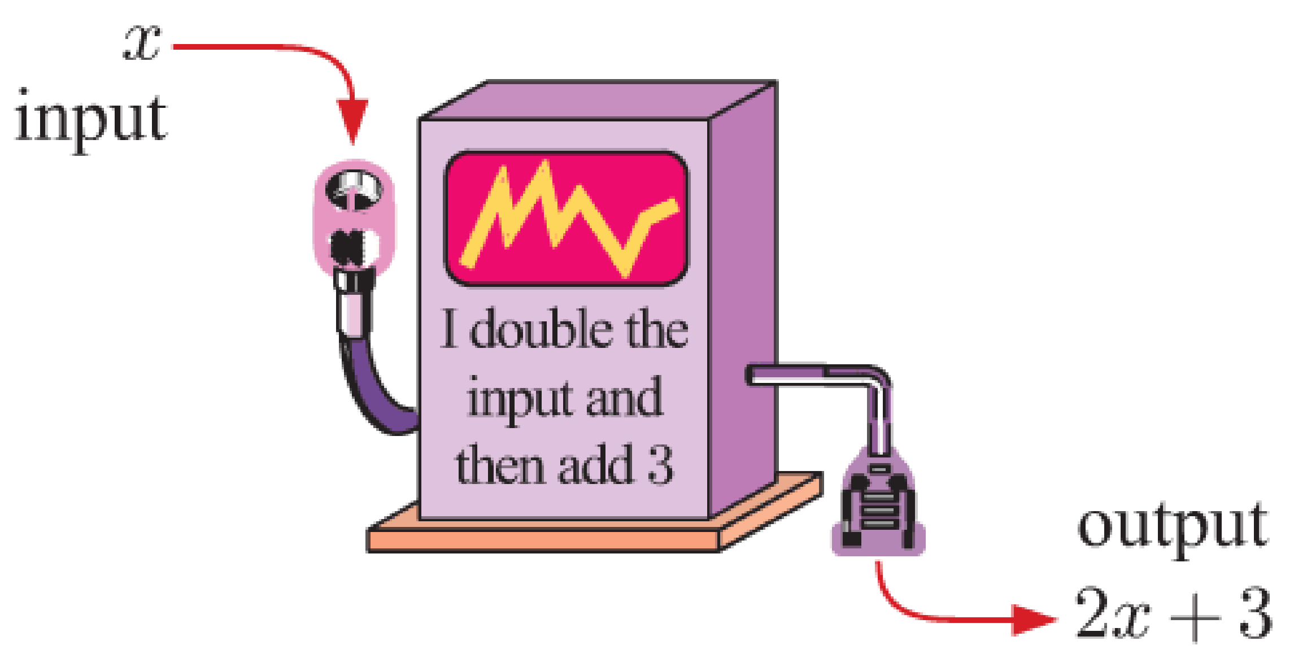
DISCUSSION

In the **Opening Problem**:

- Is the relation describing the car park charges a function?
- If we know the *time* somebody parked for, can we determine the exact *charge* they need to pay?
- If we know the *charge* somebody pays, can we determine the exact *time* they have parked for?

B FUNCTION NOTATION

Function machines are sometimes used to illustrate how functions behave.



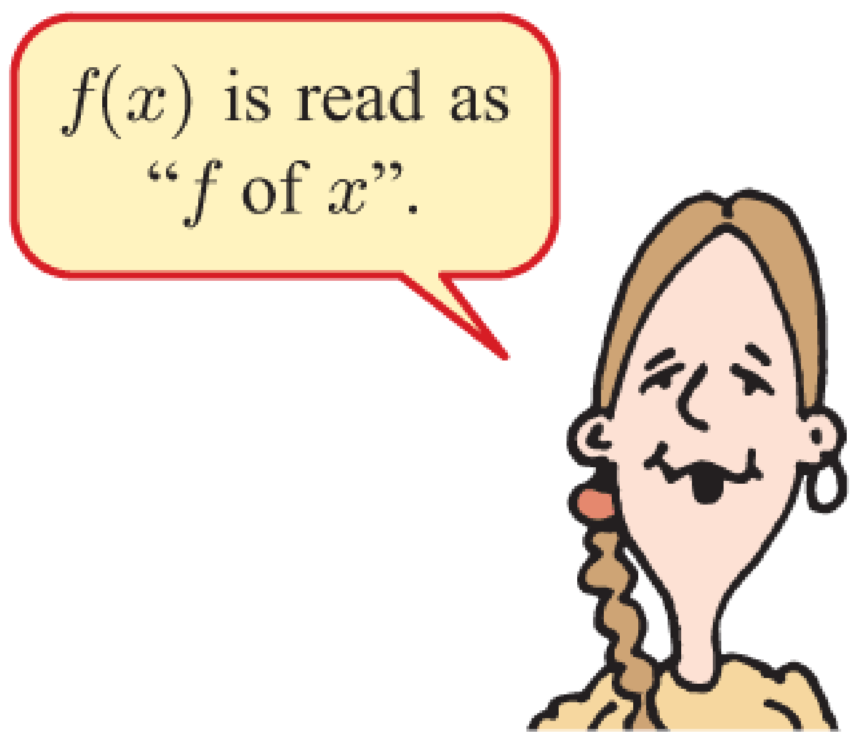
If 4 is the input fed into the machine, the output is $2(4) + 3 = 11$.

The above "machine" has been programmed to perform a particular function. If we use f to represent that particular function, we can write "f is the function that will convert x into $2x + 3$."

So, f would convert 2 into $2(2) + 3 = 7$ and
-4 into $2(-4) + 3 = -5$.

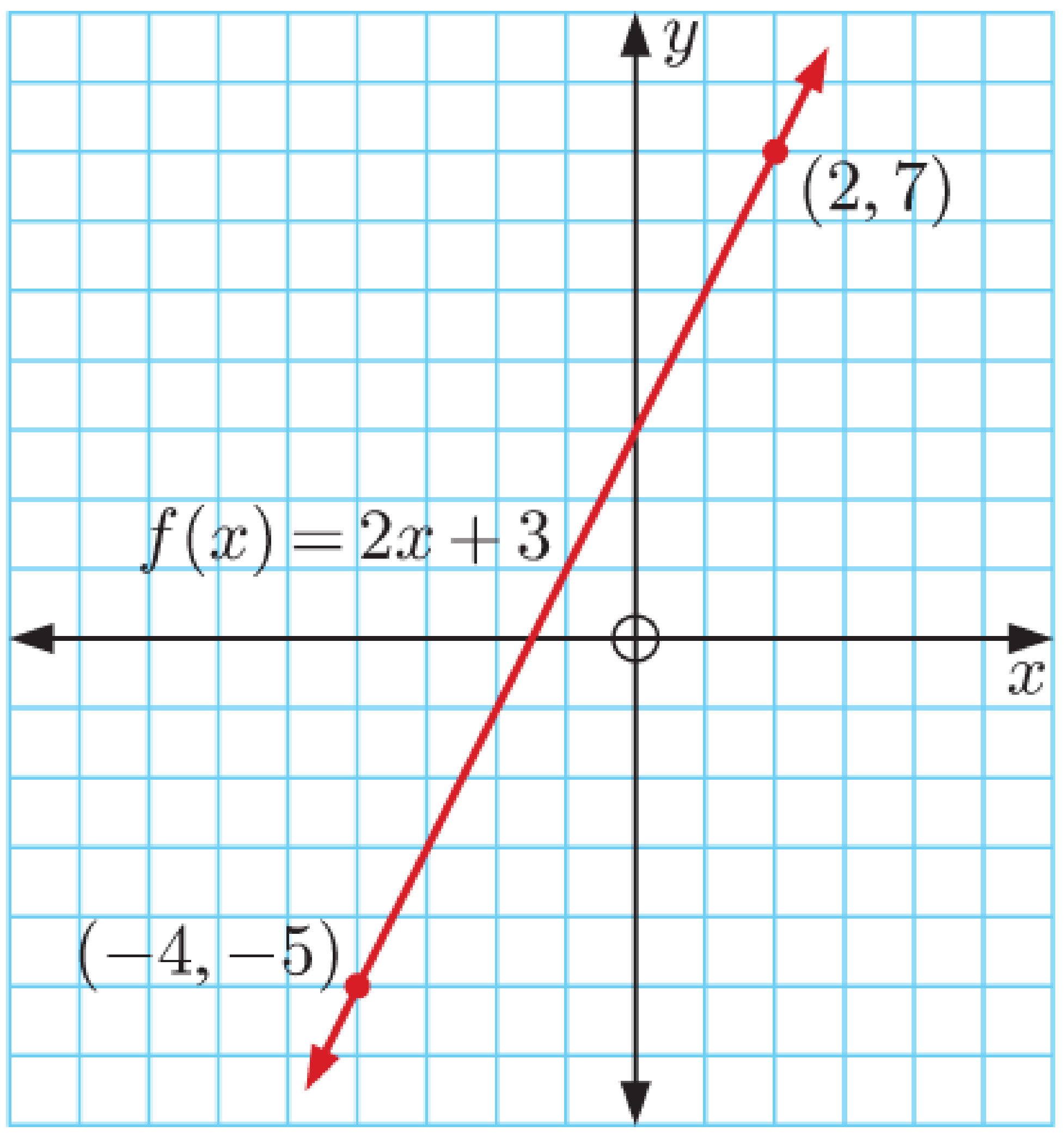
This function can be written as:

$$\underset{\text{function } f}{\boldsymbol{f}} \; \underset{\text{such that}}{\boldsymbol{:}} \; \boldsymbol{x \mapsto} \underbrace{\boldsymbol{2x+3}}_{x \text{ is converted into } 2x+3}$$



Two other equivalent forms we use are $f(x) = 2x + 3$ and $y = 2x + 3$.

$f(x)$ is the value of y for a given value of x, so $y = f(x)$.



f is the function which converts x into $f(x)$, so we write $f : x \mapsto f(x)$.

$y = f(x)$ is sometimes called the **function value** or **image** of x.

For $f(x) = 2x + 3$:

- $f(2) = 2(2) + 3 = 7$
  $\therefore$ the point $(2, 7)$ lies on the graph of the function.
- $f(-4) = 2(-4) + 3 = -5$
  $\therefore$ the point $(-4, -5)$ also lies on the graph.

Example 2 **Self Tutor**

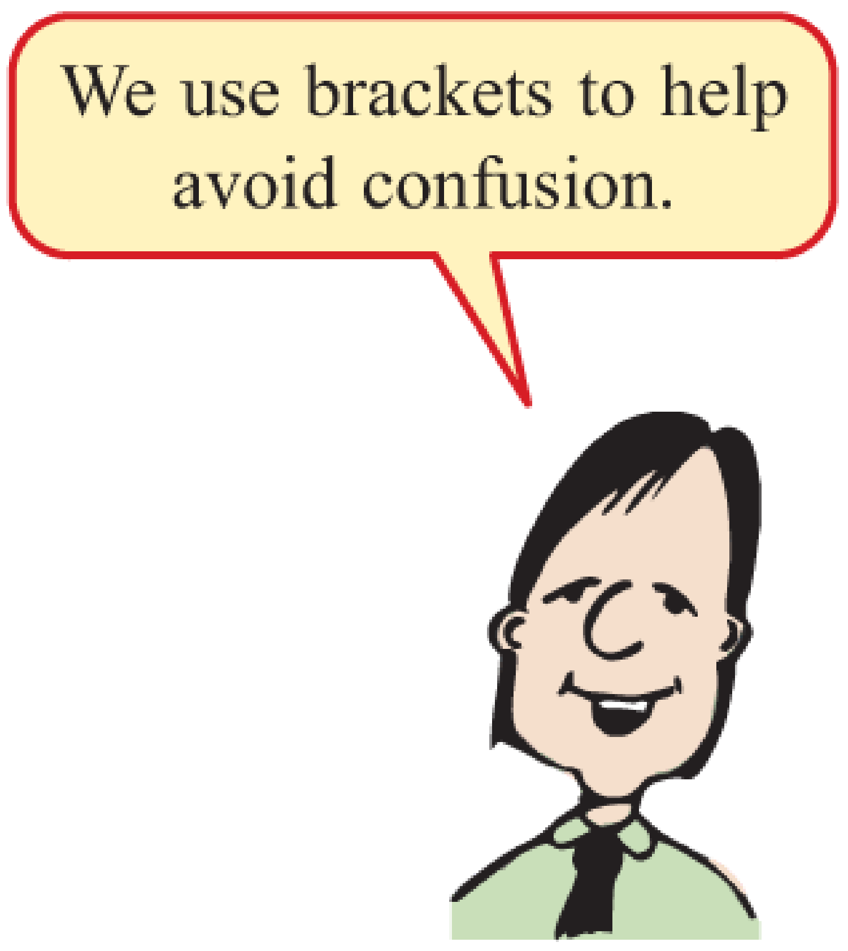
If $f : x \mapsto 2x^2 - 3x$, find the value of:

a $f(5)$ **b** $f(-4)$

$f(x) = 2x^2 - 3x$

a $f(5) = 2(5)^2 - 3(5)$ {replacing x with (5)}
$= 2 \times 25 - 15$
$= 35$

b $f(-4) = 2(-4)^2 - 3(-4)$ {replacing x with (-4)}
$= 2(16) + 12$
$= 44$



EXERCISE 15B

1 If $f(x) = 3x - x^2 + 2$, find the value of:

a $f(0)$ **b** $f(3)$ **c** $f(-3)$ **d** $f(-7)$ **e** $f(\frac{3}{2})$

2 If $g : x \mapsto x - \dfrac{4}{x}$, find the value of:

a $g(1)$ **b** $g(4)$ **c** $g(-1)$ **d** $g(-4)$ **e** $g(-\frac{1}{2})$

3 Suppose $G(x) = \dfrac{2x+3}{x-4}$.

a Evaluate: **i** $G(2)$ **ii** $G(0)$ **iii** $G(-\frac{1}{2})$

b Find a value of x such that $G(x)$ does not exist.

c Find x such that $G(x) = -3$.

Example 3 **Self Tutor**

If $f(x) = 5 - x - x^2$, find in simplest form:

a $f(-x)$ **b** $f(x+2)$ **c** $f(x-1)-5$

a $$\begin{aligned} f(-x) &= 5 - (-x) - (-x)^2 \quad \{\text{replacing } x \text{ with } (-x)\} \\ &= 5 + x - x^2 \end{aligned}$$

b $$\begin{aligned} f(x+2) &= 5 - (x+2) - (x+2)^2 \quad \{\text{replacing } x \text{ with } (x+2)\} \\ &= 5 - x - 2 - [x^2 + 4x + 4] \\ &= 3 - x - x^2 - 4x - 4 \\ &= -x^2 - 5x - 1 \end{aligned}$$

c $$\begin{aligned} f(x-1) - 5 &= (5 - (x-1) - (x-1)^2) - 5 \quad \{\text{replacing } x \text{ with } (x-1)\} \\ &= 5 - x + 1 - (x^2 - 2x + 1) - 5 \\ &= -x^2 + x \end{aligned}$$

4 If $f(x) = 7 - 3x$, find in simplest form:

a $f(a)$ **b** $f(-a)$ **c** $f(a+3)$
d $f(2a)$ **e** $f(x+2)$ **f** $f(x+h)$

5 If $F(x) = 2x^2 + 3x - 1$, find in simplest form:

a $F(x+4)$ **b** $F(2-x)$ **c** $F(-x)$
d $F(x^2)$ **e** $F(3x)$ **f** $F(x+h)$

6 If $f(x) = x^2$, find in simplest form:

a $f(3x)$ **b** $f\left(\frac{x}{2}\right)$ **c** $3\,f(x)$ **d** $2\,f(x-1)+5$

7 If $f(x) = \frac{1}{x}$, find in simplest form:

a $f(-x)$ **b** $f(\frac{1}{2}x)$ **c** $2\,f(x)+3$ **d** $3\,f(x-1)+2$

8 f represents a function. Explain the difference in meaning between f and $f(x)$.

9 On the same set of axes, draw the graphs of three different functions $f(x)$ such that $f(2) = 1$ and $f(5) = 3$.

10 Find a linear function $f(x) = ax + b$ for which $f(2) = 1$ and $f(-3) = 11$.

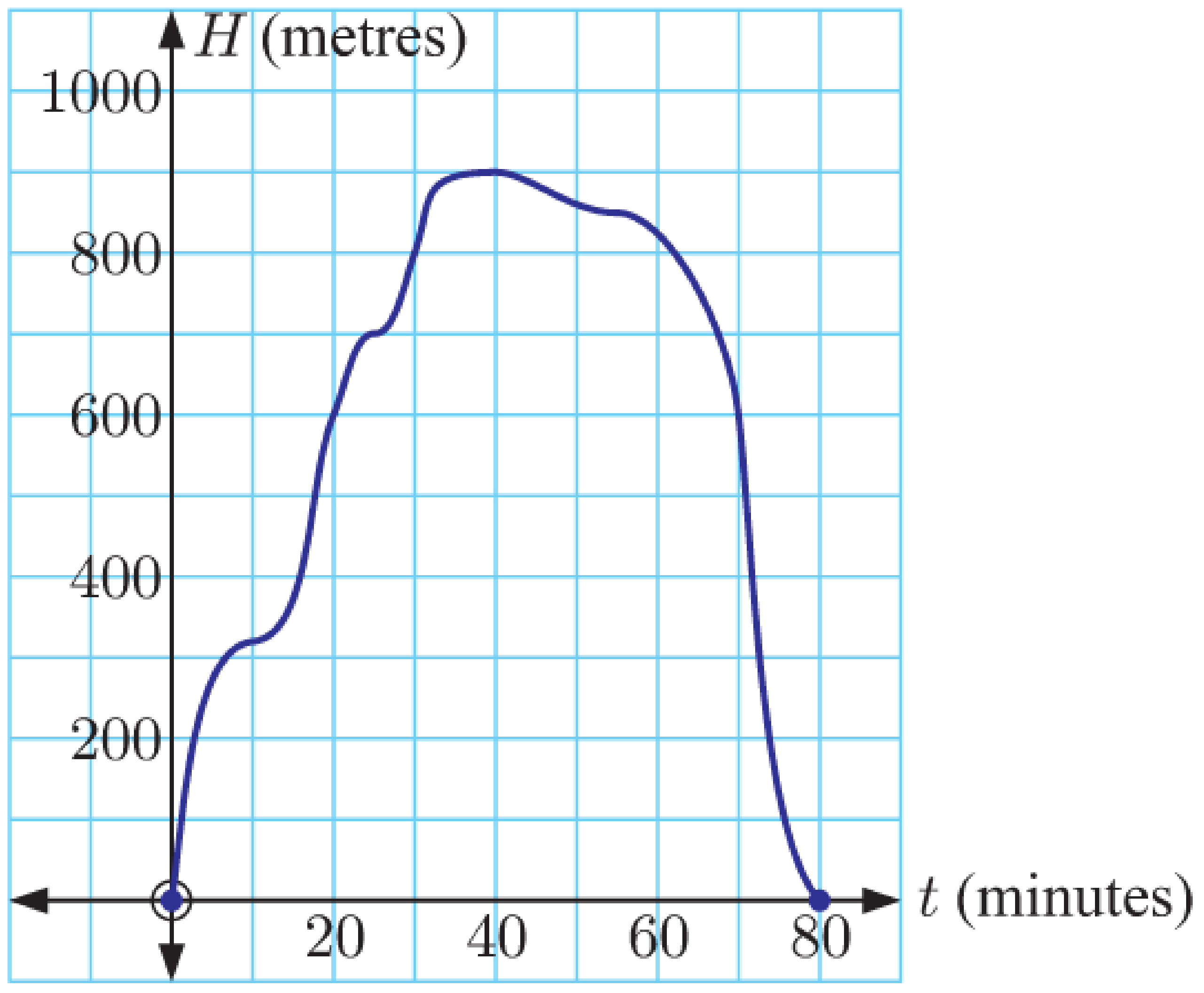
11 For a hot air balloon ride, the function $H(t)$ gives the height of the balloon after t minutes. Its graph is shown alongside.

a Find $H(30)$, and explain what your answer means.

b Find the values of t such that $H(t) = 600$. Interpret your answer.

c For what values of t was the height of the balloon recorded?

d What range of heights was recorded for the balloon?



12 Given $f(x) = ax + \frac{b}{x}$, $f(1) = 1$, and $f(2) = 5$, find constants a and b.

13 The quadratic function $T(x) = ax^2 + bx + c$ has the values $T(0) = -4$, $T(1) = -2$, and $T(2) = 6$. Find a, b, and c.

14 The value of a photocopier t years after purchase is given by $V(t) = 9000 - 900t$ pounds.

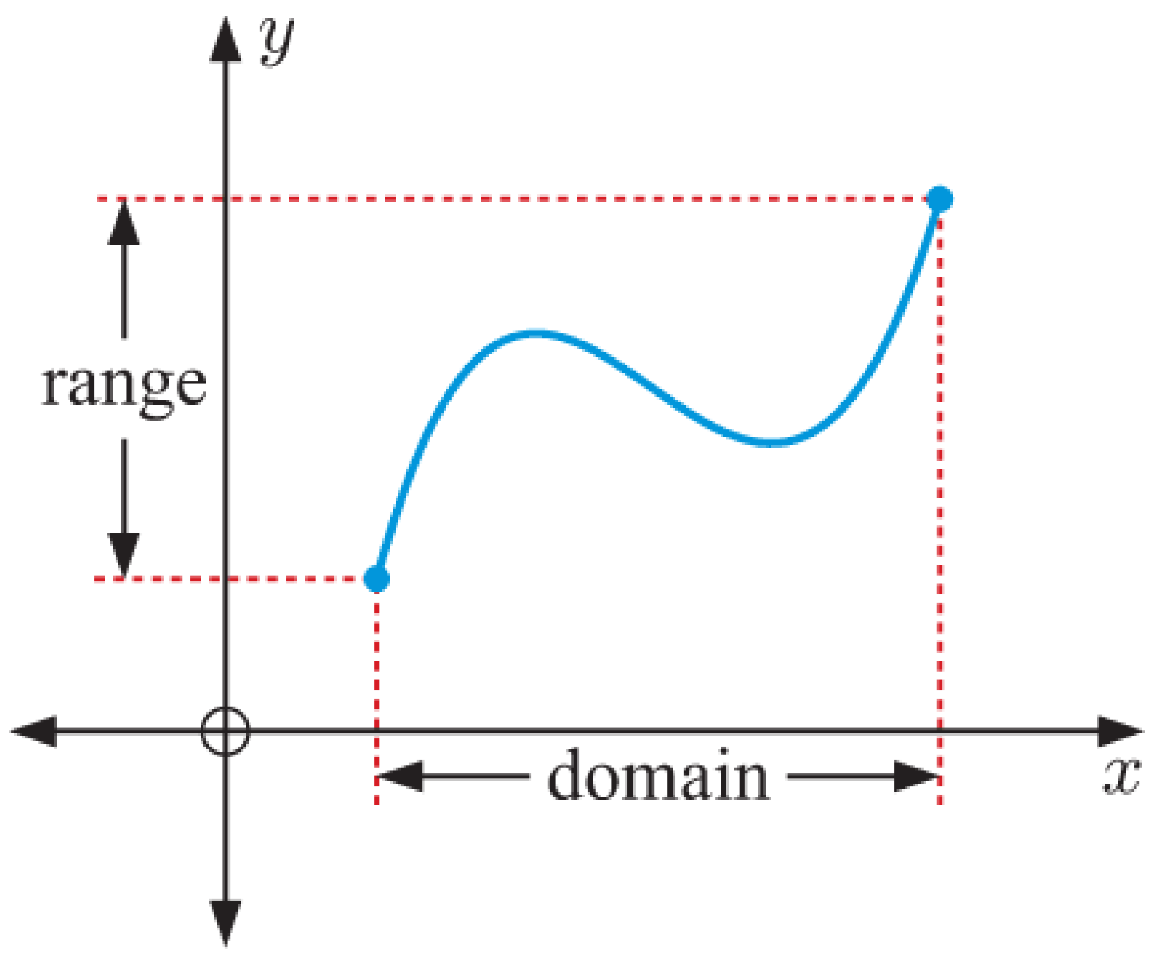
- **a** Find $V(4)$, and state what $V(4)$ means.
- **b** Find t when $V(t) = 3600$, and explain what this means.
- **c** Find the original purchase price of the photocopier.
- **d** For what values of t is it reasonable to use this function?

C DOMAIN AND RANGE

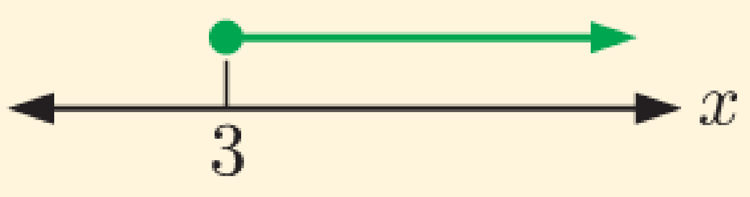
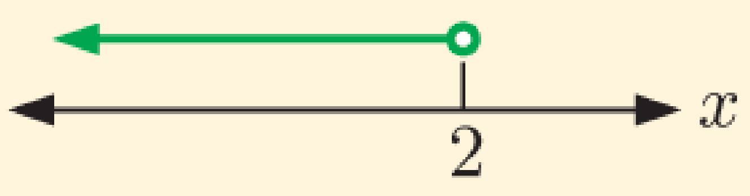
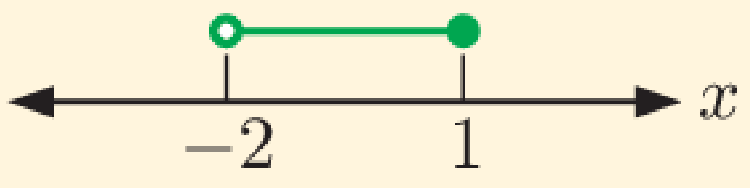
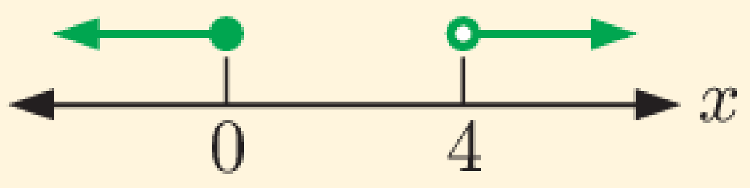
We have seen that a relation is a set of points which connects two variables.

The **domain** of a relation is the set of values which the variable on the horizontal axis can take. This variable is usually x.

The **range** of a relation is the set of values which the variable on the vertical axis can take. This variable is usually y.



The domain and range of a relation can be described in several ways. Examples are given in the table below:

Set notation	*Bracket notation*	*Number line graph*	*Meaning*
$\{x \mid x \geqslant 3\}$	$x \in [3, \infty[$		the set of all x such that x is greater than or equal to 3
$\{x \mid x < 2\}$	$x \in]-\infty, 2[$		the set of all x such that x is less than 2
$\{x \mid -2 < x \leqslant 1\}$	$x \in]-2, 1]$		the set of all x such that x is between -2 and 1, including 1
$\{x \mid x \leqslant 0 \text{ or } x > 4\}$	$x \in]-\infty, 0]$ or $]4, \infty[$		the set of all x such that x is less than or equal to 0, or greater than 4

DOMAIN AND RANGE OF FUNCTIONS

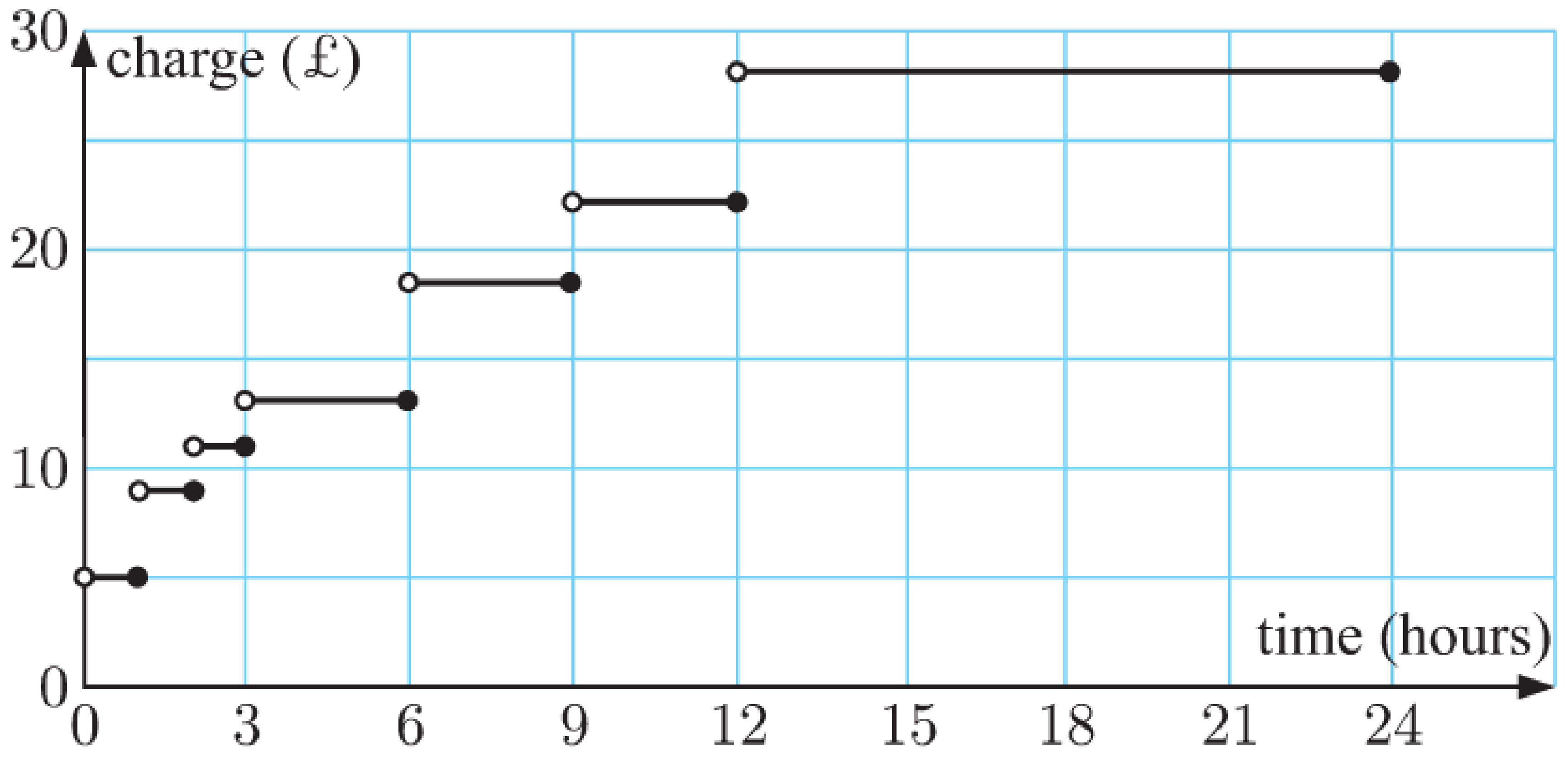
To find the domain and range of a function, we can observe its graph. For example:

(1) In the **Opening Problem**, the car park charges function is defined for all times t such that $0 < t \leqslant 24$.

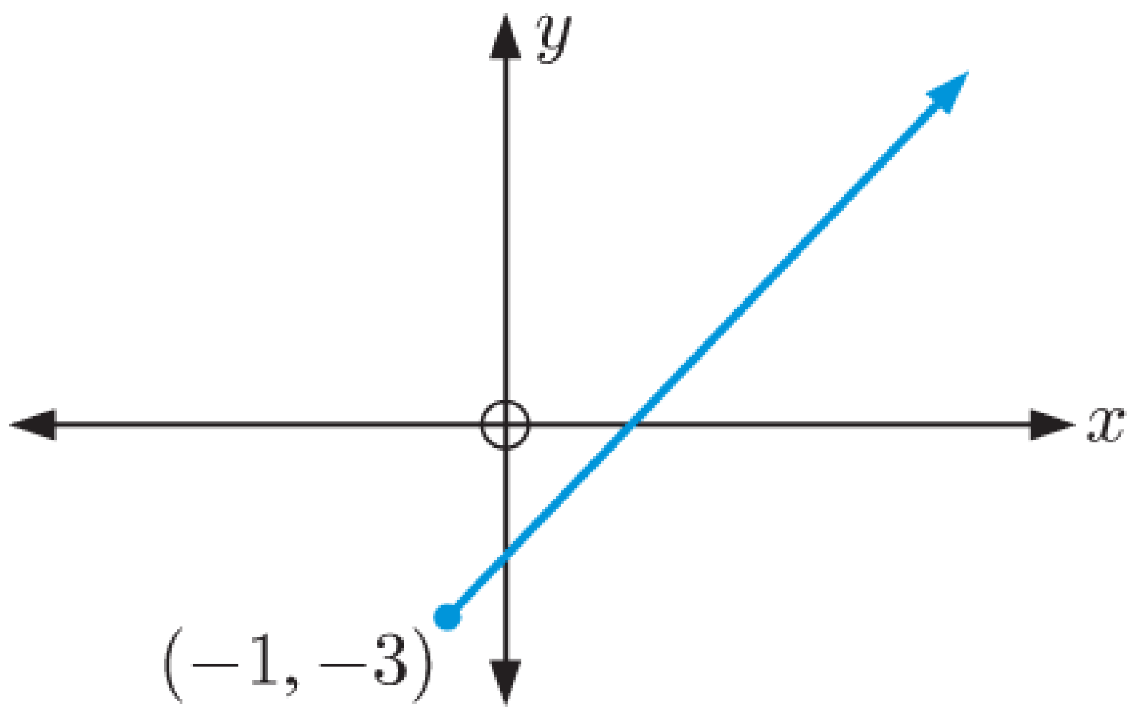
$\therefore$ the domain is $\{t \mid 0 < t \leqslant 24\}$.

The possible charges are £5, £9, £11, £13, £18, £22, and £28.

$\therefore$ the range is $\{5, 9, 11, 13, 18, 22, 28\}$.

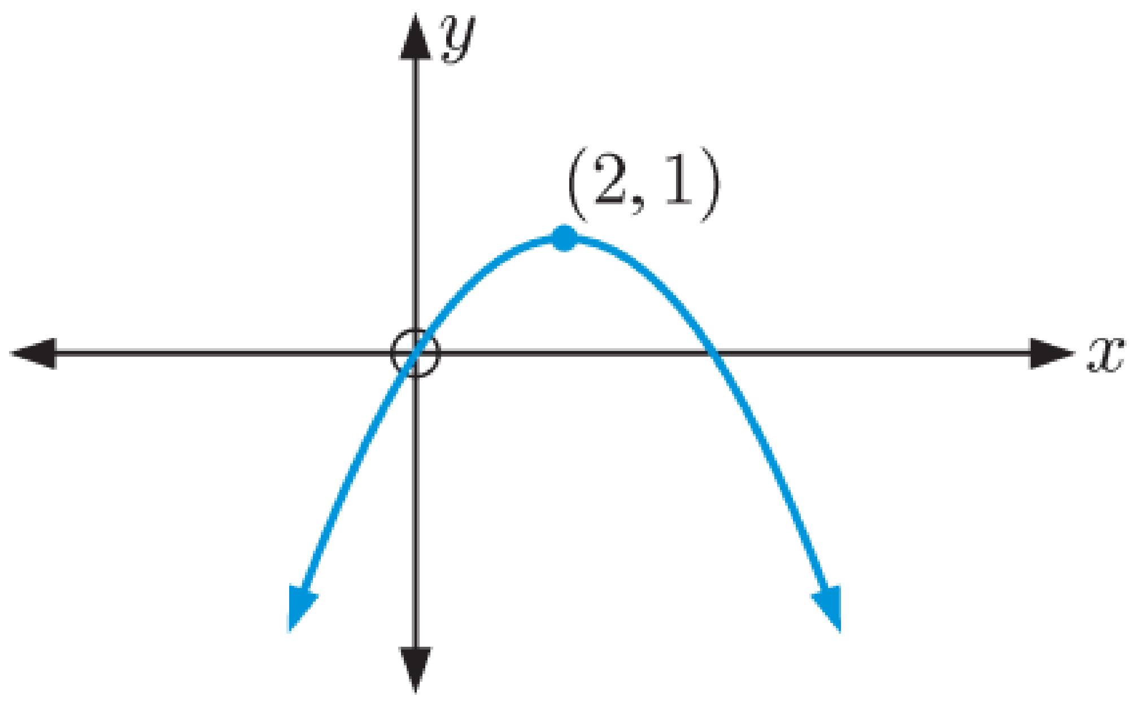


(2)



All values of $x \geqslant -1$ are included, so the domain is $\{x \mid x \geqslant -1\}$.
All values of $y \geqslant -3$ are included, so the range is $\{y \mid y \geqslant -3\}$.

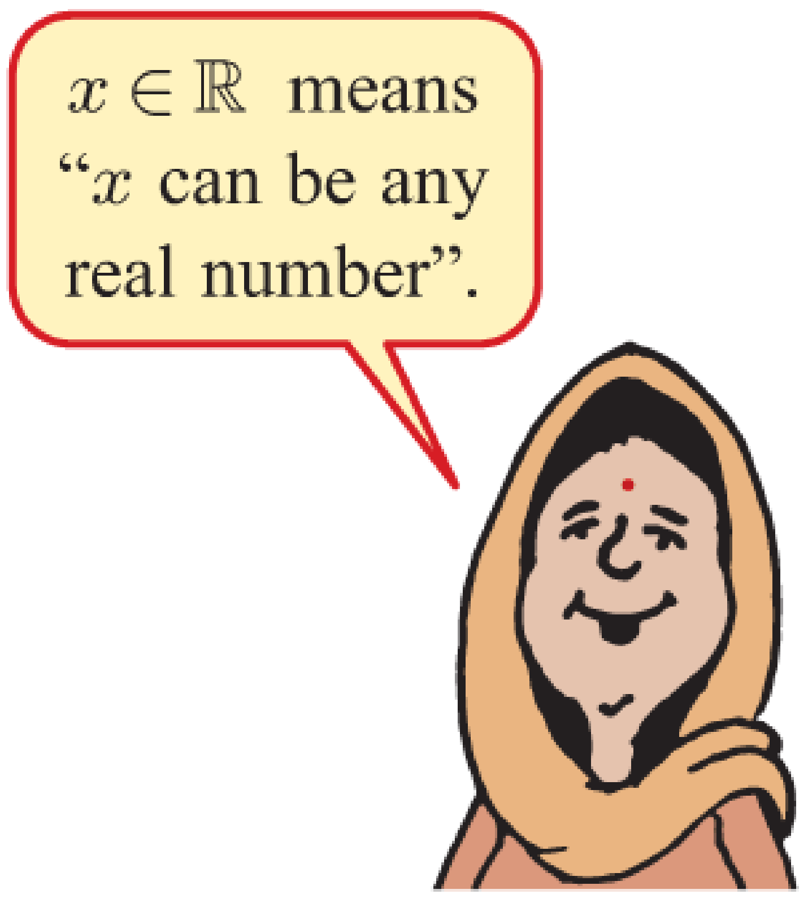
(3)



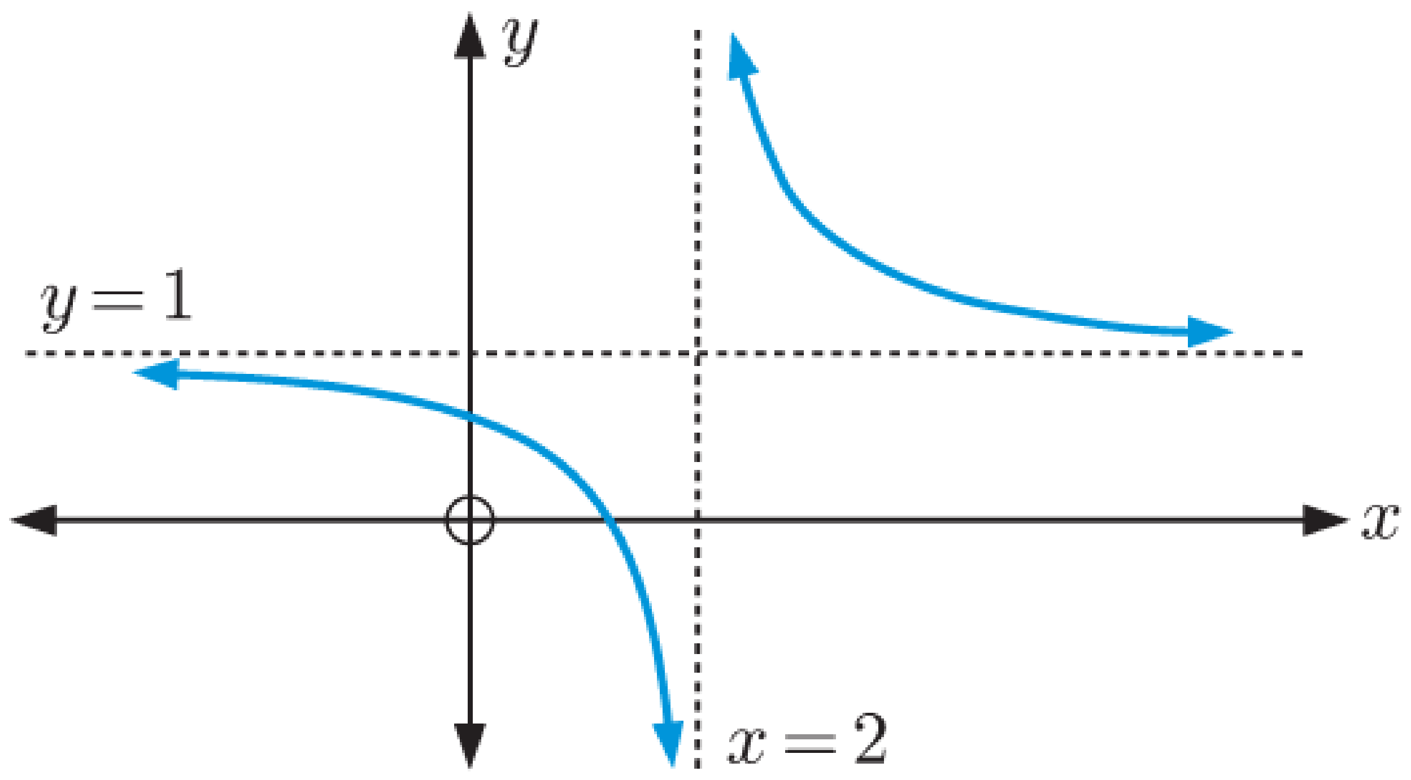
x can take any value, so the domain is

$\{x \in \mathbb{R}\}$ or $x \in \mathbb{R}$.

y cannot be > 1, so the range is $\{y \mid y \leqslant 1\}$.



(4)



x can take all values except 2, so the domain is $\{x \mid x \neq 2\}$ or $x \neq 2$.
y can take all values except 1, so the range is $\{y \mid y \neq 1\}$ or $y \neq 1$.

To fully describe a function, we need both a rule *and* a domain.

For example, we can specify $f(x) = x^2$ where $x \geqslant 0$.

If a domain is not specified, we use the **natural domain**, which is the largest part of $\mathbb{R}$ for which $f(x)$ is defined.

$f(x)$	*Natural domain*
x^2	$x \in \mathbb{R}$
$\sqrt{x}$	$x \geqslant 0$
$\frac{1}{x}$	$x \neq 0$
$\frac{1}{\sqrt{x}}$	$x > 0$

Some examples of natural domains are shown in the table opposite.

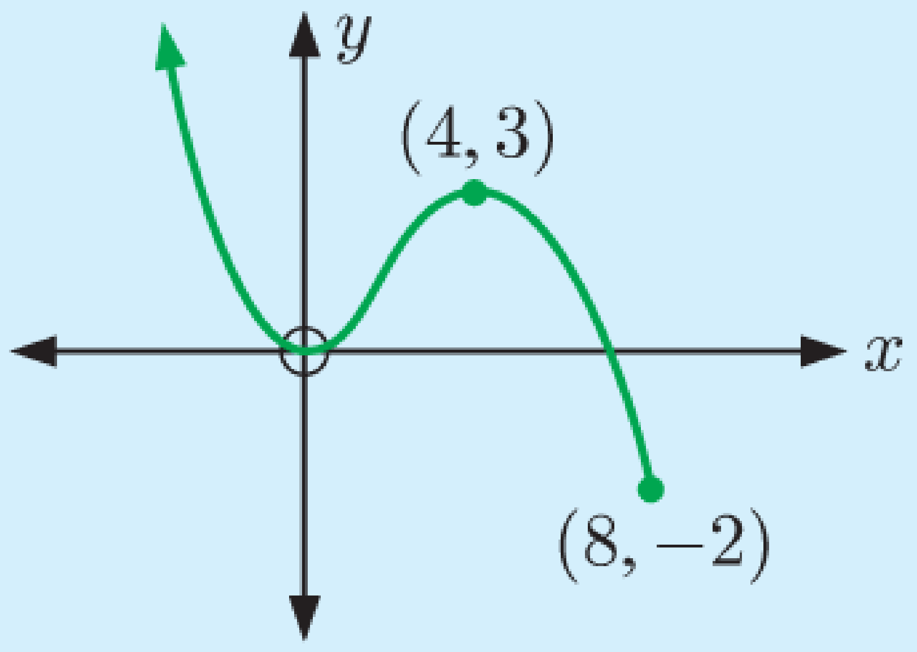
Click on the icon to obtain software for finding the natural domain and range of different functions.

DOMAIN AND RANGE

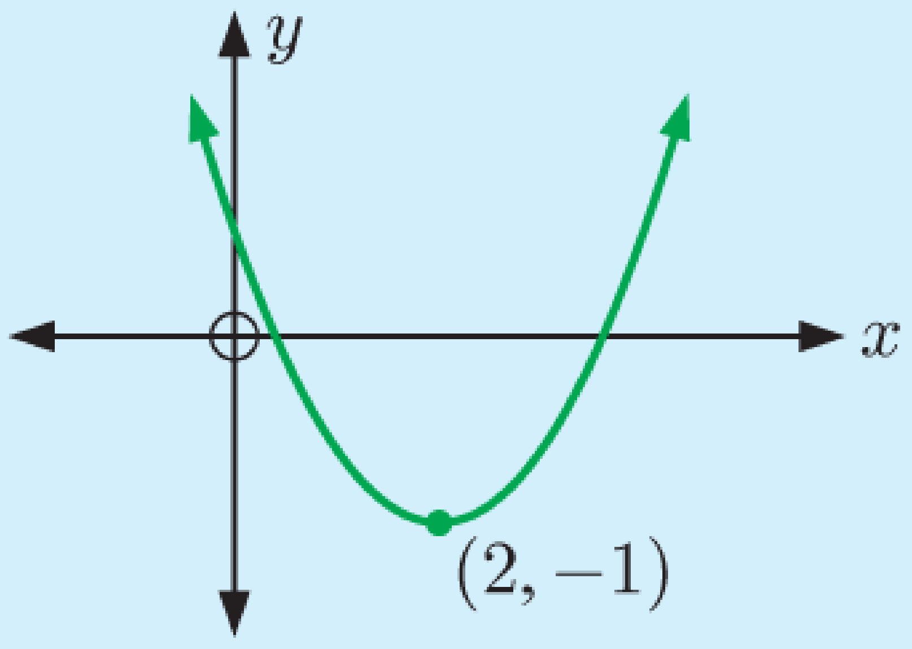
Example 4 **Self Tutor**

For each of the following graphs, state the domain and range:

a



b



a Domain is $\{x \mid x \leqslant 8\}$.
Range is $\{y \mid y \geqslant -2\}$.

b Domain is $\{x \in \mathbb{R}\}$.
Range is $\{y \mid y \geqslant -1\}$.

EXERCISE 15C

1 A driver who exceeds the speed limit receives demerit points as shown in the table.

a Draw a graph to display this information.

b Is the relation a function? Explain your answer.

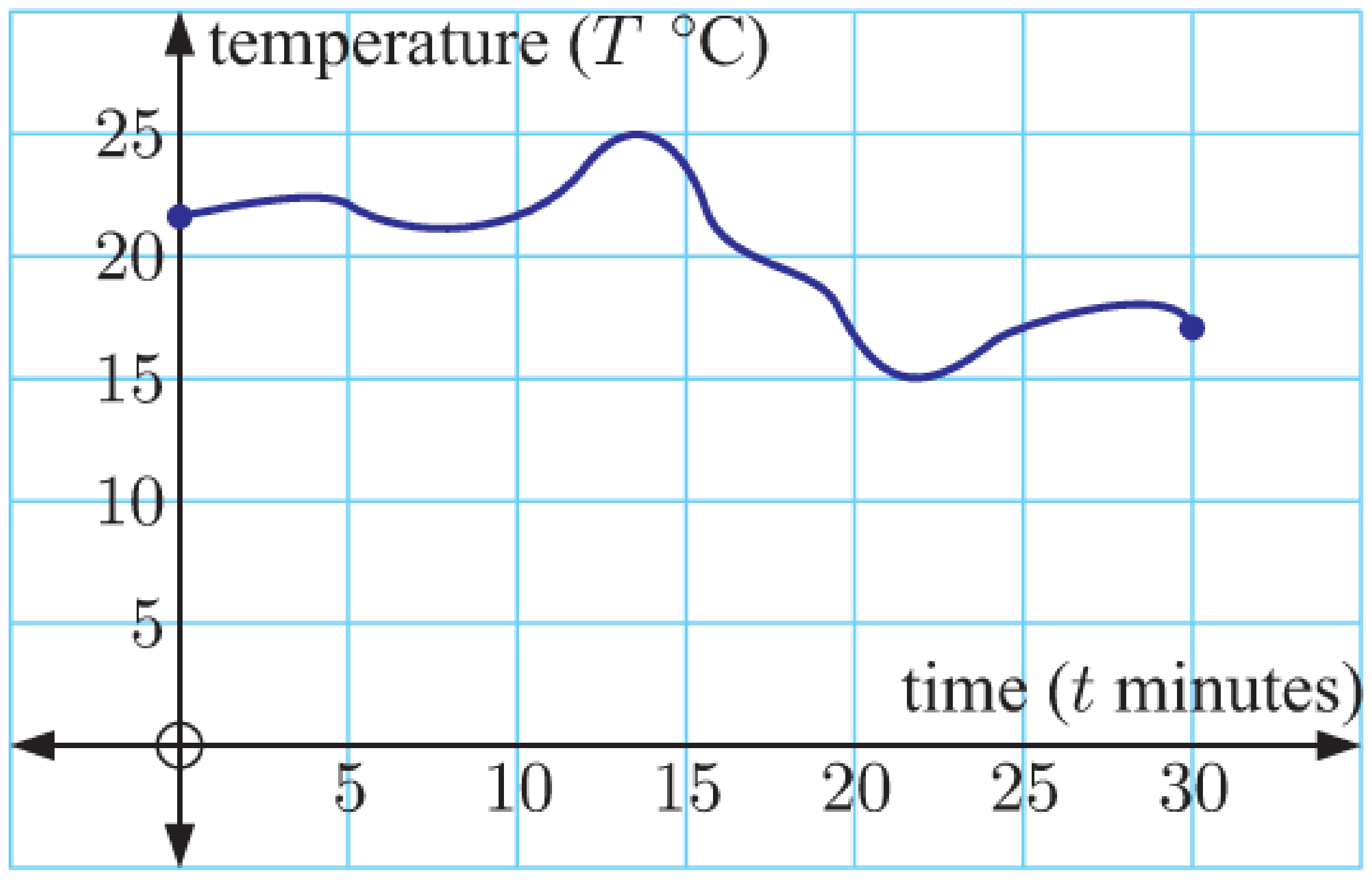
c Find the domain and range of the relation.

Amount over speed limit (x km h^{-1})	*Demerit points* (y)
$0 < x < 10$	2
$10 \leqslant x < 20$	3
$20 \leqslant x < 30$	5
$30 \leqslant x < 45$	7
$x \geqslant 45$	9

2 This graph shows the temperature in Barcelona over a 30 minute period as the wind shifts.

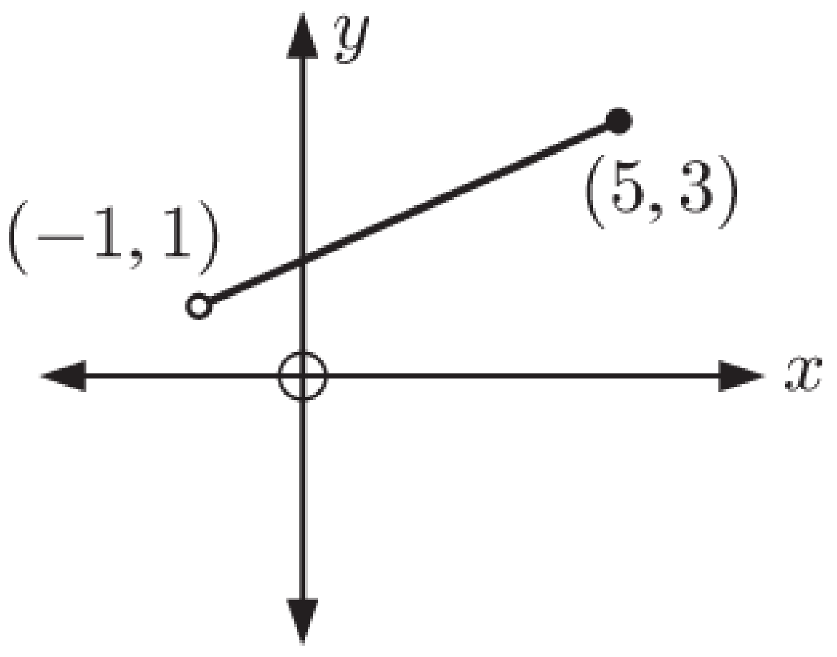
a Explain why a temperature graph like this must be a function.

b Find the domain and range of the function.

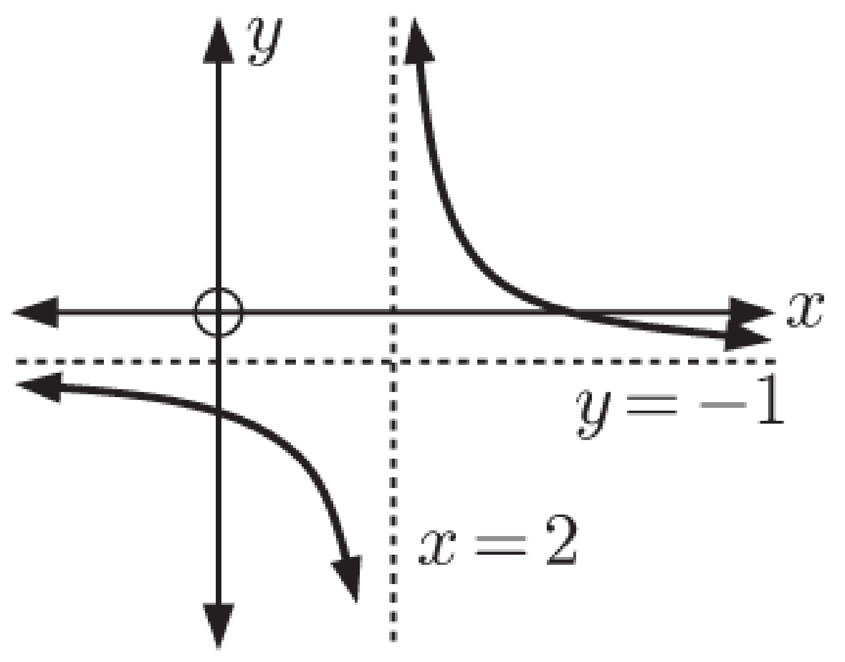


3 For each of the following graphs, find the domain and range:

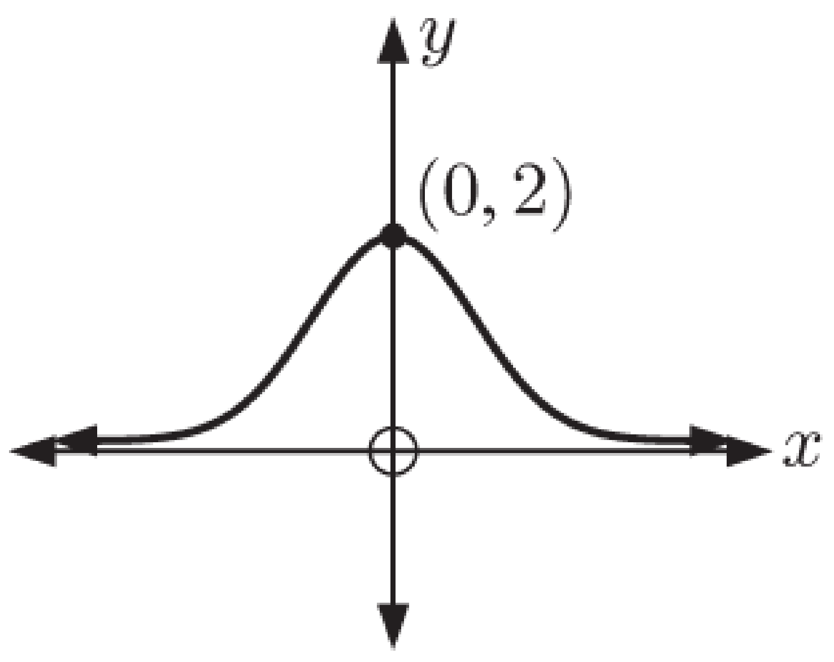
a



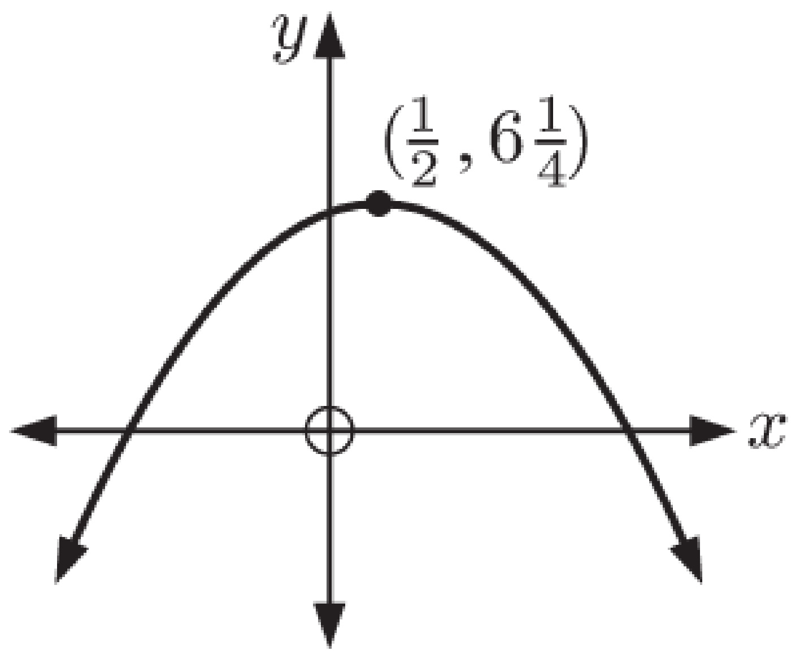
b



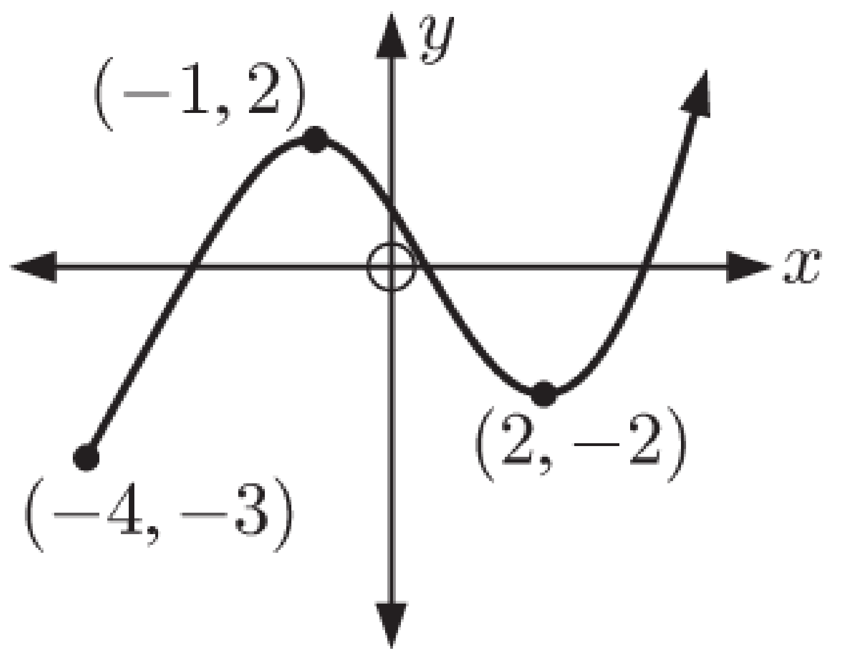
c



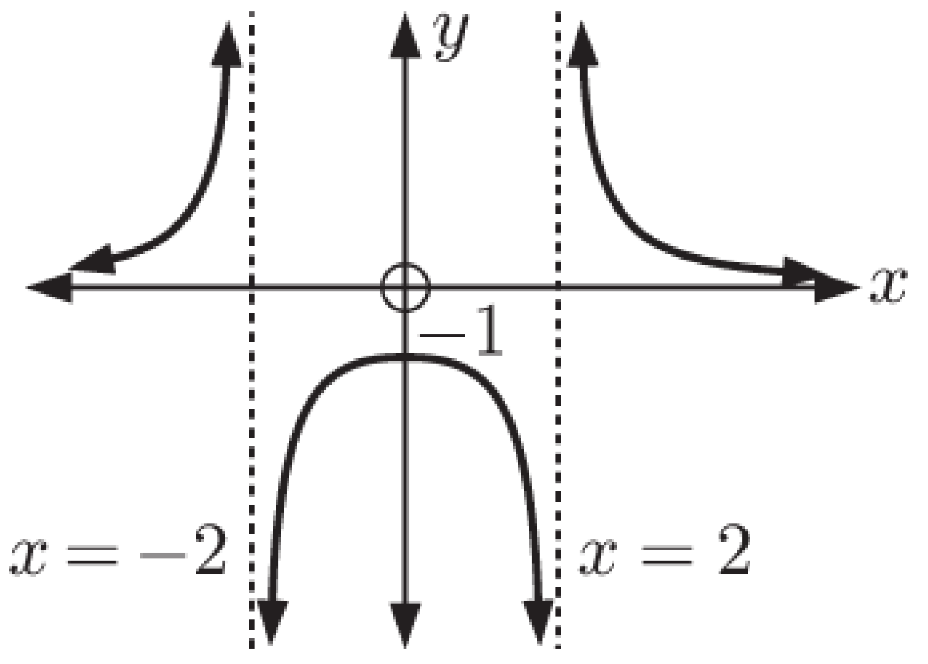
d



e

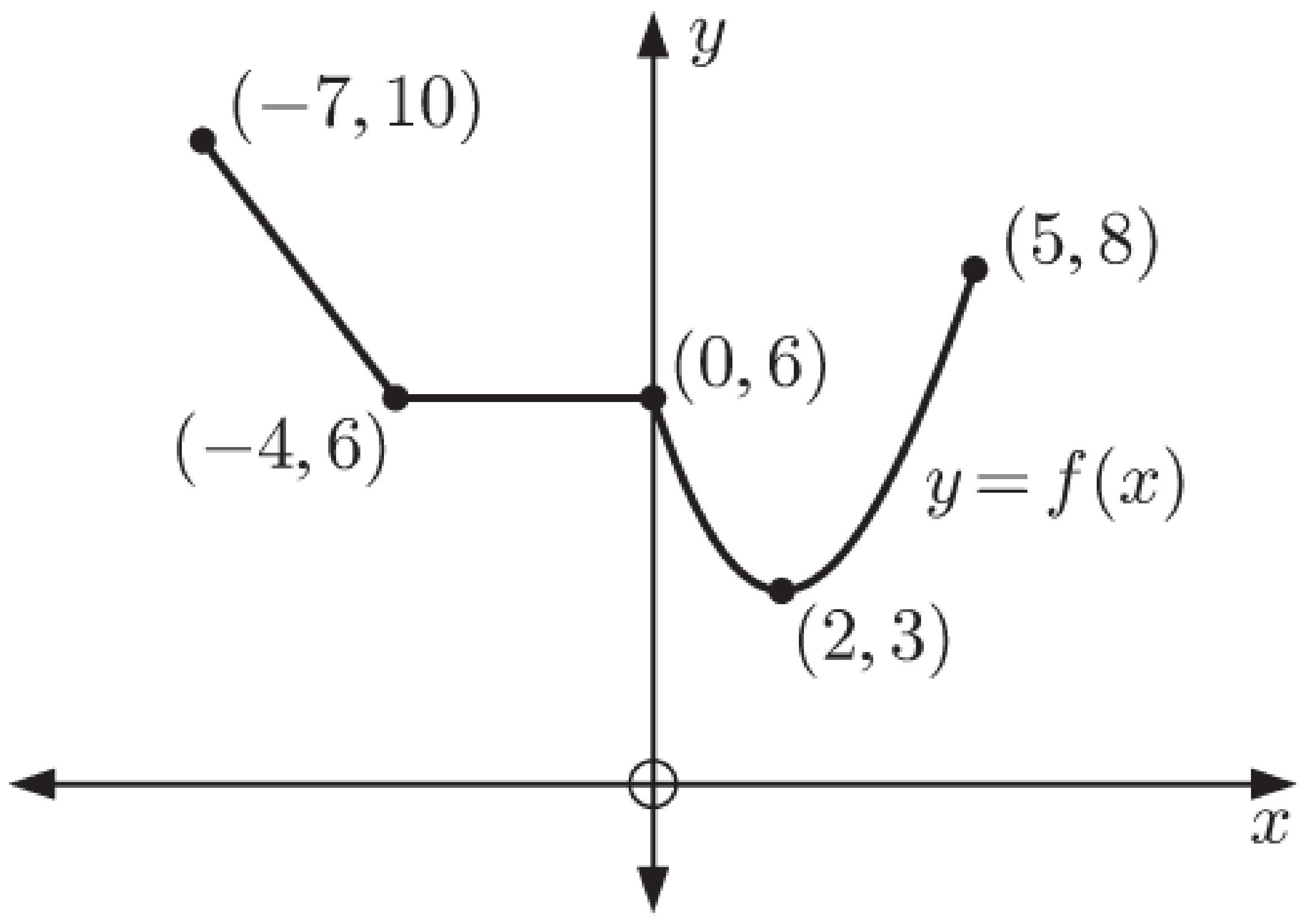


f



4 Consider the graph of $y = f(x)$ alongside.
Decide whether each statement is true or false:

- **a** -5 is in the domain of f.
- **b** 2 is in the range of f.
- **c** 9 is in the range of f.
- **d** $\sqrt{2}$ is in the domain of f.



5 Use quadratic theory to find the range of each function:

- **a** $y = x^2$
- **b** $y = -x^2$
- **c** $y = x^2 + 2$
- **d** $y = -2(x+3)^2$
- **e** $y = 1 - (x-2)^2$
- **f** $y = (2x+1)^2 + 3$
- **g** $y = x^2 - 7x + 10$
- **h** $y = -x^2 + 2x + 8$
- **i** $f : x \mapsto 5x - 3x^2$

Example 5

Self Tutor

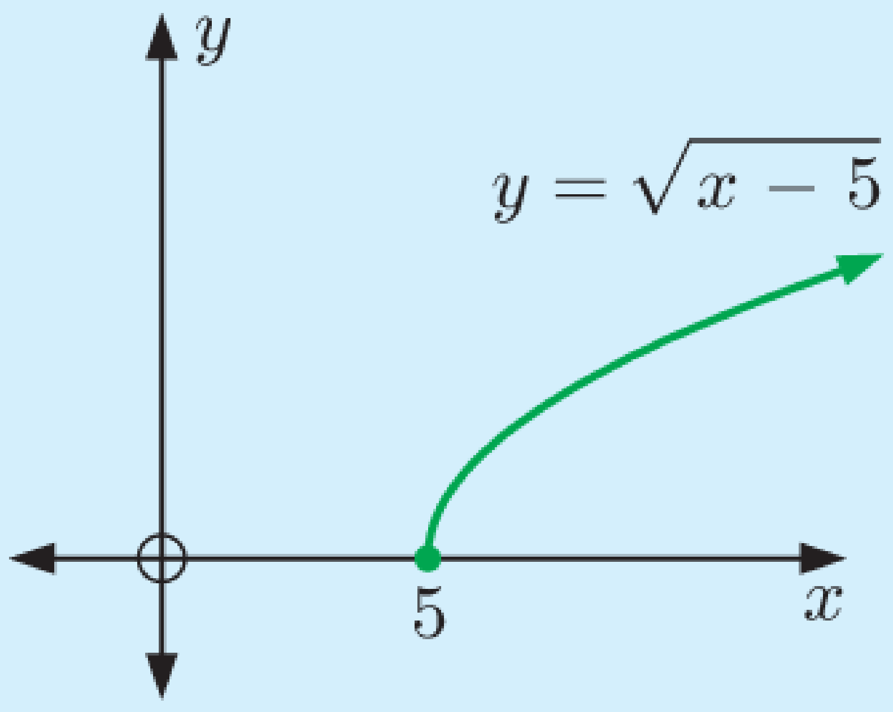
State the domain and range of each of the following functions:

a $f(x) = \sqrt{x-5}$ **b** $f(x) = \dfrac{1}{x-5}$ **c** $f(x) = \dfrac{1}{\sqrt{x-5}}$

a $\sqrt{x-5}$ is defined when $x - 5 \geqslant 0$

$\therefore \; x \geqslant 5$

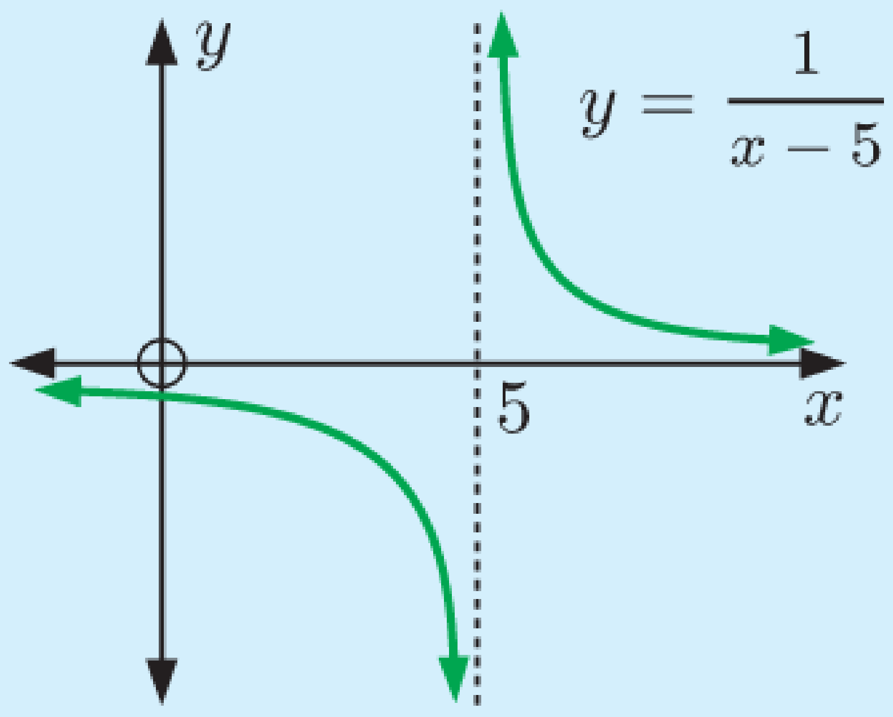
$\therefore$ the domain is $\{x \mid x \geqslant 5\}$.
A square root cannot be negative.
$\therefore$ the range is $\{y \mid y \geqslant 0\}$.



b $\dfrac{1}{x-5}$ is defined when $x - 5 \neq 0$

$\therefore \; x \neq 5$

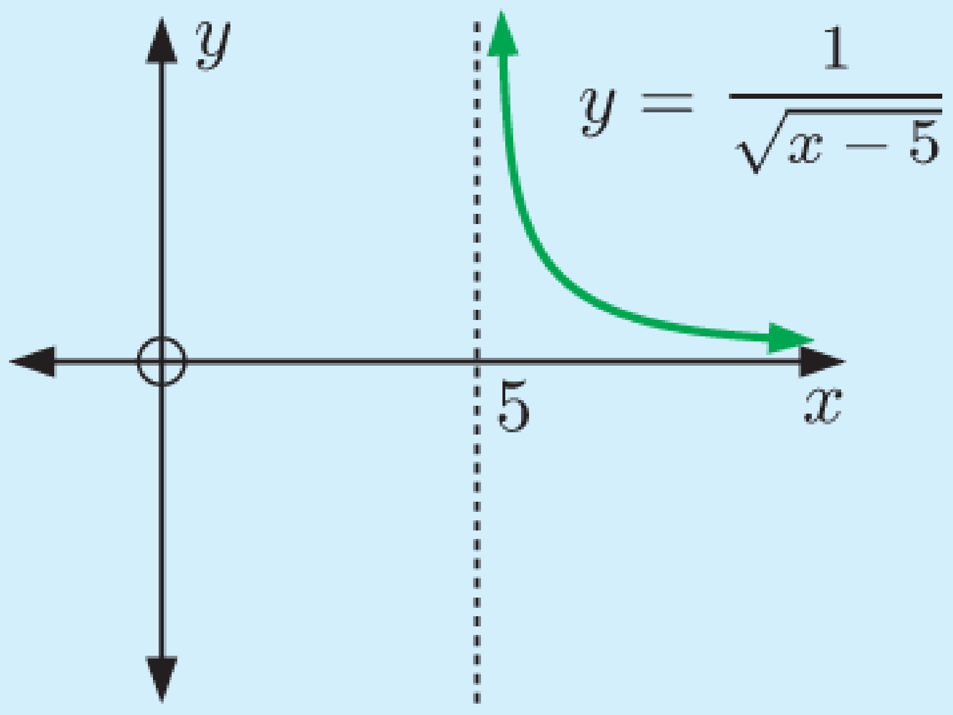
$\therefore$ the domain is $\{x \mid x \neq 5\}$.
No matter how large or small x is,
$y = f(x)$ is never zero.
$\therefore$ the range is $\{y \mid y \neq 0\}$.



c $\dfrac{1}{\sqrt{x-5}}$ is defined when $x - 5 > 0$

$\therefore \; x > 5$

$\therefore$ the domain is $\{x \mid x > 5\}$.
$y = f(x)$ is always positive and never zero.
$\therefore$ the range is $\{y \mid y > 0\}$.



6 Consider the function $f(x)=\sqrt{x}$.

a State the domain of the function.

b Copy and complete this table of values:

x	0	1	4	9	16
$f(x)$					

DOMAIN AND RANGE

c Hence sketch the graph of the function.

d Find the range of the function.

7 State the domain and range of each function:

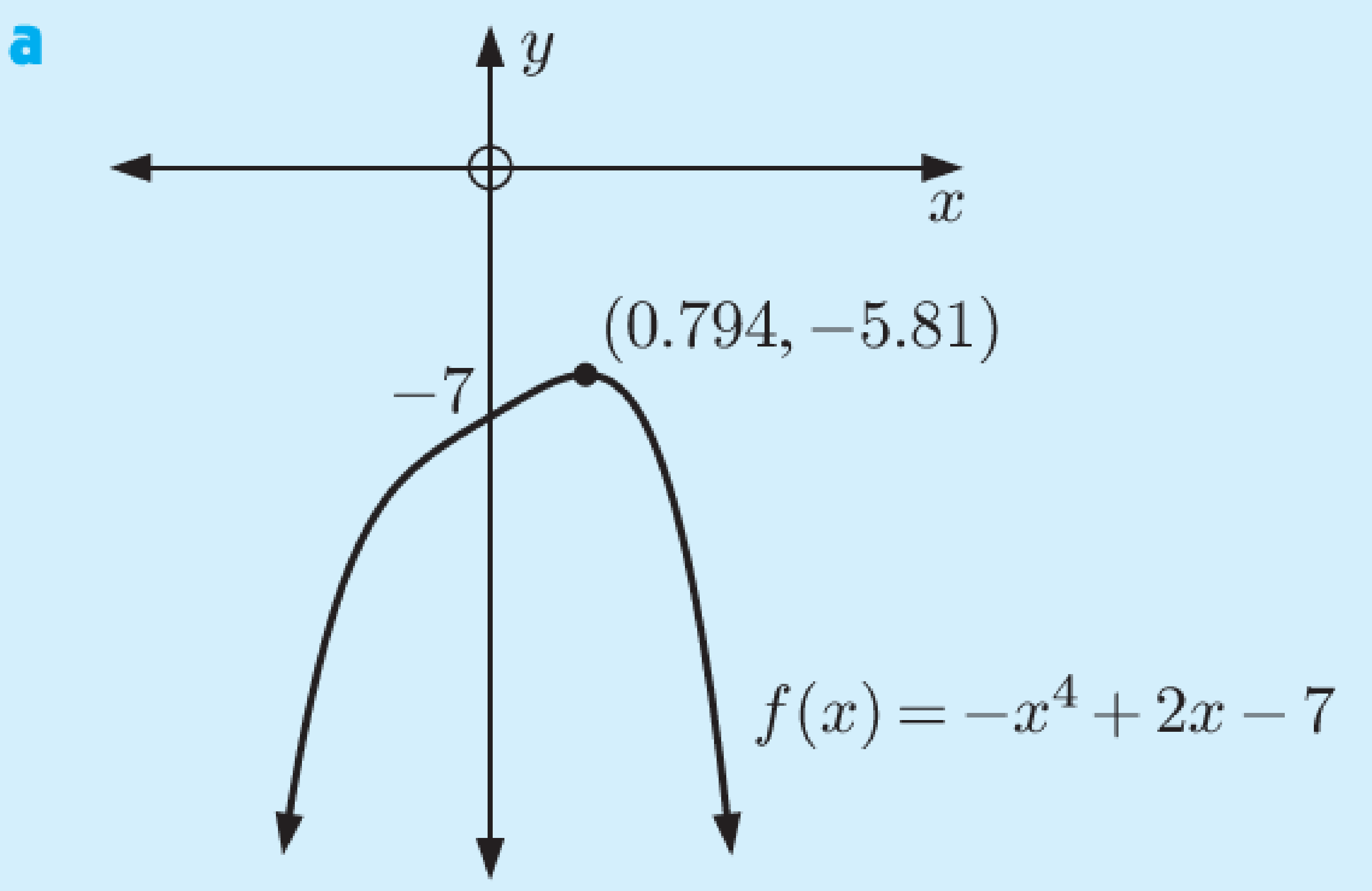
a $f(x)=\sqrt{x+6}$ **b** $f: x \mapsto \dfrac{1}{x^2}$ **c** $f(x)=\dfrac{1}{x+1}$

d $y=-\dfrac{1}{\sqrt{x}}$ **e** $f: x \mapsto \dfrac{1}{3-x}$ **f** $f: x \mapsto \sqrt{4-x}$

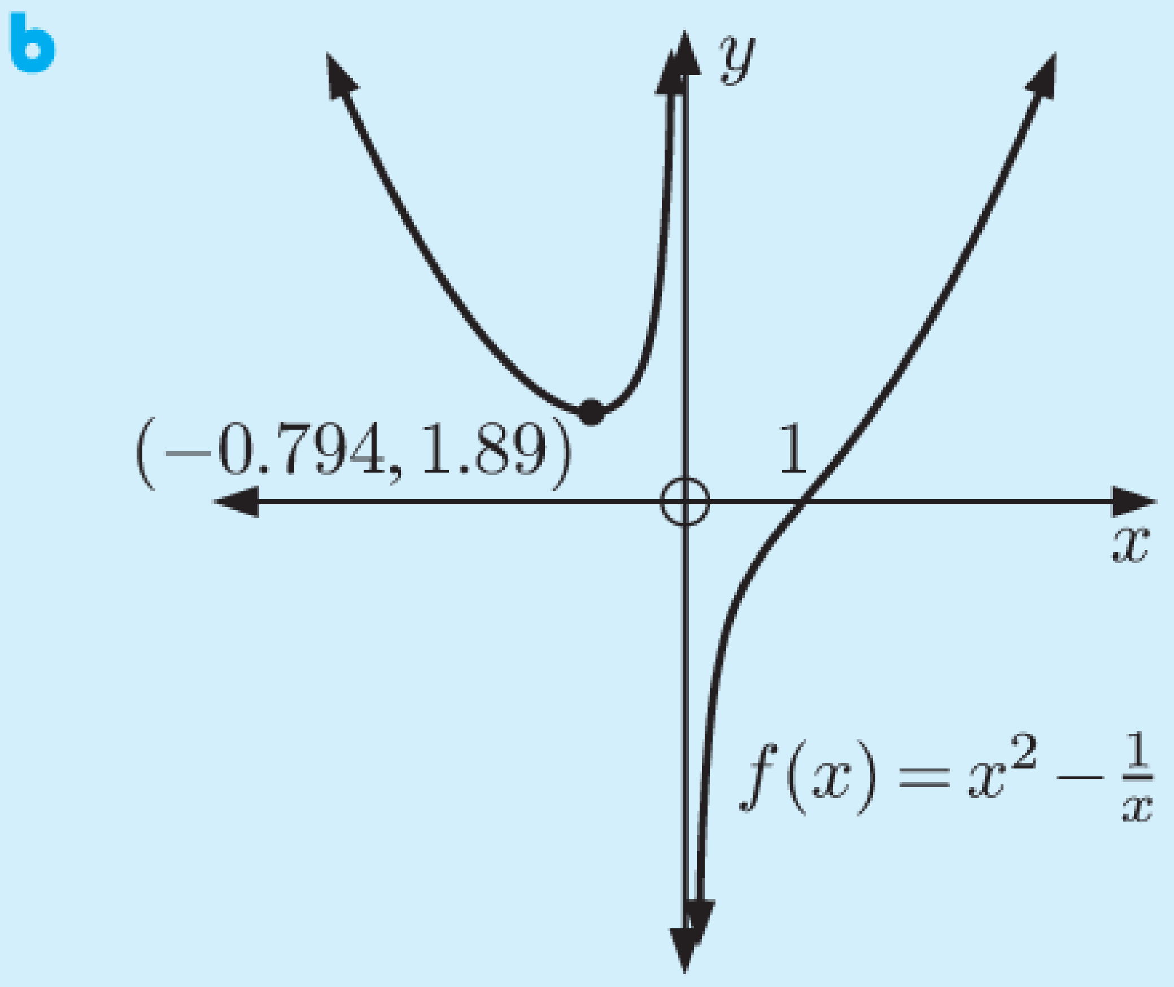
Example 6 **Self Tutor**

Use technology to help sketch these functions. Locate any turning points. Hence state the domain and range of the function.

a $f(x)=-x^4+2x-7$ **b** $f(x)=x^2-\dfrac{1}{x}$



The domain is $\{x \mid x \in \mathbb{R}\}$.
The range is $\{y \mid y \leqslant -5.81\}$.



The domain is $\{x \mid x \neq 0\}$.
The range is $\{y \mid y \in \mathbb{R}\}$.

8 Use technology to help sketch these functions. Locate any turning points. Hence state the domain and range of the function.

GRAPHING PACKAGE

a $f(x)=x^3-3x^2-9x+10$ **b** $f(x)=x^4+4x^3-16x+3$

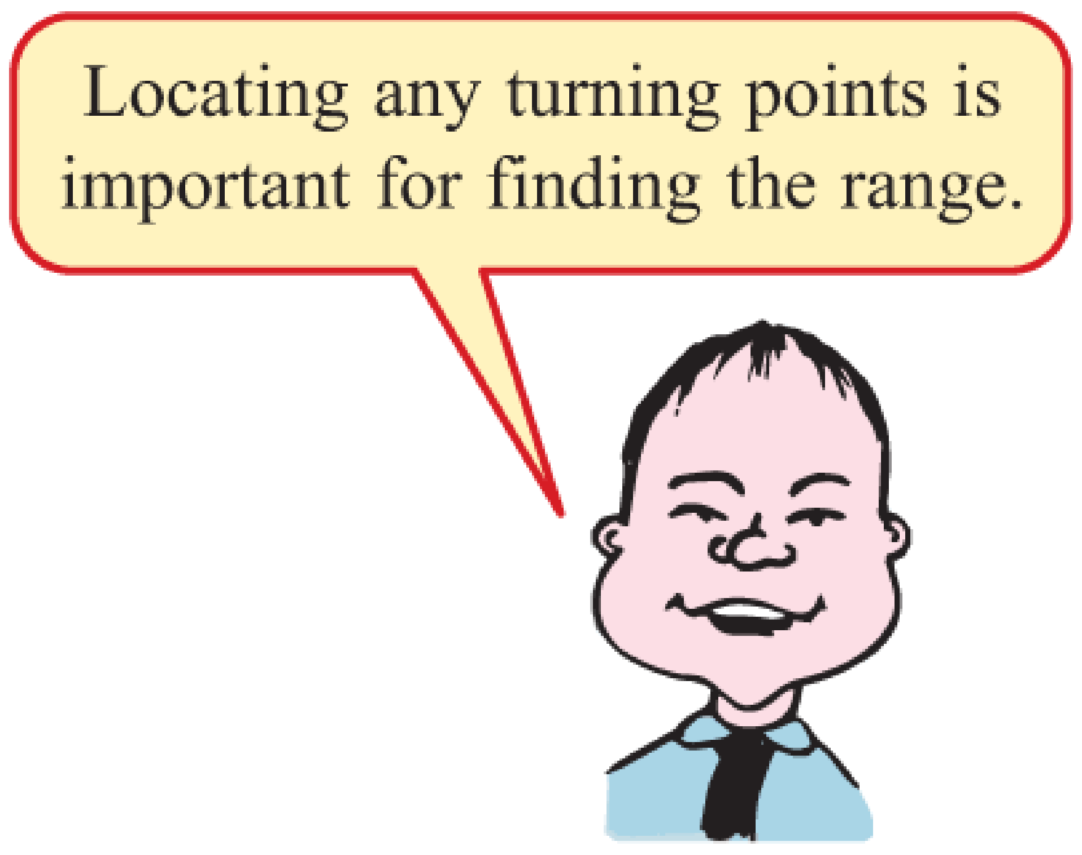
c $f(x)=\sqrt{x^2+4}$ **d** $f(x)=\sqrt{x^2-4}$

e $f(x)=\sqrt{9-x^2}$ **f** $f(x)=\dfrac{x+4}{x-2}$

g $f(x)=\dfrac{3x-9}{x^2-x-2}$ **h** $f(x)=x+\dfrac{1}{x}$

i $f(x)=x^2+\dfrac{1}{x^2}$ **j** $f(x)=x^3+\dfrac{1}{x^3}$

k $f(x)=3^x$ **l** $f(x)=x\,2^{-x}$



9 Use technology to sketch these functions on their given domain. Locate the points at the end(s) of the domain, as well as any turning points. Hence state the range of the function.

- **a** $y = -x^4 + 2x^3 + 5x^2 + x + 2, \quad 0 \leqslant x \leqslant 4$
- **b** $y = -2x^4 + 5x^2 + x + 2, \quad -2 \leqslant x \leqslant 2$
- **c** $y = \dfrac{1}{1 + 2^{-x}}, \quad x > 0$

10 The function $f(x) = \sqrt{x^2 + 5x + k}$ has natural domain $x \in \mathbb{R}$.

- **a** Find the possible values of k.
- **b** Find the range of $f(x)$ in terms of k.

11 State the domain and range of each relation:

- **a** $\{(x, y) \mid x^2 + y^2 = 4\}$
- **b** $\{(x, y) \mid x^2 + y^2 = 4, \ x \in \mathbb{Z}\}$

D RATIONAL FUNCTIONS

Linear and quadratic functions are the first members of a family called the **polynomials**. The polynomials can all be written in the form $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$

When a polynomial is divided by another polynomial, we call it a **rational function**.

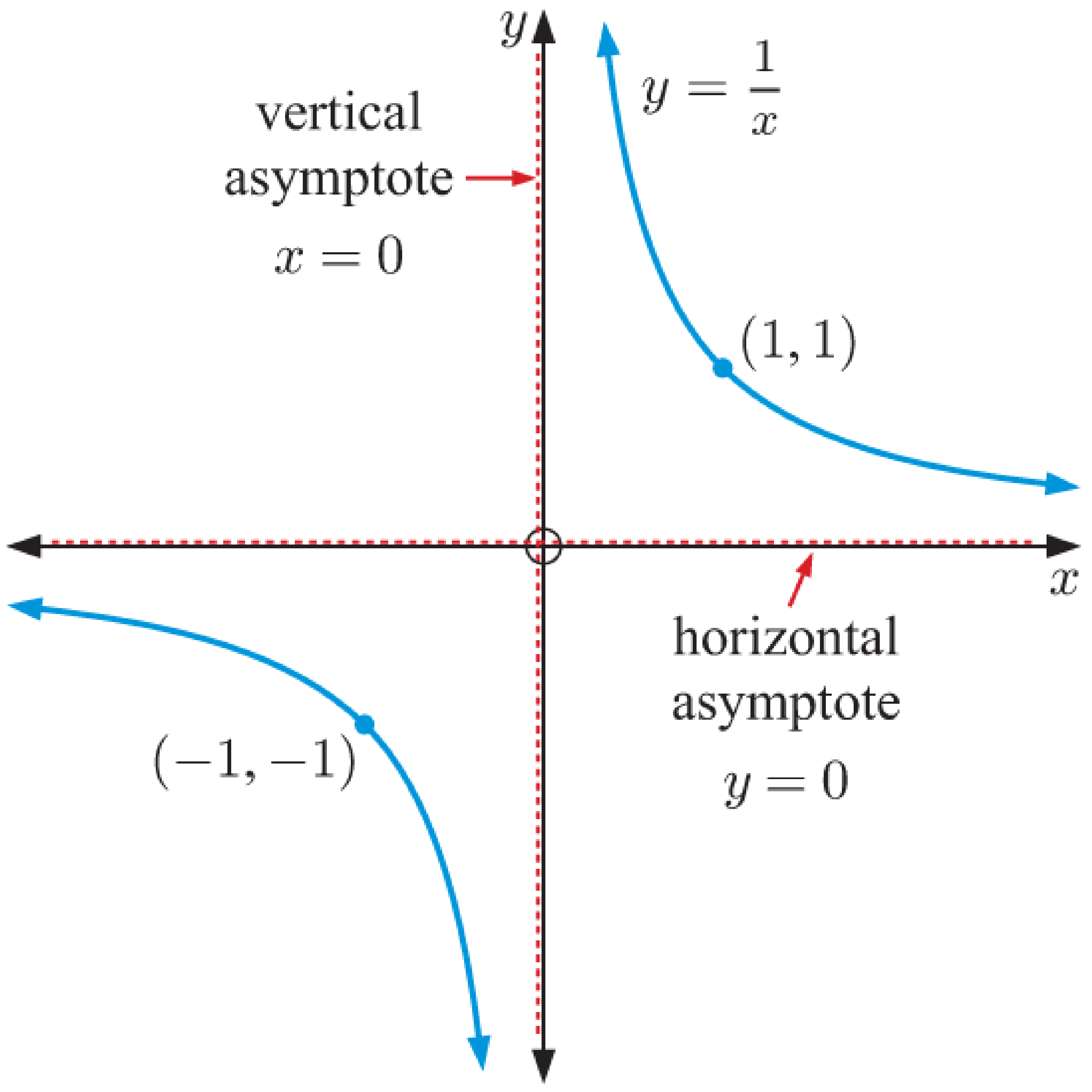
In this Section we consider only the simplest cases of a linear function divided by another linear function.

RECIPROCAL FUNCTIONS

A **reciprocal function** is a function of the form $y = \dfrac{k}{x}, \quad k \neq 0$.

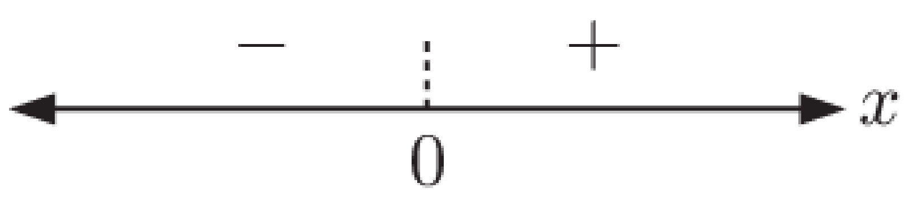
The graph of a reciprocal function is called a **rectangular hyperbola**.

The simplest example of a reciprocal function is $f(x) = \dfrac{1}{x}$. Its graph is shown below.



Notice that:

- The graph has two branches.
- $y = \dfrac{1}{x}$ is undefined when $x = 0$, so the domain is $\{x \mid x \neq 0\}$.
  On a sign diagram, we indicate this value with a dashed line.

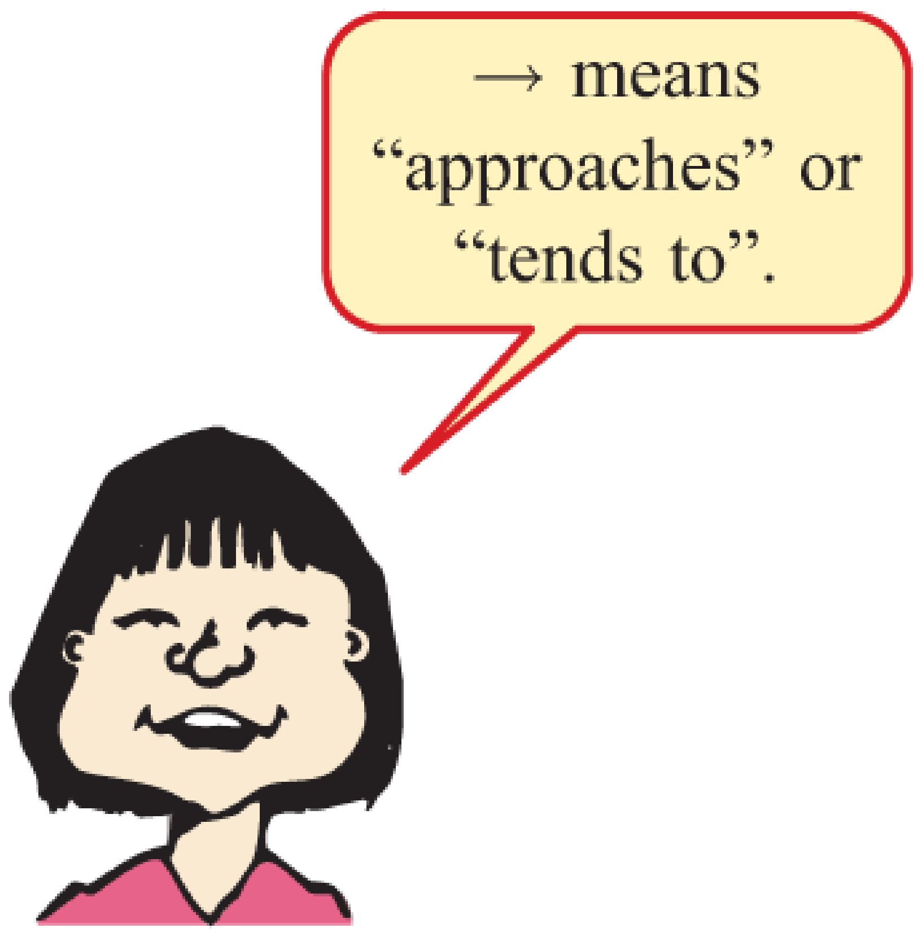


- The graph includes two **asymptotes**, which are lines the graph approaches but never reaches.
  - $x = 0$ is a **vertical asymptote**.

    We write: as $x \to 0^-$, $\dfrac{1}{x} \to -\infty$

    as $x \to 0^+$, $\dfrac{1}{x} \to \infty$

    When "as $x \to 0^+$, $\dfrac{1}{x} \to \infty$" is read out loud, we say "as x tends to zero from the right, $\dfrac{1}{x}$ tends to infinity."



- $y = 0$ is a **horizontal asymptote**.

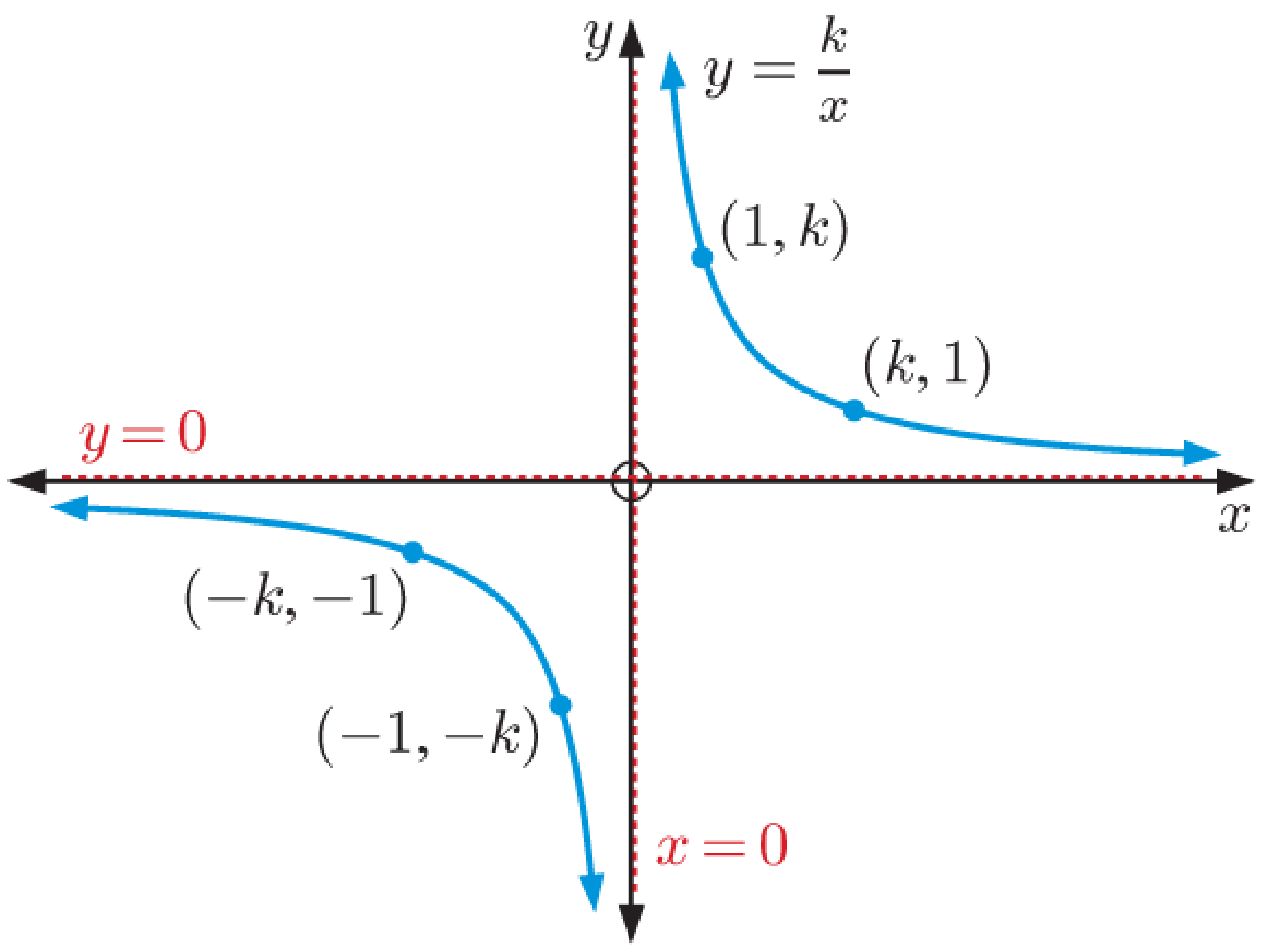
  We write: as $x \rightarrow \infty$, $\frac{1}{x} \rightarrow 0^+$

  as $x \rightarrow -\infty$, $\frac{1}{x} \rightarrow 0^-$

  When "as $x \rightarrow \infty$, $\frac{1}{x} \rightarrow 0^+$" is read out loud, we say "as x tends to infinity, $\frac{1}{x}$ tends to zero from above."

When sketching the graph of a reciprocal function, it is useful to determine some points which lie on the graph.

The reciprocal function $y = \frac{k}{x}$ passes through the points $(1, k)$, $(k, 1)$, $(-1, -k)$, and $(-k, -1)$.



EXERCISE 15D.1

1 **a** Sketch the graphs of $y = \frac{1}{x}$, $y = \frac{2}{x}$, and $y = \frac{4}{x}$ on the same set of axes.

b For the function $y = \frac{k}{x}$, $k > 0$:

- **i** Describe the effect of varying k.
- **ii** State the quadrants in which the graph lies.
- **iii** Draw a sign diagram for the function.

2 **a** Sketch the graphs of $y = -\frac{1}{x}$, $y = -\frac{2}{x}$, and $y = -\frac{4}{x}$ on the same set of axes.

b For the function $y = \frac{k}{x}$, $k < 0$:

- **i** Describe the effect of varying k.
- **ii** State the quadrants in which the graph lies.
- **iii** Draw a sign diagram for the function.

3 For the reciprocal function $y = \frac{k}{x}$, $k \neq 0$, state:

a the domain

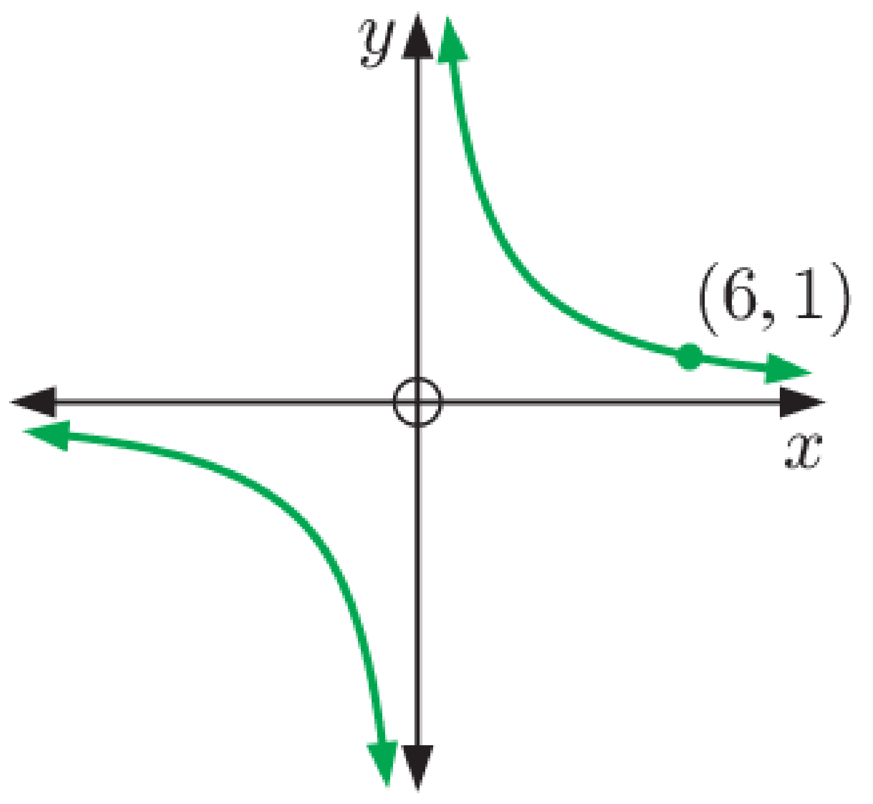
b the range

c the vertical asymptote

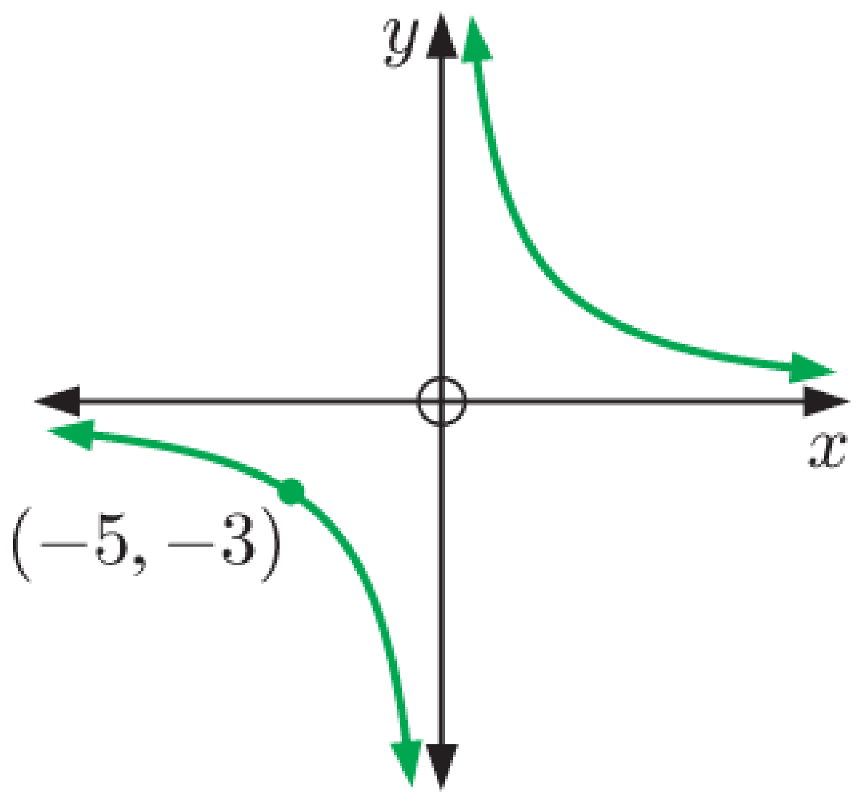
d the horizontal asymptote.

4 Determine the equation of each reciprocal function:

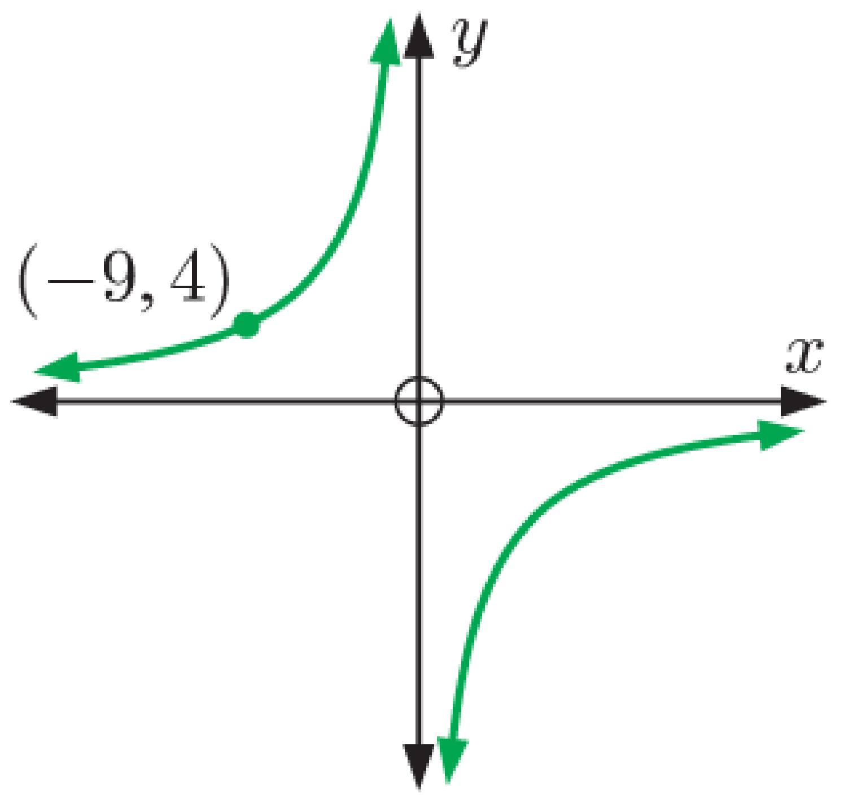
a



b



c



RATIONAL FUNCTIONS OF THE FORM $y = \frac{ax+b}{cx+d}$, $c \neq 0$

We now consider the rational functions which result when a linear function is divided by another linear function.

The graphs of these rational functions also have horizontal and vertical asymptotes.

INVESTIGATION RATIONAL FUNCTIONS

What to do:

1 Use technology to examine graphs of the following functions. For each graph:

i State the domain. **ii** Write down the equations of the asymptotes.

a $y = -1 + \frac{3}{x-2}$ **b** $y = \frac{3x+1}{x+2}$ **c** $y = \frac{2x-9}{3-x}$

GRAPHING PACKAGE

2 Experiment with functions of the form $y = \frac{b}{cx+d} + a$ where $b, c \neq 0$.

For an equation of this form, state the equation of:

a the horizontal asymptote **b** the vertical asymptote.

3 Experiment with functions of the form $y = \frac{ax+b}{cx+d}$ where $c \neq 0$.

a For an equation of this form, state the equation of the vertical asymptote.

b Can you see how to quickly write down the equation of the horizontal asymptote? Explain your answer.

- For a function written in the form $y = \frac{b}{cx+d} + a$ where $b, c \neq 0$:
  - the vertical asymptote is $x = -\frac{d}{c}$
  - the horizontal asymptote is $y = a$.
- For a function written in the form $y = \frac{ax+b}{cx+d}$ where $c \neq 0$:
  - the vertical asymptote is $x = -\frac{d}{c}$
  - the horizontal asymptote is $y = \frac{a}{c}$.

Example 7 **Self Tutor**

Consider the function $y = \dfrac{6}{x-2} + 4$.

a Find the asymptotes of the function.

b Find the axes intercepts.

c Use technology to help sketch the function, including the features found in **a** and **b**.

a The vertical asymptote is $x = 2$.
The horizontal asymptote is $y = 4$.

b When $y = 0$, $\dfrac{6}{x-2} = -4$

$\therefore\ -4(x-2) = 6$

$\therefore\ -4x + 8 = 6$

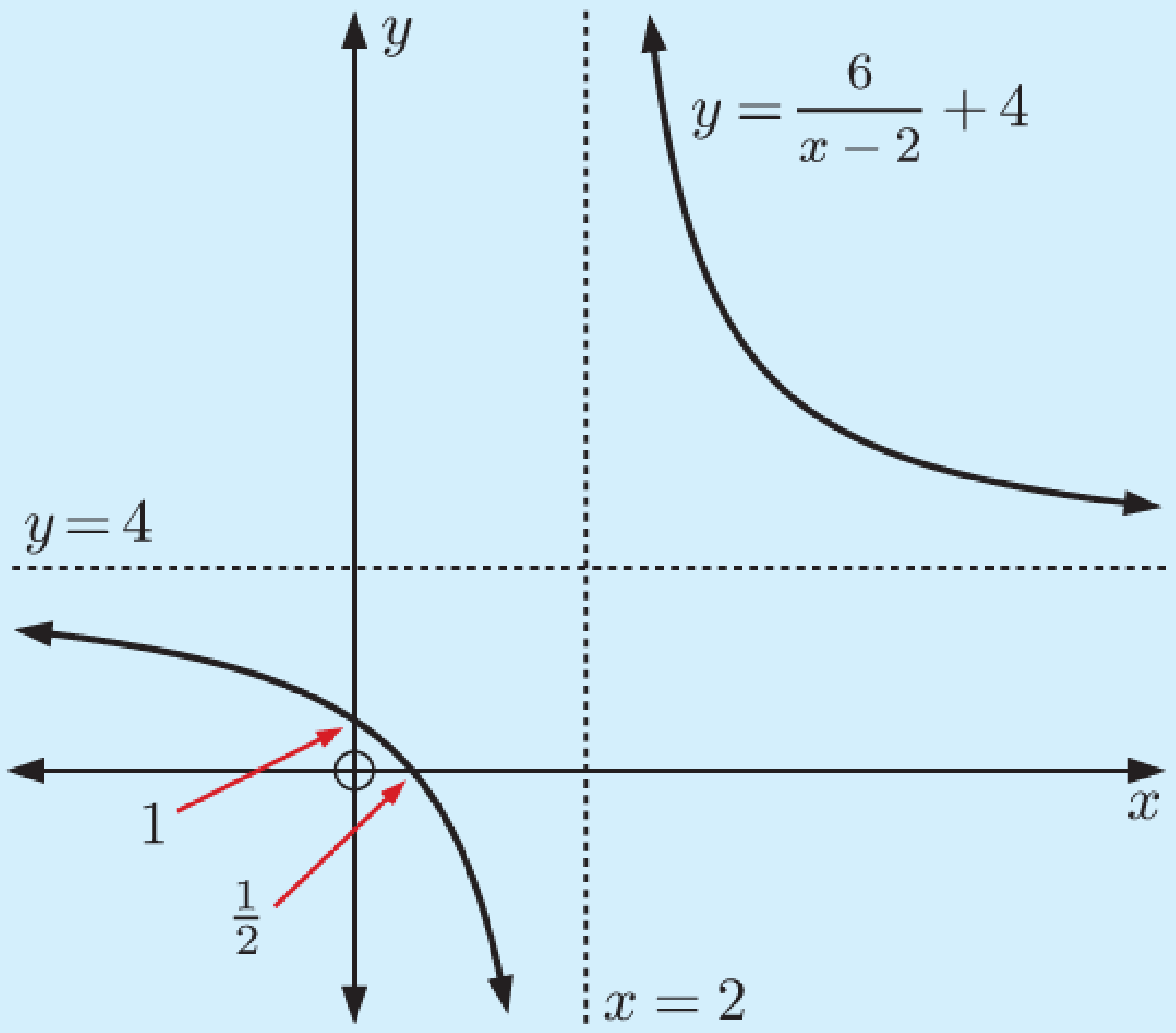
$\therefore\ -4x = -2$

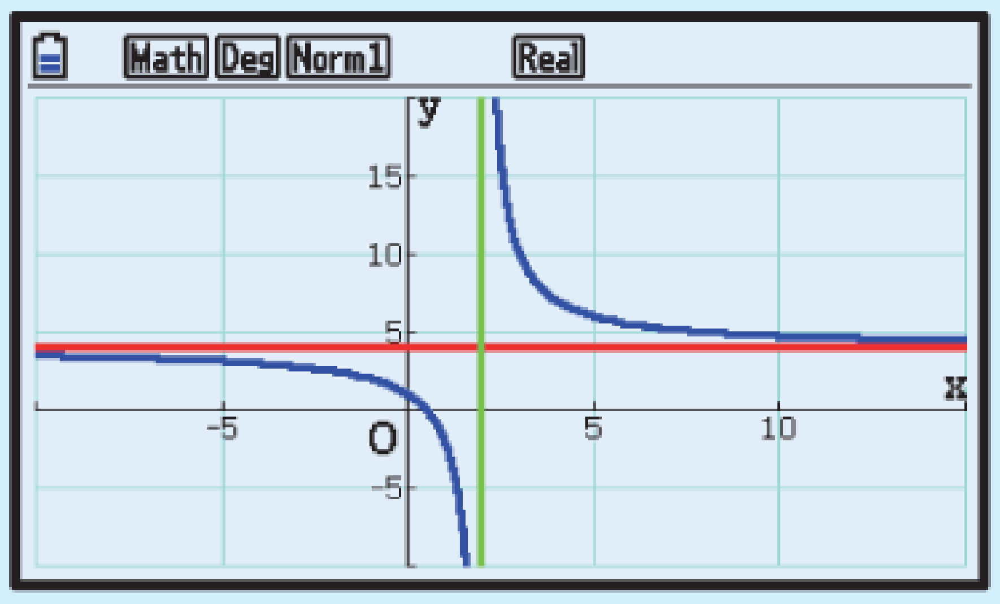
$\therefore\ x = \frac{1}{2}$

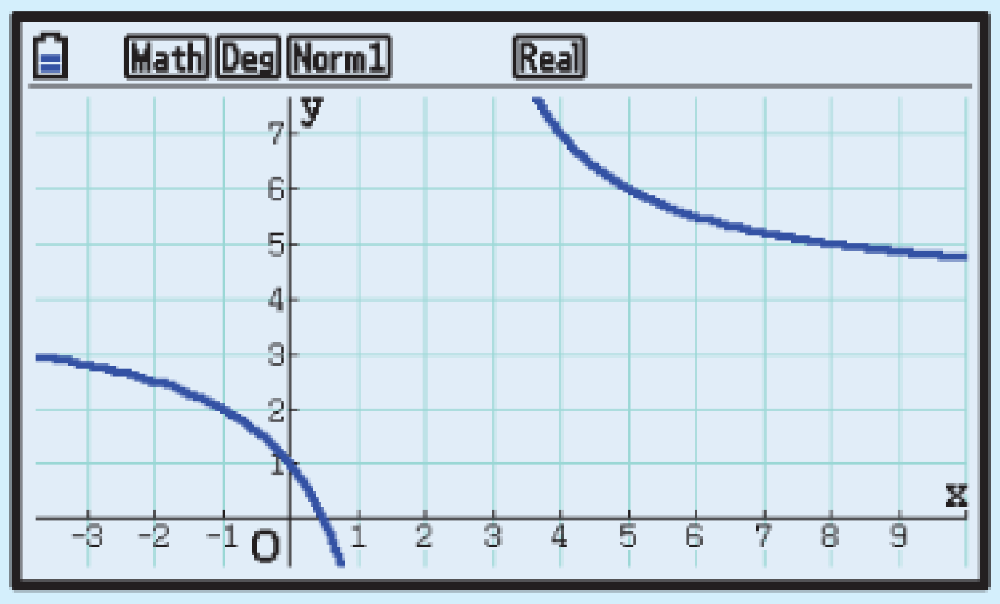
When $x = 0$, $y = \frac{6}{-2} + 4 = 1$

So, the x-intercept is $\frac{1}{2}$ and the y-intercept is 1.

c







EXERCISE 15D.2

1 For each of the following functions:

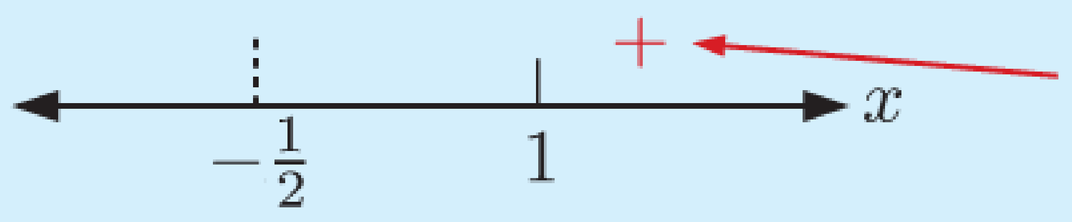
- **i** Find the equations of the asymptotes.
- **ii** State the domain and range.
- **iii** Find the axes intercepts.
- **iv** Discuss the behaviour of the function as it approaches its asymptotes.
- **v** Sketch the graph of the function.

a $f(x) = \dfrac{3}{x-2}$ **b** $f : x \mapsto 2 + \dfrac{1}{x-3}$ **c** $f(x) = 2 - \dfrac{3}{x+1}$

Example 8 **Self Tutor**

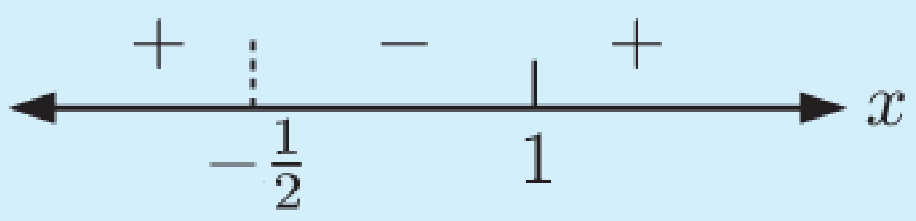
Draw a sign diagram for $\dfrac{x-1}{2x+1}$.

$\dfrac{x-1}{2x+1}$ is zero when $x = 1$ and undefined when $x = -\frac{1}{2}$.



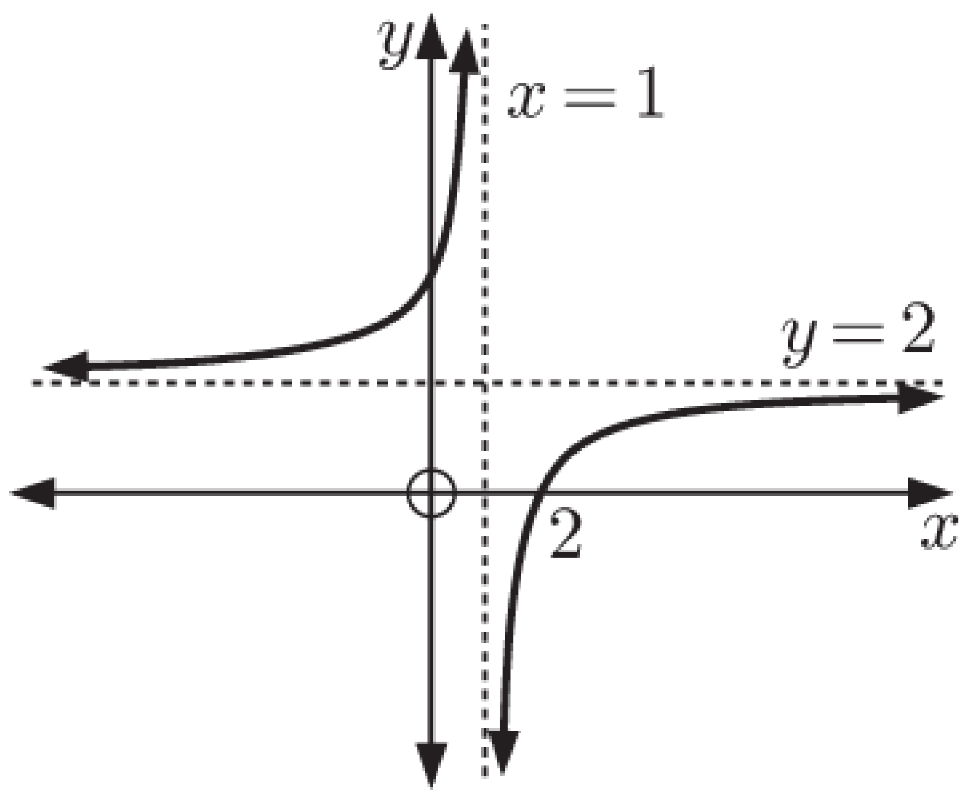
When $x = 10$, $\dfrac{x-1}{2x+1} = \dfrac{9}{21} > 0$

Since $(x-1)$ and $(2x+1)$ are single factors, the signs alternate.

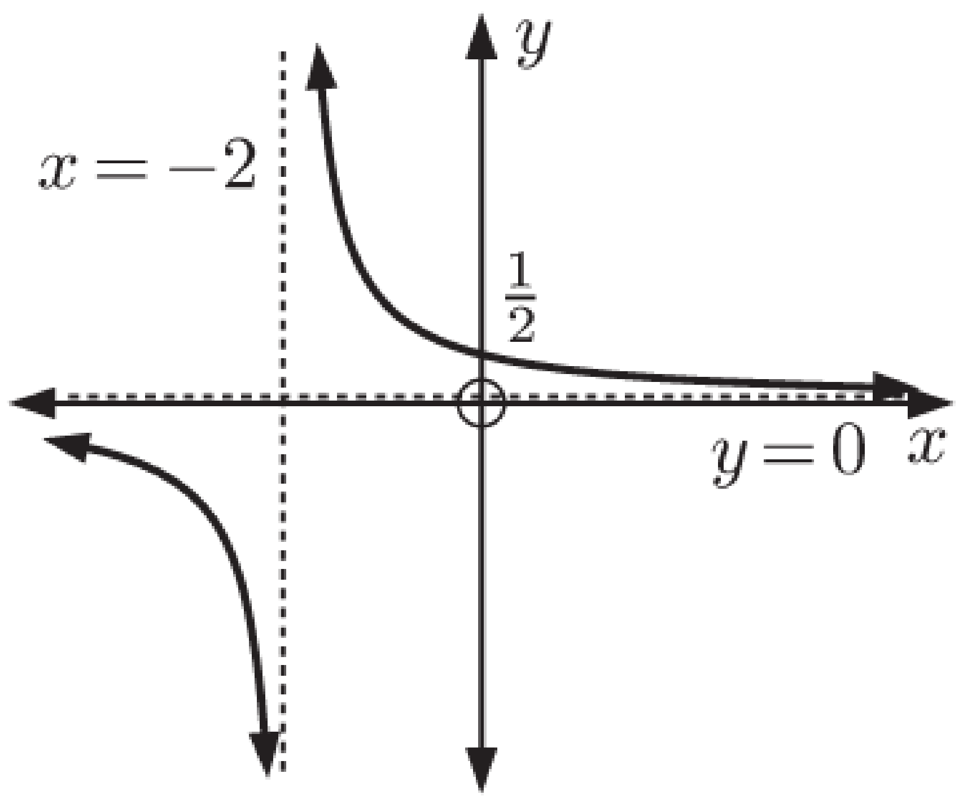


2 Draw the sign diagram for:

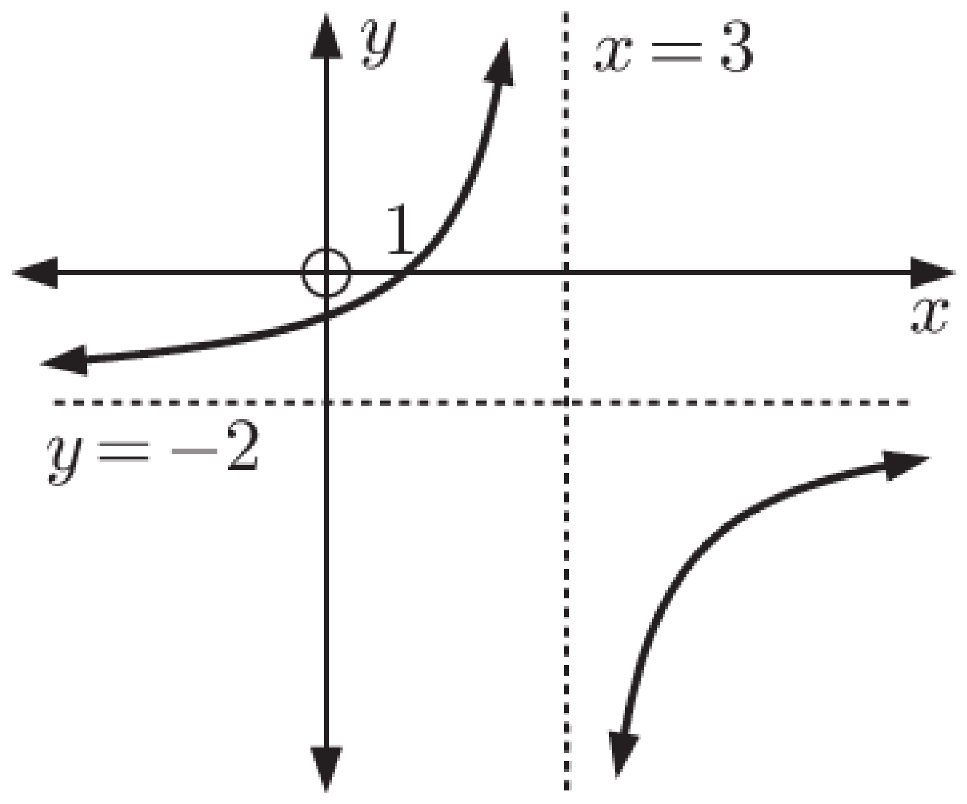
a



b



c



3 Draw a sign diagram for:

a $\dfrac{x+2}{x-1}$ **b** $\dfrac{x}{x+3}$ **c** $\dfrac{x+1}{x+5}$ **d** $\dfrac{x-2}{2x+1}$

e $\dfrac{2x+3}{4-x}$ **f** $\dfrac{4x-1}{2-x}$ **g** $\dfrac{3x}{x-2}$ **h** $\dfrac{-8x}{3-x}$

Example 9 **Self Tutor**

Consider the function $f(x)=\dfrac{2x+1}{x-1}$.

a Find the vertical asymptote of the function.

b Find the axes intercepts.

c Rearrange the function to find the horizontal asymptote.

d Draw a sign diagram of the function.

e Hence discuss the behaviour of the function near the asymptotes.

f Sketch the function, showing the features you have found.

a The vertical asymptote is $x=1$.

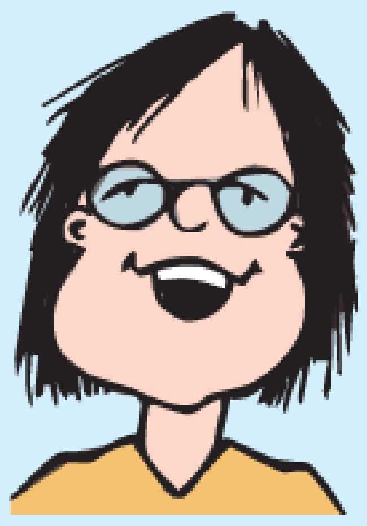
b $f(0)=\dfrac{1}{-1}=-1$, so the y-intercept is -1.

$f(x)=0$ when $2x+1=0$

$\therefore\ x=-\frac{1}{2}$

$\therefore$ the x-intercept is $-\frac{1}{2}$.

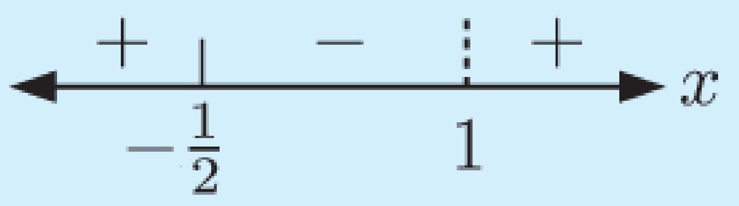
As $|x|\to\infty$, the fraction $\dfrac{3}{x-1}$ becomes infinitely small.



c
$$f(x)=\frac{2x+1}{x-1}=\frac{2(x-1)+3}{x-1}=2+\frac{3}{x-1}$$

$\therefore$ the horizontal asymptote is $y=2$.

d



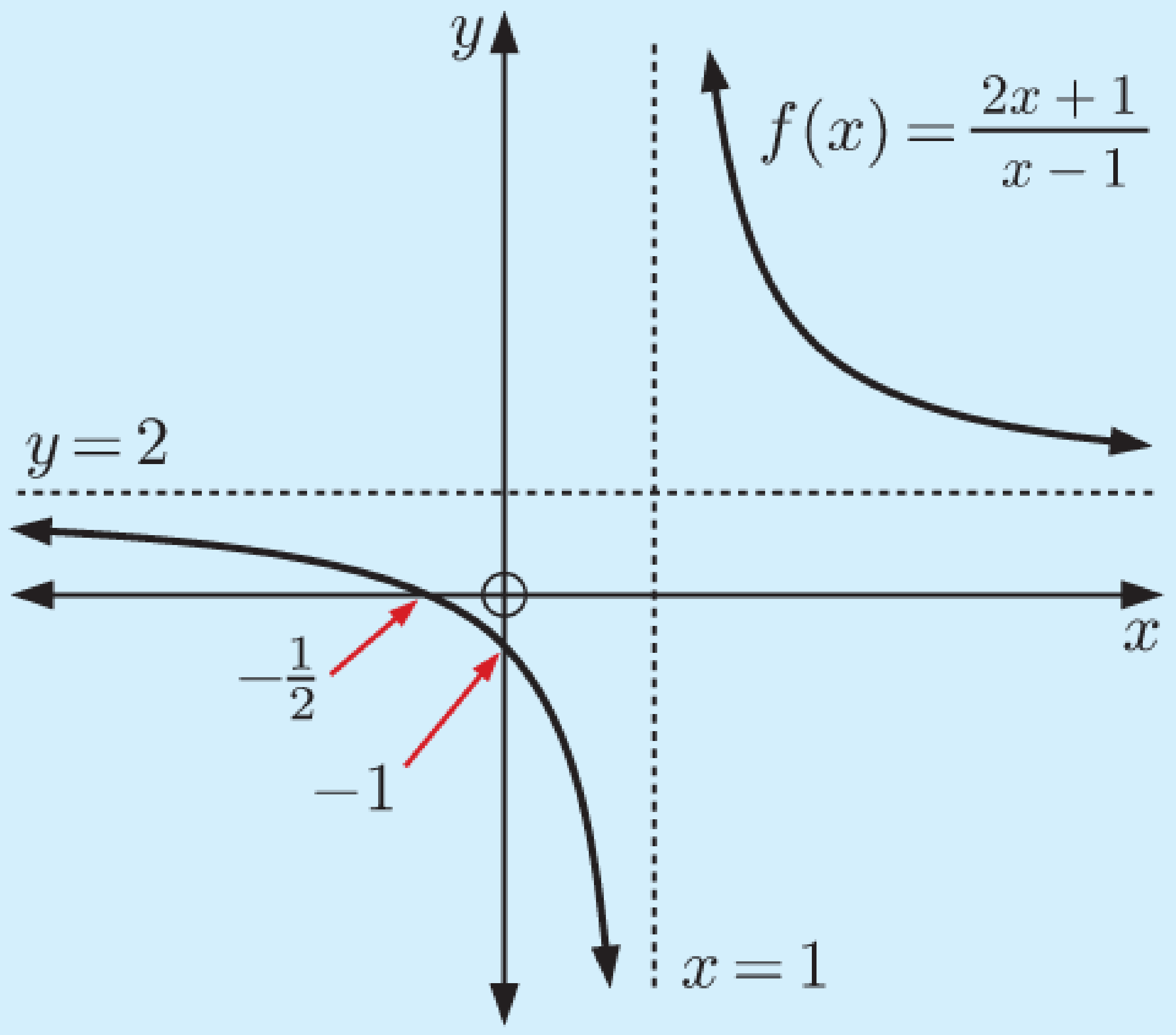
e As $x\to1^-$, $f(x)\to-\infty$

As $x\to1^+$, $f(x)\to\infty$

As $x\to-\infty$, $f(x)\to2^-$

As $x\to\infty$, $f(x)\to2^+$

f



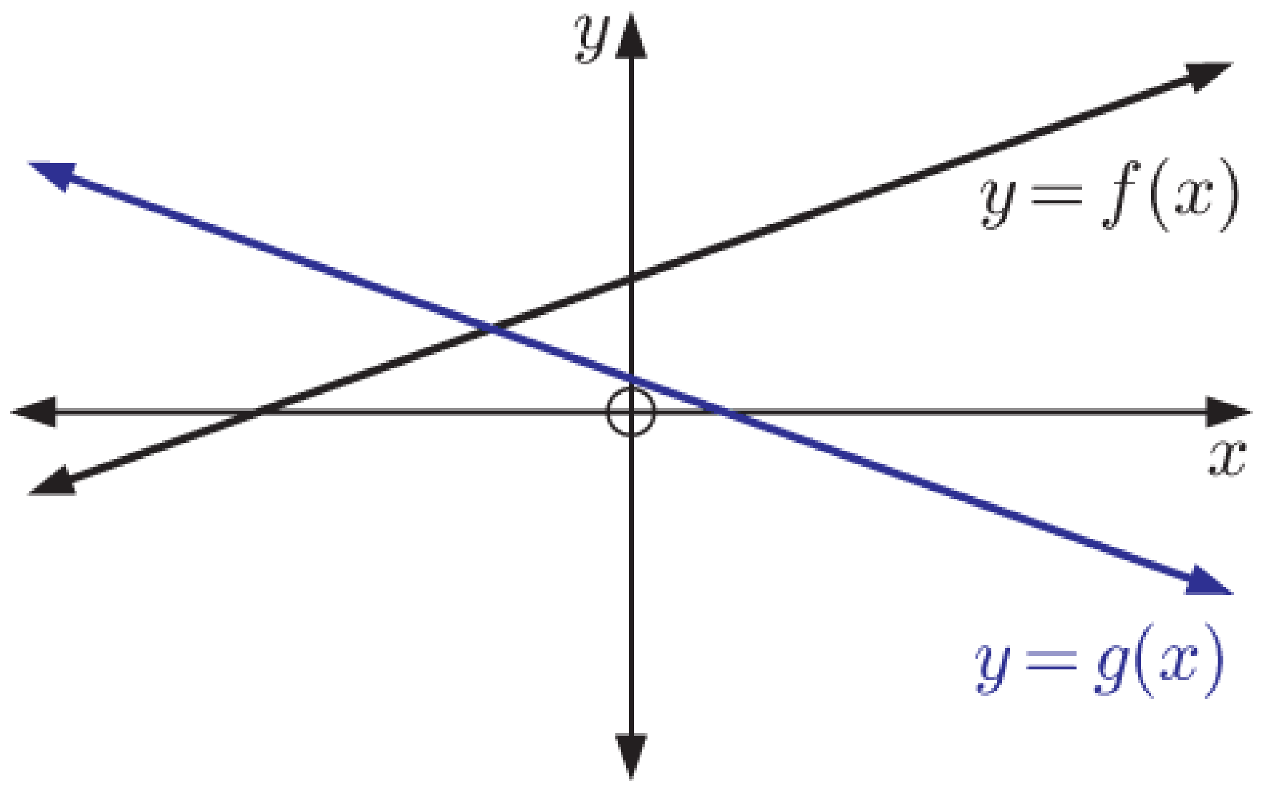
4 For each of the following functions:

- **i** Find the equation of the vertical asymptote.
- **ii** Find the axes intercepts.
- **iii** Rearrange the function to find the horizontal asymptote.
- **iv** Draw a sign diagram of the function.
- **v** Hence discuss the behaviour of the function near its asymptotes.
- **vi** Sketch the graph of the function.

a $f(x)=\dfrac{x}{x-1}$ **b** $f: x \mapsto \dfrac{x+3}{x-2}$ **c** $f(x)=\dfrac{3x-1}{x+2}$

d $f(x)=-\dfrac{2x+1}{x-3}$ **e** $f: x \mapsto \dfrac{2x+4}{3-x}$ **f** $f(x)=\dfrac{x+3}{2x-1}$

5 The graph alongside shows the linear functions $f(x)$ and $g(x)$.
Copy the graph, and on the same set of axes, graph $y=\dfrac{f(x)}{g(x)}$. Indicate clearly where any x-intercepts and asymptotes occur.



6 Consider the function $y=\dfrac{ax+b}{cx+d}$, where a, b, c, d are constants and $c \neq 0$.

a State the domain of the function.

b State the equation of the vertical asymptote.

c Find the axes intercepts.

d Show that for $c \neq 0$, $\dfrac{ax+b}{cx+d}=\dfrac{a}{c}+\dfrac{b-\frac{ad}{c}}{cx+d}$.

Hence explain why the horizontal asymptote is $y=\dfrac{a}{c}$.

ACTIVITY

Click on the icon to run a card game for rational functions.

GAME

E COMPOSITE FUNCTIONS

Given $f: x \mapsto f(x)$ and $g: x \mapsto g(x)$, the **composite function** of f and g will convert x into $f(g(x))$.

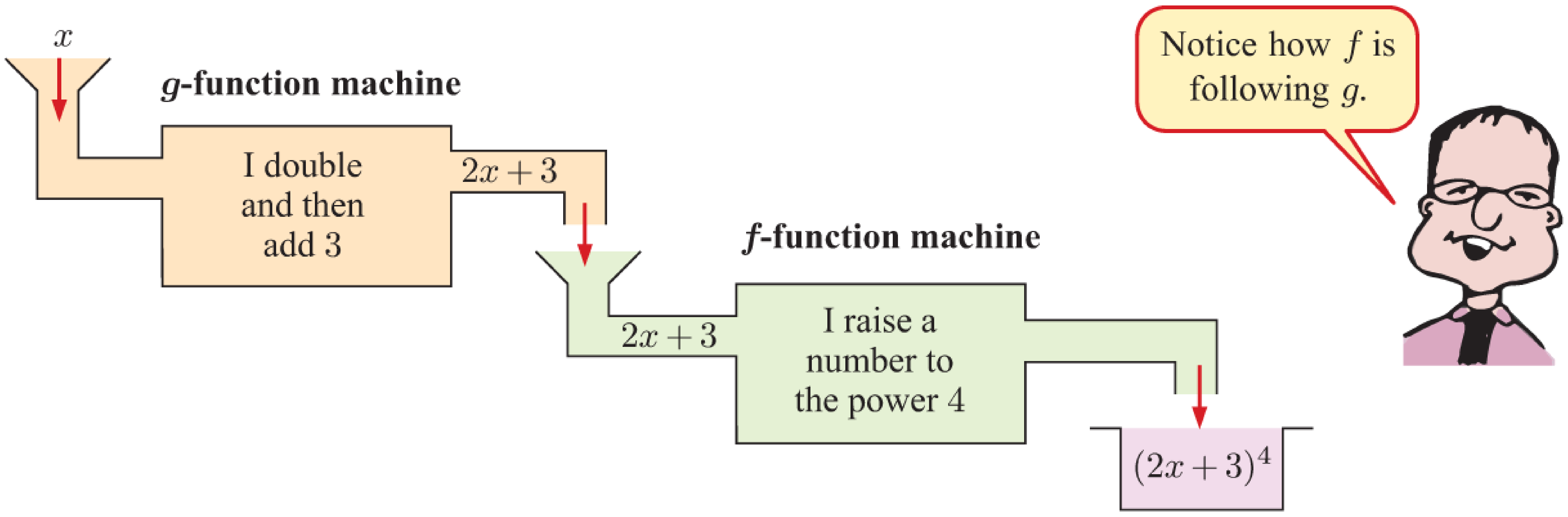
$f \circ g$ is used to represent the composite function of f and g. It means "f following g".

$$(f \circ g)(x)=f(g(x)) \quad \text{or} \quad f \circ g: x \mapsto f(g(x)).$$

Consider $f : x \mapsto x^4$ and $g : x \mapsto 2x+3$.

$f \circ g$ means that g converts x to $2x+3$ and then f converts $(2x+3)$ to $(2x+3)^4$.

This is illustrated by the two function machines below.



Algebraically, if $f(x)=x^4$ and $g(x)=2x+3$ then

$$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= f(2x+3) && \{g \text{ operates on } x \text{ first}\}\\ &= (2x+3)^4 && \{f \text{ operates on } g(x) \text{ next}\}\end{aligned}$$

and
$$\begin{aligned}(g\circ f)(x) &= g(f(x))\\ &= g(x^4) && \{f \text{ operates on } x \text{ first}\}\\ &= 2(x^4)+3 && \{g \text{ operates on } f(x) \text{ next}\}\\ &= 2x^4+3\end{aligned}$$

So, $f(g(x)) \neq g(f(x))$.

In general, $(f\circ g)(x) \neq (g\circ f)(x)$.

Example 10 **Self Tutor**

Given $f : x \mapsto 2x+1$ and $g : x \mapsto 3-4x$, find in simplest form:

a $(f\circ g)(x)$ **b** $(g\circ f)(x)$

$f(x)=2x+1$ and $g(x)=3-4x$

a
$$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= f(3-4x)\\ &= 2(3-4x)+1\\ &= 6-8x+1\\ &= 7-8x\end{aligned}$$

b
$$\begin{aligned}(g\circ f)(x) &= g(f(x))\\ &= g(2x+1)\\ &= 3-4(2x+1)\\ &= 3-8x-4\\ &= -8x-1\end{aligned}$$

In the previous Example you should have observed how we can substitute an expression into a function.

If $f(x)=2x+1$ then $f(\Delta)=2(\Delta)+1$

and so $f(3-4x)=2(3-4x)+1$.

Example 11 **Self Tutor**

Given $f(x) = 6x - 5$ and $g(x) = x^2 + x$, find:

a $(g \circ f)(-1)$ **b** $(f \circ f)(0)$

a
$$\begin{aligned} (g \circ f)(-1) &= g(f(-1)) \\ \text{Now} \quad f(-1) &= 6(-1) - 5 \\ &= -11 \\ \therefore \quad (g \circ f)(-1) &= g(-11) \\ &= (-11)^2 + (-11) \\ &= 110 \end{aligned}$$

b
$$\begin{aligned} (f \circ f)(0) &= f(f(0)) \\ \text{Now} \quad f(0) &= 6(0) - 5 \\ &= -5 \\ \therefore \quad (f \circ f)(0) &= f(-5) \\ &= 6(-5) - 5 \\ &= -35 \end{aligned}$$

You should be aware that the domain of the composite of two functions depends on the domains of the original functions.

For example, consider $f(x) = x^2$ with domain $x \in \mathbb{R}$ and $g(x) = \sqrt{x}$ with domain $x \geqslant 0$.

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= (\sqrt{x})^2 \\ &= x \end{aligned}$$

The domain of $(f \circ g)(x)$ is $x \geqslant 0$, not $\mathbb{R}$, since $(f \circ g)(x)$ is defined using function $g(x)$.

EXERCISE 15E

1 Given $f : x \mapsto -2x$ and $g : x \mapsto 1 + x^2$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $(f \circ g)(2)$ **d** $(f \circ f)(-1)$

2 Given $f(x) = 3 - x^2$ and $g(x) = 2x + 4$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $(g \circ g)(\frac{1}{2})$ **d** $(f \circ f)(-\frac{1}{2})$

3 Given $f(x) = \sqrt{6 - x}$ and $g(x) = 5x - 7$, find:

a $(g \circ g)(x)$ **b** $(f \circ g)(1)$ **c** $(g \circ f)(6)$ **d** $(f \circ f)(2)$

4 Suppose $f : x \mapsto x^2 + 1$ and $g : x \mapsto 3 - x$.

a Find in simplest form: **i** $(f \circ g)(x)$ **ii** $(g \circ f)(x)$

b Find the value(s) of x such that $(g \circ f)(x) = f(x)$.

5 Suppose $f(x) = 9 - \sqrt{x}$ and $g(x) = x^2 + 4$.

a Find $(f \circ g)(x)$ and state its domain and range. **b** Find $(g \circ f)(4)$.

c Find $(f \circ f)(x)$ and state its domain and range.

6 Suppose $f(x) = 1 - 2x$ and $g(x) = 3x + 5$.

a Find $f(g(x))$. **b** Hence solve $(f \circ g)(x) = f(x + 3)$.

7 Suppose $f : x \mapsto 2x - x^2$ and $g : x \mapsto 1 + 3x$.

a Find in simplest form: **i** $(f \circ g)(x)$ **ii** $(g \circ f)(x)$

b Find the value(s) of x such that $(f \circ g)(x) = 3(g \circ f)(x)$.

8 For each pair of functions, find $(f \circ g)(x)$ and state its domain and range:

a $f(x) = \dfrac{1}{x}$ and $g(x) = x - 3$ **b** $f(x) = -\dfrac{1}{x}$ and $g(x) = x^2 + 3x + 2$

9 Functions f and g are defined by $f = \{(0, 2), (1, 5), (2, 7), (3, 9)\}$ and $g = \{(2, 2), (5, 0), (7, 1), (9, 3)\}$. Find: **a** $f \circ g$ **b** $g \circ f$.

10 Given $f(x) = \dfrac{x+3}{x+2}$ and $g(x) = \dfrac{x+1}{x-1}$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $(g \circ g)(x)$

In each case, find the domain of the composite function.

11 **a** If $ax + b = cx + d$ for all values of x, show that $a = c$ and $b = d$.
Hint: If it is true for all x, it is true for $x = 0$ and $x = 1$.

b Given $f(x) = 2x + 3$ and $g(x) = ax + b$ and that $(f \circ g)(x) = x$ for all values of x, deduce that $a = \frac{1}{2}$ and $b = -\frac{3}{2}$.

c Is the result in **b** true if $(g \circ f)(x) = x$ for all x?

12 Suppose $f(x) = \sqrt{1-x}$ and $g(x) = x^2$. Find:

a $(f \circ g)(x)$ **b** the domain and range of $(f \circ g)(x)$

c $(g \circ f)(x)$ **d** the domain and range of $(g \circ f)(x)$.

13 Suppose $f(x)$ and $g(x)$ are functions. $f(x)$ has domain D_f and range R_f. $g(x)$ has domain D_g and range R_g.

a Under what circumstance will $(f \circ g)(x)$ be defined?

b Assuming $(f \circ g)(x)$ is defined, find its domain.

Example 12 **Self Tutor**

The outside temperature at an altitude A km above ground level in a particular city is $T(A) = 25 - 6A$ °C. The altitude of an aeroplane t minutes after taking off is $A(t) = 12 - \dfrac{60}{t+5}$ km.

Find the composite function $T \circ A$, and explain what it means.

$$\begin{aligned} T \circ A &= T(A(t)) \\ &= T\left(12 - \frac{60}{t+5}\right) \\ &= 25 - 6\left(12 - \frac{60}{t+5}\right) \\ &= \frac{360}{t+5} - 47 \end{aligned}$$

The function $T \circ A$ gives the temperature outside the aeroplane t minutes after taking off.

14 The value of Mila's car after it has been driven D thousand kilometres is $V(D) = 10\,000 - 40D$ dollars. The distance Mila's car has been driven t years after Mila purchased it is $D(t) = 80 + 10t$ thousand kilometres.

a Find the composite function $V \circ D$, and explain what it means.

b Find and interpret $(V \circ D)(6)$.

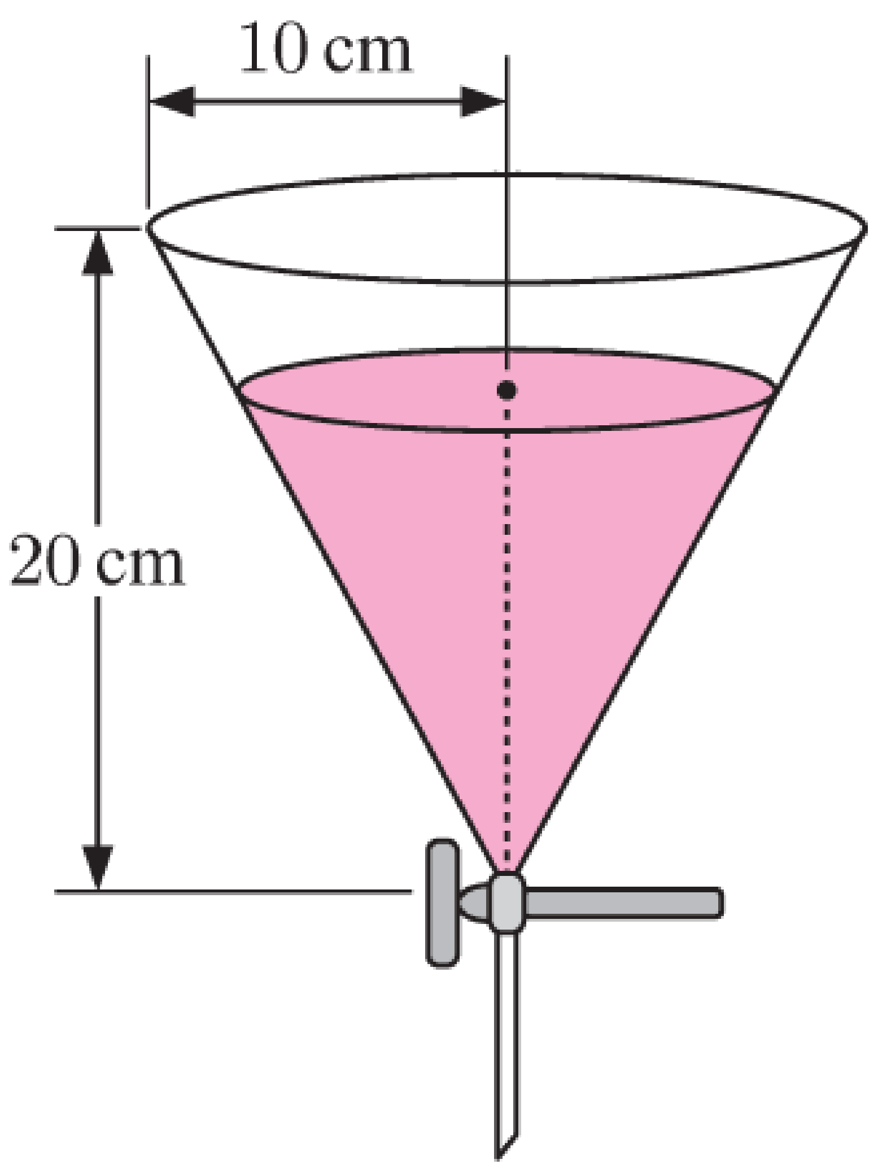
15 Diego sells sculptures online. When a sculpture with marked price €x is bought, Diego adds a €50 shipping fee and 10% tax to determine the total cost.

Let the functions $S(x) = x + 50$ and $T(x) = 1.1x$ represent the shipping and tax respectively.

- **a** What composite function should be used to determine the total cost if the tax must be paid on:
  - **i** both the marked price and the shipping fee
  - **ii** the marked price but not the shipping fee?
- **b** Suppose the tax must be paid on both the marked price and the shipping fee. Use the correct composite function to find the total cost of a sculpture with marked price €600.

16 In a laboratory, a conical funnel has radius 10 cm and height 20 cm. The funnel initially contains 2 L of solution. The tap is turned on, allowing the solution to flow from the funnel at 20 mL per minute.

- **a** Find the function for the volume V mL of solution remaining in the funnel after t minutes.
- **b** Show that if there is V mL of solution in the funnel, the height of the solution is $H(V) = \sqrt[3]{\frac{12V}{\pi}}$ cm.
- **c** Find $H \circ V$, and explain what it means.
- **d** Find and interpret $(H \circ V)(30)$.



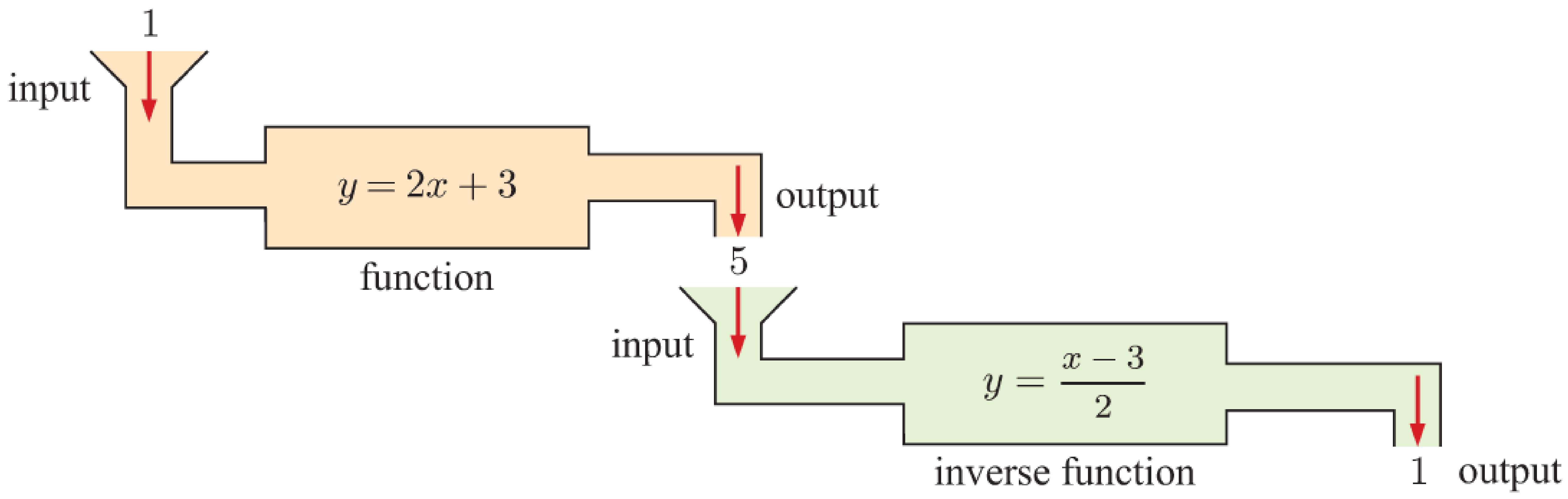
F INVERSE FUNCTIONS

The operations of + and −, × and ÷, are **inverse operations** as one "undoes" what the other does.

The function $y = 2x + 3$ can be "undone" by its *inverse* function $y = \frac{x-3}{2}$.

We can think of this as two machines. If the machines are inverses then the second machine *undoes* what the first machine does.

No matter what value of x enters the first machine, it is returned as the output from the second machine.



A function $y = f(x)$ *may or may not* have an inverse function. To understand which functions do have inverses, we need some more terminology.

ONE-TO-ONE AND MANY-TO-ONE FUNCTIONS

A **one-to-one** function is any function where:

- for each x there is only one value of y and
- for each y there is only one value of x.

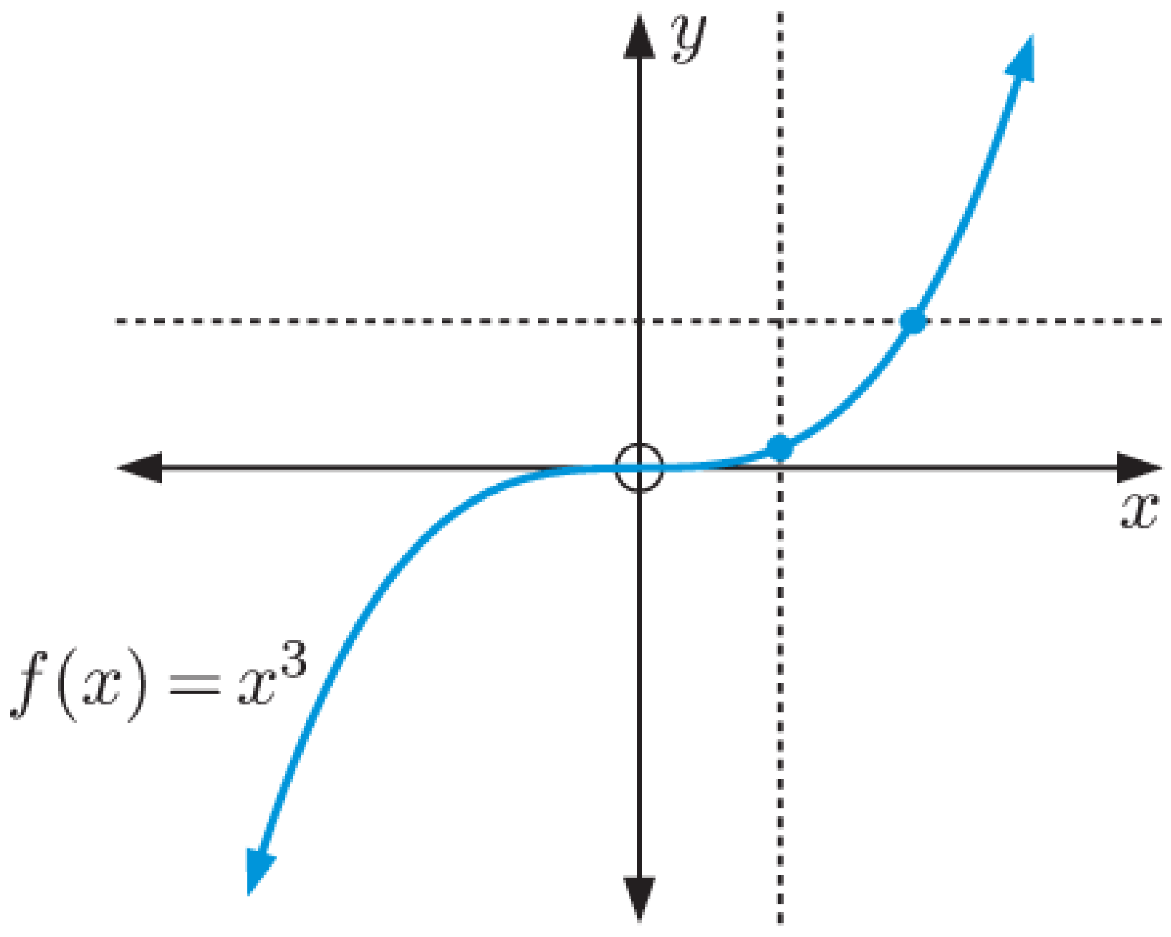
Equivalently, a function is one-to-one if $f(a) = f(b)$ only when $a = b$.

One-to-one functions satisfy both the **vertical line test** and the **horizontal line test**.

This means that:

- no vertical line can meet the graph more than once
- no horizontal line can meet the graph more than once.

For example, $f(x) = x^3$ is one-to-one since it passes both the vertical line and horizontal line tests.

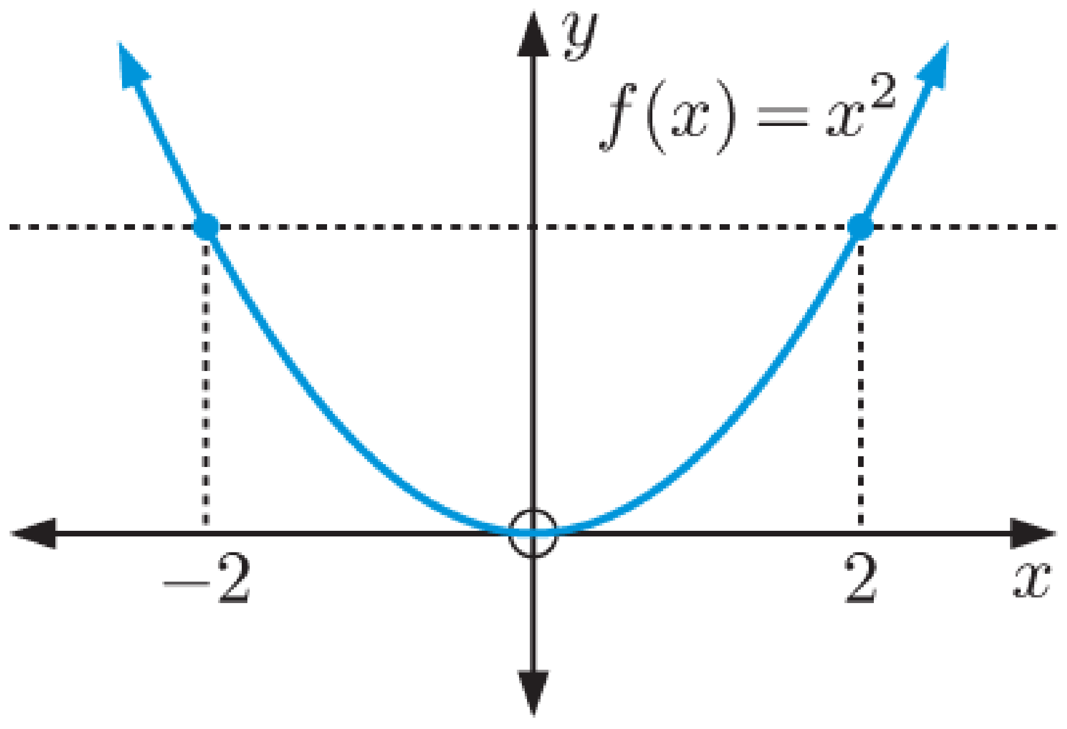


If the function $f(x)$ is **one-to-one**, it will have an inverse function which we denote $f^{-1}(x)$.

Functions that are not one-to-one are called **many-to-one**. While these functions satisfy the vertical line test, they *do not* satisfy the horizontal line test. At least one y-value has more than one corresponding x-value.

For example, $f(x) = x^2$ fails the horizontal line test, since if $f(x) = 4$ then $x = -2$ or 2.

$f(x) = x^2$ is therefore many-to-one.



If a function $f(x)$ is **many-to-one**, it *does not* have an inverse function.

However, for a many-to-one function we can often define a new function using the same formula but with a **restricted domain** to make it a one-to-one function. This new function will have an inverse function.

For example, if we restrict $f(x) = x^2$ to the domain $x \geqslant 0$ or the domain $x \leqslant 0$, then the restricted function is one-to-one, and it has an inverse function.

PROPERTIES OF THE INVERSE FUNCTION

If $f(x)$ has an **inverse function**, this new function:

- is denoted $f^{-1}(x)$
- must satisfy the vertical line test
- has a graph which is the reflection of $y = f(x)$ in the line $y = x$
- satisfies $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

The function $y = x$ is called the **identity function** because it is its own inverse, and when its inverse is found, (x, y) maps onto itself.

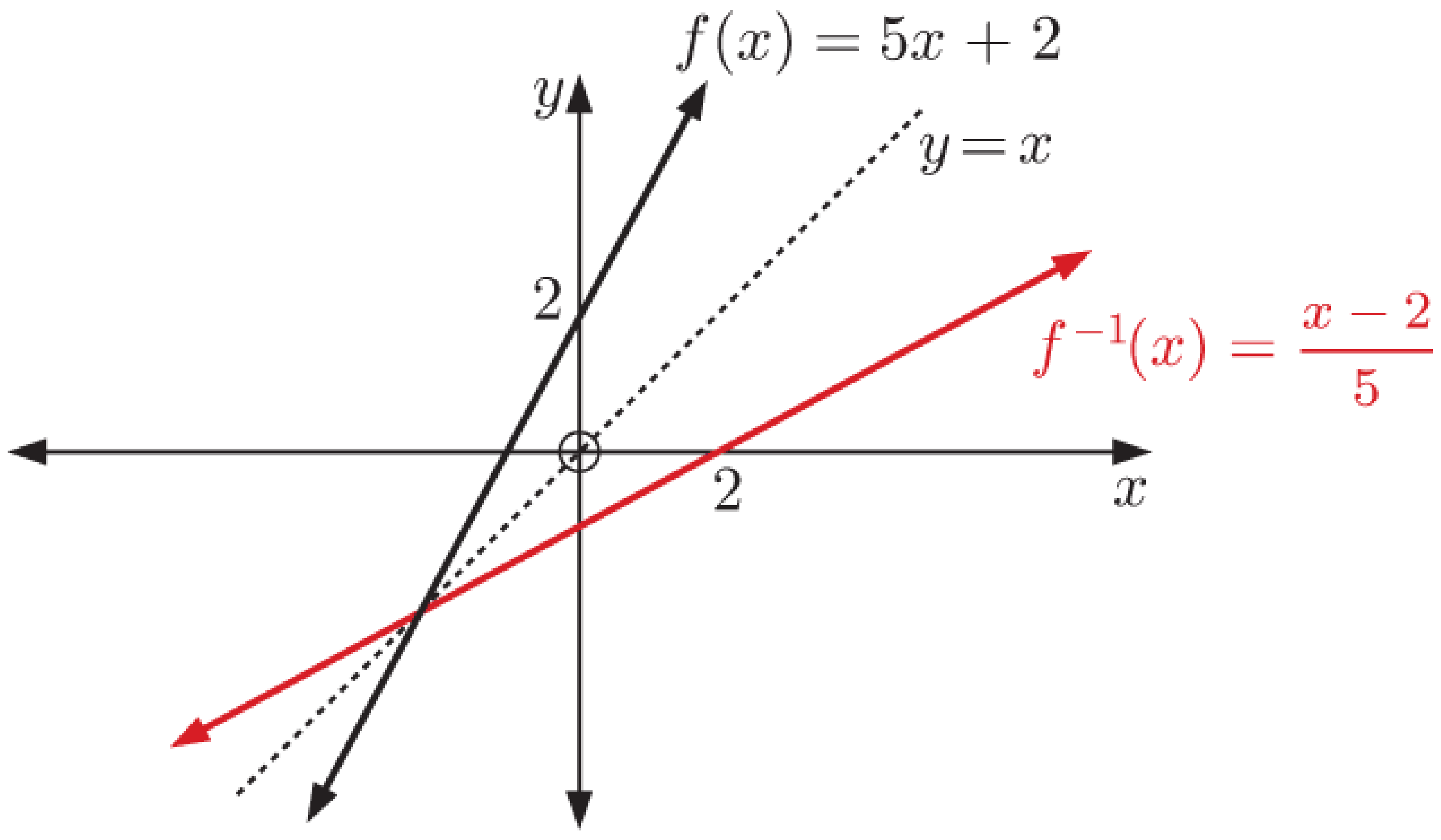
If (x, y) lies on f, then (y, x) must lie on f^{-1}.

Geometrically, this is achieved by *reflecting* the graph of $y = f(x)$ in the line $y = x$.

Algebraically, we find the formula for an inverse function by exchanging x and y.

For example, $f : y = 5x + 2$ becomes $f^{-1} : x = 5y + 2$,

which we rearrange to obtain $f^{-1} : y = \frac{x-2}{5}$.

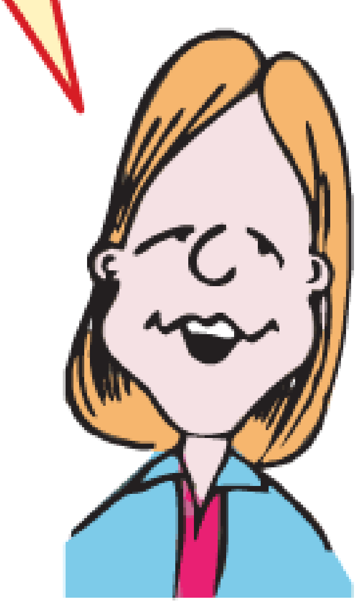


$y = f^{-1}(x)$ is the inverse of $y = f(x)$ as:

- it is also a function
- it is the reflection of $y = f(x)$ in the line $y = x$.

$f^{-1}(x)$ is the **inverse** of f, *not* its reciprocal.

In general, $f^{-1}(x) \neq \frac{1}{f(x)}$.



If $f(x)$ has an inverse function $f^{-1}(x)$, then:

The domain of f^{-1} is equal to the range of f.

The range of f^{-1} is equal to the domain of f.

Example 13

Self Tutor

Consider $f : x \mapsto 2x + 3$.

a On the same axes, graph f and its inverse function f^{-1}.

b Find $f^{-1}(x)$ using:

  i coordinate geometry and the gradient of $y = f^{-1}(x)$ from **a**

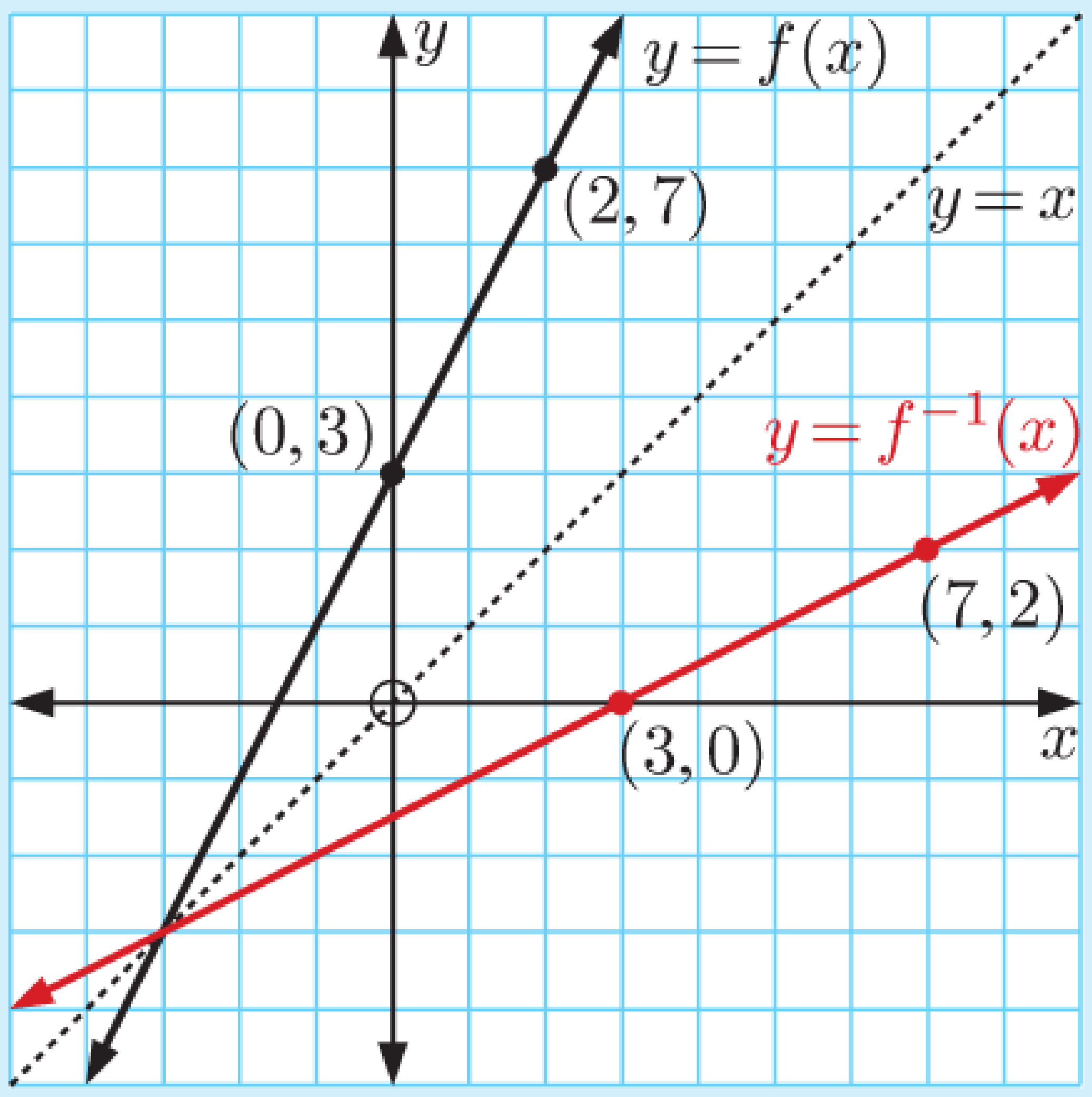
  ii variable interchange.

c Check that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$

a $f(x) = 2x + 3$ passes through $(0, 3)$ and $(2, 7)$.

$\therefore$ $f^{-1}(x)$ passes through $(3, 0)$ and $(7, 2)$.

If f includes point (a, b) then f^{-1} includes point (b, a).



b **i** $y = f^{-1}(x)$ has gradient $\dfrac{2-0}{7-3} = \dfrac{1}{2}$

Its equation is $\dfrac{y-0}{x-3} = \dfrac{1}{2}$

$$\therefore\ y = \frac{x-3}{2}$$

$$\therefore\ f^{-1}(x) = \frac{x-3}{2}$$

ii f is $y = 2x+3$,

$$\therefore\ f^{-1} \text{ is } x = 2y+3$$

$$\therefore\ x-3 = 2y$$

$$\therefore\ \frac{x-3}{2} = y$$

$$\therefore\ f^{-1}(x) = \frac{x-3}{2}$$

c $(f \circ f^{-1})(x) = f(f^{-1}(x))$

$$= f\left(\frac{x-3}{2}\right)$$

$$= 2\left(\frac{x-3}{2}\right) + 3$$

$$= x$$

and $(f^{-1} \circ f)(x) = f^{-1}(f(x))$

$$= f^{-1}(2x+3)$$

$$= \frac{(2x+3)-3}{2}$$

$$= \frac{2x}{2}$$

$$= x$$

Example 14 **Self Tutor**

Consider $f : x \mapsto x^2$.

a Explain why this function does not have an inverse function.

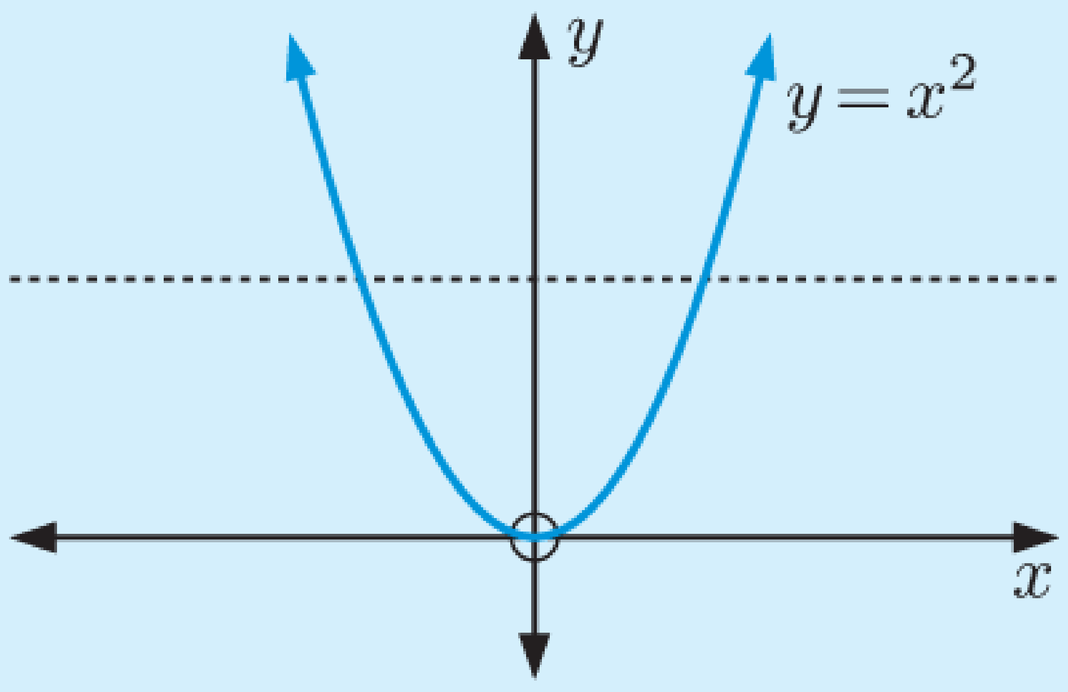
b Does $f : x \mapsto x^2$ where $x \geqslant 0$ have an inverse function?

c Find $f^{-1}(x)$ for $f : x \mapsto x^2$, $x \geqslant 0$.

d Sketch $y = f(x)$, $y = x$, and $y = f^{-1}(x)$ for f in **b** and f^{-1} in **c**.

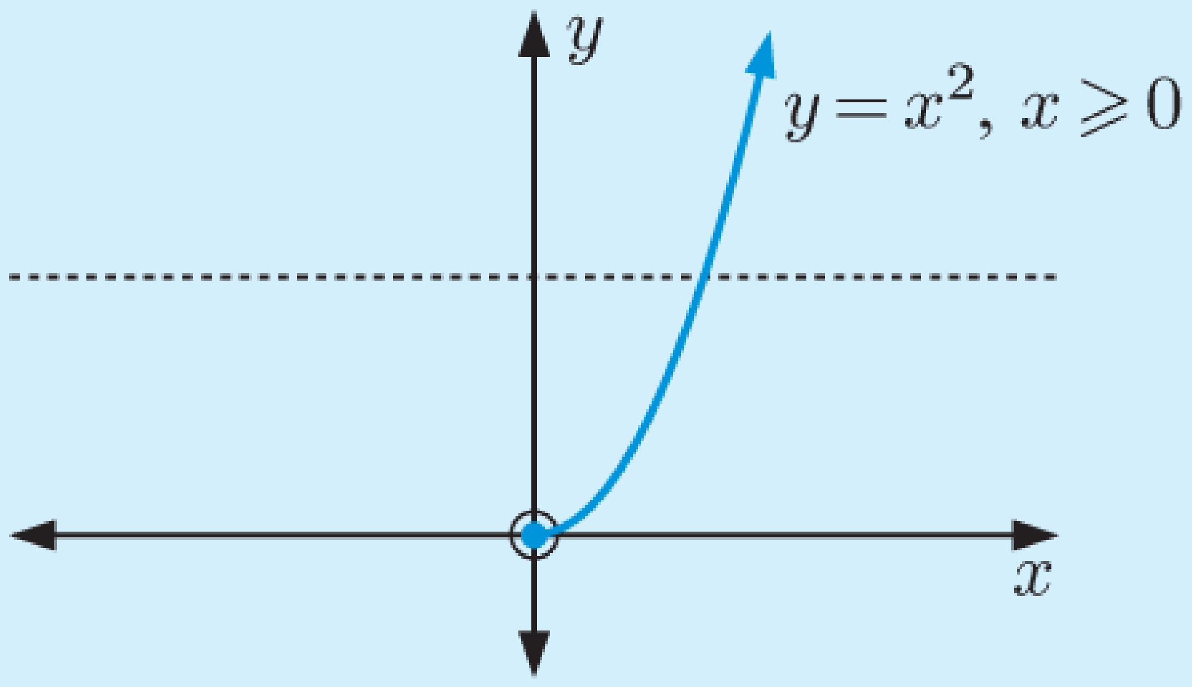
a From the graph, we can see that $f : x \mapsto x^2$ does not pass the horizontal line test.

$\therefore$ it is many-to-one and does not have an inverse function.



b If we restrict the domain to $x \geqslant 0$ or $x \in [0, \infty[$, the function now satisfies the horizontal line test.

$\therefore$ it is one-to-one and has an inverse function.



c f is defined by $y = x^2$, $x \geqslant 0$

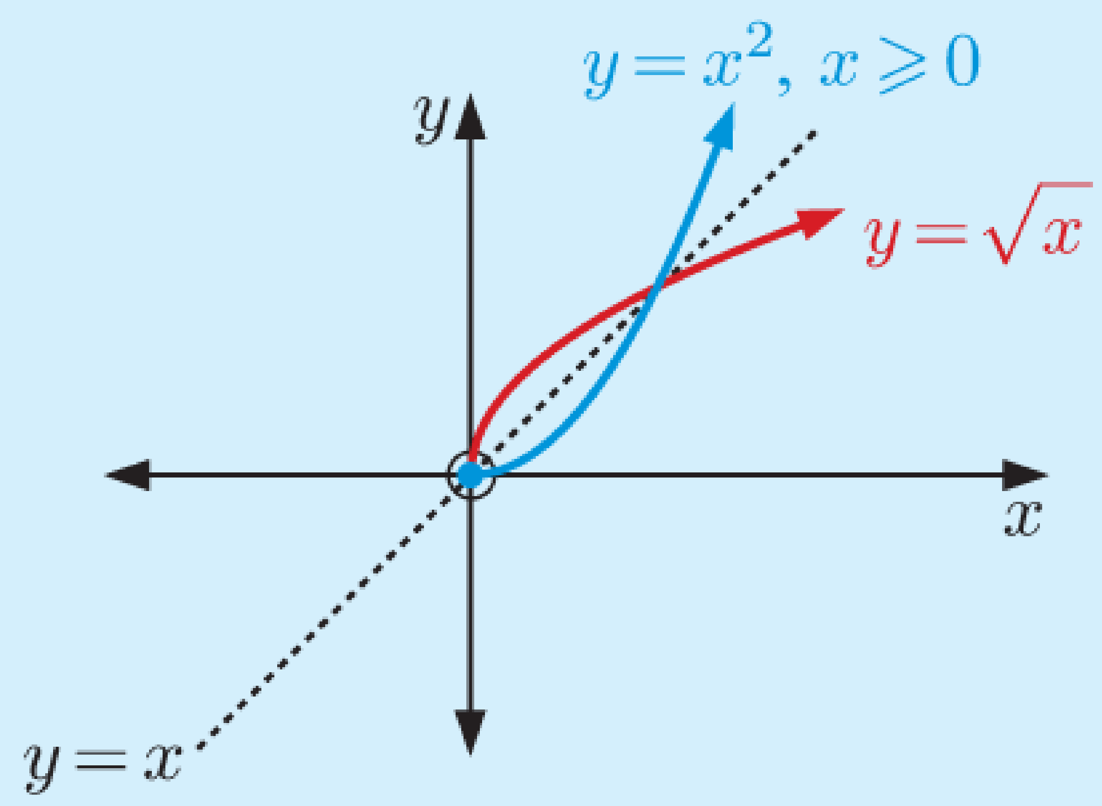
$\therefore\ f^{-1}$ is defined by $x = y^2$, $y \geqslant 0$

$$\therefore\ y = \pm\sqrt{x},\quad y \geqslant 0$$

$$\therefore\ y = \sqrt{x}\quad \{\text{as } -\sqrt{x} \text{ is } \leqslant 0\}$$

So, $f^{-1}(x) = \sqrt{x}$

d



SELF-INVERSE FUNCTIONS

Any function which has an inverse, and whose graph is symmetrical about the line $y = x$, is a **self-inverse function**.

For example:

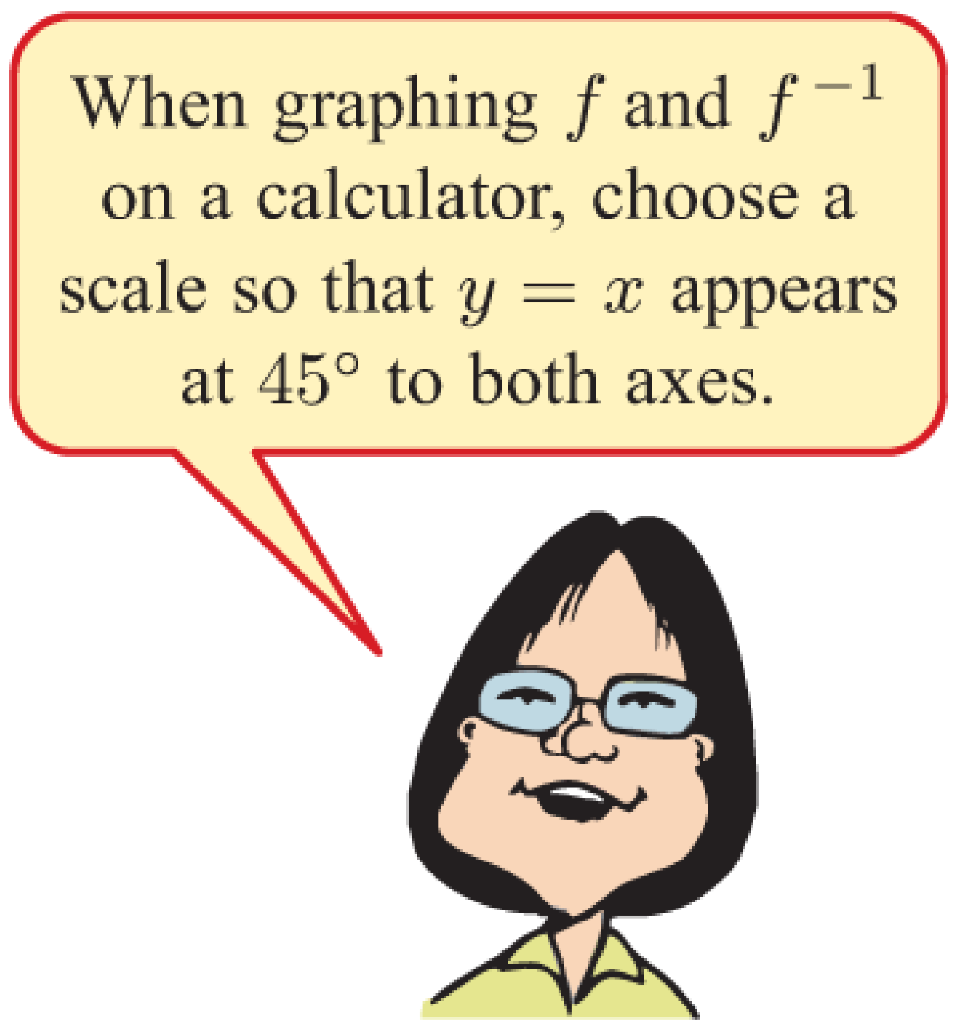
- The function $f(x) = x$ is the **identity function**, and is also a self-inverse function.
- The function $f(x) = \frac{1}{x}$, $x \neq 0$, is also a self-inverse function, as $f = f^{-1}$.

EXERCISE 15F

1 For each of the following functions f:

- **i** On the same set of axes, graph $y = x$, $y = f(x)$, and $y = f^{-1}(x)$.
- **ii** Find $f^{-1}(x)$ using coordinate geometry and the gradient of $y = f^{-1}(x)$ from **i**.
- **iii** Find $f^{-1}(x)$ using variable interchange.

a $f : x \mapsto 3x + 1$ **b** $f : x \mapsto \dfrac{x+2}{4}$



2 For each of the following functions f:

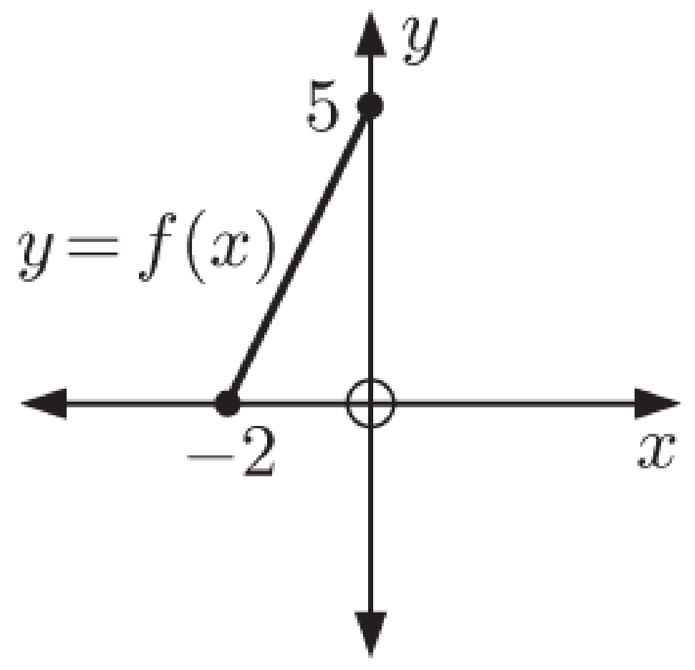
- **i** Find $f^{-1}(x)$.
- **ii** Sketch $y = f(x)$, $y = f^{-1}(x)$, and $y = x$ on the same set of axes.
- **iii** Show that $(f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$, the identity function.

a $f : x \mapsto 2x + 5$ **b** $f : x \mapsto \dfrac{3-2x}{4}$ **c** $f : x \mapsto x + 3$

3 Copy the graphs of the following functions and draw the graphs of $y = x$ and $y = f^{-1}(x)$ on the same set of axes. In each case, state the domain and range of both f and f^{-1}.

PRINTABLE GRAPHS

a



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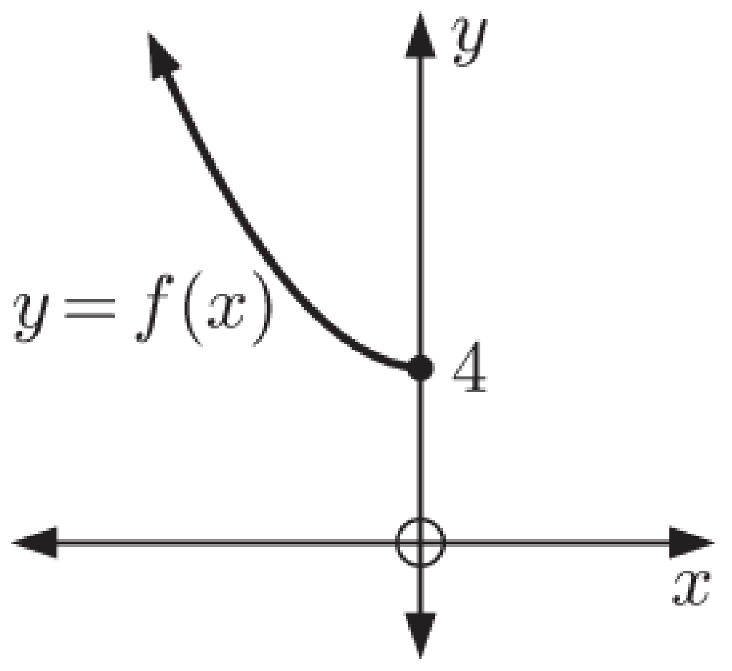

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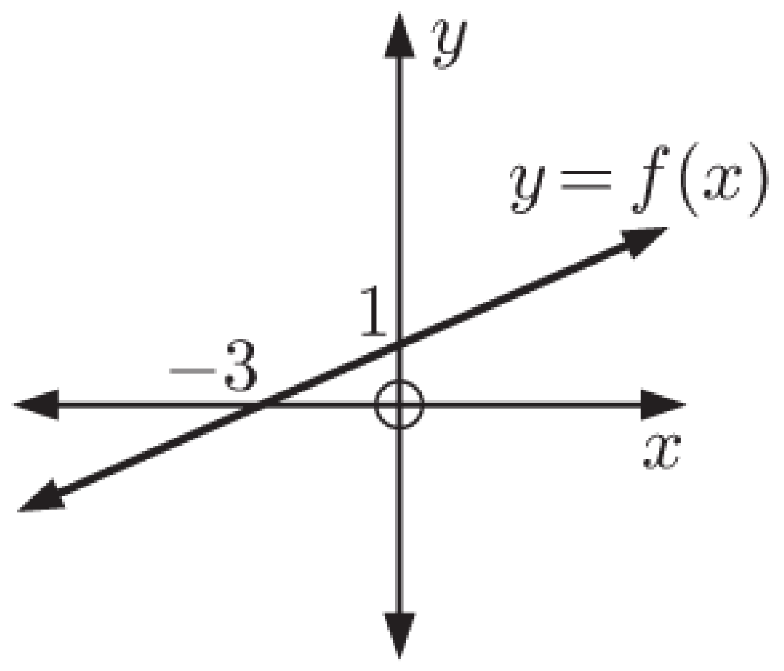

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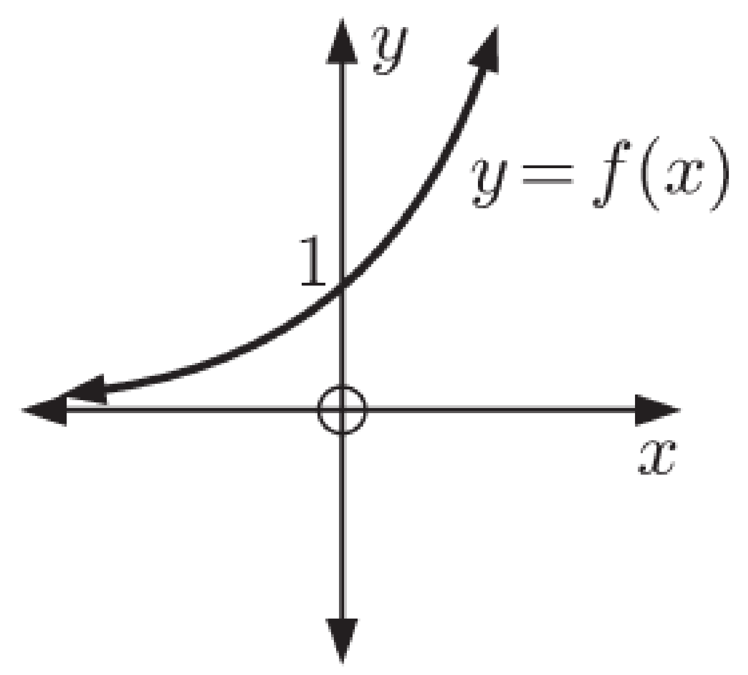

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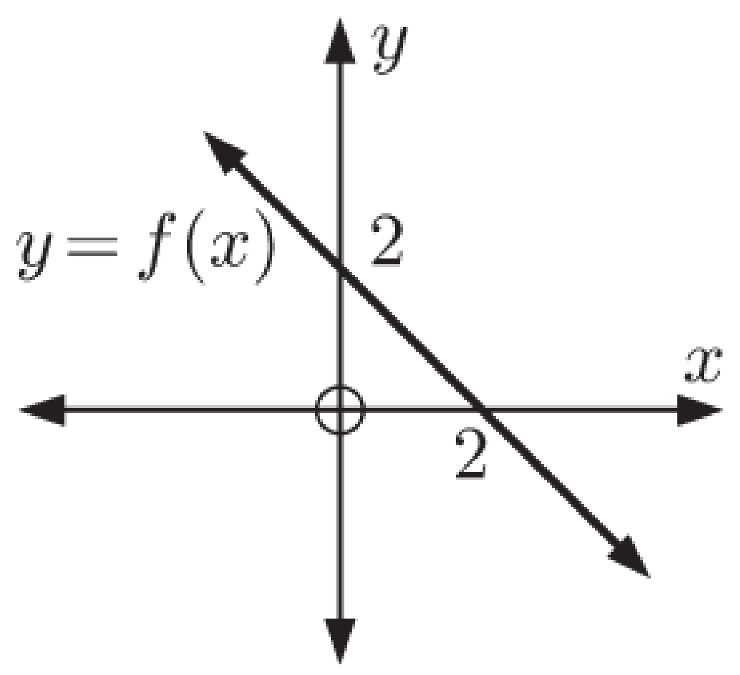

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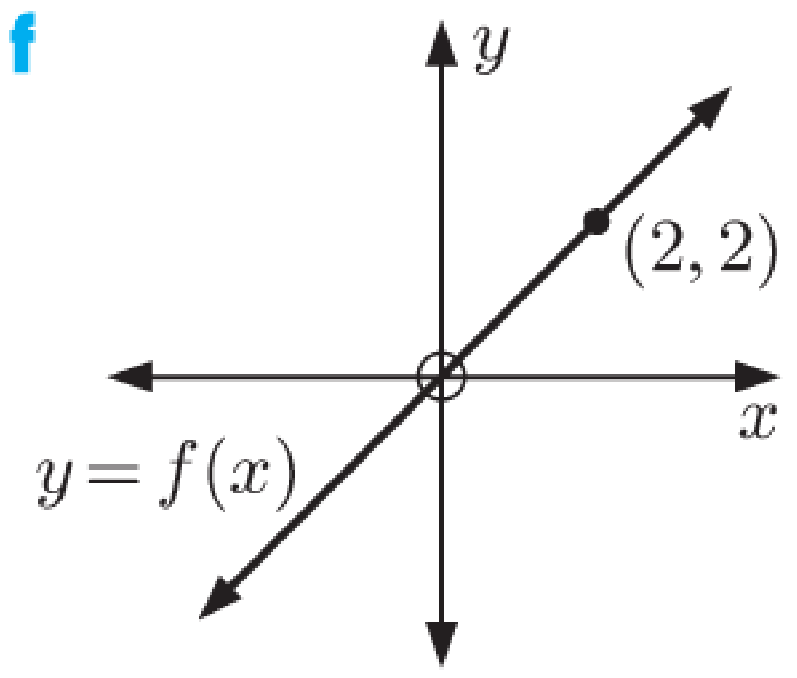


4 Given $f(x) = 2x - 5$, find $(f^{-1})^{-1}(x)$. What do you notice?

5 Which of the following functions have inverses? Where an inverse exists, write down the inverse function.

a $\{(1, 2), (2, 4), (3, 5)\}$ **b** $\{(-1, 3), (0, 2), (1, 3)\}$

c $\{(2, 1), (-1, 0), (0, 2), (1, 3)\}$ **d** $\{(-1, -1), (0, 0), (1, 1)\}$

6 Find *all* linear functions which are self-inverse.

7 If the one-to-one function $H(x)$ has domain $\{x \mid -2 \leqslant x < 3\}$, find the range of its inverse $H^{-1}(x)$.

8 **a** Sketch the graph of $f : x \mapsto x^2 - 4$ and reflect it in the line $y = x$.

b Does f have an inverse function?

c Does f with restricted domain $x \geqslant 0$ have an inverse function?

9 Sketch the graph of $f : x \mapsto x^3$ and its inverse function $f^{-1}(x)$.

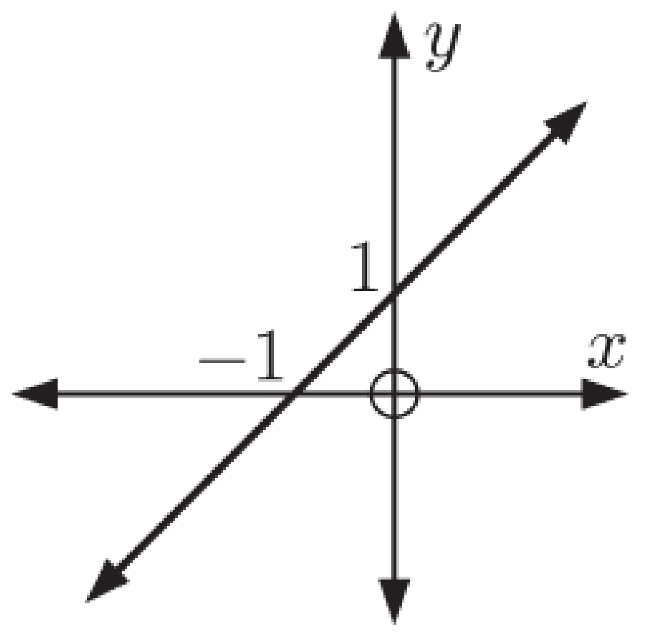
10 Given $f(x) = \frac{1}{x}$, $x \neq 0$, show that f is self-inverse.

11 The **horizontal line test** says: *For a function to have an inverse function, no horizontal line can cut its graph more than once.*

a Explain why this is a valid test for the existence of an inverse function.

b Which of the following functions have an inverse function?

i



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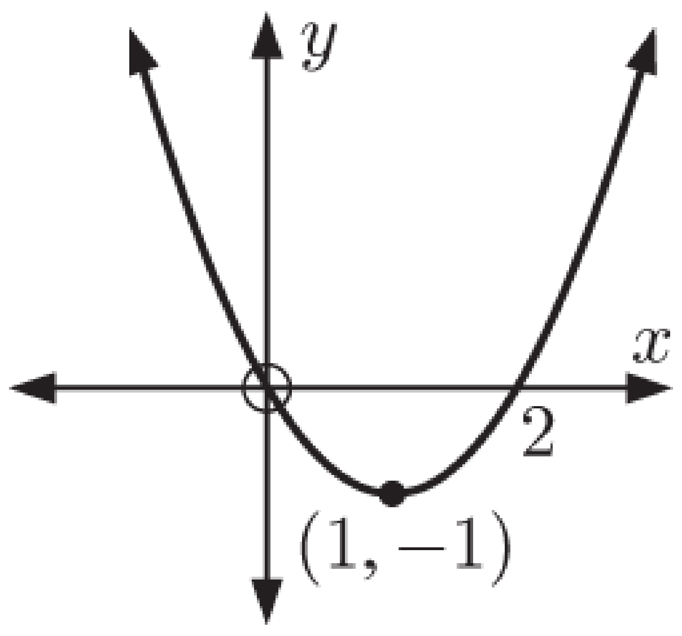

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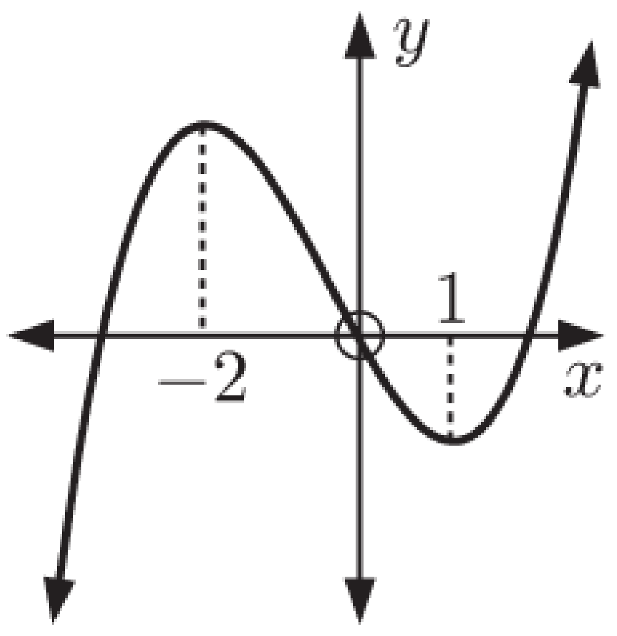


c For the functions in **b** which do not have an inverse, specify restricted domains as wide as possible such that the resulting function does have an inverse.

12 Consider $f : x \mapsto x^2$, $x \leqslant 0$.

a Find $f^{-1}(x)$.

b Sketch $y = f(x)$, $y = x$, and $y = f^{-1}(x)$ on the same set of axes.

13 **a** Explain why $f(x) = x^2 - 4x + 3$ is a function but does not have an inverse function.

b Explain why $g(x) = x^2 - 4x + 3$, $x \geqslant 2$, has the inverse function $g^{-1}(x) = 2 + \sqrt{1 + x}$.

c State the domain and range of g and g^{-1}.

d Show that $(g \circ g^{-1})(x) = (g^{-1} \circ g)(x) = x$, the identity function.

14 Consider $f : x \mapsto (x + 1)^2 + 3$, $x \geqslant -1$.

a Find $f^{-1}(x)$.

b Use technology to help sketch the graphs of $y = f(x)$, $y = x$, and $y = f^{-1}(x)$.

c State the domain and range of f and f^{-1}.

15 Consider $f(x) = 4 + 6x - x^2$, $x \leqslant 3$.

a Find $f^{-1}(x)$.

b State the domain and range of f and f^{-1}.

16 Let $f(x) = 2x^2 - 10x + 6$, $x \leqslant k$.

a Find the largest value of k such that $f^{-1}(x)$ exists.

b For this value of k:

i Find $f^{-1}(x)$.

ii State the domain and range of $f^{-1}(x)$.

17 Consider the functions $f: x \mapsto 2x+5$ and $g: x \mapsto \dfrac{8-x}{2}$.

- **a** Find $g^{-1}(x)$.
- **b** Hence solve $g(x) = -1$.
- **c** Show that $f^{-1}(-3) - g^{-1}(6) = 0$.
- **d** Find x such that $(f \circ g^{-1})(x) = 9$.

18 Consider $f: x \mapsto 2x$ and $g: x \mapsto 4x-3$.

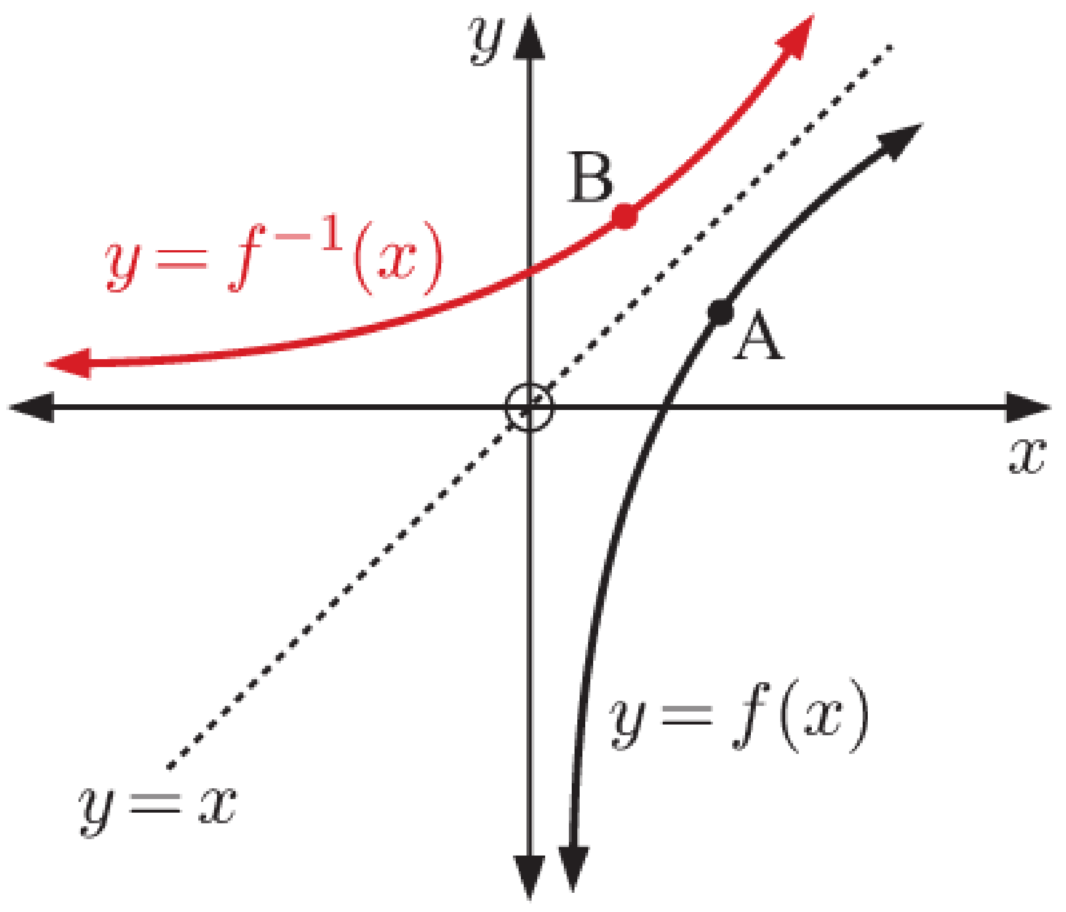
- **a** Find $(f \circ g)(x)$.
- **b** Given $(f \circ g)^{-1}(k) = 2$, find k.
- **c** Show that $(f^{-1} \circ g^{-1})(x) = (g \circ f)^{-1}(x)$.

19 Show that $f: x \mapsto \dfrac{3x-8}{x-3}$, $x \neq 3$ is self-inverse by:

- **a** referring to its graph
- **b** using algebra.

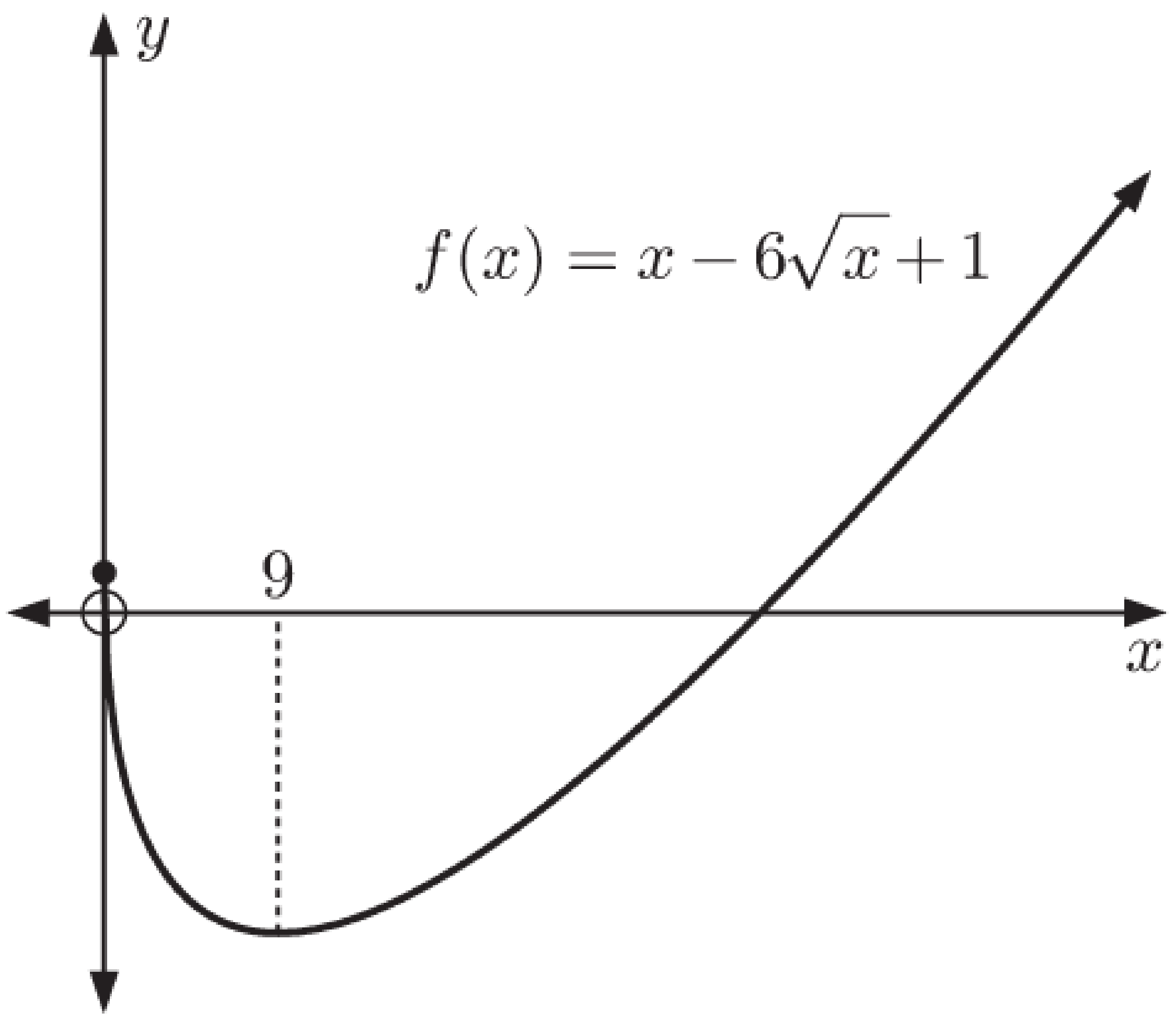
20 Under what conditions is the rational function $y = \dfrac{ax+b}{cx+d}$, $c \neq 0$, a self-inverse function?

21



- **a** B is the image of A under a reflection in the line $y = x$. If A is $(x, f(x))$, find the coordinates of B.
- **b** By substituting your result from **a** into $y = f^{-1}(x)$, show that $f^{-1}(f(x)) = x$.
- **c** Using a similar method, show that $f(f^{-1}(x)) = x$.

22 The graph of $f(x) = x - 6\sqrt{x} + 1$ is shown alongside.



- **a** State the natural domain of $f(x)$.
- **b** Does $f(x)$ have an inverse function? Explain your answer.
- **c** Let $g(x) = x - 6\sqrt{x} + 1$, $0 \leqslant x \leqslant 9$.
  - **i** Find $g^{-1}(x)$, and verify that $(g \circ g^{-1})(4) = 4$.
  - **ii** State the domain and range of g and g^{-1}.
- **d** Let $h(x) = x - 6\sqrt{x} + 1$, $x \geqslant 9$.
  - **i** Find $h^{-1}(x)$ and verify that $(h \circ h^{-1})(16) = 16$.
  - **ii** State the domain and range of h and h^{-1}.
  - **iii** Find the value of x such that $g^{-1}(x) = h^{-1}(x)$.

THEORY OF KNOWLEDGE

The notation and terminology of mathematics has rules which tell us how to construct expressions or mathematical "sentences". This allows us to communicate mathematical ideas in a written form.

For example, the expression "$1 + 1 = 2$" tells us that "one added to one is equal to two".

1. What does it mean for something to be a "language"?
2. Does mathematics have a "grammar" or **syntax** in the same sense as the English language?

In computer science, **Backus-Naur form** (BNF) is commonly used to define the syntax of programming languages. BNF can also be used to describe the rules of non-programming related languages.

3 Research how BNF works and use it to define the syntax of mathematical function notation.

4 Mathematical expressions can also be represented with diagrams such as **abstract syntax trees** and **syntax (railroad) diagrams**. Which form is more *efficient* in conveying its information? Which form is more useful?

The fact that something is *grammatically* correct does not make it *logically* true.

For example, consider the grammatically correct but illogical English sentence: "The sun is cold."

5 The **syntax** of a language refers to its structure and rules. The **semantics** of a language is all about meaning.

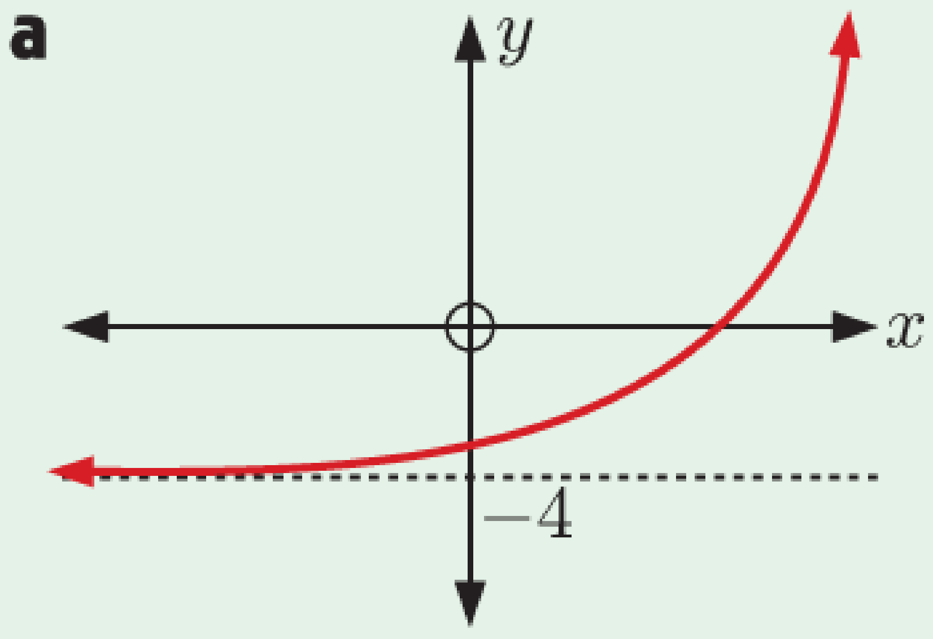
- **a** Why is it important to distinguish between these two concepts?
- **b** In mathematics, is one more important than the other?

REVIEW SET 15A

1 For each graph, state:

i the domain **ii** the range **iii** whether the graph shows a function.

a



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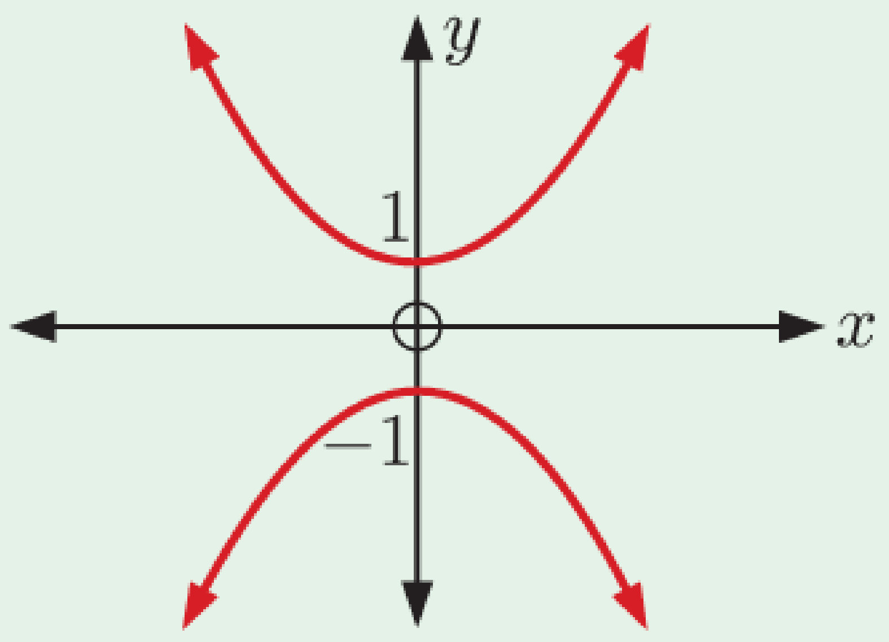
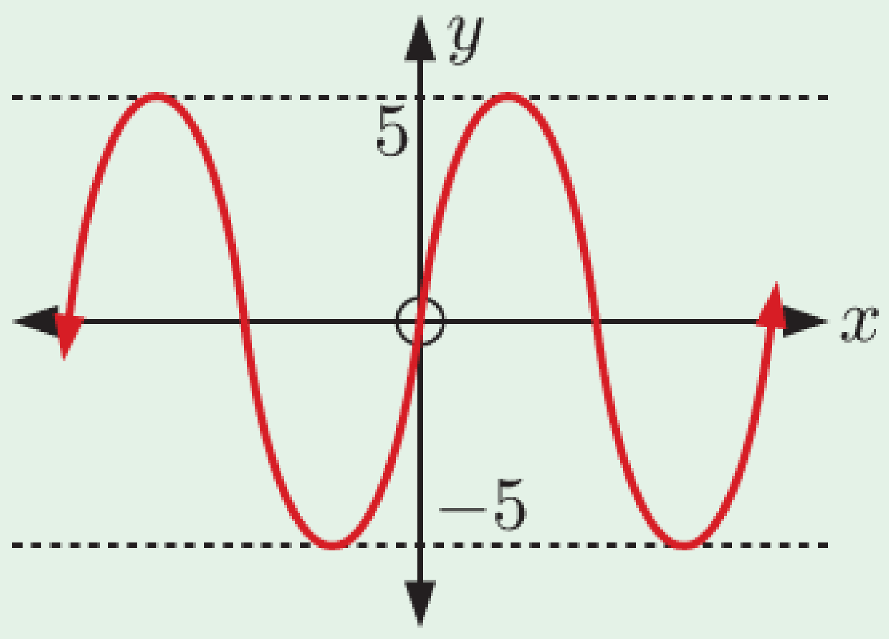


c



2 If $f(x) = 2x - x^2$, find:

- **a** $f(2)$
- **b** $f(-3)$
- **c** $f(-\frac{1}{2})$

3 Suppose $f(x) = ax + b$ where a and b are constants. If $f(1) = 7$ and $f(3) = -5$, find a and b.

4 Consider $f(x) = \frac{-2}{x^2}$.

- **a** For what value of x is $f(x)$ undefined?
- **b** Sketch the function using technology.
- **c** State the domain and range of the function.

5 Consider $f(x) = x^2$ and $g(x) = 1 - 6x$.

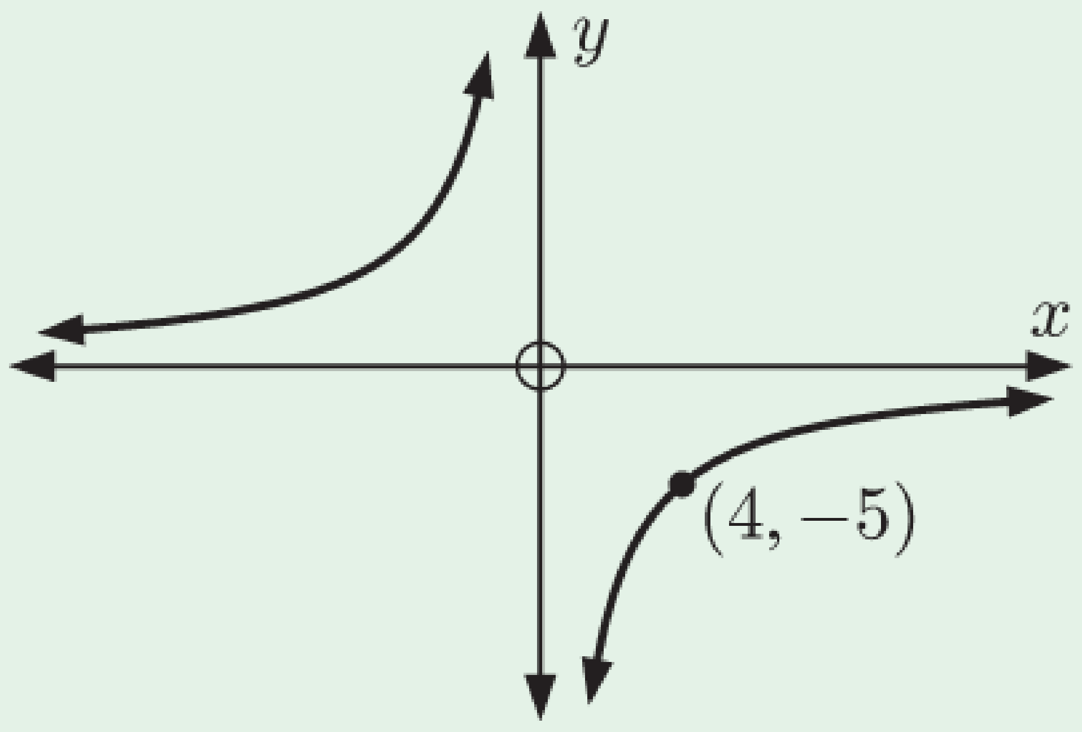
- **a** Show that $f(-3) = g(-\frac{4}{3})$.
- **b** Find x such that $g(x) = f(5)$.

6 Find the domain and range of:

- **a** $y = \sqrt{x+4}$
- **b** $y = -(1-x)^2 + 1$
- **c** $y = 2x^2 - 3x + 1$

7 Determine the equation of the reciprocal functions:

a



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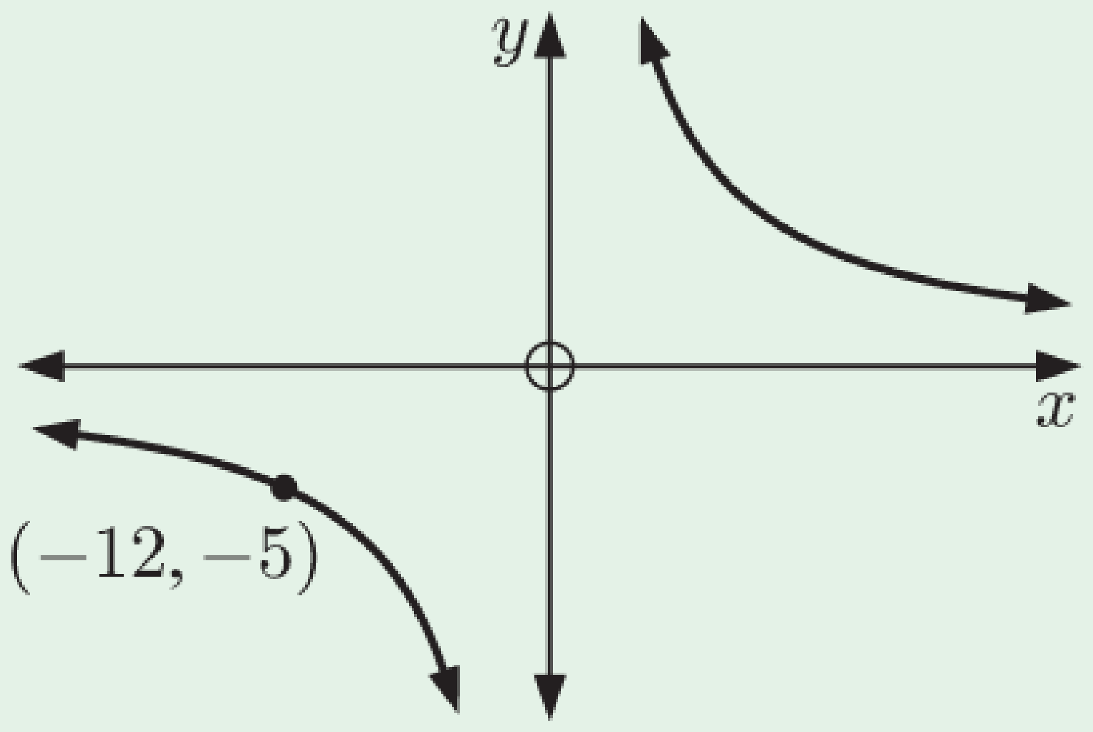


8 Consider the function $f : x \mapsto \frac{4x+1}{2-x}$.

a Find the equations of the asymptotes.

b State the domain and range of the function.

c Draw a sign diagram of the function. Hence discuss the behaviour of the function as it approaches its asymptotes.

d Find the axes intercepts. **e** Sketch the function.

9 Suppose $f(x) = 2x - 5$ and $g(x) = 3x + 1$.

a Find $(f \circ g)(x)$. **b** Solve $(f \circ g)(x) = f(x+3)$.

10 If $f(x) = 1 - 2x$ and $g(x) = \sqrt{x}$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $(g \circ g)(81)$

11 Suppose $f(x) = \sqrt{x+2}$ and $g(x) = x^2 - 3$.

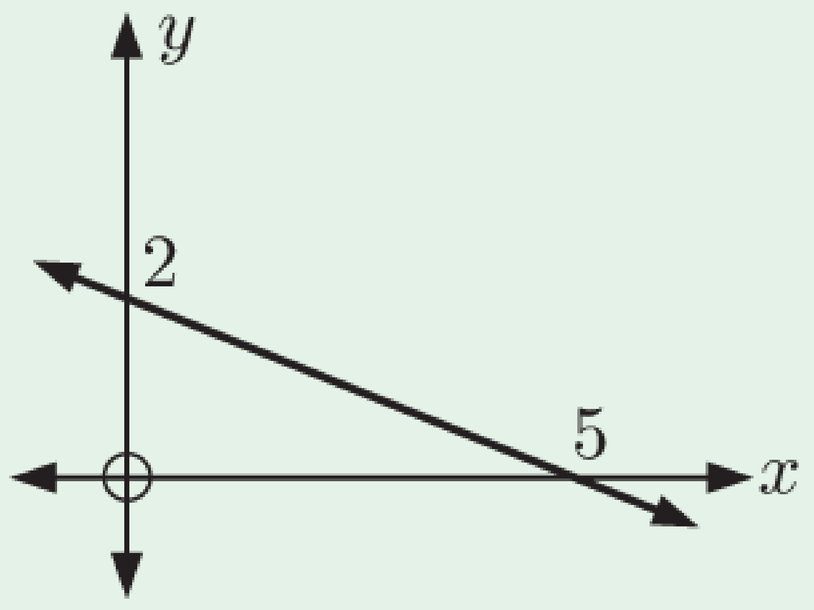
a Find $(f \circ g)(x)$, and state its domain and range.

b Find $(g \circ f)(x)$, and state its domain and range.

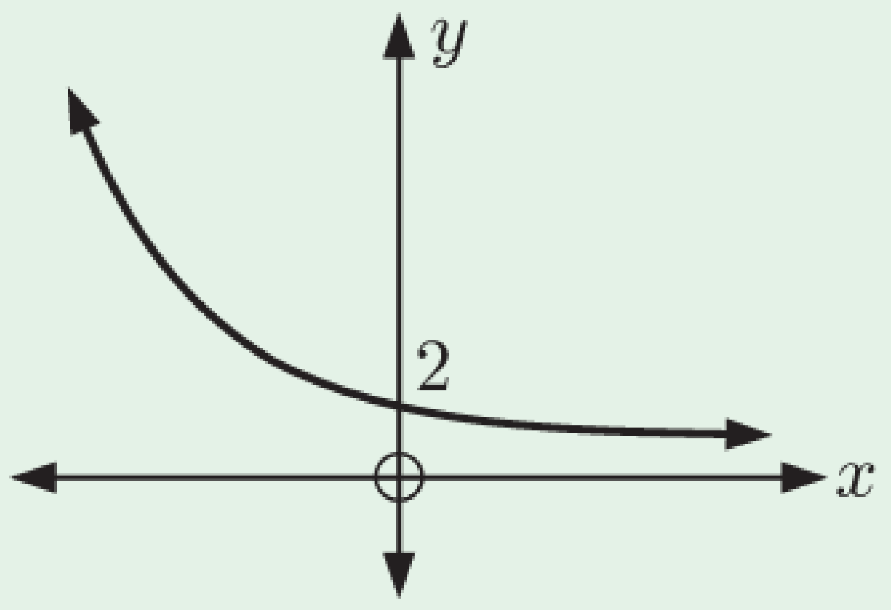
12 If $f(x) = ax + b$, $f(2) = 1$, and $f^{-1}(3) = 4$, find a and b.

13 Copy the following graphs and draw the inverse function on the same set of axes:

a



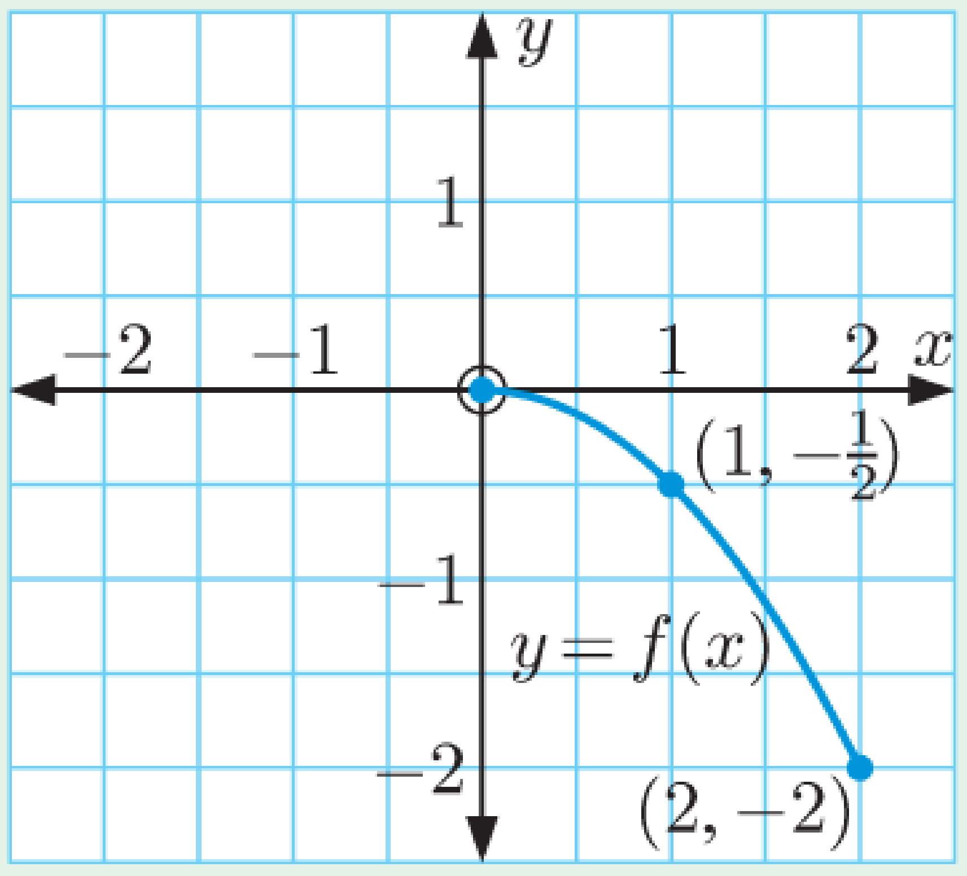
b



14 Find $f^{-1}(x)$ given that $f(x)$ is:

a $4x + 2$ **b** $\frac{3-5x}{4}$

15 The graph of the function $f(x) = -\frac{1}{2}x^2$, $0 \leqslant x \leqslant 2$ is shown alongside.



a Sketch the graph of $y = f^{-1}(x)$.

b State the range of f^{-1}.

c Solve:

i $f(x) = -\frac{3}{2}$ **ii** $f^{-1}(x) = 1$

16 Given $f : x \mapsto 3x + 6$ and $h : x \mapsto \frac{x}{3}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.

17 Show that $f : x \mapsto \frac{5x-1}{x-5}$, $x \neq 5$ is self-inverse by:

a referring to its graph **b** using algebra.

18 If $f : x \mapsto \sqrt{x}$ and $g : x \mapsto 3 + x$, find:

a $f^{-1}(2) \times g^{-1}(2)$ **b** $(f \circ g)^{-1}(2)$.

19 **a** Sketch the graph of $g : x \mapsto x^2 + 6x + 7$ for $x \in \,]-\infty, -3]$.

b Explain why g has an inverse function g^{-1}.

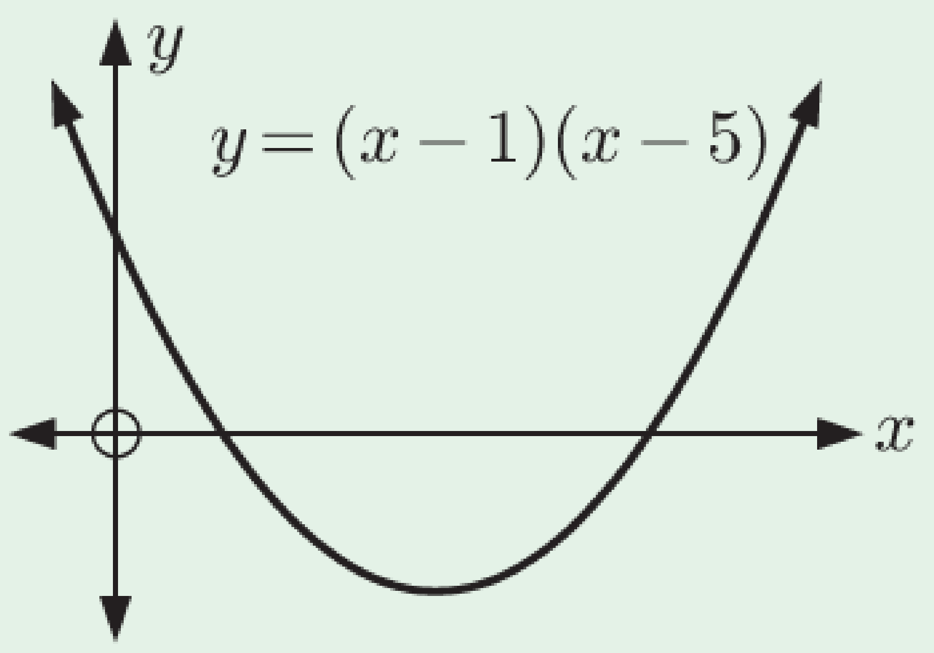
c Find algebraically, a formula for g^{-1}. **d** Sketch the graph of $y = g^{-1}(x)$.

e Find the range of g. **f** Find the domain and range of g^{-1}.

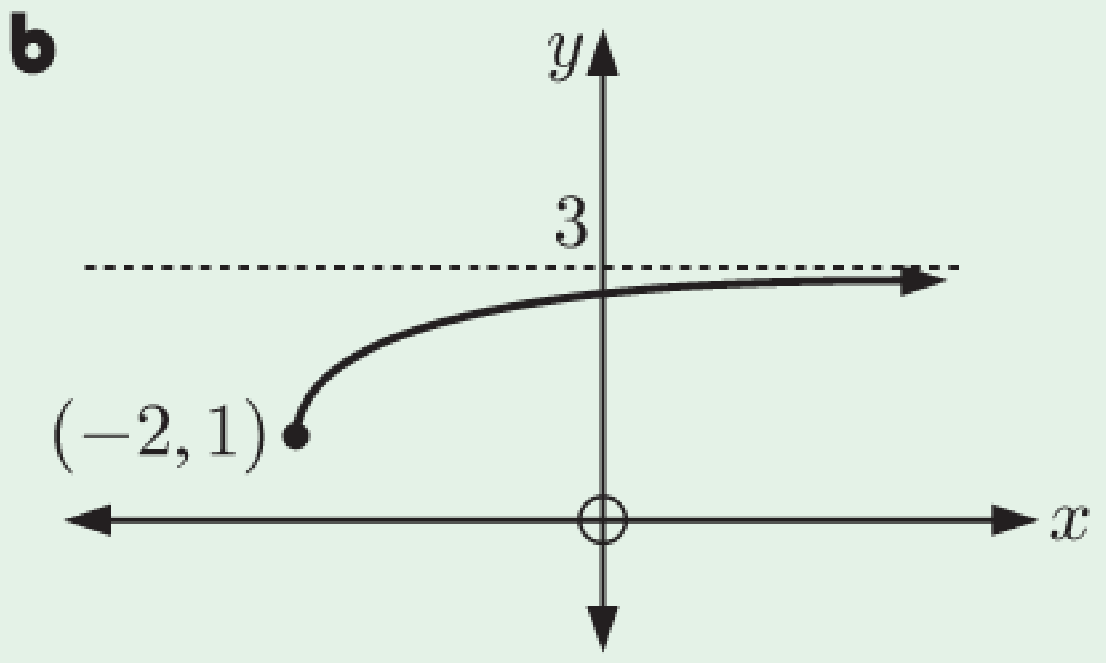
REVIEW SET 15B

1 State the domain and range of each function:

a



b



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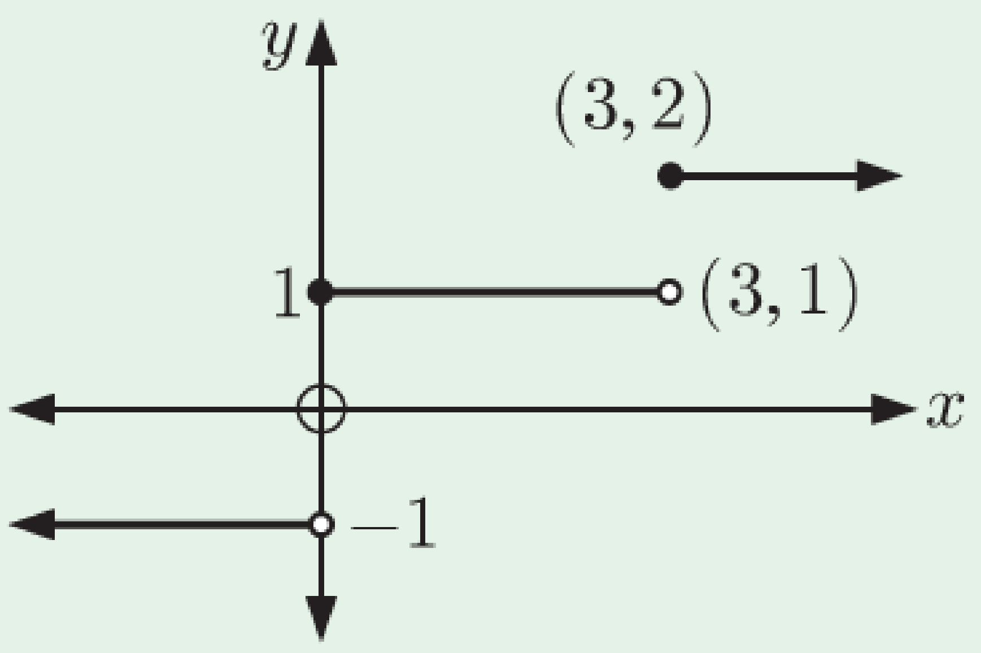


2 If $g(x) = x^2 - 3x$, find in simplest form:

a $g(x+1)$ **b** $g(4x)$

3 Use algebraic methods to determine whether these relations are functions:

a $x + 2y = 10$ **b** $x + y^2 = 10$

4 State the domain and range of:

a $f(x) = 10 + \frac{3}{2x-1}$ **b** $f(x) = \sqrt{x+7}$

5 **a** Use technology to help sketch the graph of the relation $y = \sqrt{9-x}$.

b Determine whether the relation is a function.

c Find the domain and range of the relation.

6 Find a such that the range of $y = ax^2 + 12x - 13$ is $\{y \mid y \leqslant 5\}$.

7 Suppose $f(x) = ax^2 + bx + c$. Find a, b, and c if $f(0) = 5$, $f(-2) = 21$, and $f(3) = -4$.

8 For the function $f(x) = -1 + \frac{3}{x+2}$:

a Find the equations of the asymptotes. **b** State the domain and range.

c Find the axes intercepts.

d Discuss the behaviour of the function as it approaches its asymptotes.

e Sketch the graph of the function.

9 Use technology to sketch $y = x^3 - 4x^2 + x$, $-1 \leqslant x \leqslant 4$. Include the points at the ends of the domain, and any turning points. Hence state the range of the function.

10 Given $f(x) = 3 - x^2$ and $g(x) = 2x - 1$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $(f \circ f)(-2)$

11 Suppose $f(x) = \frac{1}{x^2}$ and $g(x) = x^2 - 4x + 3$. Find $(f \circ g)(x)$ and state its domain and range.

12 Suppose $f(x) = 3x + 5$ and $g(x) = 2x^2 - x$.

a Find in simplest form: **i** $(f \circ g)(x)$ **ii** $(g \circ f)(x)$

b Hence solve $3(f \circ g)(x) = (g \circ f)(x)$.

13 An object is dropped from the top of a cliff. The distance it has travelled when it has speed S is $D(S) = \frac{S^2}{19.6}$ metres. Its speed after t seconds is given by $S(t) = 9.8t$ m s^{-1}.

a Find the composite function $D \circ S$, and explain what it means.

b Find and interpret $(D \circ S)(5)$.

14 If $f(2x + 3) = 5x - 7$, find $f^{-1}(x)$.

15 Given $f : x \mapsto 5x - 2$ and $h : x \mapsto \frac{3x}{4}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.

16 Consider the functions $f(x) = 3x + 1$ and $g(x) = \frac{2}{x}$.

a Find $(g \circ f)(x)$. **b** Given $(g \circ f)(x) = -4$, solve for x.

c Let $h(x) = (g \circ f)(x)$, $x \neq -\frac{1}{3}$.

  i Write down the equations of the asymptotes of $h(x)$.

  ii Sketch the graph of $h(x)$ for $-3 \leqslant x \leqslant 2$.

  iii State the range of $h(x)$ for the domain $-3 \leqslant x \leqslant 2$.

17 Given $f(x) = 2x + 11$ and $g(x) = x^2$, find $(g \circ f^{-1})(3)$.

18 The function $f(x) = \frac{ax + 3}{x - b}$ has asymptotes $x = -1$ and $y = 2$.

a Find a and b. **b** Find the domain and range of $f^{-1}(x)$.

19 If $f : x \mapsto 2x + 1$ and $g : x \mapsto \frac{x + 1}{x - 2}$, find:

a $(f \circ g)(x)$ **b** $g^{-1}(x)$.

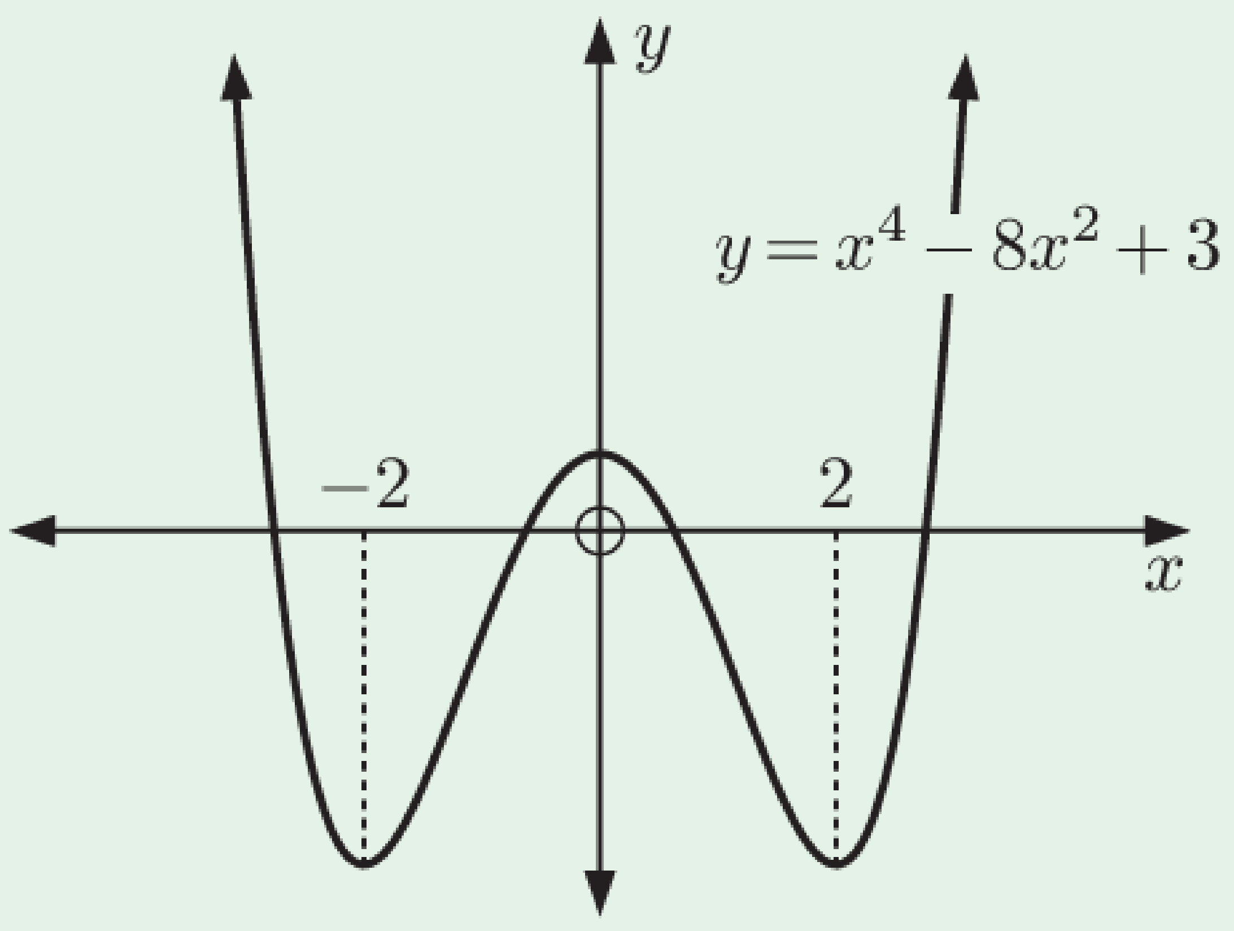
20 The graph of $y = x^4 - 8x^2 + 3$ is shown alongside. For each of the following functions, find the inverse function and state the domain and range of the inverse function:

a $f(x) = x^4 - 8x^2 + 3$, $0 \leqslant x \leqslant 2$

b $g(x) = x^4 - 8x^2 + 3$, $x \geqslant 2$

c $h(x) = x^4 - 8x^2 + 3$, $-2 \leqslant x \leqslant 0$

d $j(x) = x^4 - 8x^2 + 3$, $x \leqslant -2$



Chapter 16

Transformations of functions

Contents:

OPENING PROBLEM

In our study of quadratic functions, we saw that the completed square form $y = (x - h)^2 + k$ was extremely useful in identifying the vertex $(h,\ k)$.

Things to think about:

- **a** What transformation maps the graph $y = x^2$ onto the graph $y = (x - h)^2 + k$?
- **b** If we let $f(x) = x^2$, what function is $f(x - h) + k$?
- **c** In general terms, what transformation maps $y = f(x)$ onto $y = f(x - h) + k$?

In this Chapter we perform **transformations** of graphs to produce the graph of a related function.

The transformations of $y = f(x)$ we consider include:

- **translations** $y = f(x) + b$ and $y = f(x - a)$
- **stretches** $y = p\,f(x)$, $p > 0$ and $y = f(qx)$, $q > 0$
- **reflections** $y = -f(x)$ and $y = f(-x)$
- transformations of the form $y = \dfrac{1}{f(x)}$
- combinations of these transformations.

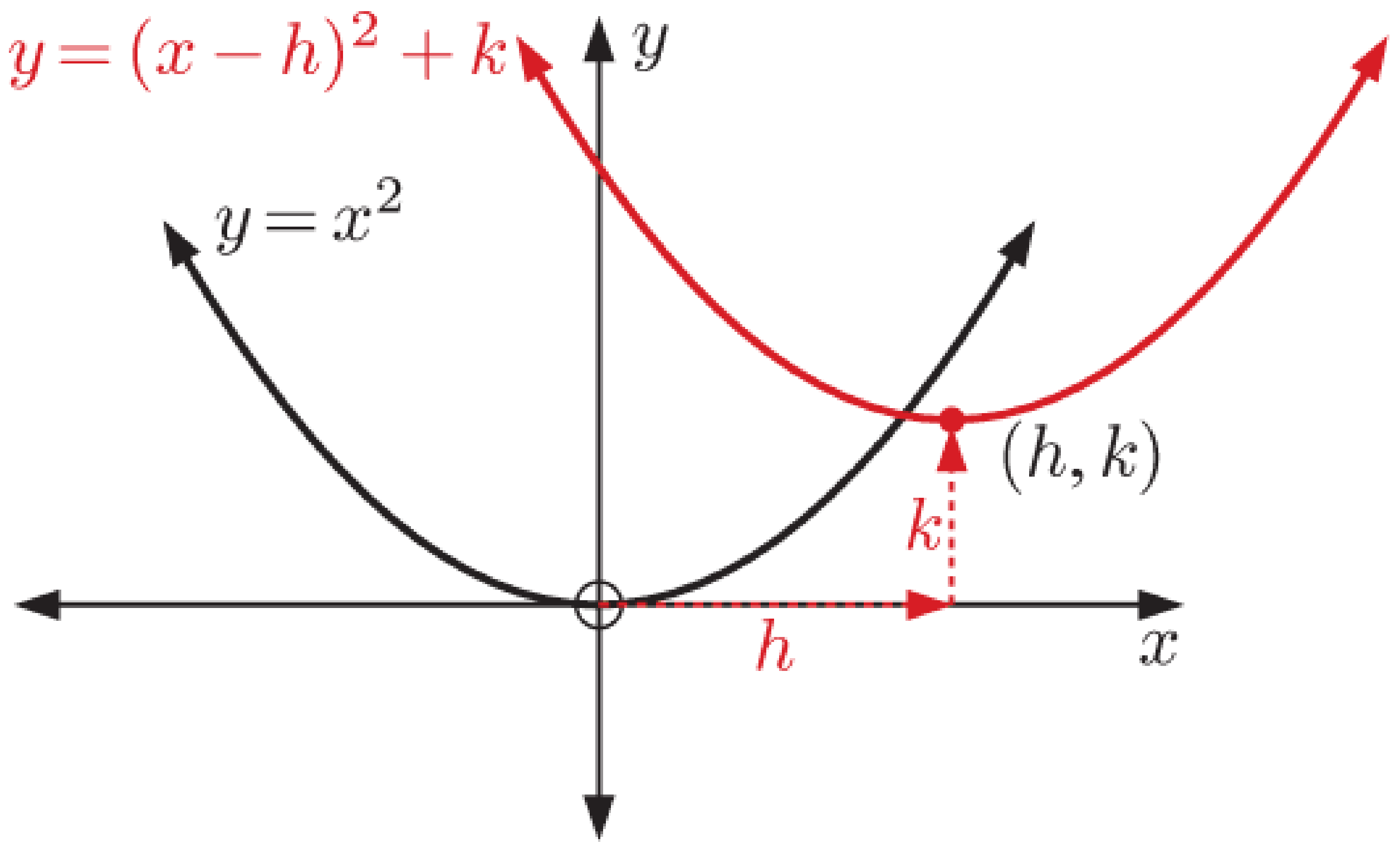
A TRANSLATIONS

If $f(x) = x^2$ then $f(x - h) + k = (x - h)^2 + k$.

The graph $y = (x - h)^2 + k$ has the same shape as $y = x^2$.

It can be produced from $y = x^2$ by a translation h units to the right and k units upwards.

This shifts the vertex of the parabola from the origin O(0, 0) to $(h,\ k)$.



INVESTIGATION 1 TRANSLATIONS

Our observations of quadratics suggest that $y = f(x)$ can be transformed into $y = f(x - a) + b$ by a *translation*. In this Investigation we test this theory with other functions.

What to do:

GRAPHING PACKAGE

1 Let $f(x) = x^3$.

- **a** Write down:
  - **i** $f(x) + 2$ **ii** $f(x) - 3$ **iii** $f(x) + 6$

  Graph $y = f(x)$ and the other three functions on the same set of axes. Record your observations.
- **b** Write down:
  - **i** $f(x - 2)$ **ii** $f(x + 3)$ **iii** $f(x - 6)$

  Graph $y = f(x)$ and the other three functions on the same set of axes. Record your observations.

c Write down:

  i $f(x-1)+3$ **ii** $f(x+2)+1$ **iii** $f(x-3)-4$

  Graph $y=f(x)$ and the other three functions on the same set of axes.

2 Repeat **1** for the function $y=\frac{1}{x}$.

3 Describe the transformation which maps $y=f(x)$ onto:

  a $y=f(x)+b$ **b** $y=f(x-a)$ **c** $y=f(x-a)+b$

4 Do any of these transformations change the *shape* of the graph?

From the **Investigation** you should have found:

- For $y=f(x)+b$, the effect of b is to **translate** the graph **vertically** through b units.
  - If $b>0$ it moves **upwards**.
  - If $b<0$ it moves **downwards**.
- For $y=f(x-a)$, the effect of a is to **translate** the graph **horizontally** through a units.
  - If $a>0$ it moves to the **right**.
  - If $a<0$ it moves to the **left**.
- For $y=f(x-a)+b$, the graph is translated horizontally a units and vertically b units.

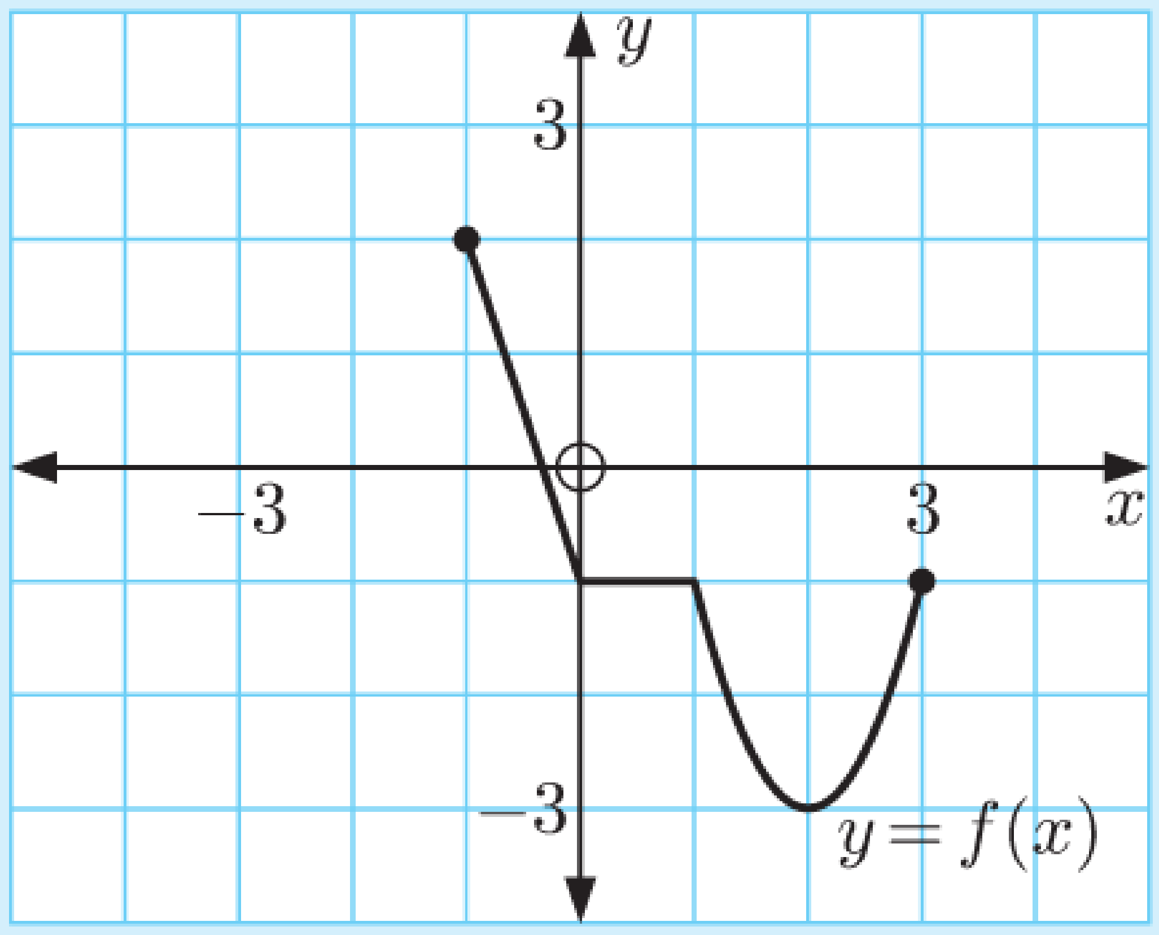
  We say it is **translated by the vector** $\begin{pmatrix} a \\ b \end{pmatrix}$.

Example 1 **Self Tutor**

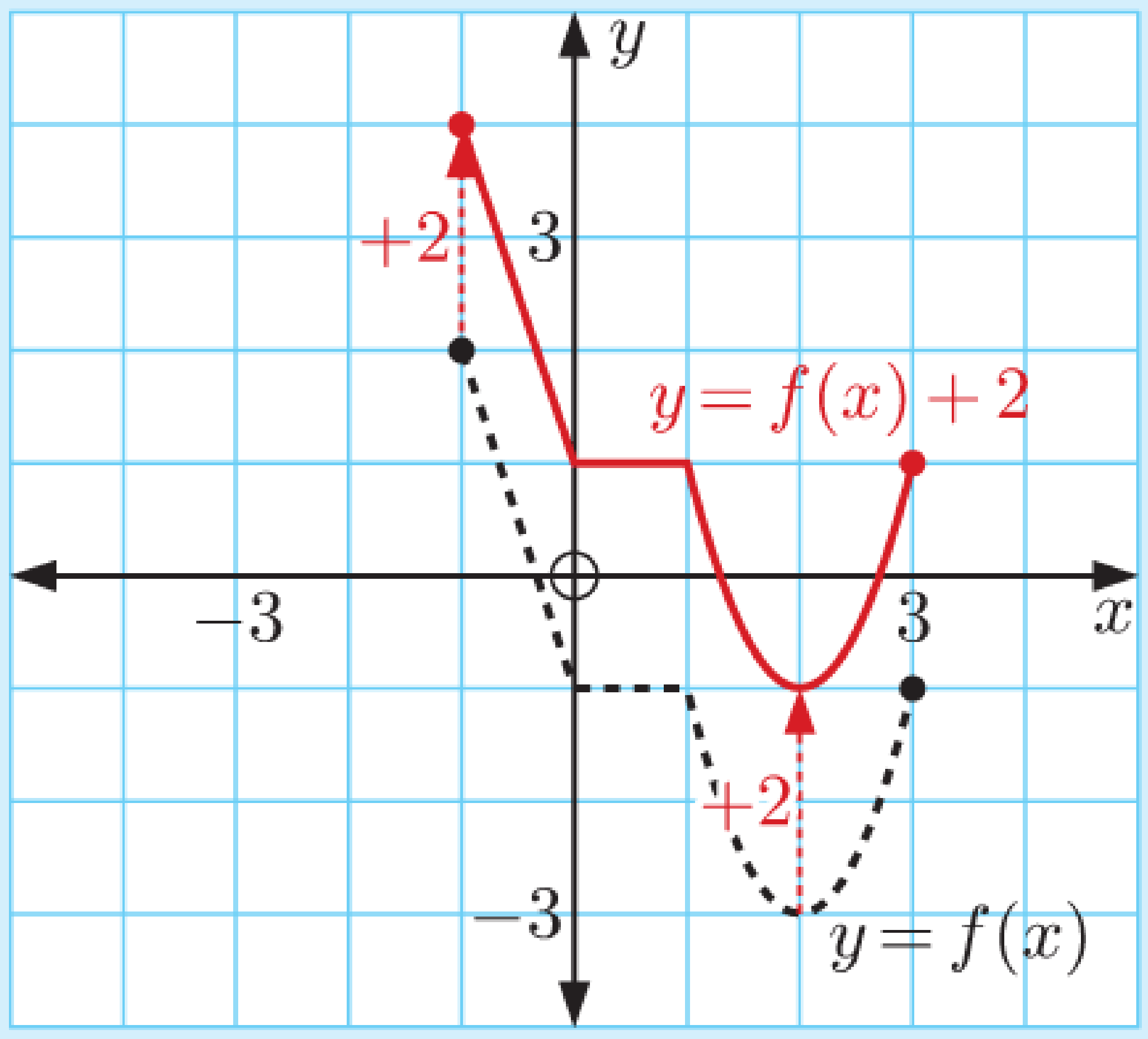
Consider the graph of $y=f(x)$ alongside.

On separate axes, draw the graphs of:

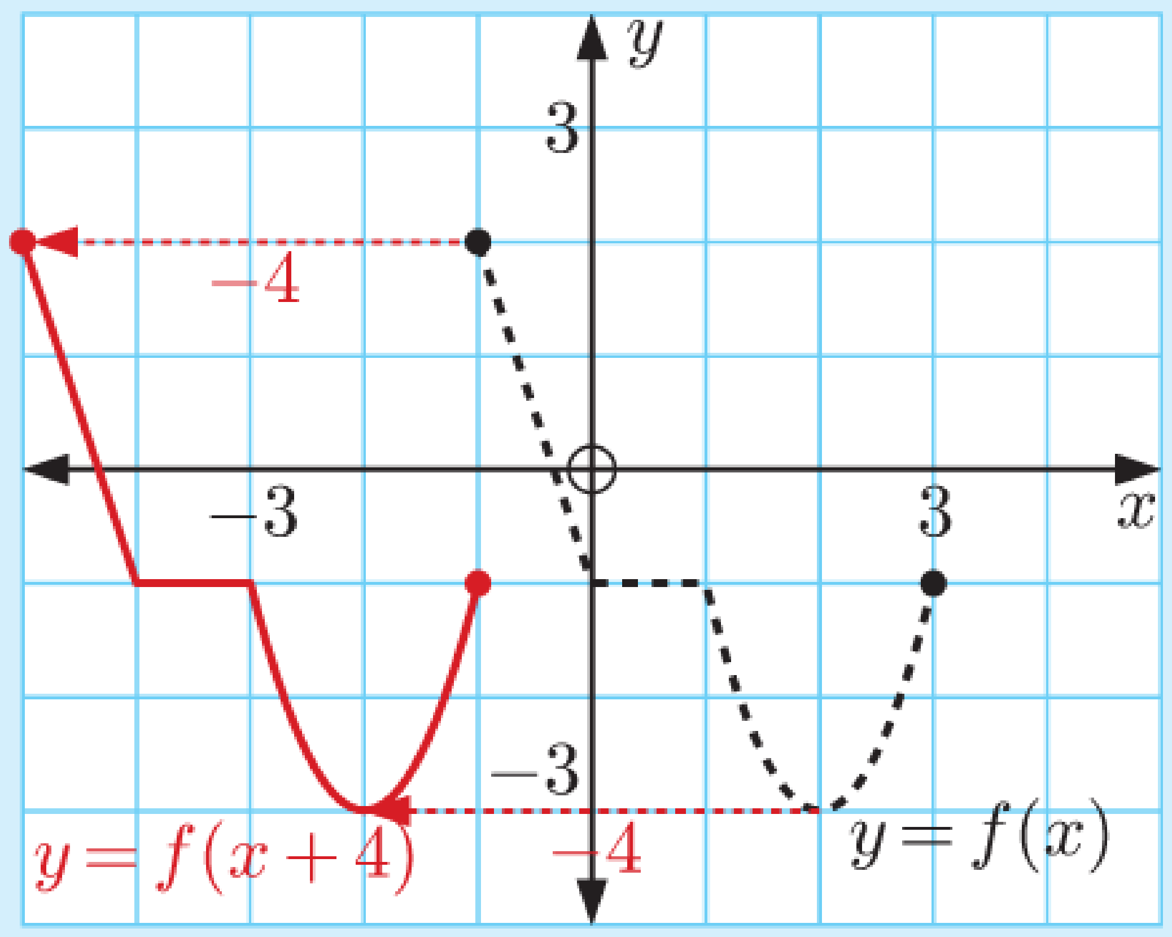
a $y=f(x)+2$ **b** $y=f(x+4)$



a The graph of $y=f(x)+2$ is found by translating $y=f(x)$ 2 units upwards.



b The graph of $y = f(x+4)$ is found by translating $y = f(x)$ 4 units to the left.

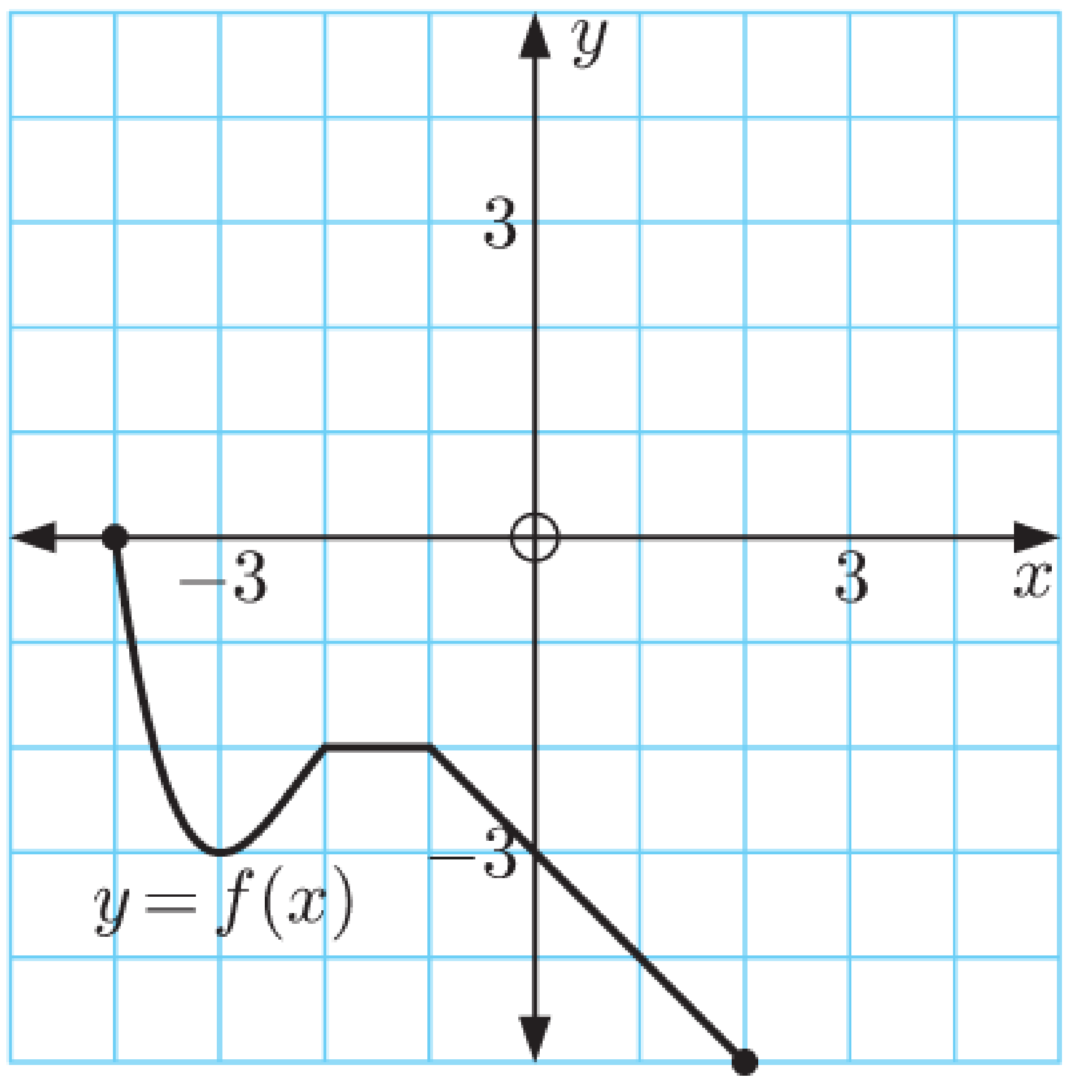


EXERCISE 16A

1 Consider the graph of $y = f(x)$ alongside.
On separate axes, draw the graphs of:

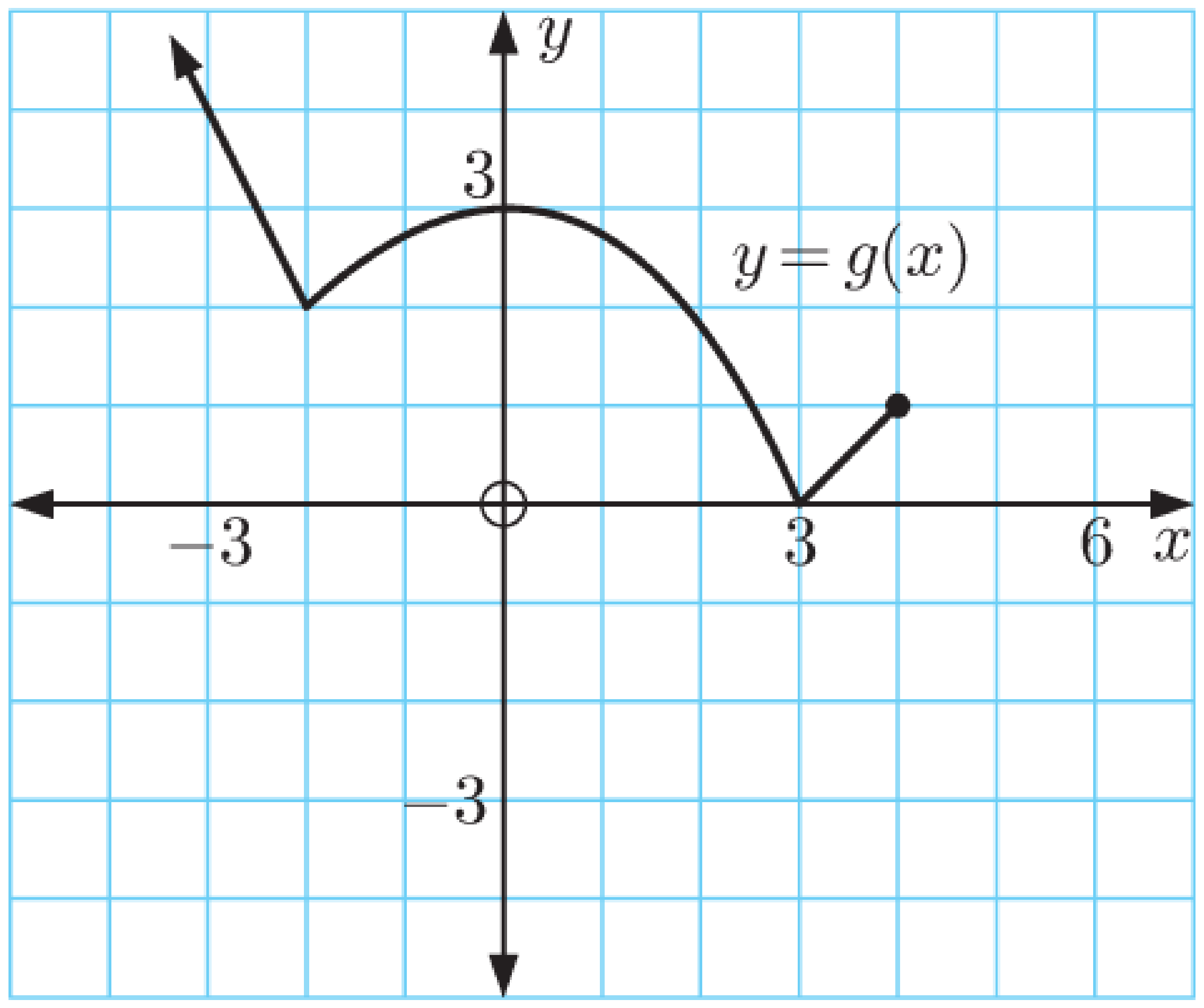
a $y = f(x) + 5$
b $y = f(x-3)$
c $y = f(x-3) + 5$

PRINTABLE GRIDS



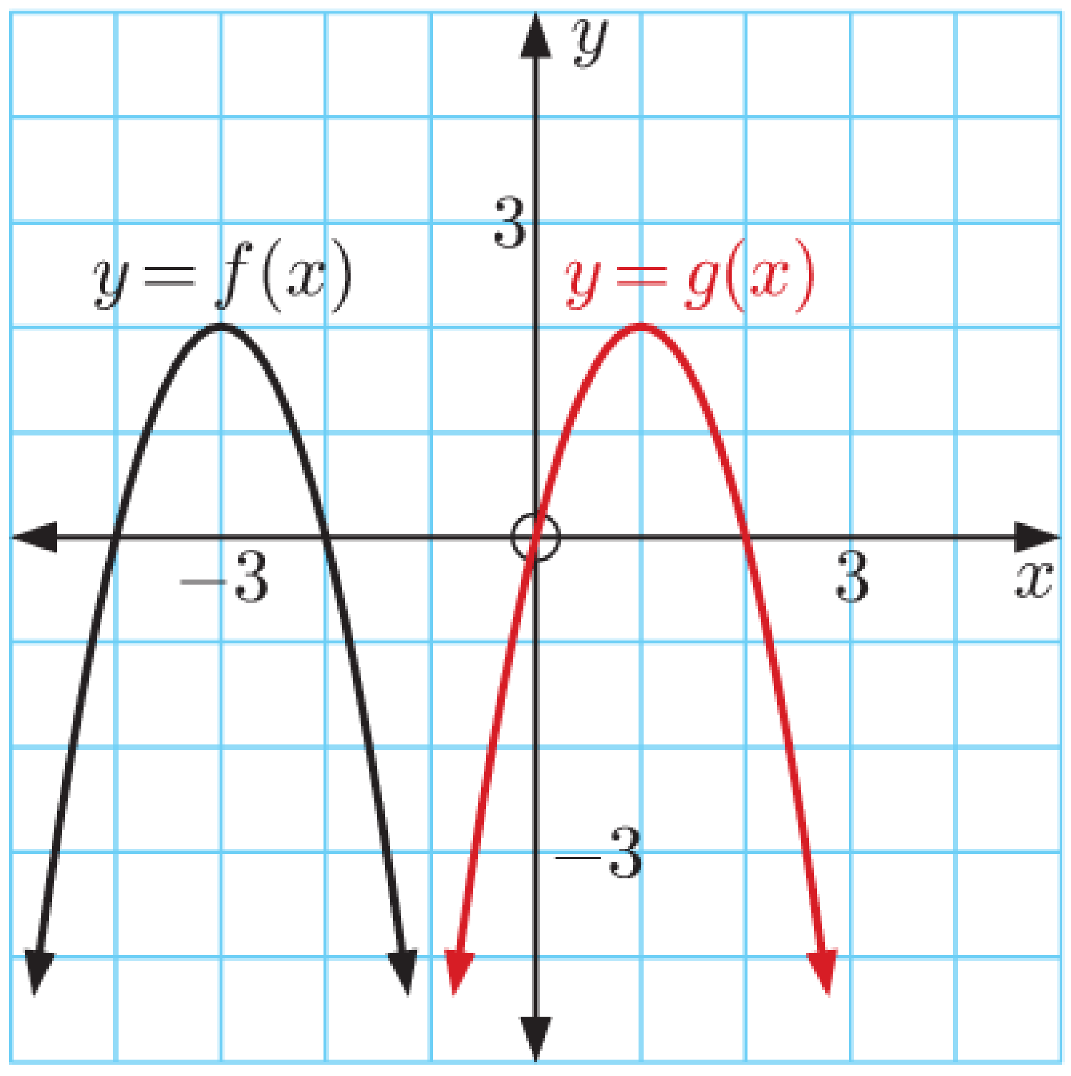
2 Consider the graph of $y = g(x)$ alongside.
On separate axes, draw the graphs of:

a $y = g(x) - 3$
b $y = g(x+1)$
c $y = g(x+1) - 3$
d $y = g(x-2) - 1$

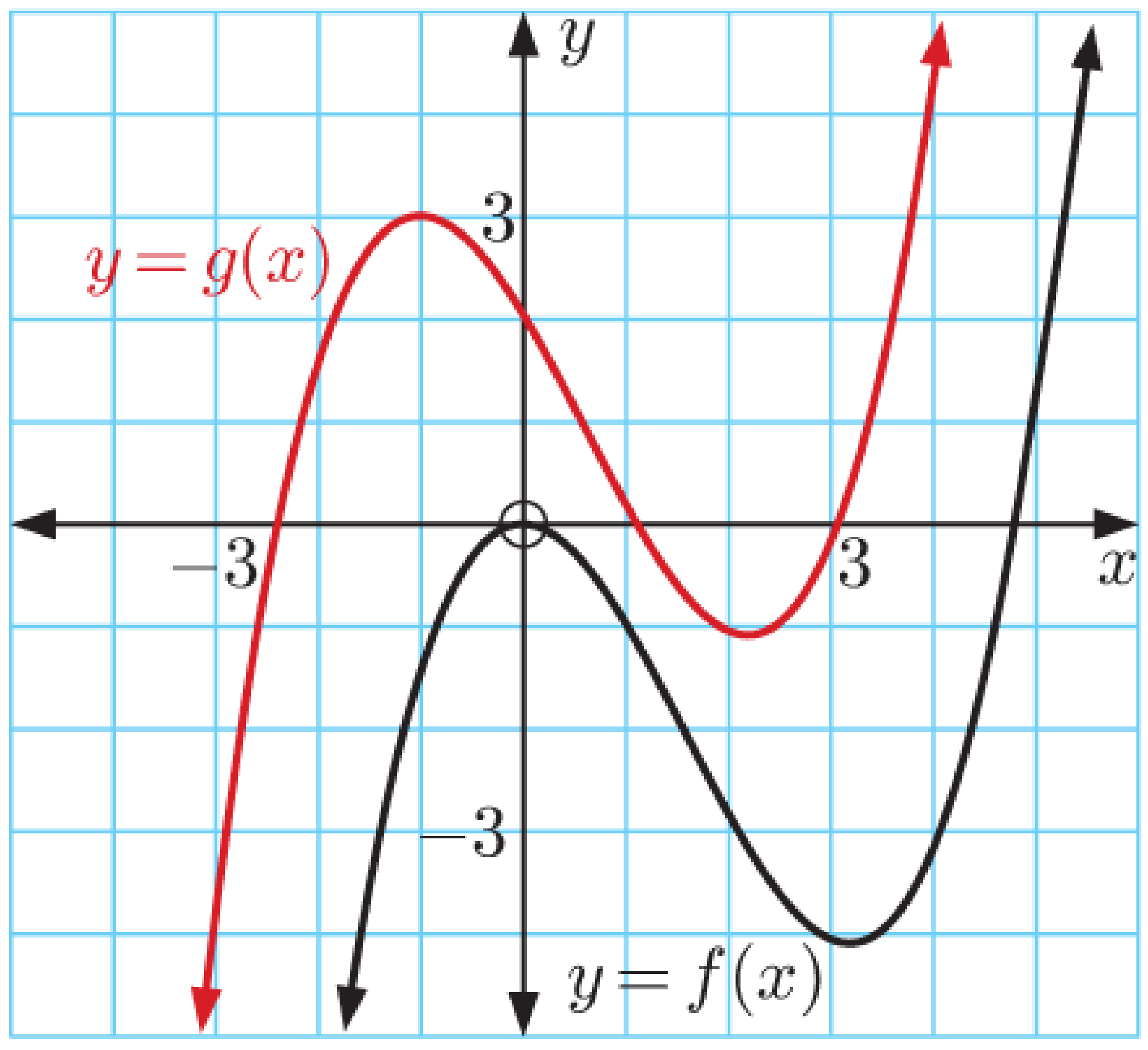


3 Write $g(x)$ in terms of $f(x)$:

a



b



4 Find the equation of the resulting graph $g(x)$ when:

- **a** $f(x) = 2x + 3$ is translated 4 units downwards
- **b** $f(x) = 3x - 4$ is translated 2 units to the left
- **c** $f(x) = -x^2 + 5x - 7$ is translated 3 units upwards
- **d** $f(x) = x^2 + 4x - 1$ is translated 5 units to the right.

5 For each of the following functions f, sketch $y = f(x)$, $y = f(x) + 1$, and $y = f(x) - 2$ on the same set of axes.

a $f(x) = x^2$ **b** $f(x) = x^3$ **c** $f(x) = \frac{1}{x}$ **d** $f(x) = (x-1)^2 + 2$

6 For each of the following functions f, sketch $y = f(x)$, $y = f(x-1)$, and $y = f(x+2)$ on the same set of axes.

a $f(x) = x^2$ **b** $f(x) = x^3$ **c** $f(x) = \frac{1}{x}$ **d** $f(x) = (x-1)^2 + 2$

7 For each of the following functions f, sketch $y = f(x)$, $y = f(x-2) + 3$, and $y = f(x+1) - 4$ on the same set of axes.

a $f(x) = x^2$ **b** $f(x) = x^3$ **c** $f(x) = \frac{1}{x}$ **d** $f(x) = (x-1)^2 + 2$

8 The point $(-2, -5)$ lies on the graph of $y = f(x)$. Find the coordinates of the corresponding point on the graph of $g(x) = f(x-3) - 4$.

9 Suppose the graph of $y = f(x)$ has x-intercepts -3 and 4, and y-intercept 2. What can you say about the axes intercepts of:

a $g(x) = f(x) - 3$ **b** $h(x) = f(x-1)$ **c** $j(x) = f(x+2) - 4$?

10 The graph of $f(x) = x^2 - 2x + 2$ is translated by $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ to form $g(x)$. Find $g(x)$ in the form $g(x) = ax^2 + bx + c$.

11 The graph of $f(x) = \frac{1}{x}$ is translated by $\begin{pmatrix} -2 \\ 7 \end{pmatrix}$ to form $g(x)$. Find $g(x)$ in the form $g(x) = \dfrac{ax+b}{cx+d}$.

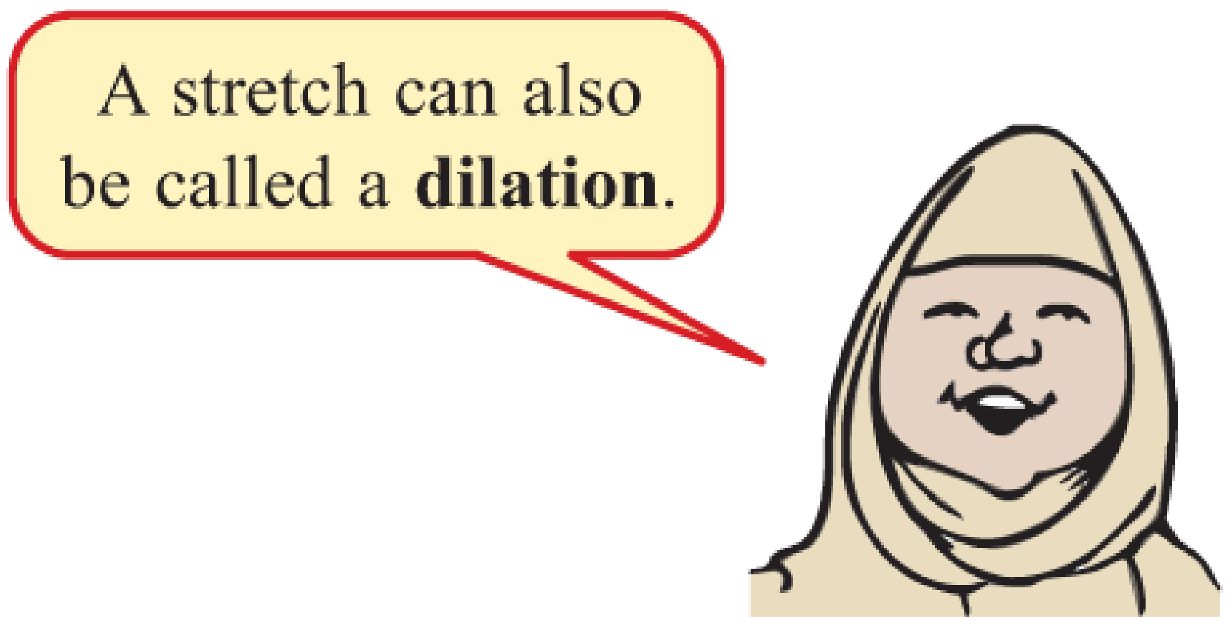
12 Suppose $f(x) = x^2$ is transformed to $g(x) = (x-3)^2 + 2$.

- **a** Find the images of the following points on $f(x)$:
  - **i** $(0, 0)$ **ii** $(-3, 9)$ **iii** $(2, 4)$
- **b** Find the points on $f(x)$ which correspond to the following points on $g(x)$:
  - **i** $(1, 6)$ **ii** $(-2, 27)$ **iii** $(1\frac{1}{2}, 4\frac{1}{4})$

B STRETCHES

In this Section we study how a function can be manipulated to *stretch* its graph.

We will consider stretches in both the horizontal and vertical directions.

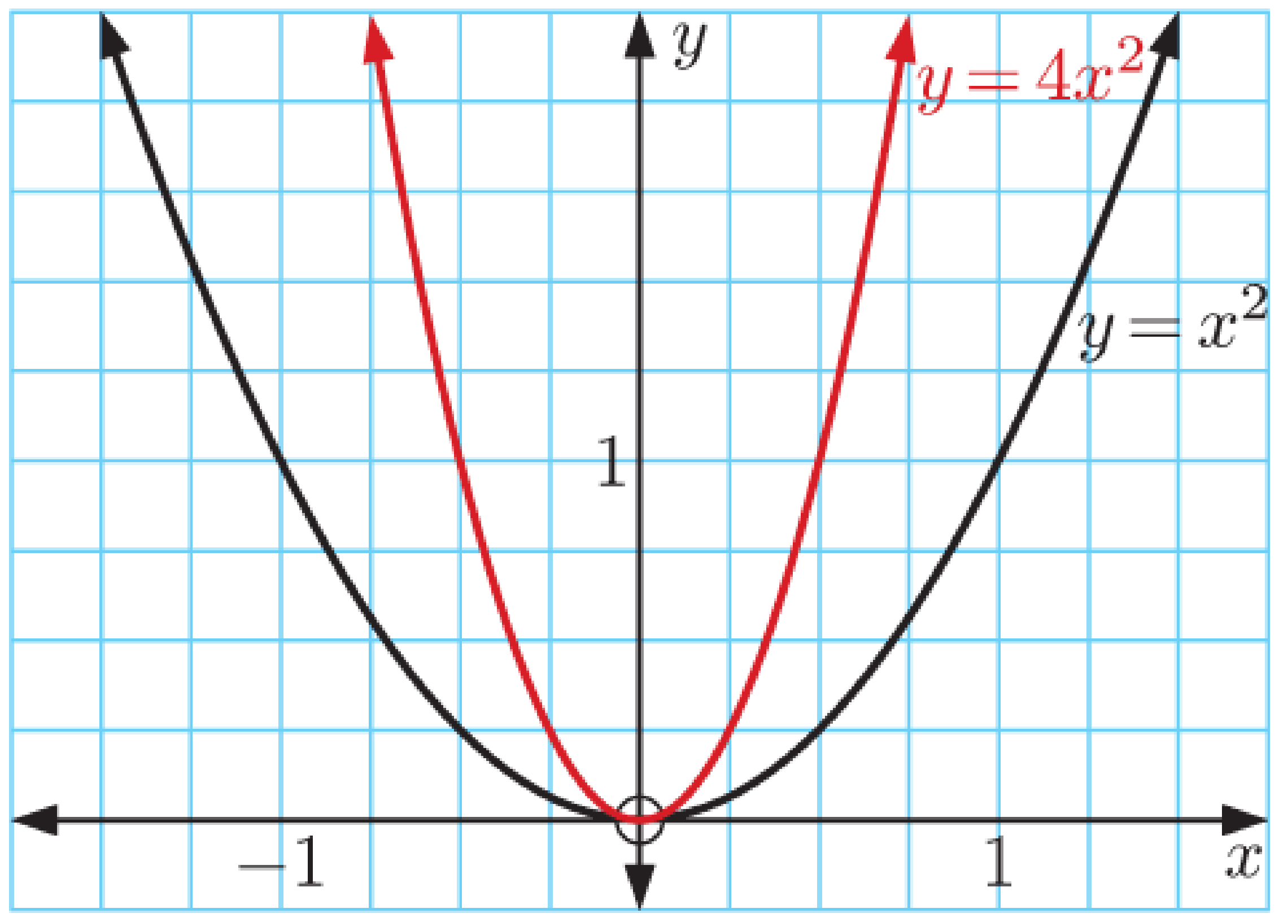


In our study of quadratic functions, we saw that the coefficient a of x^2 controls the width of the parabola.

In the case of $f(x) = x^2$,

notice that $f(2x) = (2x)^2 = 4x^2$

and $4f(x) = 4x^2$



DISCUSSION

- In what ways could $y = x^2$ be *stretched* to form $y = 4x^2$?
- Will a transformation of the form $p\,f(x)$, $p > 0$ always be equivalent to a transformation of the form $f(qx)$, $q > 0$?

INVESTIGATION 2 STRETCHES

In this Investigation we consider transformations of the form $p\,f(x)$, $p > 0$, and $f(qx)$, $q > 0$.

What to do:

1 Let $f(x) = x + 2$.

a Find, in simplest form:

i $3f(x)$ **ii** $\frac{1}{2}f(x)$ **iii** $5f(x)$

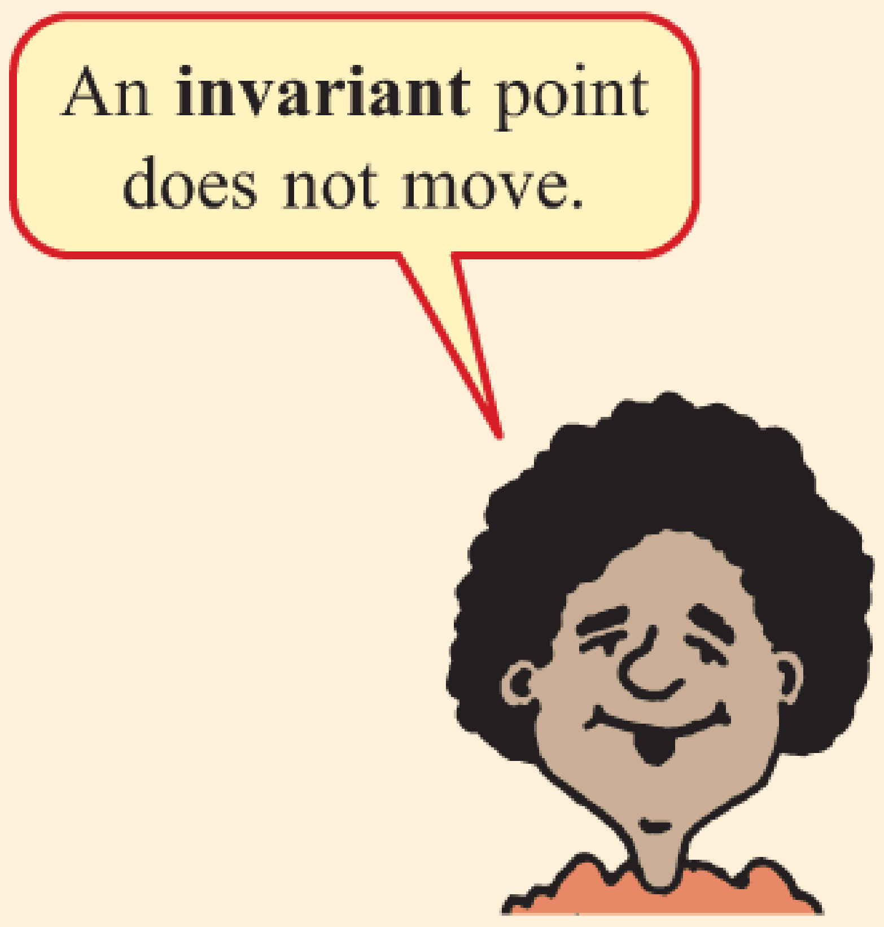
b Graph all four functions on the same set of axes.

c Which point is *invariant* under a transformation of the form $p\,f(x)$, $p > 0$?

d Copy and complete:

For the transformation $y = p\,f(x)$, each point becomes ...... times its previous distance from the x-axis.

GRAPHING PACKAGE



2 Let $f(x) = x + 2$.

a Find, in simplest form:

i $f(2x)$ **ii** $f(\frac{1}{3}x)$ **iii** $f(4x)$

b Graph all four functions on the same set of axes.

c Which point is *invariant* under a transformation of the form $f(qx)$, $q > 0$?

d Copy and complete:

For the transformation $y = f(qx)$, each point becomes ...... times its previous distance from the y-axis.

From the **Investigation** you should have found:

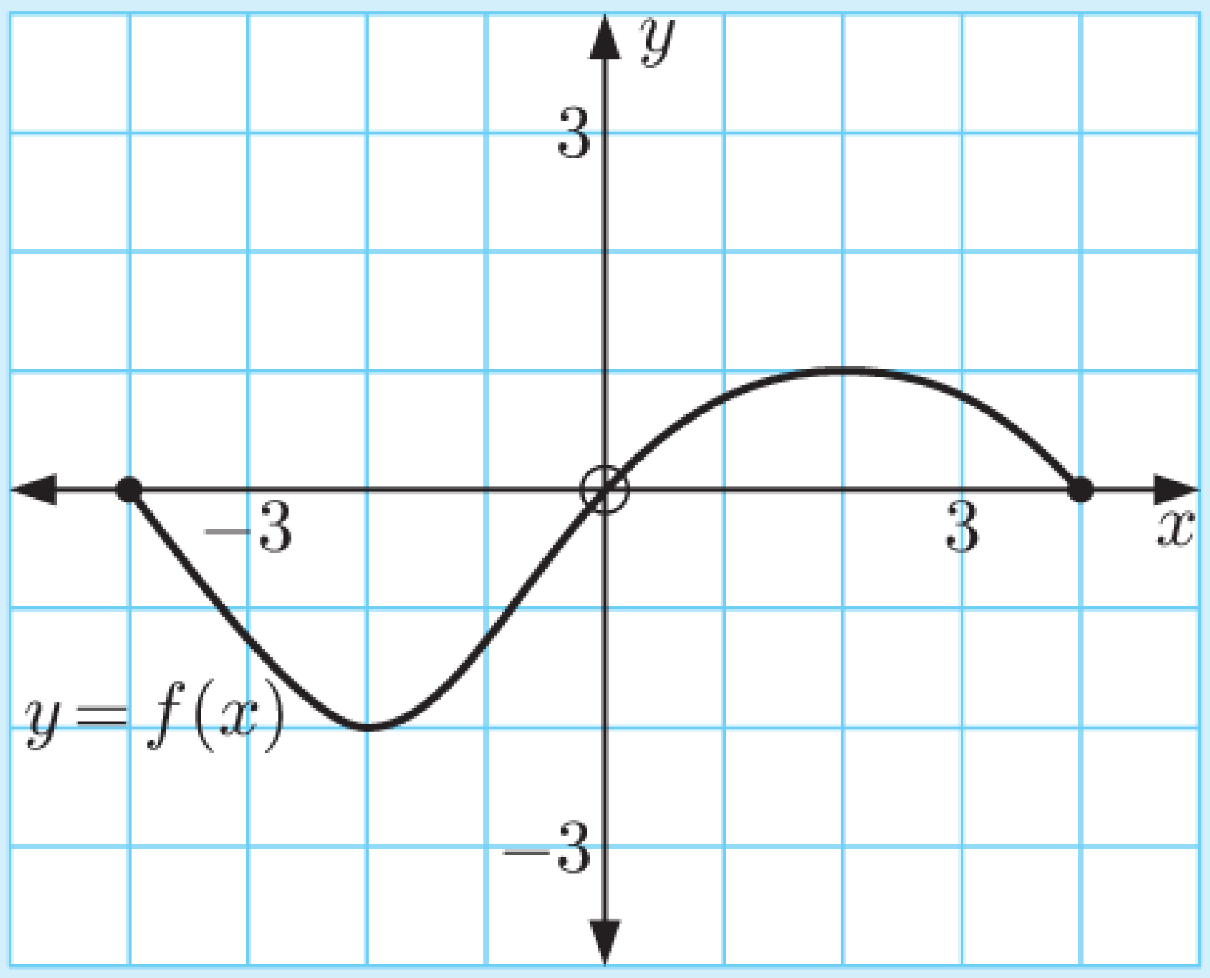
- $y = p\,f(x)$, $p > 0$ is a **vertical stretch** of $y = f(x)$ with **scale factor** p and **invariant x-axis**.
  - Each point becomes p times its previous distance from the x-axis.
  - If $p > 1$, points move further away from the x-axis.
  - If $0 < p < 1$, points move closer to the x-axis.
- $y = f(qx)$, $q > 0$ is a **horizontal stretch** of $y = f(x)$ with **scale factor** $\frac{1}{q}$ and **invariant y-axis**.
  - Each point becomes $\frac{1}{q}$ times its previous distance from the y-axis.
  - If $q > 1$, points move closer to the y-axis.
  - If $0 < q < 1$, points move further away from the y-axis.

Example 2 **Self Tutor**

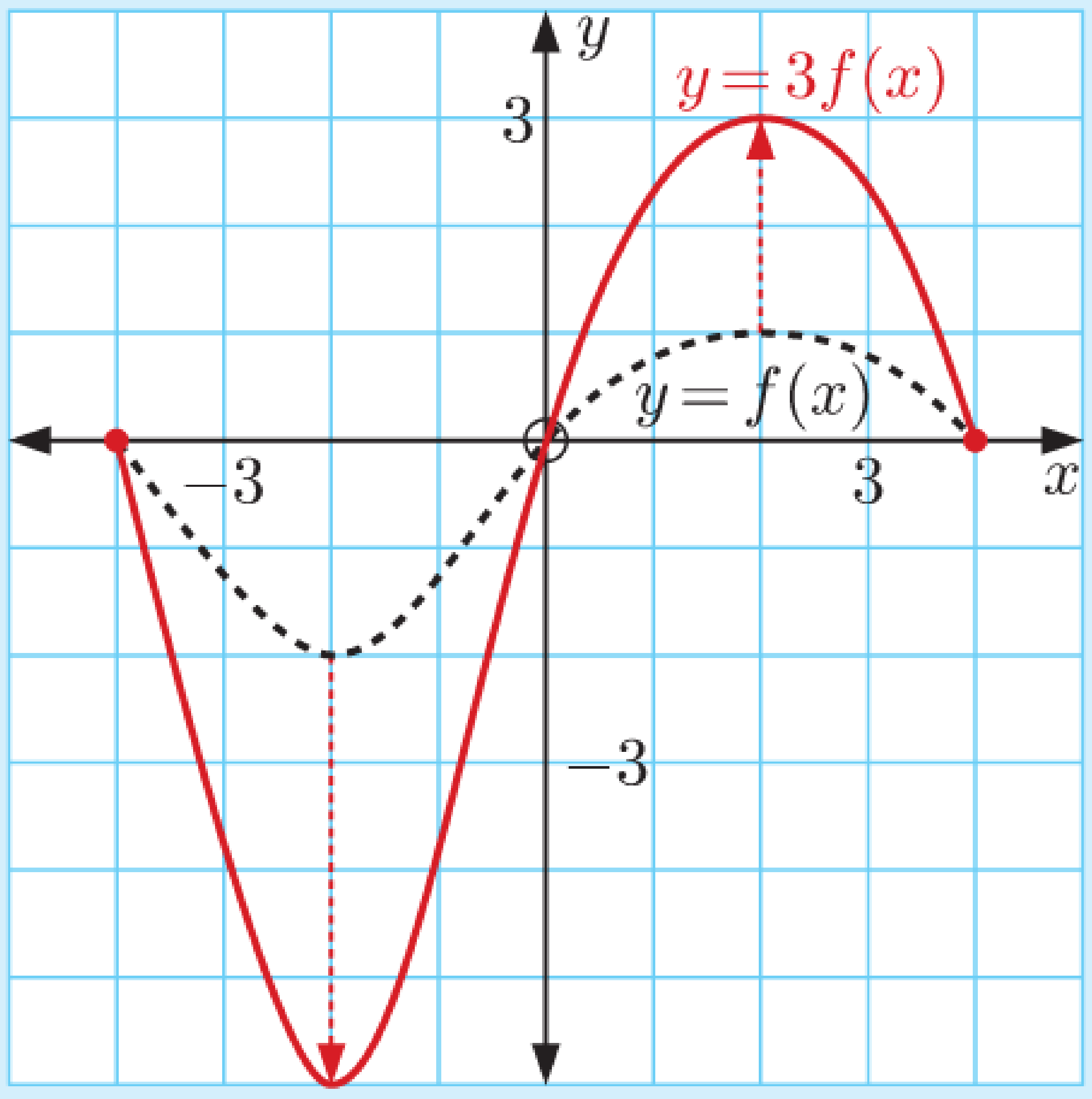
Consider the graph of $y = f(x)$ alongside.

On separate axes, draw the graphs of:

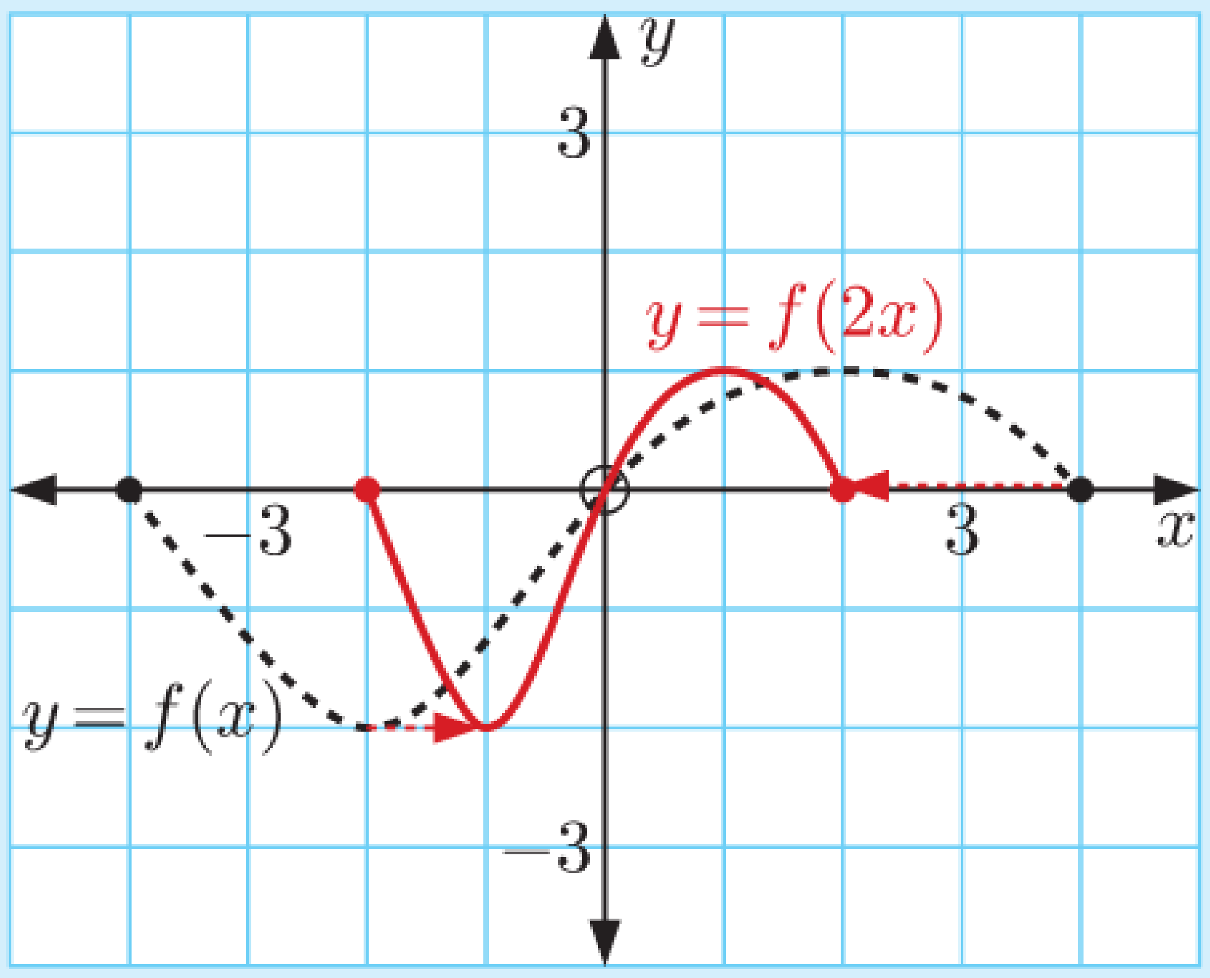
a $y = 3\,f(x)$ **b** $y = f(2x)$



a The graph of $y = 3\,f(x)$ is a vertical stretch of $y = f(x)$ with scale factor 3.



b The graph of $y = f(2x)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{2}$.



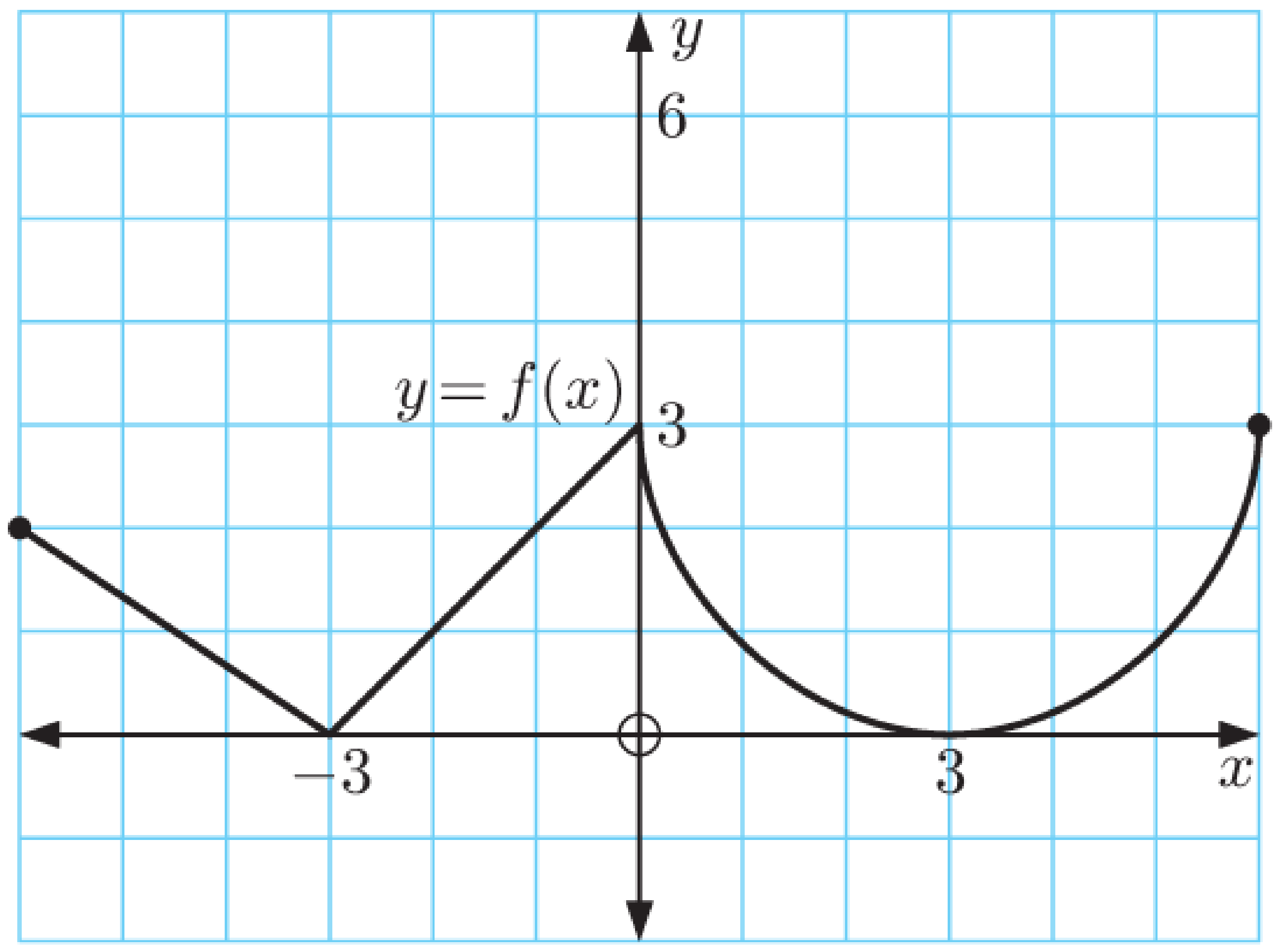
EXERCISE 16B

1 Consider the graph of $y = f(x)$ alongside.
On separate axes, draw the graphs of:

a $y = 2f(x)$

b $y = f(3x)$

PRINTABLE GRIDS



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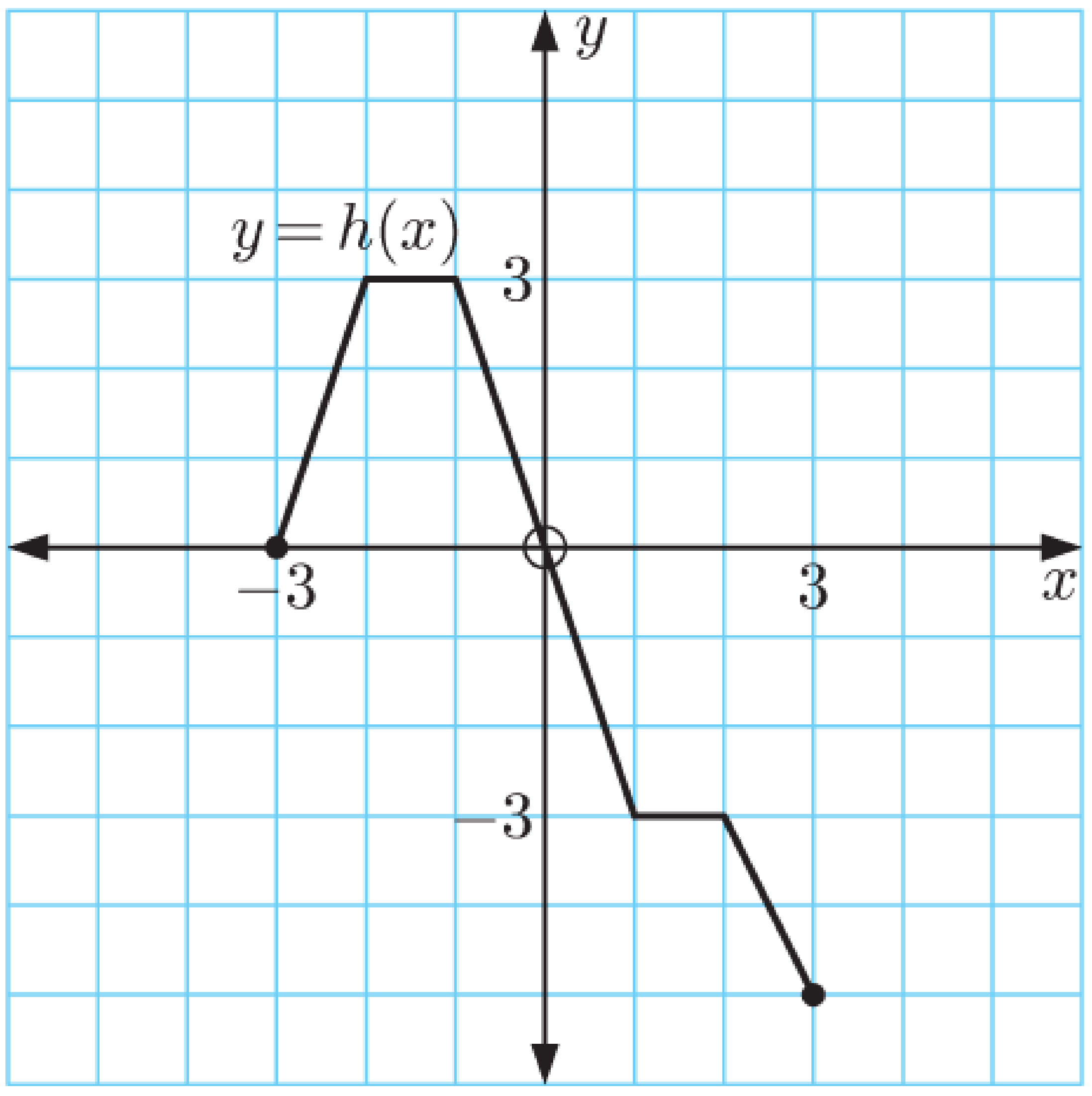
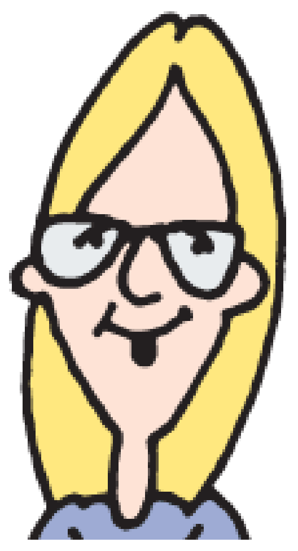


Consider the graph of $y = h(x)$ alongside.
On separate axes, draw the graphs of:

a $y = \frac{1}{3}h(x)$

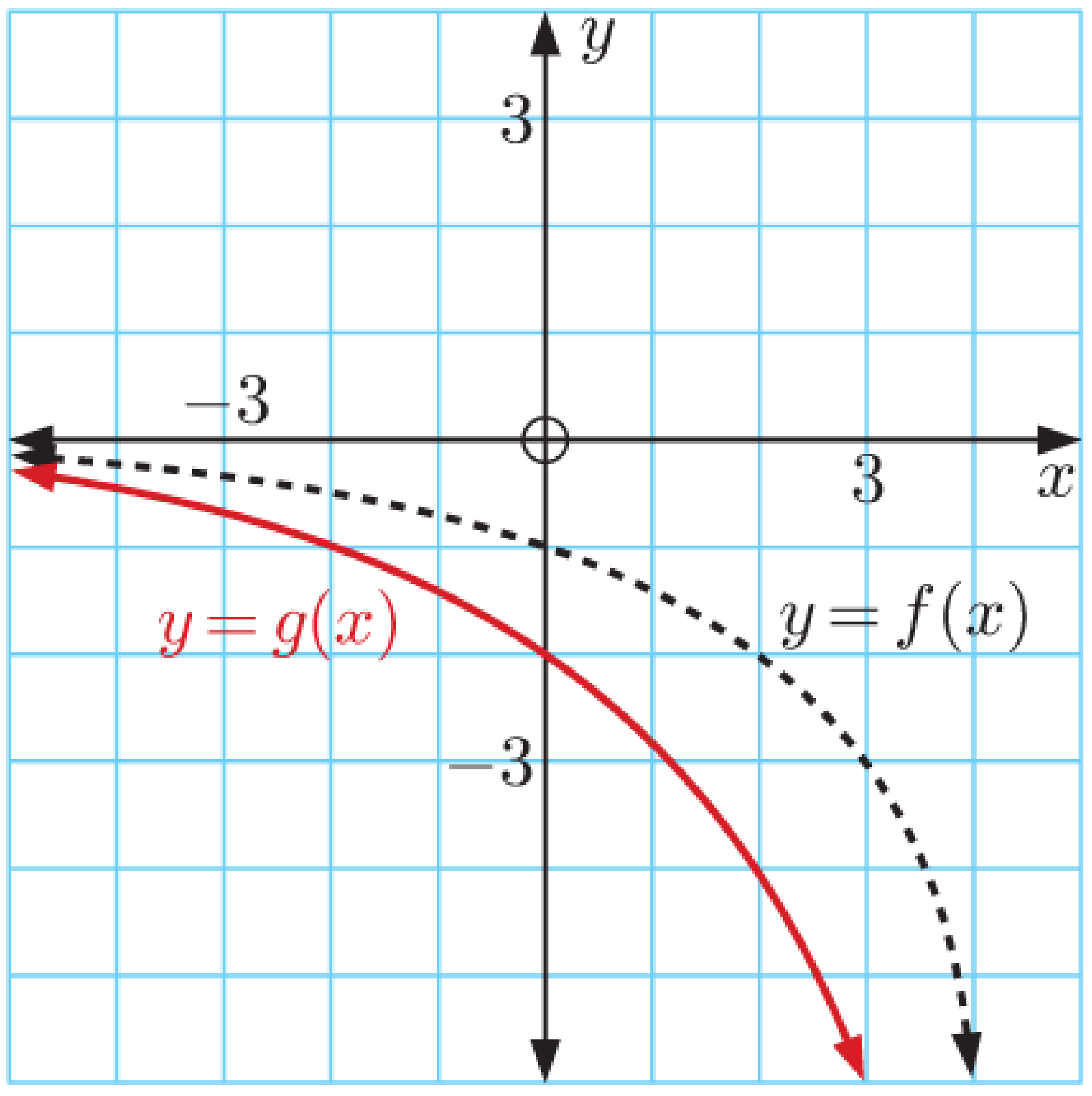
b $y = h\left(\frac{x}{2}\right)$

If scale factor > 1, the graph is *elongated*.
If $0 <$ scale factor < 1, the graph is *compressed*.

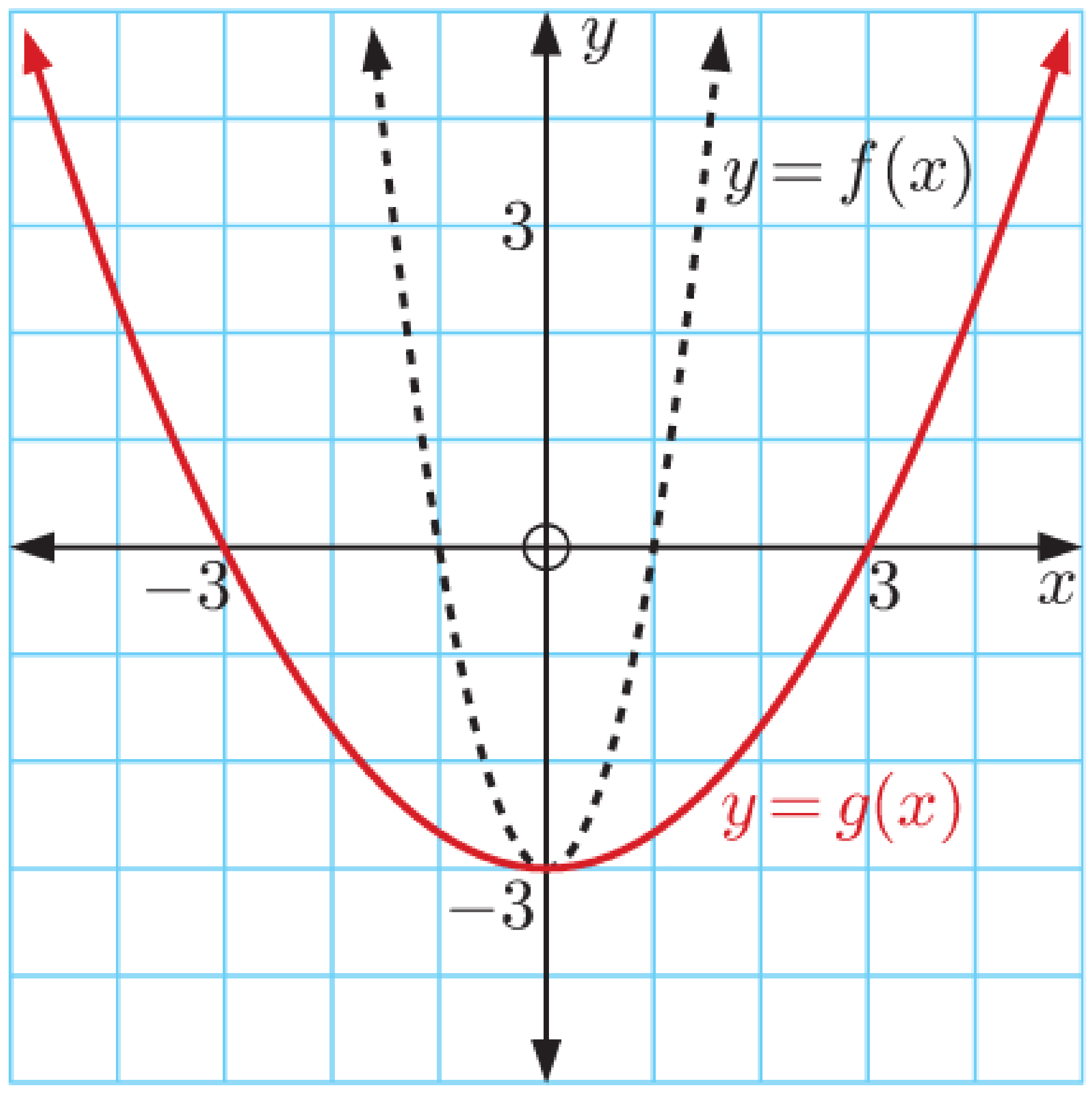


3 Write $g(x)$ in terms of $f(x)$:

a



b



4 A linear function with gradient m is vertically stretched with scale factor c. Find the gradient of the resulting line.

5 For each of the following functions f, sketch $y = f(x)$, $y = 2f(x)$, and $y = 3f(x)$ on the same set of axes:

a $f(x) = x - 1$

b $f(x) = x^2$

c $f(x) = x^3$

d $f(x) = \frac{1}{x}$

6 For each of the following functions f, sketch $y = f(x)$, $y = \frac{1}{2}f(x)$, and $y = \frac{1}{4}f(x)$ on the same set of axes:

a $f(x) = x - 1$ **b** $f(x) = x^2$ **c** $f(x) = x^3$ **d** $f(x) = \frac{1}{x}$

7 Sketch, on the same set of axes, the graphs of $y = f(x)$ and $y = f(2x)$ for:

a $y = x^2$ **b** $y = (x-1)^2$ **c** $y = (x+3)^2$

8 Sketch, on the same set of axes, the graphs of $y = f(x)$ and $y = f(\frac{x}{2})$ for:

a $y = x^2$ **b** $y = 2x$ **c** $y = (x+2)^2$

9 Suppose f and g are functions such that $g(x) = f(5x)$.

a Given that $(10, 25)$ lies on $y = f(x)$, find the coordinates of the corresponding point on $y = g(x)$.

b Given that $(-5, -15)$ lies on $y = g(x)$, find the coordinates of the corresponding point on $y = f(x)$.

10 Find the equation of the resulting graph $g(x)$ when:

a $f(x) = x^2 + 2$ is vertically stretched with scale factor 2

b $f(x) = 5 - 3x$ is horizontally stretched with scale factor 3

c $f(x) = x^3 + 8x^2 - 2$ is vertically dilated with scale factor $\frac{1}{4}$

d $f(x) = 2x^2 + x - 3$ is horizontally dilated with scale factor $\frac{1}{2}$.


A stretch can also be called a dilation.

11 Graph on the same set of axes $y = x^2$, $y = 3x^2$, and $y = 3(x+1)^2 - 2$.
Describe the combination of transformations which transform $y = x^2$ to $y = 3(x+1)^2 - 2$.

12 Graph on the same set of axes $y = x^2$, $y = \frac{1}{2}x^2$, and $y = \frac{1}{2}(x+1)^2 + 3$.
Describe the combination of transformations which transform $y = x^2$ to $y = \frac{1}{2}(x+1)^2 + 3$.

13 Graph on the same set of axes $y = x^2$, $y = 2x^2$, and $y = 2(x - \frac{3}{2})^2 + 1$.
Describe the combination of transformations which transform $y = x^2$ to $y = 2(x - \frac{3}{2})^2 + 1$.

14 Describe the combination of transformations which transform $y = x^2$ to $y = 2(x+1)^2 - 3$.
Hence sketch $y = 2(x+1)^2 - 3$.

15 Suppose f and g are functions such that $g(x) = 3f(2x)$.

a What transformations are needed to map $y = f(x)$ onto $y = g(x)$?

b Find the image of each of these points on $y = f(x)$:

i $(3, -5)$ **ii** $(1, 2)$ **iii** $(-2, 1)$

c Find the point on $y = f(x)$ which maps onto the image point:

i $(2, 1)$ **ii** $(-3, 2)$ **iii** $(-7, 3)$

16 The graph of $y = x^2$ is transformed to the graph of $y = 0.1x^2 + 5$ by a translation a units vertically followed by a horizontal stretch with scale factor b. Find a and b.

Example 3 **Self Tutor**

The function $g(x)$ results when $y = \frac{1}{x}$ is transformed by a vertical stretch with scale factor 2, followed by a translation of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

a Write an expression for $g(x)$ in the form $g(x) = \frac{ax+b}{cx+d}$.

b Find the asymptotes of $y = g(x)$.

c Sketch $y = g(x)$.

a Under a vertical stretch with scale factor 2, $f(x)$ becomes $2f(x)$.

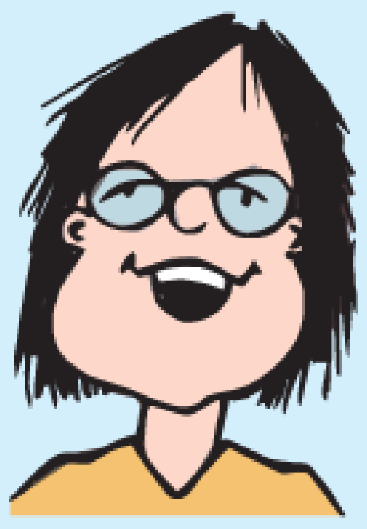
$\therefore\ \frac{1}{x}$ becomes $2\left(\frac{1}{x}\right) = \frac{2}{x}$.

Under a translation of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $f(x)$ becomes $f(x-3)-2$.

$\therefore\ \frac{2}{x}$ becomes $\frac{2}{x-3} - 2$.

So, $y = \frac{1}{x}$ becomes
$$g(x) = \frac{2}{x-3} - 2 = \frac{2-2(x-3)}{x-3} = \frac{-2x+8}{x-3}$$

$g(x)$ is a rational function which is $\frac{\text{linear}}{\text{linear}}$.

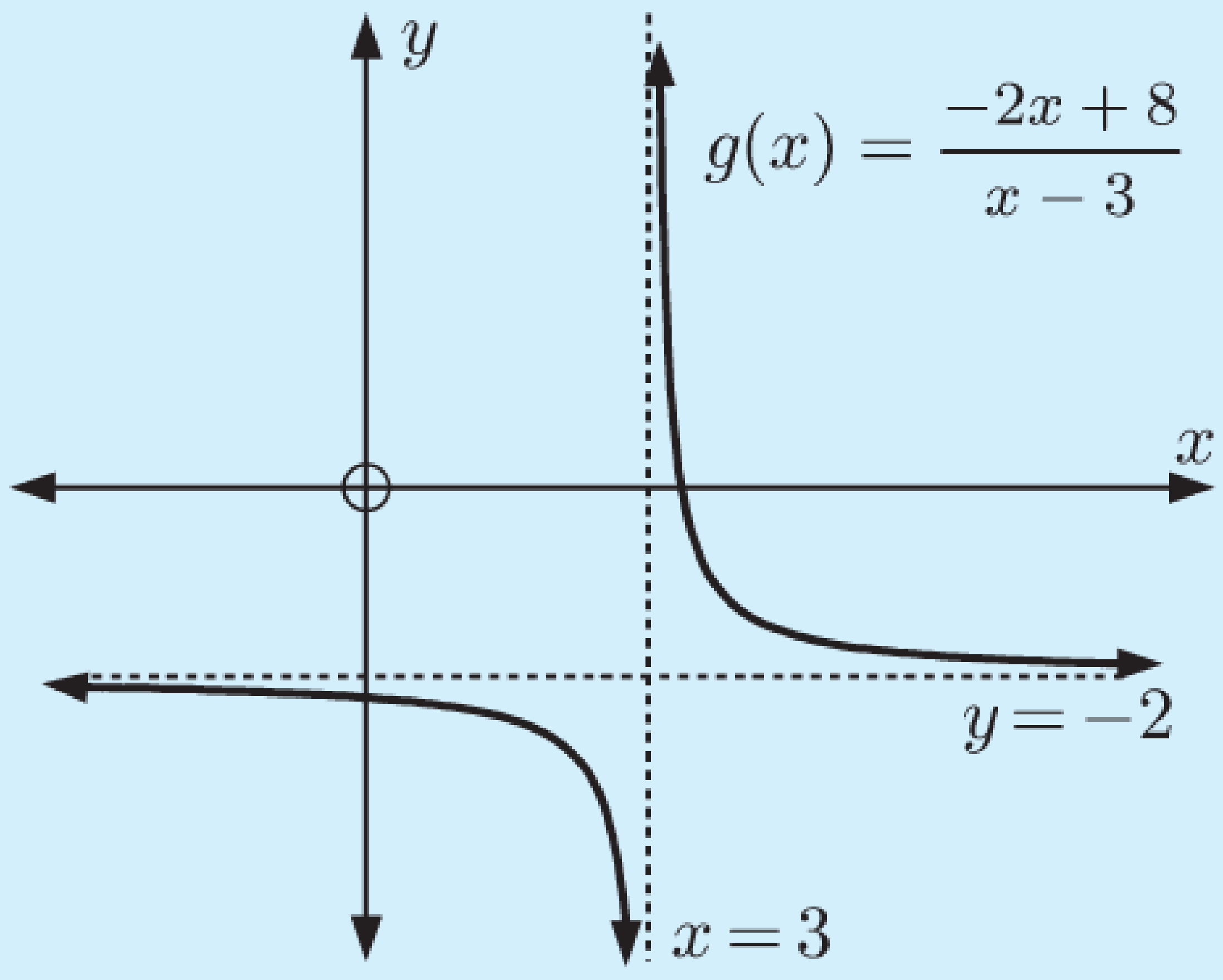


b The asymptotes of $y = \frac{1}{x}$ are $x = 0$ and $y = 0$.

These are unchanged by the stretch, and shifted $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ by the translation.

$\therefore$ the vertical asymptote is $x = 3$ and the horizontal asymptote is $y = -2$.

c



17 Write, in the form $y = \frac{ax+b}{cx+d}$, the function that results when $y = \frac{1}{x}$ is transformed by:

- **a** a vertical dilation with scale factor $\frac{1}{2}$
- **b** a horizontal dilation with scale factor 3
- **c** a horizontal translation of -3
- **d** a vertical translation of 4.

18 The function $g(x)$ results when $y = \frac{1}{x}$ is transformed by a vertical stretch with scale factor 3, followed by a translation of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

- **a** Write an expression for $g(x)$ in the form $g(x) = \frac{ax+b}{cx+d}$.
- **b** Find the asymptotes of $y = g(x)$.
- **c** State the domain and range of $g(x)$.
- **d** Sketch $y = g(x)$.

19 The function $g(x)$ results when $y = \frac{1}{x}$ is transformed by a translation of $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$, followed by a horizontal stretch with scale factor $\frac{1}{2}$.

- **a** Write an expression for $g(x)$ in the form $g(x) = \dfrac{ax+b}{cx+d}$.
- **b** Find the asymptotes of $y = g(x)$.
- **c** State the domain and range of $g(x)$.
- **d** Sketch $y = g(x)$.

20 Find *two* combinations of transformations which map $f(x) = 2x^2 + 8x - 1$ onto $g(x) = 8x^2 - 16x + 5$.

DISCUSSION

For a vertical stretch with scale factor p, each point on the function is moved vertically so it is p times as far from the x-axis.

1. Using this definition of a vertical stretch, does it make sense to talk about negative values of p?
2. If a function is transformed from $f(x)$ to $-f(x)$, what transformation has actually occurred?
3. What *combinations* of transformations would transform $f(x)$ to $-2f(x)$?
4. What can we say about $y = f(qx)$ for:
   - **a** $q = -1$
   - **b** $q < 0,\ q \neq -1$?

C REFLECTIONS

INVESTIGATION 3 REFLECTIONS

In this Investigation we consider **reflections** with the forms $y = -f(x)$ and $y = f(-x)$.

What to do:

GRAPHING PACKAGE

1. Consider $f(x) = 2x + 3$.
   - **a** Find in simplest form:
     - **i** $-f(x)$
     - **ii** $f(-x)$
   - **b** Graph $y = f(x)$, $y = -f(x)$, and $y = f(-x)$ on the same set of axes.
2. Consider $f(x) = x^3 + 1$.
   - **a** Find in simplest form:
     - **i** $-f(x)$
     - **ii** $f(-x)$
   - **b** Graph $y = f(x)$, $y = -f(x)$, and $y = f(-x)$ on the same set of axes.
3. What transformation moves:
   - **a** $y = f(x)$ to $y = -f(x)$
   - **b** $y = f(x)$ to $y = f(-x)$?

From the **Investigation** you should have discovered that:

- For $y = -f(x)$, we **reflect** $y = f(x)$ in the **x-axis**.
- For $y = f(-x)$, we **reflect** $y = f(x)$ in the **y-axis**.

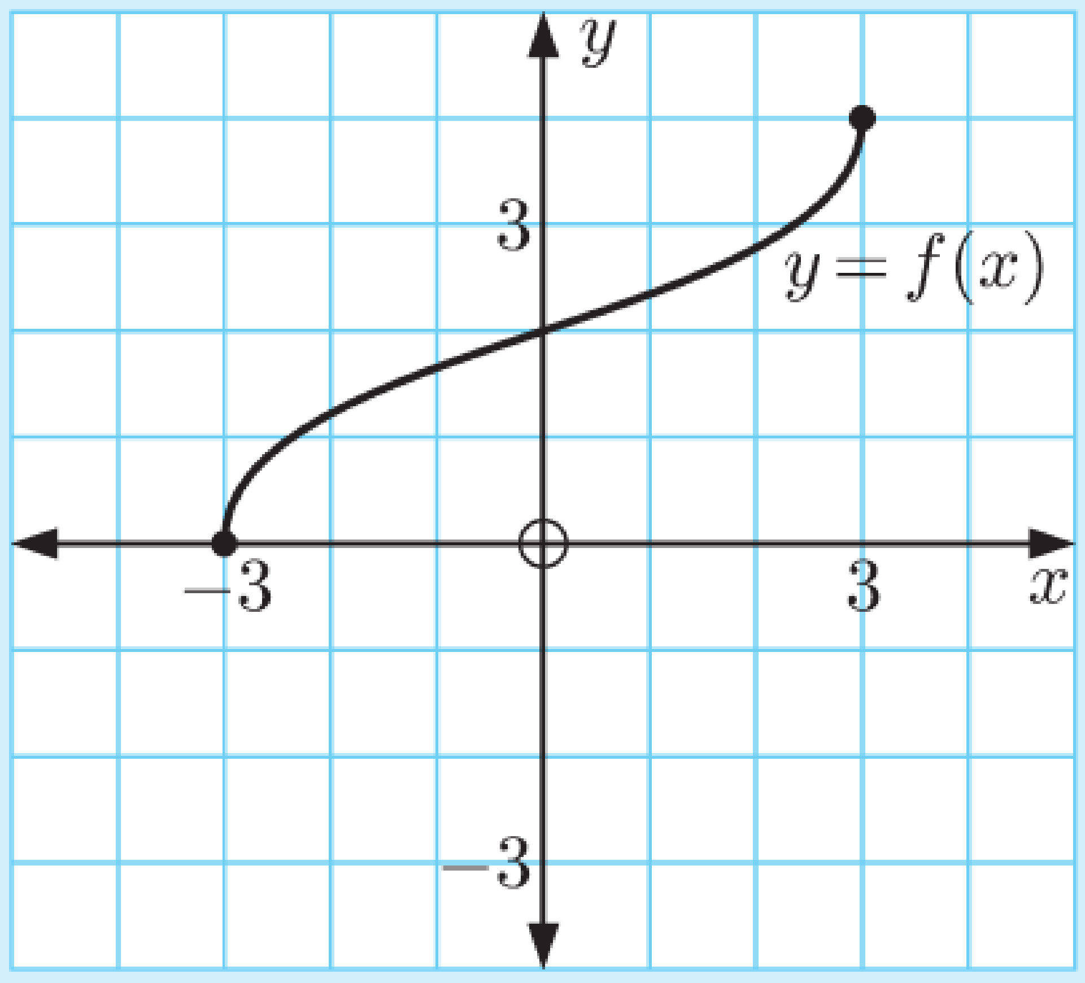
Example 4

Self Tutor

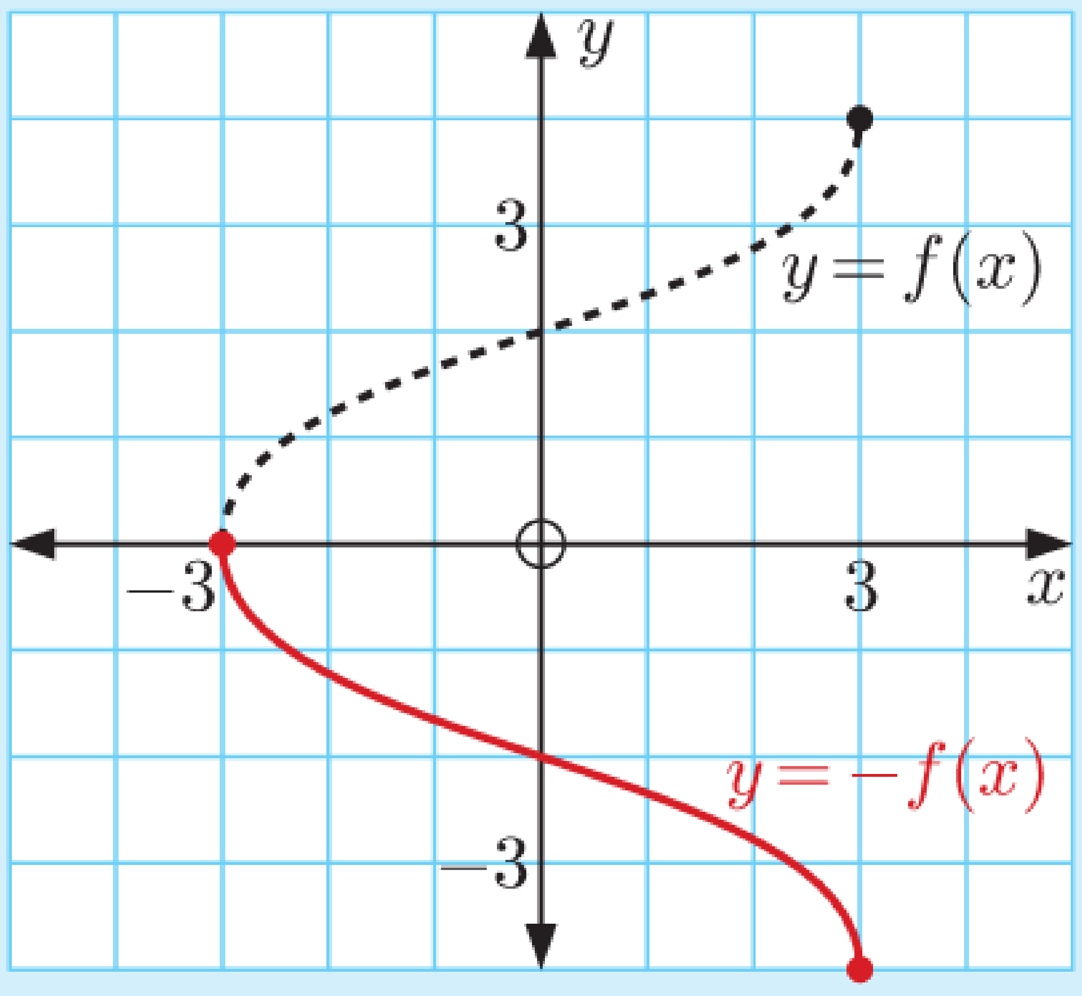
Consider the graph of $y = f(x)$ alongside.

On separate axes, draw the graphs of:

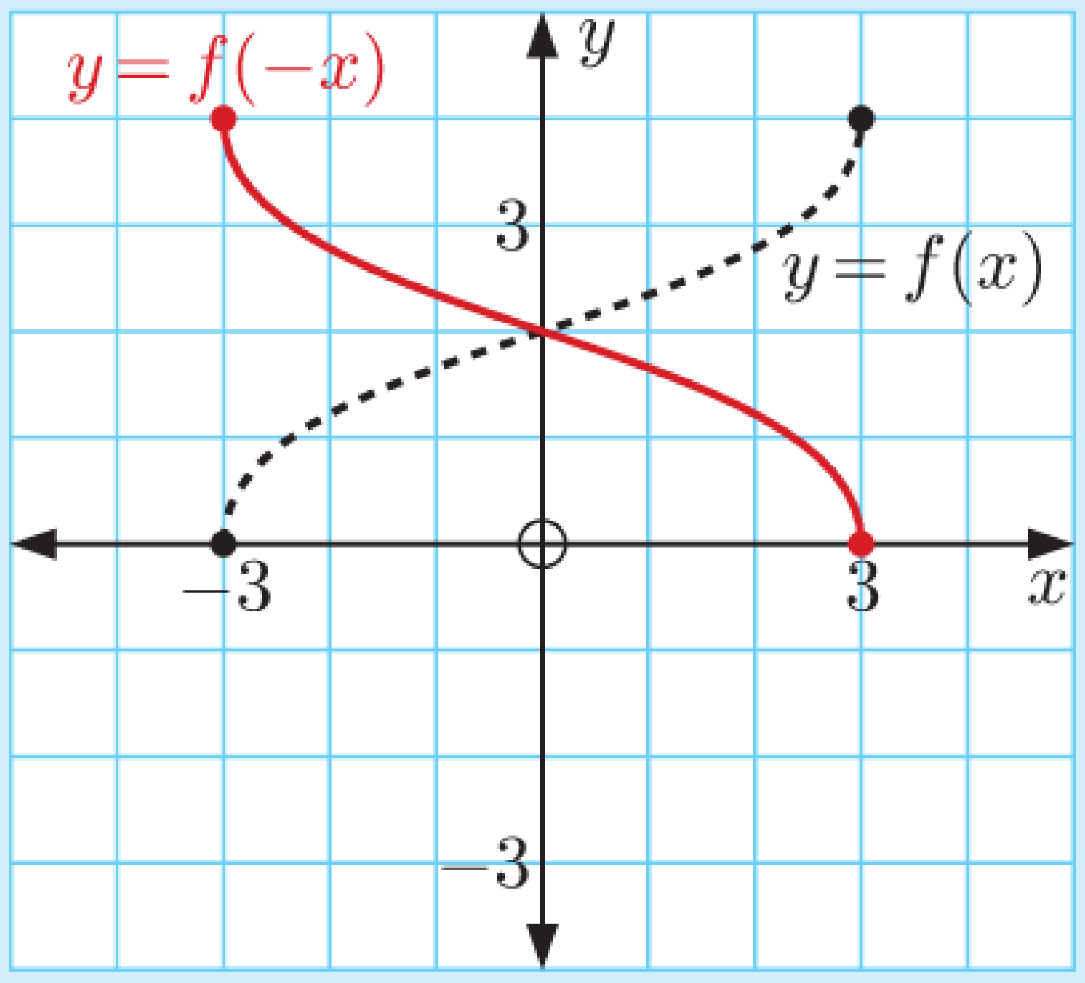
a $y = -f(x)$ **b** $y = f(-x)$



a The graph of $y = -f(x)$ is found by reflecting $y = f(x)$ in the x-axis.



b The graph of $y = f(-x)$ is found by reflecting $y = f(x)$ in the y-axis.



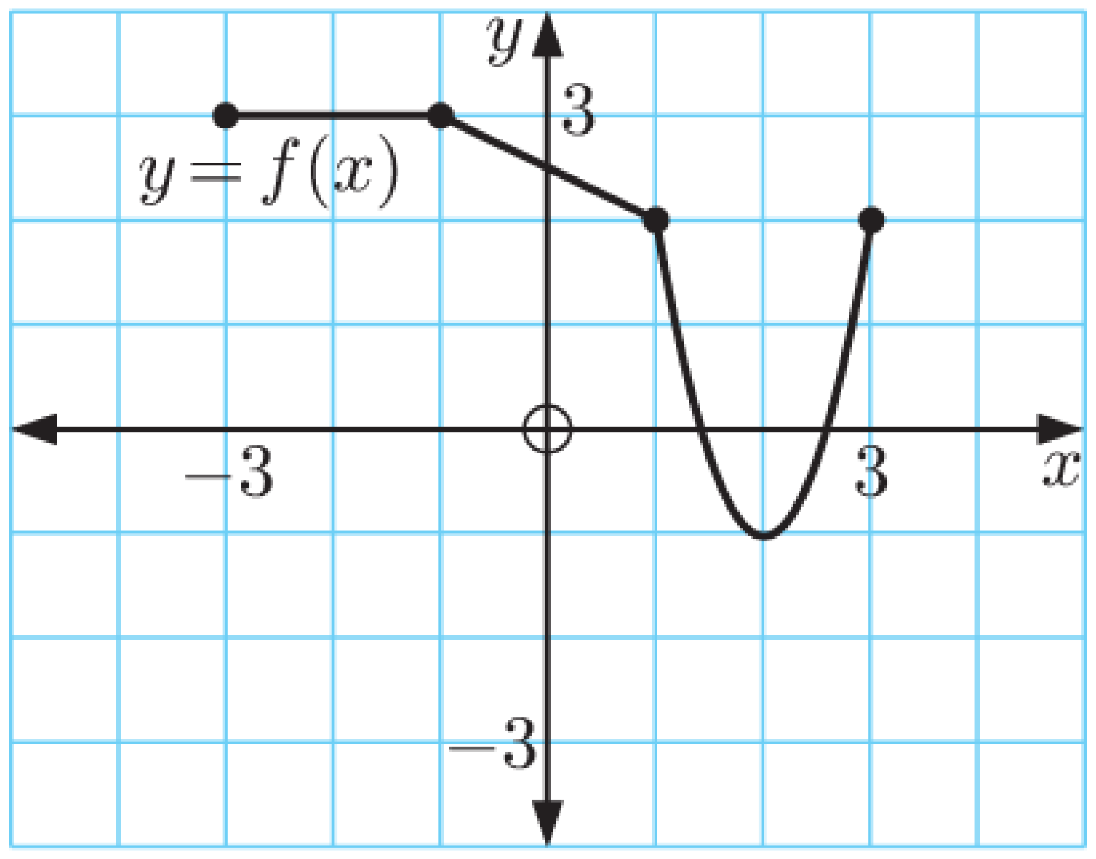
EXERCISE 16C

1 Consider the graph of $y = f(x)$ alongside.

On separate axes, draw the graphs of:

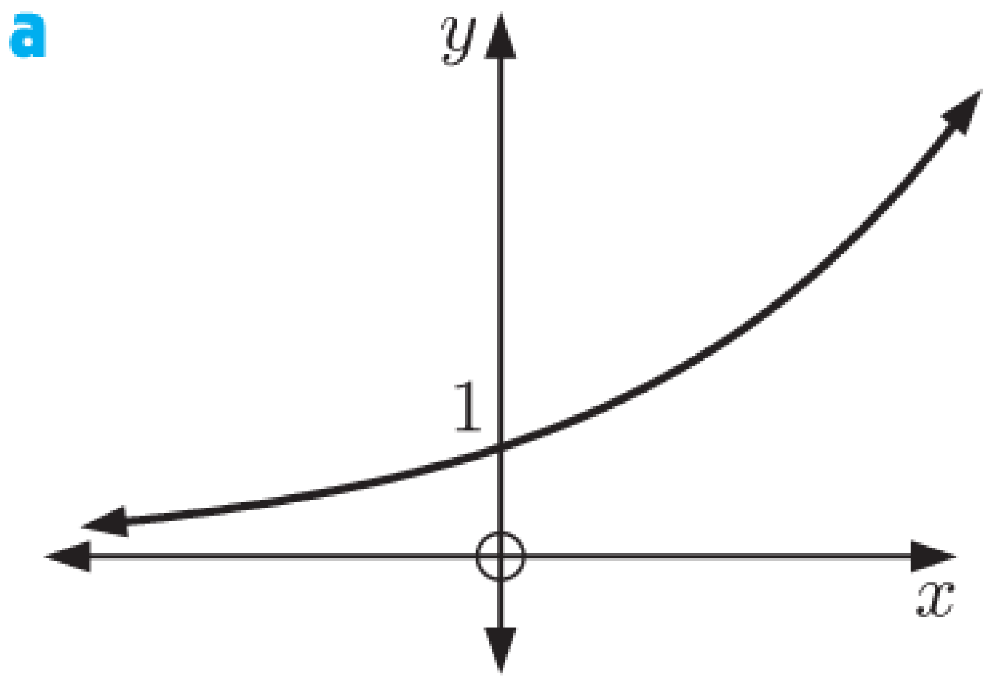
a $y = -f(x)$ **b** $y = f(-x)$

PRINTABLE GRIDS

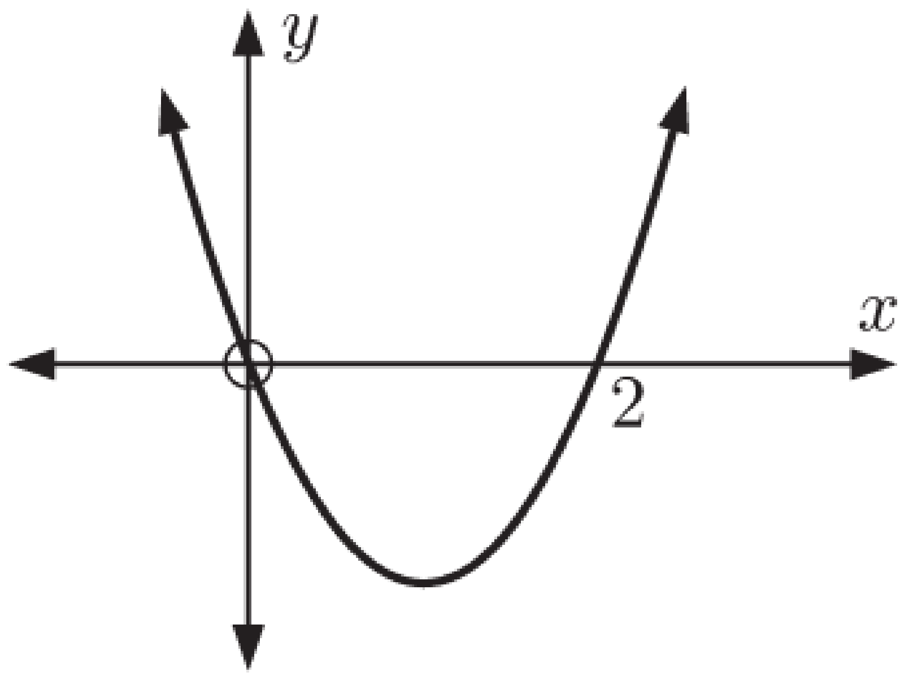


2 Copy the following graphs for $y = f(x)$ and sketch the graphs of $y = -f(x)$ on the same axes.

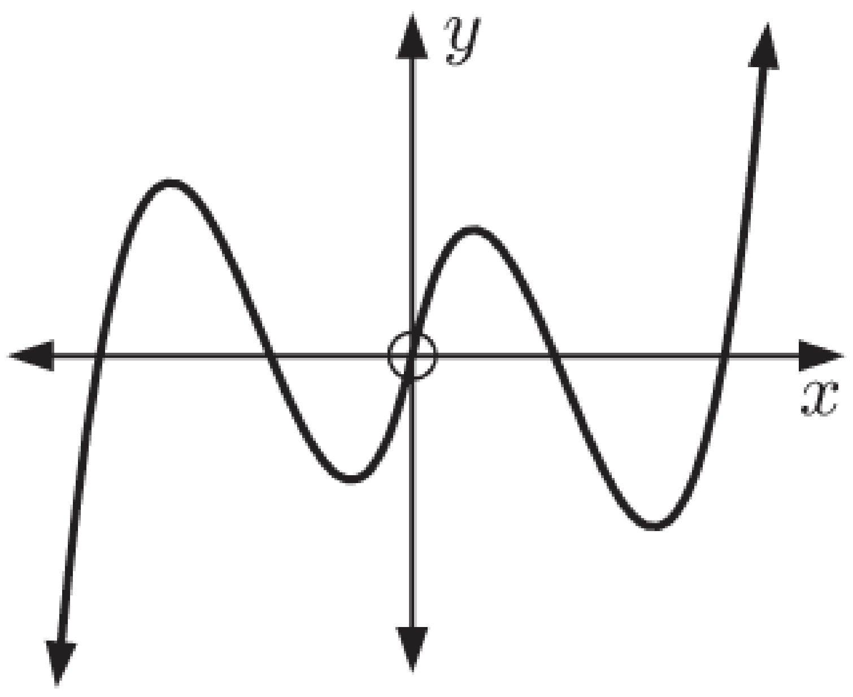
a



b

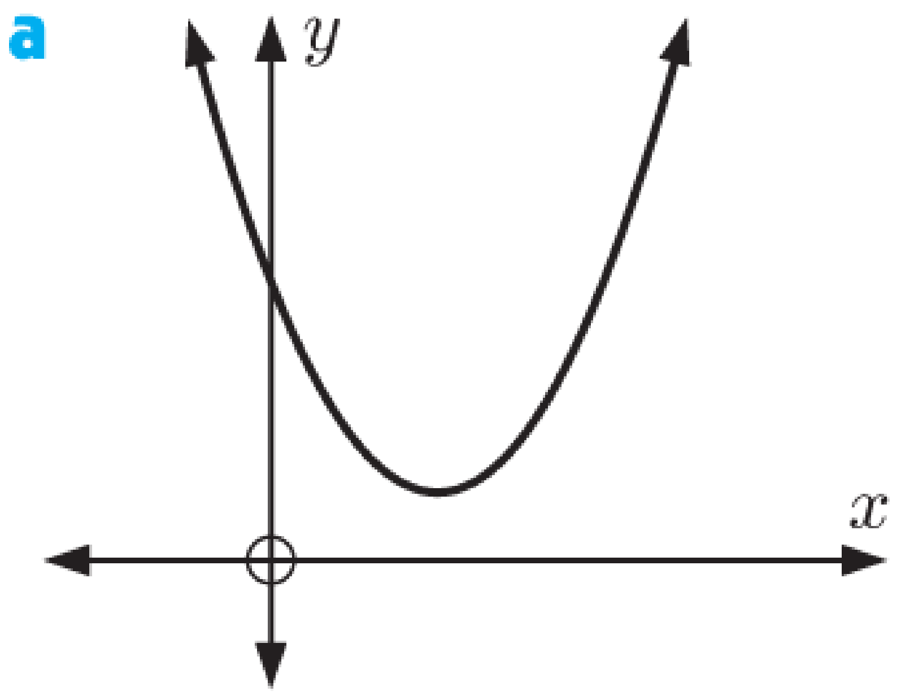


c

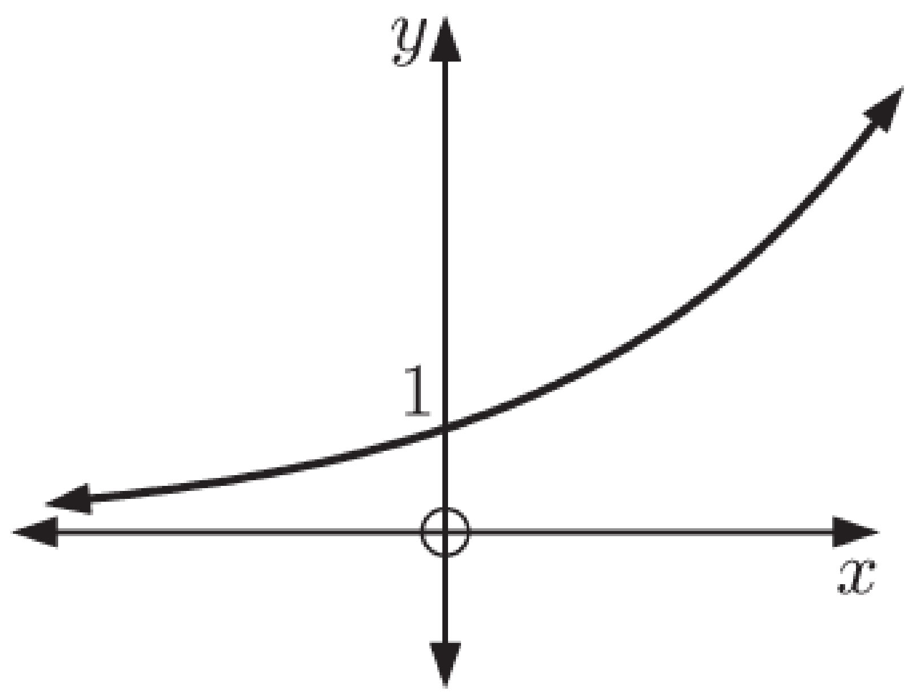


3 Copy the following graphs of $y = f(x)$ and sketch the graphs of $y = f(-x)$ on the same axes.

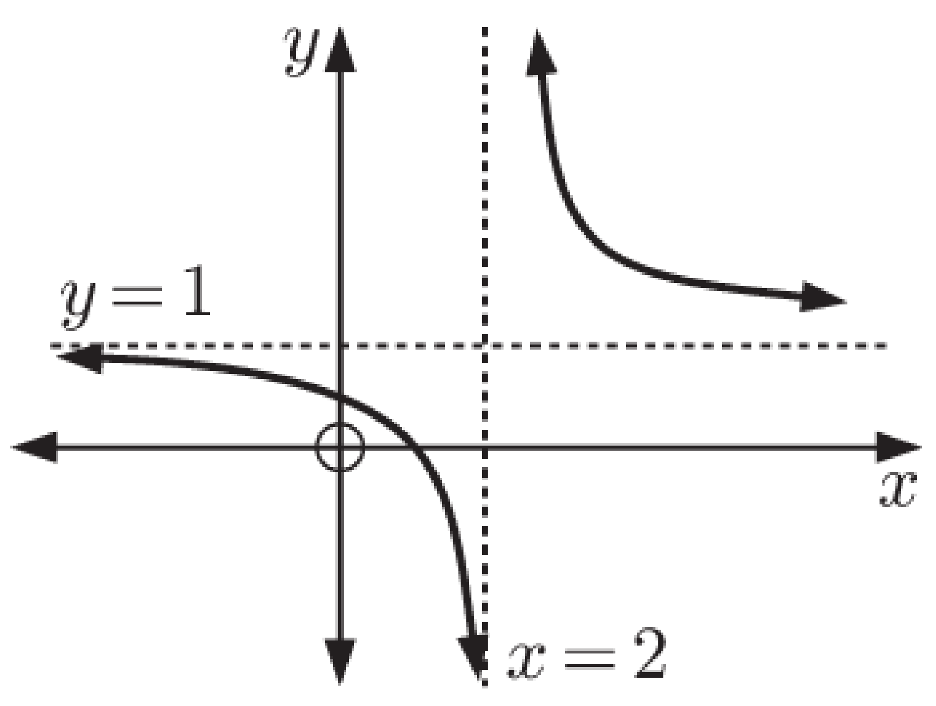
a



b



c



4 Graph $y = f(x)$ and $y = -f(x)$ for:

a $f(x) = 3x$ **b** $f(x) = x^3 - 2$ **c** $f(x) = 2(x+1)^2$

5 Graph $y = f(x)$ and $y = f(-x)$ for:

a $f(x) = 2x + 1$ **b** $f(x) = x^2 + 2x + 1$ **c** $f(x) = x^3$

6 Find the equation of the resulting graph $g(x)$ when:

a $f(x) = 5x + 7$ is reflected in the x-axis **b** $f(x) = 2^x$ is reflected in the y-axis

c $f(x) = 2x^2 + 1$ is reflected in the x-axis

d $f(x) = x^4 - 2x^3 - 3x^2 + 5x - 7$ is reflected in the y-axis.

7 The function $y = f(x)$ is transformed to $g(x) = -f(x)$.

a Find the image points on $y = g(x)$ corresponding to the following points on $y = f(x)$:

i (3, 0) **ii** (2, −1) **iii** (−3, 2)

b Find the points on $y = f(x)$ which are transformed to the following points on $y = g(x)$:

i (7, −1) **ii** (−5, 0) **iii** (−3, −2)

8 The function $y = f(x)$ is transformed to $h(x) = f(-x)$.

a Find the image points on $y = h(x)$ for the following points on $y = f(x)$:

i (2, −1) **ii** (0, 3) **iii** (−1, 2)

b Find the points on $y = f(x)$ corresponding to the following points on $y = h(x)$:

i (5, −4) **ii** (0, 3) **iii** (2, 3)

9 A function $f(x)$ is transformed to the function $g(x) = -f(-x)$.

a What combination of transformations has taken place?

b If (3, −7) lies on $y = f(x)$, find the transformed point on $y = g(x)$.

c Find the point on $f(x)$ that transforms to the point (−5, −1).

10 Let $f(x) = x + 2$.

a Describe the transformation which transforms $y = f(x)$ to $y = -f(x)$.

b Describe the transformation which transforms $y = -f(x)$ to $y = -3f(x)$.

c Hence draw the graphs of $y = f(x)$, $y = -f(x)$, and $y = -3f(x)$ on the same set of axes.

11 Let $f(x) = (x-1)^2 - 4$.

a Describe the transformation which transforms $y = f(x)$ to $y = f(-x)$.

b Describe the transformation which transforms $y = f(-x)$ to $y = f(-\frac{1}{2}x)$.

c Hence draw the graphs of $y = f(x)$, $y = f(-x)$, and $y = f(-\frac{1}{2}x)$ on the same set of axes.

12 Graph on the same set of axes $y = x^2$, $y = -x^2$, and $y = -(x+2)^2 + 3$.
Describe the combination of transformations which transform $y = x^2$ to $y = -(x+2)^2 + 3$.

13 Graph on the same set of axes $y = \frac{1}{x}$, $y = -\frac{1}{x}$, $y = -\frac{1}{x-3} + 2$.
Describe the combination of transformations which transform $y = \frac{1}{x}$ to $y = -\frac{1}{x-3} + 2$.

DISCUSSION

For which combinations of two transformations on $y = f(x)$ is the order in which the transformations are performed:

- important
- not important?

D MISCELLANEOUS TRANSFORMATIONS

A summary of all the transformations is given in the printable concept map.

CONCEPT MAP

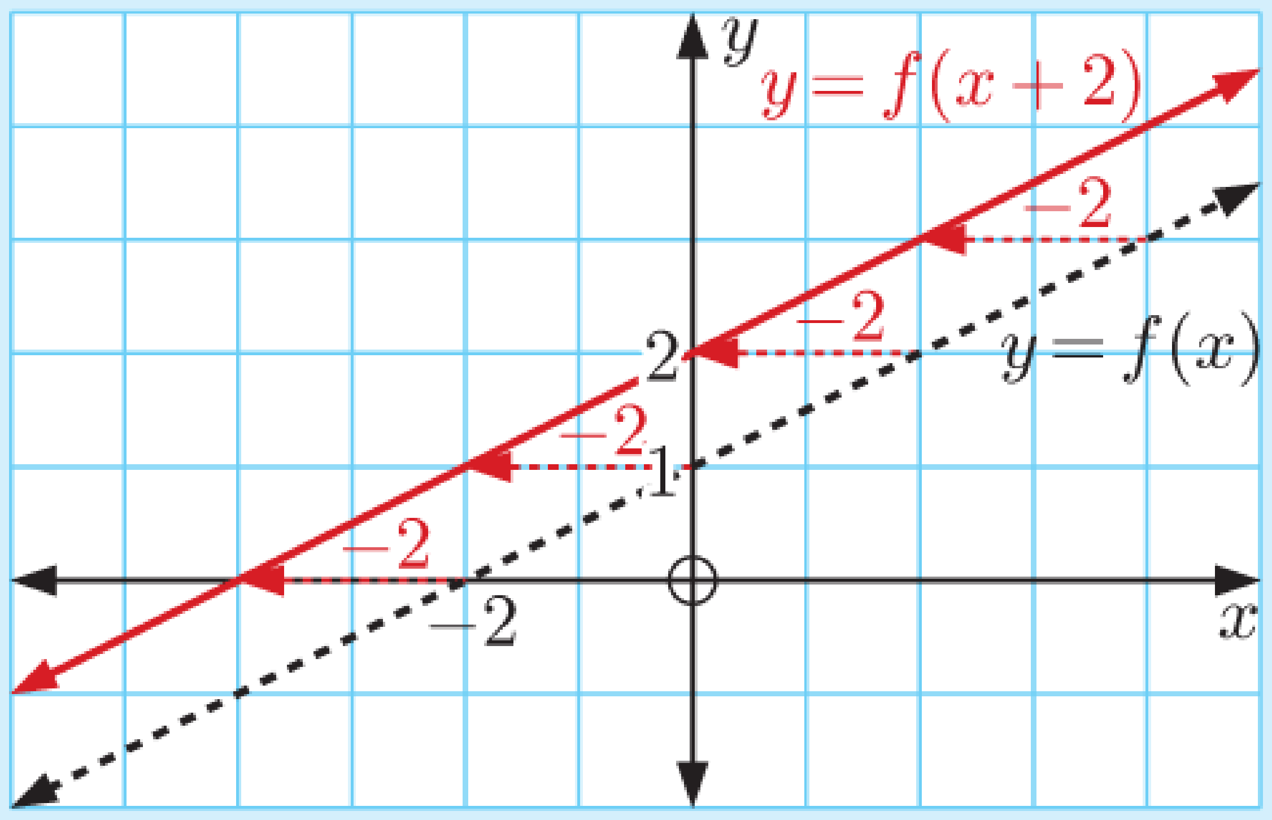
Example 5

Self Tutor

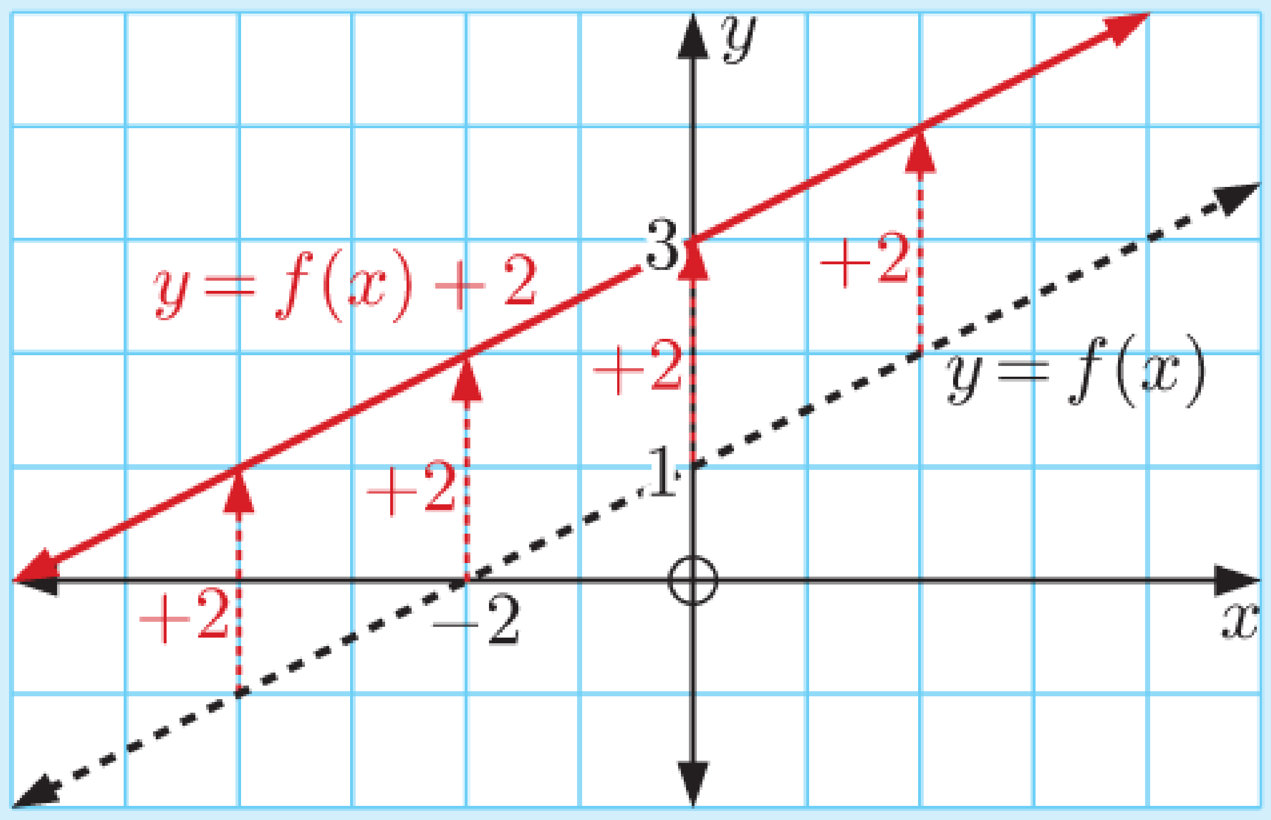
Consider $f(x) = \frac{1}{2}x + 1$. On separate sets of axes graph:

a $y = f(x)$ and $y = f(x+2)$
b $y = f(x)$ and $y = f(x) + 2$
c $y = f(x)$ and $y = 2f(x)$
d $y = f(x)$ and $y = -f(x)$

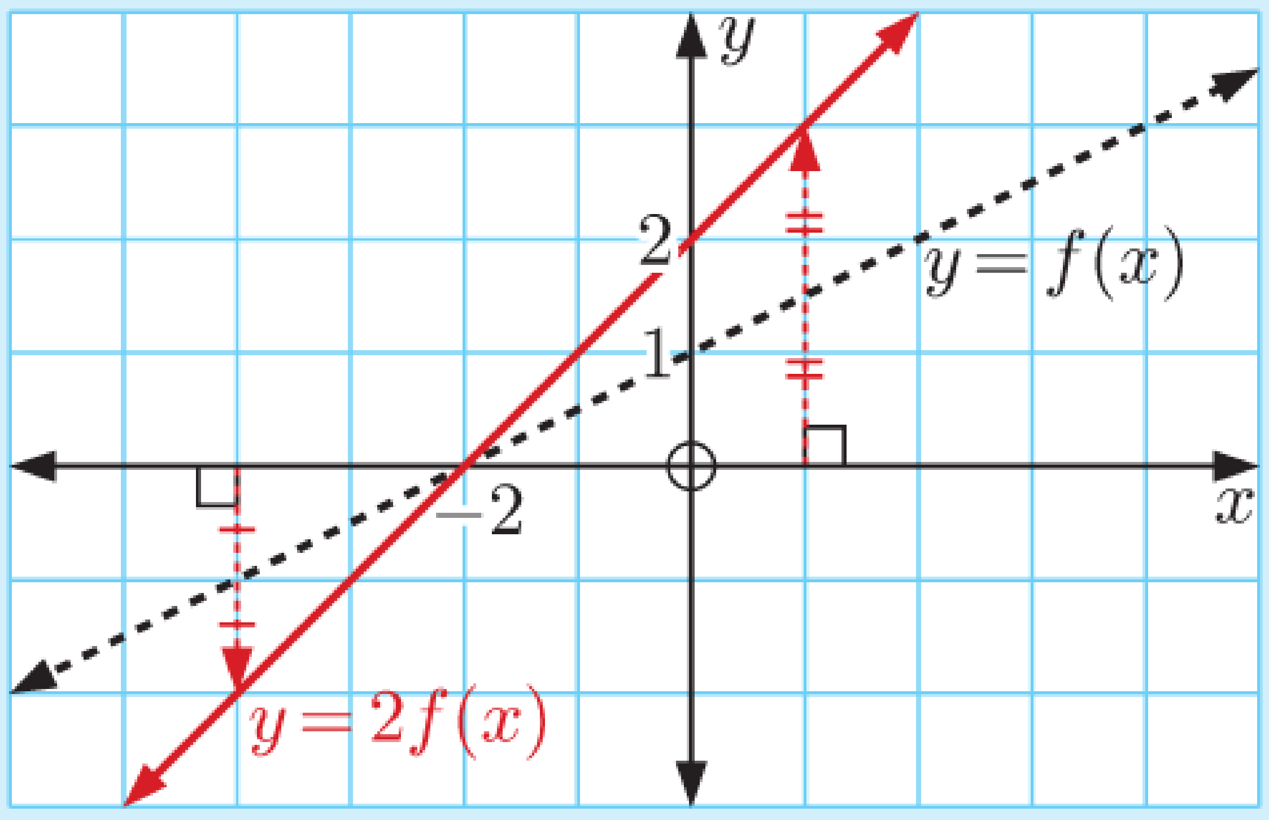
a



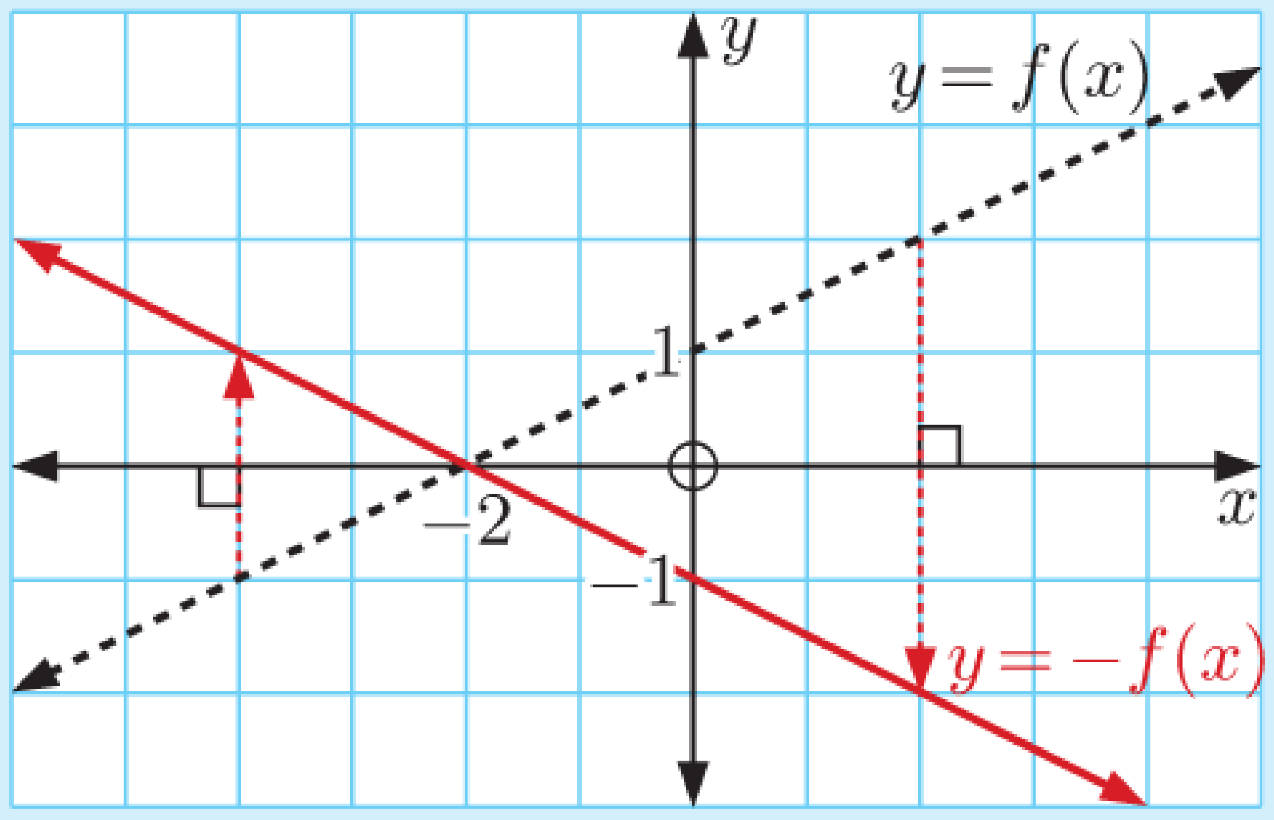
b



c



d



EXERCISE 16D

1 Consider $f(x) = x^2 - 1$.

a Graph $y = f(x)$ and state its axes intercepts.

b Graph each function and describe the transformation which has occurred:

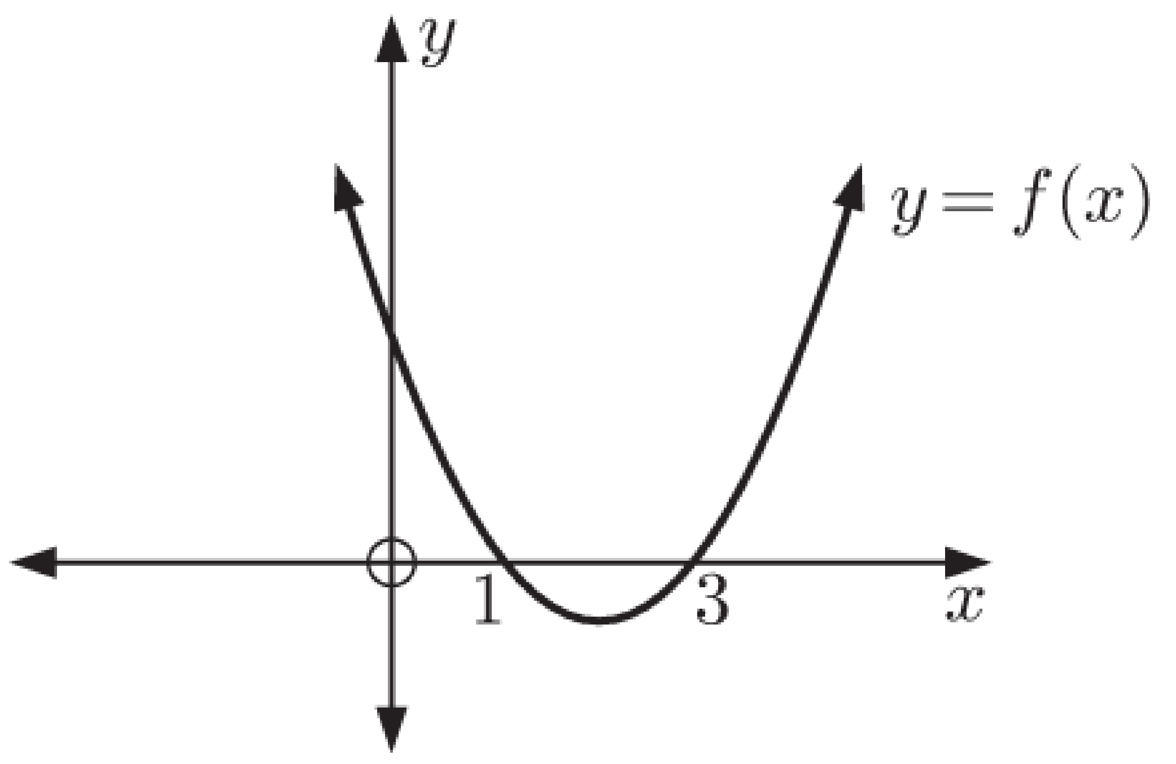
i $y = f(x) + 3$ **ii** $y = f(x - 1)$ **iii** $y = 2f(x)$ **iv** $y = -f(x)$

2 For the graph of $y = f(x)$ given, sketch the graph of:

a $y = 2f(x)$ **b** $y = \frac{1}{2}f(x)$

c $y = f(x + 2)$ **d** $y = f(2x)$

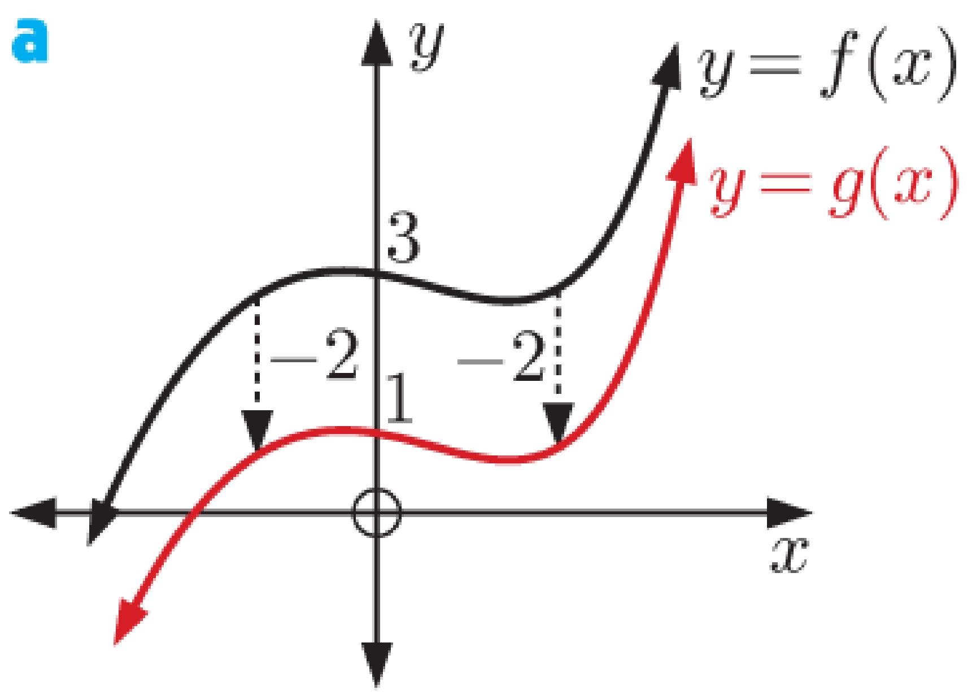
e $y = f(\frac{1}{2}x)$



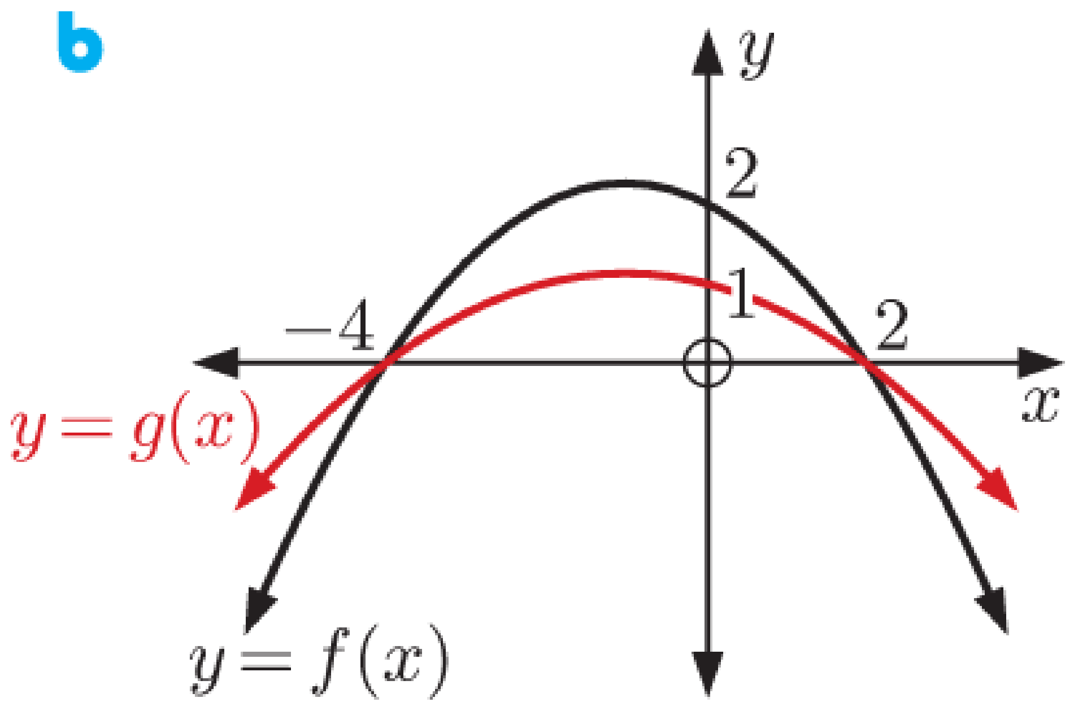
3 In each graph, $f(x)$ is transformed to $g(x)$ using a single transformation.

i Describe the transformation. **ii** Write $g(x)$ in terms of $f(x)$.

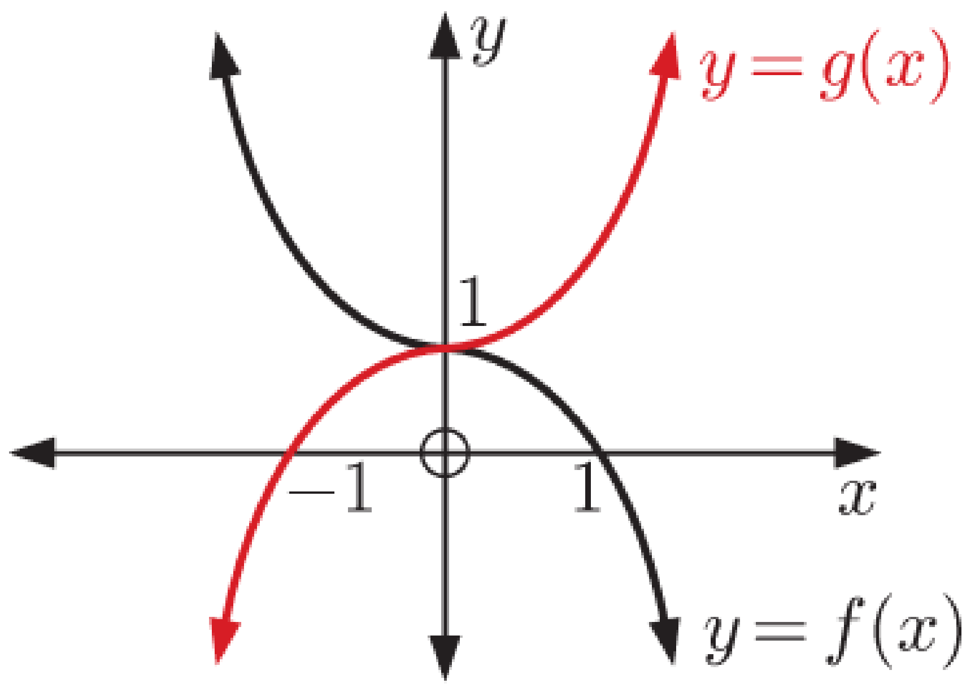
a



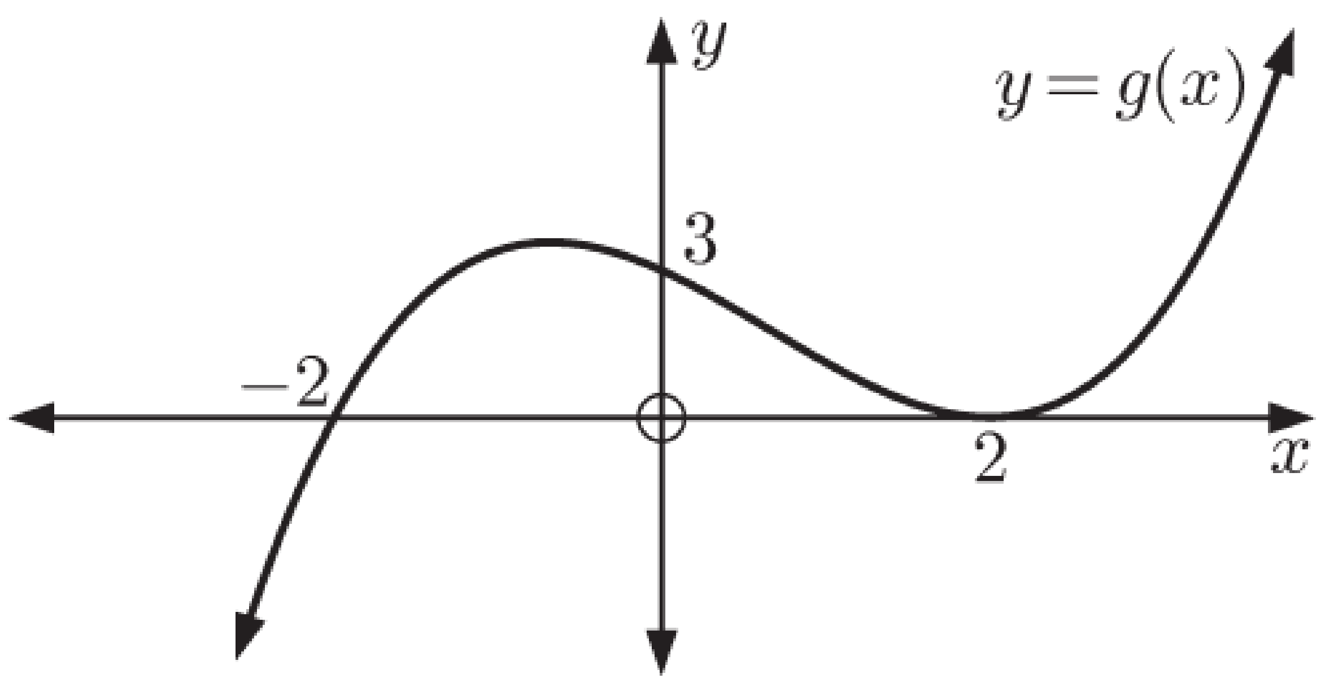
b



c



4



For the graph of $y = g(x)$ given, sketch the graph of:

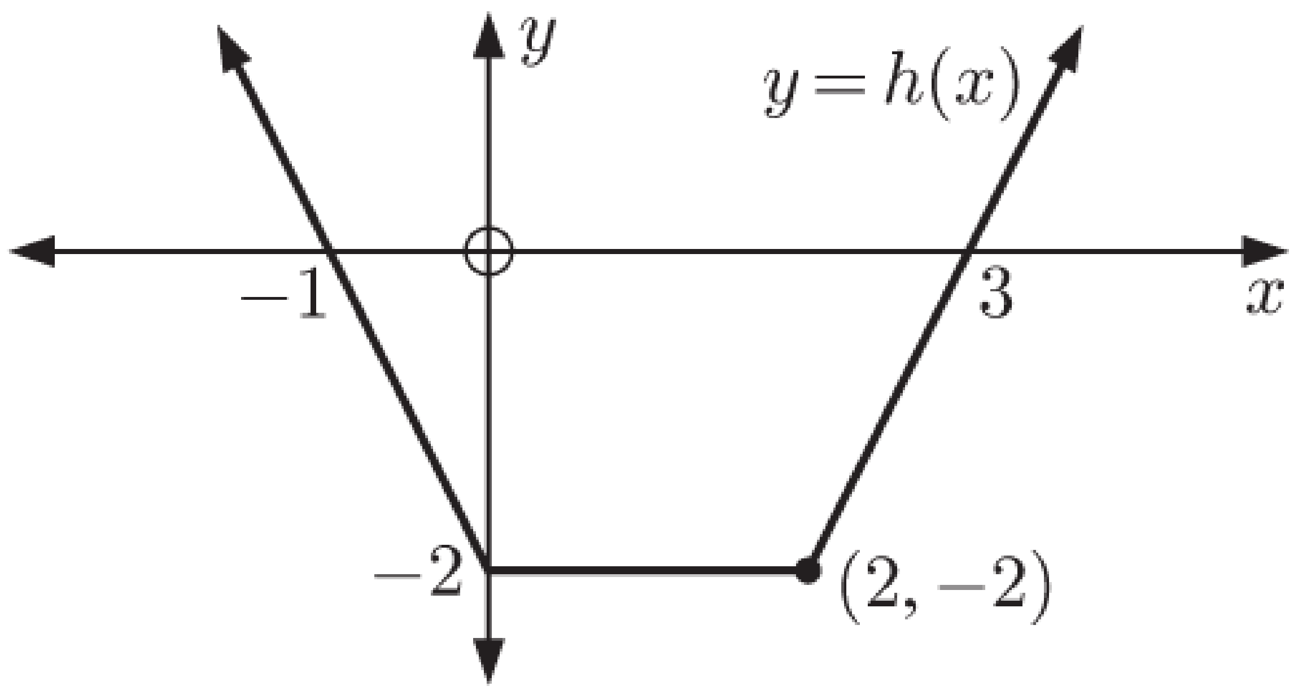
a $y = g(x) + 2$ **b** $y = -g(x)$

c $y = g(-x)$ **d** $y = g(x + 1)$

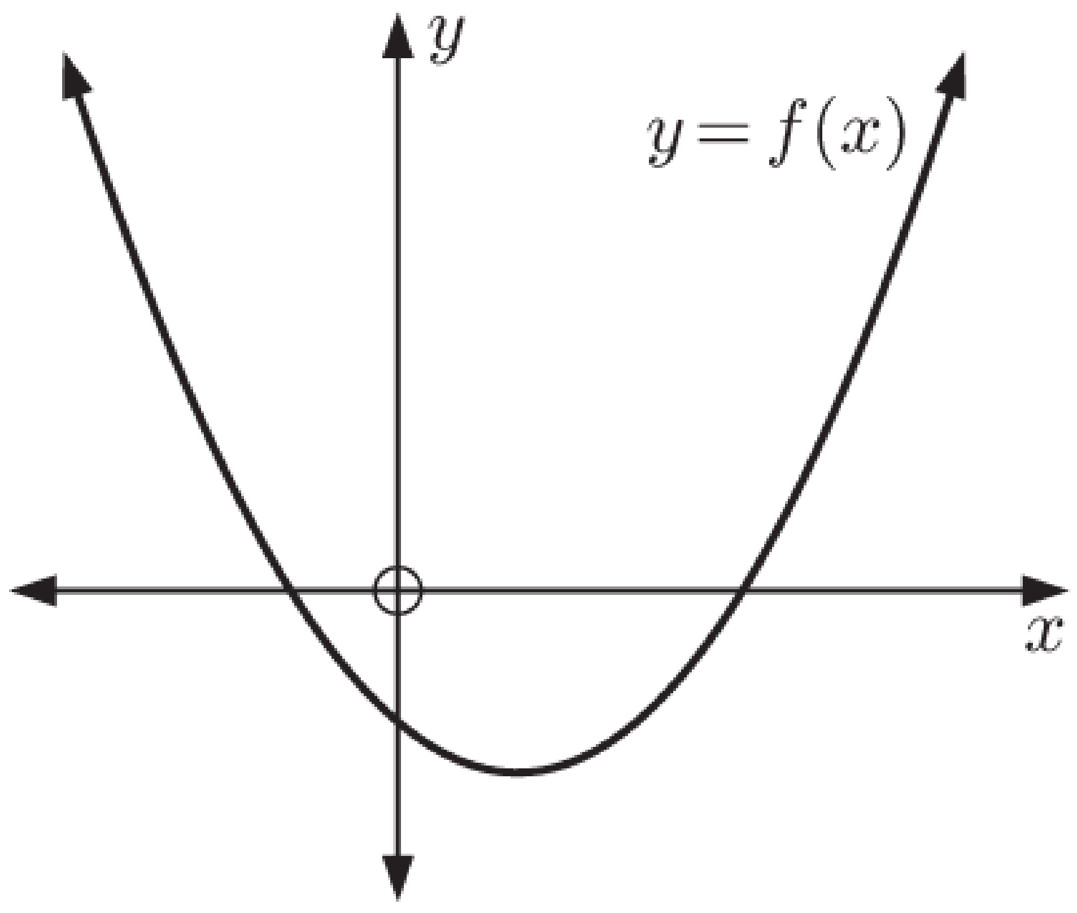
5 For the graph of $y = h(x)$ given, sketch the graph of:

a $y = h(x) + 1$ **b** $y = \frac{1}{2}h(x)$

c $y = h(-x)$ **d** $y = h\left(\frac{x}{2}\right)$



6



Consider the function $f(x) = (x + 1)(x - \beta)$ where $\beta > 0$. A sketch of the function is shown alongside.

a Determine the axes intercepts of the graph of $y = f(x)$.

b Sketch the graphs of $f(x)$ and $g(x) = -f(x - 1)$ on the same set of axes.

c Find and label the axes intercepts of $y = g(x)$.

Example 6 Self Tutor

Consider a function $f(x)$.

a What function results if $y = f(x)$ is reflected in the x-axis, then translated through $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, then stretched vertically with scale factor 2?

b Fully describe the transformations which map $y = f(x)$ onto $y = 3f(2x-1) - 2$.

a
$$f(x) \xrightarrow{\substack{\text{reflection} \\ \text{in } x\text{-axis}}} -f(x) \xrightarrow{\text{translation } \begin{pmatrix} 3 \\ -1 \end{pmatrix}} -f(x-3)-1 \xrightarrow{\substack{\text{vertical stretch} \\ \text{scale factor } 2}} 2(-f(x-3)-1)$$

The resulting function is $-2f(x-3)-2$.

b
$$f(x) \xrightarrow{\substack{\text{vertical stretch} \\ \text{scale factor } 3}} 3f(x) \xrightarrow{\text{translation } \begin{pmatrix} 1 \\ -2 \end{pmatrix}} 3f(x-1)-2 \xrightarrow{\substack{\text{horizontal stretch} \\ \text{scale factor } \frac{1}{2}}} 3f(2x-1)-2$$

7 Consider a function $f(x)$. Find the function which results if $y = f(x)$ is:

- **a** translated through $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ then reflected in the y-axis
- **b** reflected in the y-axis then translated through $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$
- **c** translated through $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ then stretched vertically with scale factor $\frac{1}{2}$
- **d** stretched vertically with scale factor $\frac{1}{2}$ then translated through $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$
- **e** translated through $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$ then stretched horizontally with scale factor 4
- **f** stretched horizontally with scale factor 4 then translated through $\begin{pmatrix} 3 \\ -5 \end{pmatrix}$.

8 Fully describe the transformations which map $y = f(x)$ onto:

- **a** $y = -f(x+1)+3$
- **b** $y = f(\frac{1}{2}x) - 7$
- **c** $y = f(3x-1)$
- **d** $y = -1 + 2f(\frac{1}{4}x - 1)$
- **e** $y = 5 + 2f(3(x-1))$
- **f** $y = -4f(\frac{1}{2}(x+3)) - 1$

9 The function $f(x)$ has domain $\{x \mid x \geqslant 1\}$ and range $\{y \mid -2 \leqslant y < 5\}$.
Find the domain and range of:

- **a** $g(x) = f(x+4) - 1$
- **b** $g(x) = -2f(3x)$
- **c** $g(x) = \frac{1}{3}f(2x-5) + 4$

10 Let T_A be a translation through $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$,

T_B be a reflection in the y-axis, and
T_C be a vertical stretch with scale factor 5.

Find the resulting function when $f(x) = \sqrt{x}$ has the following transformations applied:

- **a** T_A then T_B then T_C
- **b** T_C then T_A then T_B
- **c** T_C then T_B then T_A.

In each case state the domain and range of the transformed function.

11 The graph of $y = x^2$ is transformed into $y = a(x-h)^2 + k$ using three transformations:

- a vertical stretch with invariant x-axis
- a translation with vector $\begin{pmatrix} h \\ k \end{pmatrix}$
- a reflection in the x-axis.

Discuss what you know about:

a the transformations **b** the function.

12 **a** Write $\dfrac{10x+11}{2x+3}$ in the form $a + \dfrac{b}{2x+3}$, where a and b are constants.

b Hence describe the combination of transformations which map $y = \dfrac{1}{x}$ onto $y = \dfrac{10x+11}{2x+3}$.

E THE GRAPH OF $y = \dfrac{1}{f(x)}$

The **reciprocal** of a function $y = f(x)$ is the function $y = \dfrac{1}{f(x)}$.

INVESTIGATION 4 THE GRAPH OF $y = \dfrac{1}{f(x)}$

In this Investigation we will examine the reciprocals of various functions using technology.

GRAPHING PACKAGE

What to do:

1 Sketch each pair of functions on the same set of axes. Include all axes intercepts and asymptotes.

a $y = x$ and $y = \dfrac{1}{x}$

b $y = x + 2$ and $y = \dfrac{1}{x+2}$

c $y = x - 2$ and $y = \dfrac{1}{x-2}$

d $y = 3x + 4$ and $y = \dfrac{1}{3x+4}$.

What do you notice regarding intercepts and asymptotes?

2 **a** Sketch each pair of functions on the same set of axes. Include all axes intercepts and asymptotes.

i $y = x^2$ and $y = \dfrac{1}{x^2}$

ii $y = -(x-1)^2$ and $y = -\dfrac{1}{(x-1)^2}$

iii $y = x^2 + 4$ and $y = \dfrac{1}{x^2+4}$

iv $y = -(x^2-4)$ and $y = -\dfrac{1}{x^2-4}$

v $y = (x-1)(x-3)$ and $y = \dfrac{1}{(x-1)(x-3)}$

b How can the vertical asymptotes of $y = \dfrac{1}{f(x)}$ be established from $f(x)$ without first viewing its graph?

c What other observations can you make about the graph of $y = \dfrac{1}{f(x)}$?

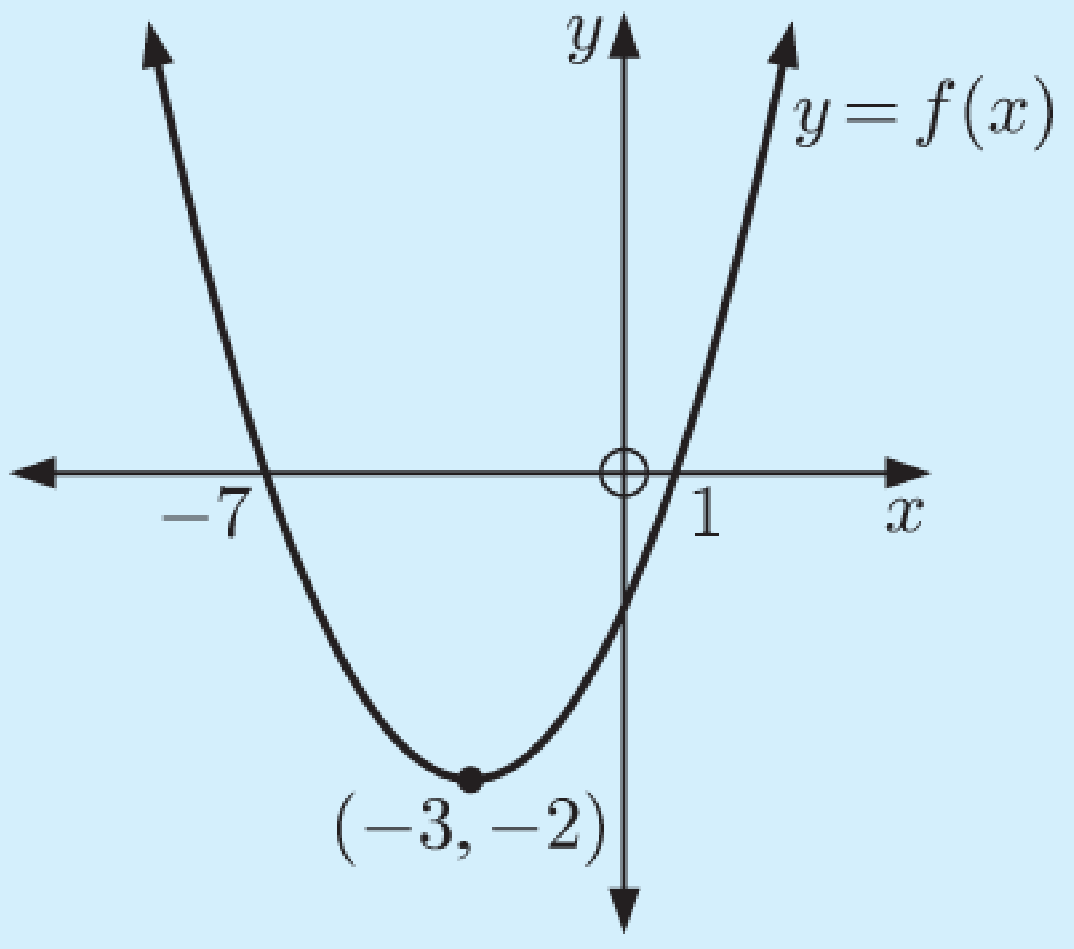
From the **Investigation**, you should have observed that:

When $y = \dfrac{1}{f(x)}$ is graphed from $y = f(x)$:

- the zeros of $y = f(x)$ become vertical asymptotes of $y = \dfrac{1}{f(x)}$
- the vertical asymptotes of $y = f(x)$ become zeros of $y = \dfrac{1}{f(x)}$
- the local maxima of $y = f(x)$ correspond to local minima of $y = \dfrac{1}{f(x)}$
- the local minima of $y = f(x)$ correspond to local maxima of $y = \dfrac{1}{f(x)}$
- when $f(x) > 0$, $\dfrac{1}{f(x)} > 0$ and when $f(x) < 0$, $\dfrac{1}{f(x)} < 0$
- when $f(x) \to 0$, $\dfrac{1}{f(x)} \to \pm\infty$ and when $f(x) \to \pm\infty$, $\dfrac{1}{f(x)} \to 0$.

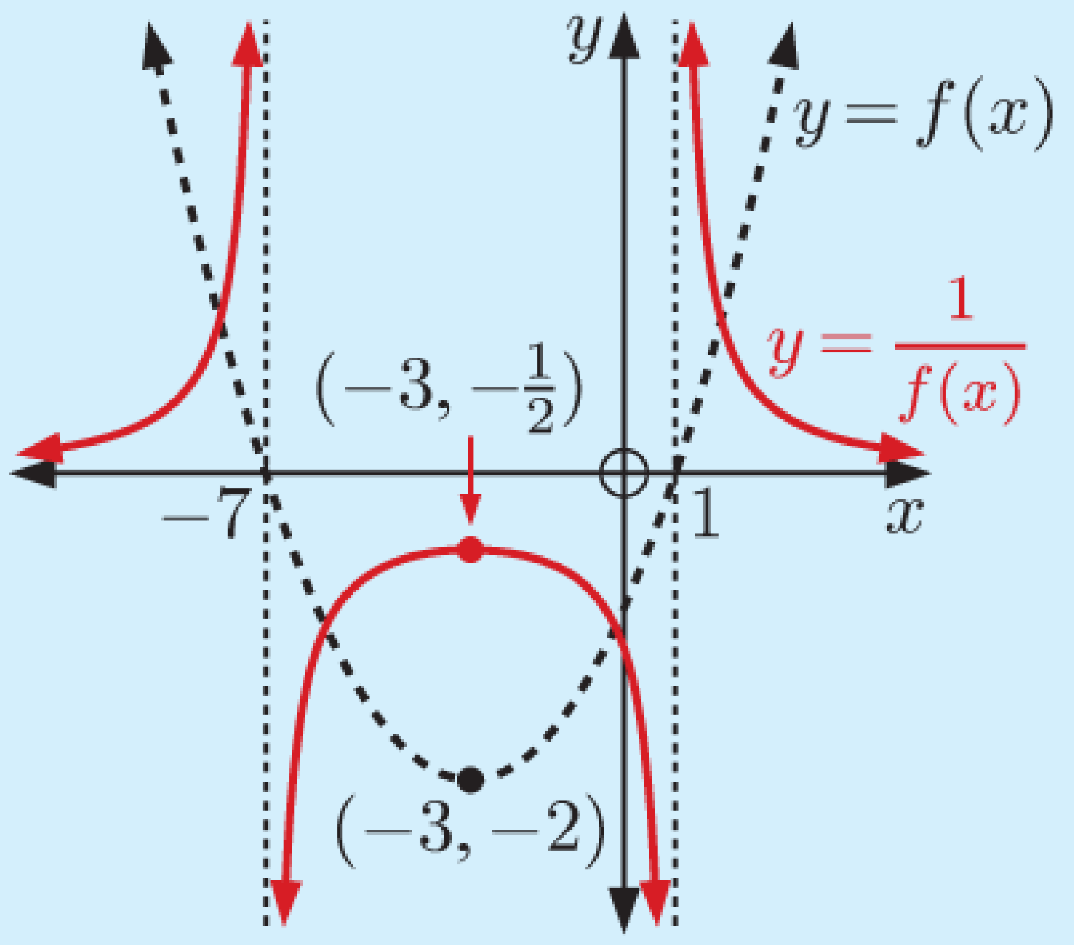
Example 7 **Self Tutor**

For the graph of $y = f(x)$ alongside, draw the graph of $y = \dfrac{1}{f(x)}$.



$y = f(x)$ has x-intercepts -7 and 1, so $y = \dfrac{1}{f(x)}$ has vertical asymptotes $x = -7$ and $x = 1$.

$y = f(x)$ has a local minimum at $(-3, -2)$, so $y = \dfrac{1}{f(x)}$ has a local maximum at $(-3, -\frac{1}{2})$.



EXERCISE 16E

1 Graph on the same set of axes:

a $y = x + 3$ and $y = \dfrac{1}{x+3}$

b $y = -x^2$ and $y = -\dfrac{1}{x^2}$

c $y = \sqrt{x}$ and $y = \dfrac{1}{\sqrt{x}}$

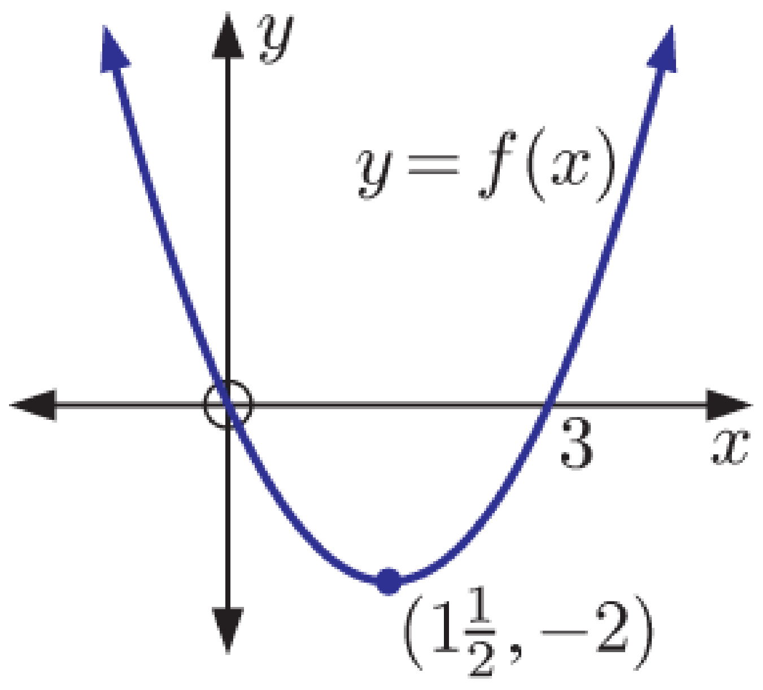
d $y = (x+1)(x-3)$ and $y = \dfrac{1}{(x+1)(x-3)}$

2 Show that if $y = f(x)$ is transformed to $y = \dfrac{1}{f(x)}$, invariant points occur at $y = \pm 1$.

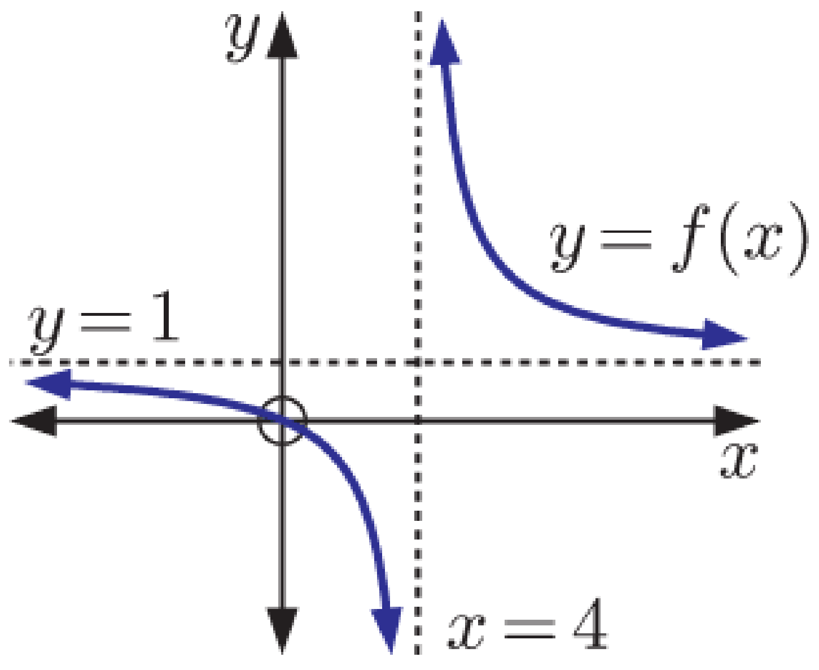
Check your results in question **1** for invariant points.

3 Copy the following graphs for $y = f(x)$ and on the same axes graph $y = \dfrac{1}{f(x)}$:

a



b



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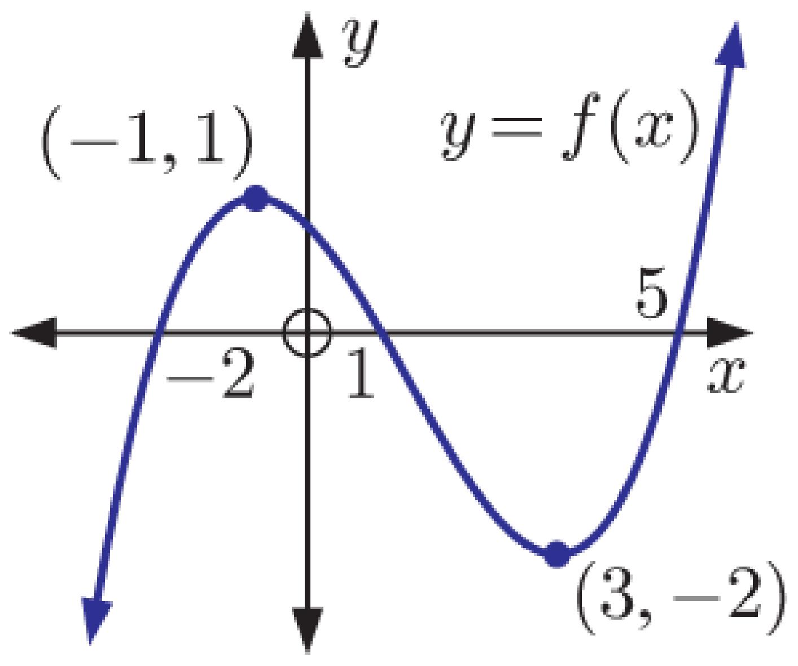
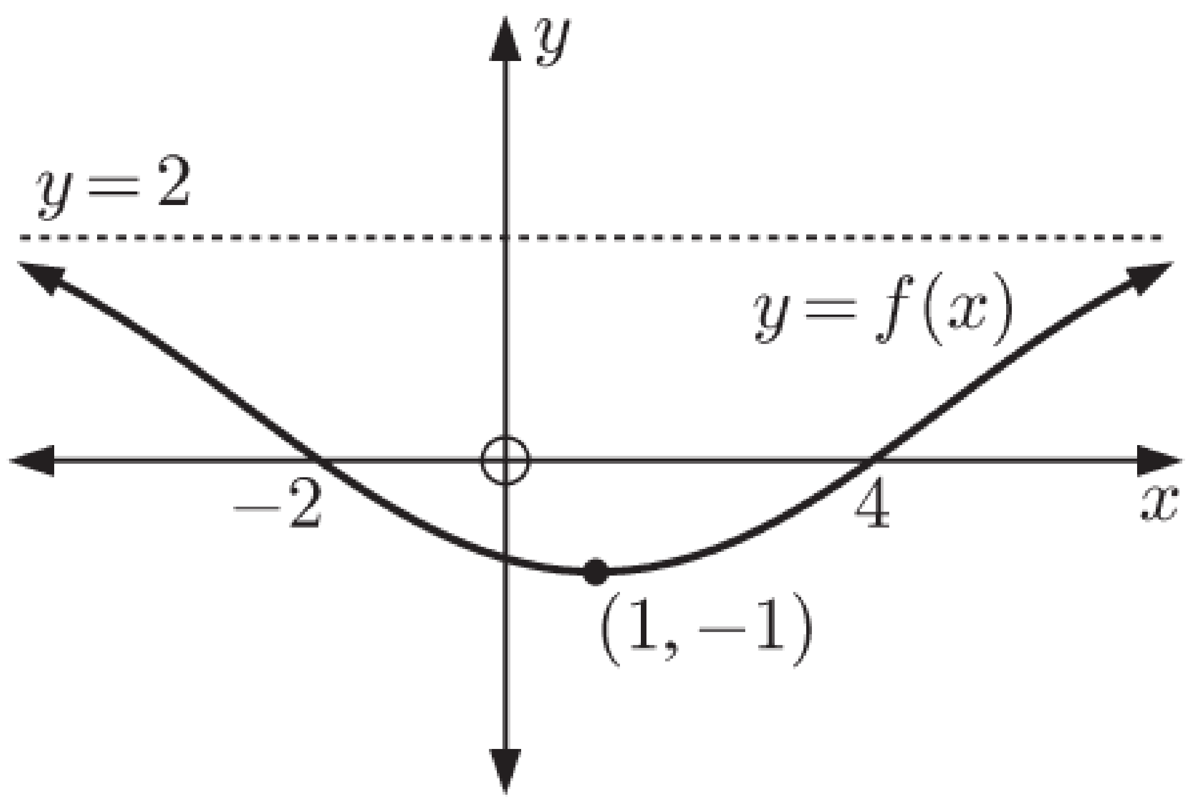


4 Copy the graph of $y = f(x)$ alongside, and graph $y = \dfrac{1}{f(x)} - 3$ on the same set of axes. Clearly show the asymptotes and turning points.



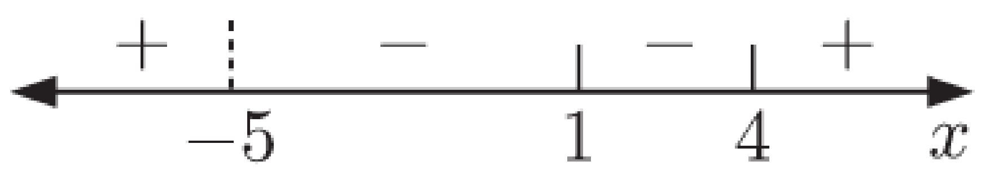
PRINTABLE GRAPHS

5 Let $f(x) = x^2 + 4x + 3$.

- **a** Find the axes intercepts and vertex of $f(x)$.
- **b** Sketch $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.
- **c** Solve for x: $\dfrac{1}{f(x)} = \dfrac{4}{21}$

6 The sign diagram of $f(x)$ is shown alongside.

Draw the sign diagram of $\dfrac{1}{f(x)}$.



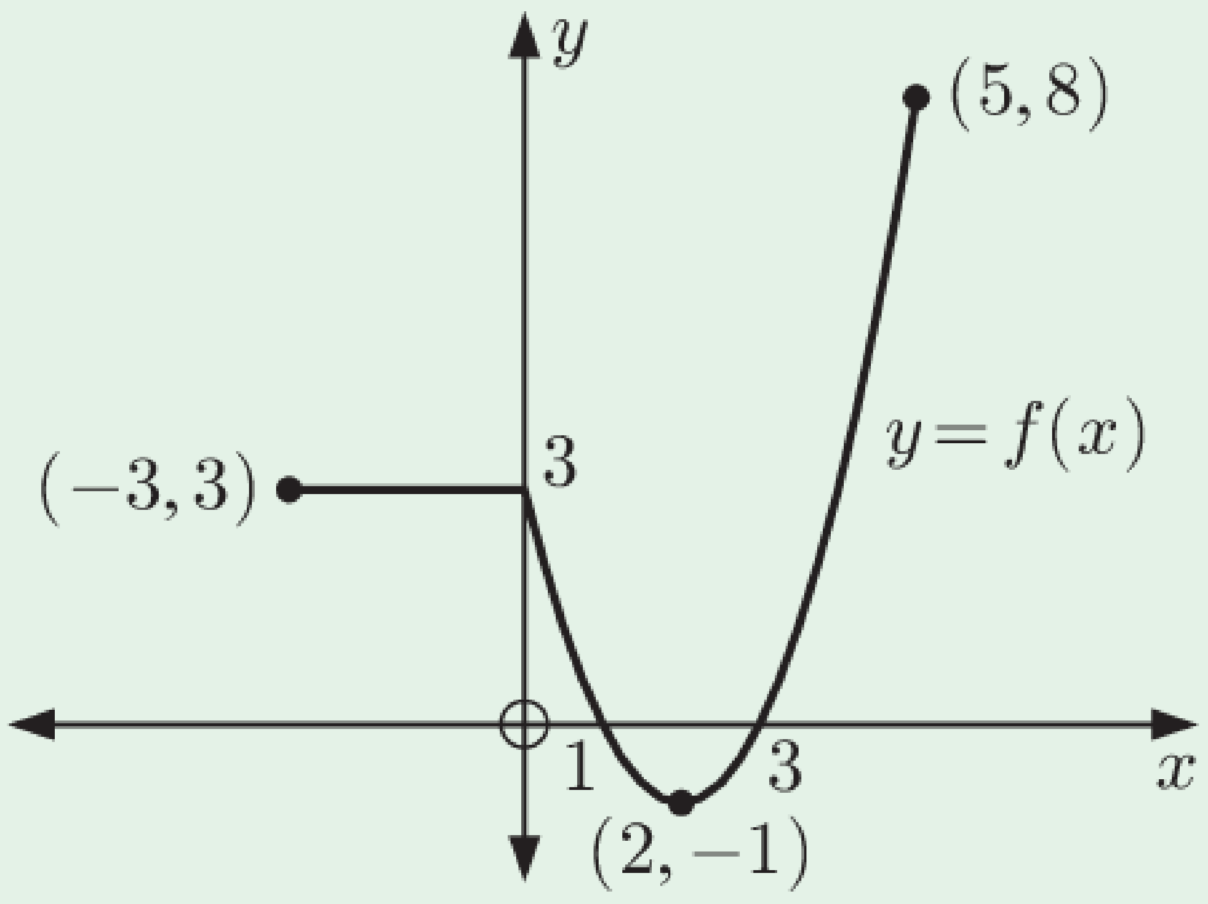
7 If possible, find the domain and range of $\dfrac{1}{f(x)}$ given that:

- **a** $f(x)$ has domain $-1 \leqslant x \leqslant 6$ and range $2 \leqslant y < 5$
- **b** $f(x)$ has domain $2 \leqslant x \leqslant 8$ and range $-3 \leqslant y \leqslant 3$.

REVIEW SET 16A

1 For the graph of $y = f(x)$, sketch graphs of:

- **a** $y = f(-x)$
- **b** $y = -f(x)$
- **c** $y = f(x + 2)$
- **d** $y = f(x) + 2$



2 Consider the function $f : x \mapsto x^2$.
On the same set of axes, graph $y = f(x)$, $y = 3f(x)$, and $y = 3f(x-1)+2$.

3 Find the equation of the resulting graph $g(x)$ when:

- **a** $f(x) = 4x - 7$ is translated 3 units downwards
- **b** $f(x) = x^2 + 6$ is vertically stretched with scale factor 5
- **c** $f(x) = 7 - 3x$ is translated 4 units to the left
- **d** $f(x) = 2x^2 - x + 4$ is horizontally stretched with scale factor 3
- **e** $f(x) = x^3$ is reflected in the y-axis.

4 Sketch the graph of $f(x) = x^2 + 1$, and on the same set of axes sketch the graph of:

a $y = -f(x)$ **b** $y = f(2x)$ **c** $y = f(x) + 3$

5 The function $f(x)$ has domain $\{x \mid -2 \leqslant x \leqslant 3\}$ and range $\{y \mid -1 \leqslant y \leqslant 7\}$.
Find the domain and range of $g(x) = f(x+3) - 4$. Explain your answers.

6 The graph of the function $f(x) = (x+1)^2 + 4$ is translated 2 units to the right and 4 units up.

- **a** Find the function $g(x)$ corresponding to the translated graph.
- **b** State the range of: **i** $f(x)$ **ii** $g(x)$

7 Show that the discriminant of a quadratic function is unchanged when the graph of the function is:

a reflected in the x-axis **b** reflected in the y-axis
c translated h units to the right.

8 The graph of $f(x) = 3x^2 - x + 4$ is translated by the vector $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Write the equation of the image in the form $g(x) = ax^2 + bx + c$.

9 Consider a function $f(x)$. Find the function which results if $y = f(x)$ is:

- **a** reflected in the x-axis then translated through $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$
- **b** translated through $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ then vertically stretched with scale factor 2.

10 The point A$(-2,\ 3)$ lies on the graph of $y = f(x)$. Find the image of A under the transformation:

a $y = f(x-2) + 1$ **b** $y = 2f(x-2)$ **c** $y = f(2x-3)$

11 Suppose the graph of $y = f(x)$ has x-intercepts -5 and 1, and y-intercept -3. What can you say about the axes intercepts of:

a $y = f(x+4)$ **b** $y = 3f(x)$ **c** $y = f\left(\frac{x}{2}\right)$ **d** $y = -f(x)$?

12 The function $g(x)$ results when $y = \frac{1}{x}$ is transformed by a translation through $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ followed by a reflection in the y-axis.

- **a** Write an expression for $g(x)$ in the form $g(x) = \dfrac{ax+b}{cx+d}$.
- **b** Find the asymptotes of $y = g(x)$.
- **c** State the domain and range of $g(x)$.
- **d** Sketch $y = g(x)$.

13 Graph on the same set of axes $y = x^2$, $y = \frac{1}{4}x^2$, and $y = \frac{1}{4}(x-2)^2 - 1$.
Describe the combination of transformations which transform $y = x^2$ to $y = \frac{1}{4}(x-2)^2 - 1$.

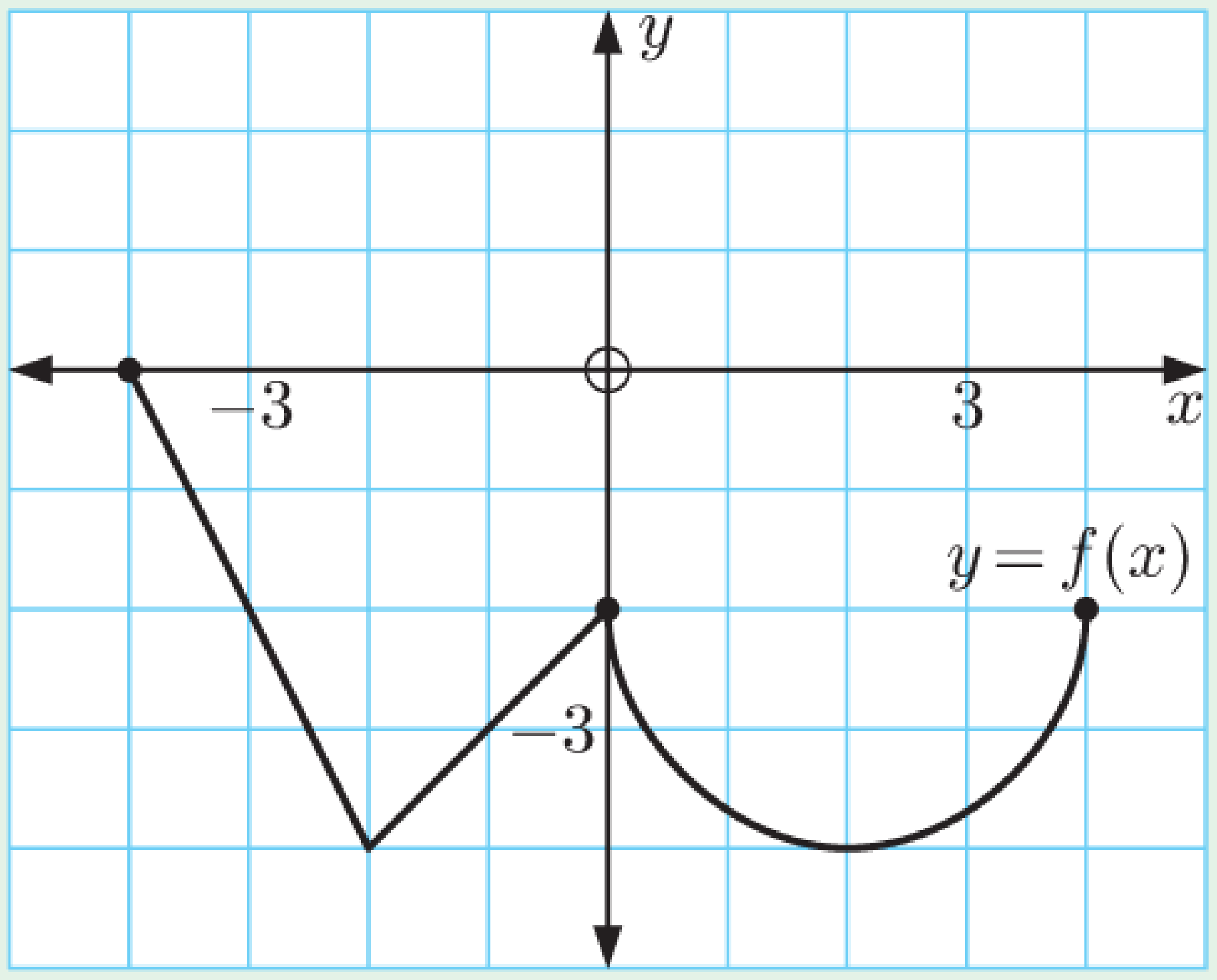
14 Sketch $y = (x-2)(x+3)$ and $y = \dfrac{1}{(x-2)(x+3)}$ on the same set of axes. Clearly label all axes intercepts and asymptotes.

REVIEW SET 16B

1 Consider the graph of $y = f(x)$ alongside.
On separate axes, draw the graphs of:

a $y = f(x-1)$ **b** $y = f(2x)$
c $y = f(x) + 3$ **d** $y = 2f(x)$
e $y = f(-x)$ **f** $y = -f(x)$

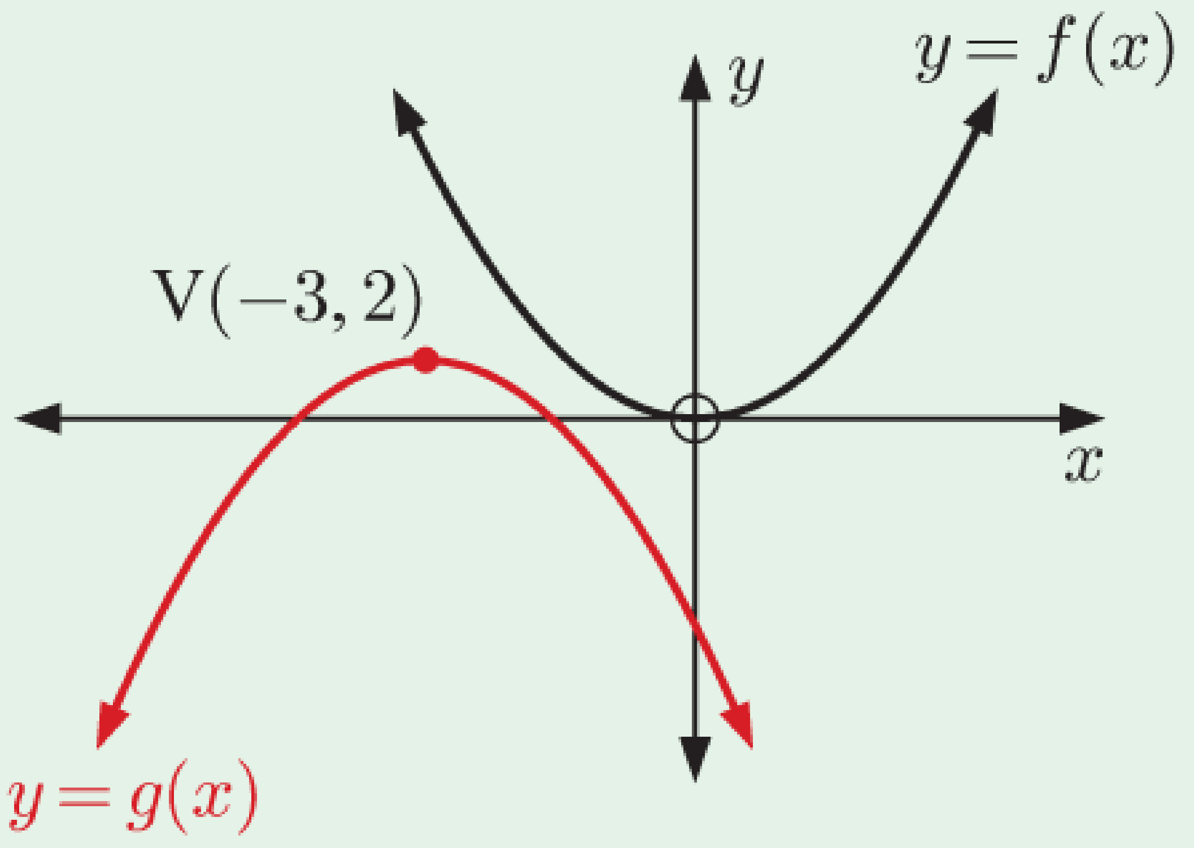
PRINTABLE GRIDS



2 Find the equation of the resulting graph $g(x)$ when:

a $f(x) = x^2 - 3x$ is reflected in the x-axis
b $f(x) = 14 - x$ is translated 2 units upwards
c $f(x) = \frac{1}{3}x + 2$ is horizontally stretched with scale factor 4.

3 The graph of $f(x) = x^2$ is transformed to the graph of $g(x)$ by a reflection and a translation as illustrated. Find the formula for $g(x)$ in the form $g(x) = ax^2 + bx + c$.

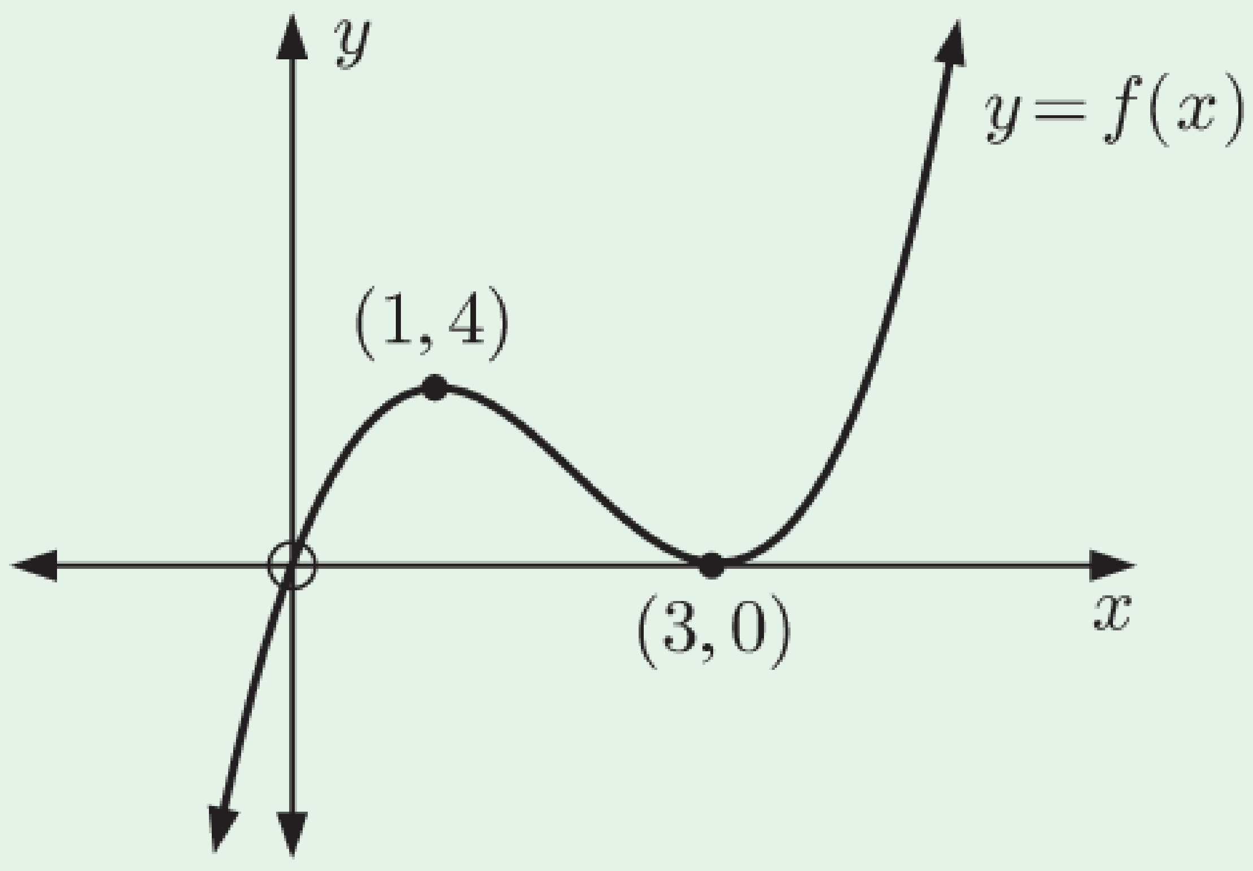


4 Sketch the graph of $f(x) = -x^2$, and on the same set of axes sketch the graph of:

a $y = f(-x)$ **b** $y = -f(x)$ **c** $y = f(2x)$ **d** $y = f(x-2)$

5 The graph of a cubic function $y = f(x)$ is shown alongside.

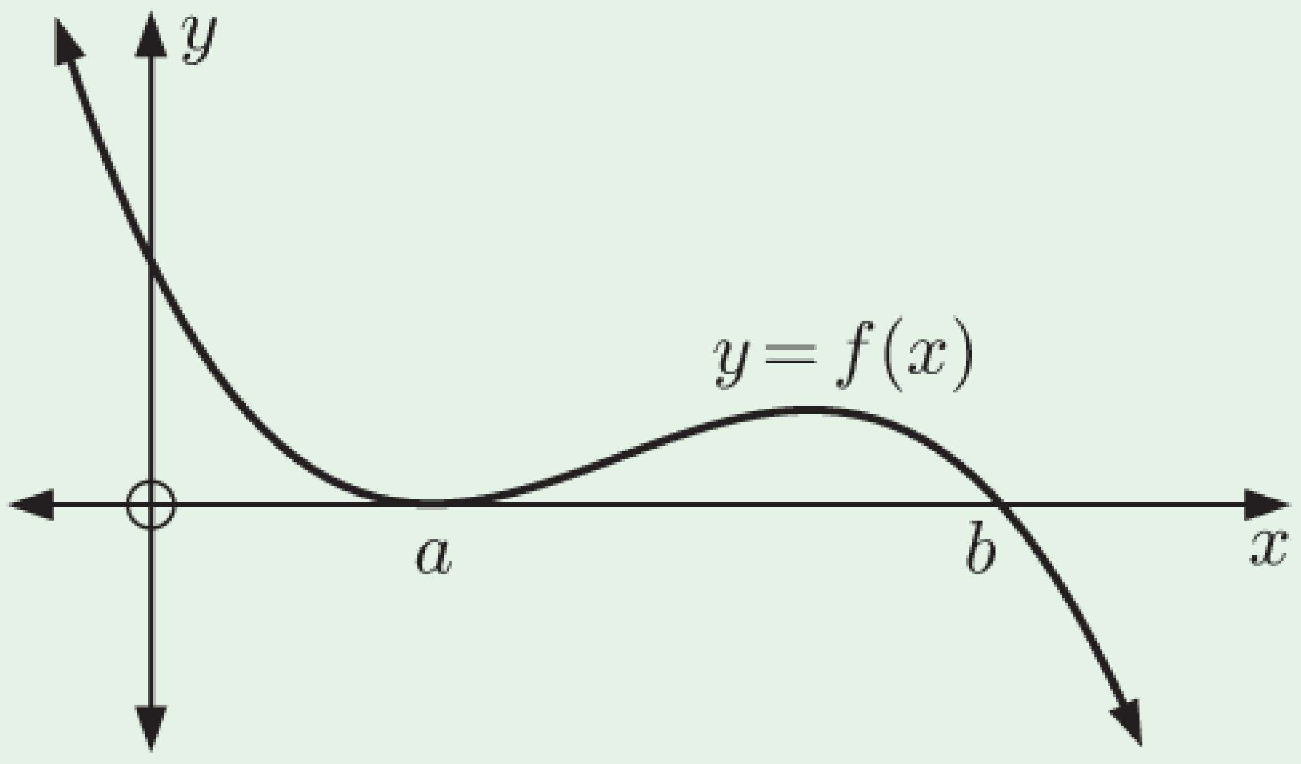
a Sketch the graph of $g(x) = -f(x-1)$.
b State the coordinates of the turning points of $y = g(x)$.



6 The graph of $f(x) = -2x^2 + x + 2$ is translated by the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Write the equation of the image in the form $y = ax^2 + bx + c$.

7 The graph of $y = f(x)$ is shown alongside.
The x-axis is a tangent to $f(x)$ at $x = a$ and $f(x)$ cuts the x-axis at $x = b$.
On the same diagram, sketch the graph of $y = f(x - c)$ where $0 < c < b - a$.
Indicate the x-intercepts of $y = f(x - c)$.



8 Find the combination of transformations which maps $f(x) = 2x^2 + 8x - 3$ onto $g(x) = -2x^2 + 2x + 7$.

9 The point $(-1, 6)$ lies on the graph of $y = f(x)$. Find the corresponding point on the graph of $y = \frac{1}{2}f(x - 2) + 3$.

10 Fully describe the transformations which map $y = f(x)$ onto:

a $y = 2f(x + 1) + 3$ **b** $y = -f(\frac{2}{3}x) - 6$ **c** $y = \frac{1}{3}f(-x + 2)$

11 The quadratic function $f(x) = x^2 + bx + c$ is reflected in the y-axis, stretched horizontally with scale factor $\frac{3}{2}$, then translated through $\begin{pmatrix} -10 \\ 20 \end{pmatrix}$. The resulting quadratic function has the same x-intercepts as $f(x)$. Find b and c.

12 **a** Graph on the same set of axes $y = \dfrac{1}{x}$, $y = -\dfrac{1}{x}$, $y = -\dfrac{1}{2x}$, and $y = -\dfrac{1}{2(x+1)} - 2$.

b Describe the combination of transformations which transform $y = \dfrac{1}{x}$ into $y = -\dfrac{1}{2(x+1)} - 2$.

c Write the resulting function in the form $y = \dfrac{ax + b}{cx + d}$, and state its domain and range.

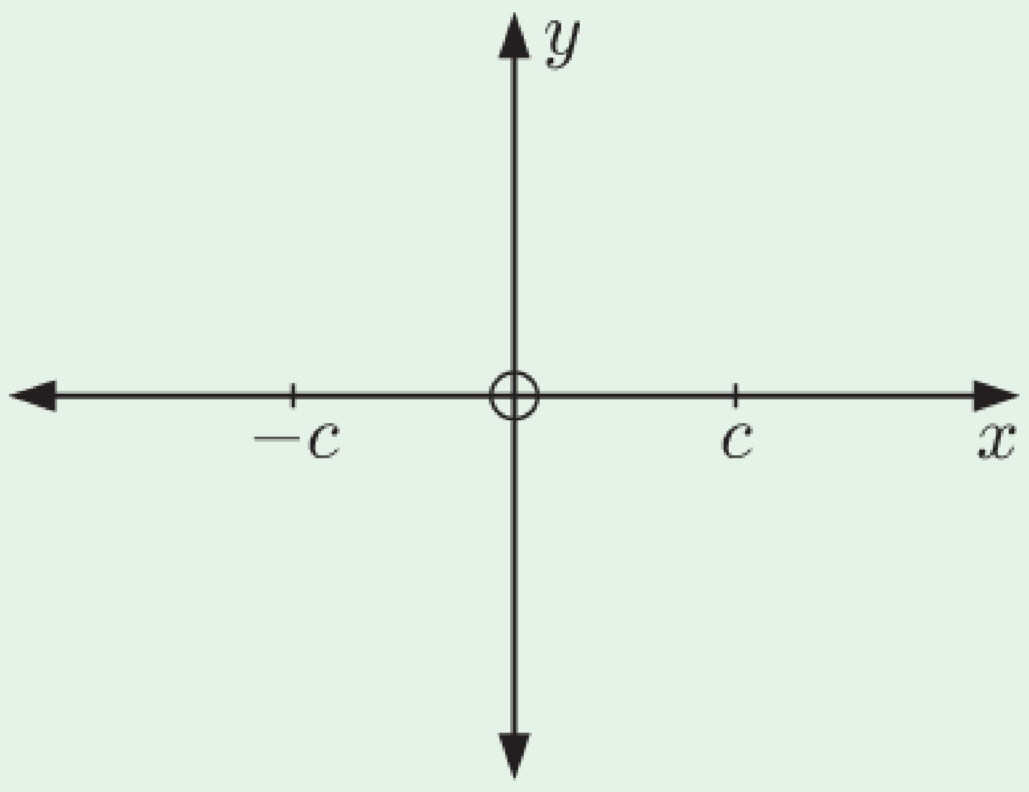
13 **a** Sketch the graph of $f(x) = -2x + 3$, clearly showing the axes intercepts.

b Find the invariant points for the graph of $y = \dfrac{1}{f(x)}$.

c State the y-intercept and vertical asymptote of $y = \dfrac{1}{f(x)}$.

d Sketch the graph of $y = \dfrac{1}{f(x)}$ on the same axes as in part **a**, showing clearly the information you have found.

14 Let $f(x) = \dfrac{c}{x + c}$, $x \neq -c$, $c > 0$.
On a set of axes like those shown, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$. Clearly label any points of intersection with the axes and any asymptotes.



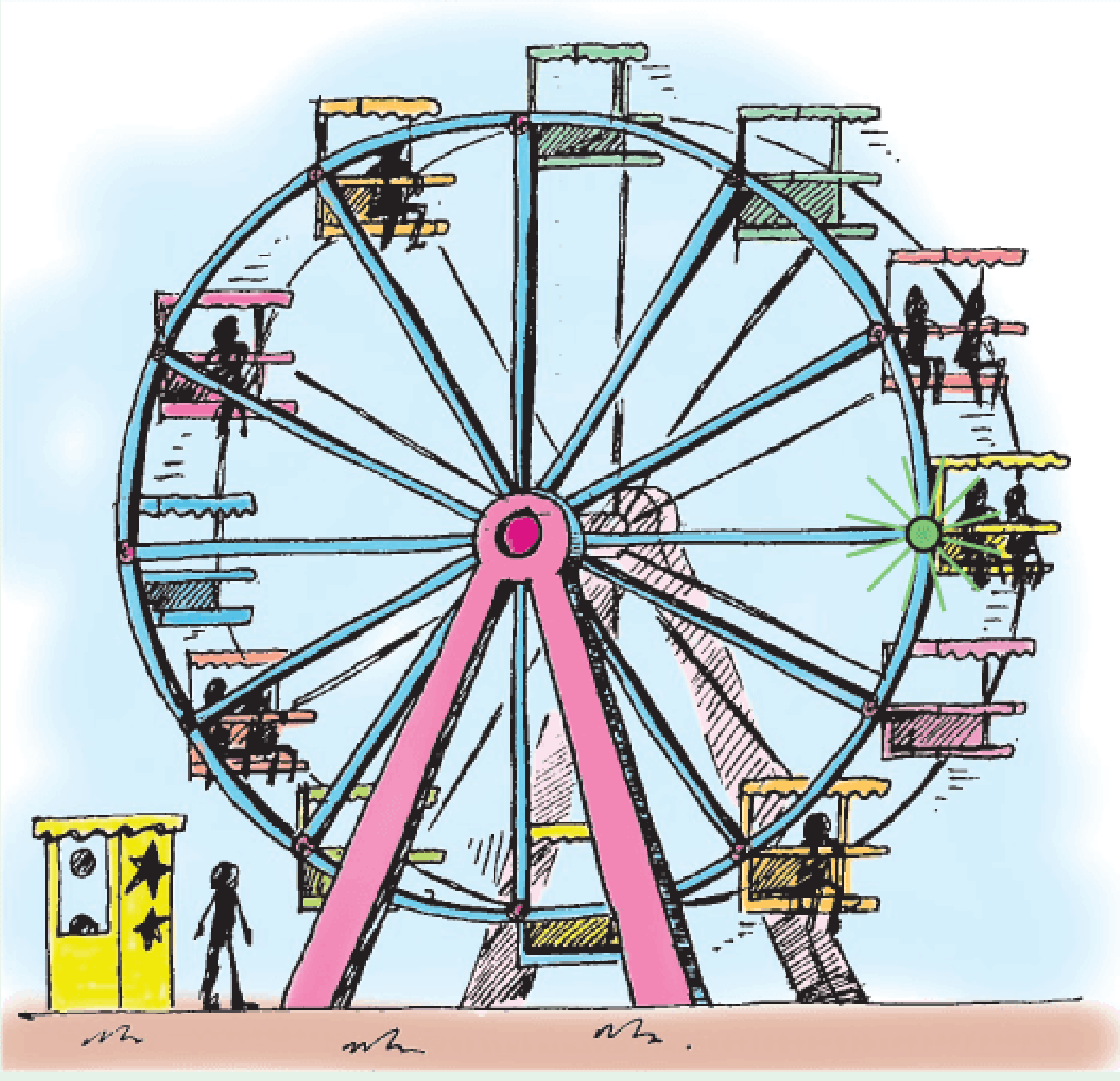
Chapter 17

Trigonometric functions

Contents:

OPENING PROBLEM

A Ferris wheel rotates anticlockwise at a constant speed. The wheel's radius is 10 m and the bottom of the wheel is 2 m above ground level. From his viewing point next to the ticket booth, Andrew is watching a green light on the perimeter of the wheel. He notices that the green light moves in a circle. It takes 100 seconds for a full revolution.



Click on the icon to visit a simulation of the Ferris wheel. You will be able to view the light from:

- in front of the wheel
- a side-on position
- above the wheel.

DEMO

You can then observe graphs of the green light's position as the wheel rotates at a constant rate.

Things to think about:

a Andrew estimates how high the light is above ground level at two second intervals. What will a graph of this data look like? Assume that the light is initially in the position shown.

b Andrew then estimates the horizontal position of the light at two second intervals. What will a graph of this data look like?

c What similarities and differences will there be between your two graphs?

d Can you write a function which will give the:

  i height of the light at any time t seconds

  ii horizontal displacement of the light at any time t seconds?

A PERIODIC BEHAVIOUR

Periodic phenomena occur all the time in the physical world. For example, in:

- seasonal variations in our climate
- variations in average maximum and minimum monthly temperatures
- the number of daylight hours at a particular location
- tidal variations in the depth of water in a harbour
- the phases of the moon
- animal populations.

These phenomena illustrate variable behaviour which is repeated over time. The repetition may be called **periodic**, **oscillatory**, or **cyclic** in different situations.

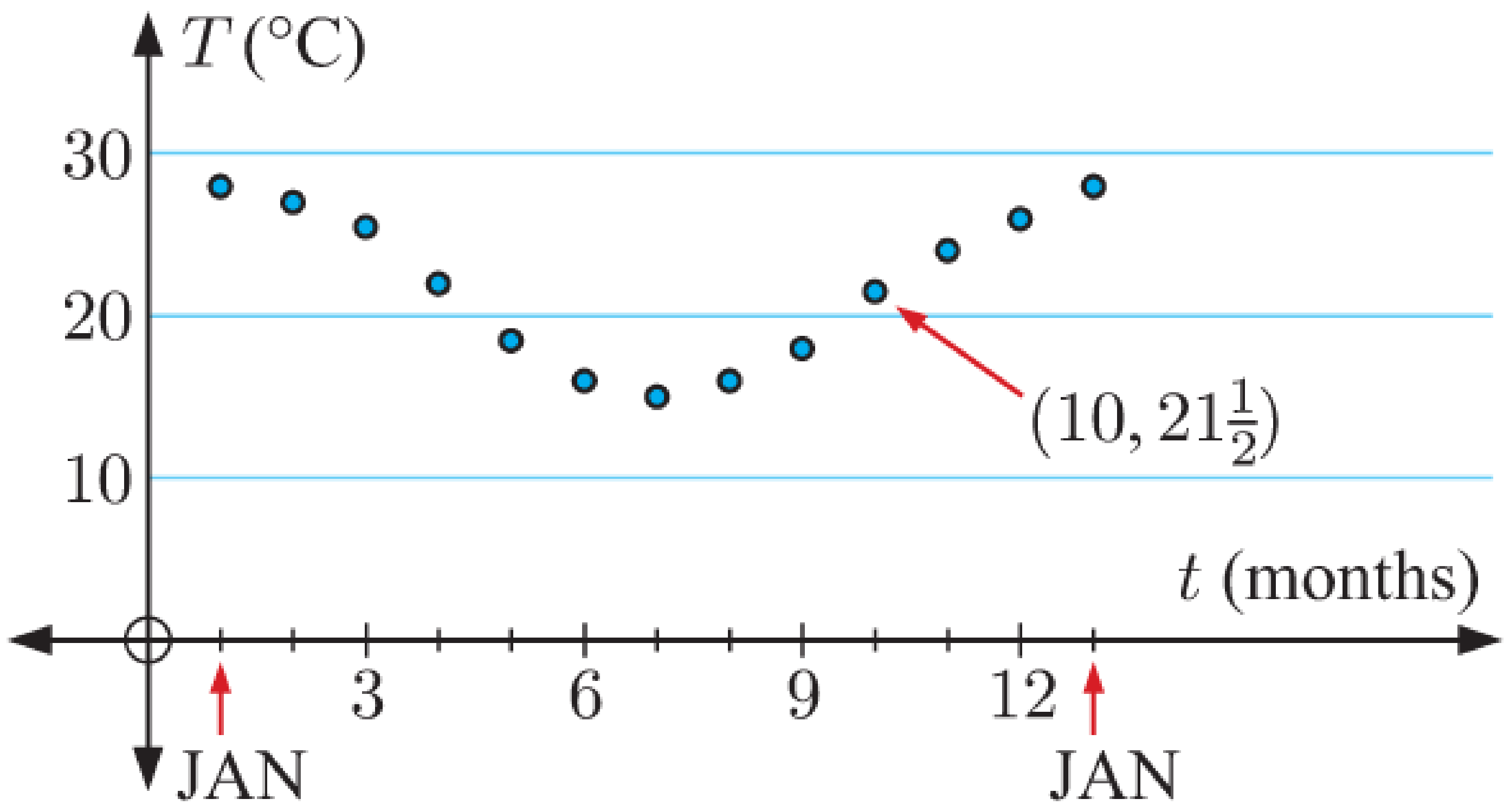
In this Chapter we will see how trigonometric functions can be used to model periodic phenomena.

OBSERVING PERIODIC BEHAVIOUR

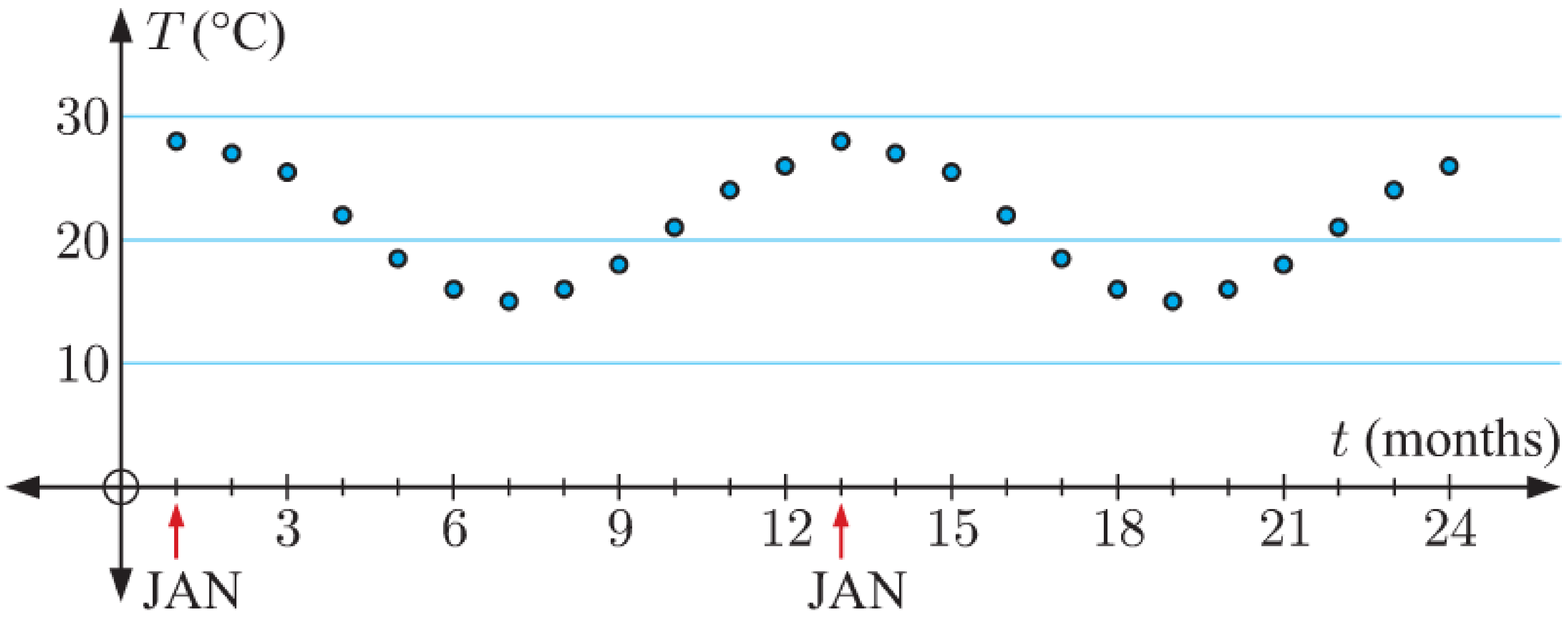
The table below shows the mean monthly maximum temperature for Cape Town, South Africa.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temperature T (°C)	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18	$21\frac{1}{2}$	24	26

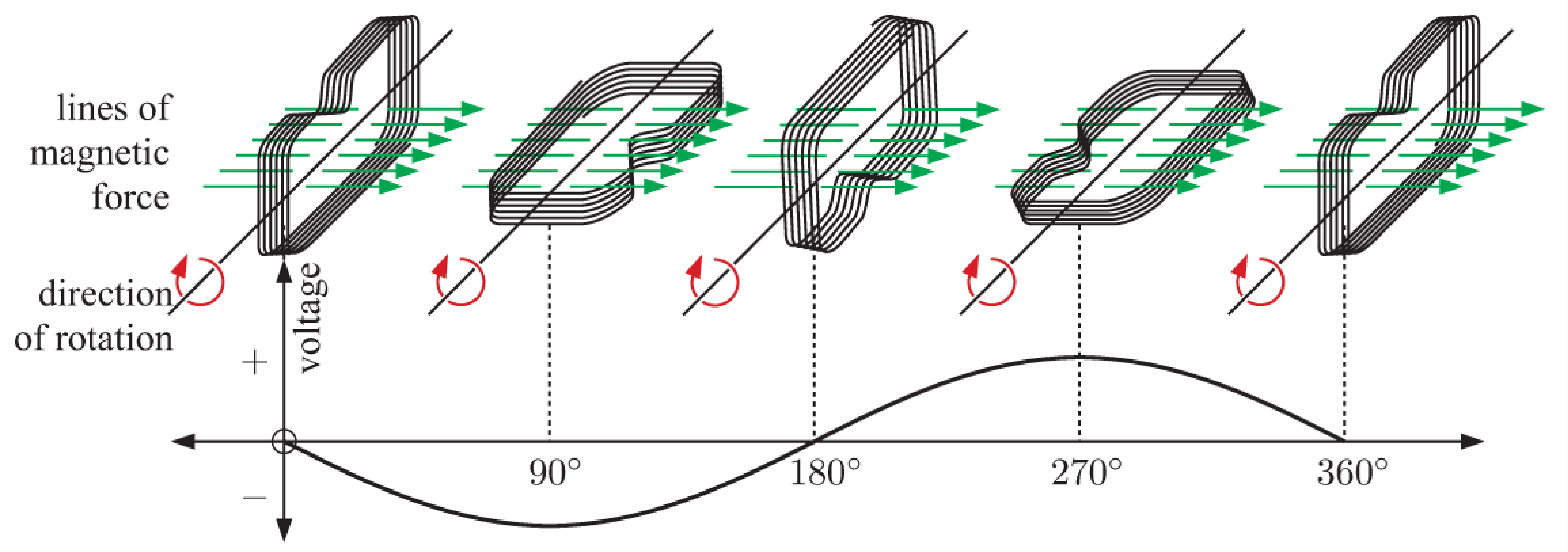
On the graph alongside we plot the temperature T on the vertical axis. We assign January as $t = 1$ month, February as $t = 2$ months, and so on for the 12 months of the year.



The temperature shows a variation from an average of 28°C in January through a range of values across the months. The cycle will approximately repeat itself for each subsequent 12 month period. By the end of the Chapter we will be able to establish a **periodic function** which approximately fits this set of points.



HISTORICAL NOTE



In 1831, **Michael Faraday** discovered that an electric current was generated by rotating a coil of wire at a constant speed through 360° in a magnetic field. The electric current produced showed a voltage which varied between positive and negative values in a periodic function called a **sine wave**.

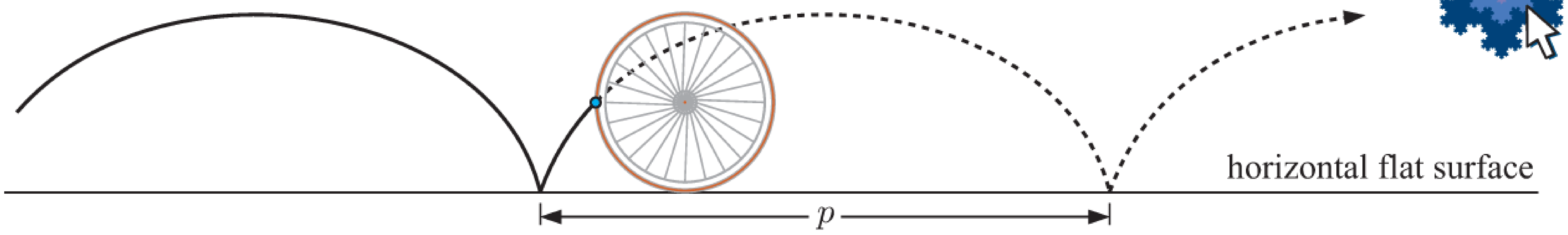
TERMINOLOGY USED TO DESCRIBE PERIODICITY

A **periodic function** is one which repeats itself over and over in a horizontal direction, in intervals of the same length. The **period** of a periodic function is the length of one repetition or cycle.

$f(x)$ is a periodic function with period p if $\boldsymbol{f(x+p)=f(x)}$ for all x, and p is the smallest positive value for this to be true.

A **cycloid** is an example of a periodic function. It is the curve traced out by a point on a circle as the circle rolls across a flat surface in a straight line.

DEMO



ACTIVITY 1 PERIODIC FUNCTIONS

GRAPHING PACKAGE

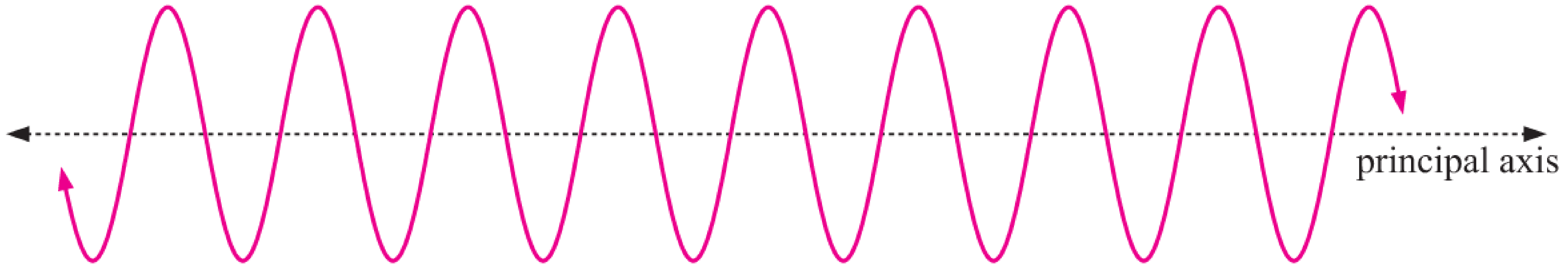
Use a **graphing package** to examine the function $f(x) = x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is "the largest integer less than or equal to x".

In the graphing package, you type $\lfloor x \rfloor$ as $\text{floor}(x)$.

Is $f(x)$ periodic? What is its period?

WAVES

In this course we are mainly concerned with periodic phenomena which show a wave pattern:



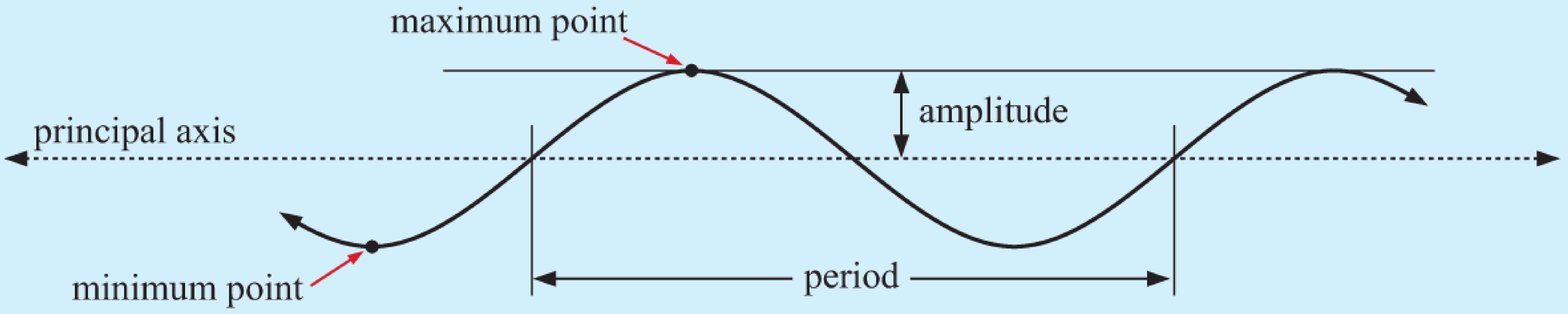
A **wave** oscillates about a horizontal line called the **principal axis** or **mean line**.

A **maximum point** occurs at the top of a crest, and a **minimum point** at the bottom of a trough.

If the maximum and minimum values of the wave are **max** and **min** respectively, then the principal axis has equation $y = \dfrac{\textbf{max} + \textbf{min}}{2}$.

The **amplitude** is the distance between a maximum (or minimum) point and the principal axis.

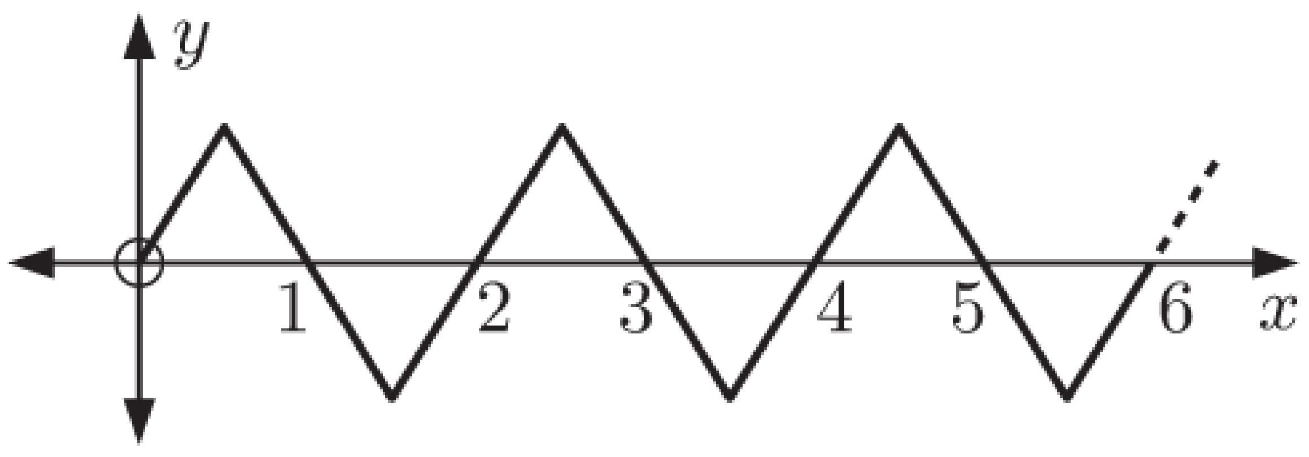
$$\textbf{amplitude} = \frac{\textbf{max} - \textbf{min}}{2}$$



EXERCISE 17A

1 Which of these graphs show periodic behaviour?

a



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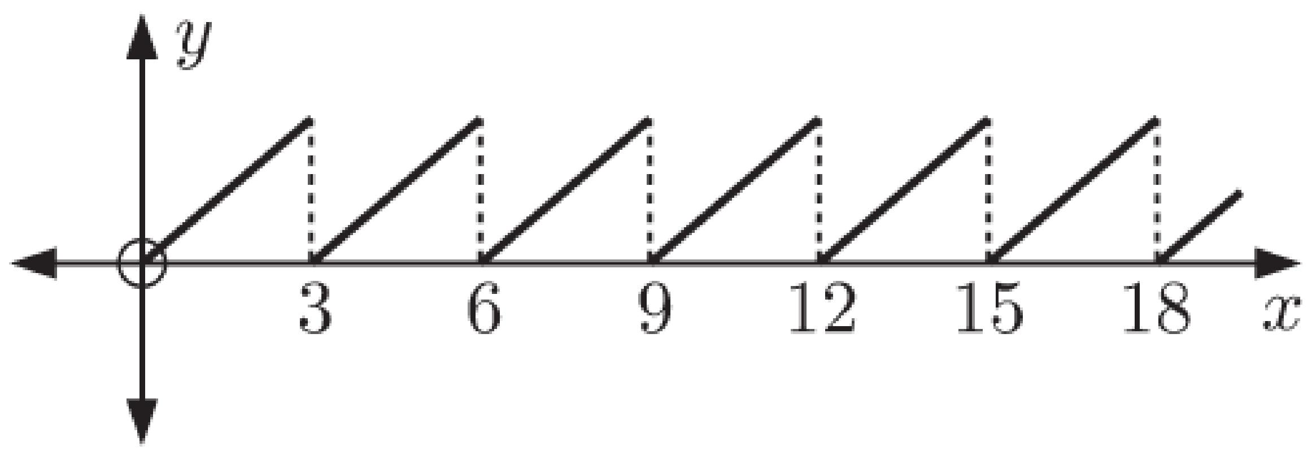

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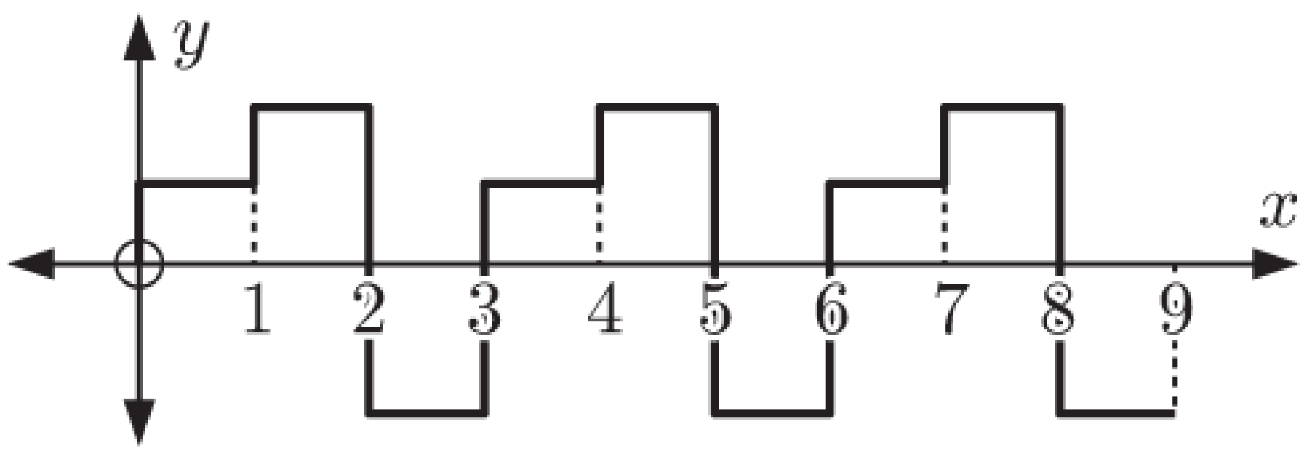

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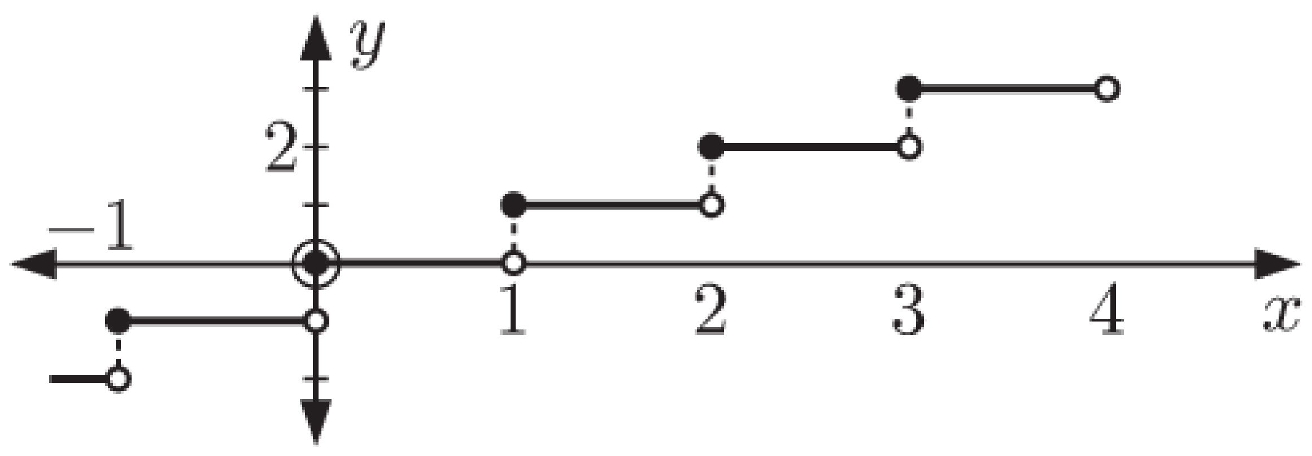

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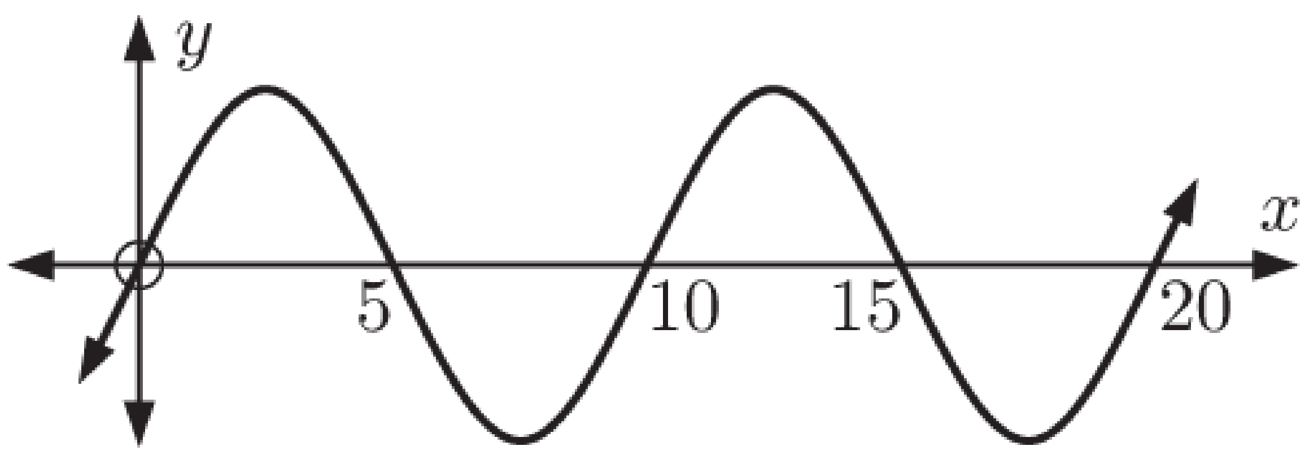

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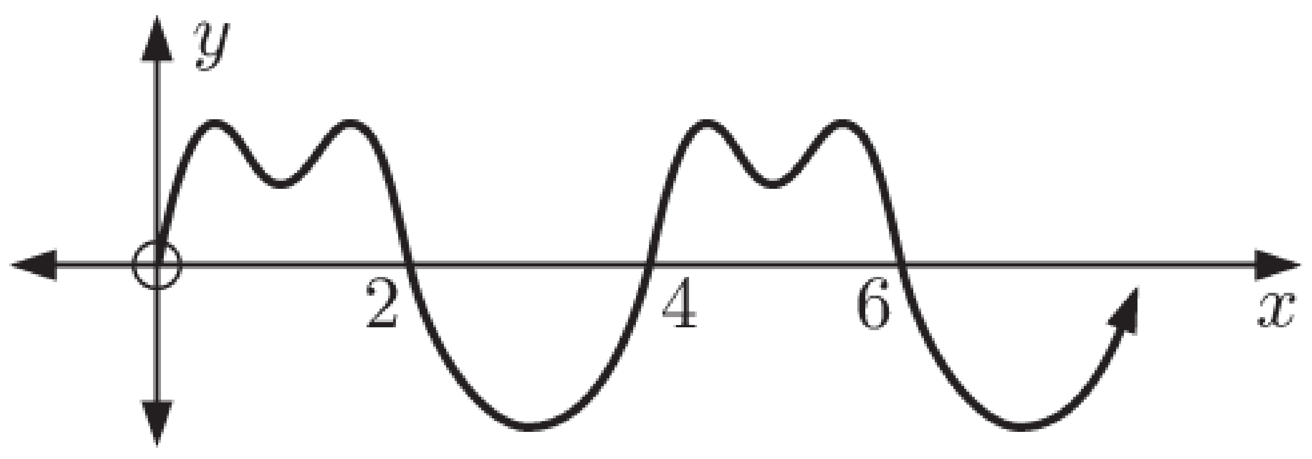

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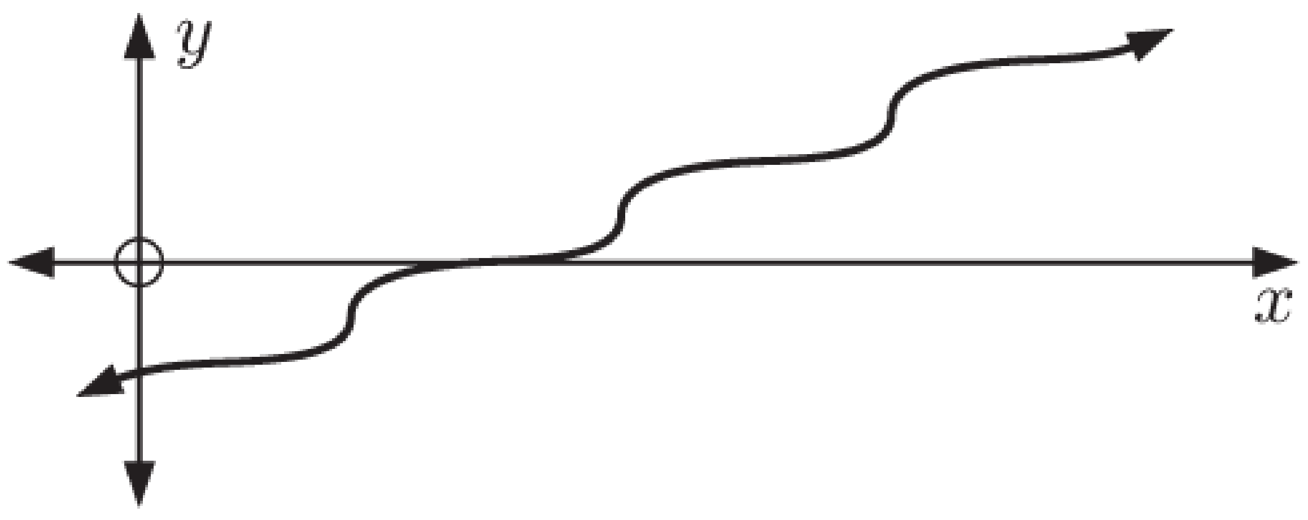

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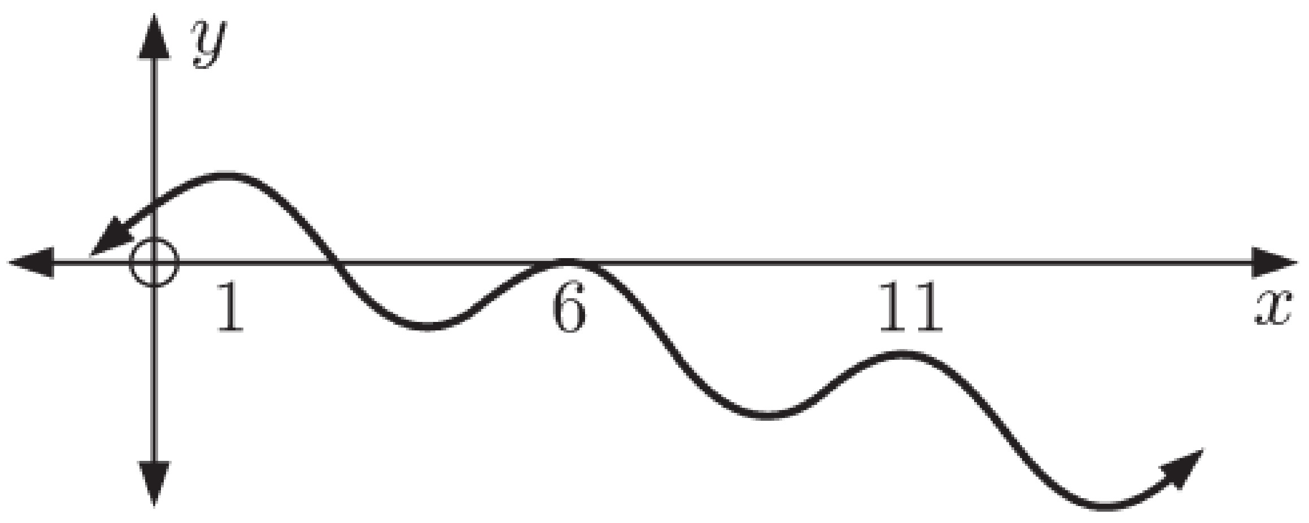


2 Paul spun the wheel of his bicycle. The following tabled values show the height above the ground of a point on the wheel at various times.

Time (seconds)	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
Height above ground (cm)	0	6	23	42	57	64	59	43	23	7	1

Time (seconds)	2.2	2.4	2.6	2.8	3	3.2	3.4	3.6	3.8	4
Height above ground (cm)	5	27	40	55	63	60	44	24	9	3

a Plot the graph of height against time.

b Is it reasonable to fit a curve to this data, or should we leave it as discrete points?

c Is the data periodic? If so, estimate:

- **i** the equation of the principal axis
- **ii** the maximum value
- **iii** the period
- **iv** the amplitude.

3 Plot the points for each data set below. Is there any evidence to suggest the data is periodic?

a

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	0	1	1.4	1	0	−1	−1.4	−1	0	1	1.4	1	0

b

x	0	2	3	4	5	6	7	8	9	10	12
y	0	4.7	3.4	1.7	2.1	5.2	8.9	10.9	10.2	8.4	10.4

GRAPHICS CALCULATOR INSTRUCTIONS

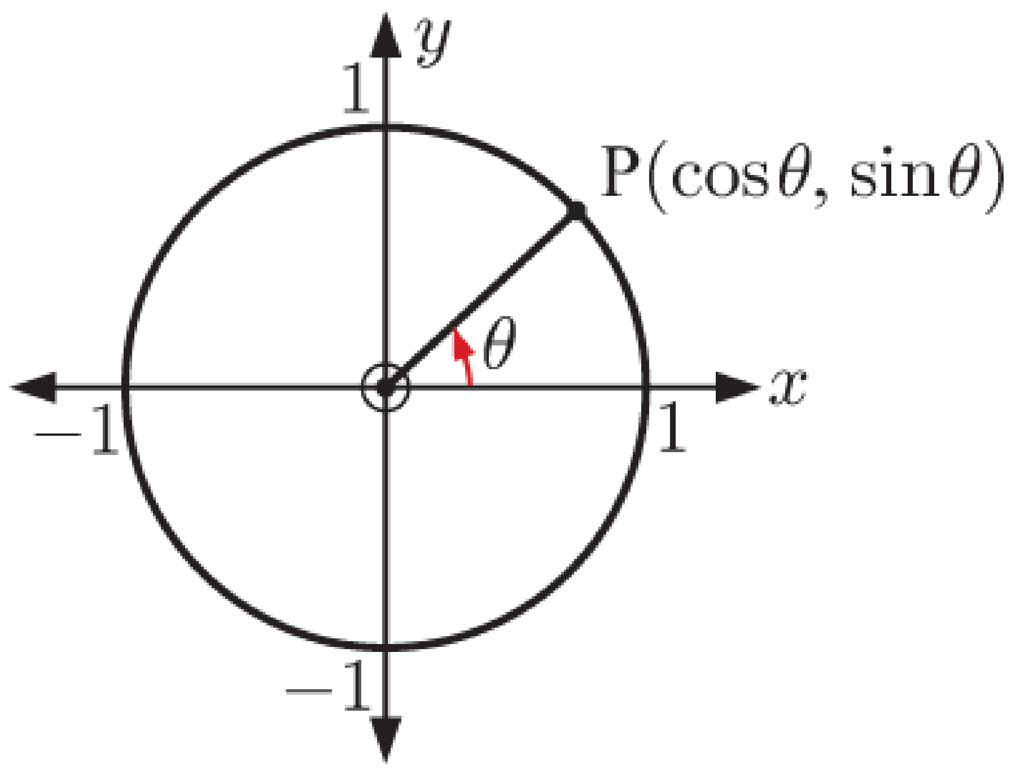
B THE SINE AND COSINE FUNCTIONS

A **trigonometric function** is a function which involves one of the trigonometric ratios.

Consider the point P$(\cos\theta, \sin\theta)$ on the unit circle.

As θ increases, the point P moves around the unit circle, and the values of $\cos\theta$ and $\sin\theta$ change.

We can draw the graphs of $y = \sin\theta$ and $y = \cos\theta$ by plotting the values of $\sin\theta$ and $\cos\theta$ against θ.

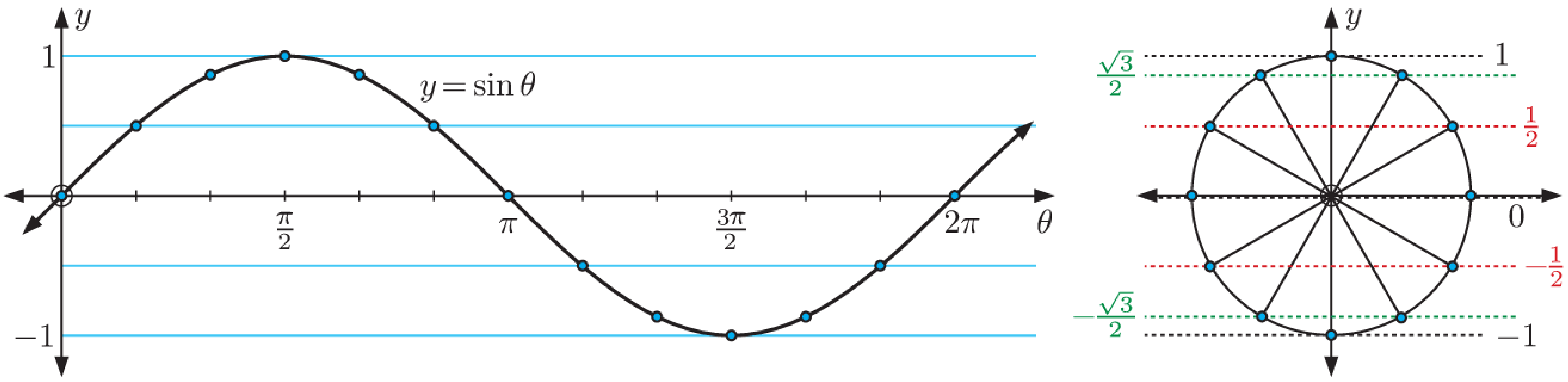


THE GRAPH OF $y = \sin\theta$

By considering the y-coordinates of the points on the unit circle at intervals of $\frac{\pi}{6}$, we can create a table of values for $\sin\theta$:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

Plotting $\sin\theta$ against θ gives:



Once we reach 2π, P has completed a full revolution of the unit circle, and so this pattern repeats itself.

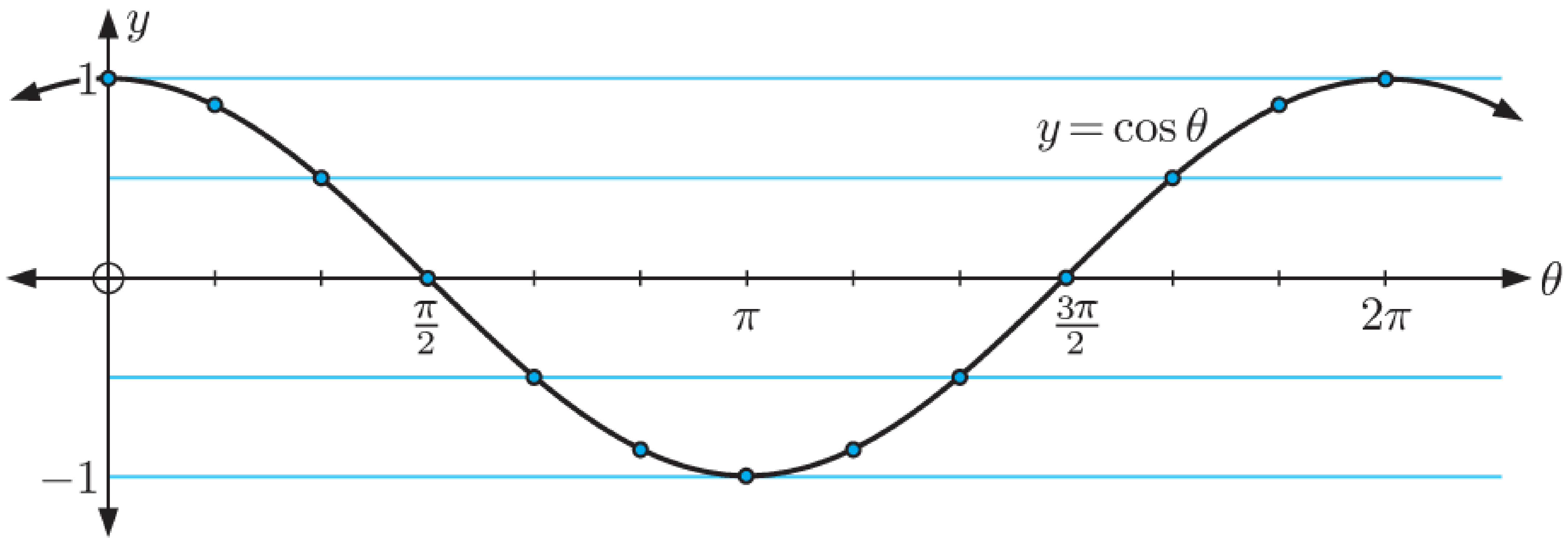
DEMO

THE GRAPH OF $y = \cos\theta$

By considering the x-coordinates of the points on the unit circle at intervals of $\frac{\pi}{6}$, we can create a table of values for $\cos\theta$:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

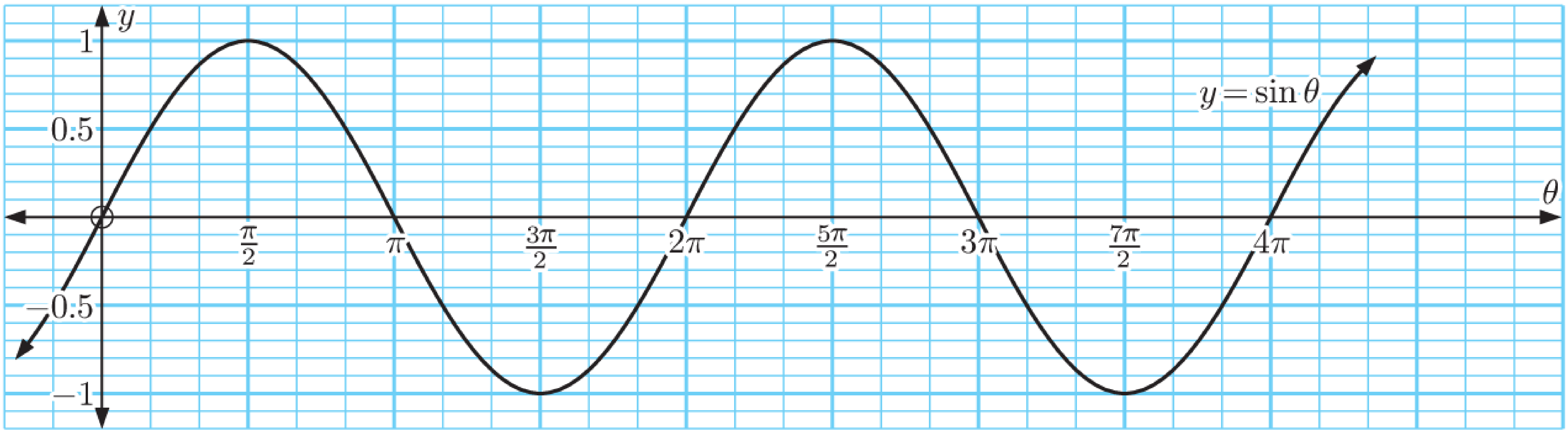
Plotting $\cos\theta$ against θ gives:



The graph of $y = \cos\theta$ shows the x-coordinate of P as P moves around the unit circle.

EXERCISE 17B

1 Below is an accurate graph of $y = \sin\theta$.



a Find the y-intercept of the graph.

b Find the values of θ on $0 \leqslant \theta \leqslant 4\pi$ for which:

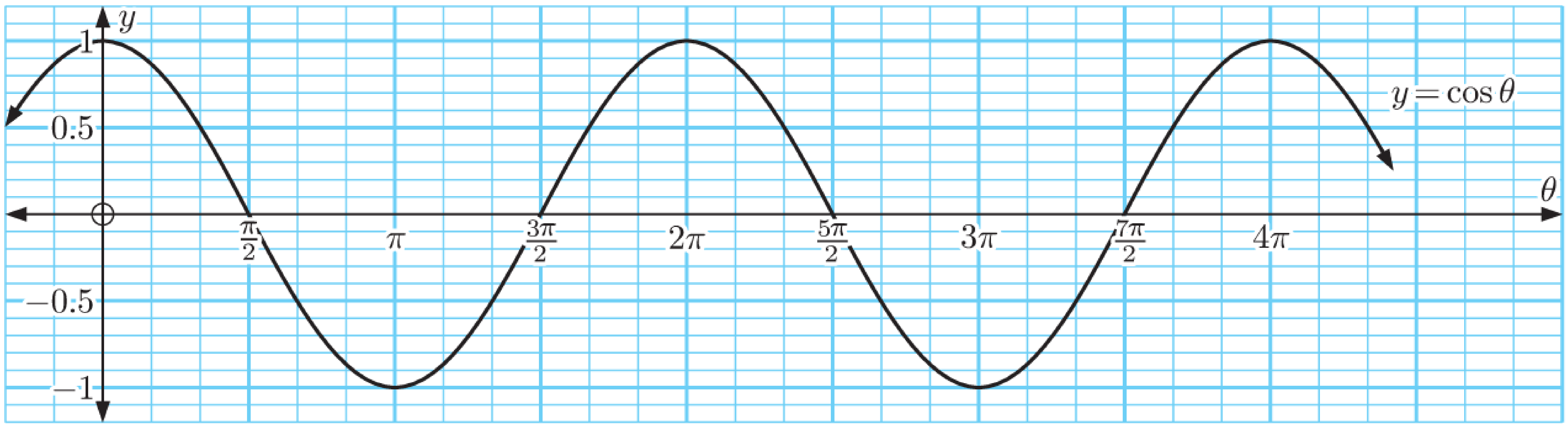
i $\sin\theta = 0$ **ii** $\sin\theta = -1$ **iii** $\sin\theta = \frac{1}{2}$ **iv** $\sin\theta = \frac{\sqrt{3}}{2}$

c Find the intervals on $0 \leqslant \theta \leqslant 4\pi$ where $\sin\theta$ is:

i positive **ii** negative.

d Find the range of the function.

2 Below is an accurate graph of $y = \cos\theta$.



a Find the y-intercept of the graph.

b Find the values of θ on $0 \leqslant \theta \leqslant 4\pi$ for which:

i $\cos\theta = 0$ **ii** $\cos\theta = 1$ **iii** $\cos\theta = -\frac{1}{2}$ **iv** $\cos\theta = -\frac{1}{\sqrt{2}}$

c Find the intervals on $0 \leqslant \theta \leqslant 4\pi$ where $\cos\theta$ is:

i positive **ii** negative.

d Find the range of the function.

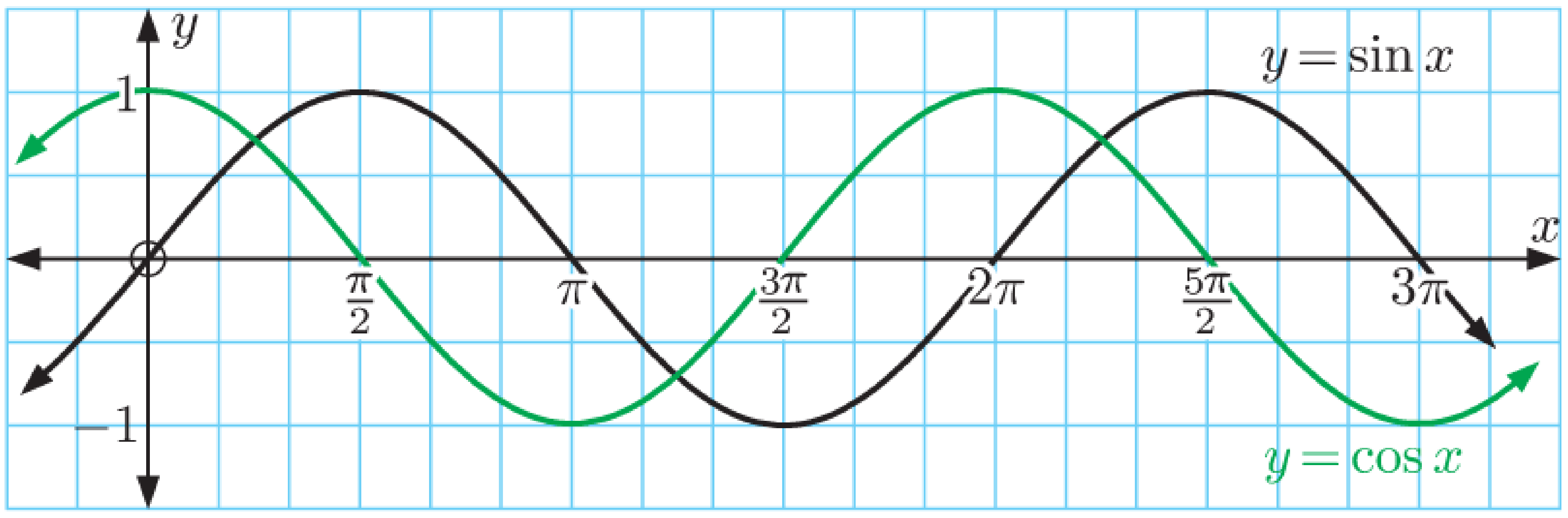
C GENERAL SINE AND COSINE FUNCTIONS

Now that we are familiar with the graphs of $y = \sin\theta$ and $y = \cos\theta$, we can use transformations to graph more complicated trigonometric functions.

Instead of using θ, we will now use x to represent the angle variable. This is just for convenience, so we are dealing with the familiar function form $y = f(x)$.

For the graphs of $y = \sin x$ and $y = \cos x$:

- the **period** is 2π
- the **amplitude** is 1
- the **principal axis** is the line $y = 0$.



We immediately notice that $y = \sin x$ is a horizontal translation of $y = \cos x$ by $\frac{\pi}{2}$ units to the right.

For all x, $\quad \boldsymbol{\sin x = \cos\left(x - \frac{\pi}{2}\right)}$

and $\quad \boldsymbol{\cos x = \sin\left(x + \frac{\pi}{2}\right)}$

INVESTIGATION FAMILIES OF TRIGONOMETRIC FUNCTIONS

GRAPHING PACKAGE

What to do:

1 **a** Use the graphing package to graph on the same set of axes:

i $y = \sin x$ **ii** $y = 2\sin x$ **iii** $y = \frac{1}{2}\sin x$

iv $y = -\sin x$ **v** $y = -\frac{1}{3}\sin x$ **vi** $y = -\frac{3}{2}\sin x$

b For graphs of the form $y = a\sin x$, comment on the significance of:

i the sign of a **ii** the size of a, or $|a|$.

2 **a** Use the graphing package to graph on the same set of axes:

i $y = \sin x$ **ii** $y = \sin 2x$ **iii** $y = \sin\left(\frac{1}{2}x\right)$ **iv** $y = \sin 3x$

b For graphs of the form $y = \sin bx$, $b > 0$, what is the period?

3 **a** Graph on the same set of axes:

i $y = \sin x$ **ii** $y = \sin\left(x - \frac{\pi}{3}\right)$ **iii** $y = \sin\left(x + \frac{\pi}{6}\right)$

b What translation moves $y = \sin x$ to $y = \sin(x - c)$?

4 **a** Graph on the same set of axes:

i $y = \sin x$ **ii** $y = \sin x + 2$ **iii** $y = \sin x - 2$

b What translation moves $y = \sin x$ to $y = \sin x + d$?

c What is the principal axis of $y = \sin x + d$?

5 What sequence of transformations maps $y = \sin x$ onto $y = a\sin b(x - c) + d$?

From the **Investigation** you should have observed the following properties of the general sine function:

For the **general sine function**

$$y = a\sin(b(x-c)) + d$$

affects amplitude (a)

affects period (b)

affects horizontal translation (c)

affects vertical translation (d)

- the amplitude is $|\,a\,|$
- the period is $\frac{2\pi}{b}$ for $b > 0$
- the principal axis is $y = d$
- $y = a\sin(b(x-c)) + d$ is obtained from $y = \sin x$ by a vertical stretch with scale factor a and a horizontal stretch with scale factor $\frac{1}{b}$, followed by a horizontal translation of c units and a vertical translation of d units.

The properties of the **general cosine function** $y = a\cos(b(x-c)) + d$ are the same as those of the general sine function.

DEMO

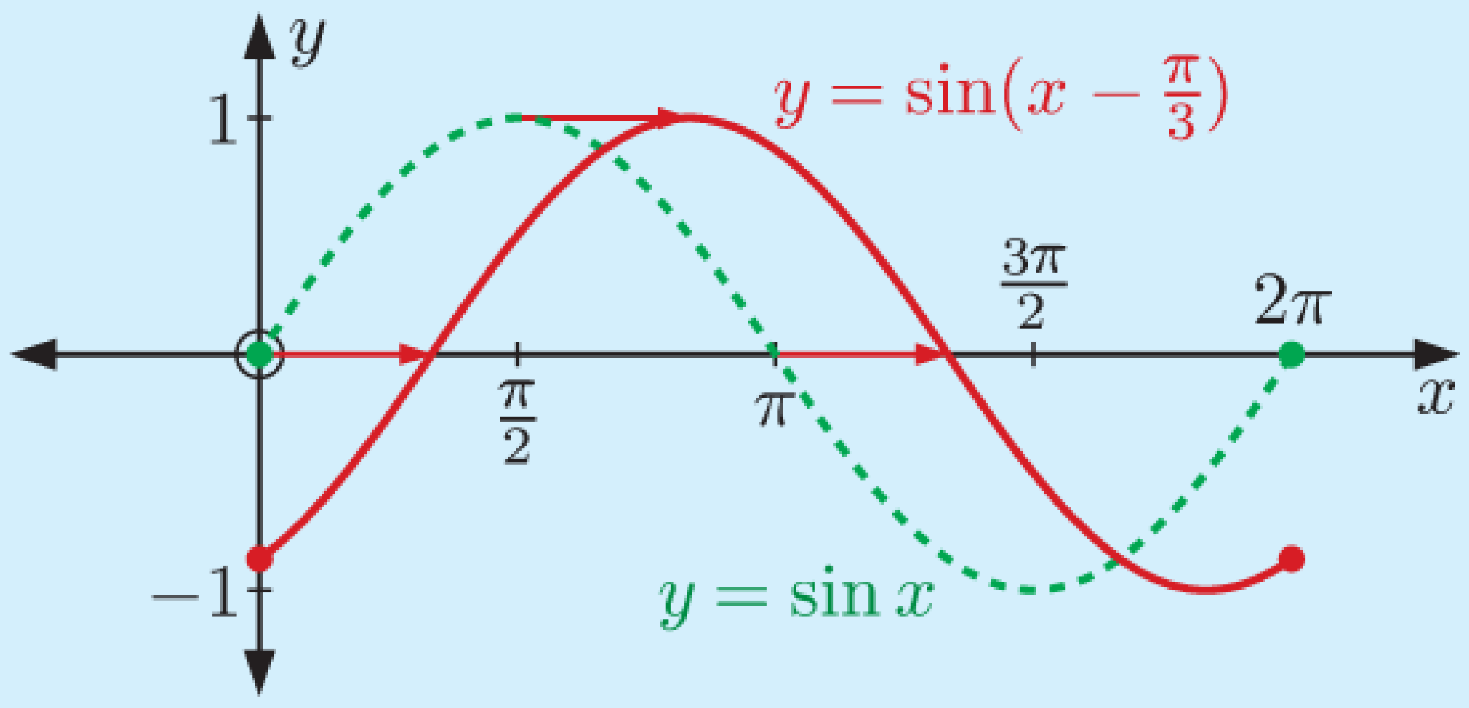
Example 1

Self Tutor

Sketch the graphs of the following on $0 \leqslant x \leqslant 2\pi$:

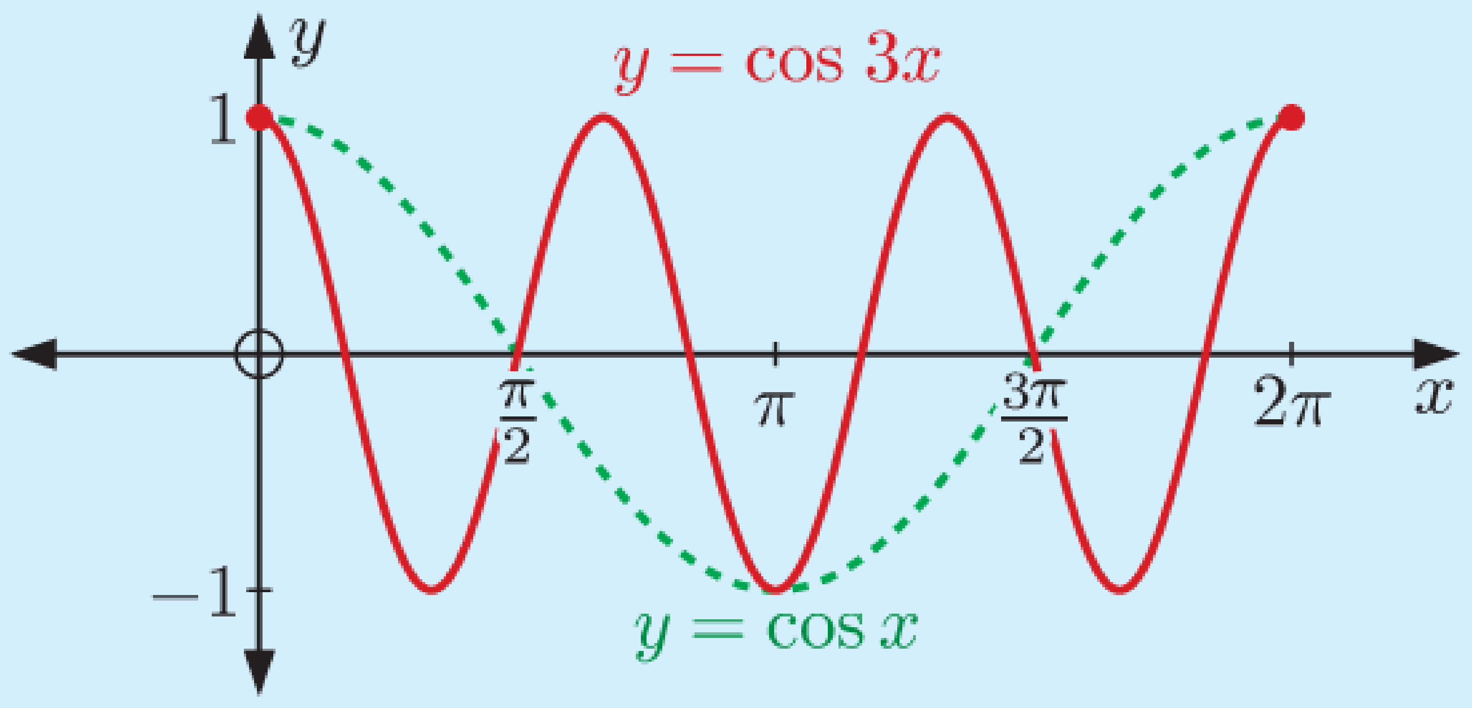
a $y = \sin\left(x - \frac{\pi}{3}\right)$ **b** $y = \cos 3x$ **c** $y = \cos\left(x + \frac{\pi}{6}\right) + 1$ **d** $y = -\sin x$

a We translate $y = \sin x$ horizontally $\frac{\pi}{3}$ units to the right.

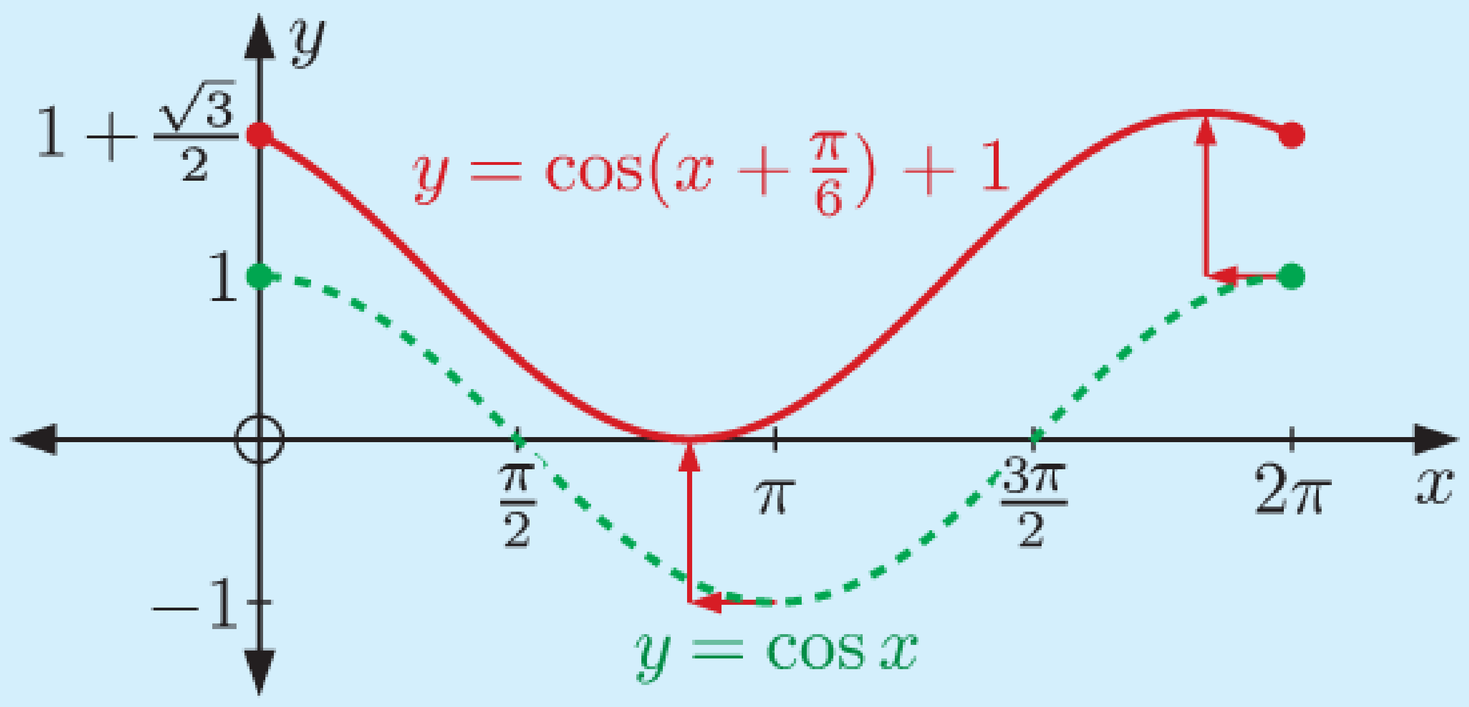


b We stretch $y = \cos x$ horizontally with scale factor $\frac{1}{3}$.

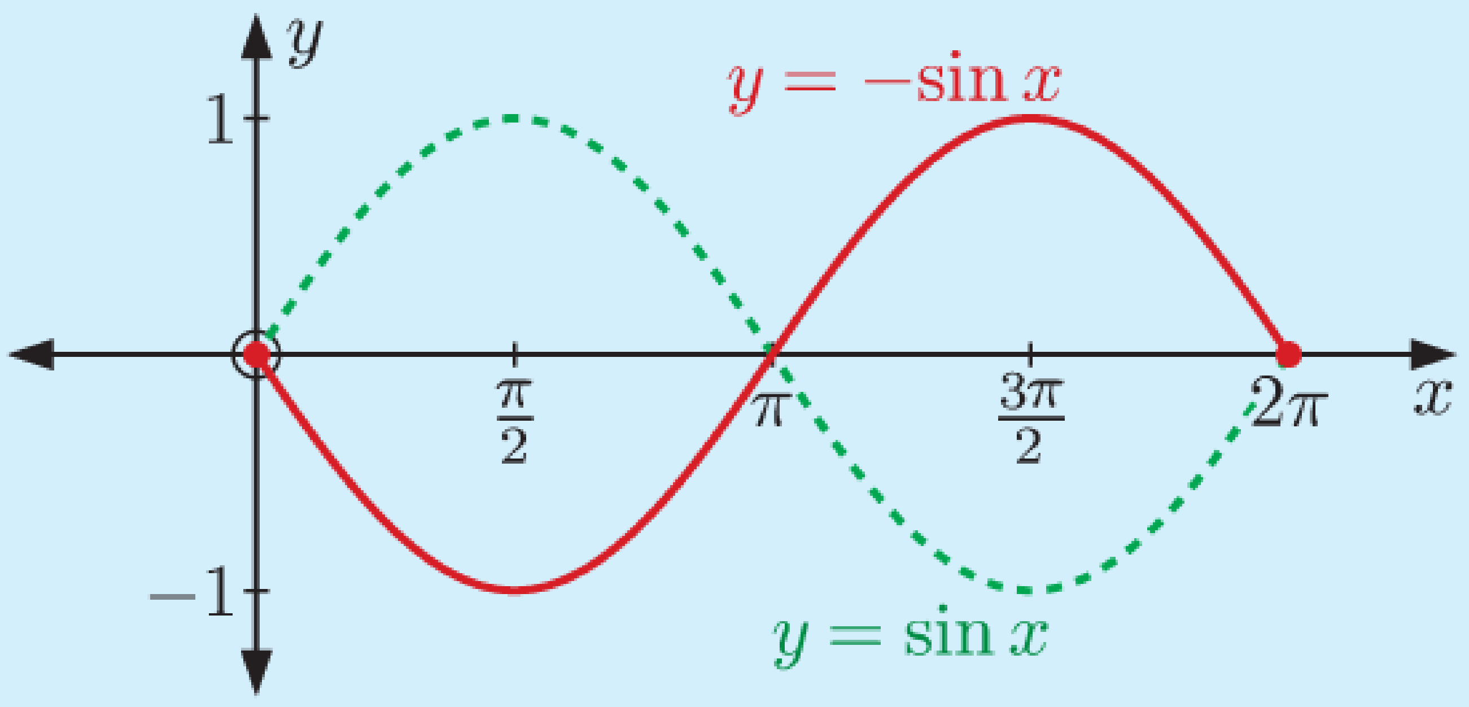
$\therefore\ y = \cos 3x$ has period $\frac{2\pi}{3}$.



c We translate $y = \cos x$ horizontally $\frac{\pi}{6}$ units to the left, and 1 unit upwards.



d We reflect $y = \sin x$ in the x-axis.



EXERCISE 17C

1 State the transformation which maps $y = \sin x$ onto:

a $y = \sin x - 1$ **b** $y = \sin\left(x - \frac{\pi}{4}\right)$ **c** $y = 2\sin x$

d $y = \sin 4x$ **e** $y = \sin \frac{x}{4}$ **f** $y = \sin\left(x - \frac{\pi}{3}\right) + 2$

2 State the transformation which maps $y = \cos x$ onto:

a $y = \frac{1}{2}\cos x$ **b** $y = -\cos x$ **c** $y = \cos\left(x + \frac{\pi}{6}\right) - 2$

3 State the period of:

a $y = \sin 5x$ **b** $y = \sin(0.6x)$ **c** $y = \sin \pi x$

d $y = \cos 3x$ **e** $y = \cos \frac{x}{3}$ **f** $y = \cos \frac{\pi x}{50}$

4 Find b given that the function $y = \sin bx$, $b > 0$ has period:

a 5π **b** $\frac{2\pi}{3}$ **c** 12π **d** 4 **e** 100

5 State the maximum and minimum value of:

a $y = 4\cos 2x$ **b** $y = 3\cos x + 5$ **c** $y = -2\cos(x - 3) - 4$

6 For the function $y = 4\sin 3x + 2$, state the:

a amplitude **b** period **c** range.

7 The general cosine function is $y = a\cos(b(x - c)) + d$.
State the geometrical significance of a, b, c, and d.

8 Sketch the graphs of the following for $0 \leqslant x \leqslant 4\pi$:

a $y = \sin x - 2$ **b** $y = \sin x + 3$ **c** $y = \sin x - 0.5$

d $y = \sin(x - 2)$ **e** $y = \sin(x + 2)$ **f** $y = \sin\left(x - \frac{\pi}{4}\right)$

g $y = \sin\left(x - \frac{\pi}{6}\right) + 1$ **h** $y = \sin(x - 1) - 2$ **i** $y = \sin\left(x + \frac{\pi}{4}\right) + 2$

j $y = 3\sin x$ **k** $y = \frac{1}{2}\sin x$ **l** $y = \frac{3}{2}\sin x$

m $y = \sin 3x$ **n** $y = \sin \frac{x}{2}$ **o** $y = \sin 4x$

9 Sketch the graphs of the following for $-2\pi \leqslant x \leqslant 2\pi$:

a $y = \cos x + 2$ **b** $y = \cos\left(x - \frac{\pi}{4}\right)$ **c** $y = \cos\left(x + \frac{\pi}{6}\right)$

d $y = \frac{3}{2}\cos x$ **e** $y = -\cos x$ **f** $y = \cos\left(x - \frac{\pi}{6}\right) + 1$

g $y = \cos\left(x + \frac{\pi}{4}\right) - 1$ **h** $y = \cos 2x$ **i** $y = \cos \frac{x}{2}$

10 **a** Sketch the curve $y = 4\sin x$ for $0 \leqslant x \leqslant 2\pi$.

b Find the value of y when: **i** $x = \frac{5\pi}{6}$ **ii** $x = \frac{7\pi}{4}$
Mark these points on your graph in **a**.

11 For what values of d does the graph of $y = 3\cos x + d$ lie:

a entirely above the x-axis

b entirely below the x-axis

c partially above and partially below the x-axis?

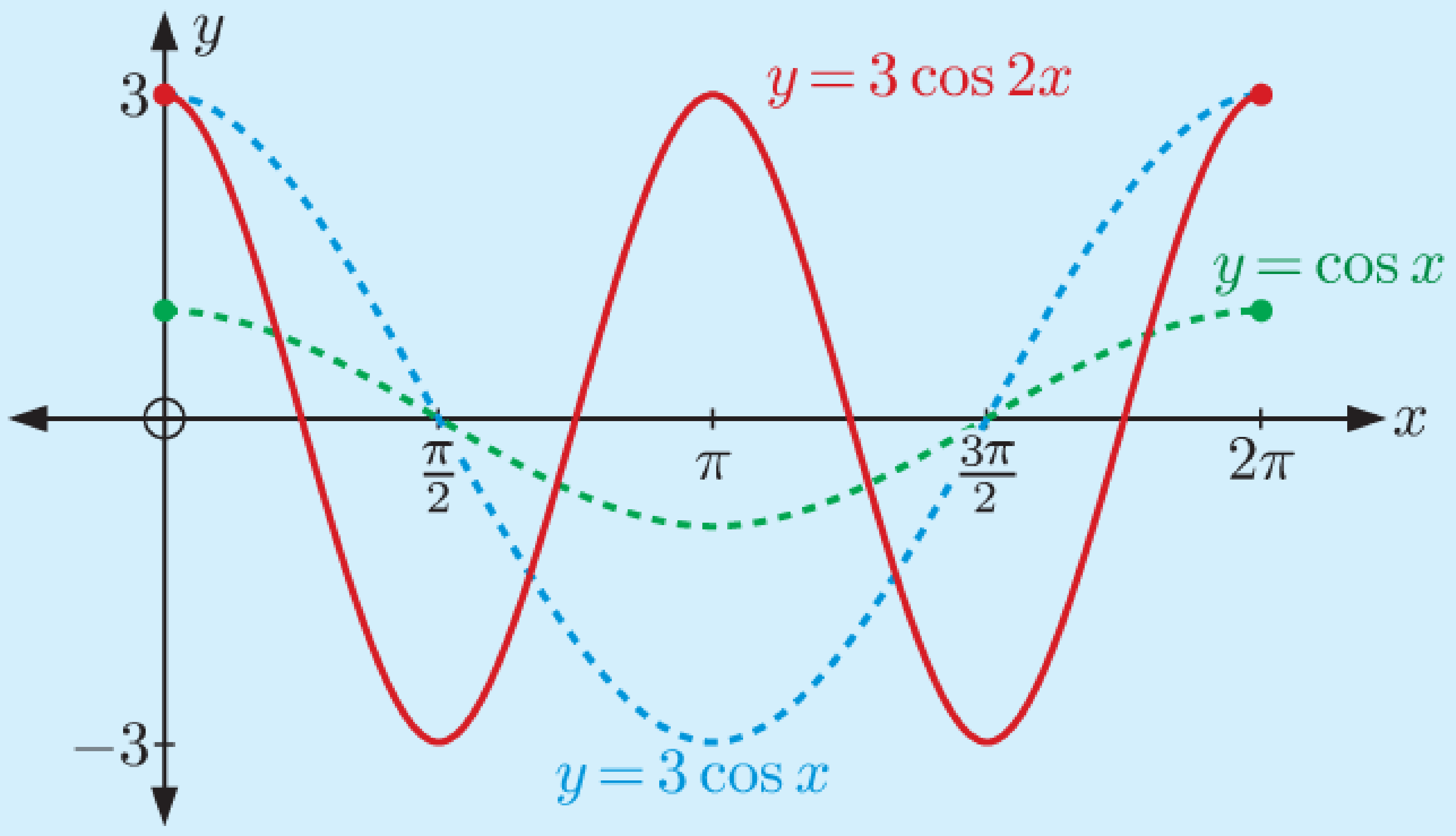
Example 2 **Self Tutor**

Sketch the graph of $y = 3\cos 2x$ for $0 \leqslant x \leqslant 2\pi$.

$a = 3$, so the amplitude is $|\,3\,| = 3$.

$b = 2$, so the period is $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$.

We stretch $y = \cos x$ vertically with scale factor 3 to give $y = 3\cos x$, then stretch $y = 3\cos x$ horizontally with scale factor $\frac{1}{2}$ to give $y = 3\cos 2x$.



12 State the transformations which map:

a $y = \sin x$ onto $y = 2\sin 3x$

b $y = \cos x$ onto $y = -2\cos x$

c $y = \sin x$ onto $y = 3\sin x - 5$

d $y = \cos x$ onto $y = \cos\left(2\left(x + \frac{\pi}{6}\right)\right)$

13 Sketch the graphs of the following for $-2\pi \leqslant x \leqslant 2\pi$:

a $y = -3\sin x$

b $y = \cos 2x + 1$

c $y = \frac{1}{2}\sin\left(x + \frac{\pi}{6}\right) - \frac{1}{3}$

d $y = \frac{1}{3}\cos\left(x + \frac{\pi}{4}\right) + 1$

e $y = 3\sin\left(x - \frac{\pi}{3}\right) - 1$

f $y = -\cos\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right)$

14 Consider the general sine function $y = a\sin(b(x - c)) + d$. State which of the variables a, b, c, and d can be changed to always produce a change in:

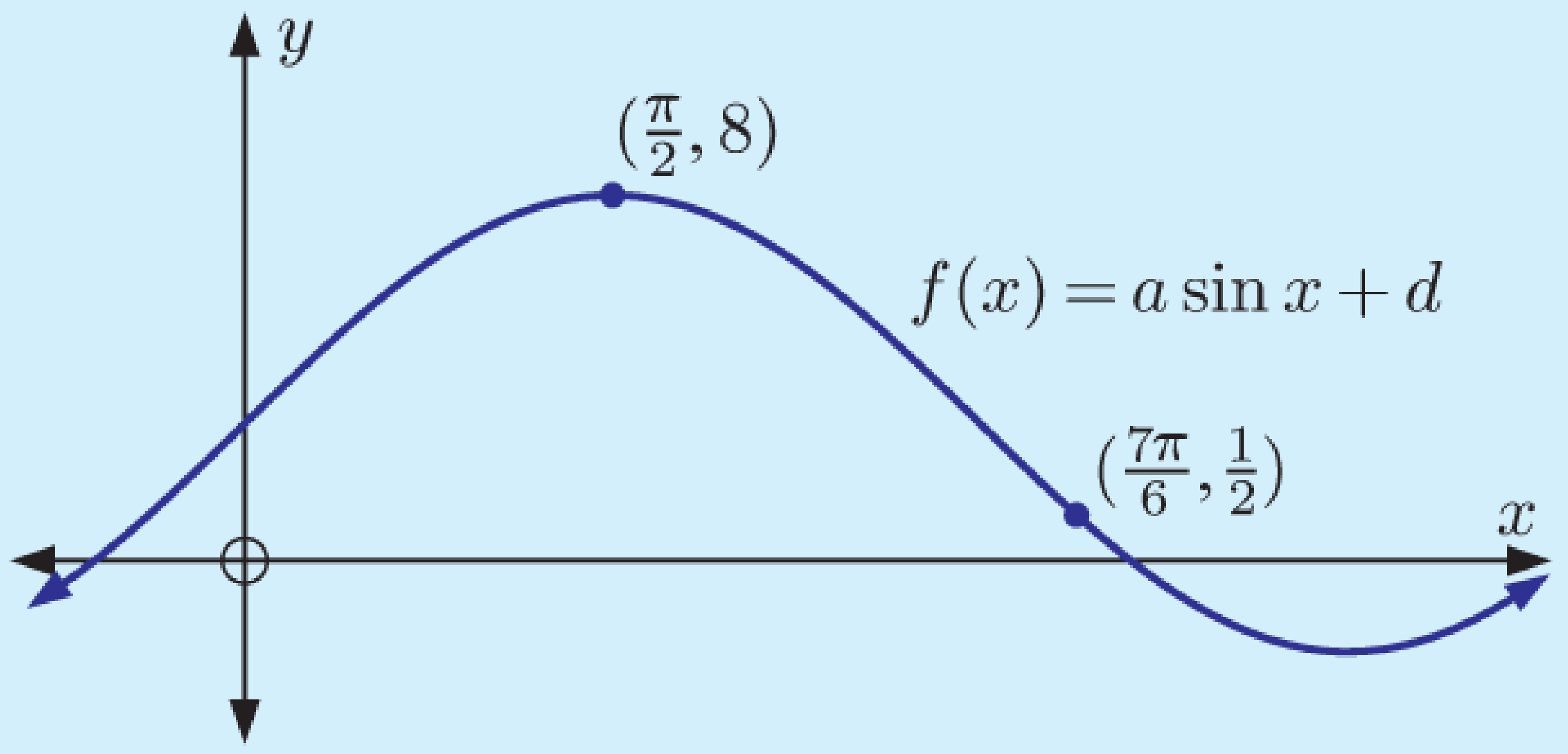
a the x-intercepts of the function

b the y-intercept of the function

c the range of the function.

Example 3 **Self Tutor**

Find the unknowns in this function:



$f(\frac{\pi}{2}) = 8$, so $a\sin\frac{\pi}{2} + d = 8$

$\therefore\ a + d = 8$ .... (1)

$f(\frac{7\pi}{6}) = \frac{1}{2}$, so $a\sin\frac{7\pi}{6} + d = \frac{1}{2}$

$\therefore\ -\frac{1}{2}a + d = \frac{1}{2}$ .... (2)

So, we have

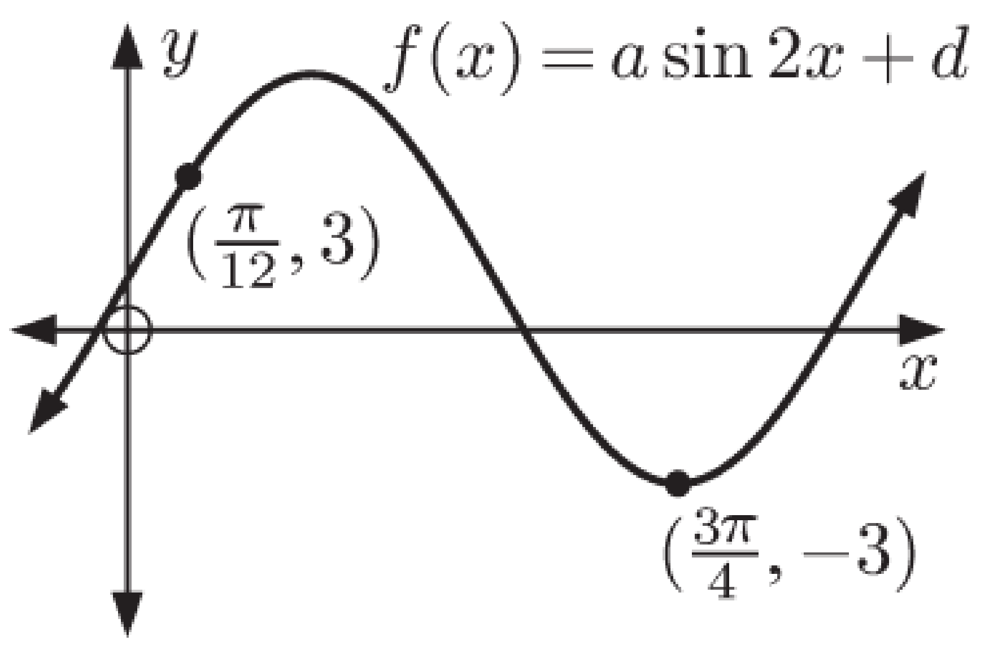
$$\begin{aligned} a + d &= 8 \quad \{(1)\} \\ -a + 2d &= 1 \quad \{2 \times (2)\} \\ \hline \text{Adding,} \quad 3d &= 9 \quad \text{and so } d = 3 \end{aligned}$$

Substituting $d = 3$ into (1) gives $a + 3 = 8$

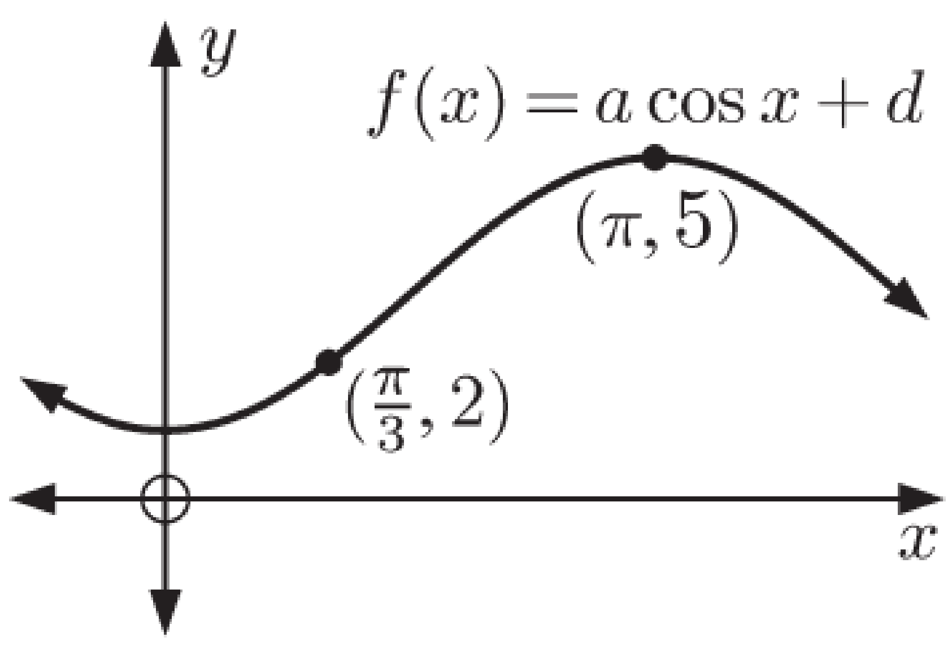
$\therefore\ a = 5$

15 Find the unknowns in each function:

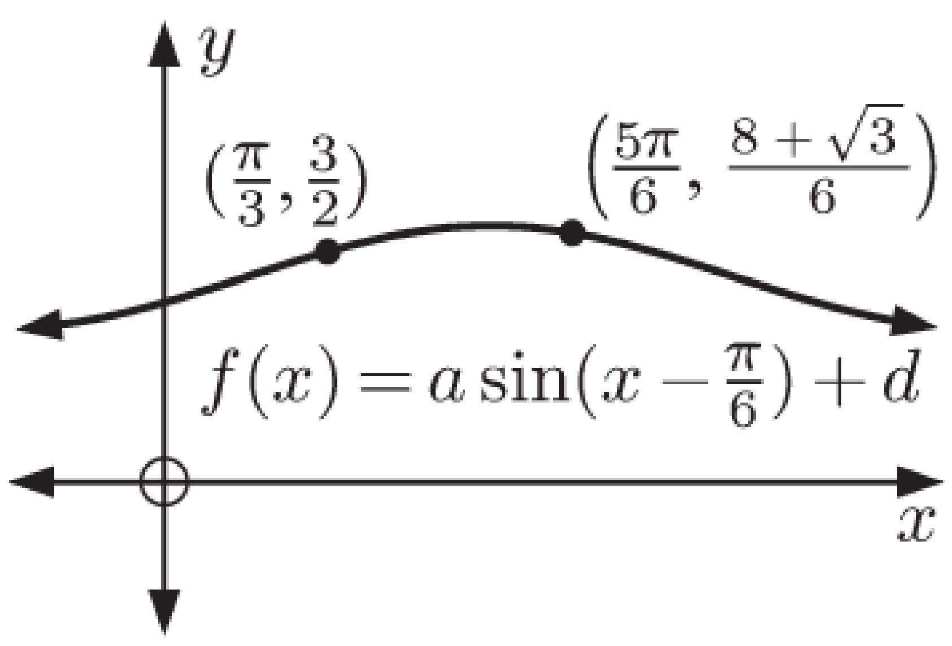
a



b



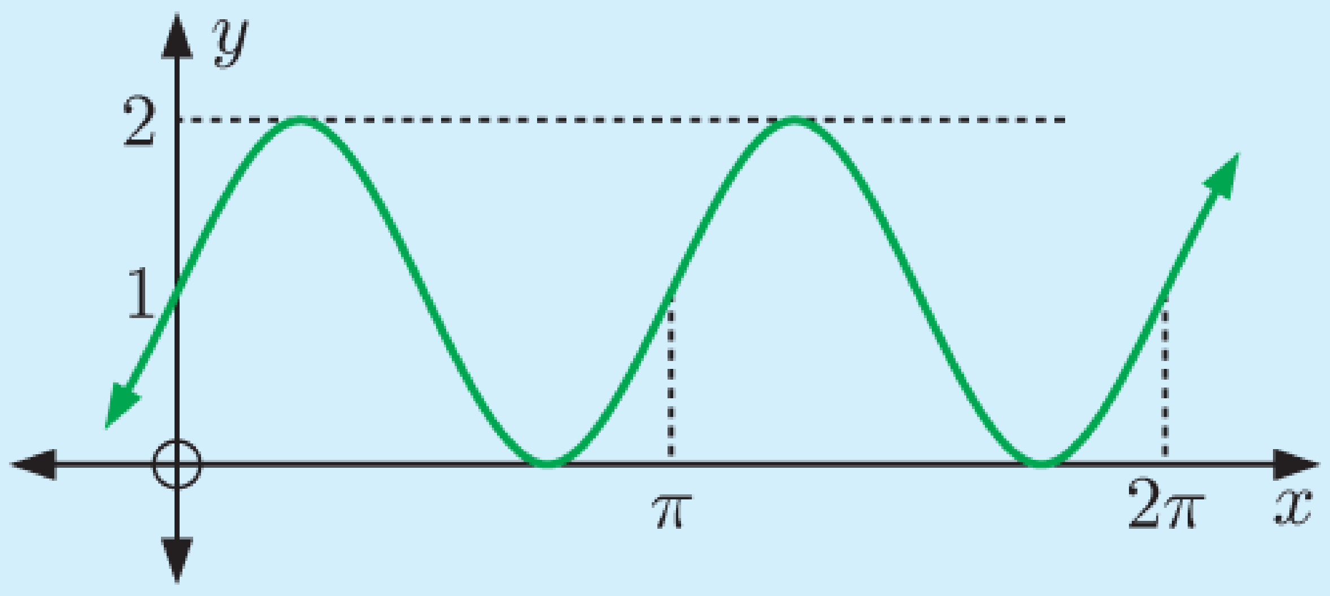
c



Example 4

Self Tutor

Find the equation of this sine function.



The amplitude is 1, so $a=1$.

The period is π, so $\frac{2\pi}{b}=\pi$ and $\therefore\ b=2$.

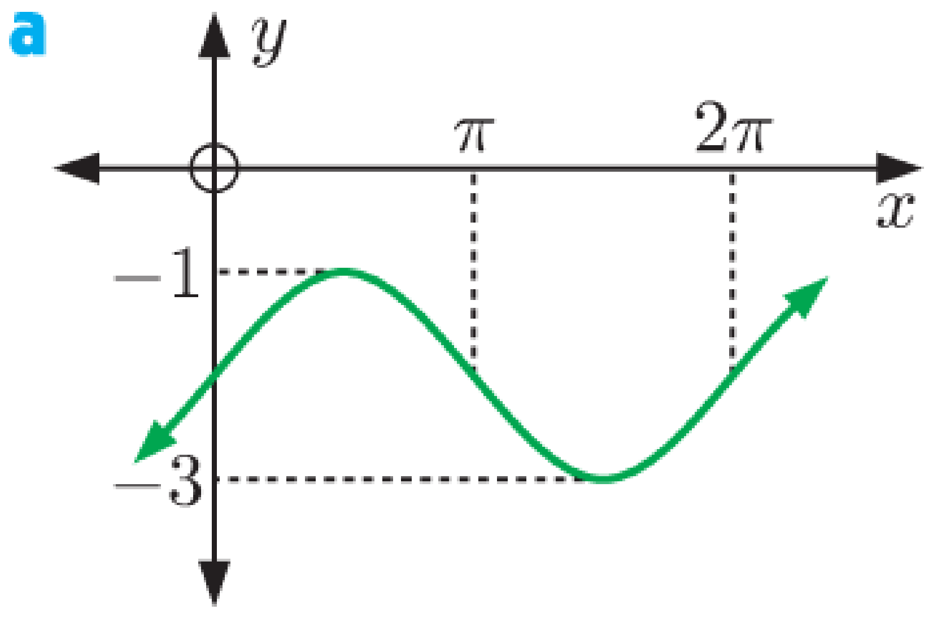
There is no horizontal translation, so $c=0$.

The principal axis is $y=1$, so $d=1$.

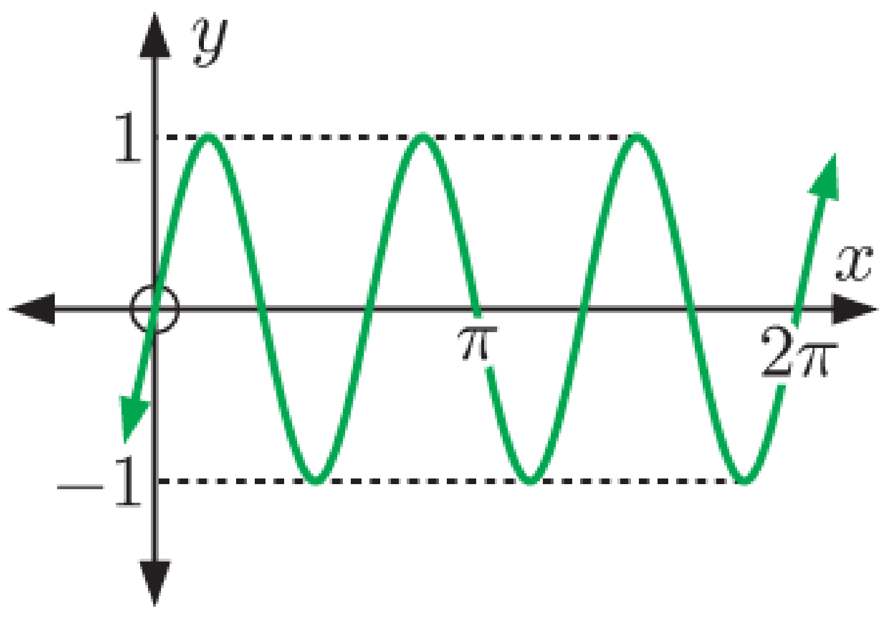
The equation of the function is $y=\sin 2x+1$.

16 Find the equation of each sine function:

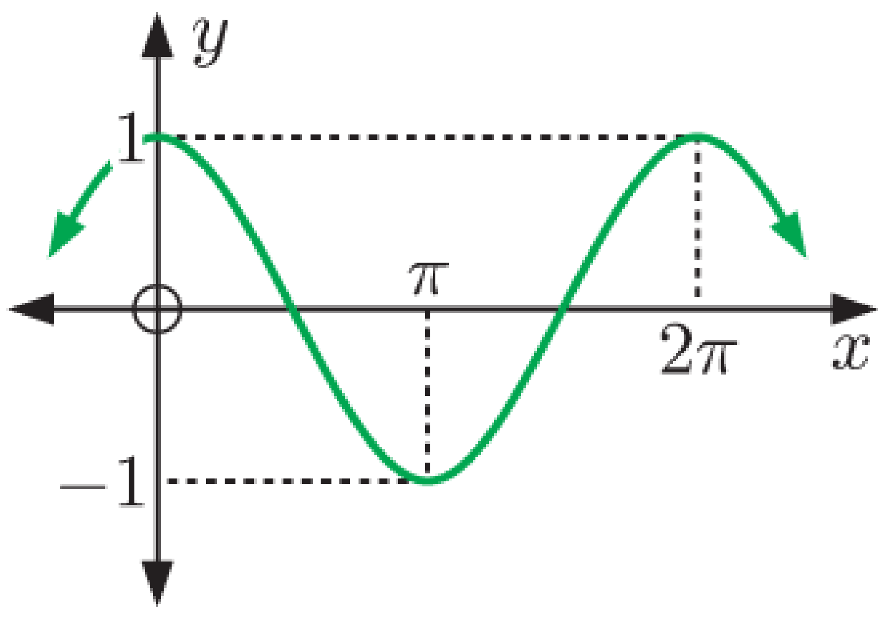
a



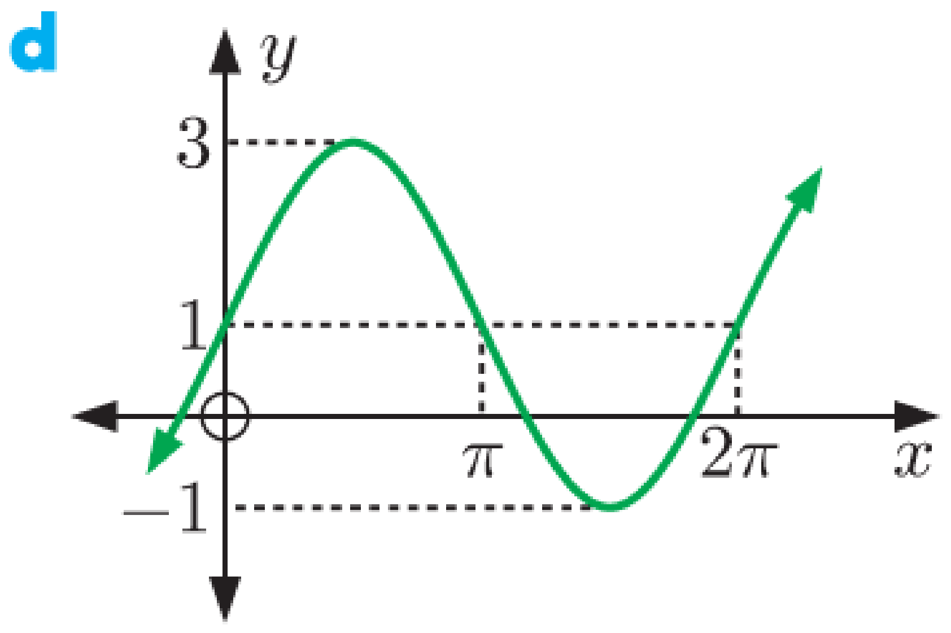
b



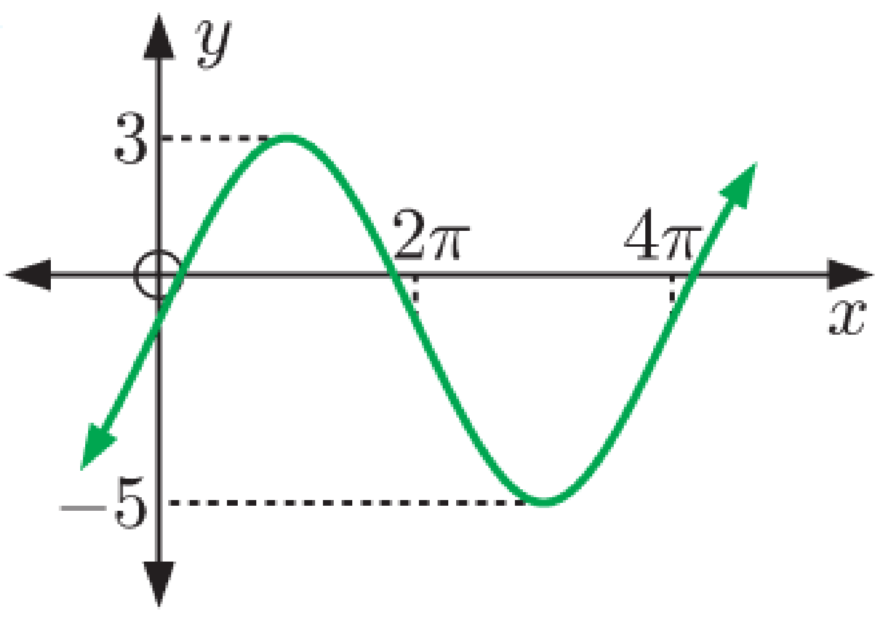
c



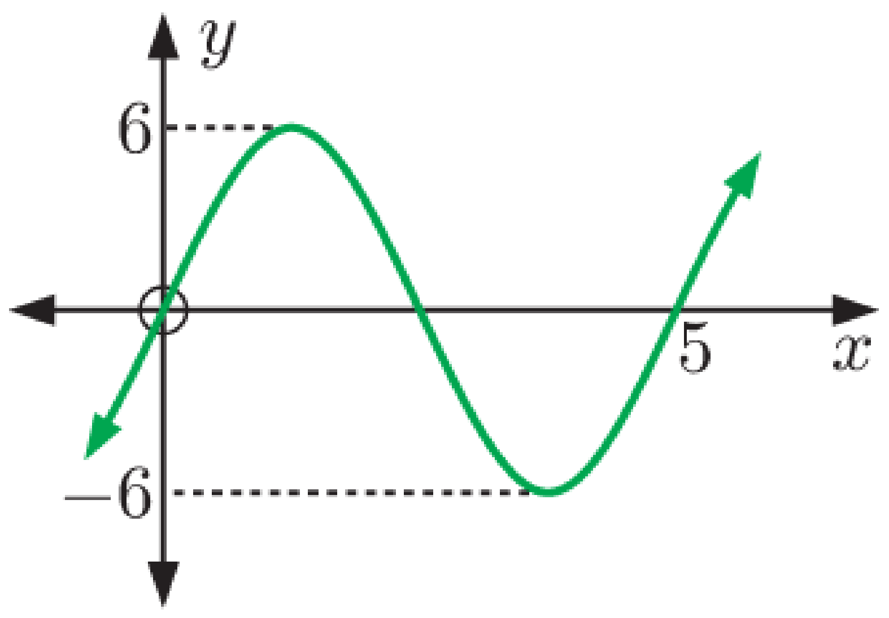
d



e

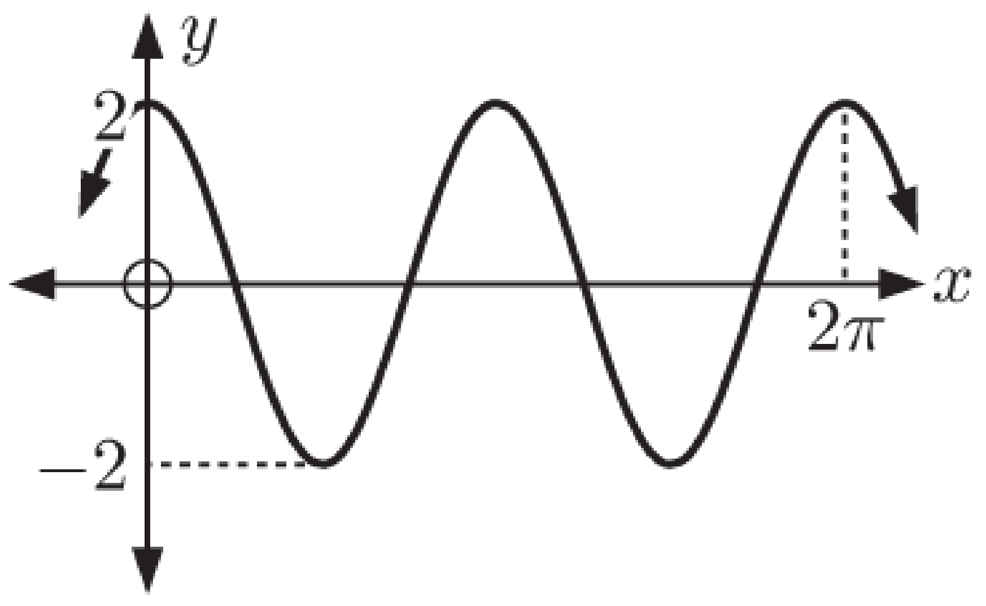


f

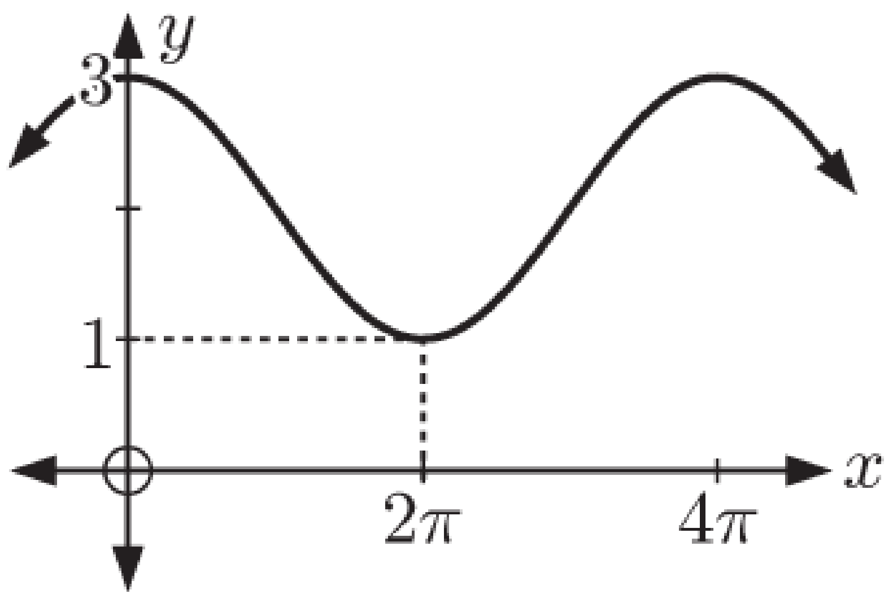


17 Find the cosine function shown in the graph:

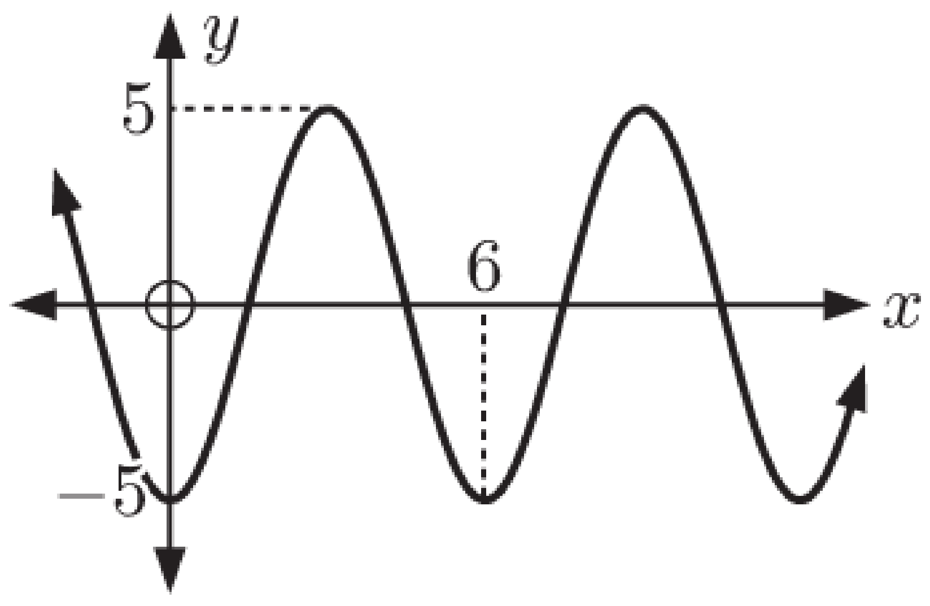
a



b



c



D MODELLING PERIODIC BEHAVIOUR

The sine and cosine functions are both referred to as **sinusoidal functions**. They can be used to model many periodic phenomena in the real world. In some cases, such as the movement of the hands on a clock, the models we find will be almost exact. In other cases, such as the maximum daily temperature of a city over a year, the model will be less accurate.

Example 5 **Self Tutor**

The average daytime temperature for a city is given by the function $D(t) = 5\cos\left(\frac{\pi}{6}t\right) + 20$ °C, where t is the time in months after January.

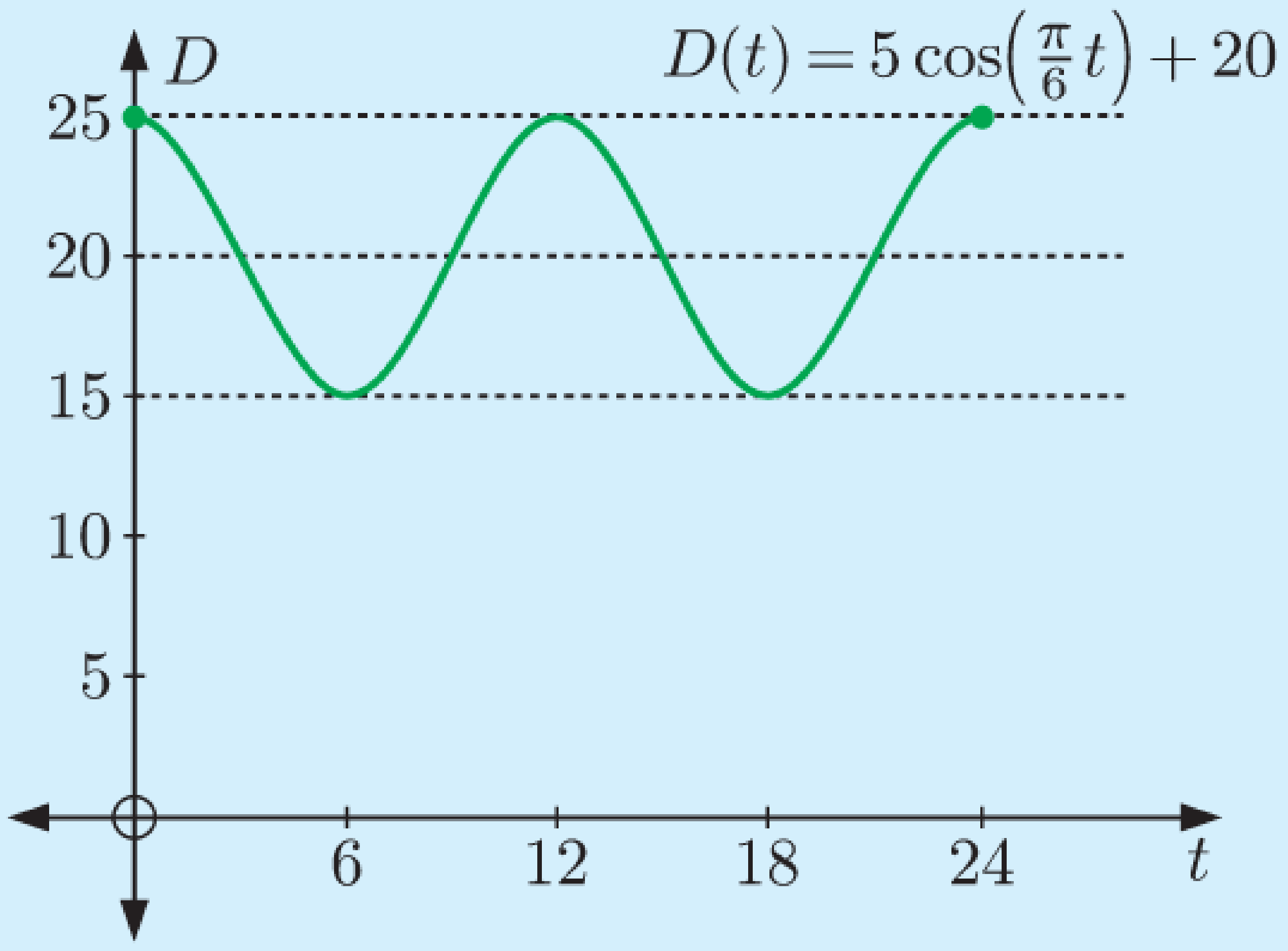
a Sketch the graph of D against t for $0 \leqslant t \leqslant 24$.

b Find the average daytime temperature during May.

c Find the minimum average daytime temperature, and the month in which it occurs.

a For $D(t) = 5\cos\left(\frac{\pi}{6}t\right) + 20$:

- the amplitude is 5
- the period is $\dfrac{2\pi}{\left(\frac{\pi}{6}\right)} = 12$ months
- the principal axis is $D = 20$.



b May is 4 months after January.

When $t = 4$, $D = 5 \times \cos\frac{4\pi}{6} + 20$

$= 5 \times \left(-\frac{1}{2}\right) + 20$

$= 17.5$

So, the average daytime temperature during May is 17.5°C.

c The minimum average daytime temperature is $20 - 5 = 15$°C, which occurs when $t = 6$ or 18.

So, the minimum average daytime temperature occurs during July.

EXERCISE 17D

1 The temperature inside Vanessa's house t hours after midday is given by the function $T(t) = 6\sin\left(\frac{\pi}{12}t\right) + 26$ °C.

a Sketch the graph of T against t for $0 \leqslant t \leqslant 24$.

b Find the temperature inside Vanessa's house at:

i midnight **ii** 2 pm.

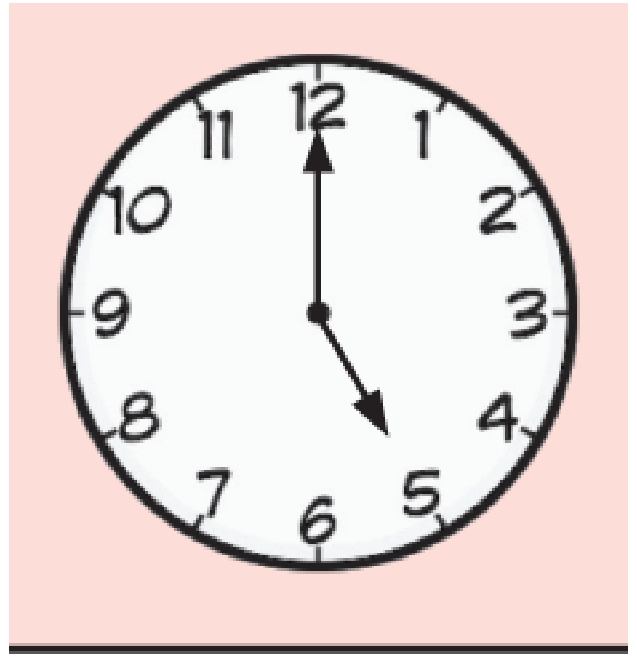
c Find the maximum temperature inside Vanessa's house, and the time at which it occurs.

2 The depth of water in a harbour t hours after midnight is $D(t) = 4\cos\left(\frac{\pi}{6}t\right) + 6$ metres.

a Sketch the graph of D against t for $0 \leqslant t \leqslant 24$.

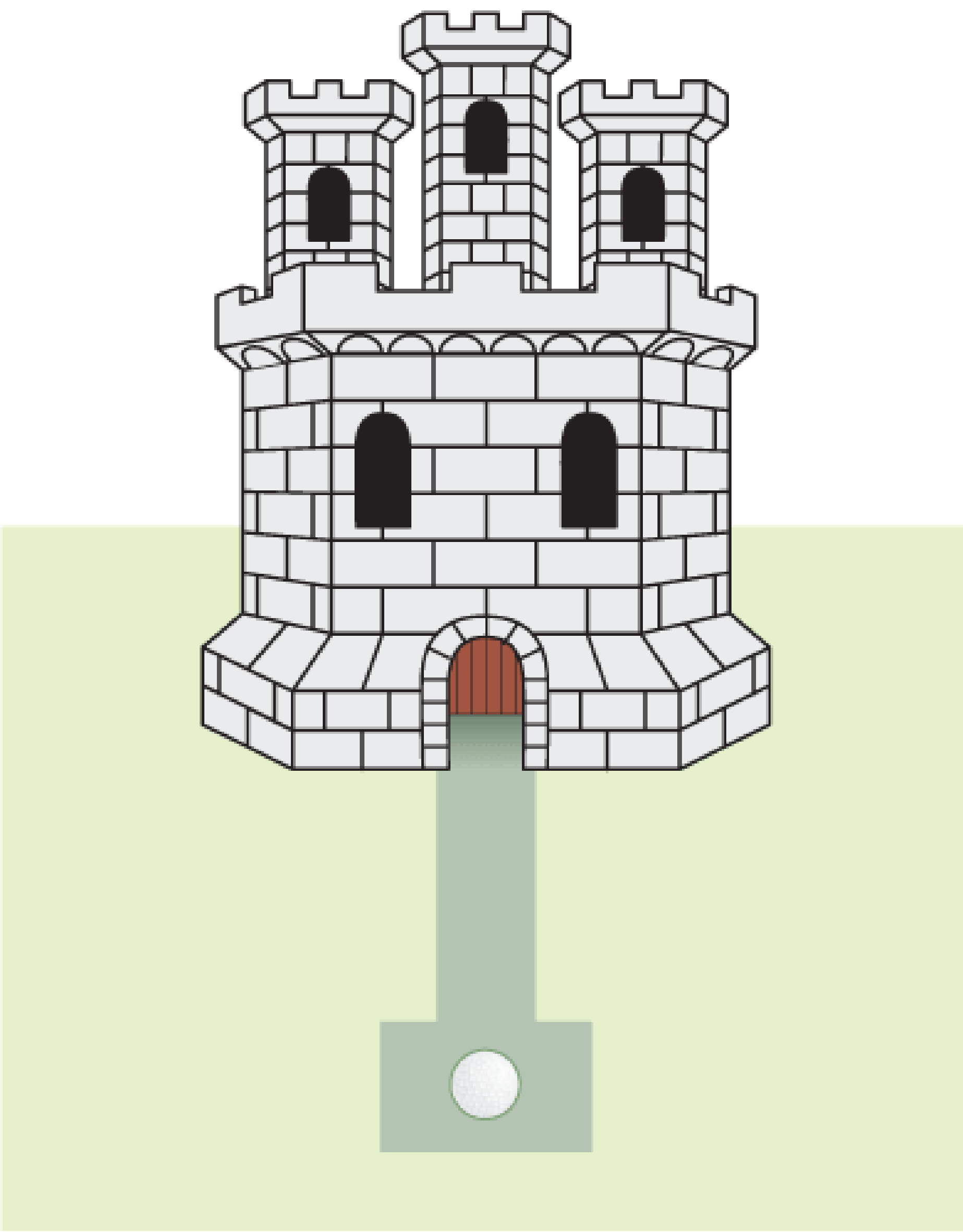
b Find the highest and lowest depths of the water, and the times at which they occur.

c A boat requires a water depth of 5 metres to sail in. Will the boat be able to enter the harbour at 8 pm?



3 The tip of a clock's minute hand is $H(t) = 15\cos\left(\frac{\pi}{30}t\right) + 150$ cm above ground level, where t is the time in minutes after 5 pm.

- **a** Sketch the graph of H against t for $0 \leqslant t \leqslant 180$.
- **b** Find the length of the minute hand.
- **c** Find, rounded to 1 decimal place, the height of the minute hand's tip at:
  - **i** 5:08 pm
  - **ii** 5:37 pm
  - **iii** 5:51 pm
  - **iv** 6:23 pm



4 On a mini-golf hole, golfers must putt the ball through a castle's entrance. The entrance is protected by a gate which moves up and down.

The height of the gate above the ground t seconds after it touches the ground is $H(t) = 4\sin\left(\frac{\pi}{4}(t-2)\right) + 4$ cm.

- **a** Sketch the graph of H against t for $0 \leqslant t \leqslant 16$.
- **b** Find the height of the gate above the ground 2 seconds after the gate touches the ground.
- **c** Eric is using a golf ball with radius 2.14 cm. He putts the ball 1 second after the gate touches the ground, and the ball takes 5.3 seconds to reach the castle's entrance. Will the ball pass through the entrance?

Example 6 **Self Tutor**

On a hot summer day in Madrid, Antonio pays careful attention to the temperature. The maximum of 41.8°C occurs at 3:30 pm. The minimum was 27.5°C. Suggest a sine function to model the temperature for that day.

The mean temperature $= \dfrac{41.8 + 27.5}{2} = 34.65°\text{C}$, so $d = 34.65$.

The amplitude $= \dfrac{41.8 - 27.5}{2} = 7.15°\text{C}$

$\therefore \quad a = 7.15$

The period is 24 hours, so $b = \frac{2\pi}{24} = \frac{\pi}{12}$.

The maximum occurs at 3:30 pm, so we assume the temperature passed its mean value 6 hours earlier, at 9:30 am.

The day begins at midnight, so the function is shifted $9\frac{1}{2}$ hours to the right, thus $c = 9.5$.

If t is the number of hours after midnight, the temperature T is modelled by

$$T(t) = 7.15\sin\left(\tfrac{\pi}{12}(t - 9.5)\right) + 34.65 \text{ °C}.$$

5 On a September day in Moscow, the maximum temperature 15.8°C occurred at 2 pm. The minimum was 5.4°C. Suggest a sine function to model the temperature for that day. Let T be the temperature and t be the time in hours after midnight.

6 The ferry operator at Picton, New Zealand, is studying the tides. High tides occur every 12.4 hours. The first high tide tomorrow will be at 1:30 am. The high tide will be 1.36 m and the low tide will be 0.16 m. Find a cosine function to model the tide height for the day. Let H be the tide height and t be the time in hours after midnight.

7 Answer the **Opening Problem** on page **448**.

8 Some of the largest tides in the world are observed in Canada's Bay of Fundy. The difference between high and low tides is 14 metres, and the average time difference between high tides is about 12.4 hours. On a particular day, the first high tide is 16.2 m, occurring at 9 am.

- **a** Find a sine model for the height of the tide H in terms of the time t.
- **b** Sketch the graph of the function for that day.

9 On an analogue clock, the hour hand is 6 cm long and the minute hand is 12 cm long. Let t be the time in hours after midnight.

- **a** Write a cosine function for the height of the tip of the hour hand relative to the centre of the clock.
- **b** Write a sine function for the horizontal displacement of the tip of the minute hand relative to the centre of the clock.

E FITTING TRIGONOMETRIC MODELS TO DATA

Suppose we have **data** in which we observe periodic behavior. In such cases, we usually cannot fit an *exact* model. However, we can still apply the same principles to estimate the period, amplitude, and principal axis from the data.

You can check your models using your graphics calculator. Click on this icon for instructions.

GRAPHICS CALCULATOR INSTRUCTIONS

Example 7 **Self Tutor**

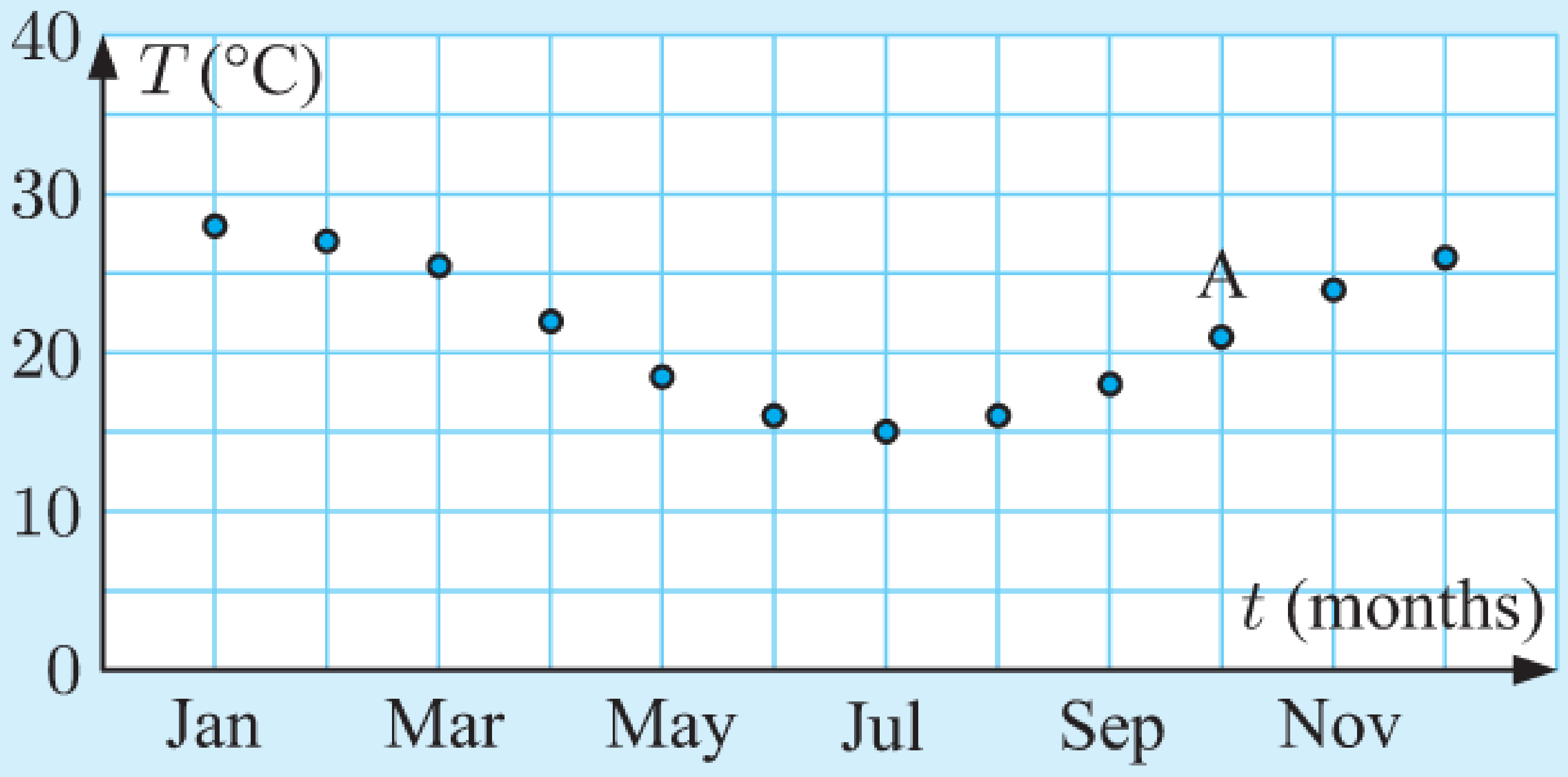
The mean monthly maximum temperatures for Cape Town, South Africa are shown below:

Month (t)	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temperature (T °C)	28	27	25.5	22	18.5	16	15	16	18	21.5	24	26

We want to model the data with a trigonometric function of the form $T = a\sin(b(t-c))+d$ where Jan $\equiv 1$, Feb $\equiv 2$, and so on.

- **a** Draw a scatter diagram of the data.
- **b** Without using technology, estimate:
  - **i** b
  - **ii** a
  - **iii** d
  - **iv** c
- **c** Check your answers using technology.

a



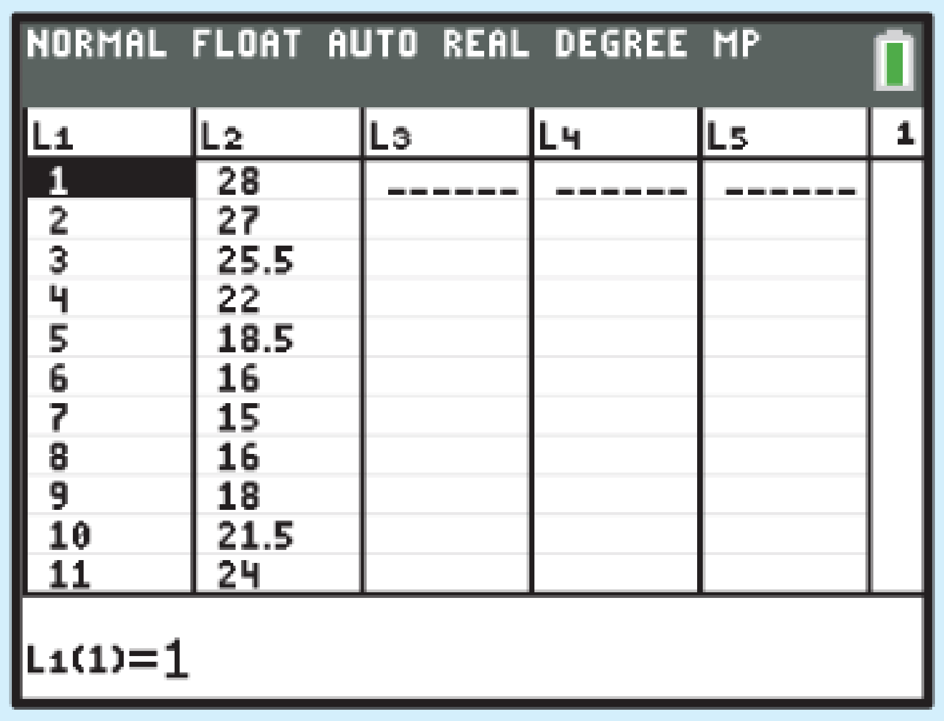
b **i** The period is 12 months, so $\frac{2\pi}{b} = 12$ and $\therefore$ $b = \frac{\pi}{6}$.

ii The amplitude $= \frac{\text{max} - \text{min}}{2} \approx \frac{28 - 15}{2} \approx 6.5$, so $a \approx 6.5$.

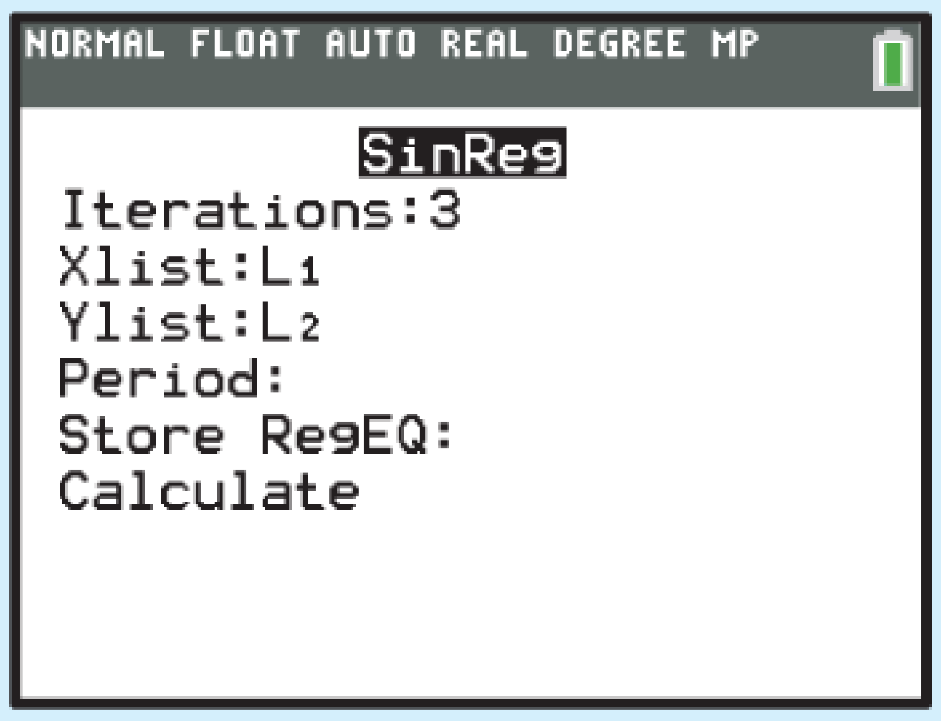
iii The principal axis is midway between the maximum and minimum, so
$d \approx \frac{28 + 15}{2} \approx 21.5$.

iv The model is $T \approx 6.5 \sin\left(\frac{\pi}{6}(t - c)\right) + 21.5$ for some constant c.
On the original graph, point A is the first point shown at which the sine function starts a new period. Since A is at (10, 21.5), $c = 10$.

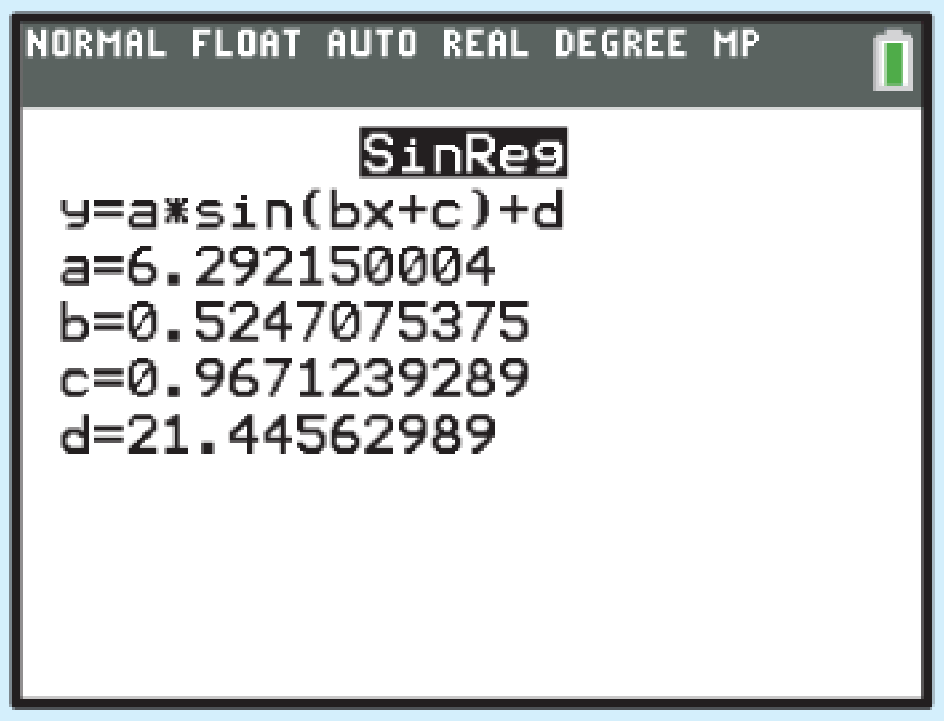
c From **b**, our model is
$$\begin{aligned} T &\approx 6.5 \sin\left(\tfrac{\pi}{6}(t - 10)\right) + 21.5 \\ &\approx 6.5 \sin(0.524t - 5.24) + 21.5 \end{aligned}$$



```
NORMAL FLOAT AUTO REAL DEGREE MP
L1    L2    L3     L4     L5     1
1     28    ------ ------ ------
2     27
3     25.5
4     22
5     18.5
6     16
7     15
8     16
9     18
10    21.5
11    24
L1(1)=1
```

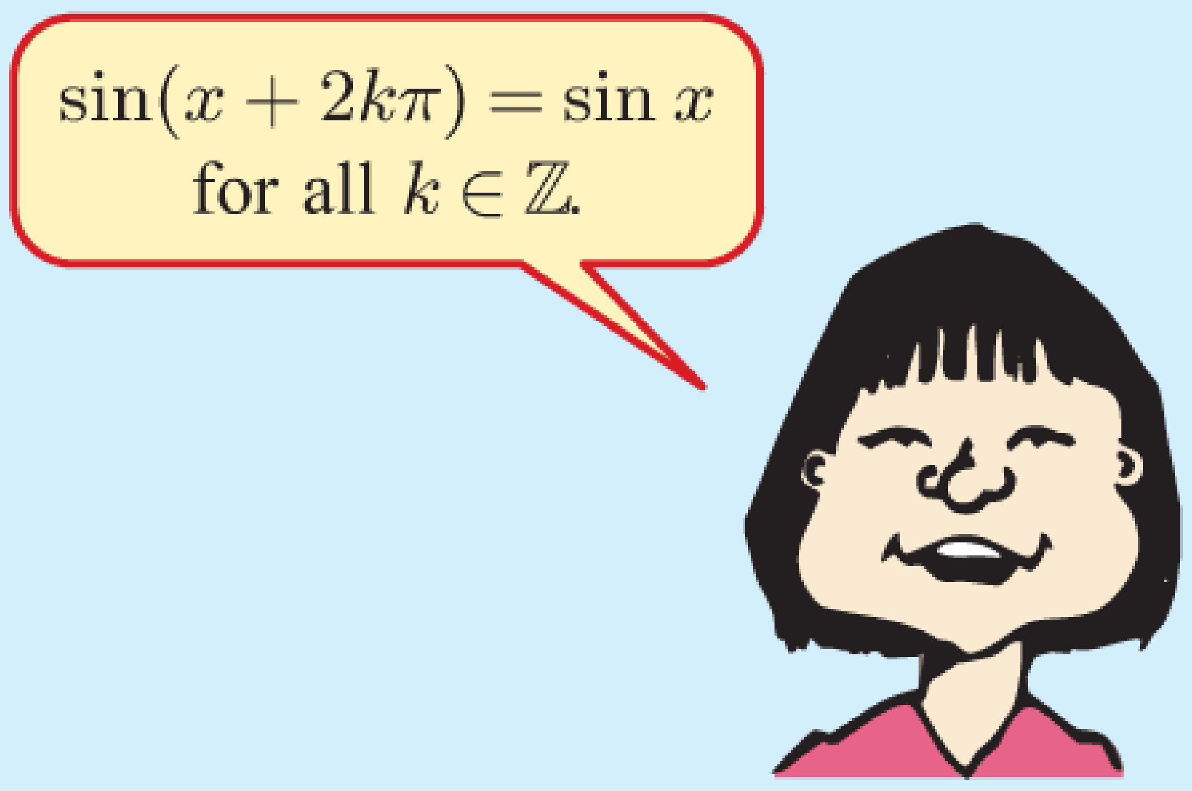


```
NORMAL FLOAT AUTO REAL DEGREE MP
          SinReg
 Iterations:3
 Xlist:L1
 Ylist:L2
 Period:
 Store RegEQ:
 Calculate
```



```
NORMAL FLOAT AUTO REAL DEGREE MP
          SinReg
 y=a*sin(bx+c)+d
 a=6.292150004
 b=0.5247075375
 c=0.9671239289
 d=21.44562989
```

Using technology,
$$\begin{aligned} T &\approx 6.29 \sin(0.525t + 0.967) + 21.4 \\ &\approx 6.29 \sin(0.525t + 0.967 - 2\pi) + 21.4 \\ &\approx 6.29 \sin(0.525t - 5.32) + 21.4 \end{aligned}$$



EXERCISE 17E

1 Below is a table which shows the mean monthly maximum temperatures for a city in Greece.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	15	14	15	18	21	25	27	26	24	20	18	16

a Draw a scatter diagram of the data.

b What features of the data suggest a trigonometric model is appropriate?

c Your task is to model the data with a sine function of the form $T \approx a \sin(b(t - c)) + d$, where Jan $\equiv 1$, Feb $\equiv 2$, and so on.
Without using technology, estimate:

i b **ii** a **iii** d **iv** c

d Use technology to check your model. How well does your model fit?

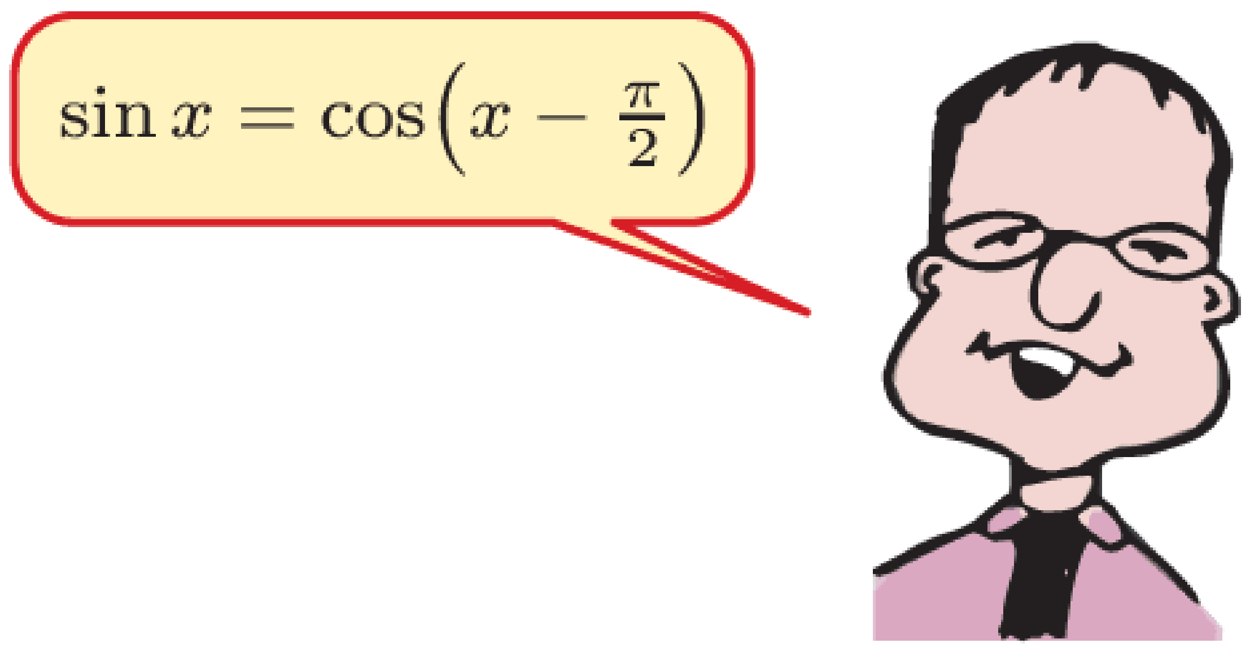
2 The data in the table shows the mean monthly temperatures for Christchurch, New Zealand.

Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	15	16	$14\frac{1}{2}$	12	10	$7\frac{1}{2}$	7	$7\frac{1}{2}$	$8\frac{1}{2}$	$10\frac{1}{2}$	$12\frac{1}{2}$	14

a Find a cosine model for this data in the form $T \approx a\cos(b(t-c))+d$ without using technology. Let Jan $\equiv 1$, Feb $\equiv 2$, and so on.

b Draw a scatter diagram of the data and sketch the graph of your model on the same set of axes.

c Use technology to check your answer to **a**.



3 At the Mawson base in Antarctica, the mean monthly temperatures for the last 30 years are:

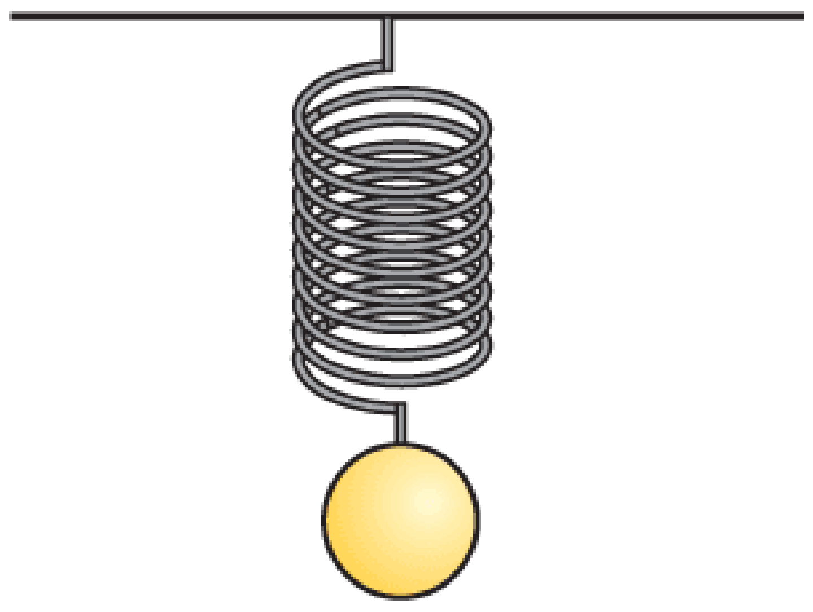
Month	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sept	Oct	Nov	Dec
Temperature (°C)	0	0	-4	-9	-14	-17	-18	-19	-17	-13	-6	-2

a Find a sine model for this data without using technology. Use Jan $\equiv 1$, Feb $\equiv 2$, and so on.

b Draw a scatter diagram of the data and sketch the graph of your model on the same set of axes.

c How appropriate is the model?

4 An object is suspended from a spring. If the object is pulled below its resting position and then released, it will oscillate up and down. The data below shows the height of the object relative to its rest position, at different times.



Time (t seconds)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Height (H cm)	-15	-13	-7.5	0	7.5	13	15	13	7.5	0

Time (t seconds)	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
Height (H cm)	-7.5	-13	-15	-13	-7.5	0	7.5	13	15	13	7.5

a Draw a scatter diagram of the data.

b Find a trigonometric function which models the height of the object over time.

c Use your model to predict the height of the object after 4.25 seconds.

d What do you think is unrealistic about this model? What would happen differently in reality?

RESEARCH

1 How accurately will a trigonometric function model the phases of the moon?

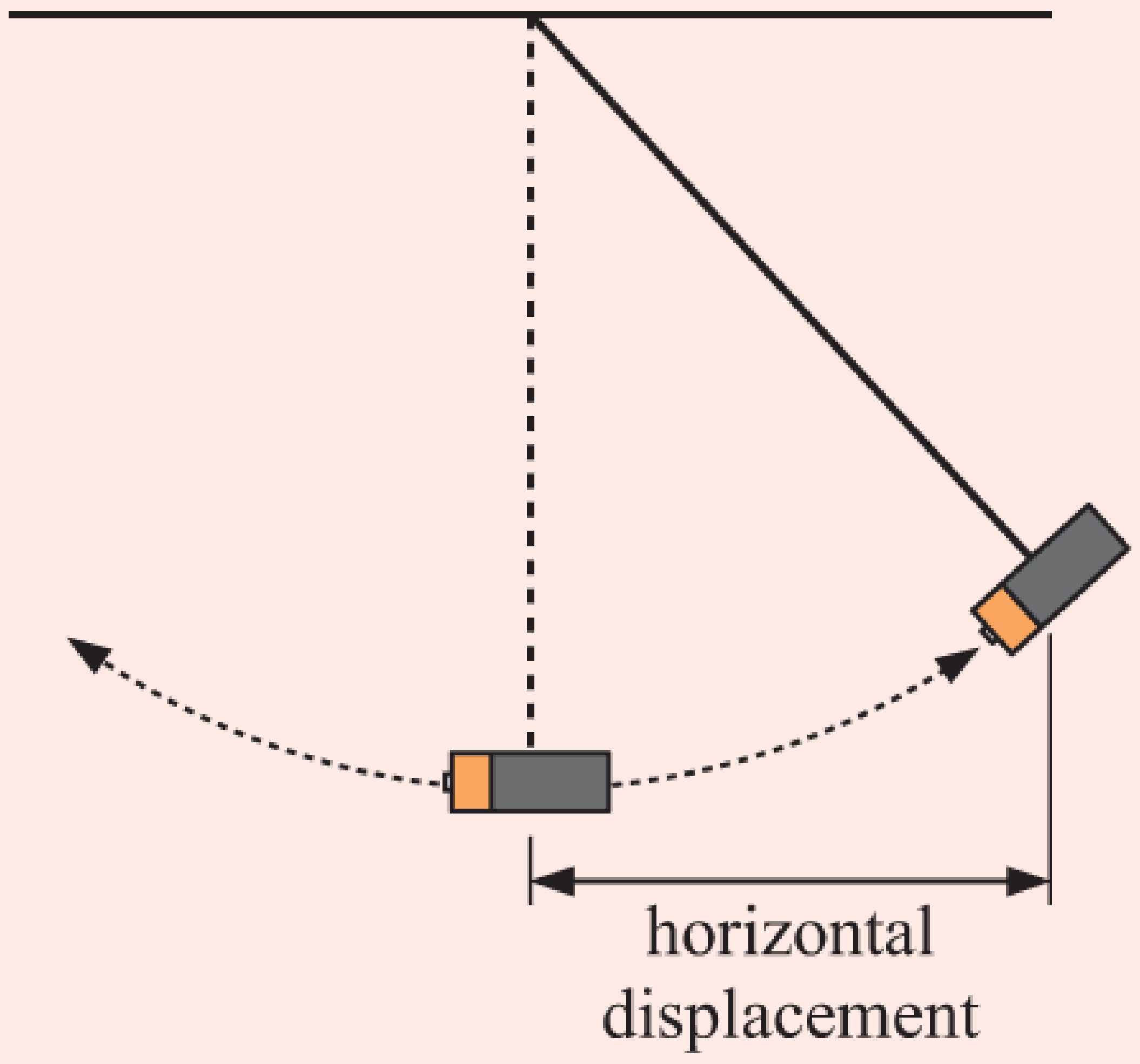
2 Are there any periodic phenomena which can be modelled by the *sum* of trigonometric functions?

ACTIVITY 2 THE PENDULUM

In this Activity you will work in small groups to model the behaviour of a pendulum.

You will need: string, sticky tape, a ruler, a stopwatch, and a AA battery.

What to do:



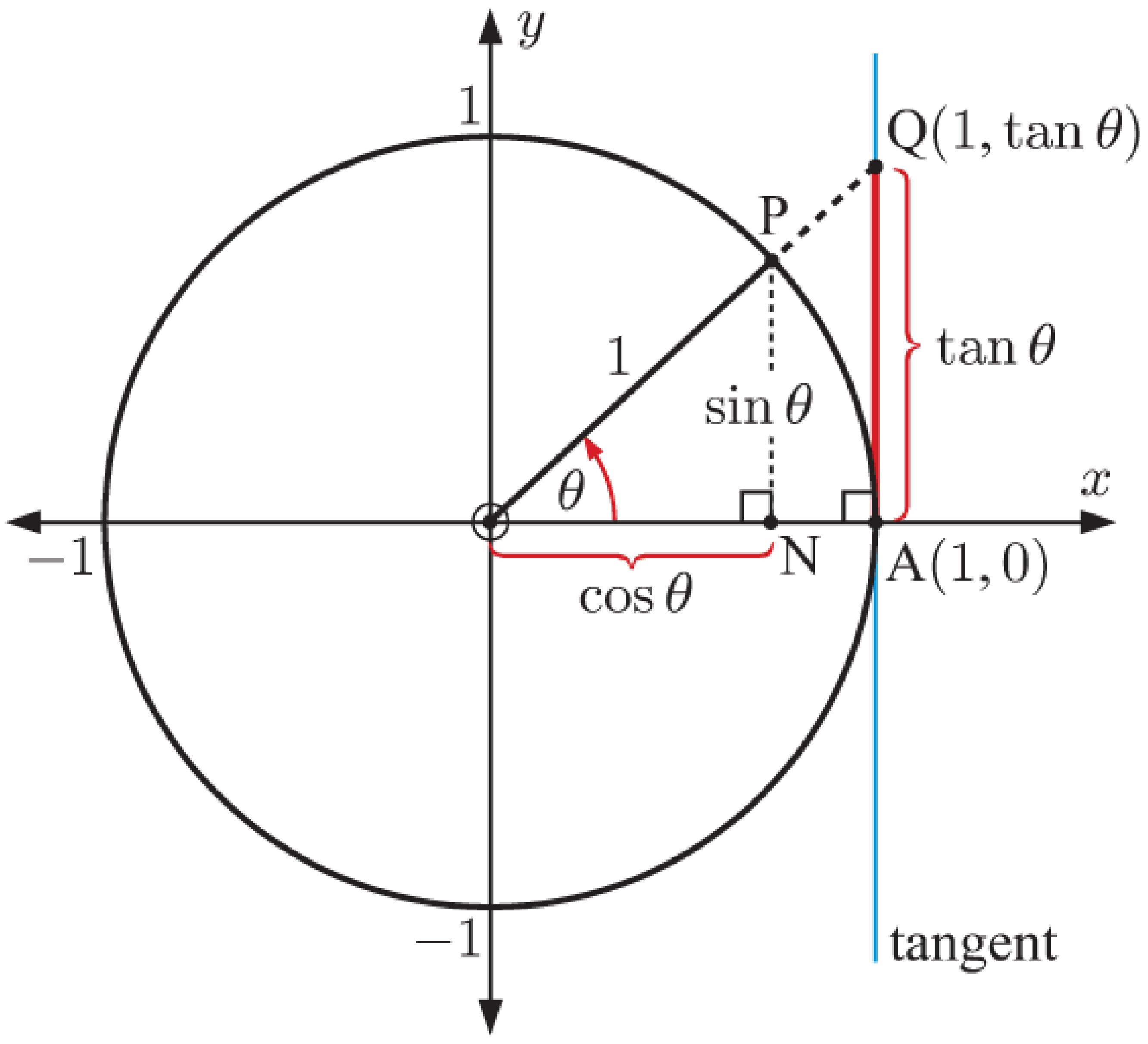
1. Cut a piece of string of length 75 cm. Attach one end of the string to the battery, and the other end to your desk.
2. Hold the battery to one side, then release it, causing the battery to swing back and forth like a pendulum.
3. Using your stopwatch and ruler, measure the maximum and minimum horizontal displacement reached by the battery, and the times at which they occurred. You may need to repeat the experiment several times, but make sure the battery is released from the same position each time.
4. Use your data to find a trigonometric function which models the horizontal displacement of the battery over time.
5. What part of the function affects the *period* of the pendulum?
6. Repeat the experiment with strings of different length. Explore the relationship between the length of the string and the period of the pendulum.

F THE TANGENT FUNCTION

We have seen that if $P(\cos\theta, \sin\theta)$ is a point which is free to move around the unit circle, and if [OP] is extended to meet the tangent at $A(1, 0)$, the intersection between these lines occurs at $Q(1, \tan\theta)$.

This enables us to define the **tangent function**

$$\boldsymbol{\tan\theta = \frac{\sin\theta}{\cos\theta}}.$$



We have also seen that $\tan\theta$ is:

- positive in quadrants 1 and 3
- negative in quadrants 2 and 4
- periodic with period π.

DISCUSSION

What happens to $\tan\theta$ when P is at:

a $(1, 0)$ and $(-1, 0)$

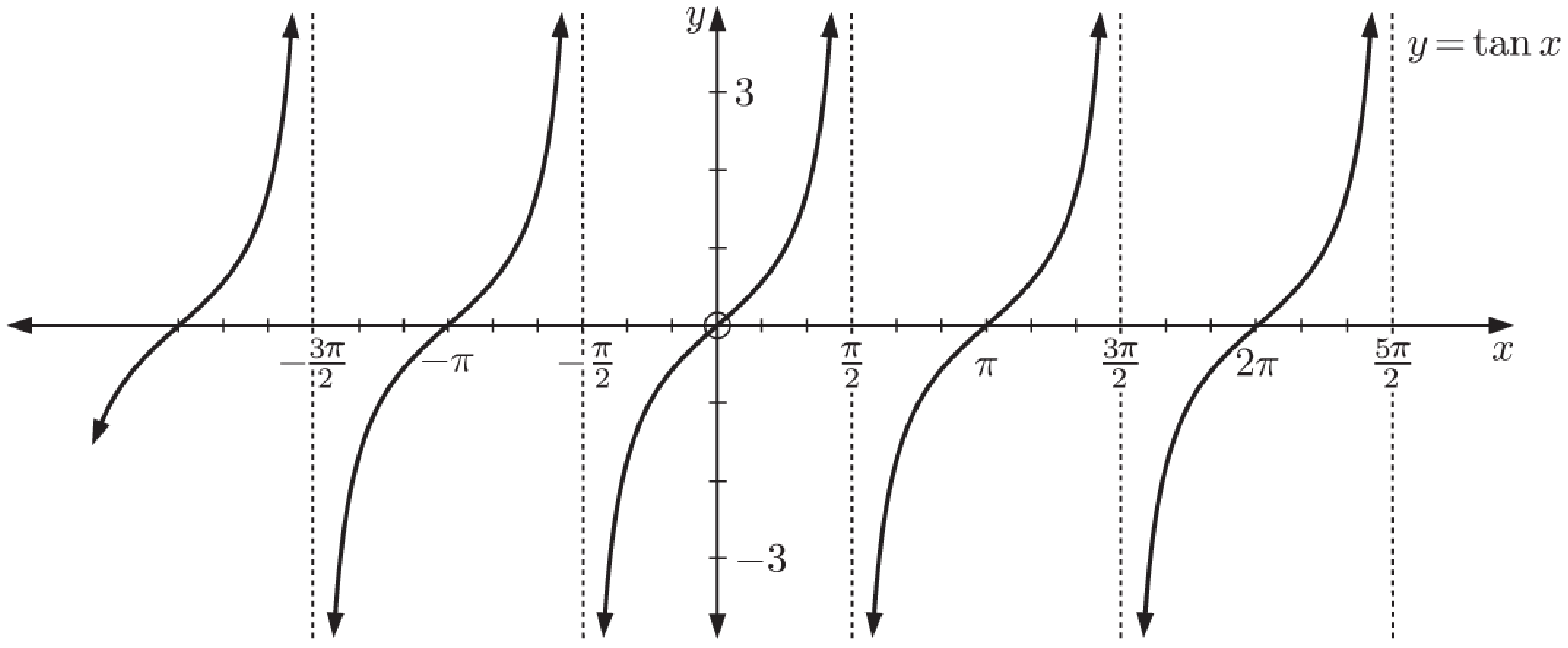
b $(0, 1)$ and $(0, -1)$?

THE GRAPH OF $y = \tan x$

$\tan x$ is zero whenever $\sin x = 0$, so the **zeros** of $y = \tan x$ are $k\pi$, $k \in \mathbb{Z}$.

$\tan x$ is undefined whenever $\cos x = 0$, so the **vertical asymptotes** of $y = \tan x$ are $x = \frac{\pi}{2} + k\pi$ for all $k \in \mathbb{Z}$.

$\tan x$ has period $= \pi$ and range $y \in \mathbb{R}$.



Click on the icon to explore how the tangent function is produced from the unit circle.

TANGENT FUNCTION

THE GENERAL TANGENT FUNCTION

The **general tangent function** is $y = a\tan(b(x - c)) + d$, $a \neq 0$, $b > 0$.

- The **principal axis** is $y = d$.
- The **period** of this function is $\dfrac{\pi}{b}$.
- The **amplitude** of this function is undefined.
- There are infinitely many vertical asymptotes.

DYNAMIC TANGENT FUNCTION

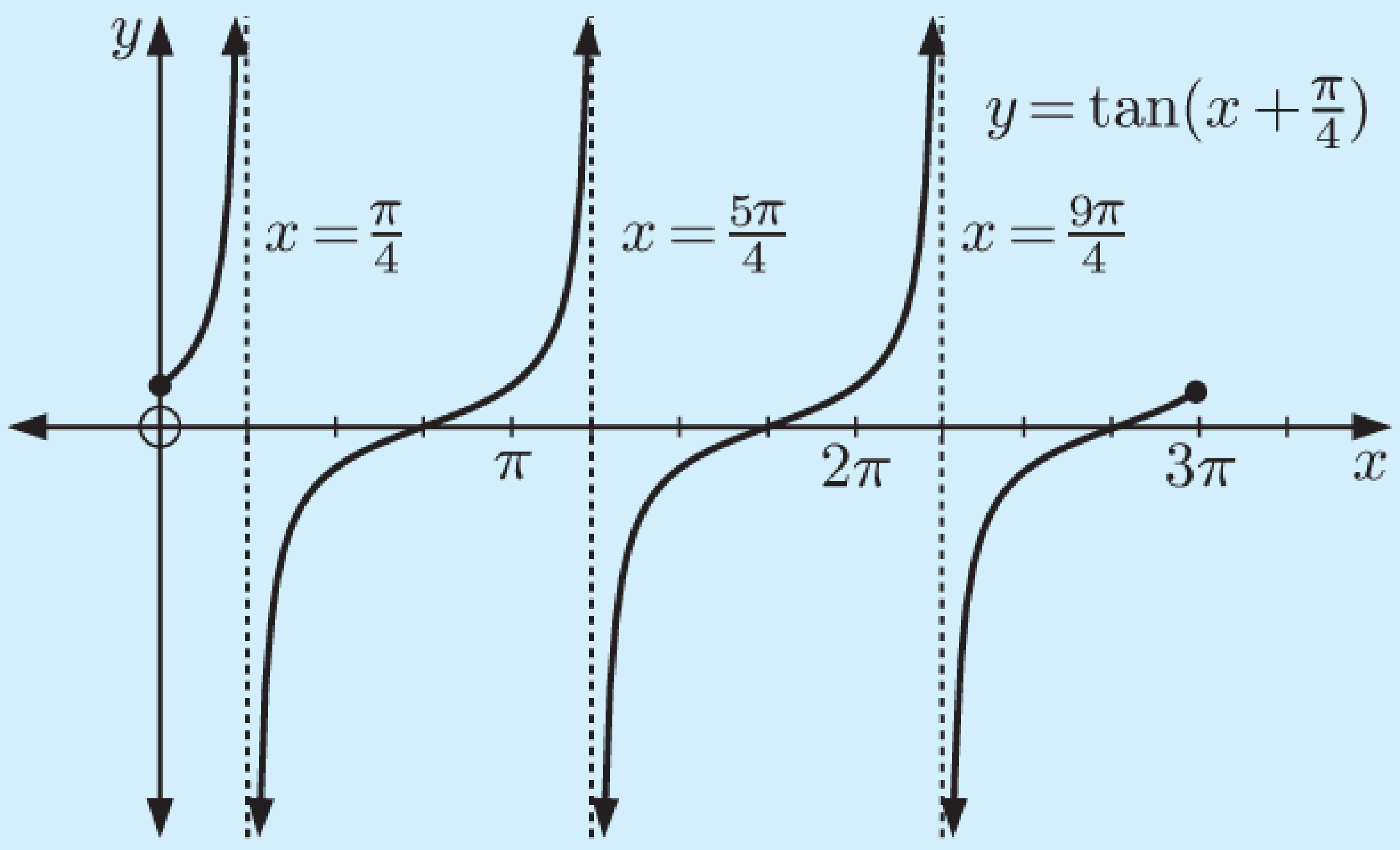
Example 8 **Self Tutor**

Without using technology, sketch the graph of $y = \tan\left(x + \frac{\pi}{4}\right)$ for $0 \leqslant x \leqslant 3\pi$.

$y = \tan\left(x + \frac{\pi}{4}\right)$ is a horizontal translation of $y = \tan x$ to the left by $\frac{\pi}{4}$ units.

$y = \tan x$ has vertical asymptotes $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$, $x = \frac{5\pi}{2}$, and its x-intercepts are 0, π, 2π, and 3π.

$\therefore$ $y = \tan\left(x + \frac{\pi}{4}\right)$ has vertical asymptotes $x = \frac{\pi}{4}$, $x = \frac{5\pi}{4}$, $x = \frac{9\pi}{4}$, and x-intercepts $\frac{3\pi}{4}$, $\frac{7\pi}{4}$, and $\frac{11\pi}{4}$.



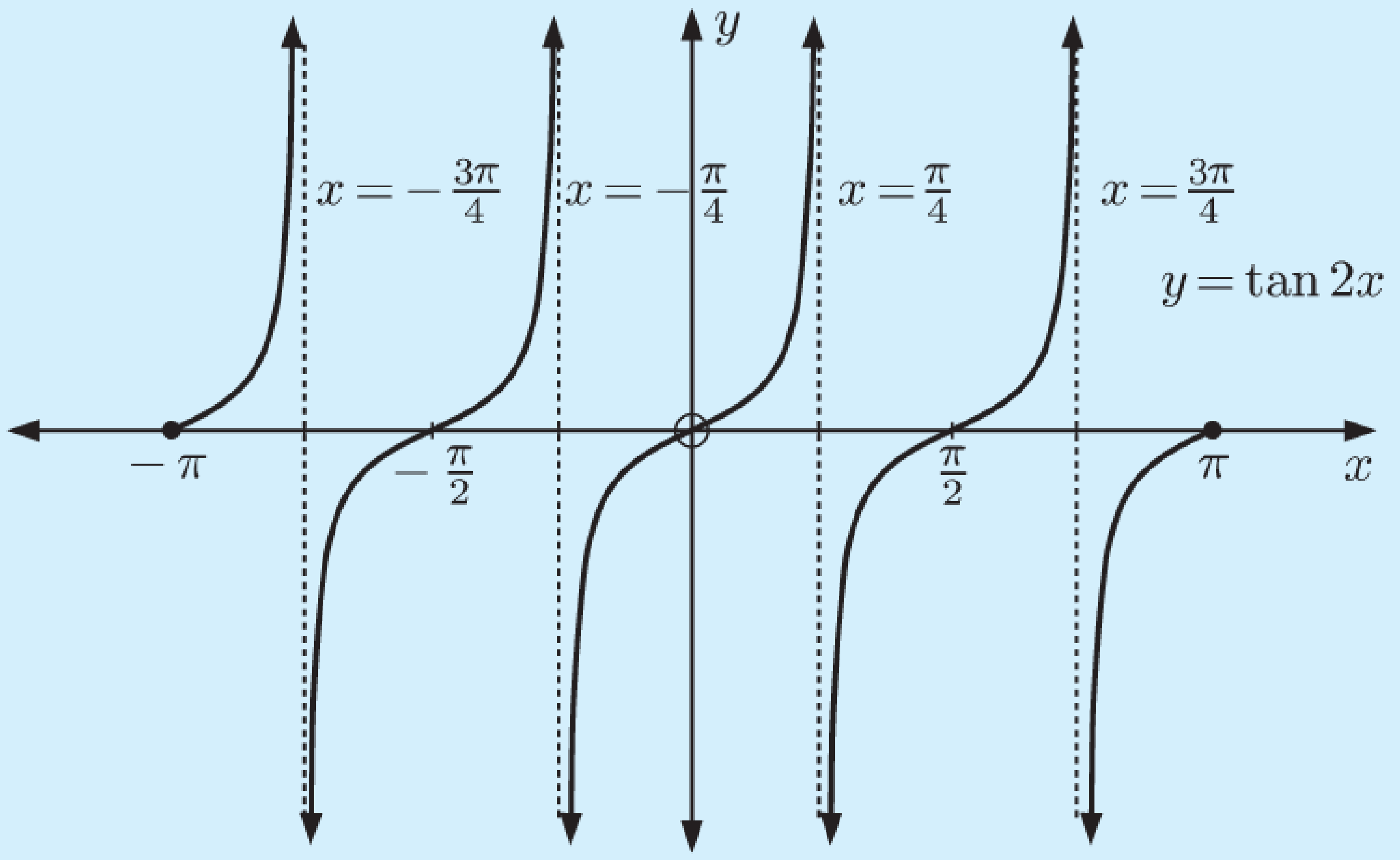
Example 9 **Self Tutor**

Without using technology, sketch the graph of $y = \tan 2x$ for $-\pi \leqslant x \leqslant \pi$.

$y = \tan 2x$ is a horizontal stretch of $y = \tan x$ with scale factor $\frac{1}{2}$.

Since $b = 2$, the period is $\frac{\pi}{2}$.

$y = \tan 2x$ has vertical asymptotes $x = \pm\frac{\pi}{4}$, $x = \pm\frac{3\pi}{4}$, and x-intercepts $0, \pm\frac{\pi}{2}, \pm\pi$.



EXERCISE 17F

1 State the transformations which map $y = \tan x$ onto:

a $y = \tan\left(x - \frac{\pi}{2}\right)$ **b** $y = 4\tan x$ **c** $y = \tan\left(\frac{\pi}{2}x\right)$

d $y = \tan 2x - 1$ **e** $y = -\frac{1}{2}\tan x$ **f** $y = \tan(x + \pi) + 2$

2 State the period of:

a $y = \tan 3x$ **b** $y = \tan\frac{x}{4}$ **c** $y = \tan \pi x$

d $y = -\tan\left(\frac{\pi}{2}x\right)$ **e** $y = \tan\left(\frac{2x}{3} - \frac{\pi}{3}\right)$ **f** $y = \tan nx$, $n \neq 0$

3 For each function, write down the:

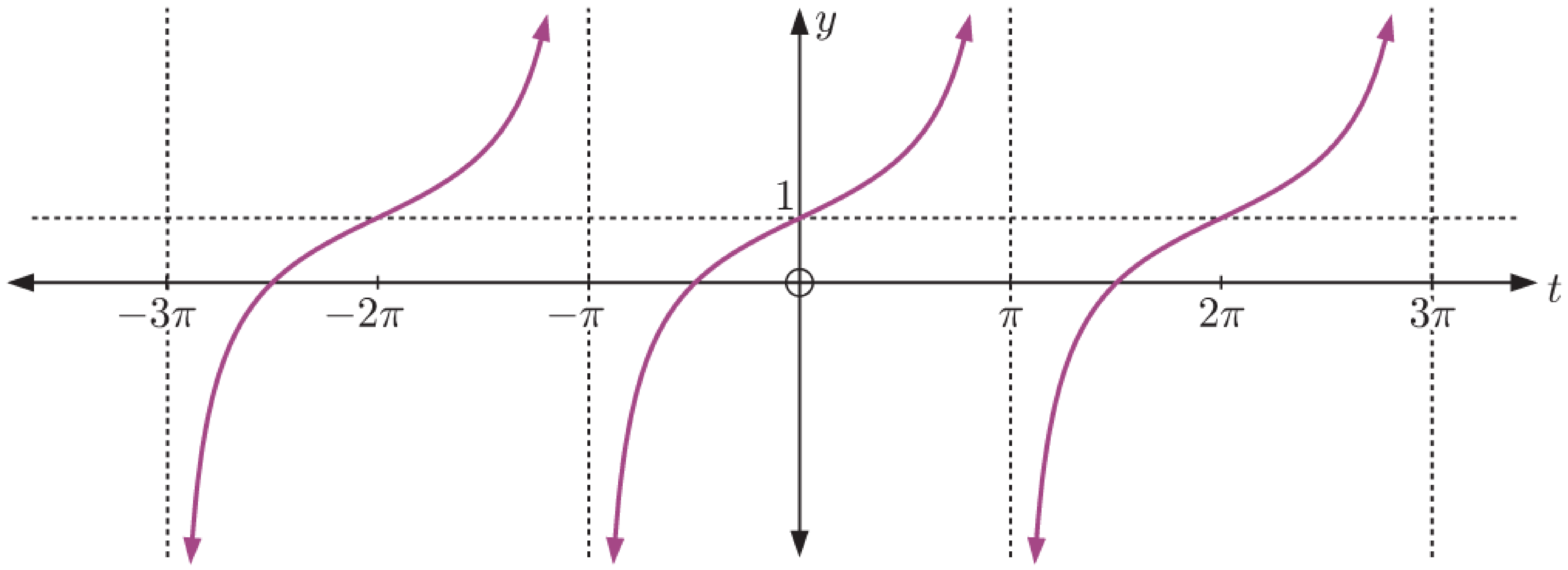
i zeros **ii** vertical asymptotes.

a $y = \tan 2x$ **b** $y = \tan\left(x + \frac{\pi}{3}\right)$ **c** $y = \frac{1}{2}\tan\left(\frac{1}{2}\left(x - \frac{\pi}{6}\right)\right)$

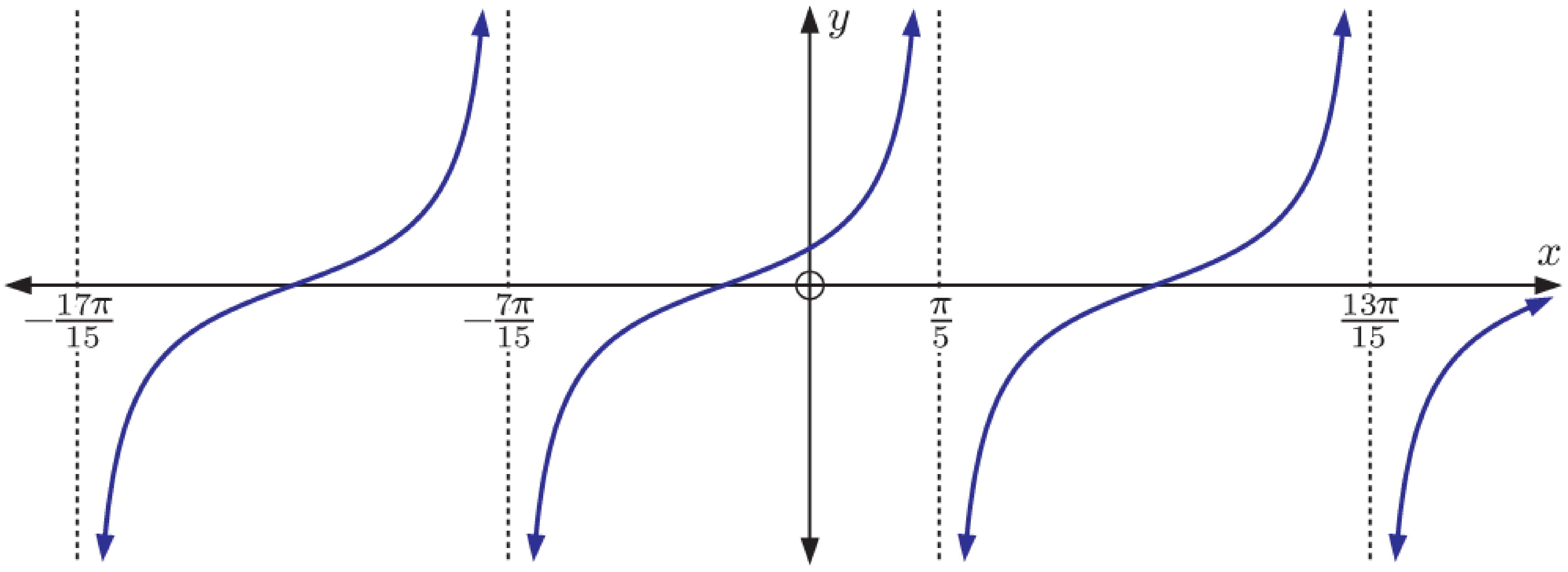
4 Sketch the graph of the following for $-2\pi \leqslant x \leqslant 2\pi$:

a $y = \tan\left(x - \frac{\pi}{4}\right)$ **b** $y = \frac{1}{2}\tan\frac{x}{4}$ **c** $y = 3\tan\left(x - \frac{\pi}{9}\right)$

5 Find p and q given the following graph is of the function $y = \tan pt + q$.



6 Find the possible values of a and b given the following graph is of the function $y = \tan a(x-b)$.



7 **a** Describe the sequence of transformations used to transform $y = \tan x$ into $y = 2\tan\left(x + \frac{\pi}{4}\right) - 1$.

b Sketch $y = 2\tan\left(x + \frac{\pi}{4}\right) - 1$ for $-2\pi \leqslant x \leqslant 2\pi$.

8 Show that the general tangent function $y = a\tan(b(x-c)) + d$, $a \neq 0$, $b > 0$, has asymptotes $x = \frac{\pi}{2b}(2k+1) + c$ for all $k \in \mathbb{Z}$.

9 Consider the functions $f(x) = \tan x$ and $g(x) = 2x - \frac{\pi}{2}$.

a Find:

i $(f \circ g)(x)$ **ii** $(g \circ f)(x)$

b Find the value of:

i $(f \circ g)\left(\frac{\pi}{3}\right)$ **ii** $(g \circ f)(\pi)$

c Write down the period and vertical asymptotes of:

i $(f \circ g)(x)$ **ii** $(g \circ f)(x)$

d Sketch the graphs of $(f \circ g)(x)$ and $(g \circ f)(x)$ for $-2\pi \leqslant x \leqslant 2\pi$.

ACTIVITY 3

Click on the icon to run a card game for trigonometric functions.

CARD GAME

G TRIGONOMETRIC EQUATIONS

Trigonometric equations will often have infinitely many solutions unless a restricted domain such as $0 \leqslant x \leqslant 3\pi$ is given.

In the **Opening Problem**, the height of the green light after t seconds is given by $H(t) = 10\sin\left(\frac{\pi}{50}t\right) + 12$ metres. So, the green light will be 16 metres above the ground when $10\sin\left(\frac{\pi}{50}t\right) + 12 = 16$.

This trigonometric equation has infinitely many solutions provided the wheel keeps rotating. For this reason we would normally specify a time interval for the solution. For example, if we are interested in the first three minutes of its rotation, we specify the domain $0 \leqslant t \leqslant 180$.

We will examine solving trigonometric equations using:

- pre-prepared graphs
- technology
- algebra.

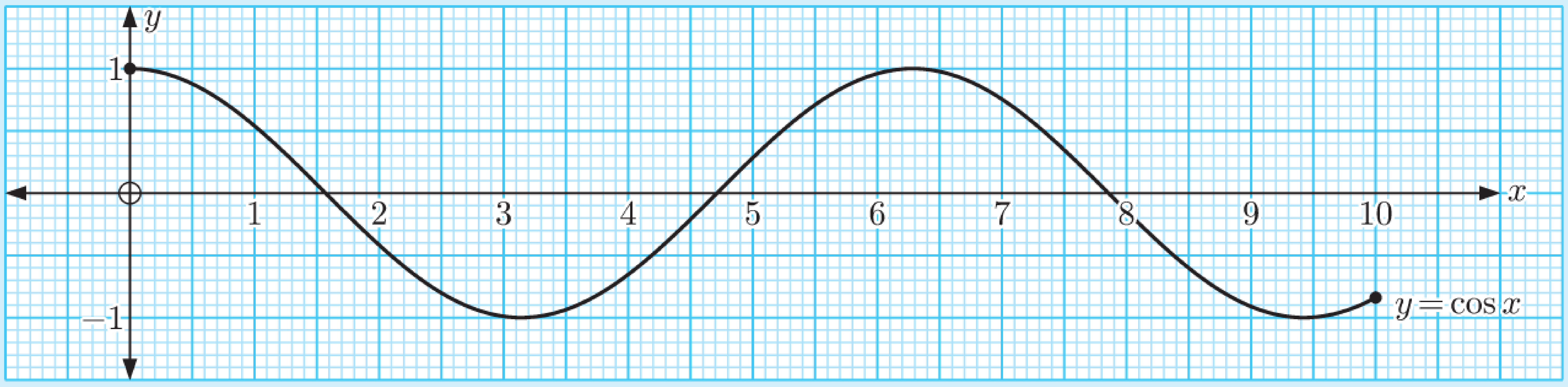
GRAPHICAL SOLUTION OF TRIGONOMETRIC EQUATIONS

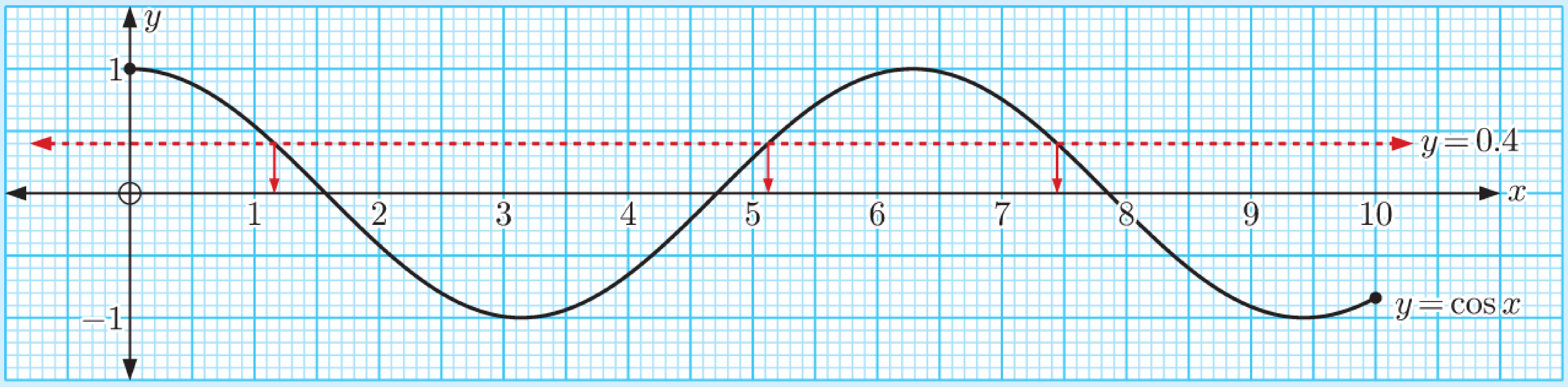
If we are given a graph with sufficient accuracy, we can use it to estimate solutions.

Example 10

Self Tutor

Solve $\cos x = 0.4$ for $0 \leqslant x \leqslant 10$ radians using the graph of $y = \cos x$.



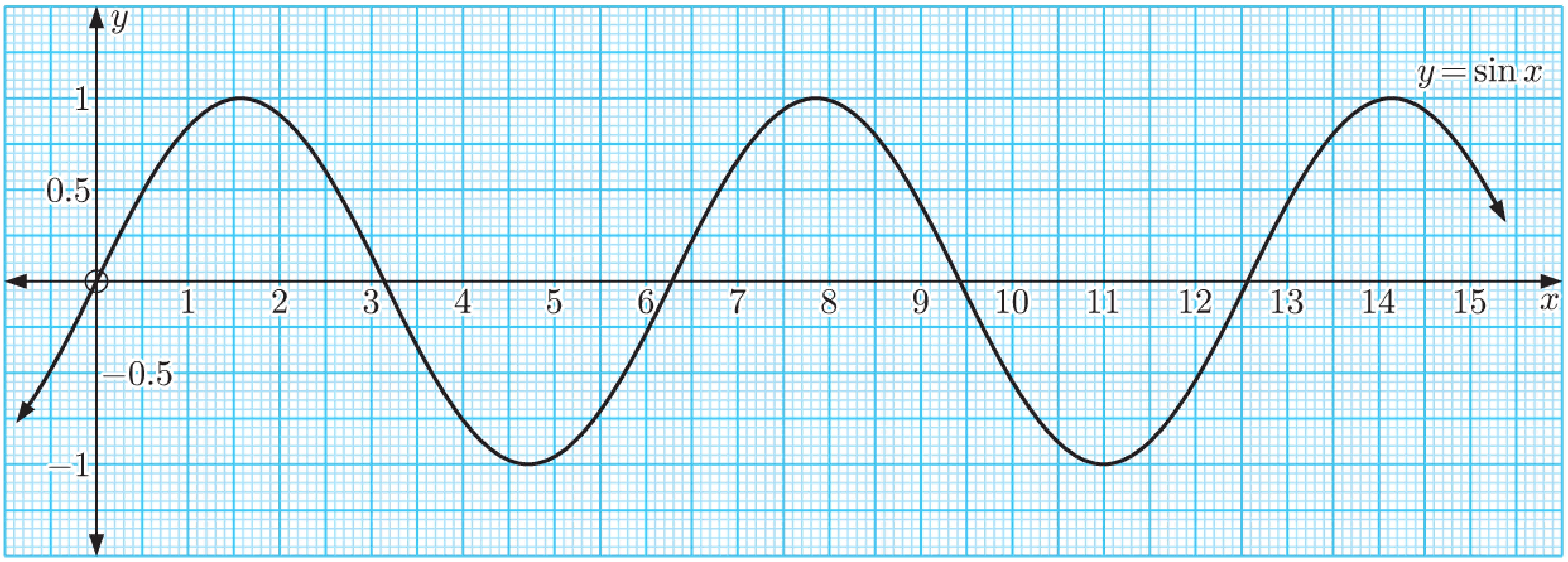


$y = 0.4$ meets $y = \cos x$ when $x \approx 1.2$, 5.1, or 7.4.

The solutions of $\cos x = 0.4$ for $0 \leqslant x \leqslant 10$ radians are 1.2, 5.1, and 7.4.

EXERCISE 17G.1

1



Use the graph of $y = \sin x$ to solve, correct to 1 decimal place:

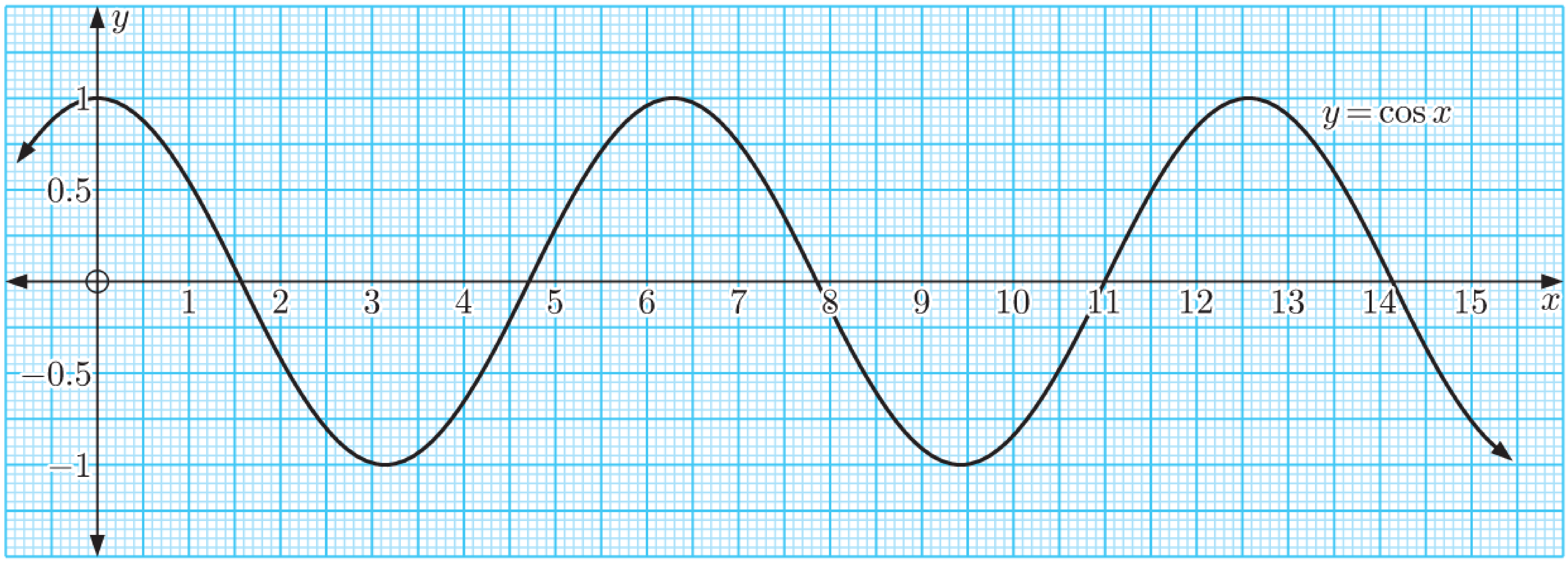
a $\sin x = 0.3$ for $0 \leqslant x \leqslant 15$

b $\sin x = -0.4$ for $5 \leqslant x \leqslant 15$

c $\sin x = 0.3$ or $0 \leqslant x \leqslant 2\pi$

d $\sin x = -0.6$ for $\pi \leqslant x \leqslant 2\pi$

2



Use the graph of $y = \cos x$ to solve, correct to 1 decimal place:

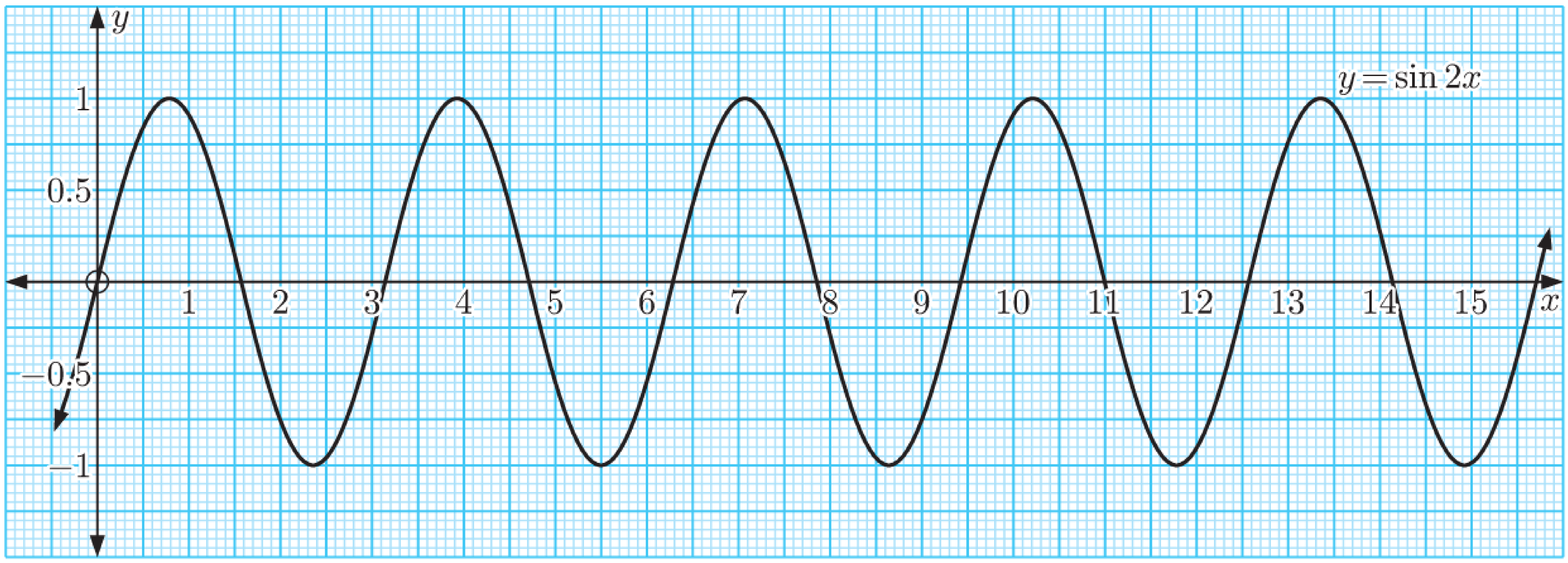
a $\cos x = 0.4$ for $0 \leqslant x \leqslant 10$

b $\cos x = -0.3$ for $4 \leqslant x \leqslant 12$

c $\cos x = 0.5$ for $\pi \leqslant x \leqslant 2\pi$

d $\cos x = -0.8$ for $\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$

3



Use the graph of $y = \sin 2x$ to solve, correct to 1 decimal place:

a $\sin 2x = 0.7$ for $0 \leqslant x \leqslant 16$

b $\sin 2x = -0.3$ for $0 \leqslant x \leqslant 16$

c $\sin 2x = 0.2$ for $\pi \leqslant x \leqslant 2\pi$

d $\sin 2x = -0.1$ for $0 \leqslant x \leqslant 2\pi$

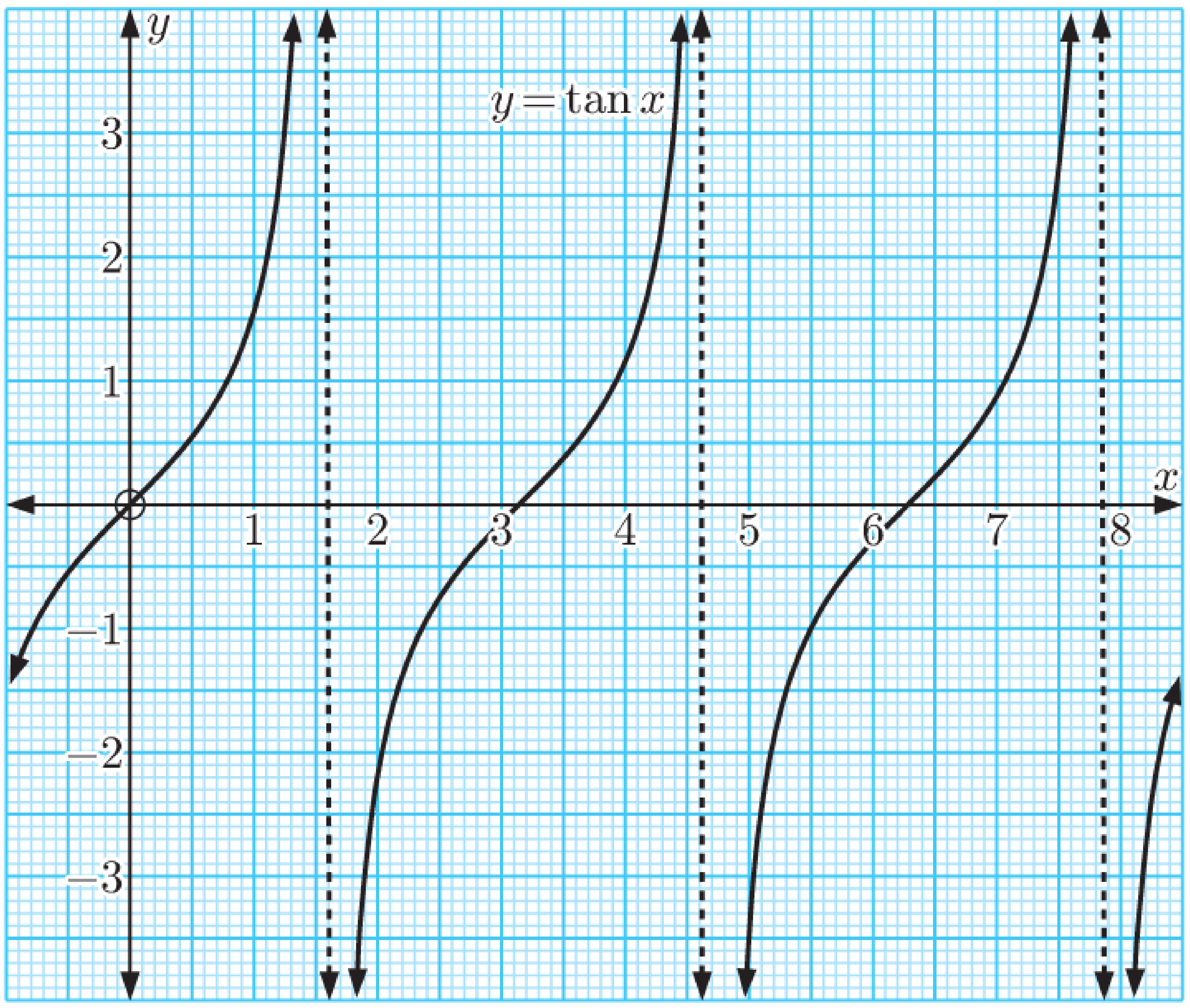
4 Use the graph of $y = \tan x$ to solve, correct to 1 decimal place:

a $\tan x = 2$ for $0 \leqslant x \leqslant 8$

b $\tan x = -1.4$ for $2 \leqslant x \leqslant 7$

c $\tan x = 3.5$ for $0 \leqslant x \leqslant 2\pi$

d $\tan x = -2.4$ for $\frac{\pi}{2} \leqslant x \leqslant \frac{3\pi}{2}$



SOLVING TRIGONOMETRIC EQUATIONS USING TECHNOLOGY

Trigonometric equations may be solved *numerically* using either a **graphing package** or a **graphics calculator**. In most cases the answers will not be exact, but rather a decimal approximation.

When using a graphics calculator, make sure that the **mode** is set to **radians**.

Example 11

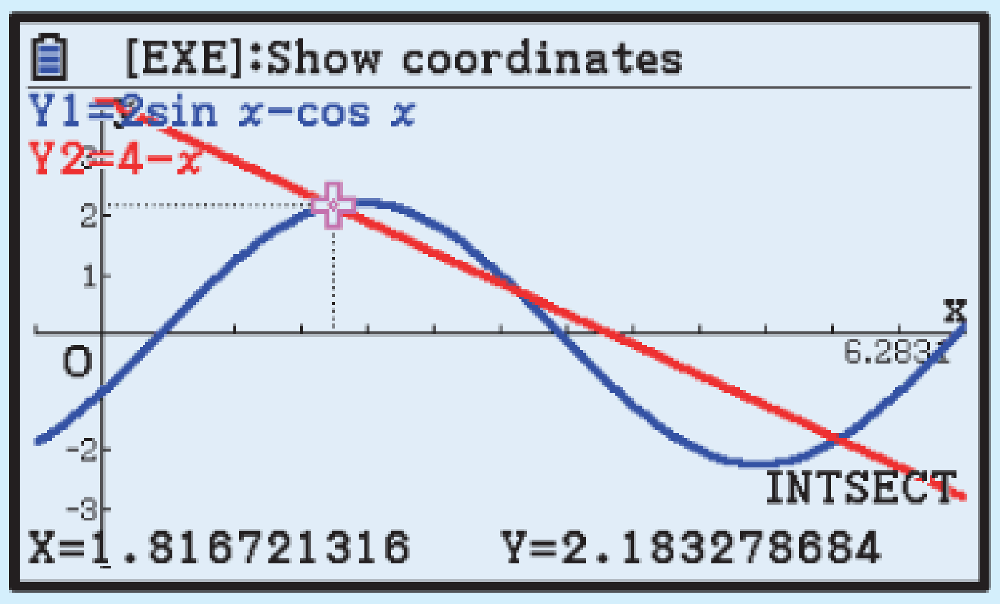
Solve $2\sin x - \cos x = 4 - x$ for $0 \leqslant x \leqslant 2\pi$.

We graph the functions $\mathrm{Y}_1 = 2\sin \mathrm{X} - \cos \mathrm{X}$ and $\mathrm{Y}_2 = 4 - \mathrm{X}$ on the same set of axes.

We use **window** settings just larger than the domain:

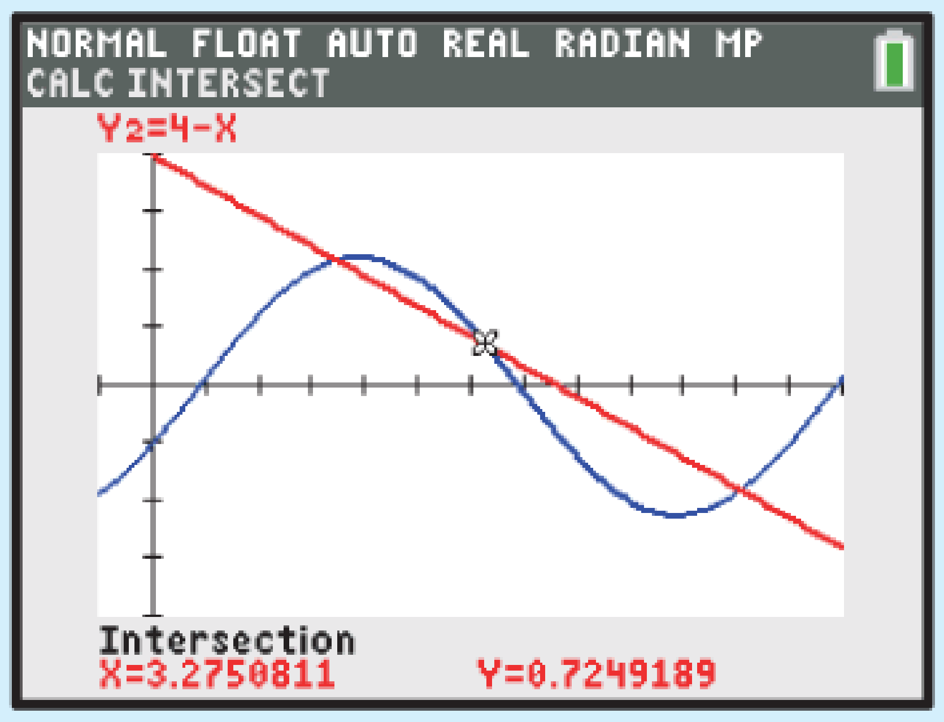
$$X\text{min} = -\tfrac{\pi}{6} \quad X\text{max} = \tfrac{13\pi}{6} \quad X\text{scale} = \tfrac{\pi}{6}$$

Casio fx-CG50



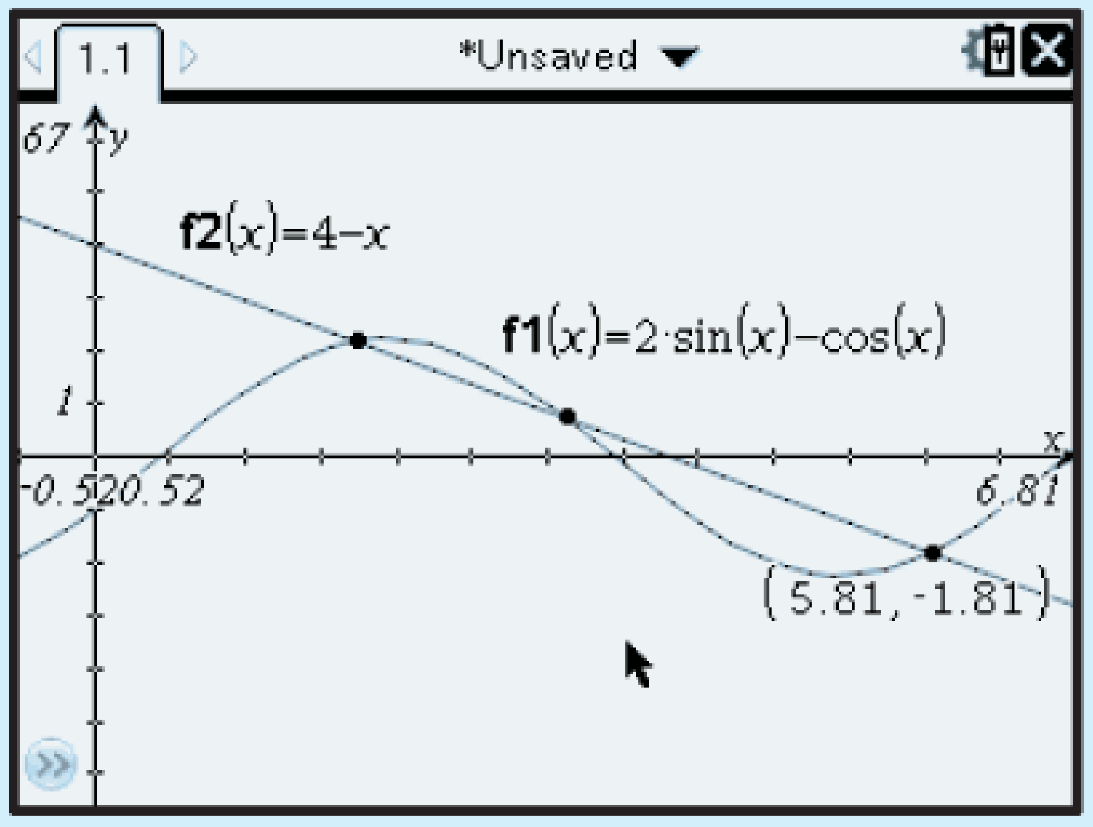
```
[EXE]:Show coordinates
Y1=2sin x−cos x
Y2=4−x
INTSECT
X=1.816721316    Y=2.183278684
```

TI-84 Plus CE



```
NORMAL FLOAT AUTO REAL RADIAN MP
CALC INTERSECT
Y2=4-X
Intersection
X=3.2750811      Y=0.7249189
```

TI-nspire



The solutions are $x \approx 1.82$, 3.28, and 5.81 .

EXERCISE 17G.2

1 Solve for x on the domain $0 < x < 12$:

a $\sin x = 0.431$ **b** $\cos x = -0.814$ **c** $3\tan x - 2 = 0$

2 Solve for x on the domain $-5 \leqslant x \leqslant 5$:

a $5\cos x - 4 = 0$ **b** $2\tan x + 13 = 0$ **c** $8\sin x + 3 = 0$

3 Solve for $0 \leqslant x \leqslant 2\pi$:

a $\sin(x + 2) = 0.0652$ **b** $\sin^2 x + \sin x - 1 = 0$

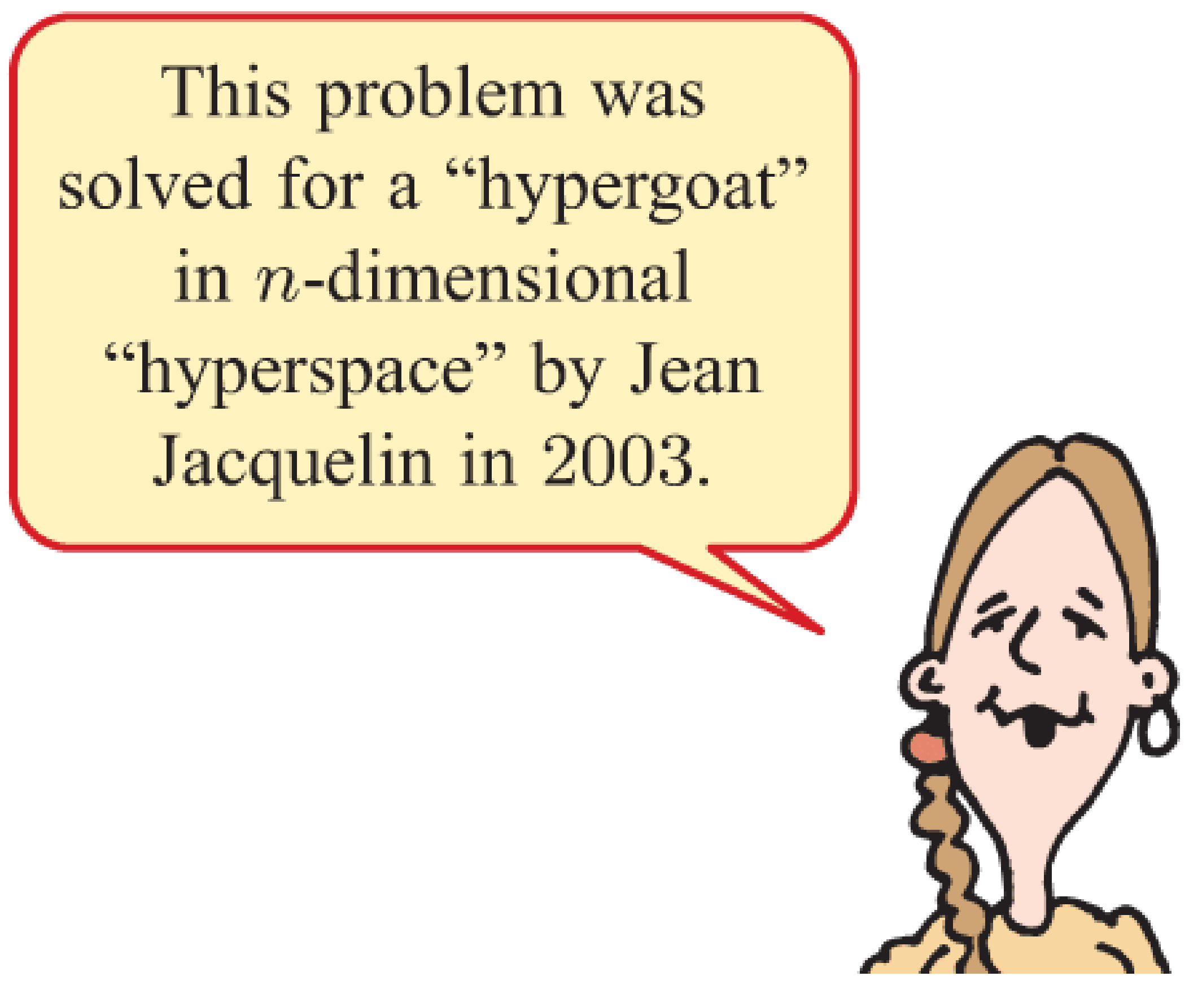
4 Solve for x: $\cos(x - 1) + \sin(x + 1) = 6x + 5x^2 - x^3$ for $-2 \leqslant x \leqslant 6$.

5 A goat is tethered by a rope to the edge of a circular grass field. The ratio of the rope length to the radius of the field is x, where $0 < x < 2$.

a Write a function $P(x)$ for the proportion of the field which the goat can graze.

b Sketch $P(x)$.

c Find, to 3 decimal places, the value of x such that the goat can graze exactly half of the field.



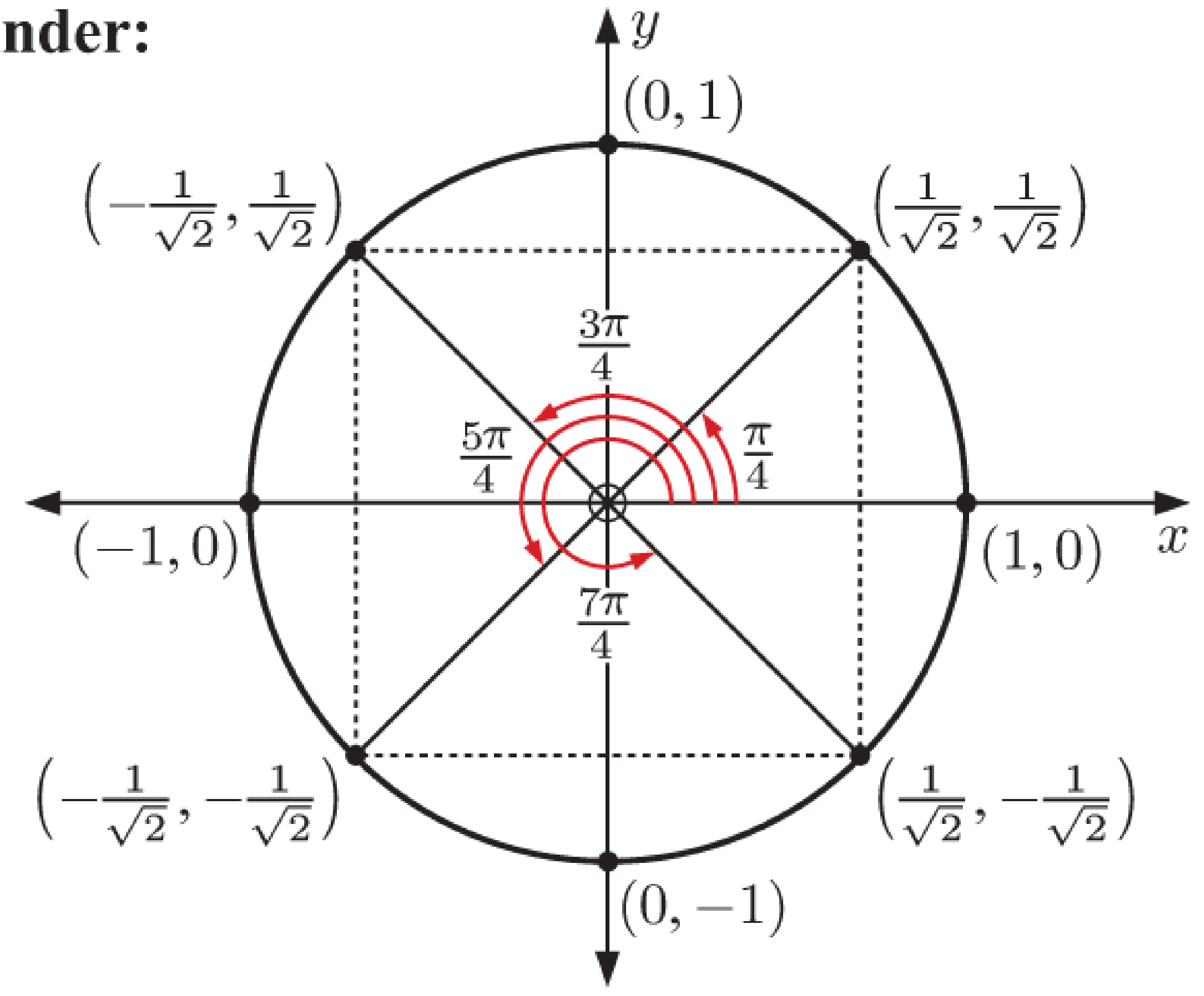
SOLVING TRIGONOMETRIC EQUATIONS USING ALGEBRA

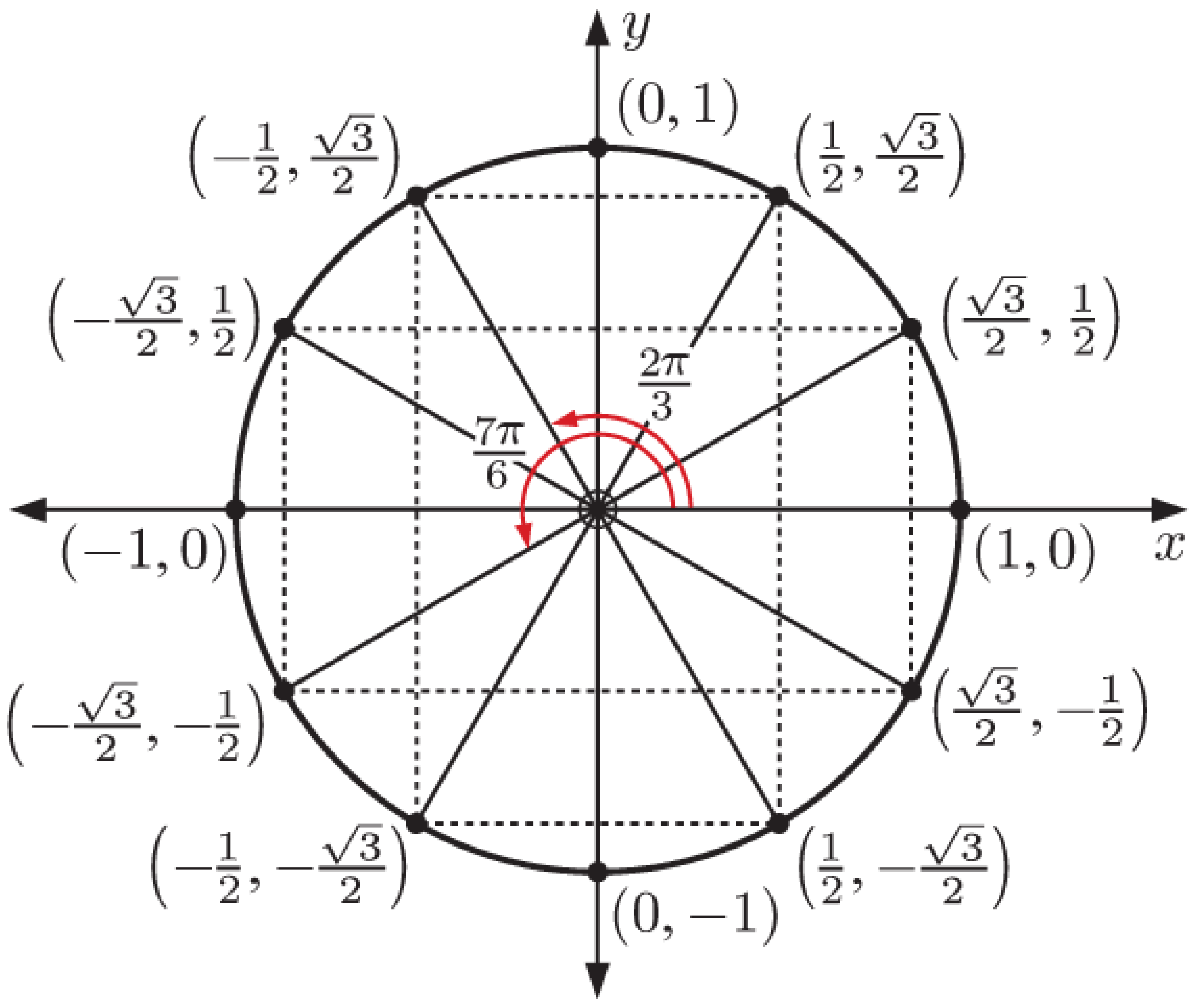
Exact solutions obtained using algebra are called **analytic** solutions. We can find analytic solutions to *some* trigonometric equations, but only if they correspond to angles for which the trigonometric ratios can be expressed exactly.

We use the *periodicity* of the trigonometric functions to give us all solutions in the required domain. Remember that $\sin x$ and $\cos x$ both have period 2π, and $\tan x$ has period π.

For an equation such as $\sin 2x = \frac{1}{2}$ on the domain $0 \leqslant x \leqslant 2\pi$, we need to understand that if $0 \leqslant x \leqslant 2\pi$ then $0 \leqslant 2x \leqslant 4\pi$. So, when we consider points on the unit circle with sine $\frac{1}{2}$, we need to consider angles from 0 to 4π.

Reminder:



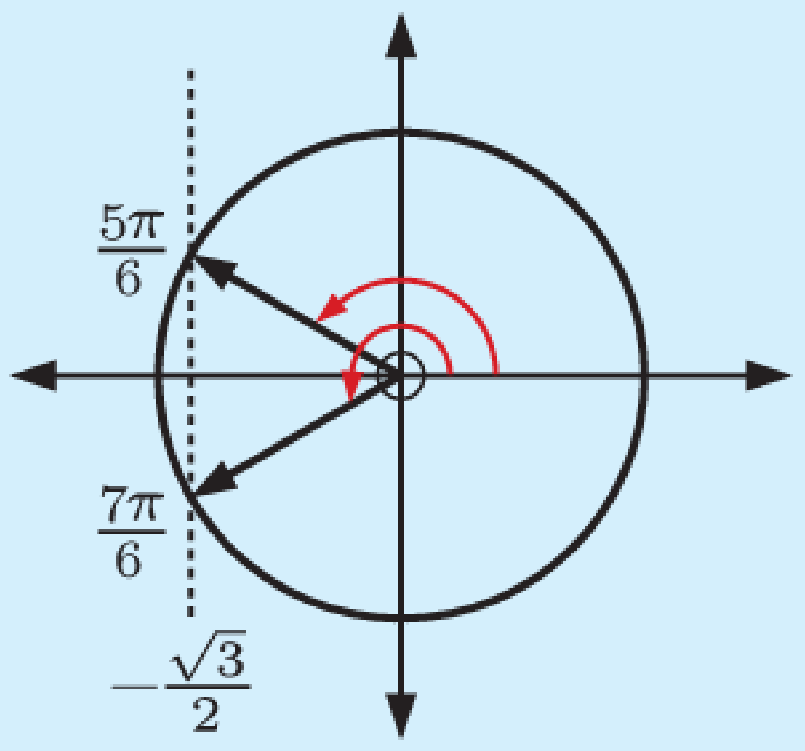


Example 12 **Self Tutor**

Solve for x on the domain $0 \leqslant x \leqslant 2\pi$:

a $\cos x = -\frac{\sqrt{3}}{2}$ **b** $2\sin x - 1 = 0$ **c** $\tan x + \sqrt{3} = 0$

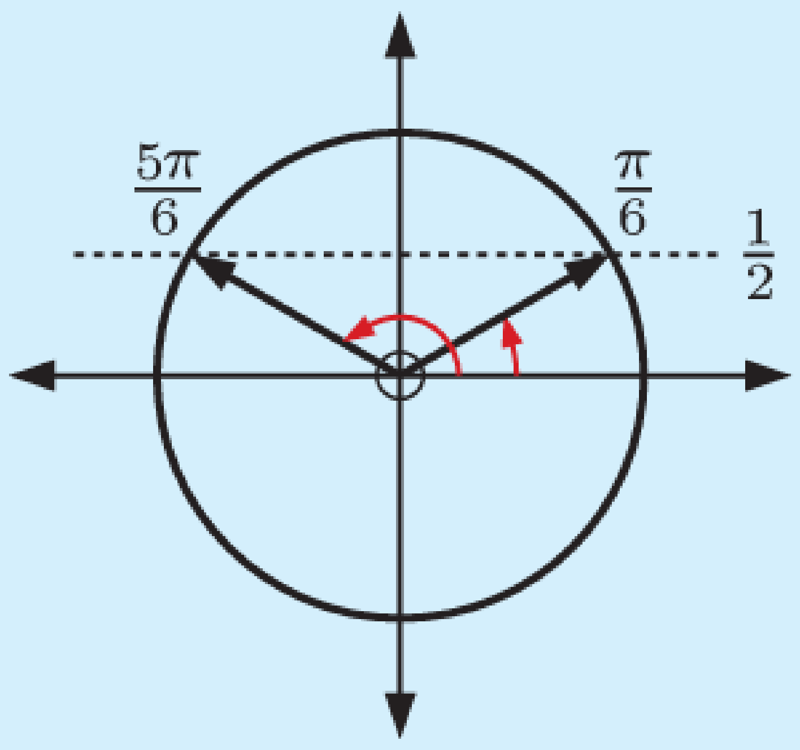
a $\cos x = -\frac{\sqrt{3}}{2}$



$\therefore \quad x = \frac{5\pi}{6}$ or $\frac{7\pi}{6}$

b $2\sin x - 1 = 0$

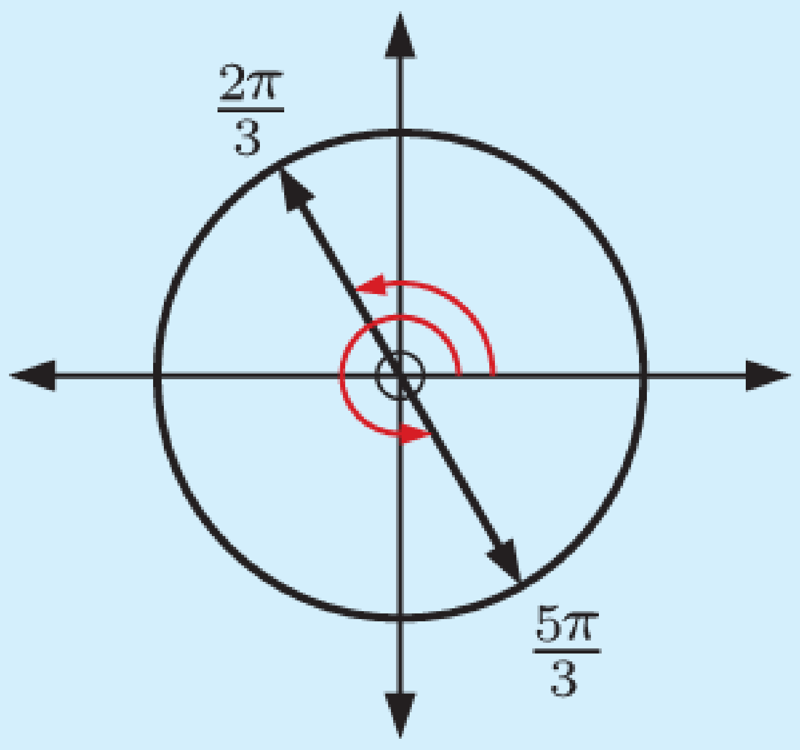
$\therefore \quad \sin x = \frac{1}{2}$



$\therefore \quad x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$

c $\tan x + \sqrt{3} = 0$

$\therefore \quad \tan x = -\sqrt{3}$



$\therefore \quad x = \frac{2\pi}{3}$ or $\frac{5\pi}{3}$

EXERCISE 17G.3

1 Solve for x on the domain $0 \leqslant x \leqslant 2\pi$:

a $\cos x = \frac{1}{2}$ **b** $\sin x = -\frac{1}{\sqrt{2}}$ **c** $\tan x = \frac{1}{\sqrt{3}}$

d $\sin x = -1$ **e** $\cos x = 0$ **f** $\tan x = 0$

2 Solve for x on the domain $0 \leqslant x \leqslant 2\pi$:

a $2\sin x = \sqrt{3}$ **b** $3\cos x + 3 = 0$ **c** $2\tan x - 2 = 0$

3 Solve for x on the domain $0 \leqslant x \leqslant 4\pi$:

a $2\cos x + 1 = 0$ **b** $\sqrt{2}\sin x = 1$ **c** $\tan x = 1$

4 Solve for x on the domain $-2\pi \leqslant x \leqslant 2\pi$:

a $2\sin x + \sqrt{3} = 0$ **b** $\sqrt{2}\cos x + 1 = 0$ **c** $\tan x = -1$

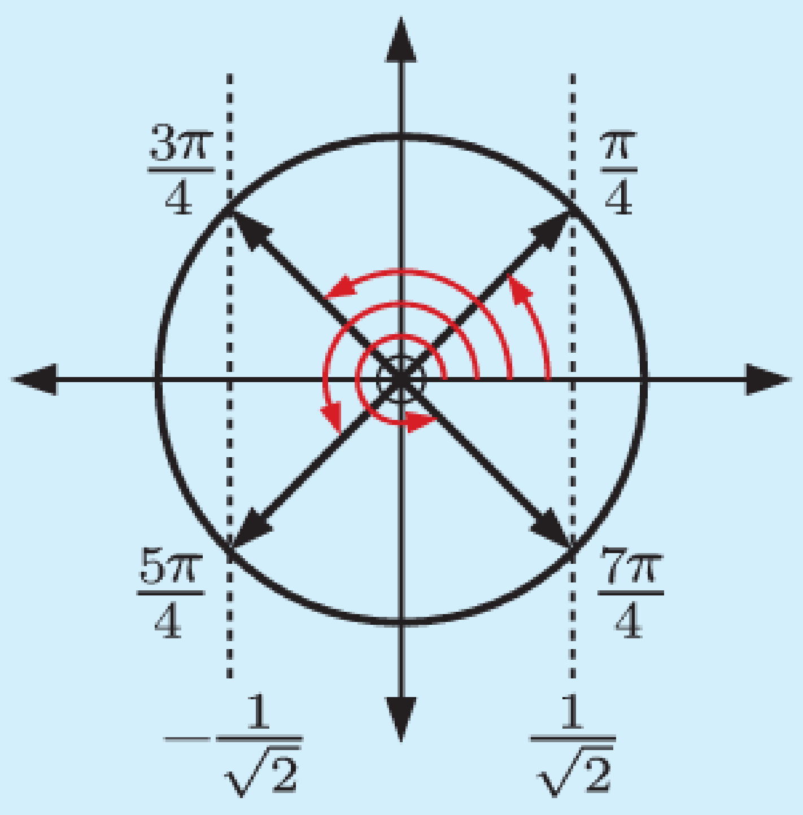
Example 13 **Self Tutor**

Solve $\cos^2 x = \frac{1}{2}$ on $0 \leqslant x \leqslant 2\pi$.

$\cos^2 x = \frac{1}{2}$

$\therefore \cos x = \pm\frac{1}{\sqrt{2}}$

$\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$



5 Solve for x on $0 \leqslant x \leqslant 2\pi$:

a $\cos^2 x = \frac{3}{4}$ **b** $\sin^2 x = 1$ **c** $\tan^2 x = 3$

Example 14 **Self Tutor**

Solve exactly for $0 \leqslant x \leqslant 3\pi$:

a $\sin x = -\frac{1}{2}$ **b** $\sin 2x = -\frac{1}{2}$ **c** $\sin\left(x - \frac{\pi}{6}\right) = -\frac{1}{2}$

The three equations all have the form $\sin\theta = -\frac{1}{2}$.

a $0 \leqslant x \leqslant 3\pi$

$\therefore x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

b In this case θ is $2x$.

If $0 \leqslant x \leqslant 3\pi$ then $0 \leqslant 2x \leqslant 6\pi$.

$\therefore 2x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \frac{31\pi}{6}$, or $\frac{35\pi}{6}$

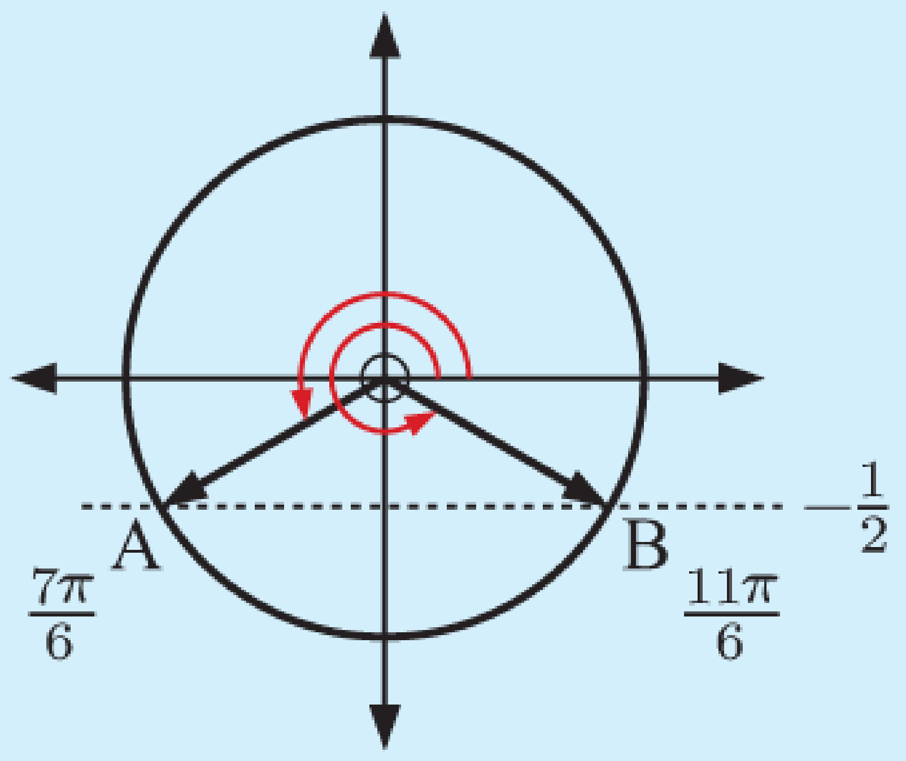
$\therefore x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}, \frac{31\pi}{12}$, or $\frac{35\pi}{12}$

c In this case θ is $x - \frac{\pi}{6}$.

If $0 \leqslant x \leqslant 3\pi$ then $-\frac{\pi}{6} \leqslant x - \frac{\pi}{6} \leqslant \frac{17\pi}{6}$.

$\therefore x - \frac{\pi}{6} = -\frac{\pi}{6}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$

$\therefore x = 0, \frac{4\pi}{3}$, or 2π



Start at $-\frac{\pi}{6}$ and work around to $\frac{17\pi}{6}$, recording the angle every time you reach points A and B.

6 If $0 \leqslant x \leqslant 2\pi$, state the domain of:

a $2x$ **b** $\dfrac{x}{4}$ **c** $x + \frac{\pi}{2}$ **d** $x - \frac{\pi}{6}$ **e** $2\left(x - \frac{\pi}{4}\right)$ **f** $-x$

7 If $-\pi \leqslant x \leqslant \pi$, state the domain of:

a $3x$ **b** $\dfrac{x}{4}$ **c** $x - \frac{\pi}{2}$ **d** $2x + \frac{\pi}{2}$ **e** $-2x$ **f** $\pi - x$

8 Solve exactly for $0 \leqslant x \leqslant 3\pi$:

a $\cos x = \frac{1}{2}$ **b** $\cos 2x = \frac{1}{2}$ **c** $\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$

9 Solve for x on $0 \leqslant x \leqslant 2\pi$:

a $\sin 2x = -\frac{1}{2}$ **b** $\cos 3x = \frac{\sqrt{3}}{2}$ **c** $\tan 2x - \sqrt{3} = 0$

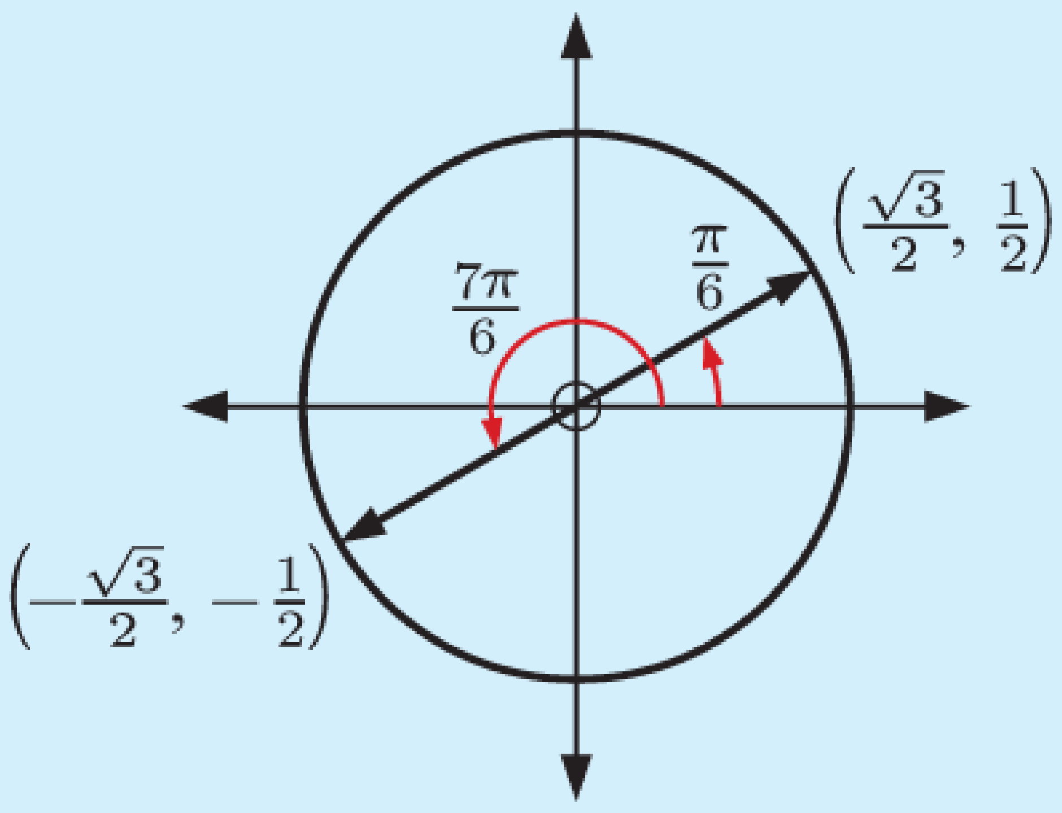
d $\sin \frac{x}{2} = \frac{1}{\sqrt{2}}$ **e** $2\cos \frac{x}{2} + 1 = 0$ **f** $3\tan \frac{x}{3} - 3 = 0$

10 Solve for x on $0 \leqslant x \leqslant 2\pi$:

a $\cos^2 3x = \frac{1}{4}$ **b** $\sin^2 2x = 1$ **c** $\tan^2\left(\frac{x}{2}\right) = \frac{1}{3}$

Example 15 **Self Tutor**

Find the exact solutions of $\sqrt{3}\sin x = \cos x$ for $0 \leqslant x \leqslant 2\pi$.



$$\sqrt{3}\sin x = \cos x$$

$$\therefore \quad \frac{\sin x}{\cos x} = \frac{1}{\sqrt{3}} \qquad \{\text{dividing both sides by } \sqrt{3}\cos x\}$$

$$\therefore \quad \tan x = \frac{1}{\sqrt{3}}$$

$$\therefore \quad x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6}$$

11 Find the exact solutions for $0 \leqslant x \leqslant 2\pi$:

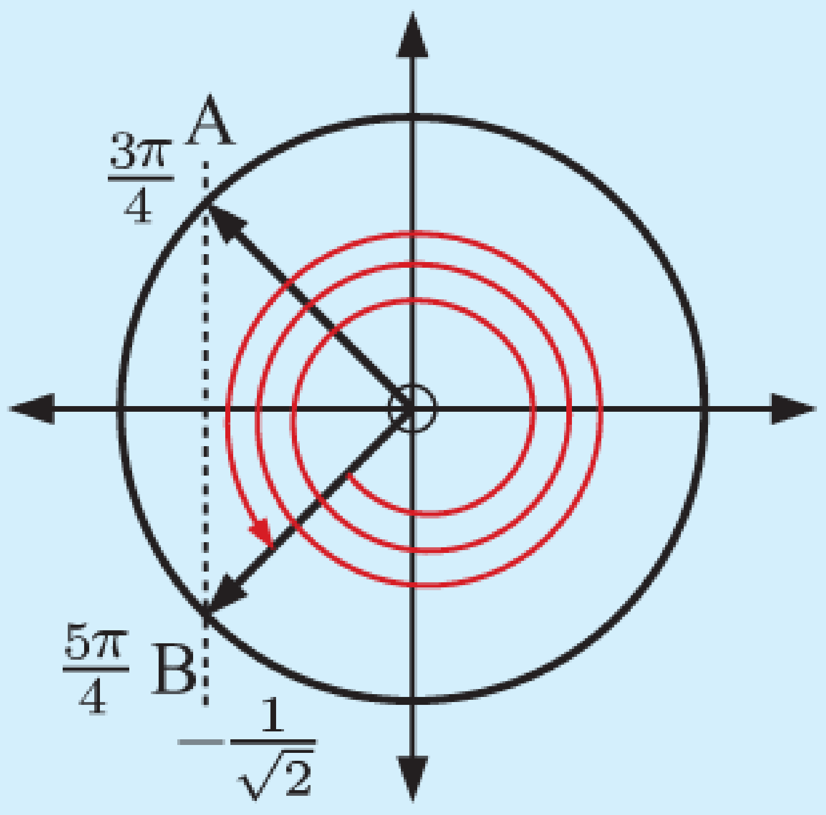
a $\sin x = -\cos x$ **b** $\sin 3x = \cos 3x$ **c** $\sin 2x = \sqrt{3}\cos 2x$

Example 16 **Self Tutor**

Solve $\sqrt{2}\cos\left(x - \frac{3\pi}{4}\right) + 1 = 0$ for $0 \leqslant x \leqslant 6\pi$.

$$\sqrt{2}\cos\left(x - \frac{3\pi}{4}\right) + 1 = 0$$

$$\therefore \quad \cos\left(x - \frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}.$$



Start at $-\frac{3\pi}{4}$ and work around to $\frac{21\pi}{4}$, recording the angle every time you reach points A and B.

Since $0 \leqslant x \leqslant 6\pi$,

$$-\frac{3\pi}{4} \leqslant x - \frac{3\pi}{4} \leqslant \frac{21\pi}{4}$$

So, $x - \frac{3\pi}{4} = -\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{11\pi}{4}, \frac{13\pi}{4}, \frac{19\pi}{4}$, or $\frac{21\pi}{4}$

$$\therefore \quad x = 0, \frac{3\pi}{2}, 2\pi, \frac{7\pi}{2}, 4\pi, \frac{11\pi}{2}, \text{ or } 6\pi$$

12 Solve exactly:

a $\cos\left(x - \frac{2\pi}{3}\right) = \frac{1}{2}, \quad -2\pi \leqslant x \leqslant 2\pi$

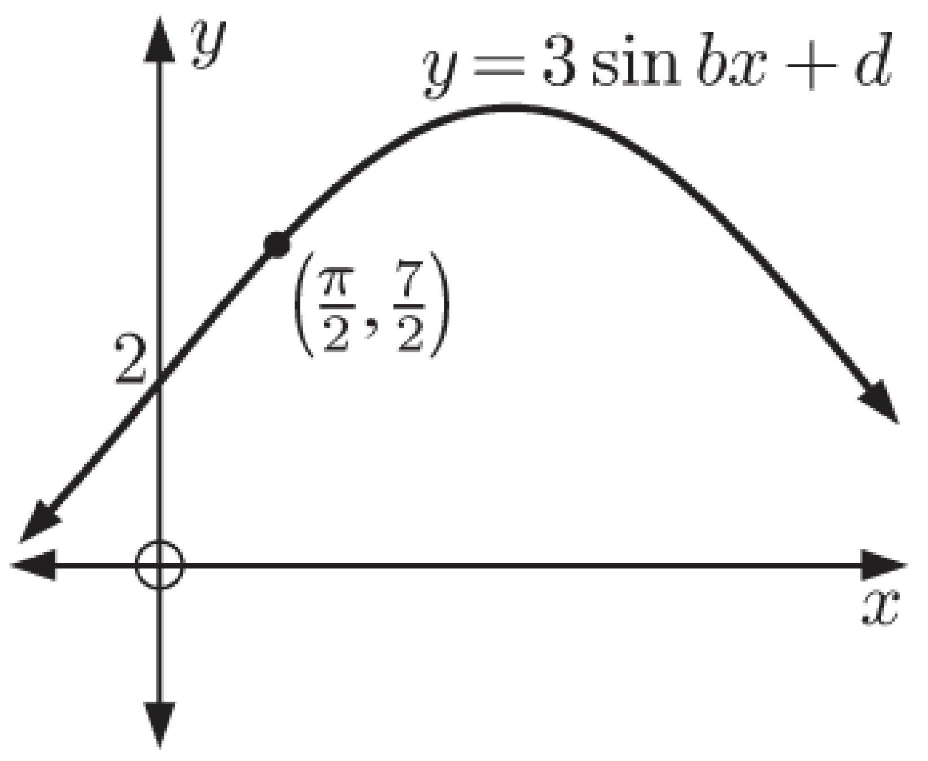
b $\sqrt{2}\sin\left(x - \frac{\pi}{4}\right) + 1 = 0, \quad 0 \leqslant x \leqslant 3\pi$

c $\sin\left(4\left(x - \frac{\pi}{4}\right)\right) = 0, \quad 0 \leqslant x \leqslant \pi$

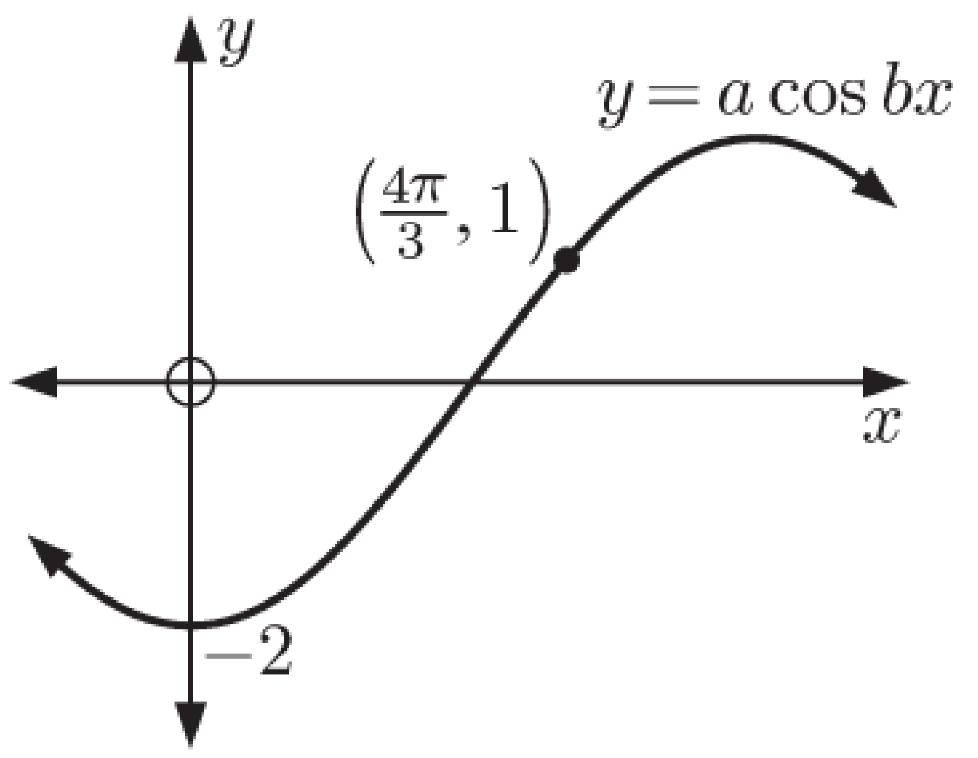
d $2\sin\left(2\left(x - \frac{\pi}{3}\right)\right) = -\sqrt{3}, \quad 0 \leqslant x \leqslant 2\pi$

13 Find the unknowns in each function:

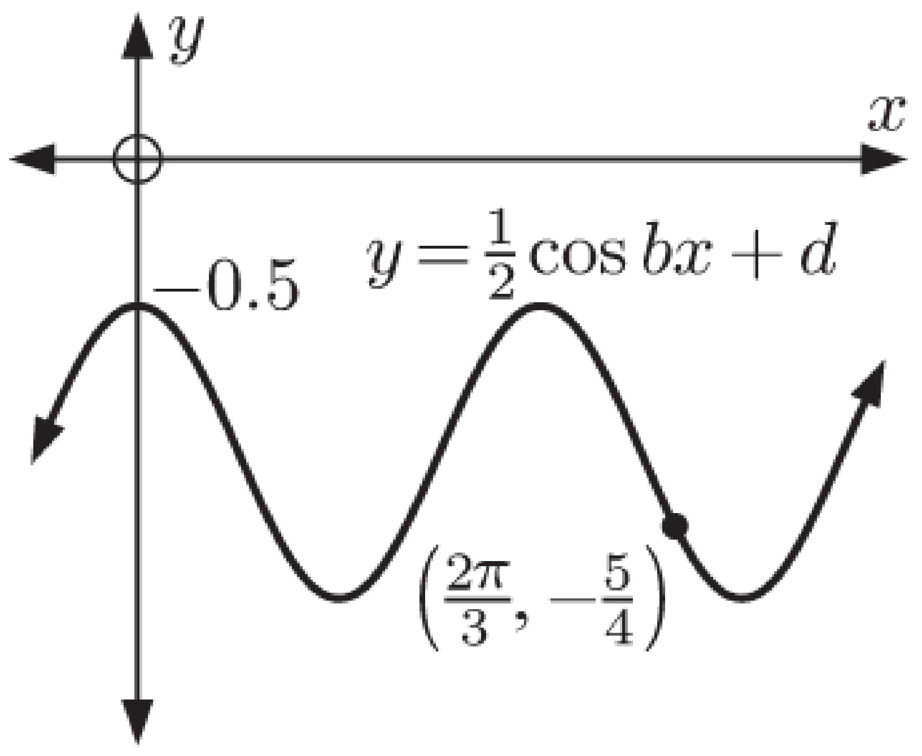
a



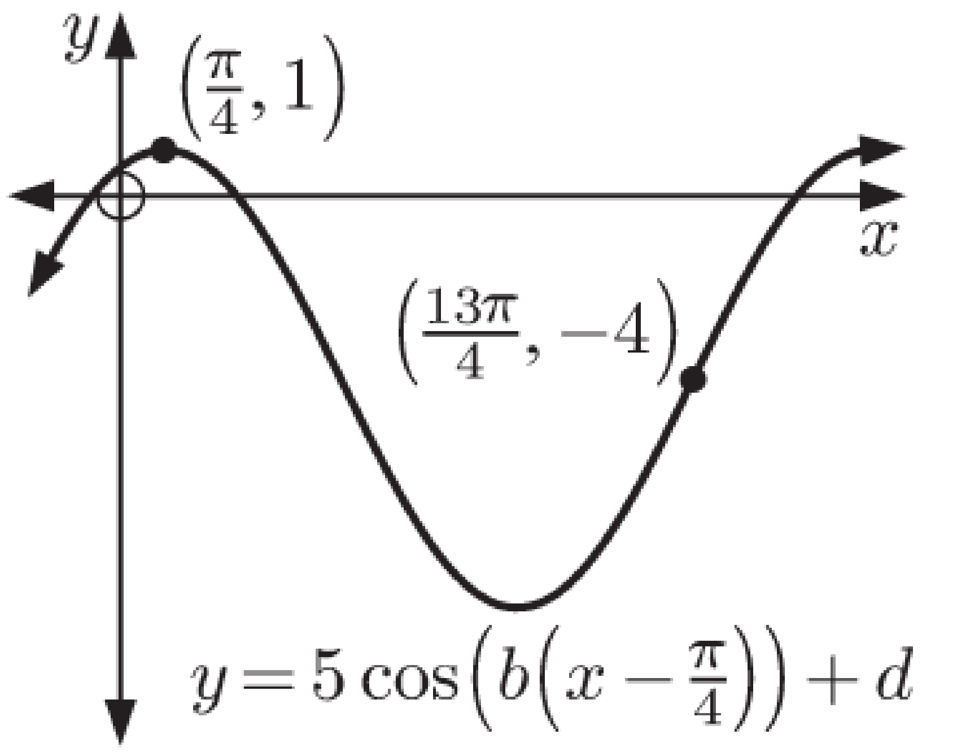
b



c



d



14 Find the exact solutions of $\tan x = \sqrt{3}$ for $0 \leqslant x \leqslant 2\pi$. Hence solve the following equations for $0 \leqslant x \leqslant 2\pi$:

a $\tan\left(x - \frac{\pi}{6}\right) = \sqrt{3}$

b $\tan 4x = \sqrt{3}$

c $\tan^2 x = 3$

Example 17 **Self Tutor**

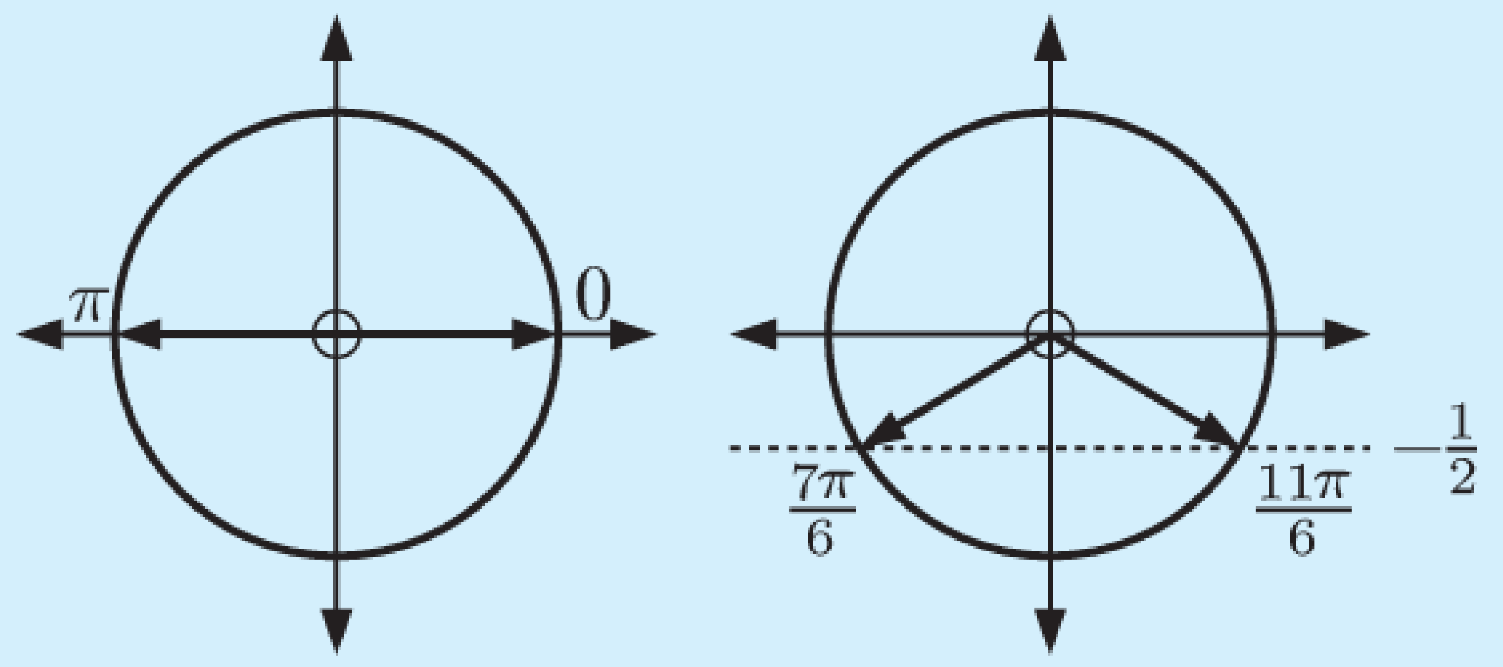
Solve for x on $0 \leqslant x \leqslant 2\pi$, giving your answers as exact values:

a $2\sin^2 x + \sin x = 0$

b $2\cos^2 x + \cos x - 1 = 0$

a

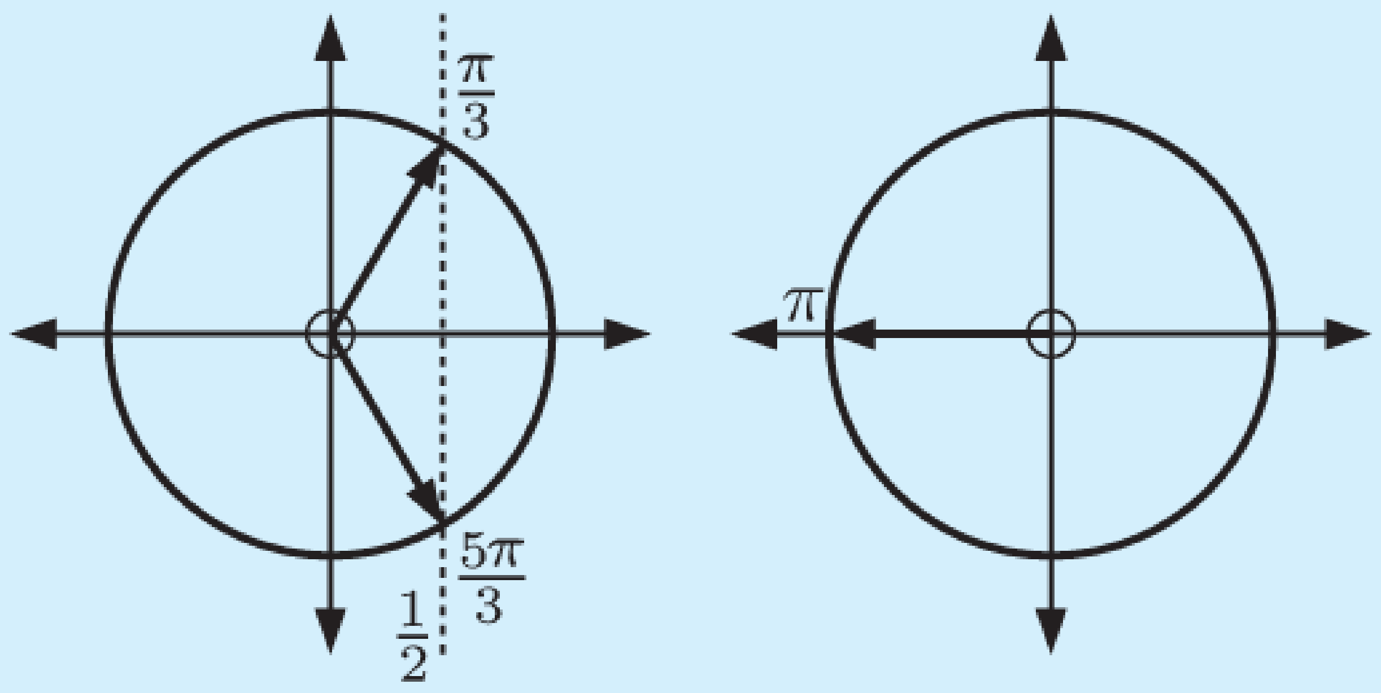
$$2\sin^2 x + \sin x = 0$$
$$\therefore \sin x(2\sin x + 1) = 0$$
$$\therefore \sin x = 0 \text{ or } -\tfrac{1}{2}$$



$\therefore \quad x = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, \text{ or } 2\pi.$

b

$$2\cos^2 x + \cos x - 1 = 0$$
$$\therefore (2\cos x - 1)(\cos x + 1) = 0$$
$$\therefore \cos x = \tfrac{1}{2} \text{ or } -1$$



$\therefore \quad x = \frac{\pi}{3}, \pi, \text{ or } \frac{5\pi}{3}.$

15 Solve for $0 \leqslant x \leqslant 2\pi$ giving your answers as exact values:

a $2\sin^2 x - \sin x = 0$

b $2\cos^2 x = \cos x$

c $2\cos^2 x - \cos x - 1 = 0$

d $2\sin^2 x + 3\sin x + 1 = 0$

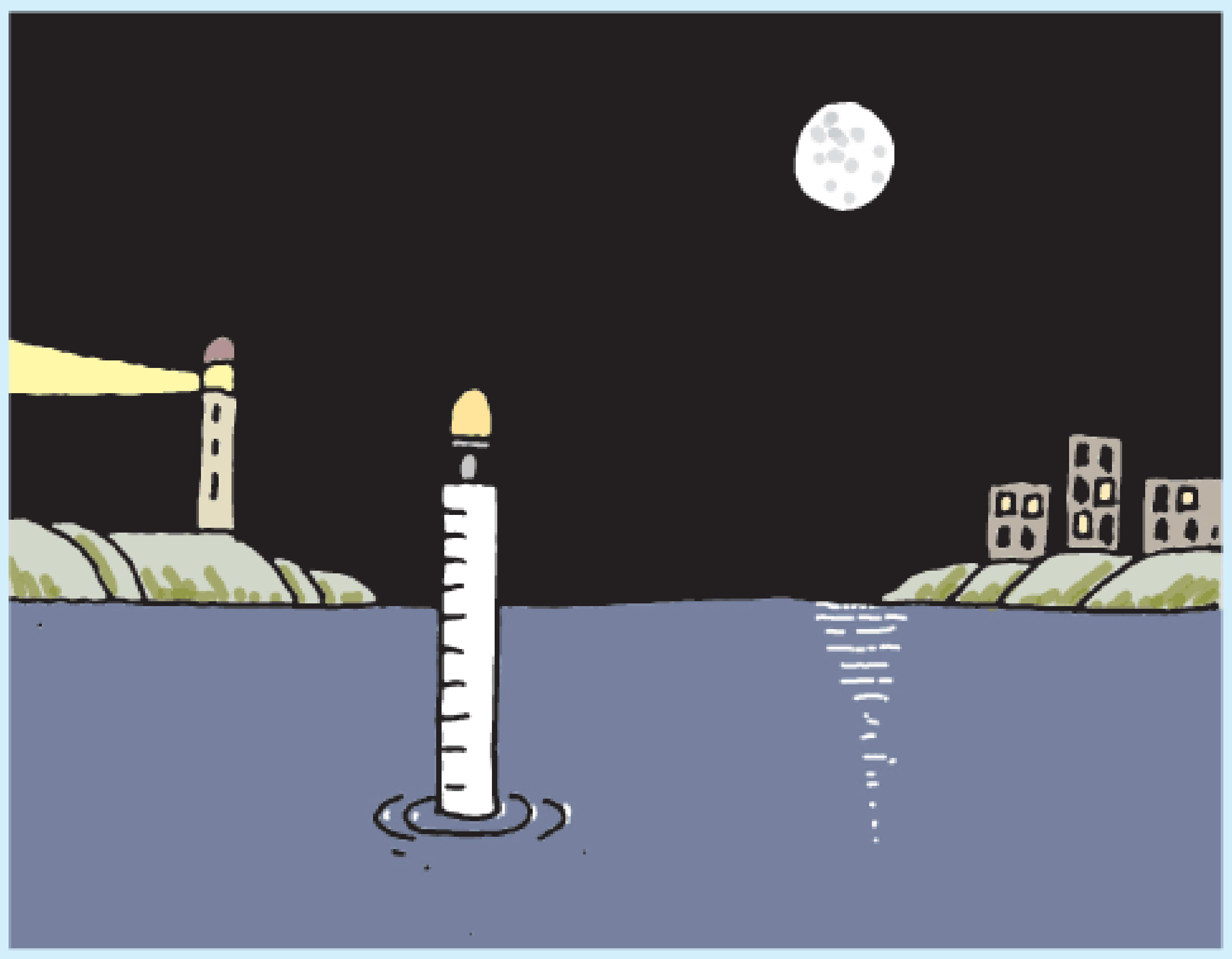
In **a** we cannot simply divide through by $\sin x$, or we will lose the solutions corresponding to $\sin x = 0$.

H USING TRIGONOMETRIC MODELS

Having studied trigonometric equations, we can now apply them to the trigonometric models studied in **Section D**.

Example 18 Self Tutor

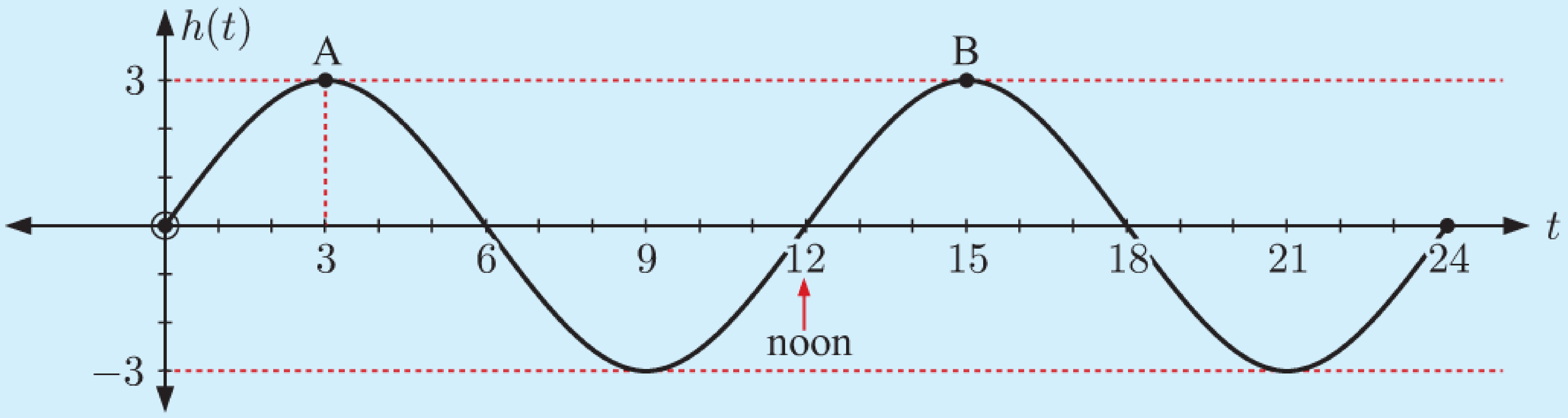
The height of the tide above mean sea level on January 24th at Cape Town is modelled by $h(t) = 3\sin\frac{\pi t}{6}$ metres, where t is the number of hours after midnight.



- **a** Graph $y = h(t)$ for $0 \leqslant t \leqslant 24$.
- **b** When is high tide and what is the maximum height?
- **c** What is the height of the tide at 2 pm?
- **d** A ship can cross the harbour provided the tide is at least 2 m above mean sea level. When is crossing possible on January 24th?

a $h(0) = 0$

$h(t) = 3\sin\frac{\pi t}{6}$ has period $= \dfrac{2\pi}{\frac{\pi}{6}} = 2\pi \times \frac{6}{\pi} = 12$ hours

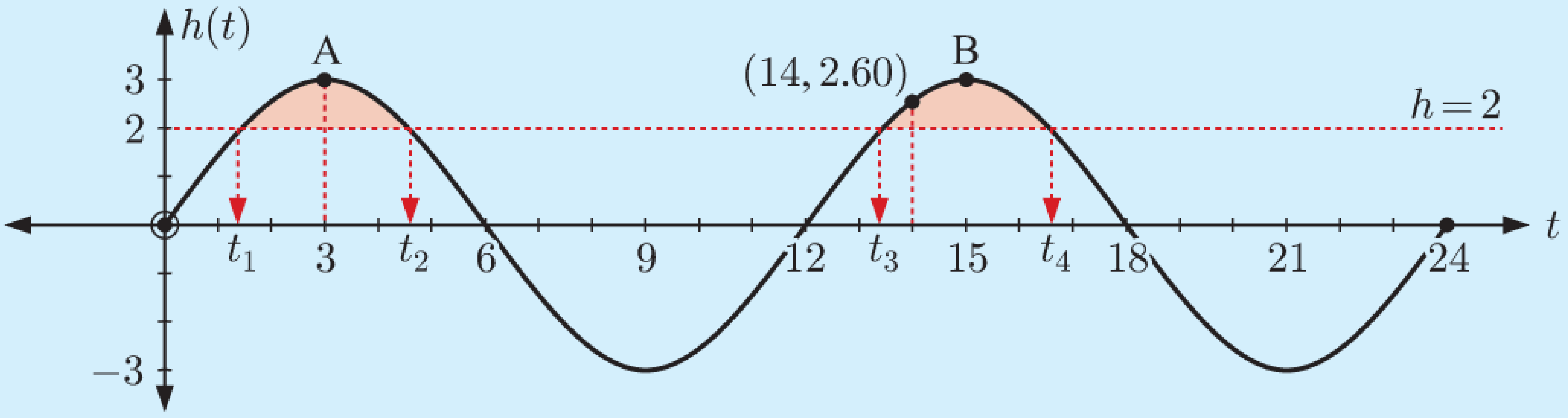


b High tide is at 3 am and 3 pm. The maximum height is 3 m above the mean as seen at points A and B.

c At 2 pm, $t = 14$ and $h(14) = 3\sin\frac{14\pi}{6} \approx 2.60$ m.

So, the tide is 2.6 m above the mean.

d



We need to solve $h(t) = 2$, so $3\sin\frac{\pi t}{6} = 2$.

Using a graphics calculator with $Y_1 = 3\sin\frac{\pi X}{6}$ and $Y_2 = 2$

we obtain $t_1 \approx 1.39$, $t_2 \approx 4.61$, $t_3 \approx 13.39$, $t_4 \approx 16.61$

Now 1.39 hours = 1 hour 23 minutes, and so on.

So, the ship can cross between 1:23 am and 4:37 am or 1:23 pm and 4:37 pm.

EXERCISE 17H

1 The population of grasshoppers after t weeks is $P(t) = 7500 + 3000 \sin \frac{\pi t}{8}$ for $0 \leqslant t \leqslant 12$.

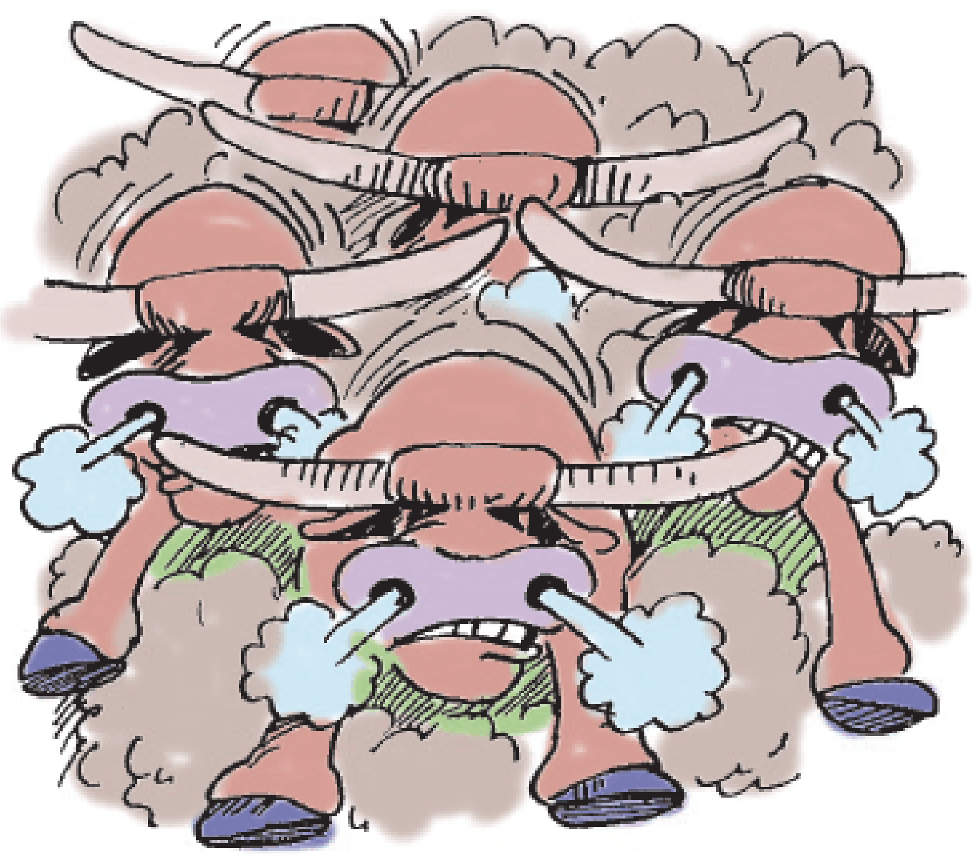
- **a** Find:
  - **i** the initial population
  - **ii** the population after 5 weeks.
- **b** What is the greatest population size over this interval and when does it occur?
- **c** When is the population: **i** 9000 **ii** 6000?
- **d** During what time interval(s) does the population size exceed 10 000?

2 The model for the height of a passenger on a Ferris wheel is $H(t) = 20 - 19 \cos \frac{2\pi t}{3}$, where H is the height in metres above the ground, and t is in minutes.

- **a** Where is the passenger at time $t = 0$?
- **b** At what time is the passenger at the maximum height in the first revolution of the wheel?
- **c** How long does the wheel take to complete one revolution?
- **d** Sketch the graph of the function $H(t)$ over one revolution.
- **e** The passenger can see his friend when he is at least 13 m above the ground. During what times in the first revolution can the passenger see his friend?

3 The population of water buffalo is given by $P(t) = 400 + 250 \sin \frac{\pi t}{2}$ where t is the number of years since the first estimate was made.

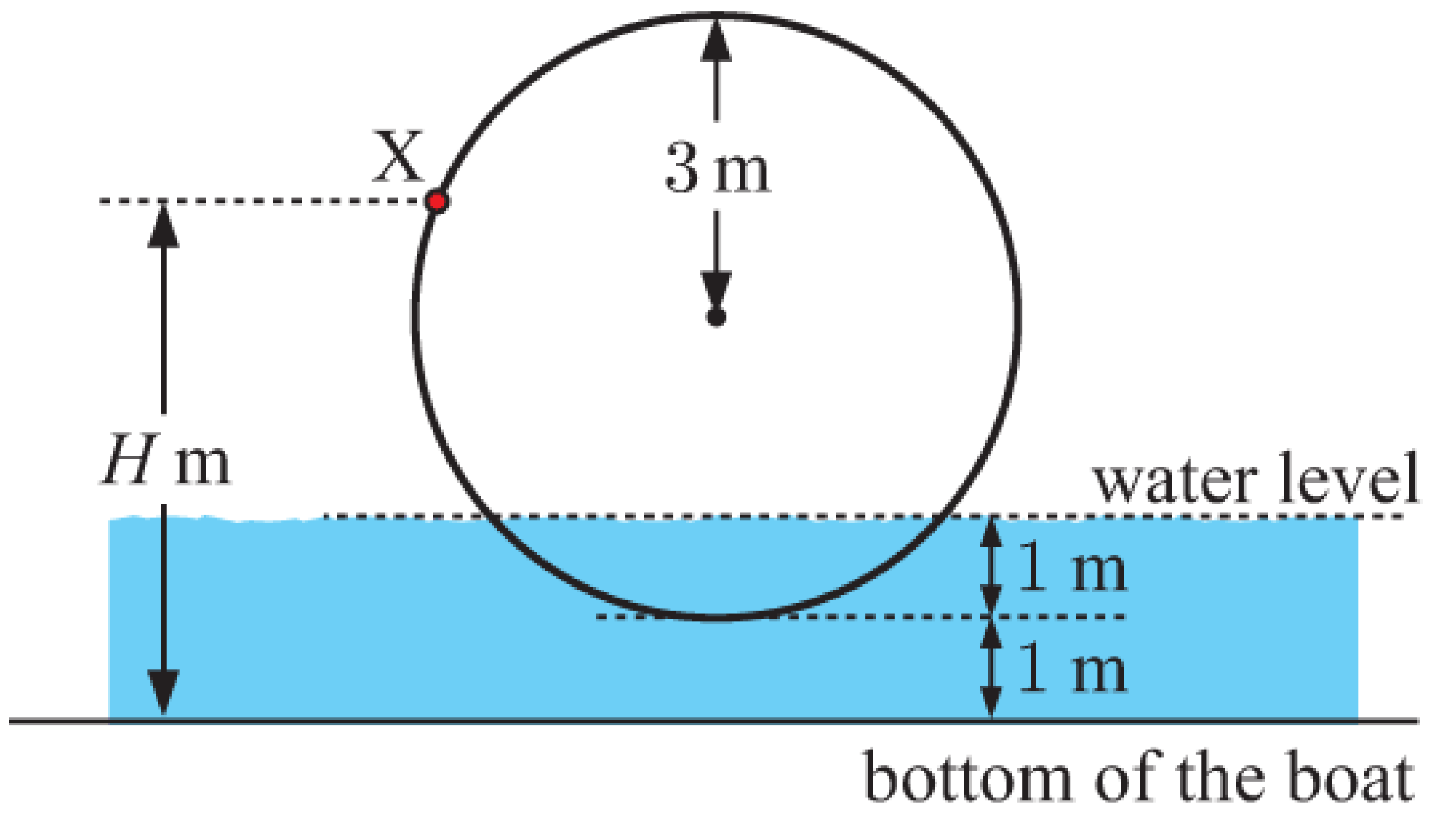
- **a** What was the initial estimate?
- **b** What was the population size after:
  - **i** 6 months
  - **ii** two years?
- **c** Find $P(1)$. What is the significance of this value?
- **d** Find the smallest population size and when it first occurred.
- **e** Find the first time when the herd exceeded 500.



4 Over a 28 day period, the cost per litre of petrol was modelled by $C(t) = 9.2 \sin\left(\frac{\pi}{7}(t-4)\right) + 107.8$ cents L^{-1}.

- **a** True or false?
  - **i** "The cost per litre oscillates about 107.8 cents with maximum price \$1.17 per litre."
  - **ii** "Every 14 days, the cycle repeats itself."
- **b** What was the cost of petrol on day 7, to the nearest tenth of a cent per litre?
- **c** On which days was the petrol priced at \$1.10 per litre?
- **d** What was the minimum cost per litre and when did it occur?

5 A paint spot X lies on the outer rim of the wheel of a paddle-steamer. The wheel has radius 3 m and rotates anticlockwise at a constant rate. X is seen entering the water every 4 seconds.
H is the distance of X above the bottom of the boat. At time $t = 0$, X is at its highest point.

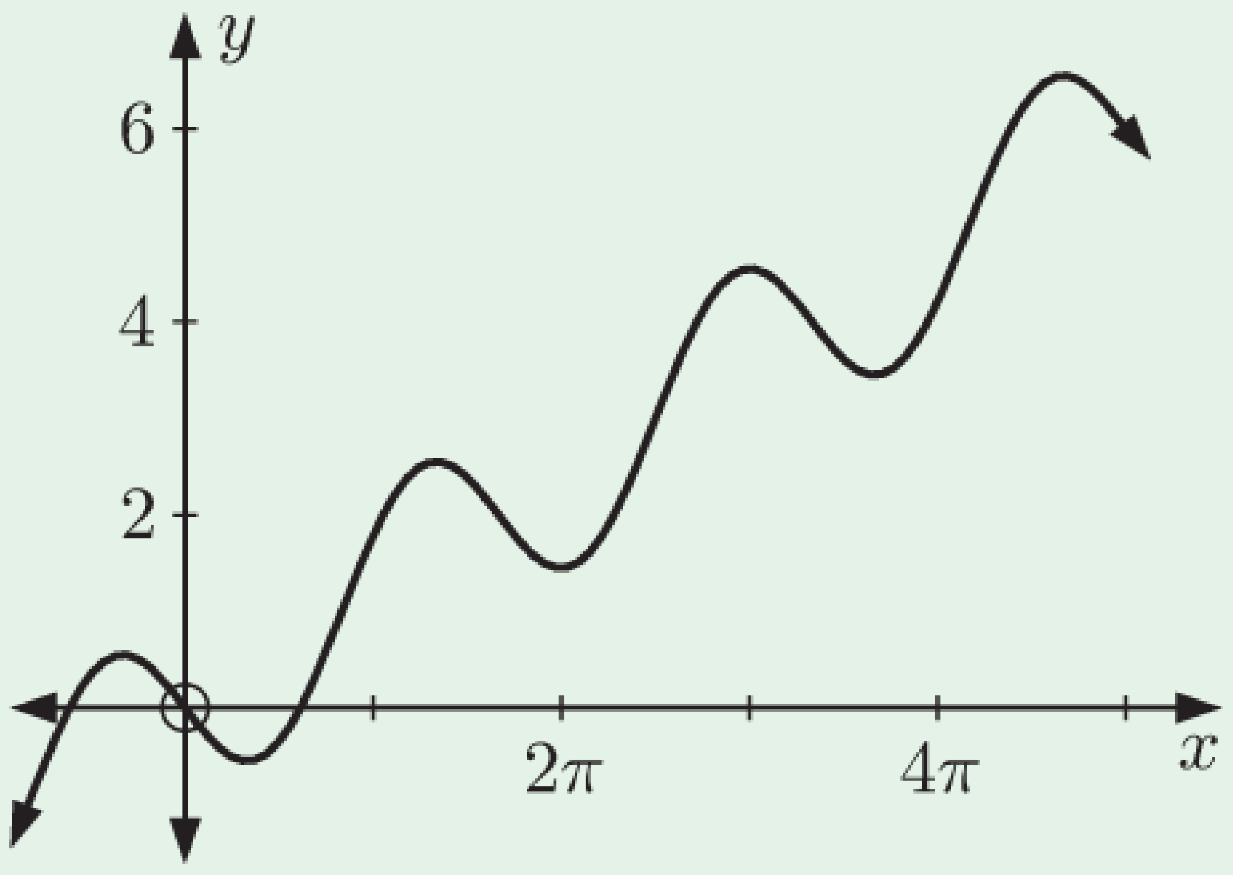


- **a** Find a cosine model for H in the form $H(t) = a \cos(b(t - c)) + d$.
- **b** At what time t does X first enter the water?

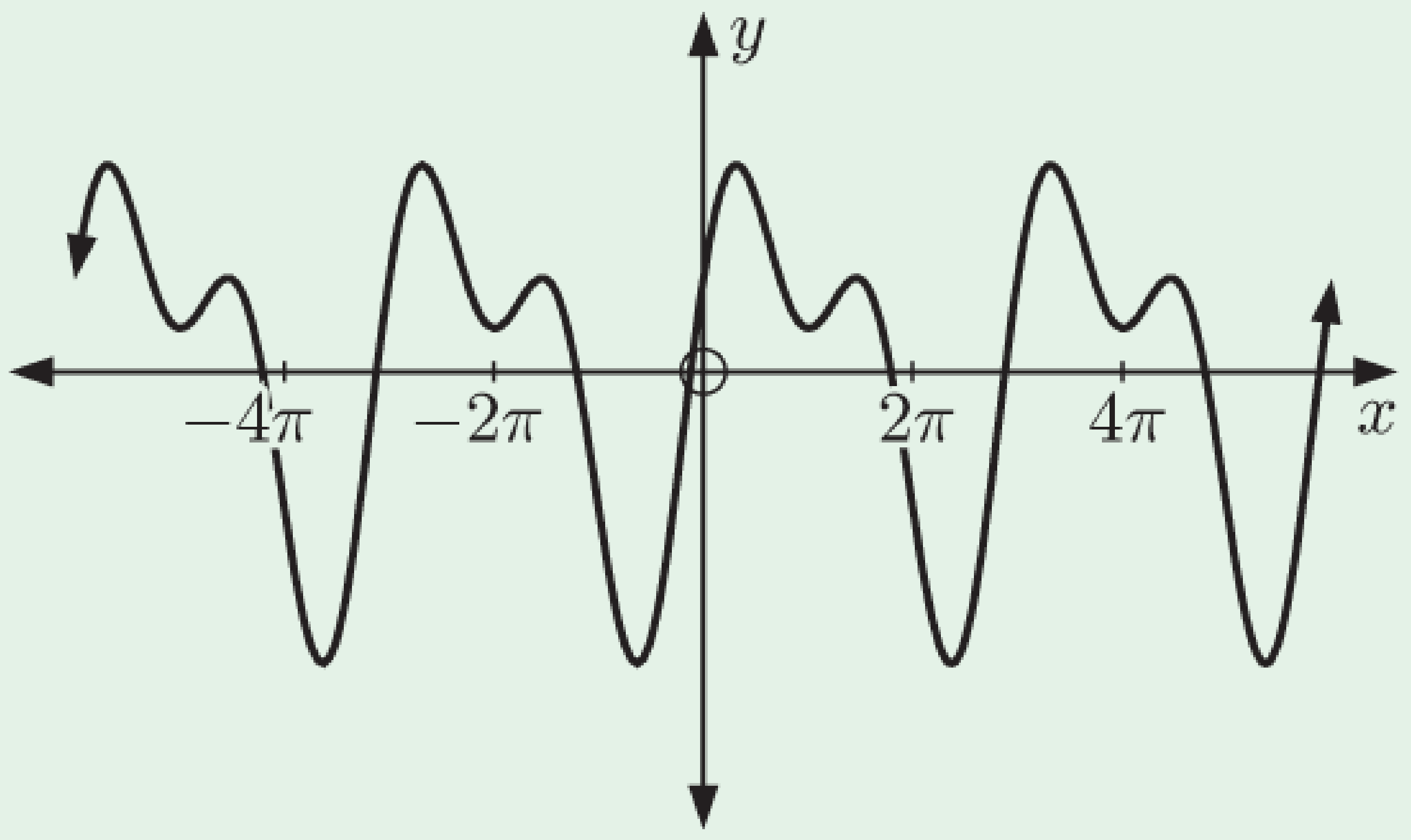
REVIEW SET 17A

1 Which of the following graphs display periodic behaviour?

a



b



2 State the minimum and maximum values of:

a $1+\sin x$

b $-2\cos 3x$

3 State the period of:

a $y=4\sin\frac{x}{5}$

b $y=-2\cos 4x$

c $y=4\cos\frac{x}{2}+4$

d $y=\frac{1}{2}\tan 3x$

4 Copy and complete:

Function	*Period*	*Amplitude*	*Range*
$y=-3\sin\frac{x}{4}+1$			
$y=3\cos\pi x$			

5 **a** Draw the graph of $y=\cos 3x$ for $0\leqslant x\leqslant 2\pi$.

b Find the value of y when $x=\frac{3\pi}{4}$. Mark this point on your graph.

6 Sketch the graphs of the following for $-2\pi\leqslant x\leqslant 2\pi$:

a $y=4\sin x$

b $y=\sin\left(x-\frac{\pi}{3}\right)+2$

c $y=\sin\frac{3x}{2}$

d $y=\cos\left(x+\frac{\pi}{4}\right)$

e $y=\frac{3}{4}\cos x$

f $y=\cos 4x$

7 State the transformations which map:

a $y=\sin x$ onto $y=3\sin 2x$

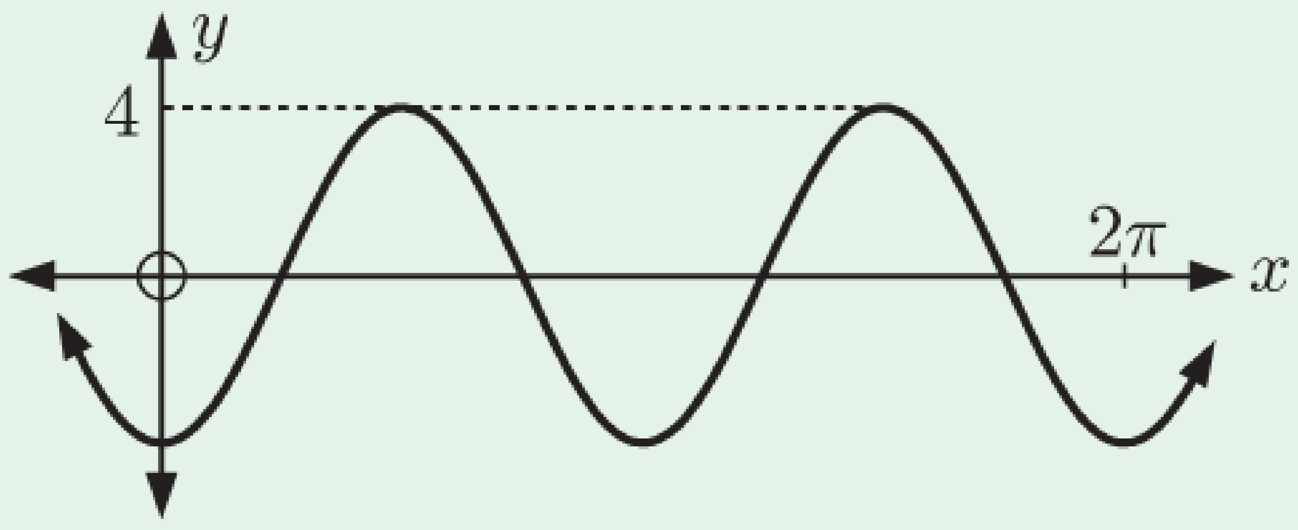
b $y=\cos x$ onto $y=\cos\left(x-\frac{\pi}{3}\right)-1$

c $y=\tan x$ onto $y=-\tan 2x$

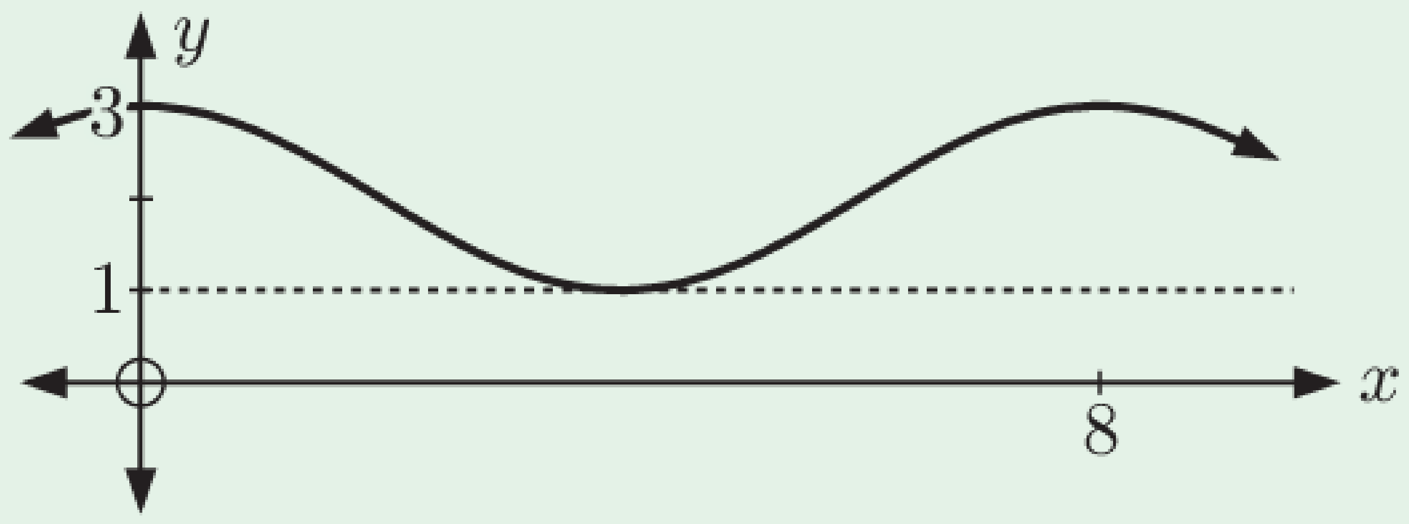
d $y=\sin x$ onto $y=2\sin\left(\frac{x}{2}-\frac{\pi}{4}\right)+\frac{1}{2}$.

8 Find the cosine function represented in each of the following graphs:

a



b



9 Sketch for $0\leqslant x\leqslant 4\pi$:

a $y=\tan\frac{x}{4}$

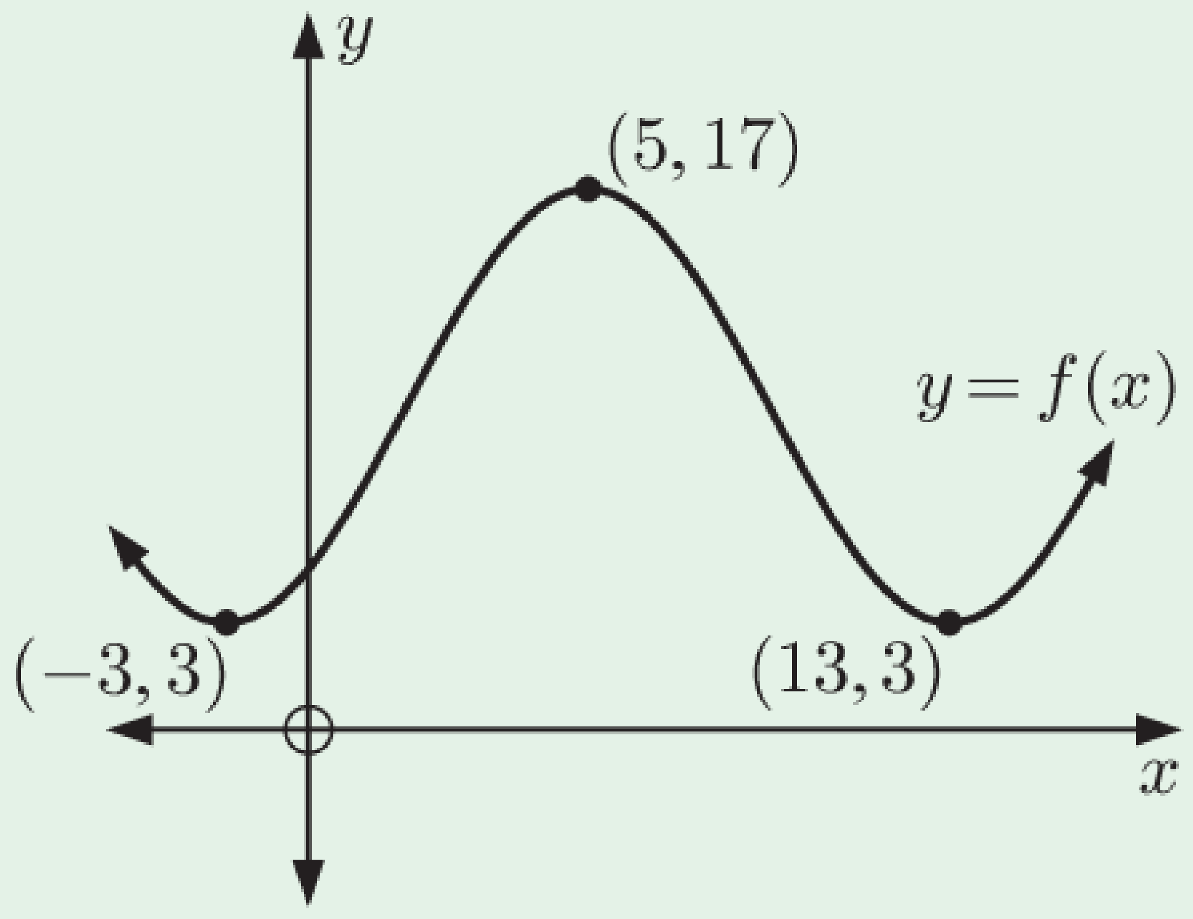
b $y=\frac{1}{4}\tan\frac{x}{2}$

10 **a** Describe the sequence of transformations which maps $y = \tan x$ onto $y = \tan 3x + 2$.

b State the period of $y = \tan 3x + 2$.

c Sketch $y = \tan 3x + 2$ for $-\pi \leqslant x \leqslant \pi$.

11 The graph of $f(x) = a\sin(b(x - c)) + d$ is shown alongside.



a Find the values of a, b, c, and d.

b The function $g(x)$ is obtained by translating $f(x)$ 2 units right and 3 units down, followed by a vertical stretch with scale factor 2.
Find $g(x)$ in the form $g(x) = p\sin(q(x-r))+s$.

12 The proportion of the Moon which is illuminated each night is given by the function $M(t) = \frac{1}{2}\cos\left(\frac{\pi}{15}t\right) + \frac{1}{2}$, where t is the time in days after January 1st.

a Sketch the graph of M against t for $0 \leqslant t \leqslant 60$.

b Find the proportion of the Moon which is illuminated on the night of:

i January 6th **ii** January 21st **iii** January 27th **iv** February 19th.

c How often does a full moon occur?

d On what dates during January and February is the Moon not illuminated at all?

13 On an April day in Kyoto, the maximum temperature 14.1°C occurred at 2:30 pm. The minimum was 6.7°C.

a Suggest a sine function to model the temperature for that day. Let T be the temperature and t be the time in hours after midnight.

b Graph $T(t)$ for $0 \leqslant t \leqslant 24$.

14 A robot on Mars records the temperature every Mars day. A summary series, showing every one hundredth Mars day, is shown in the table below.

Number of Mars days (n)	0	100	200	300	400	500	600	700	800	900	1000	1100	1200	1300
Temp. (°C)	−43	−15	−5	−21	−59	−79	−68	−50	−27	−8	−15	−70	−78	−68

a Find the maximum and minimum temperatures recorded by the robot.

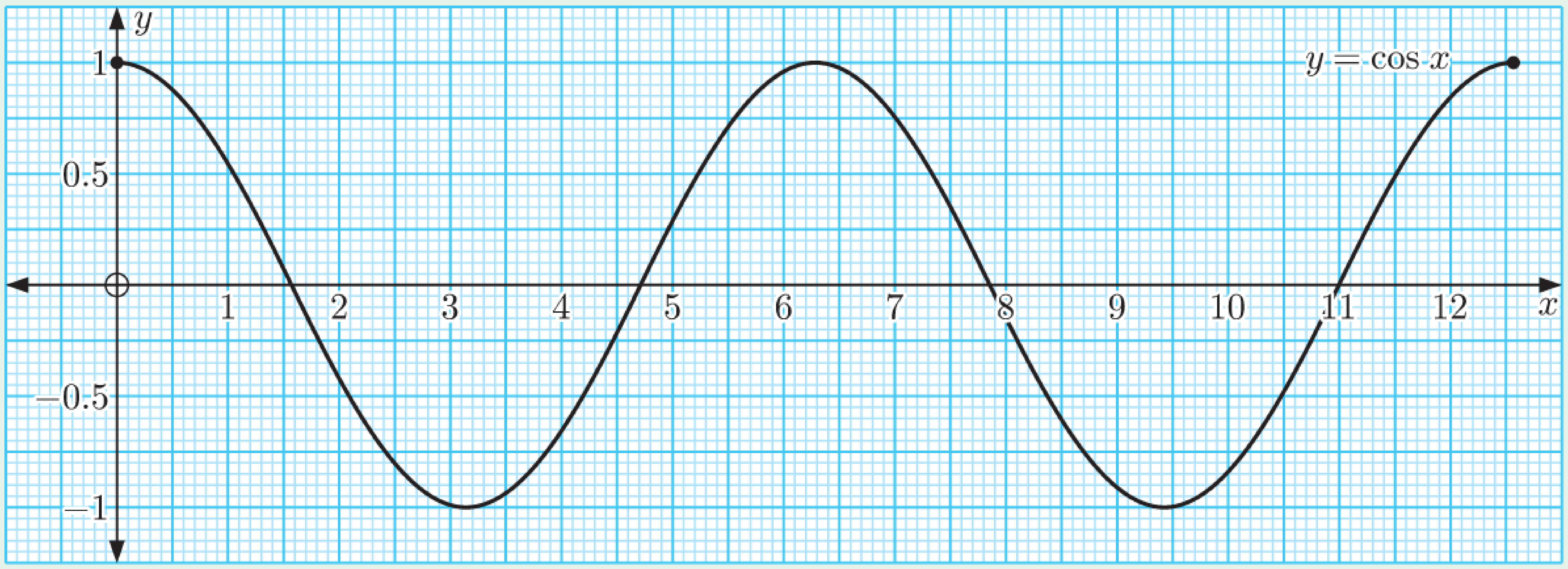
b Use the data to estimate the length of a Mars year.

c Without using technology, find a sine model for the temperature T in terms of the number of Mars days n.

d Draw a scatter diagram of the data and sketch the graph of your model on the same set of axes.

e Check your answer to **c** using technology. How well does your model fit?

15



Use the graph of $y = \cos x$ on the domain $0 \leqslant x \leqslant 4\pi$ to solve, correct to 1 decimal place:

a $\cos x = -0.4$ **b** $\cos x = 0.9$

16 Solve for $0 \leqslant x \leqslant 8$:

a $\sin x = 0.382$ **b** $\tan \frac{x}{2} = -0.458$

17 Solve:

a $\cos x = 0.4379$ for $0 \leqslant x \leqslant 10$ **b** $\cos(x - 2.4) = -0.6014$ for $0 \leqslant x \leqslant 6$

18 Solve for $0 \leqslant x \leqslant 2\pi$:

a $2\sin x = -1$ **b** $\sqrt{2}\cos x - 1 = 0$ **c** $2\cos 2x + 1 = 0$

19 Solve algebraically for $0 \leqslant x \leqslant 2\pi$, giving answers in terms of π:

a $\tan^2 2x = 1$ **b** $\sin^2 x - \sin x - 2 = 0$ **c** $4\sin^2 x = 1$

20 Find the exact solutions of:

a $\sqrt{2}\cos\left(x + \frac{\pi}{4}\right) - 1 = 0,\ \ 0 \leqslant x \leqslant 4\pi$ **b** $\tan 2x - \sqrt{3} = 0,\ \ 0 \leqslant x \leqslant 2\pi$

21 An ecologist studying a species of water beetle estimates the population of a colony over an eight week period. If t is the number of weeks after the initial estimate is made, then the population in thousands can be modelled by $P(t) = 5 + 2\sin \frac{\pi t}{3}$ where $0 \leqslant t \leqslant 8$.

a What was the initial population?

b What were the smallest and largest populations?

c During what time interval(s) did the population exceed 6000?

REVIEW SET 17B

1 Consider the graph alongside.

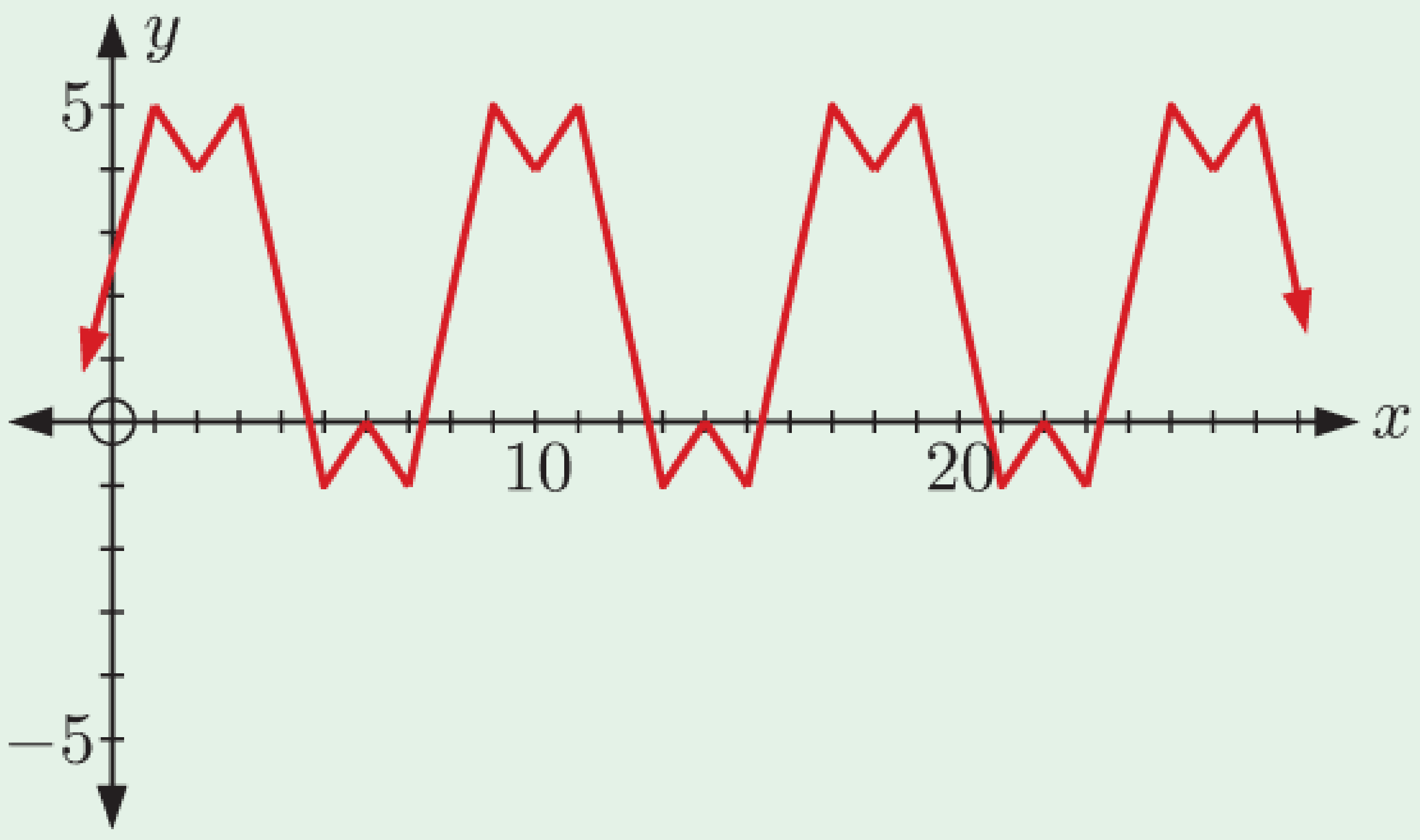
a Explain why this graph shows periodic behaviour.

b State:

i the period

ii the maximum value

iii the minimum value.



2 State the transformation which maps:

a $y = \cos x$ onto $y = \cos\left(x - \frac{\pi}{3}\right) + 1$ **b** $y = \sin x$ onto $y = \sin 3x$.

3 State the period of:

a $y = 4\sin\frac{x}{3}$ **b** $y = \tan 4x$

4 Find b given that the function $y = \sin bx$, $b > 0$ has period:

a 6π **b** $\frac{\pi}{12}$ **c** 9

5 State the minimum and maximum values of:

a $y = 5\sin x - 3$ **b** $y = \frac{1}{3}\cos x + 1$

6 Find the principal axis of:

a $y = -\frac{1}{3}\sin\left(x - \frac{\pi}{4}\right) + 5$ **b** $y = 2\cos\frac{x}{3} - 4$

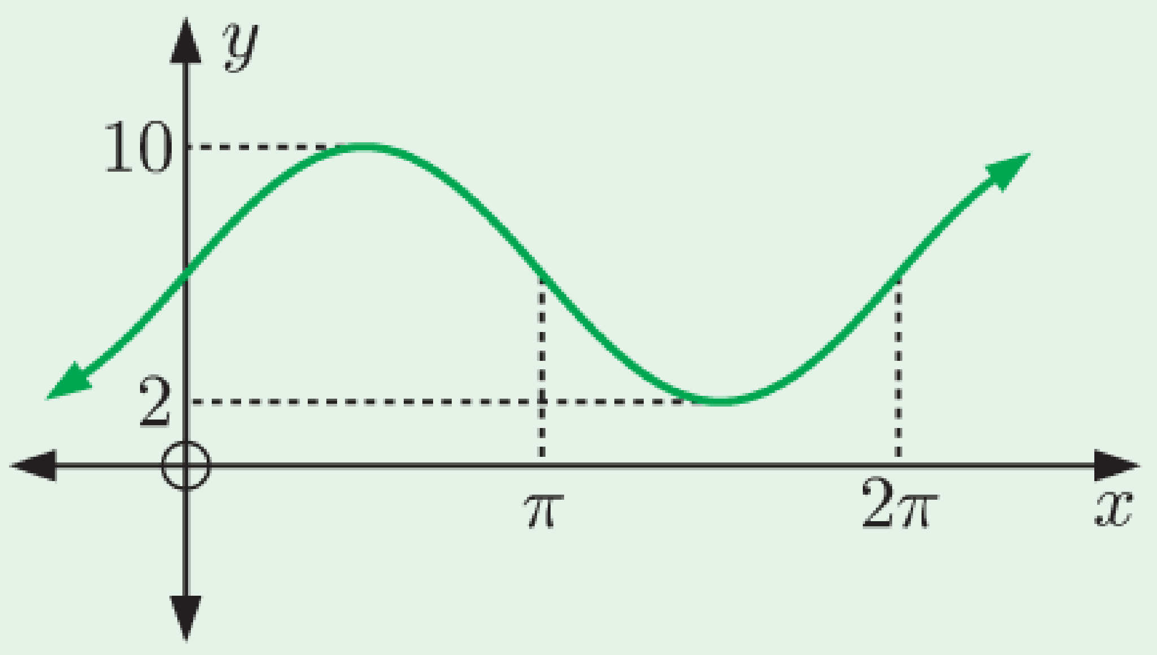
7 Sketch the graphs of the following for $0 \leqslant x \leqslant 2\pi$:

a $y = 2\cos 3x$ **b** $y = 2\sin\left(x - \frac{\pi}{3}\right) + 3$ **c** $y = -\cos\left(x + \frac{\pi}{4}\right)$

d $y = 2\sin x - \frac{1}{2}$ **e** $y = \frac{3}{2}\tan\left(x - \frac{\pi}{6}\right)$ **f** $y = 2\tan\frac{x}{2}$

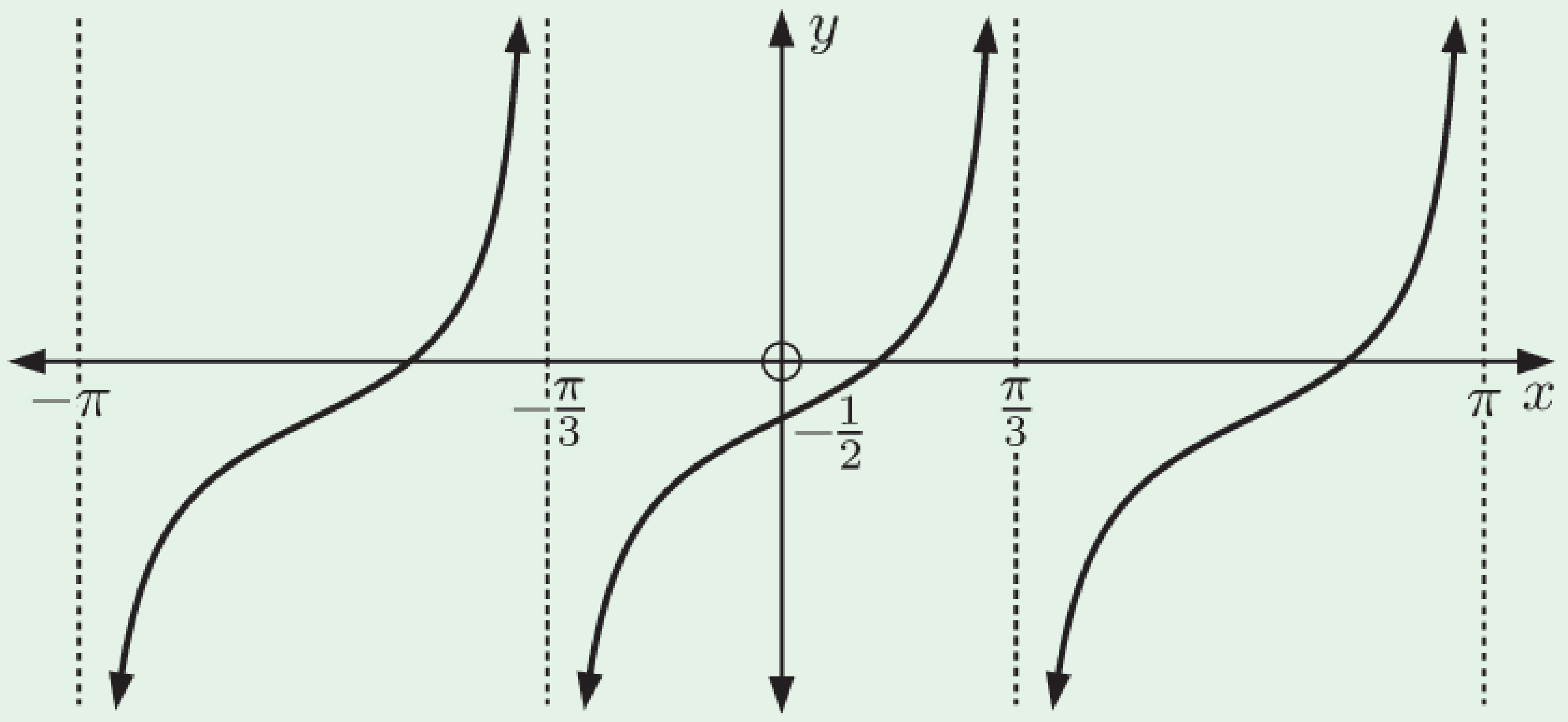
8 **a** Find the sine function shown in this graph.

b Write down the equivalent cosine function for this graph.



9 Draw the graph of $y = 0.6\cos(2.3x)$ for $0 \leqslant x \leqslant 5$.

10 Find a and b given the graph of $y = \tan ax + b$ shown.

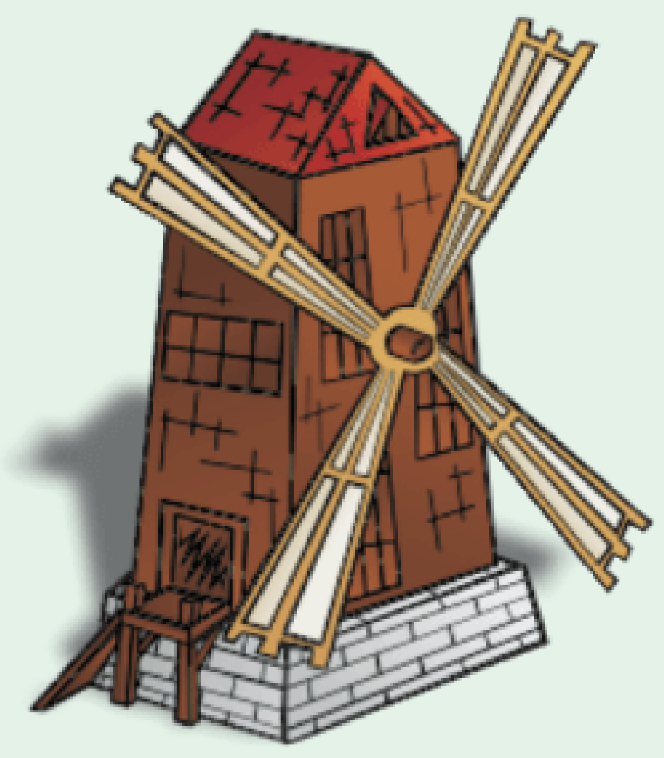


11 State the transformations which map:

a $y = \tan x$ onto $y = -\tan 2x$

b $y = \sin x$ onto $y = 2\sin\left(\frac{1}{2}\left(x - \frac{\pi}{2}\right)\right) + \frac{1}{2}$.

12 As the tip of a windmill's blade rotates, its height above ground is given by $H(t) = 10\cos\left(\frac{\pi}{6}t\right) + 20$ metres, where t is the time in seconds.



a Sketch the graph of H against t for $0 \leqslant t \leqslant 36$.

b Find the height of the blade's tip after 9 seconds.

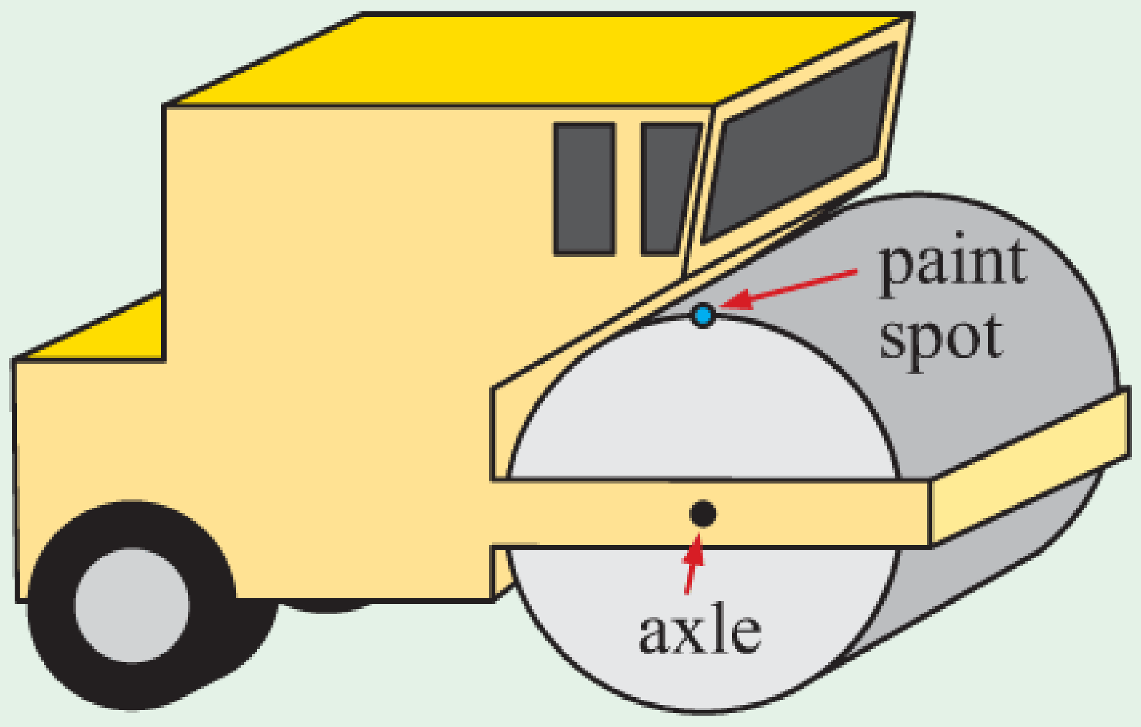
c Find the minimum height of the blade's tip.

d How long does the blade take to complete a full revolution?

13 A steamroller has a spot of paint on its roller. As the steamroller moves, the spot rotates around the axle. The roller has radius 1 metre and completes one full revolution every 2 seconds.

a What does the graph of the spot's height over time look like?

b What function gives the height of the paint spot over time?



14 The table below gives the mean monthly maximum temperature for Perth Airport in Australia.

Month (t)	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Temperature (°C)	31.5	31.8	29.5	25.4	21.5	18.8	17.7	18.3	20.1	22.4	25.5	28.8

a A sine function of the form $T \approx a\sin(b(t-c))+d$ is used to model the data. Find good estimates of the constants a, b, c, and d without using technology. Use Jan $\equiv 1$, Feb $\equiv 2$, and so on.

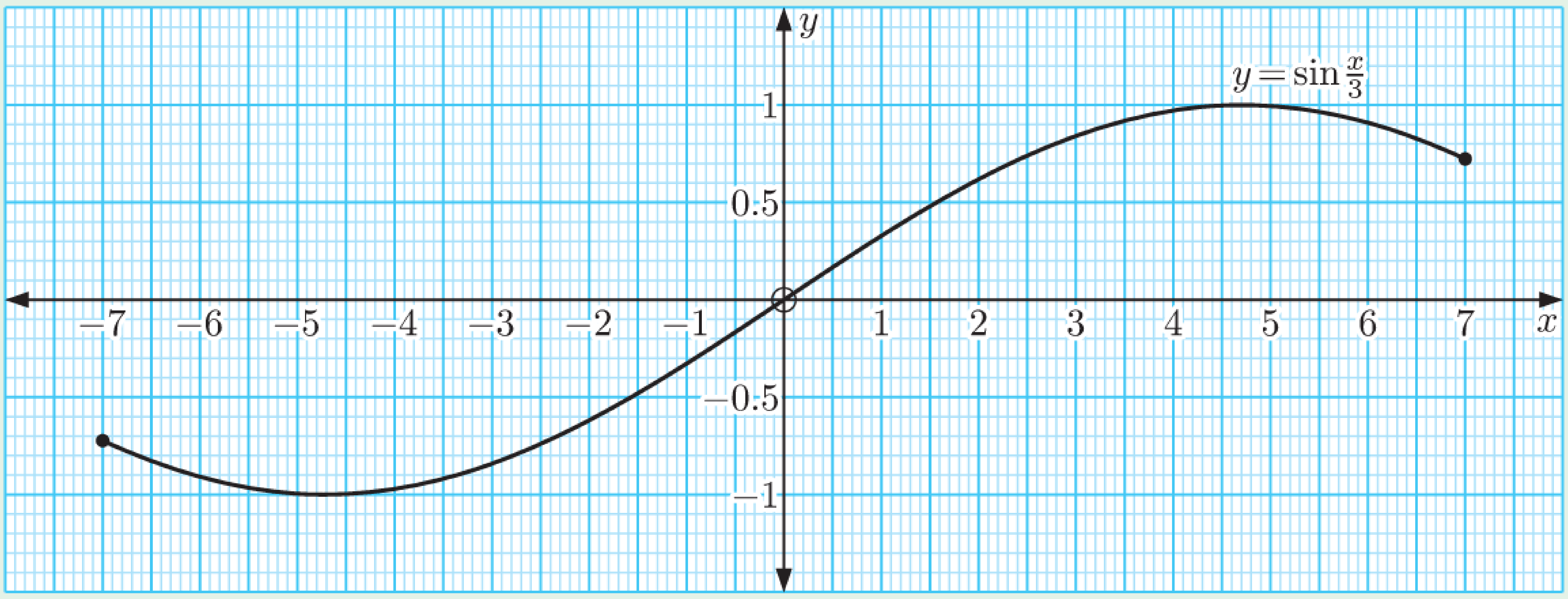
b Draw a scatter diagram of the data and the graph of your model on the same set of axes.

c Check your answer to **a** using technology. How well does your model fit?

15 Consider $y=\sin\frac{x}{3}$ on the domain $-7 \leqslant x \leqslant 7$. Use the graph to solve, correct to 1 decimal place:

a $\sin\frac{x}{3} = -0.9$

b $\sin\frac{x}{3} = \frac{1}{4}$



16 Solve for $0 \leqslant x \leqslant 2\pi$:

a $\cos x = 0.3$

b $43 + 8\sin x = 50.1$

17 Solve for $0 \leqslant x \leqslant 10$:

a $\tan x = 4$

b $\tan\frac{x}{4} = 4$

c $\tan(x-1.5) = 4$

18 Solve for $0 \leqslant x \leqslant 2\pi$:

a $2\sin 3x = -\sqrt{3}$

b $\sqrt{3}\tan\frac{x}{2} = -1$

c $\cos 2x = \sqrt{3}\sin 2x$

19 Suppose $f(x)=\cos x$ and $g(x)=2x$. Solve for $0 \leqslant x \leqslant 2\pi$:

a $(f \circ g)(x) = 1$

b $(g \circ f)(x) = 1$

20 Find exact solutions for x given $-\pi \leqslant x \leqslant \pi$:

a $\tan\left(x+\frac{\pi}{6}\right) = -\sqrt{3}$

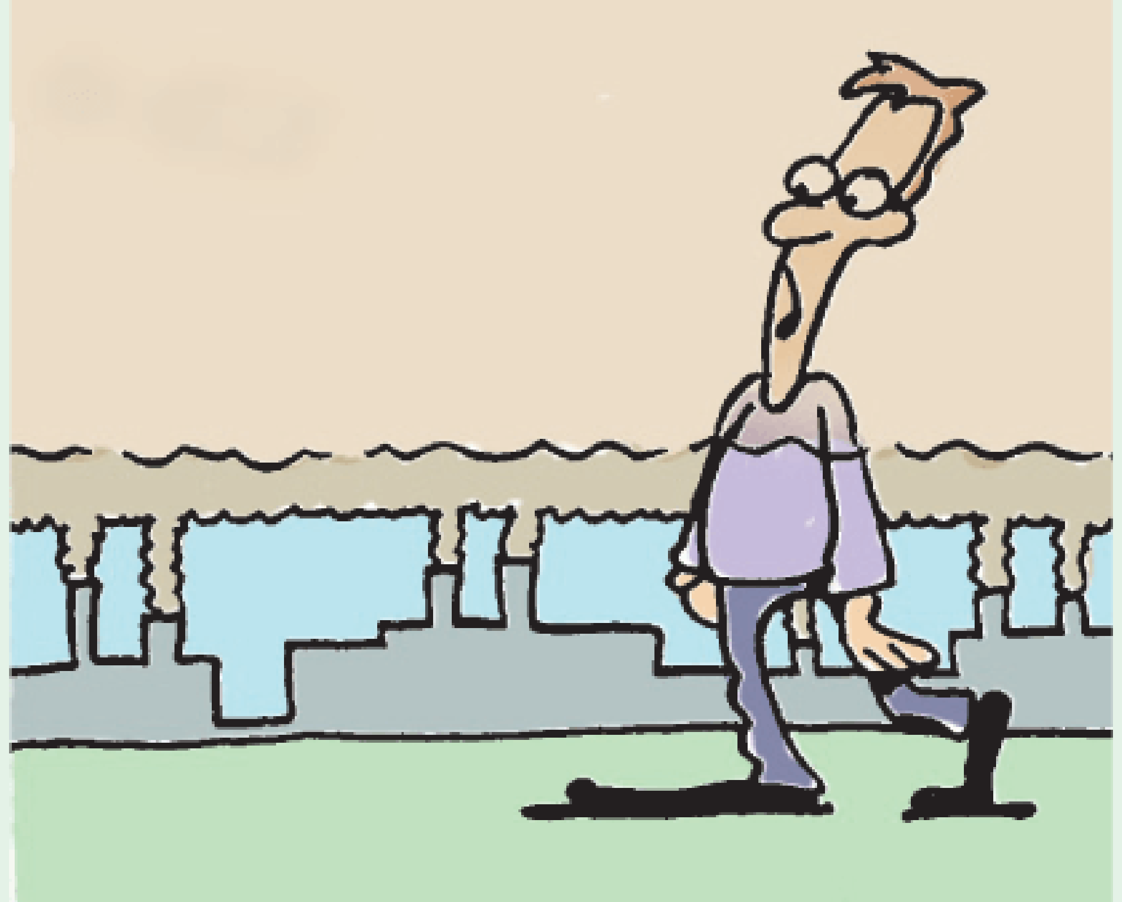
b $\tan 2x = -\sqrt{3}$

c $\tan^2 x - 3 = 0$

21 Find the x-intercepts of:

a $y = 2\sin 3x + \sqrt{3}$ for $0 \leqslant x \leqslant 2\pi$ **b** $y = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$ for $0 \leqslant x \leqslant 3\pi$

22 In an industrial city, the amount of pollution in the air becomes greater during the working week when factories are operating, and lessens over the weekend. The number of milligrams of pollutants in a cubic metre of air is given by $P(t) = 40 + 12\sin\left(\frac{2\pi}{7}\left(t - \frac{37}{12}\right)\right)$ where t is the number of days after midnight on Saturday night.



a What is the minimum level of pollution?

b At what time during the week does this minimum level occur?

ANSWERS

EXERCISE 1A

1 **a** $m = 3,\ c = 7$ **b** $m = -2,\ c = -5$
c $m = \frac{2}{3},\ c = -\frac{1}{3}$ **d** $m = \frac{7}{9},\ c = \frac{2}{9}$
e $m = \frac{1}{3},\ c = -\frac{1}{2}$ **f** $m = -\frac{5}{8},\ c = \frac{3}{8}$

2 **a** $y = 3x - 11$ **b** $y = -2x - 1$ **c** $y = \frac{1}{4}x - 4$
d $y = -\frac{3}{4}x + 4$

3 **a** The gradient is -10 which means that the balance in the account decreases by \$10 each year.
The y-intercept is 90 which means that the initial balance was \$90.
b $y = -10x + 90$ **c** 9 years

4 **a** $-\frac{23}{910}$ **b** $y = -\frac{23}{910}x + 46$

5 **a** 150 metres
b The height of the helicopter above sea level increases by 120 metres each minute after taking off.
c 390 metres **d** 4 minutes 10 seconds

6 **a** $4x + y = 6$ **b** $5x - y = 3$ **c** $3x + 4y = 5$
d $3x - 5y = 1$

7 **a** $y = -5x + 2$ **b** $y = -\frac{3}{7}x - \frac{2}{7}$ **c** $y = 2x - 6$
d $y = \frac{3}{13}x + \frac{4}{13}$

8 $ax + by = d$ can be written as $y = -\frac{a}{b}x + \frac{d}{b}$ which has the form $y = mx + c$. $\therefore\ m = -\frac{a}{b}$

9 **a** $4x + y = 6$ **b** $x - 2y = 13$ **c** $5x + 3y = 8$
d $7x - 6y = 17$

10 **a** $y = 2x + 5$ **b** $y = -x + 9$ **c** $y = \frac{7}{5}x - \frac{11}{5}$
d $y = -\frac{5}{6}x + \frac{19}{6}$

11 **a** $2x - y = -2$ **b** $3x + 10y = 8$ **c** $8x + 5y = -13$

12 **a** $y = \frac{3}{4}x - \frac{5}{4}$ **b** $-\frac{5}{4}$

13 **a** $y = 3x + 1$ **b** $2x - y = 7$ **c** $y = \frac{1}{2}x + \frac{11}{2}$
d $2x - y = -3$

14 *Line 1*: $y = \frac{2}{3}x + \frac{1}{3}$, *Line 2*: $y = -\frac{3}{2}x - 4$

15 **a** yes **b** no **c** yes **d** yes

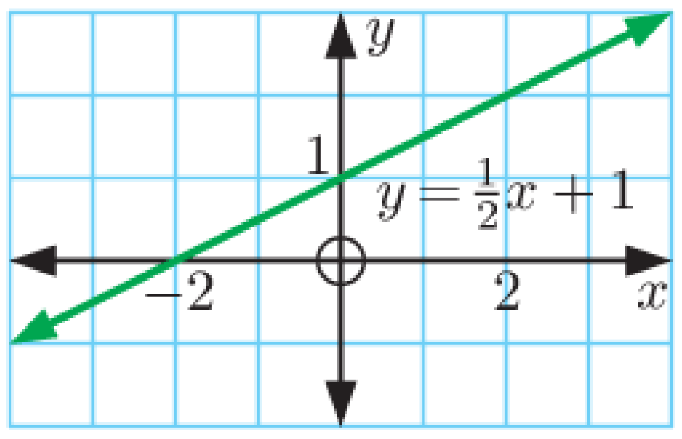
16 **a** $c = 7$ **b** $m = 11$ **c** $t = 8$

17 **a** $k = -3$ **b** $k = -51$ **c** $k = -12$

18 **a** $x - y + 2 = 0$ **b** -2 **19** 126 units^2

EXERCISE 1B

1 **a**



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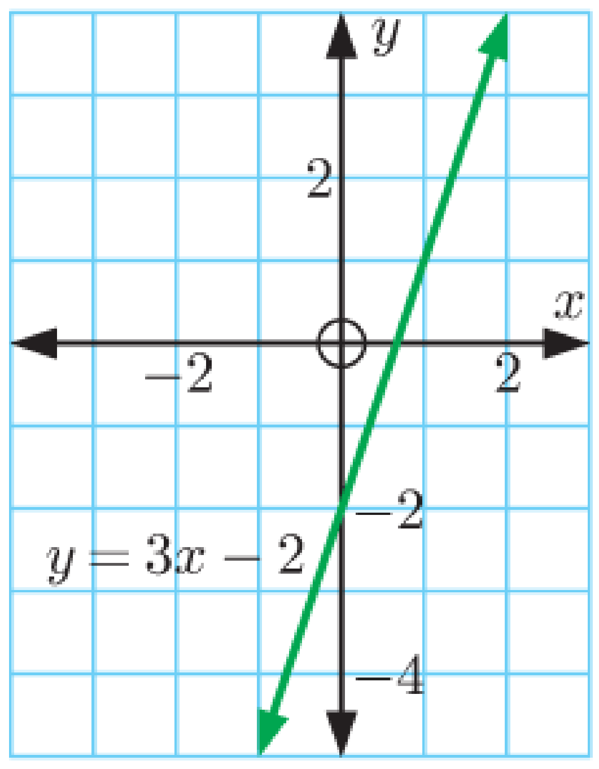

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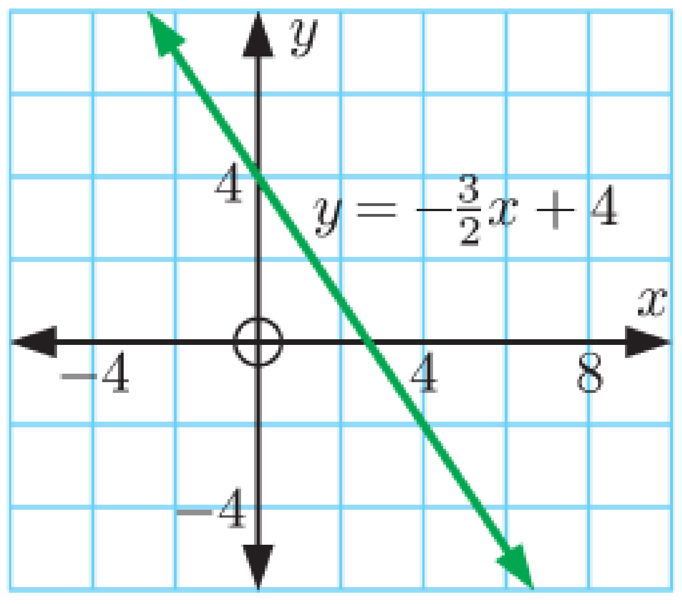

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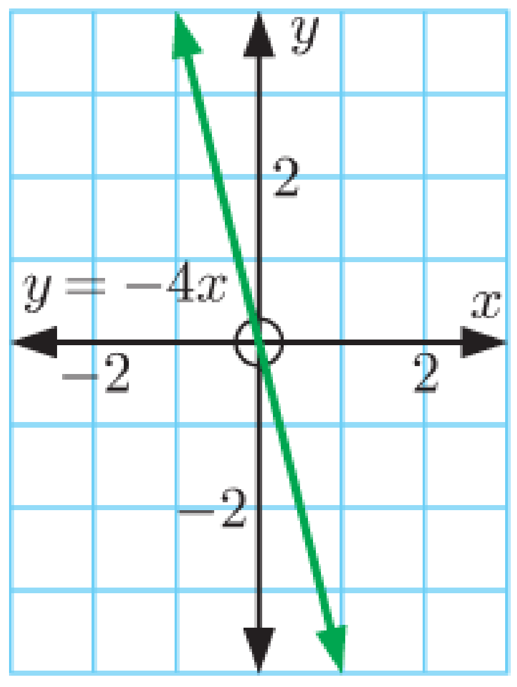

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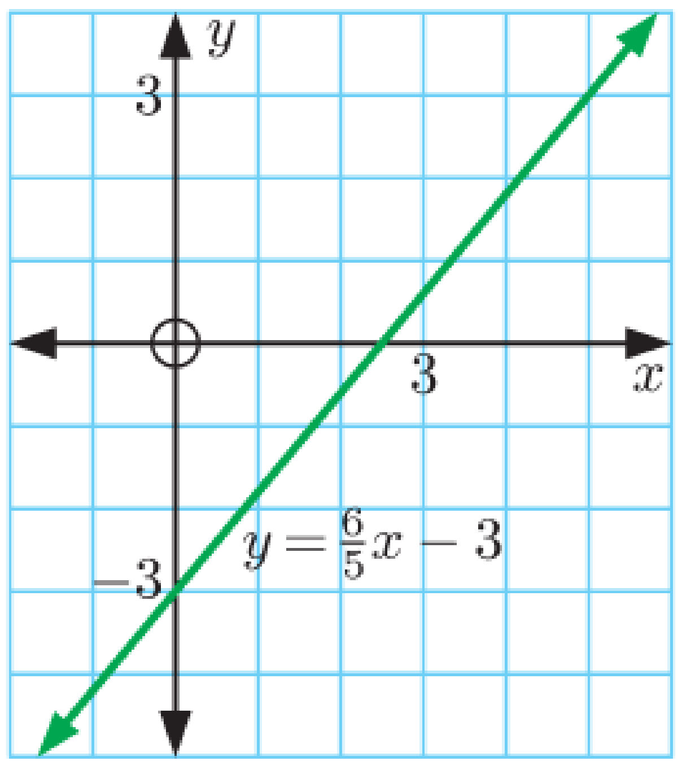

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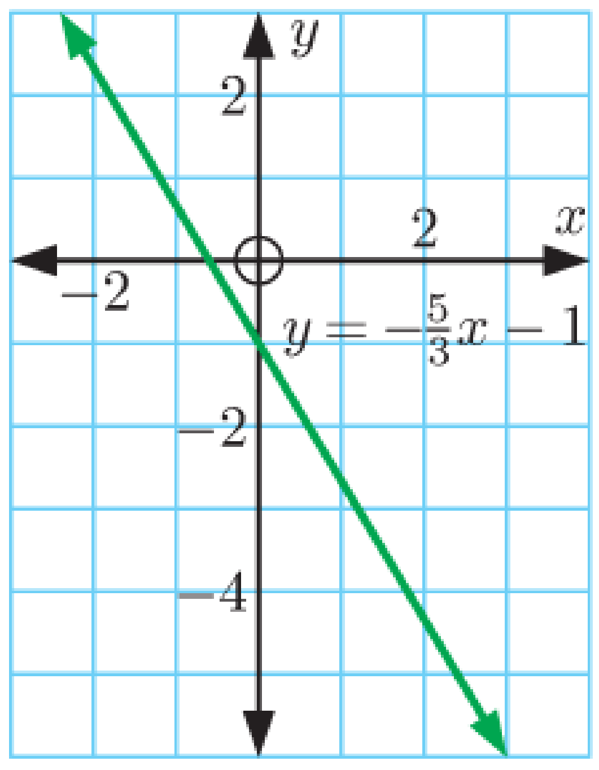
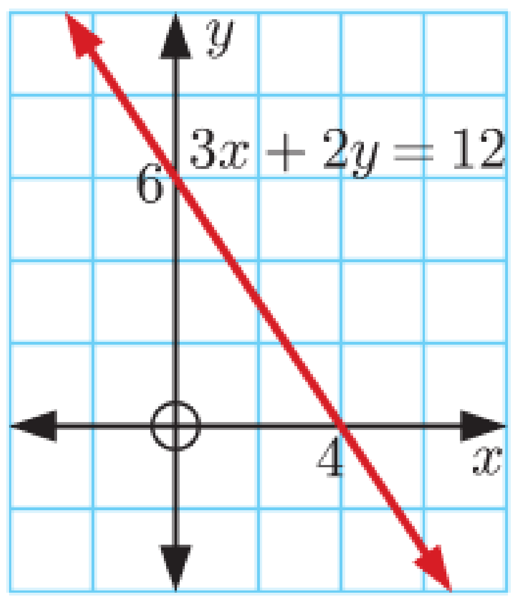


2 **a**



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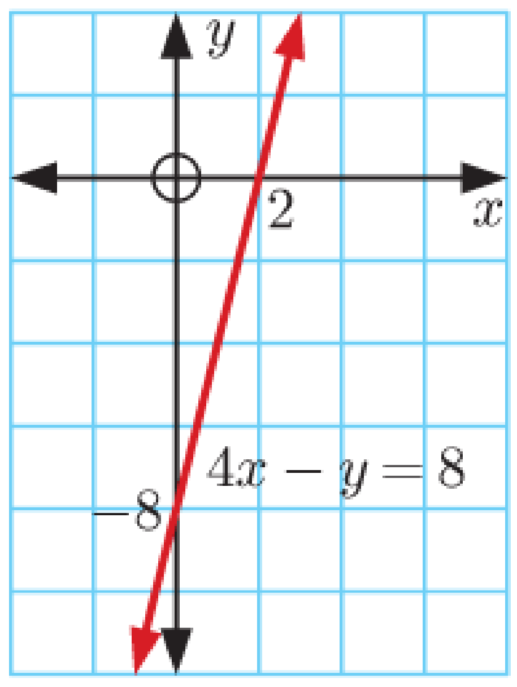

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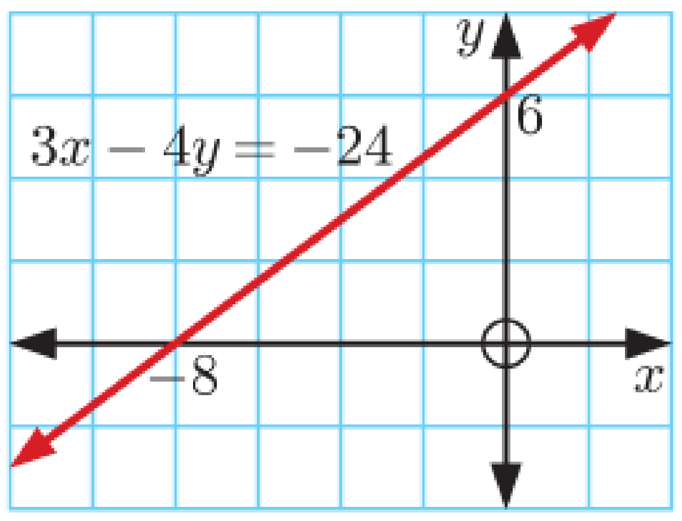

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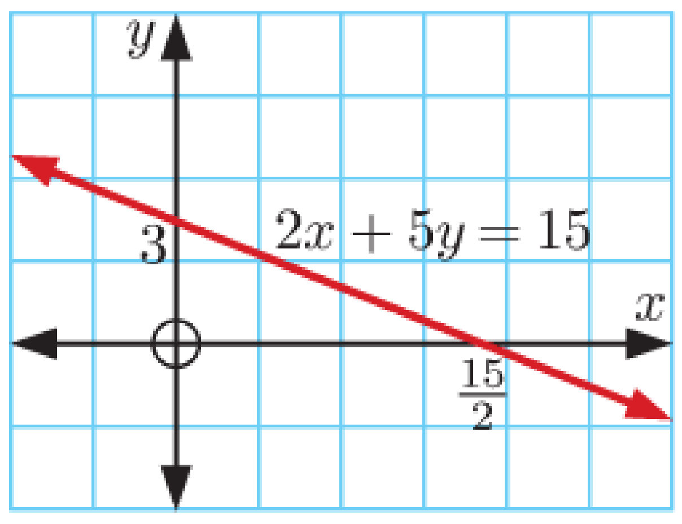

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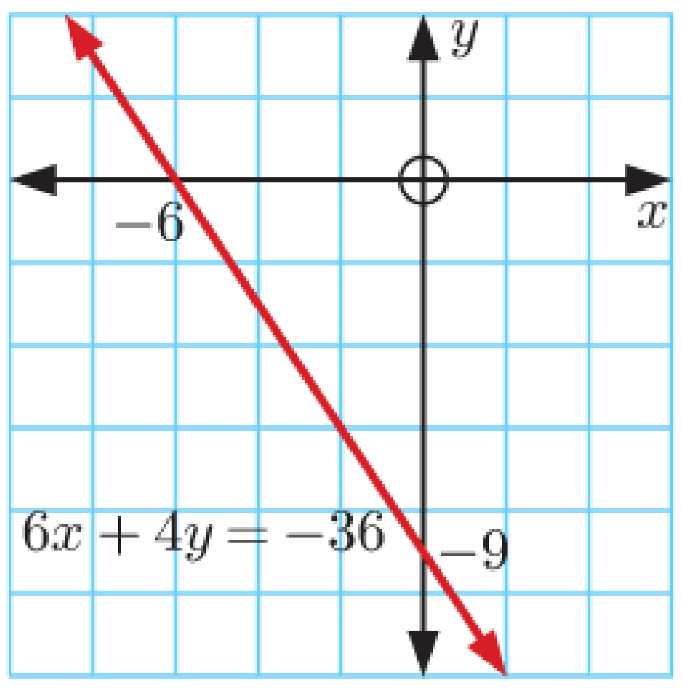

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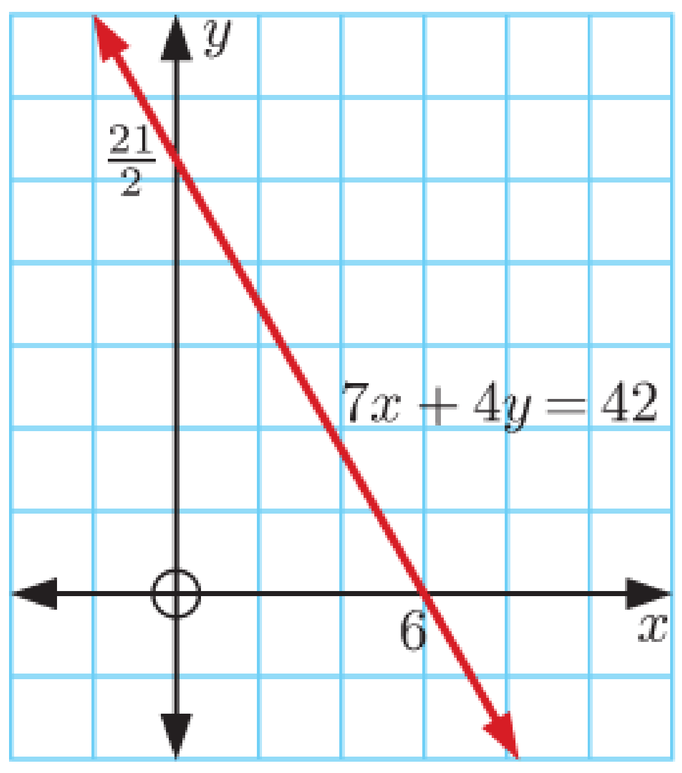
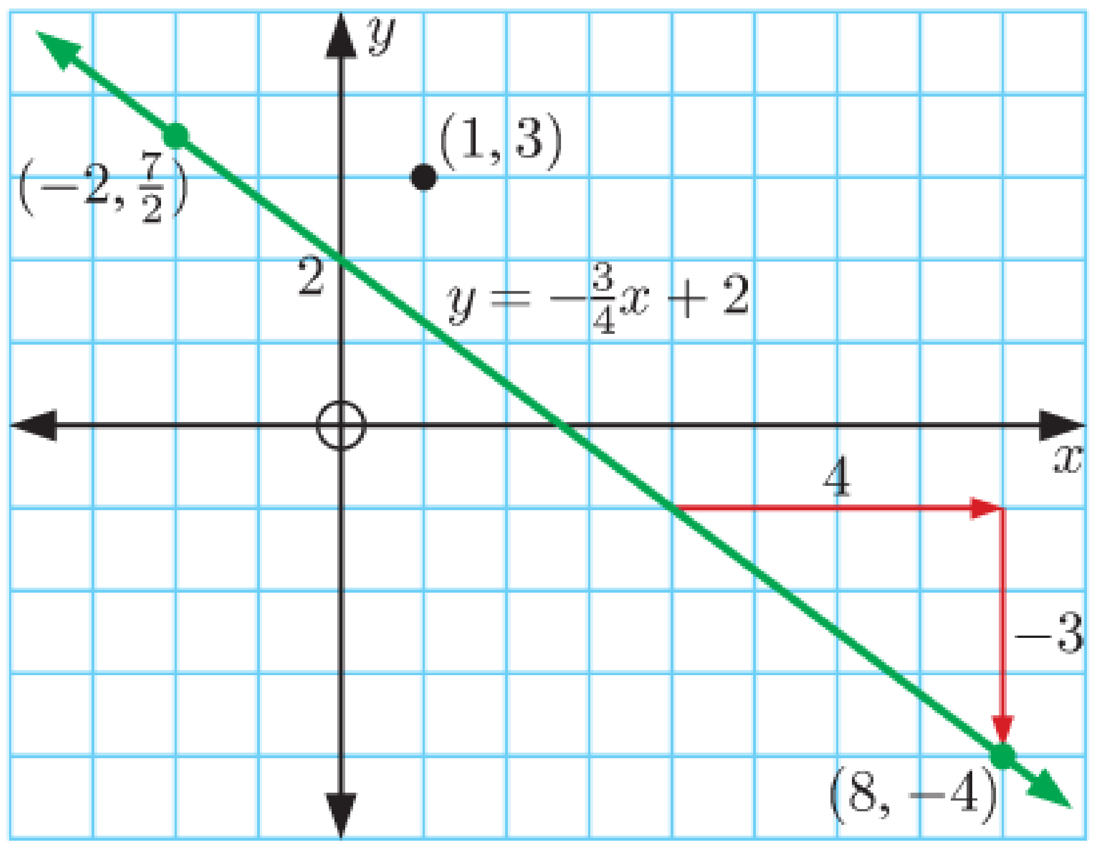
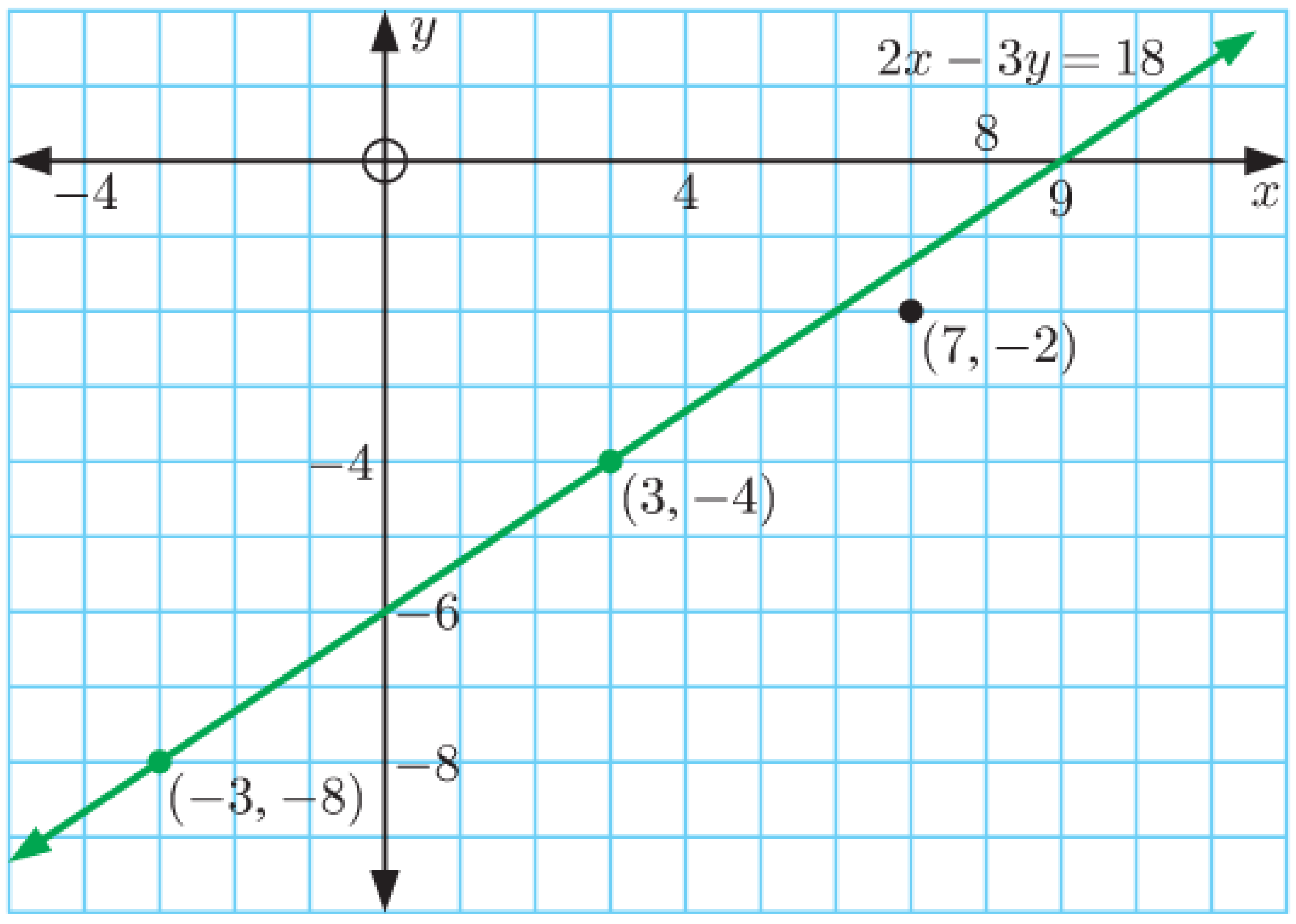


3 **a** $m = -\frac{3}{4},\ c = 2$ **b** **i** yes **ii** no **iii** yes
c



4 **a** x-intercept 9, y-intercept -6
b **i** yes **ii** no **c** $c = -8$
d

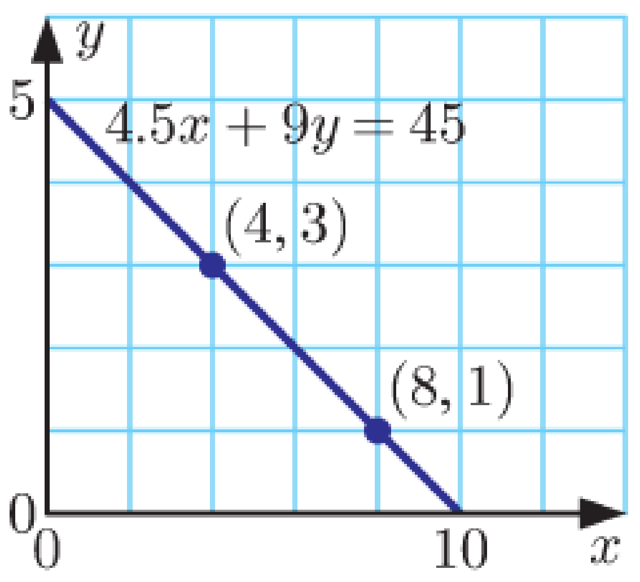


5 **a** x serves of nigiri at \$4.50 each and y serves of sashimi at \$9 each adds up to a total of \$45. $\therefore$ $4.5x + 9y = 45$

b 3 serves of sashimi

c 8 serves of nigiri

d



EXERCISE 1C

1 **a** (4, 4) **b** 3 **c** $-\frac{1}{3}$ **d** $x + 3y = 16$

2 **a** $2x - y = 3$ **b** $3x - y = 2$ **c** $x - y = 5$

d $2x - y = 2$ **e** $x = \frac{9}{2}$ **f** $8x + 6y = 35$

3 **a** $2x - 3y = 2$ **b** $2(1) - 3(0) = 2$ ✓

c PR = QR = $\sqrt{26}$ units

4 **a** AB = BC = CD = AD = $\sqrt{29}$ units $\therefore$ ABCD is a rhombus.

b $y = -x$ **c** B: $2 = -(-2)$ ✓ D: $-1 = -(1)$ ✓

5 **a** **i** $\frac{3}{2}$ **ii** $-\frac{2}{3}$ **b** $2x + 3y - 21 = 0$

6 **a** $(-\frac{1}{2}, 1)$

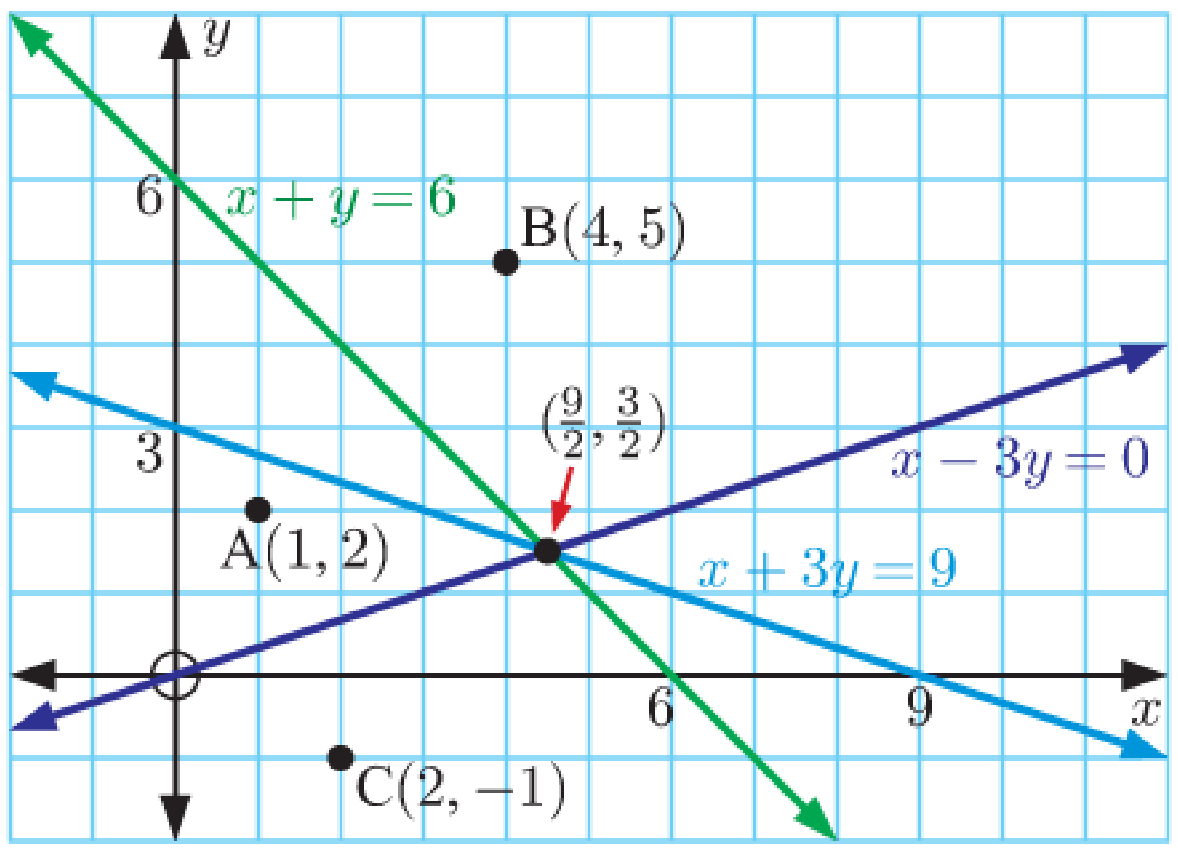
b The perpendicular bisector of the line joining the two hospitals is $10x + 12y = 7$. An ambulance crew should be sent from A to locations below this line, and from B to locations above this line.

7 **a** **Hint:** Start by finding the gradient and midpoint of [AB].

b We can find the perpendicular bisector of any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ by substituting in the values of x_1, x_2, y_1, and y_2.

8 **a** **i** $x + y = 6$ **ii** $x - 3y = 0$ **iii** $x + 3y = 9$

b



The perpendicular bisectors all intersect at $(\frac{9}{2}, \frac{3}{2})$.

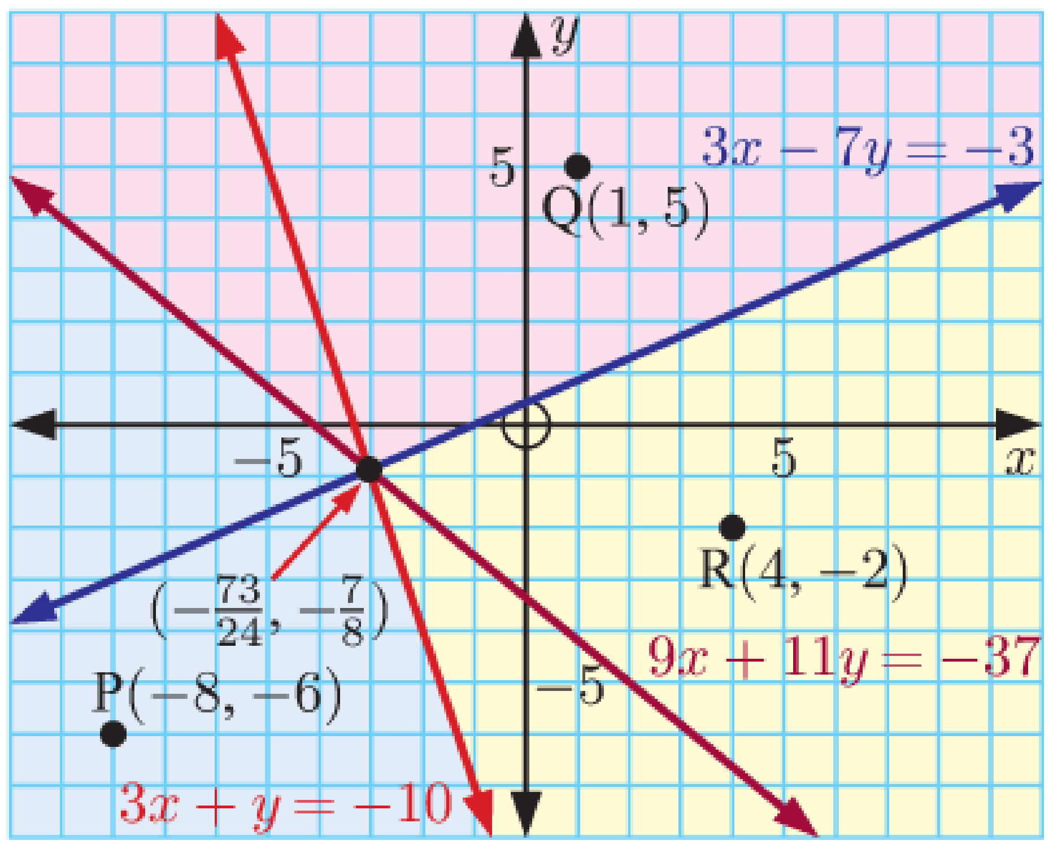
A, B, and C are all equidistant from this point.

c The perpendicular bisectors of each pair of points will meet at a single point. As the three points are equidistant from the point of intersection, a circle centred at the point of intersection that passes through one of them will pass through all of them.

9 **a** **i** $9x + 11y = -37$ **ii** $3x + y = -10$

iii $3x - 7y = -3$

b



EXERCISE 1D

1 **a** $x = -2,\ y = -4$ **b** $x = 1,\ y = -3$

c $x = 6,\ y = 7$ **d** $x = 3,\ y = 2$

e $x = 6,\ y = 1$ **f** $x = 2,\ y = -12$

2 **a** $x = 3,\ y = 5$ **b** $x = 1,\ y = -1$

c $x = -1,\ y = 8$ **d** $x = 5,\ y = 8$

e $x = -4,\ y = -\frac{1}{4}$ **f** $x = -1\frac{11}{31},\ y = -\frac{4}{31}$

g $x = -3,\ y = 3\frac{1}{2}$ **h** $x = -2\frac{1}{4},\ y = 3$

i $x = \frac{1}{4},\ y = \frac{3}{4}$

3 **a** $x = 2,\ y = 1$ **b** $x = 3,\ y = -1$

c $x = 3,\ y = 7$ **d** $x = \frac{1}{3},\ y = 4$

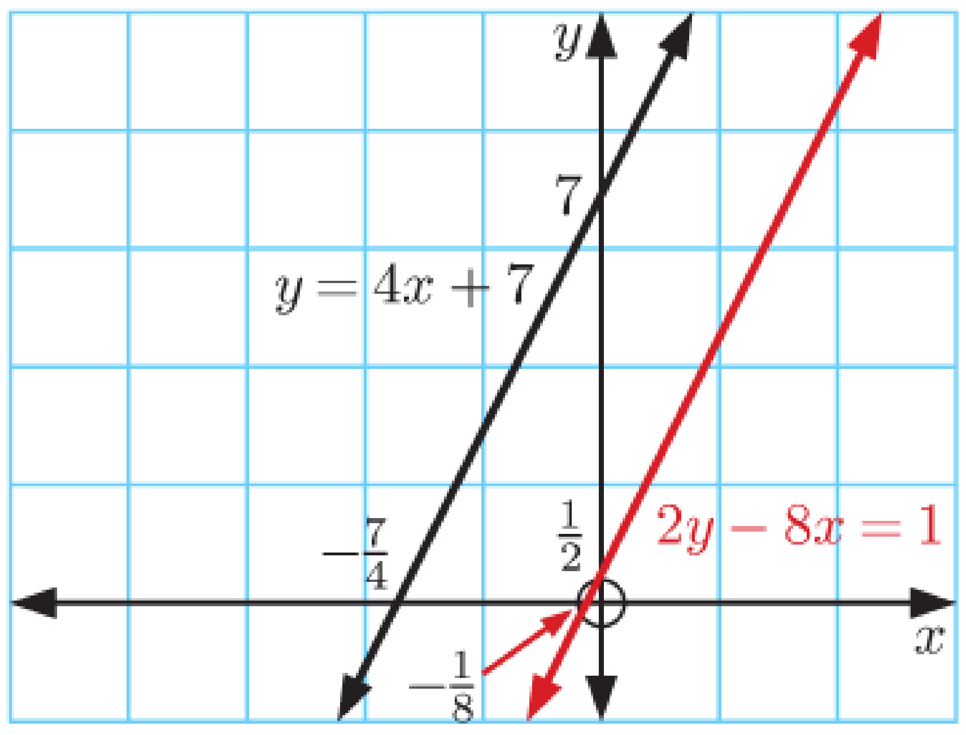
e $x = \frac{1}{4},\ y = 1\frac{1}{4}$ **f** $x = 5,\ y = -2$

g $x = -3,\ y = -4$ **h** $x = -4\frac{1}{2},\ y = -2\frac{1}{2}$

i $x = -28\frac{2}{3},\ y = -17\frac{2}{3}$

4 **a** $12\frac{1}{4}$ units^2 **b** $1\frac{4}{25}$ units^2

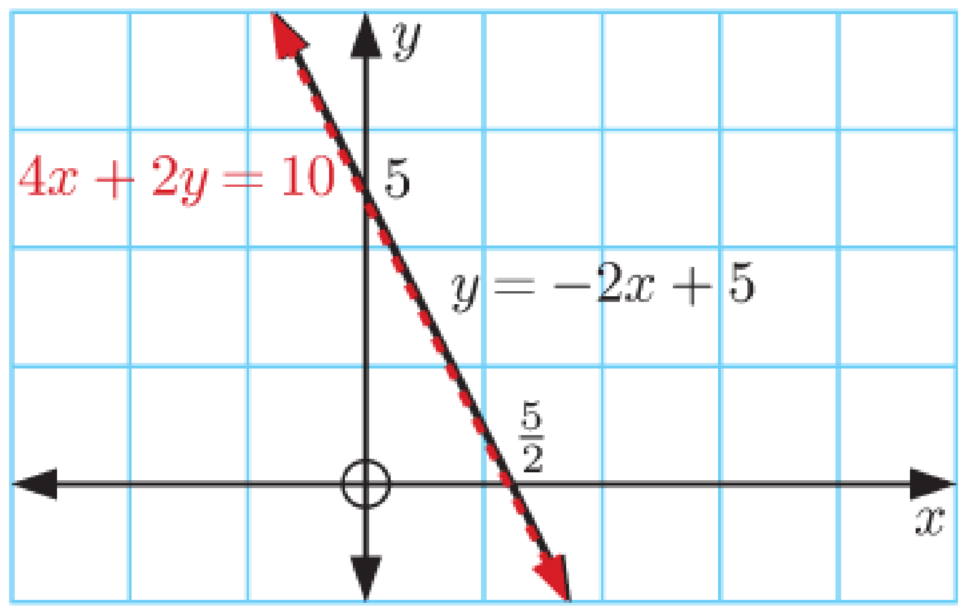
5 **a**



The lines are parallel.

c no solutions

6 **a**



The lines are coincident.

c infinitely many solutions

7 **a** $\frac{3}{2}$ and m

b $m = \frac{3}{2}$, in this case the lines are coincident and hence there are infinitely many solutions.

c $x = 0,\ y = -6$

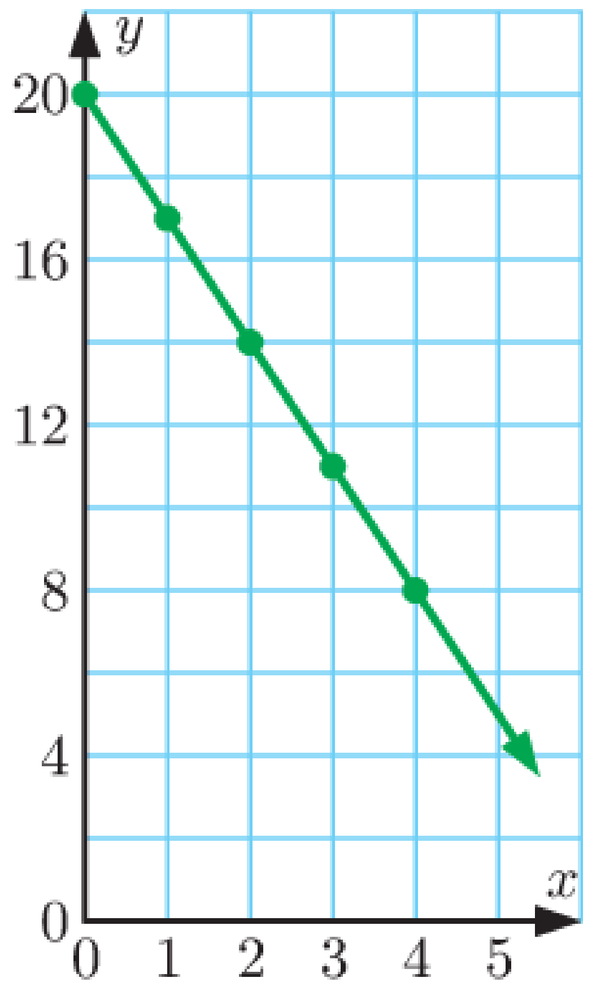
8 **a** $\dfrac{4}{c}$ and $\frac{2}{3}$

b $c = 6$, in this case the lines are parallel and hence there are no solutions.

c $x = \dfrac{18 + 3c}{6 - c},\quad y = \dfrac{24}{6 - c}$

REVIEW SET 1A

1 **a**



b Yes, the variables are linearly related as the points all lie on a straight line.

c gradient is -3, y-intercept is 20

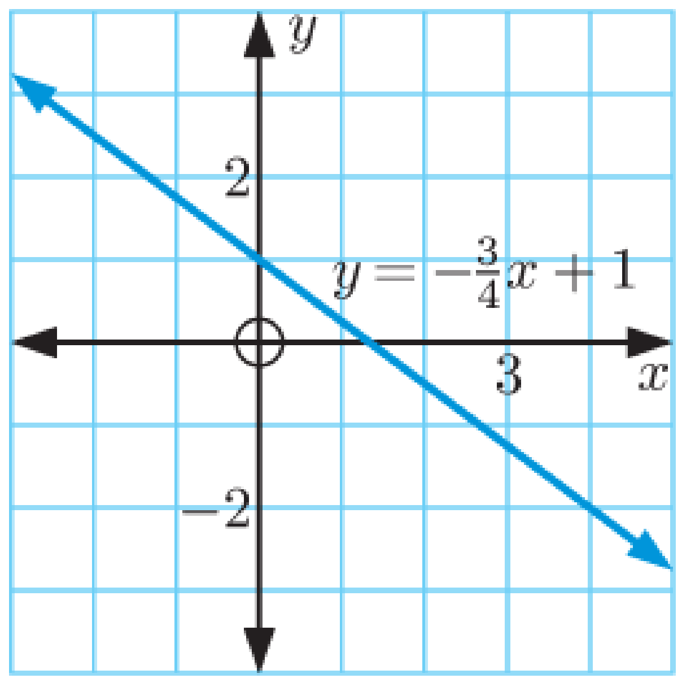
d $y = -3x + 20$

e $y = -1$

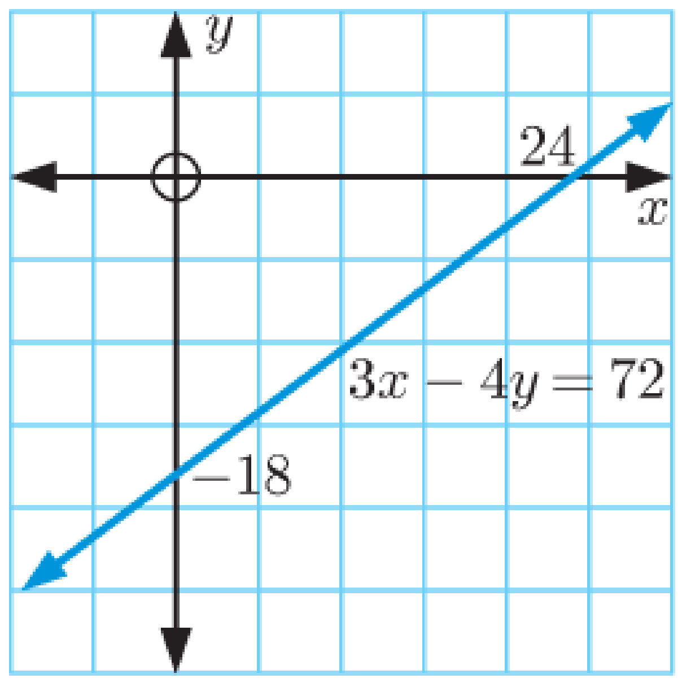
2 **a** $y = -\frac{1}{3}x + 4$ **b** $x + 3y - 12 = 0$

3 **a** $3x - 2y = 12$ **b** 4 **4** **a** yes **b** yes

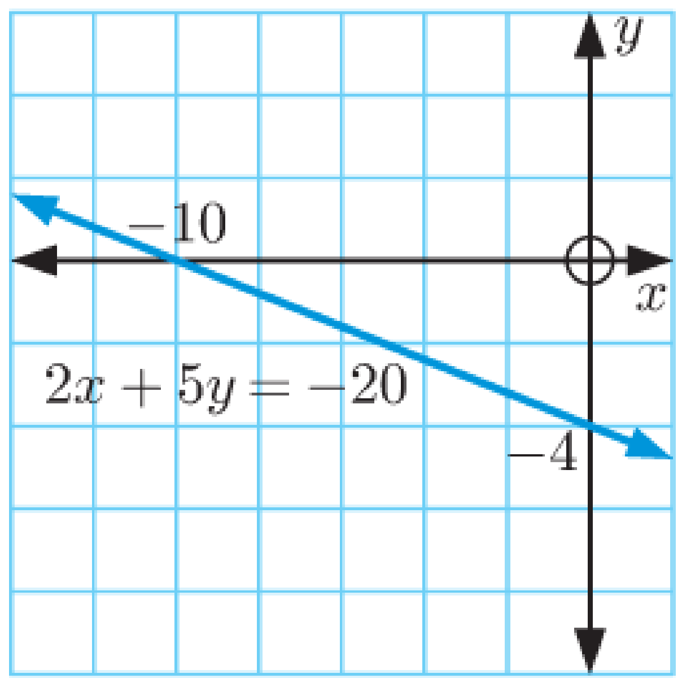
5 **a**



b



c



6 **a** $y = -1$ **b** $3x - 2y = 9$

7 **a** **i** $7x + 5y = -6$ **ii** $5x - 7y = 1$

b ABCD is a square.

8 **a** $x = -2,\ y = -5$ **b** $x = 4,\ y = -2$

9 **a** $x = 1,\ y = 7$ **b** $x = -1,\ y = 2$

10 **a** $x = 3,\ y = -1$ **b** $x = -4,\ y = 3$

11 **a** $-\frac{1}{2}$ and $-\frac{1}{2}$

b **i** $k \neq 4$ **ii** $k = 4$

If $k \neq 4$, the lines are parallel and hence there are no solutions.

If $k = 4$, the lines are coincident and hence there are infinitely many solutions.

REVIEW SET 1B

1 **a** The gradient is 10 which means that the speed increases by 10 m s^{-1} each second.

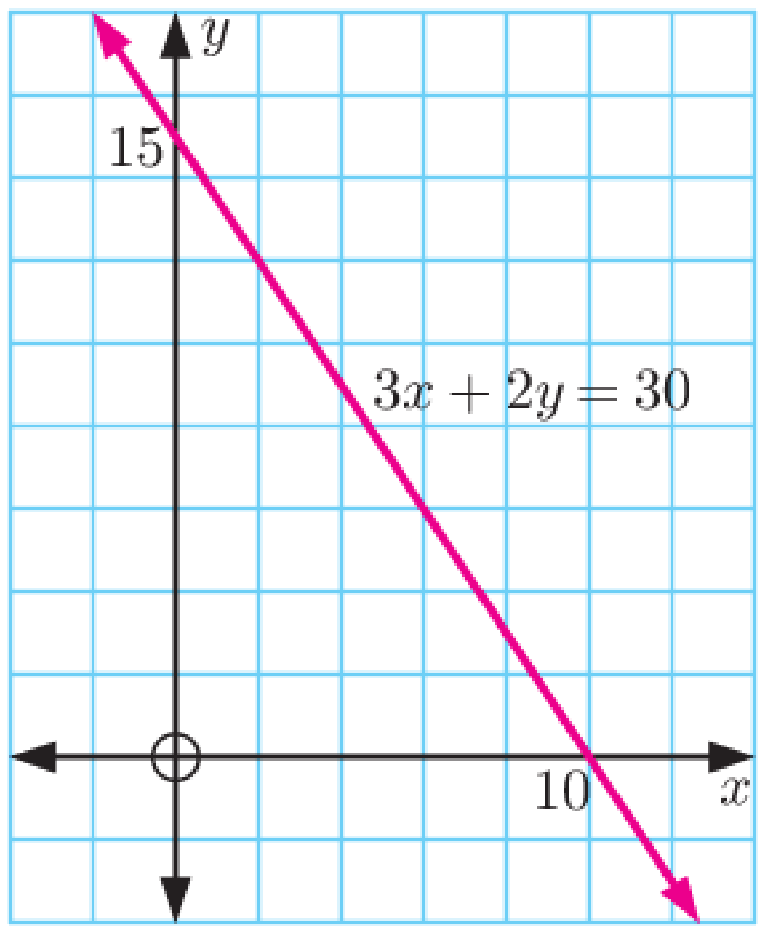
The y-intercept is 5 which means that the initial speed was 5 m s^{-1}.

b $y = 10x + 5$ **c** 85 m s^{-1}

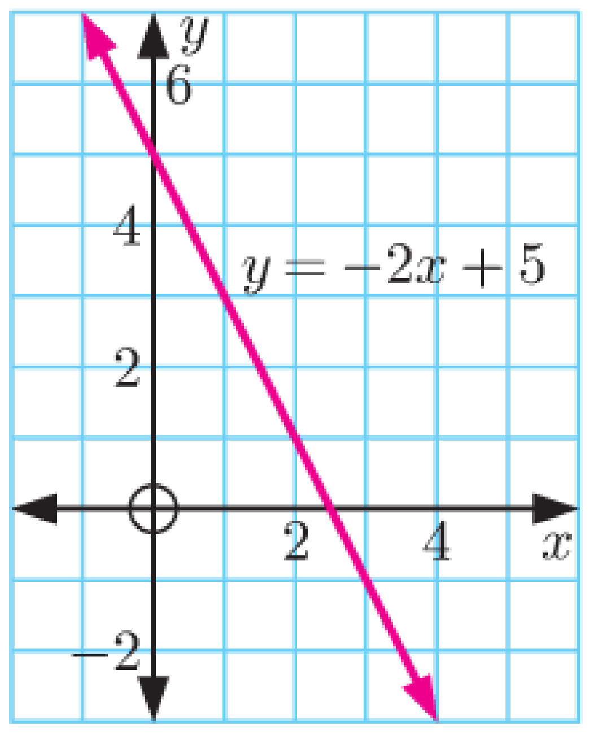
2 **a** $y = 3x + 1$ **b** $5x - 2y = -3$

3 **a** $k = 7$ **b** $k = -11$

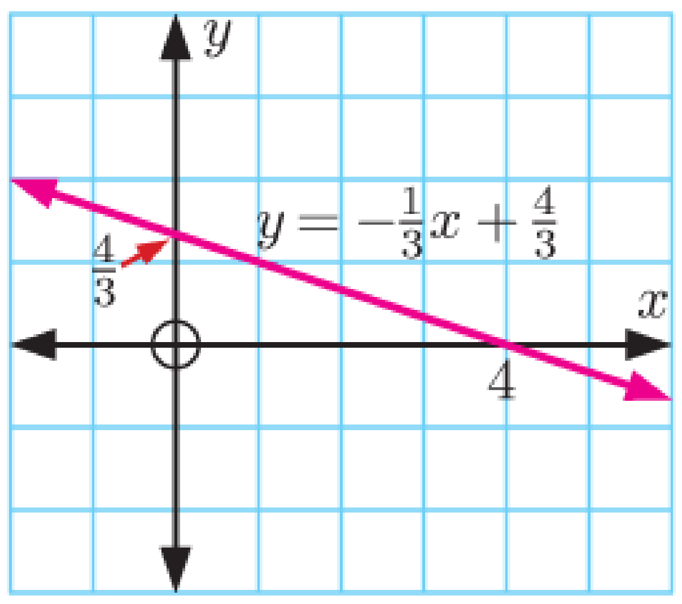
4 **a**



b



c

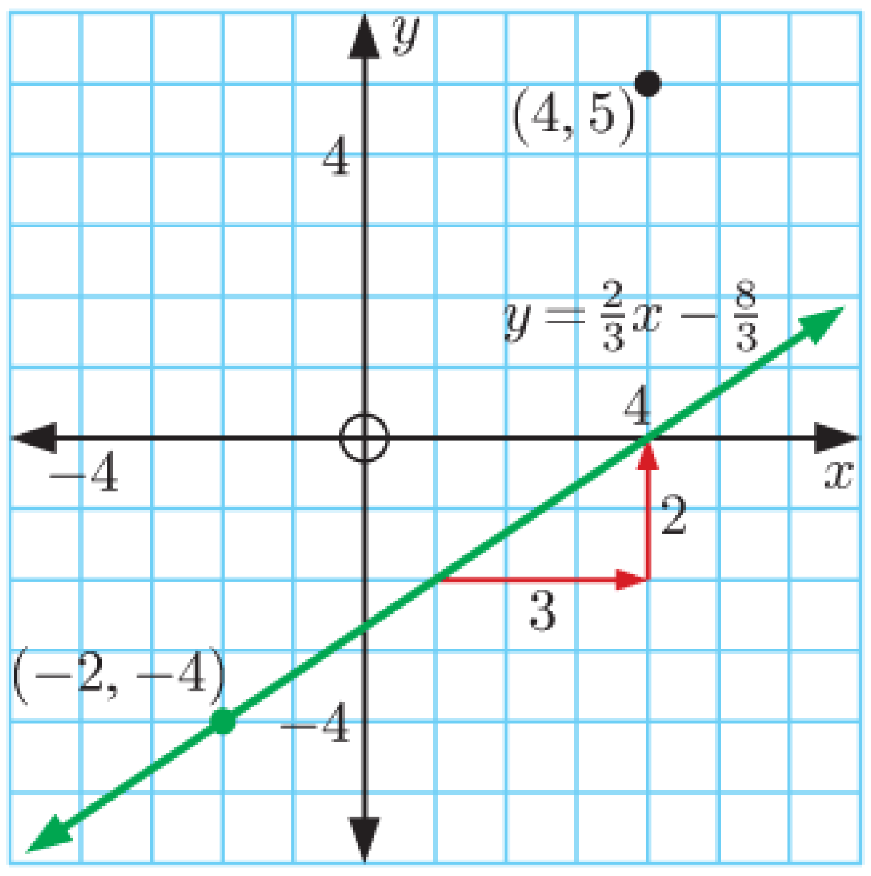


5 **a** $m = \frac{2}{3}$

b **i** yes

ii no

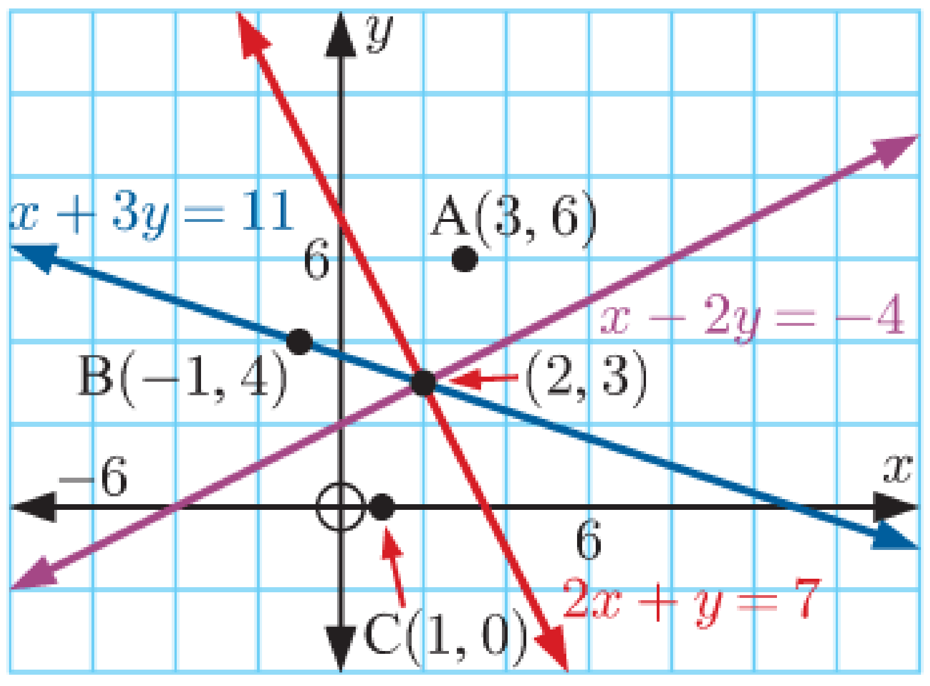
c



6 $108\frac{1}{3}$ units^2 **7** **a** **i** $\frac{1}{5}$ **ii** -5 **b** $5x + y = 22$

8 **a** **i** $2x + y = 7$ **ii** $x + 3y = 11$ **iii** $x - 2y = -4$

b



All three perpendicular bisectors intersect at (2, 3).
A, B, and C are all equidistant from this point.

9 **a** $x = \frac{1}{3},\ y = 4$ **b** $x = -2,\ y = 4$

10 **a** $x = 3,\ y = -\frac{1}{2}$ **b** $x = 1\frac{1}{2},\ y = -3\frac{1}{2}$

11 **a** $-\dfrac{a}{4}$ and $\frac{1}{2}$

b $a = -2$, in this case the lines are parallel and hence there are no solutions.

c $x = \dfrac{2}{a+2},\quad y = \dfrac{a+3}{a+2}$

EXERCISE 2A

1 **a** $A = \{1, 2, 4, 8\},\quad n(A) = 4$

b $A = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18\},\quad n(A) = 10$

c $A = \{\text{A, R, D, V, K}\},\quad n(A) = 5$

d $A = \{41, 43, 47\},\quad n(A) = 3$

2 **a** finite **b** infinite **c** infinite

3 **a** **i** 6 **ii** 3

b **i** true **ii** false **iii** true **iv** true **v** true

4 **a** subsets of S: $\varnothing$, $\{1\}$, $\{2\}$, $\{1, 2\}$

subsets of T: $\varnothing$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$

b yes **c** $\frac{1}{2}$

5 Each of the four elements can be *included* or *not included* in a subset.

$\therefore$ the set has $2 \times 2 \times 2 \times 2 = 16$ subsets.

6 $x = 3$

7 If $A \subseteq B$, then all elements of A are in B.

If $B \subseteq A$, then all elements of B are in A.

This is only possible if A and B contain exactly the same elements.

$\therefore$ $A = B$.

EXERCISE 2B

1 **a** **i** $A \cap B = \{9\}$

ii $A \cup B = \{5, 6, 7, 8, 9, 10, 11, 12, 13\}$

b **i** $A \cap B = \varnothing$ **ii** $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$

c **i** $A \cap B = \{1, 3, 5, 7\}$

ii $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

d **i** $A \cap B = \{5, 8\}$

ii $A \cup B = \{0, 1, 3, 4, 5, 8, 11, 13, 14\}$

2 **a** disjoint **b** not disjoint

3 **a** True, $R \cap S = \varnothing$ tells us that R and S have no elements in common, and hence are disjoint.

b True, every element of $A \cap B$ is an element of A, and every element of $A \cap B$ is an element of B.

c True, if $A \cap B = A \cup B$ then there are no elements that are in only A or only B. $\therefore$ $A = B$.

d False, consider $A = \{1, 2, 3, ...\}$ and $B = \{-1, -2, -3, ...\}$ which are infinite but $A \cap B = \varnothing$ which is finite.

4 Not necessarily, consider $A = \{1, 2\}$, $B = \{3, 4\}$, and $C = \{1, 6\}$. $A \cap B = \varnothing$ and $B \cap C = \varnothing$, but $A \cap C = \{1\}$, so A and C are not disjoint sets.

5 **a** $n(A \cap B) = 0, 1, 2, 3, 4, 5, 6, 7,$ or 8

b $n(A \cup B) = 11, 12, 13, 14, 15, 16, 17, 18,$ or 19

6 Each element in $A \cup B$ must be in A or B, or both. It is not possible that $n(A \cup B) > n(A) + n(B)$

$\therefore$ $n(A \cup B) \leqslant n(A) + n(B)$

EXERCISE 2C

1 **a** $A' = \{1, 4, 5, 9\}$

b No, since 1 is neither prime nor composite.

2 **a** $A = \{10, 12, 15, 20\}$ **b** $B = \{12, 15, 18\}$

c $A' = \{11, 13, 14, 16, 17, 18, 19\}$

d $B' = \{10, 11, 13, 14, 16, 17, 19, 20\}$

e $A \cap B = \{12, 15\}$ **f** $A \cup B = \{10, 12, 15, 18, 20\}$

g $A' \cap B = \{18\}$

h $A' \cup B = \{11, 12, 13, 14, 15, 16, 17, 18, 19\}$

i $A \cap B' = \{10, 20\}$

j $A \cup B' = \{10, 11, 12, 13, 14, 15, 16, 17, 19, 20\}$

k $A' \cap B' = \{11, 13, 14, 16, 17, 19\}$

l $A' \cup B' = \{10, 11, 13, 14, 16, 17, 18, 19, 20\}$

3 **a** **i** 7 **ii** 3 **iii** 4 **iv** 4 **v** 3

b For any set S within a universal set U, $n(S) + n(S') = n(U)$.

4 **a** 9 **b** 11

5 As $P \subseteq Q$, then all elements of P are in Q.

$\therefore$ if an element is not in Q then it is not in P. $\therefore$ $Q' \subseteq P'$

6 Let $U = \{2, 3, 4, ...\}$ and $P = \{\text{primes}\}$.

$P' = \{\text{composites}\}$ which is an infinite set.

Let $U = \{0, 1, 2, 3, ...\}$ and $P = \{1, 2, 3, ...\}$.

$P' = \{0\}$ which is a finite set.

EXERCISE 2D

1

Number	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
6	✓	✓	✓	✓
$-\frac{3}{8}$	✗	✗	✓	✓
1.8	✗	✗	✓	✓
$1.\overline{8}$	✗	✗	✓	✓
-17	✗	✓	✓	✓
$\sqrt{64}$	✓	✓	✓	✓
$\frac{\pi}{2}$	✗	✗	✗	✓
$\sqrt{-3}$	✗	✗	✗	✗
$-\sqrt{3}$	✗	✗	✗	✓

2 **a** false **b** true **c** false **d** true

e false **f** true **g** true **h** false

3 **a** true **b** true **c** false **d** true

e false **f** true **g** true **h** false

4 **a** finite **b** infinite **c** infinite **d** infinite

5 $\mathbb{Z}^- \cup \{0\}$

EXERCISE 2E

1 **a** **i** The set of all x such that x is an integer between -1 and 7, including -1 and 7.

ii $A = \{-1, 0, 1, 2, 3, 4, 5, 6, 7\}$ **iii** 9

b **i** The set of all x such that x is a natural number between -2 and 8.

ii $A = \{0, 1, 2, 3, 4, 5, 6, 7\}$ **iii** 8

c **i** The set of all real x such that x is greater than or equal to 0, and less than or equal to 1.

ii not possible

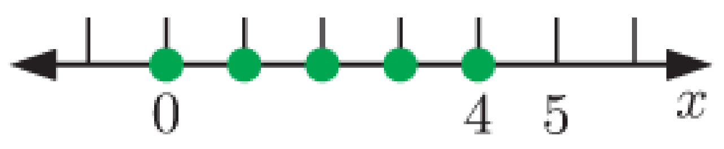
iii A is an infinite set, $n(A)$ is undefined.

d **i** The set of all x such that x is a rational number greater than or equal to 5, and less than or equal to 6.

ii not possible

iii A is an infinite set, $n(A)$ is undefined.

2 **a**



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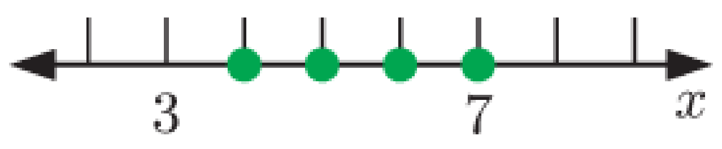

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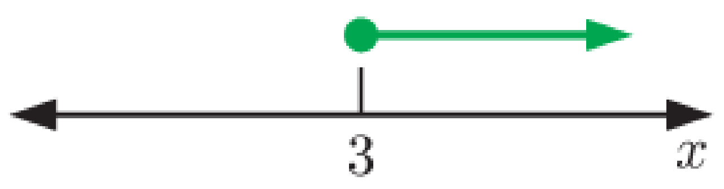

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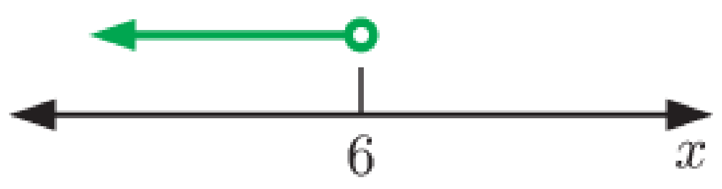

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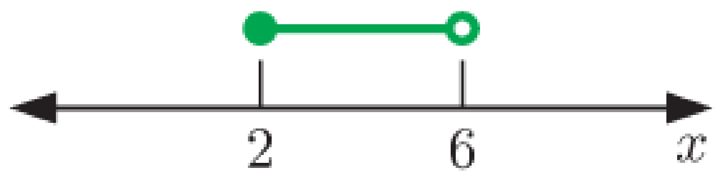

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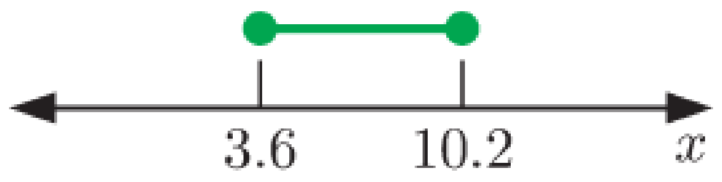

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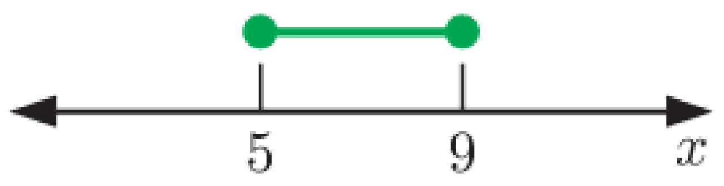

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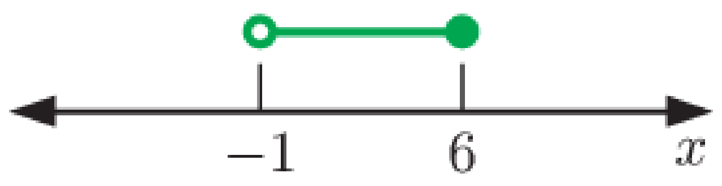

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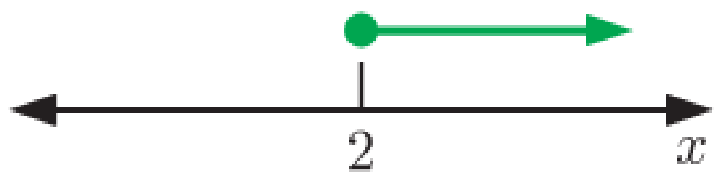

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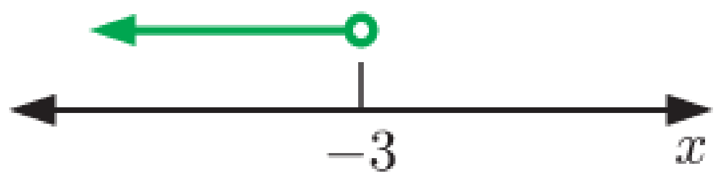

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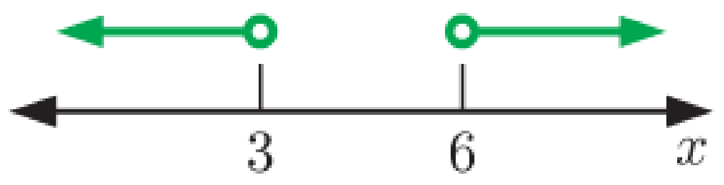

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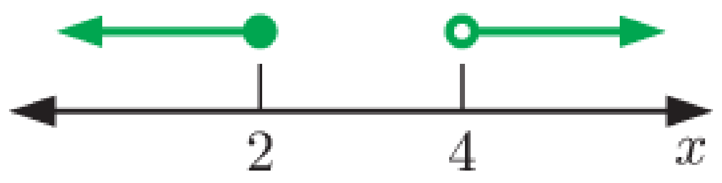

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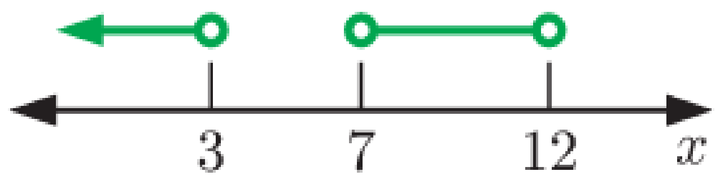

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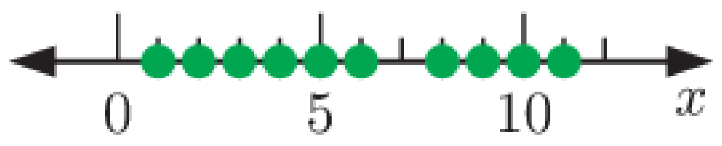

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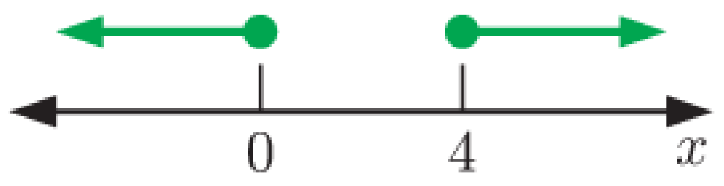

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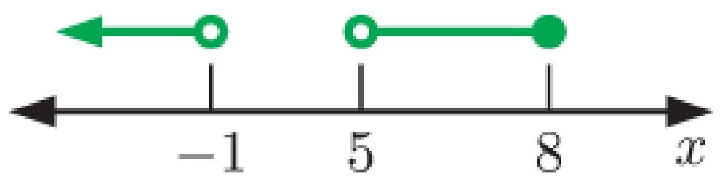


3 **a** $\{x \in \mathbb{Z} \mid -100 < x < 100\}$ **b** $\{x \in \mathbb{R} \mid x > 1000\}$

c $\{x \in \mathbb{Q} \mid 2 \leqslant x \leqslant 3\}$

4 **a** $\{x \mid x \geqslant 8\}$ **b** $\{x \mid -1 \leqslant x < 4\}$

c $\{x \in \mathbb{Z} \mid -3 < x < 4\}$

d $\{x \in \mathbb{N} \mid x \leqslant 4\} \cup \{x \in \mathbb{N} \mid x = 6\}$

5 **a** $x \in [-3, 2[$ **b** $x \in [3, \infty[$ **c** $x \in]0, 2[$

d $x \in [1, 4] \cup [6, \infty[$

6 **a** $A \subseteq B$ **b** $A \not\subseteq B$ **c** $A \subseteq B$ **d** $A \subseteq B$

e $A \not\subseteq B$ **f** $A \not\subseteq B$

7 **a** $A = \{2, 3, 4, 5, 6, 7\}$ **b** $A' = \{0, 1, 8\}$

c $B = \{5, 6, 7, 8\}$ **d** $B' = \{0, 1, 2, 3, 4\}$

e $A \cap B = \{5, 6, 7\}$ **f** $A \cup B = \{2, 3, 4, 5, 6, 7, 8\}$

g $A \cap B' = \{2, 3, 4\}$

8 **a** $P = \{1, 2, 4, 7, 14, 28\}$, $Q = \{1, 2, 4, 5, 8, 10, 20, 40\}$

b $P \cap Q = \{1, 2, 4\}$

c $P \cup Q = \{1, 2, 4, 5, 7, 8, 10, 14, 20, 28, 40\}$

d $n(P) + n(Q) - n(P \cap Q) = 6 + 8 - 3 = 11$
$= n(P \cup Q)$

9 **a** $C = \{-4, -3, -2, -1\}$
$D = \{-7, -6, -5, -4, -3, -2, -1\}$

b $C \cap D = \{-4, -3, -2, -1\}$

c $C \cup D = \{-7, -6, -5, -4, -3, -2, -1\}$

d $n(C) + n(D) - n(C \cap D) = 4 + 7 - 4 = 7$
$= n(C \cup D)$

10 **a** $A = \{6, 12, 18, 24, 30\}$, $B = \{1, 2, 3, 5, 6, 10, 15, 30\}$,
$C = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$

b **i** $A \cap B = \{6, 30\}$ **ii** $B \cap C = \{2, 3, 5\}$

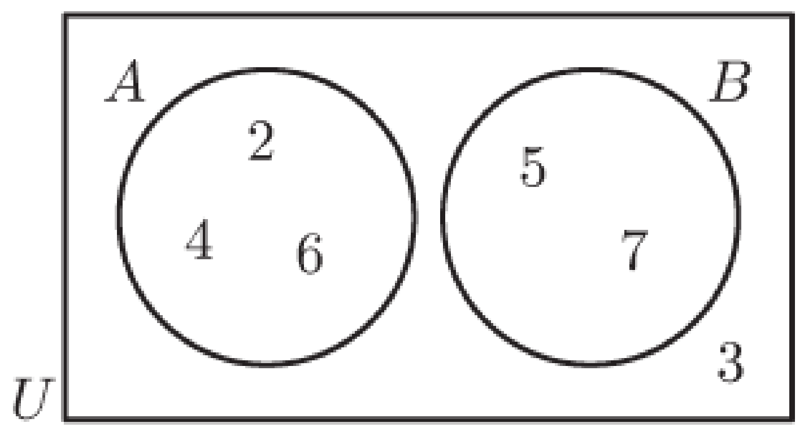
iii $A \cap C = \varnothing$ **iv** $A \cap B \cap C = \varnothing$

v $A \cup B \cup C = \{1, 2, 3, 5, 6, 7, 10, 11, 12, 13, 15, 17, 18, 19, 23, 24, 29, 30\}$

c $n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$
$- n(A \cap C) + n(A \cap B \cap C)$
$= 5 + 8 + 10 - 2 - 3 - 0 + 0$
$= 18$
$= n(A \cup B \cup C)$

EXERCISE 2F

1 **a**



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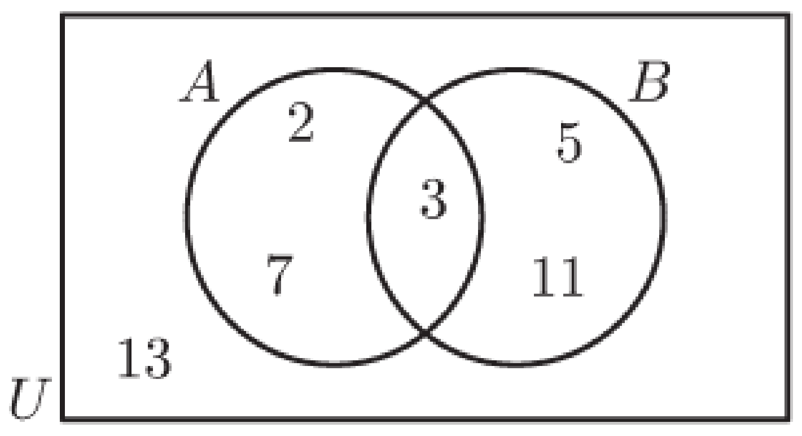

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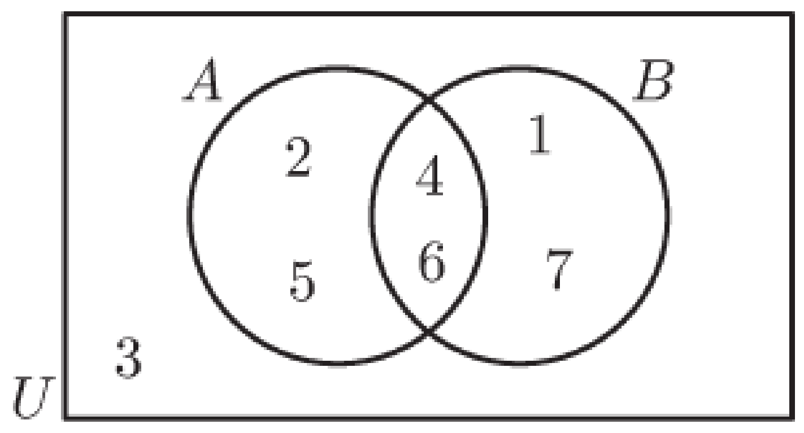

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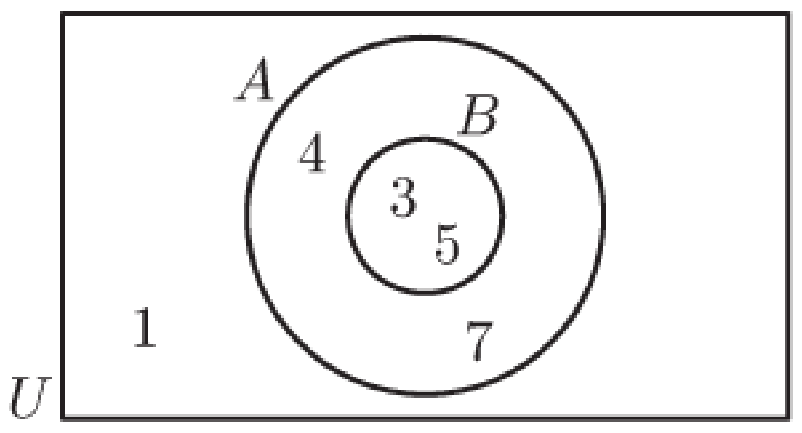
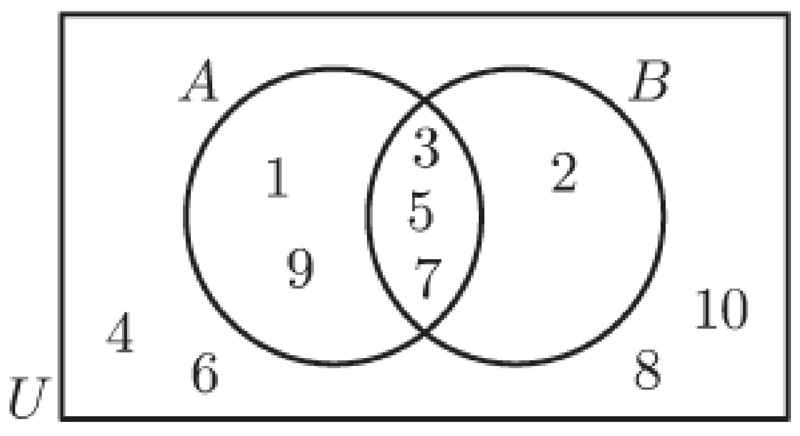


2 **a** $A = \{1, 3, 5, 7, 9\}$
$B = \{2, 3, 5, 7\}$

b $A \cap B = \{3, 5, 7\}$
$A \cup B = \{1, 2, 3, 5, 7, 9\}$

c



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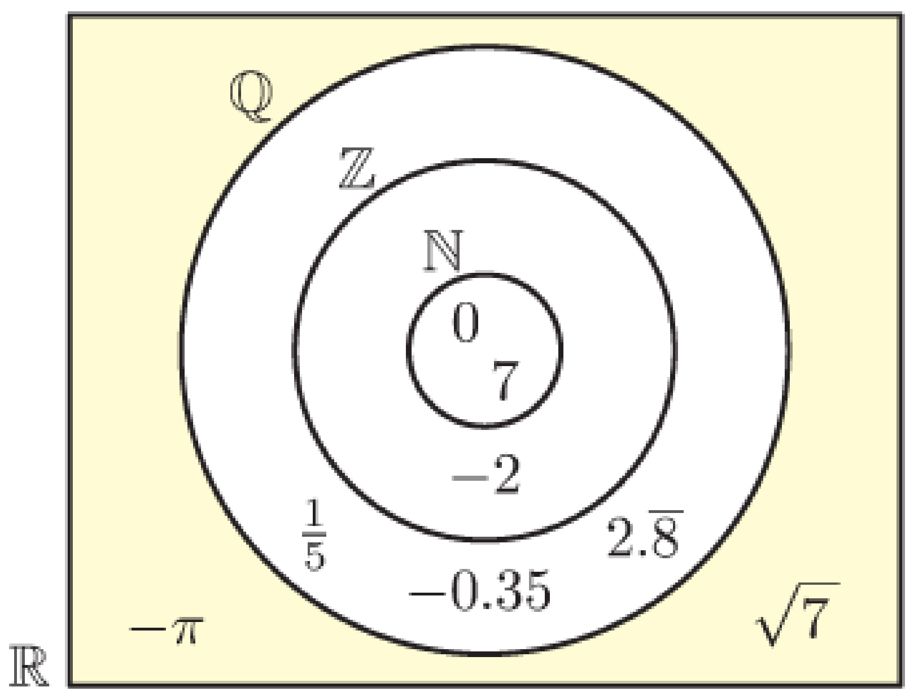
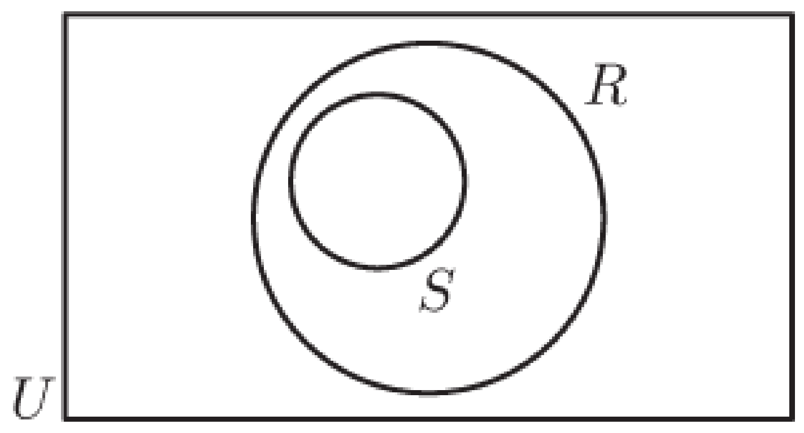


4 **a**



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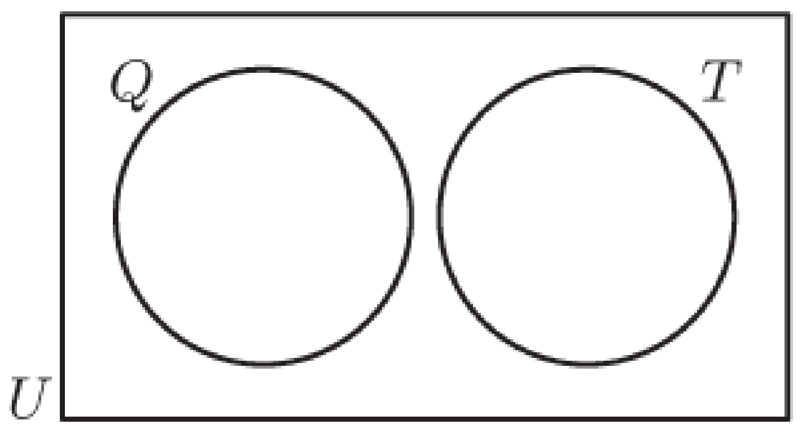


5 **a** **i** $A = \{a, b, c, d, h, j\}$ **ii** $B = \{a, c, d, e, f, g, k\}$

iii $C = \{a, b, e, f, i, l\}$ **iv** $A \cap B = \{a, c, d\}$

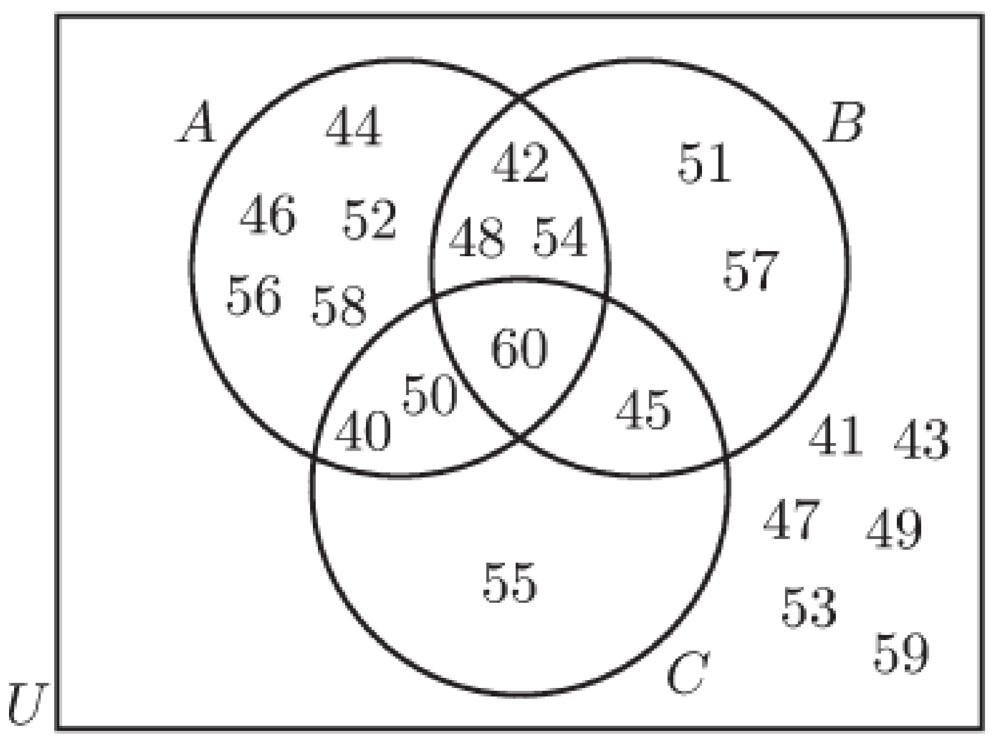
v $A \cup B = \{a, b, c, d, e, f, g, h, j, k\}$

vi $B \cap C = \{a, e, f\}$ **vii** $A \cap B \cap C = \{a\}$

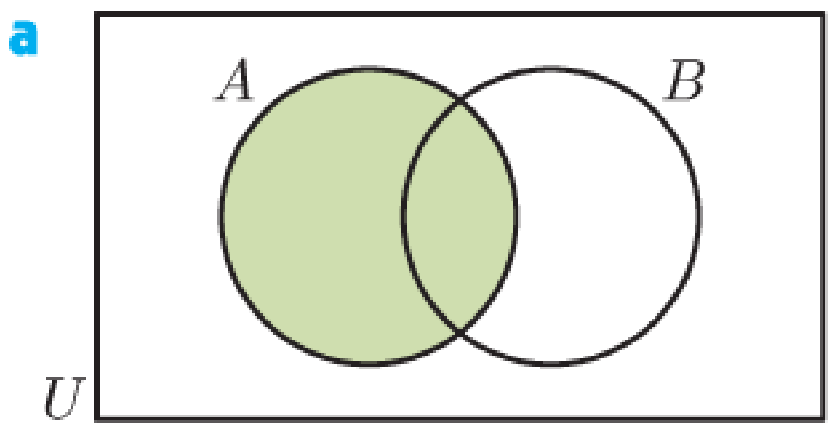
viii $A \cup B \cup C = \{a, b, c, d, e, f, g, h, i, j, k, l\}$

b $n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C)$
$- n(B \cap C) + n(A \cap B \cap C)$
$= 6 + 7 + 6 - 3 - 2 - 3 + 1$
$= 12$
$= n(A \cup B \cup C)$

6



7 **a**



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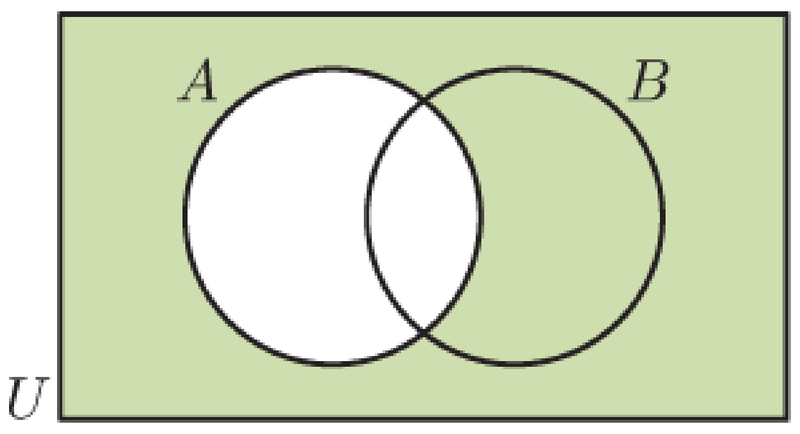

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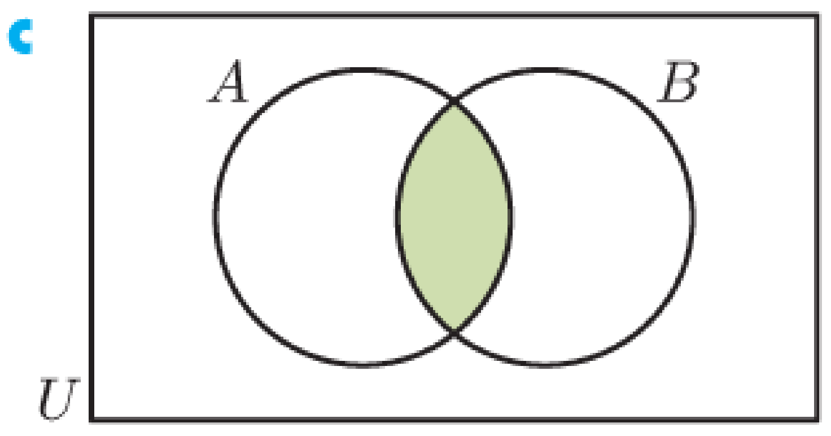

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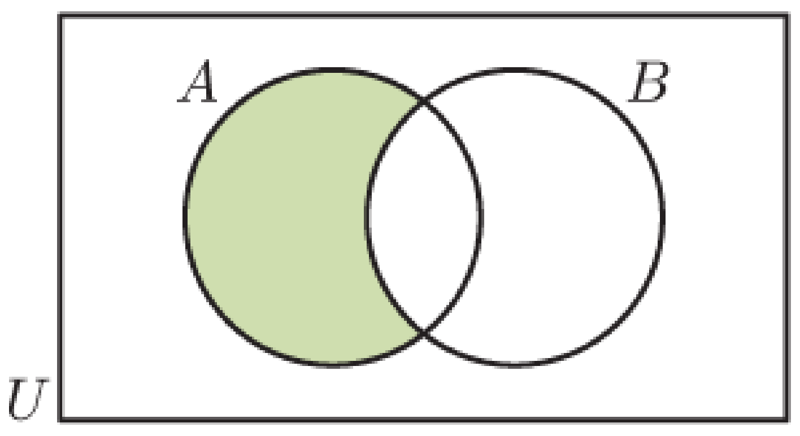

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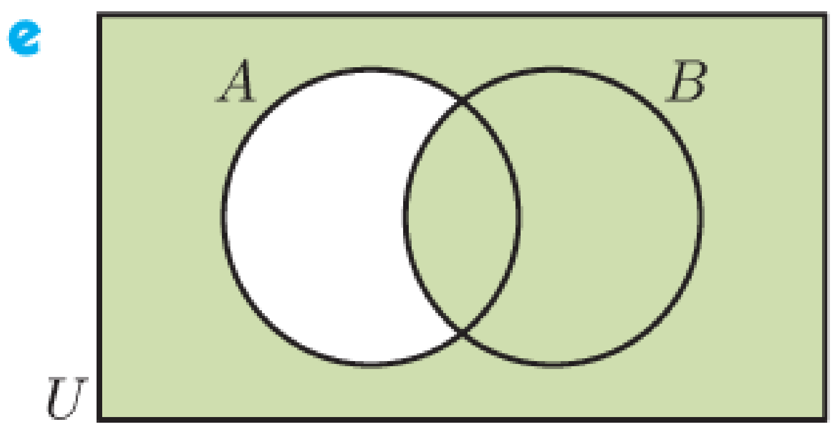

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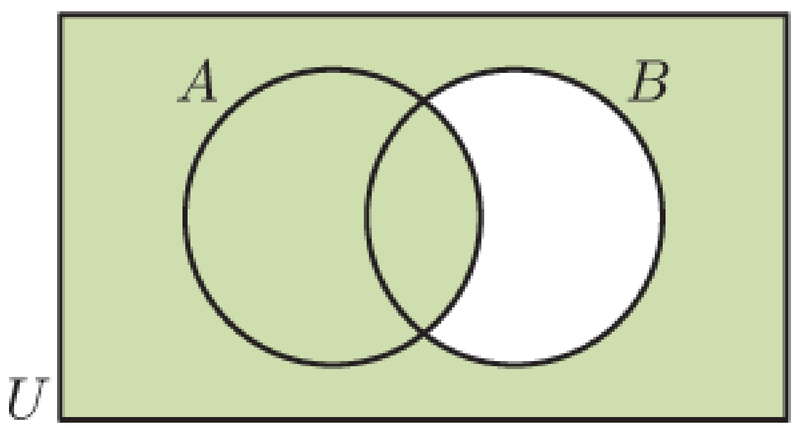

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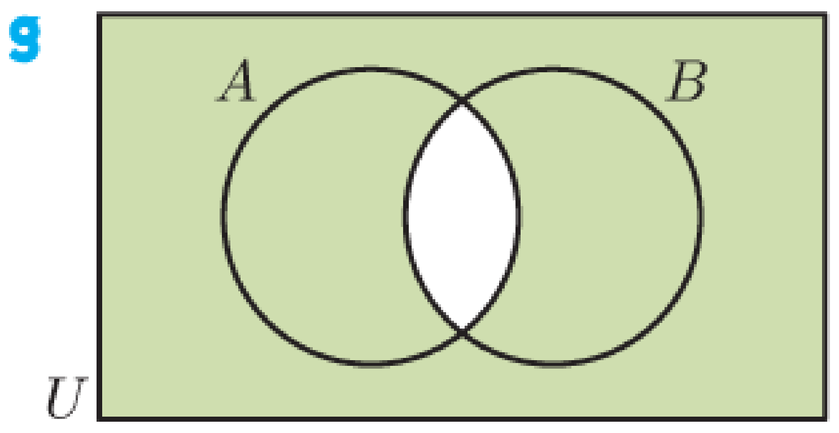

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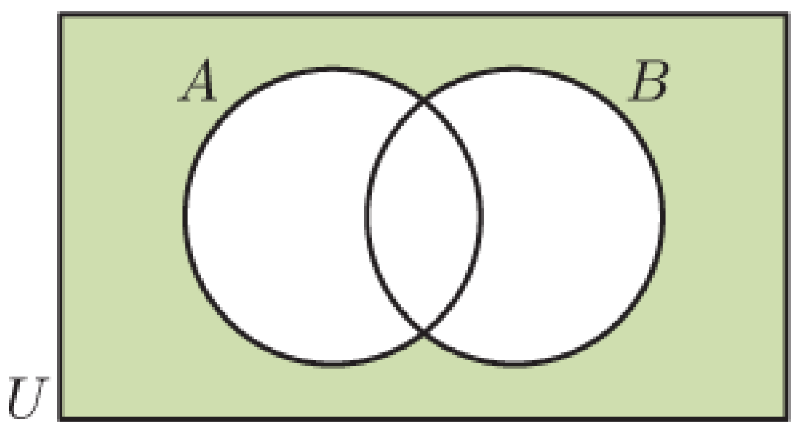
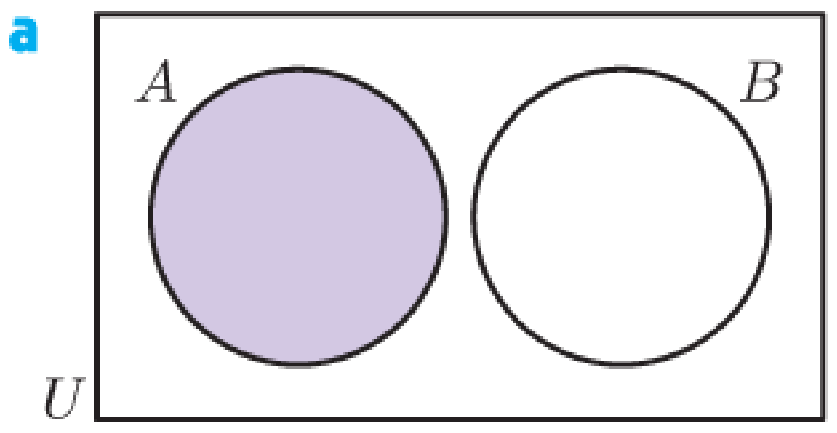


8 **a**



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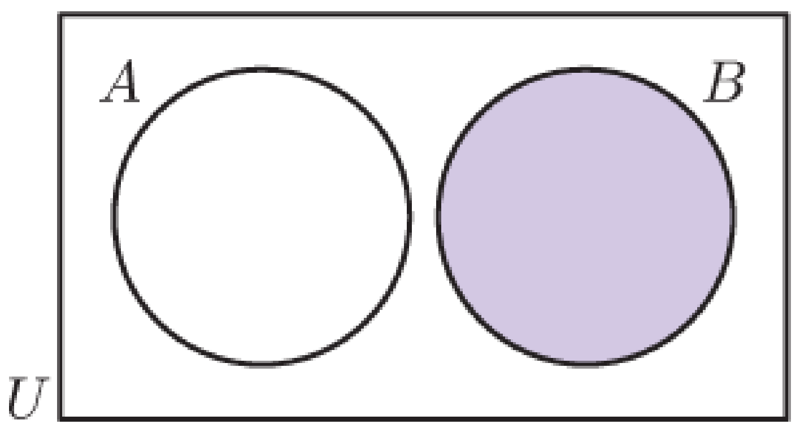

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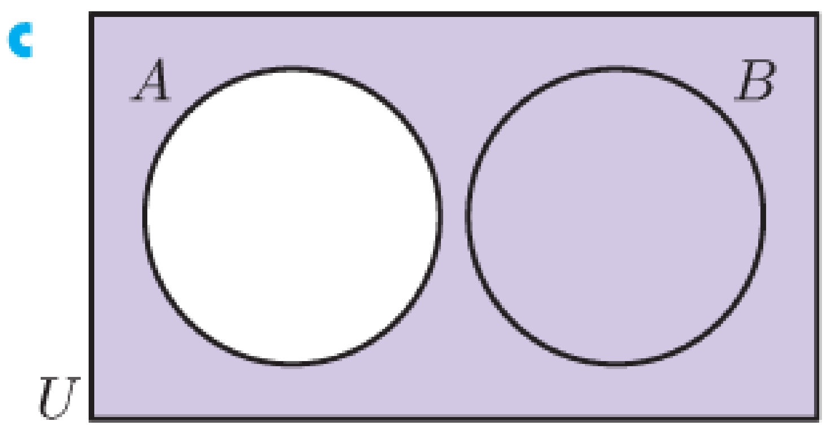

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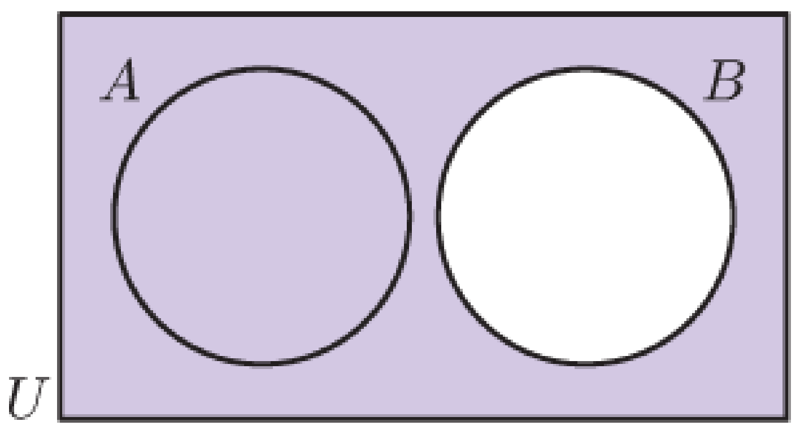

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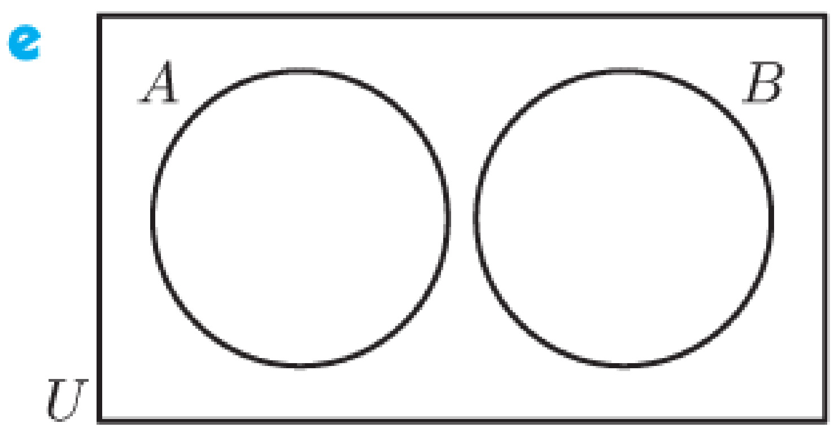

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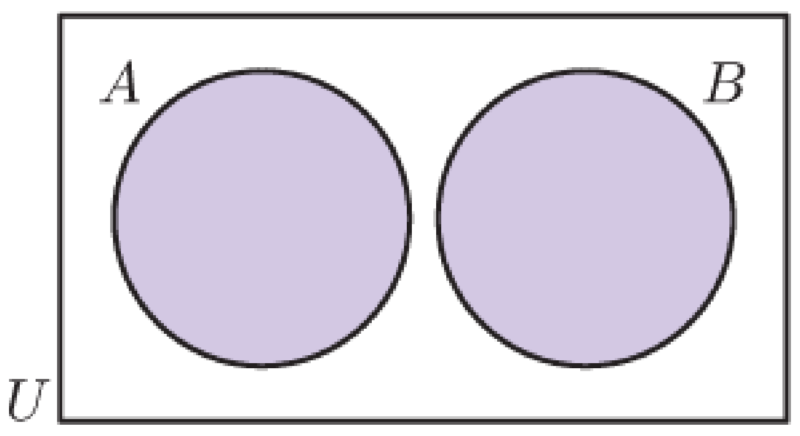

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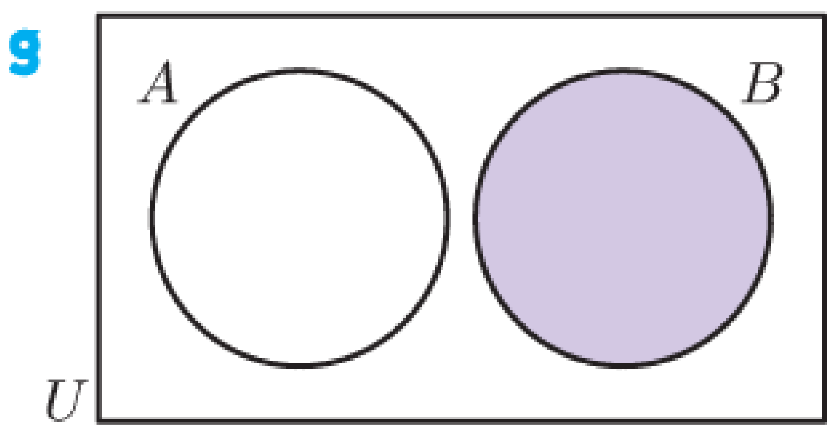

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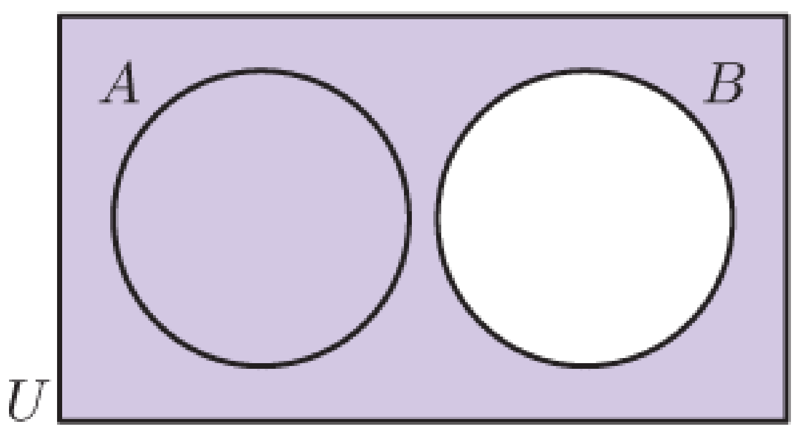

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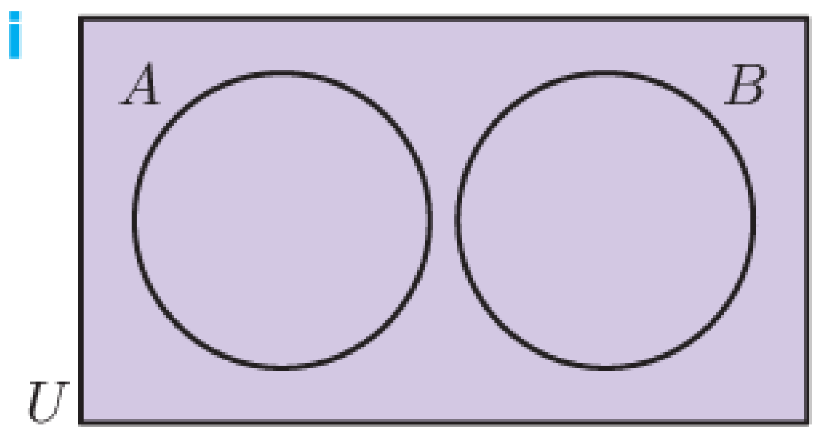
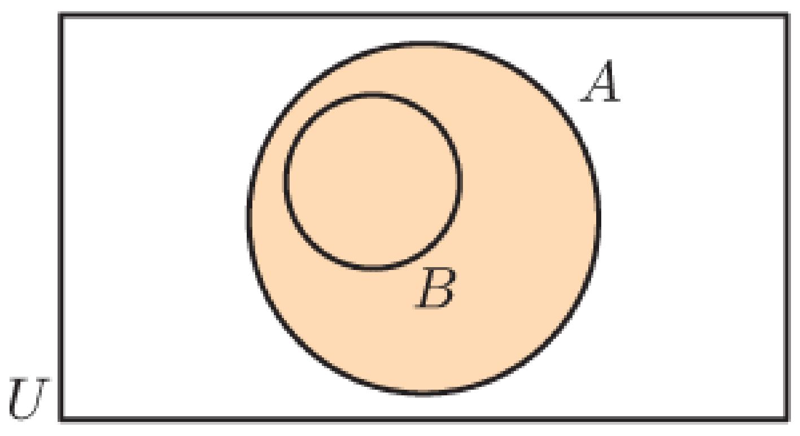


9 **a**



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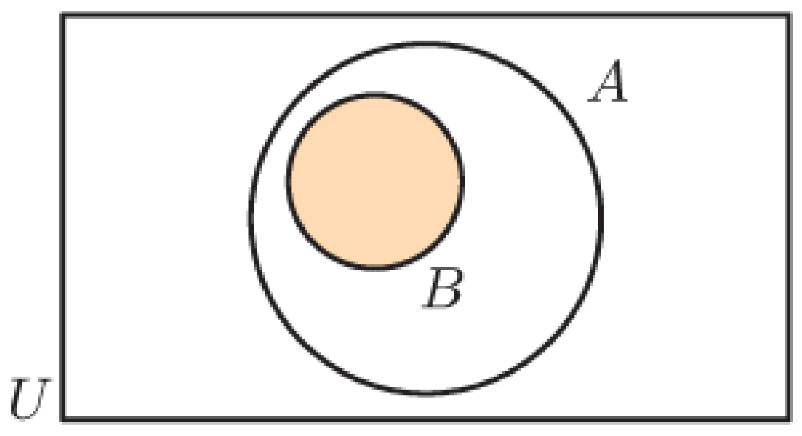
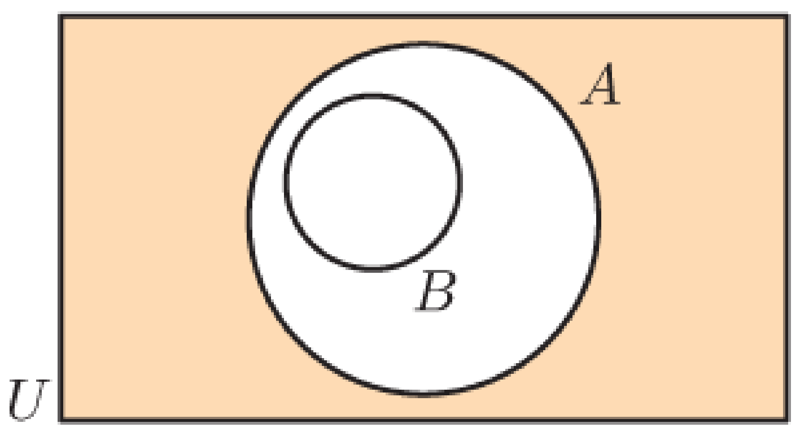


c



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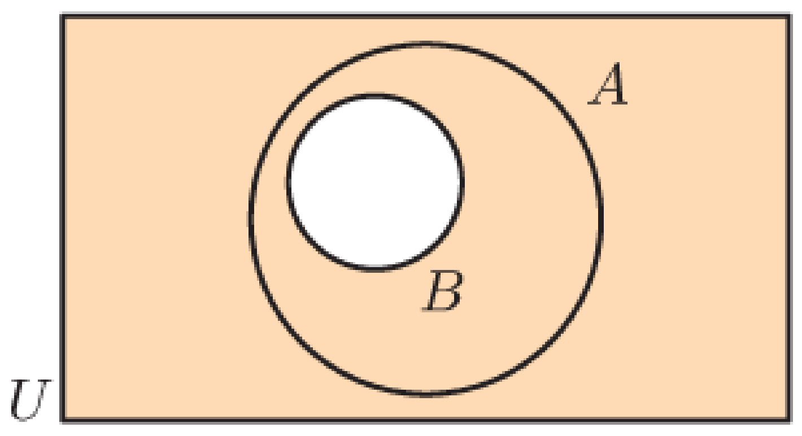

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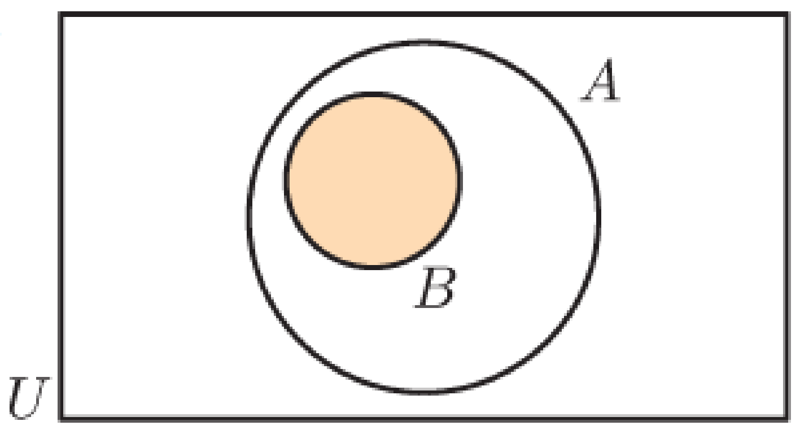
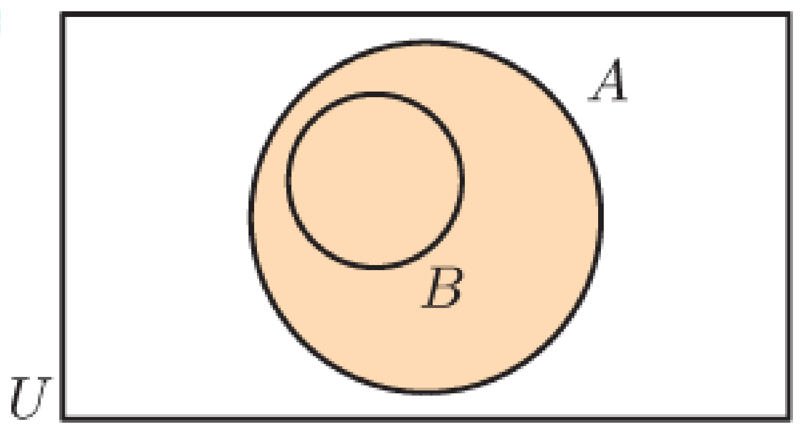


f



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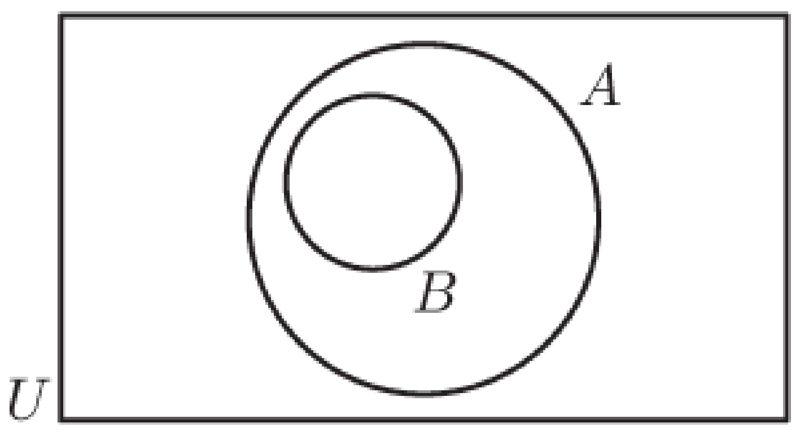
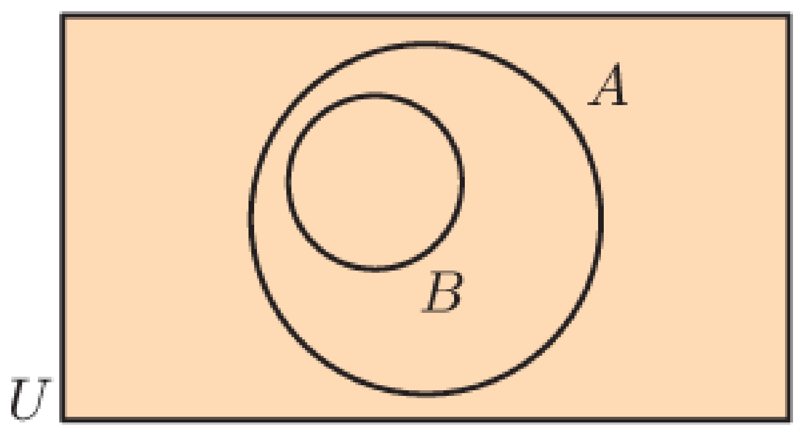


h



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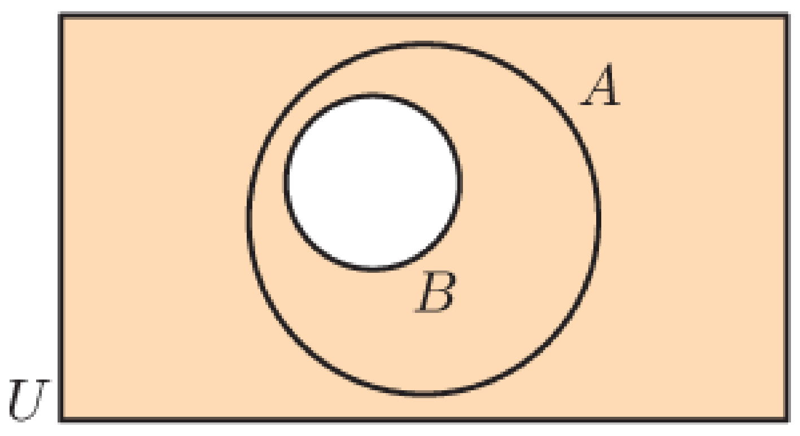
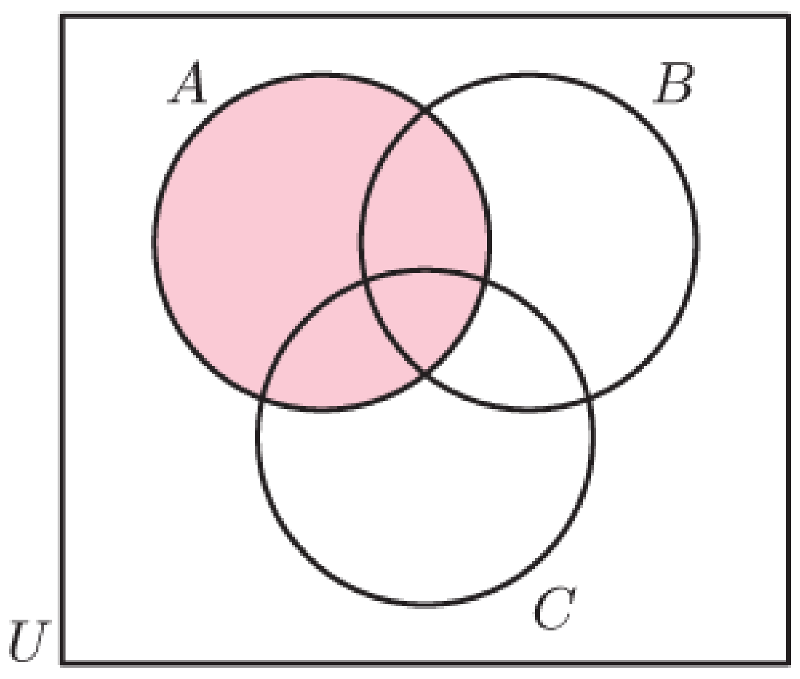


10 **a**



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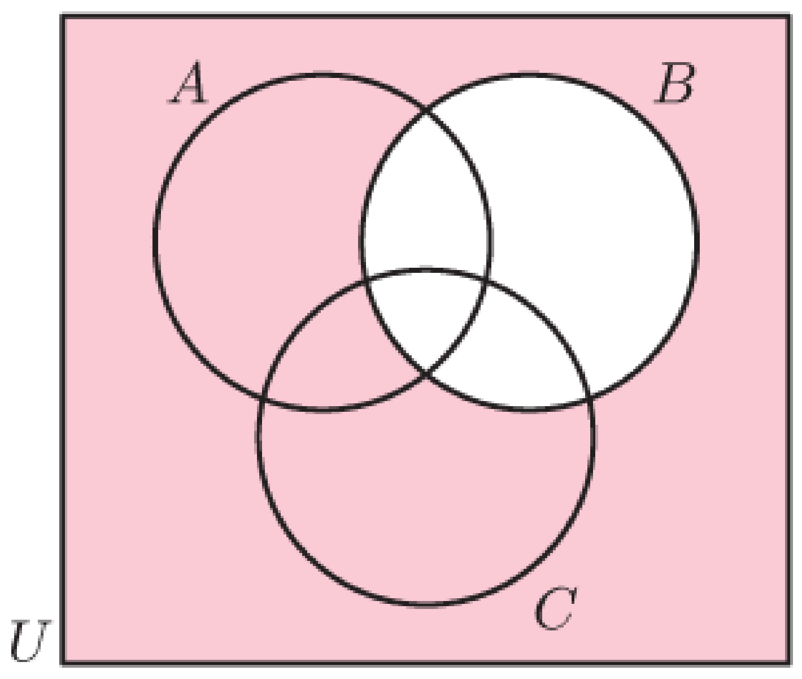
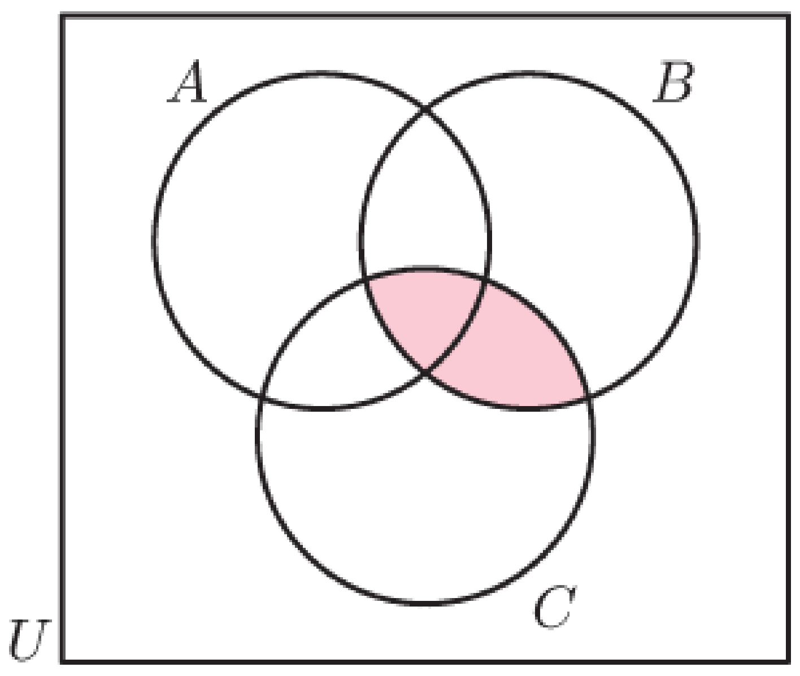
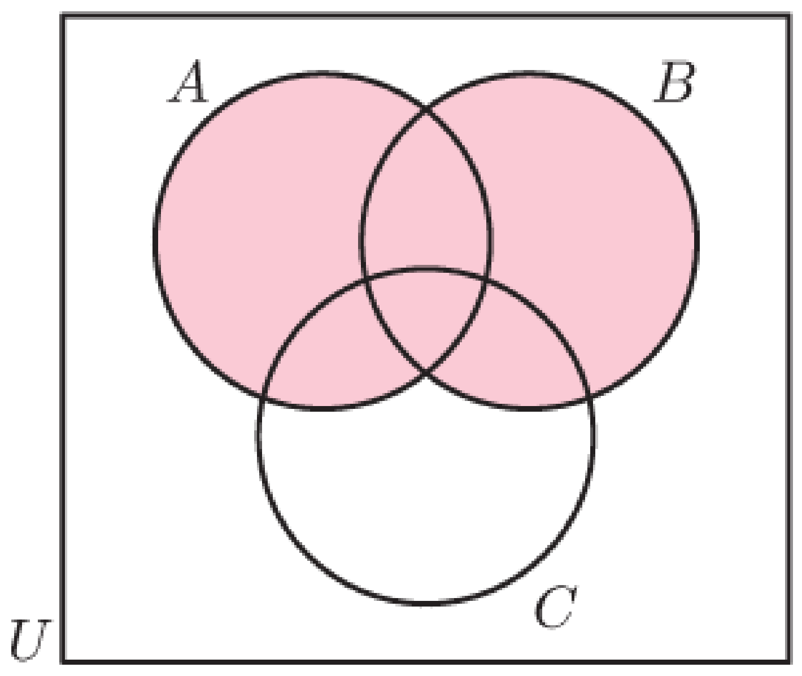


c



d



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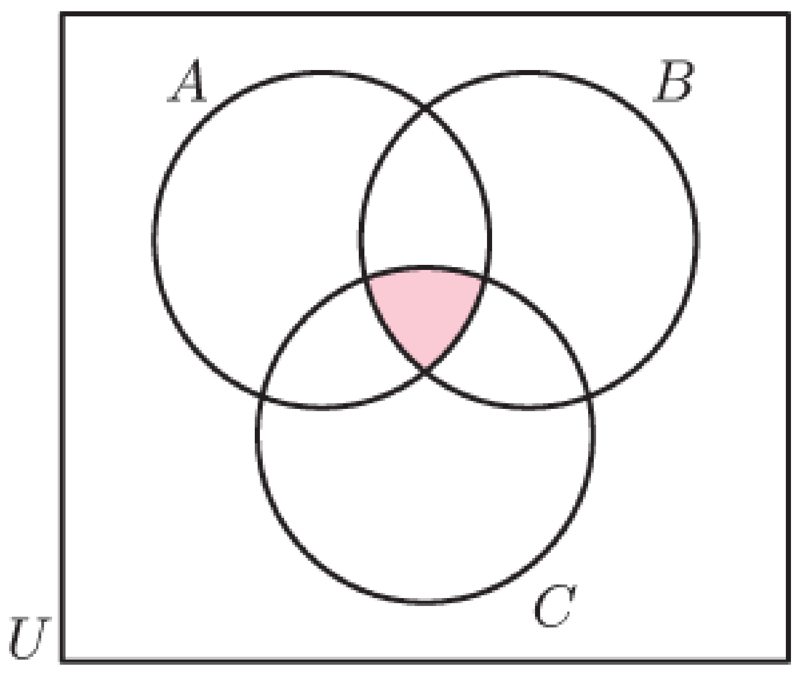

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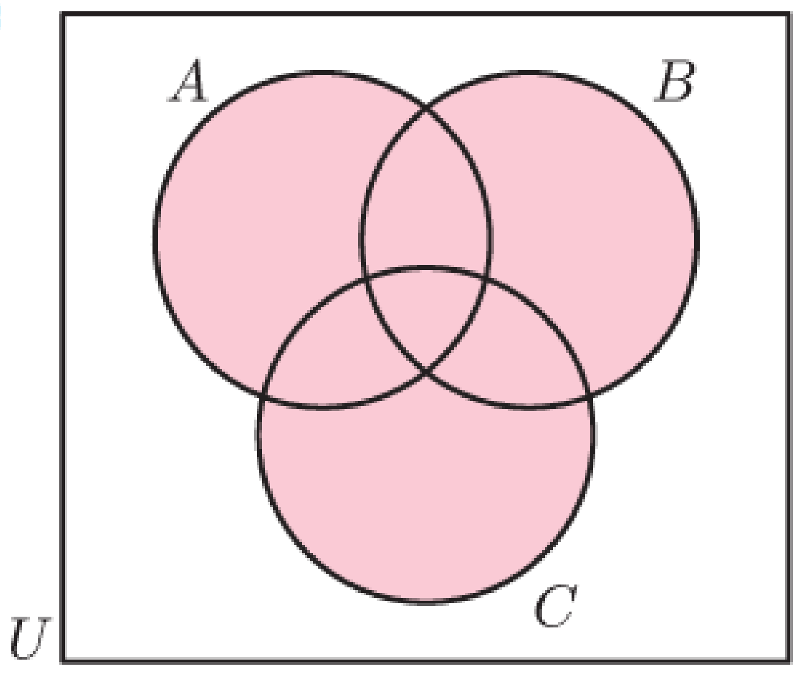
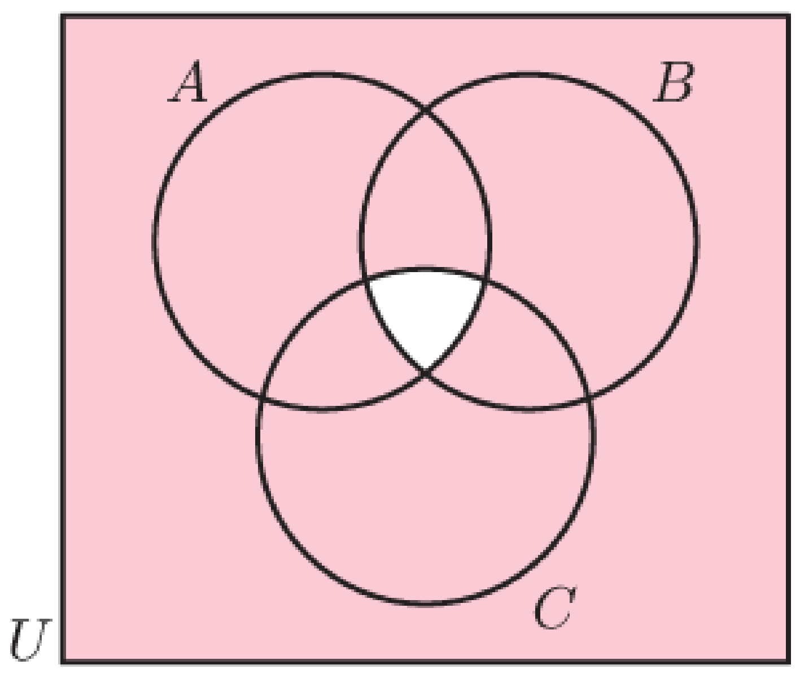


g



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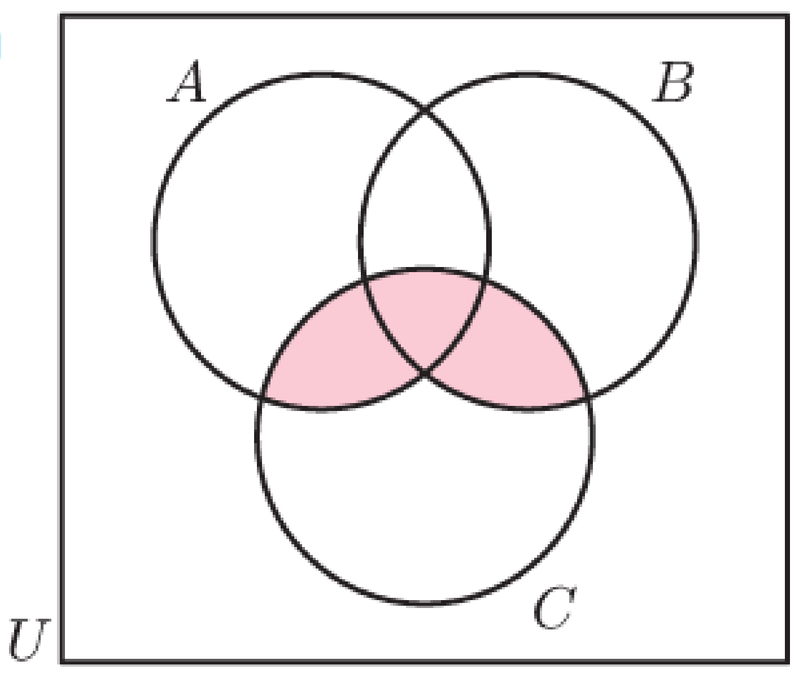
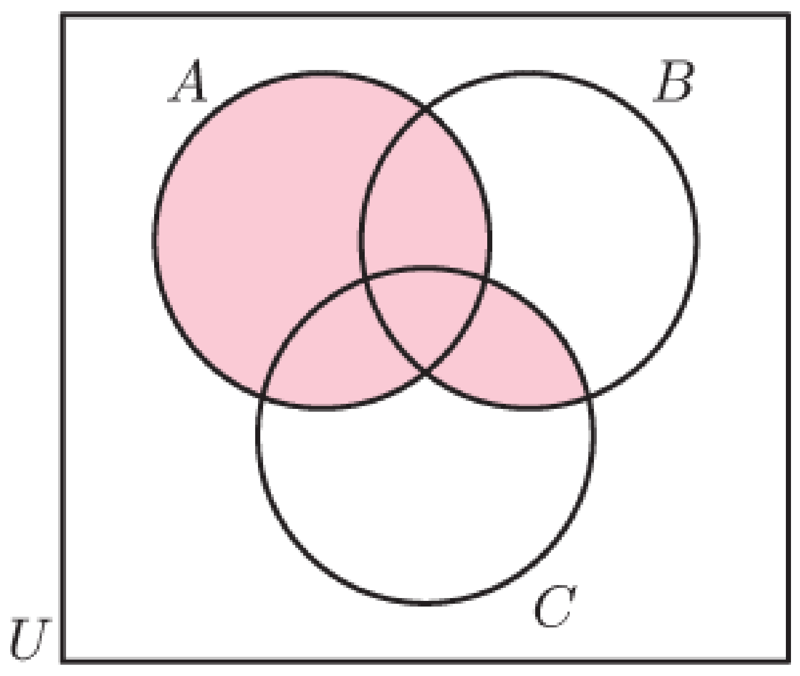
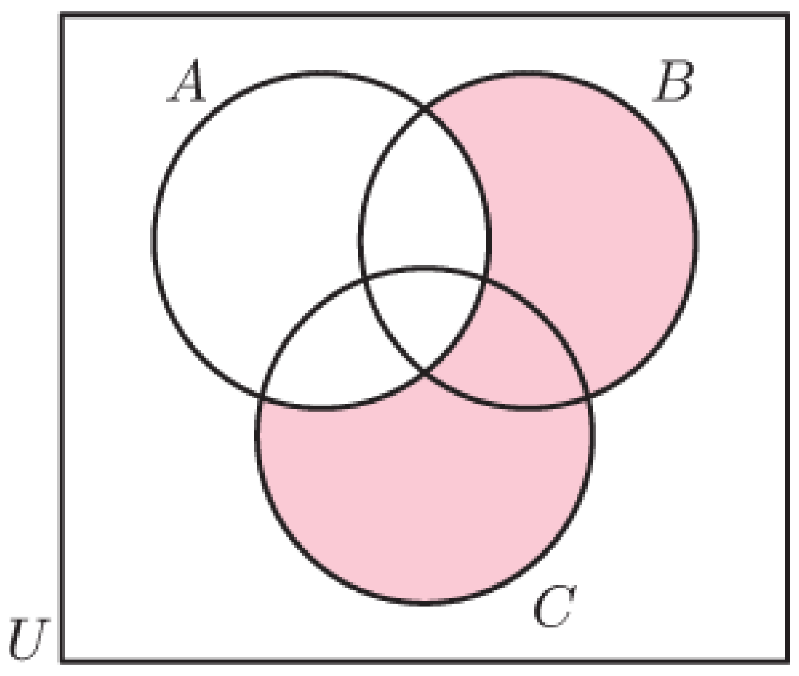


i



j



EXERCISE 2G

1 **a** 5 **b** 6 **c** 17 **d** 8 **e** 3 **f** 2

2 **a** **i** a **ii** $3a$ **iii** $2a+4$ **iv** $4a+4$
v $3a-5$ **vi** $5a-1$

b **i** $a=6$ **ii** $a=\frac{32}{5}=6.4$

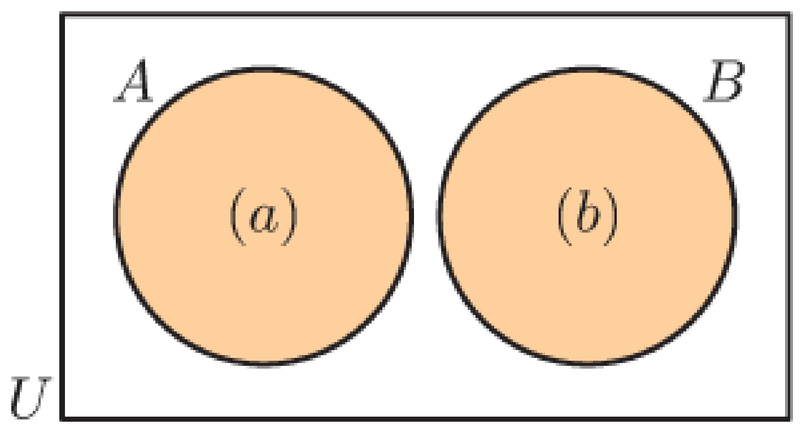
It is not possible to have a non-integer number of elements, as we have in **ii**.

$\therefore$ $n(U)$ can be equal to 29, but not equal to 31.

3 **a**
$$\begin{aligned} n(A)+n(B)-n(A\cap B) &= (a+b)+(b+c)-b \\ &= a+b+c \\ &= n(A\cup B) \end{aligned}$$

b $n(A)-n(A\cap B)=(a+b)-b=a=n(A\cap B')$

4

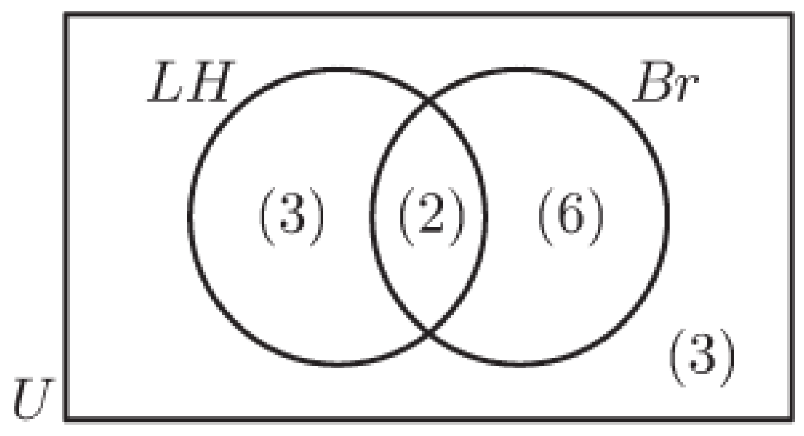


$$\begin{aligned} n(A)+n(B) &= a+b \\ &= n(A\cup B) \end{aligned}$$

5 **a** 15 **b** 4 **6** **a** 18 **b** 6 **7** **a** 7 **b** 23

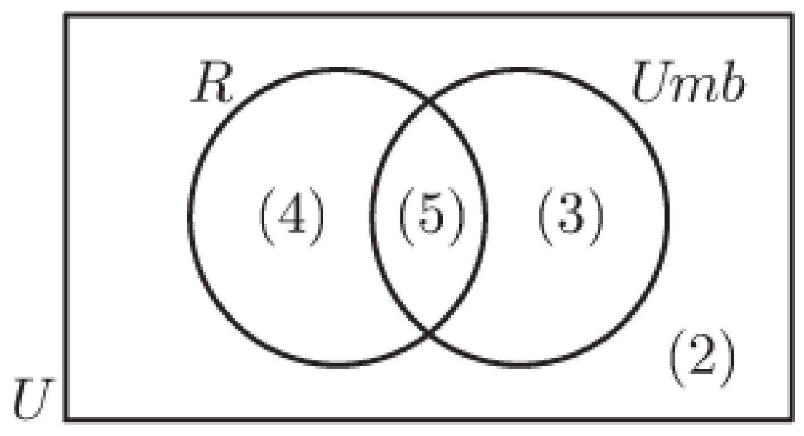
EXERCISE 2H

1 **a**



b **i** 9 cavies
ii 3 cavies
iii 3 cavies

2 **a**

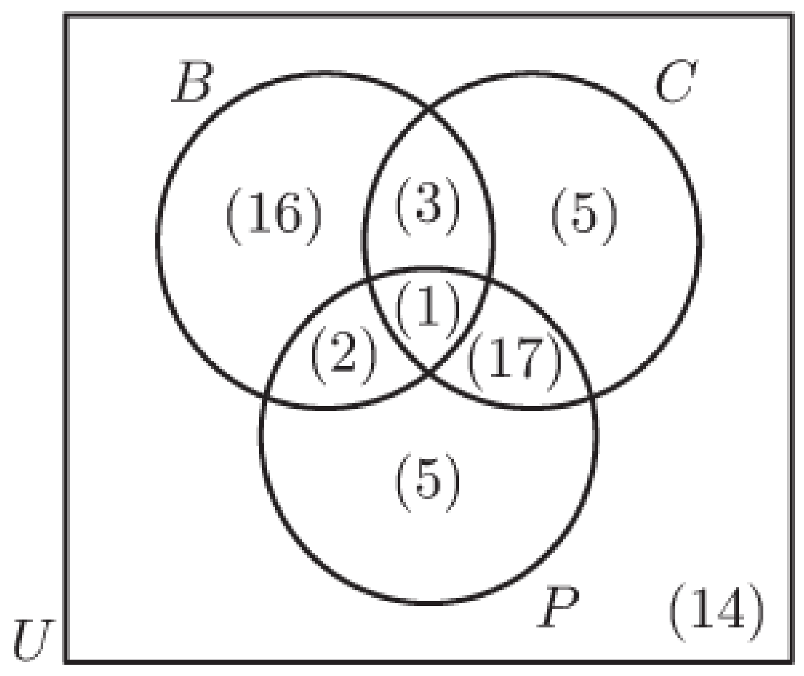


b **i** 4 days
ii 2 days

3 20 people **4** **a** 4 stalls **b** 27 stalls

5 **a** 10 movies **b** 4 movies

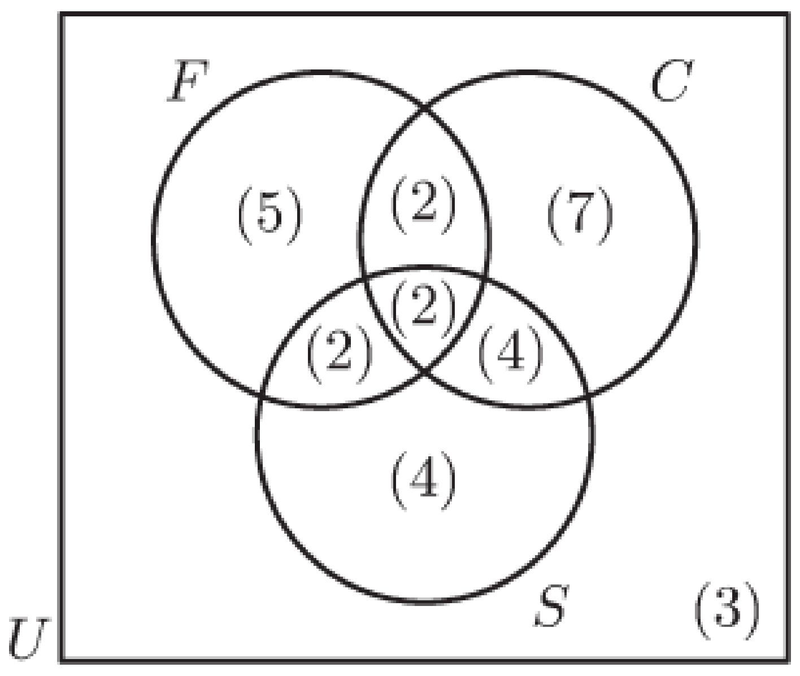
6 **a**



b **i** 16 students
ii 33 students
iii 14 students
iv 7 students

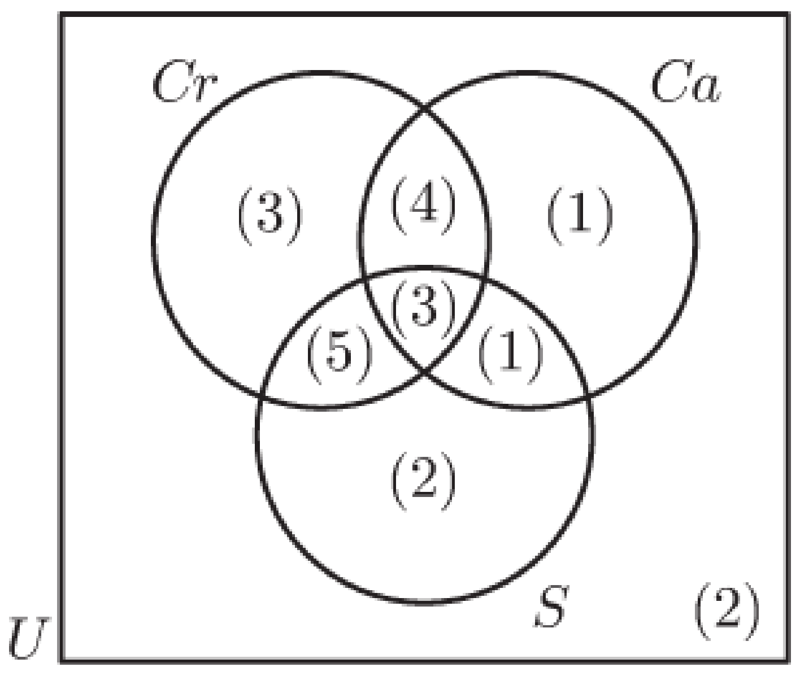
7 **a** 29 students **b** 6 students **c** 1 student **d** 11 students

8 **a**



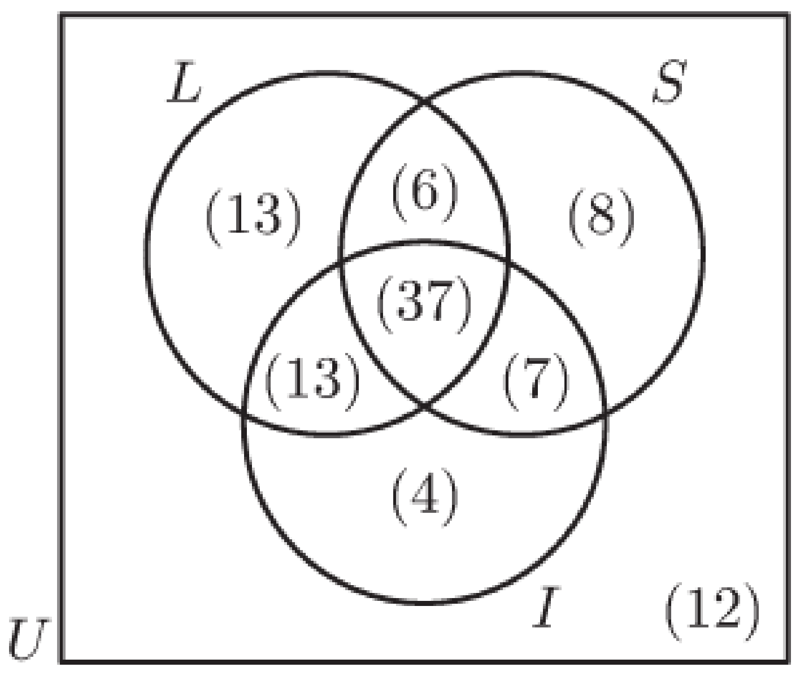
b **i** 2 students
ii 4 students
iii 4 students
iv 16 students

9 **a**



b **i** 3 farms
ii 4 farms
iii 9 farms

10 **a**



b Yes, $n(U) - n(L \cup S \cup I) = 100 - 88 = 12$ nations.

c **i** 8 nations **ii** 30 nations **iii** 7 nations

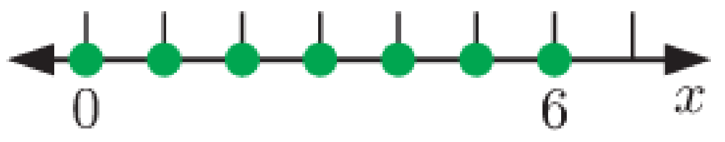
REVIEW SET 2A

1 **a** $A = \{\text{V, E, N}\}$, $B = \{\text{D, I, A, G, R, M}\}$
b $n(A) = 3$, $n(B) = 6$
c $A \cap B = \varnothing$, "VENN" and "DIAGRAM" have no letters in common.
d **i** false **ii** true **iii** true

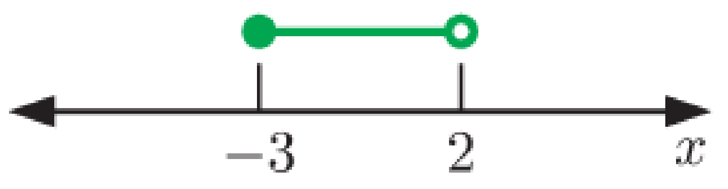
2 $A' = \{12, 18, 24, 30, 42, 48, 54, 60\}$

3 **a** true **b** false **c** true **d** false

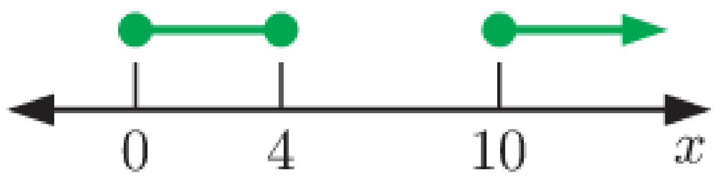
4 **a**



b



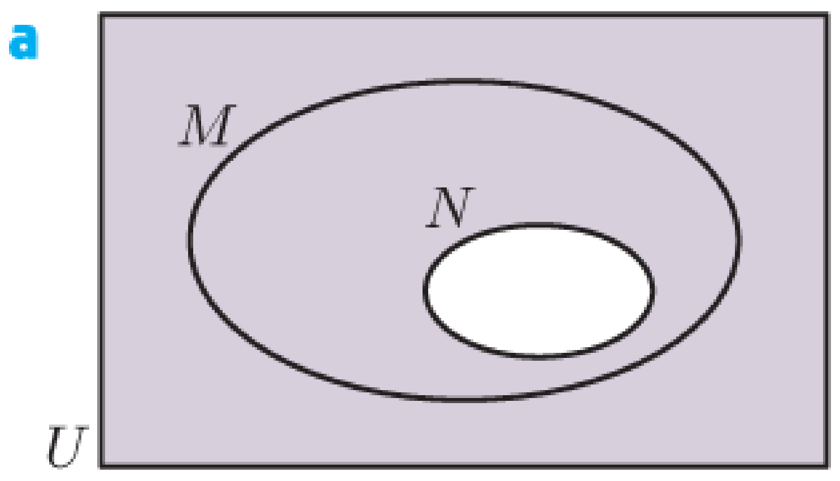
c



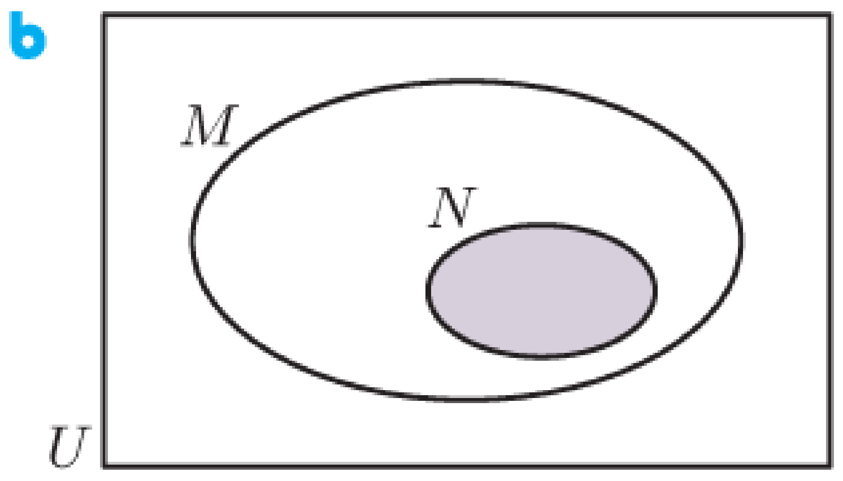
5 **a** **i** $P = \{1, 2, 3, 4, 6, 8, 12, 24\}$
ii $Q = \{1, 2, 3, 5, 6, 10, 15, 30\}$
iii $P \cap Q = \{1, 2, 3, 6\}$
iv $P \cup Q = \{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 24, 30\}$

b $n(P) + n(Q) - n(P \cap Q) = 8 + 8 - 4 = 12$
$= n(P \cup Q)$

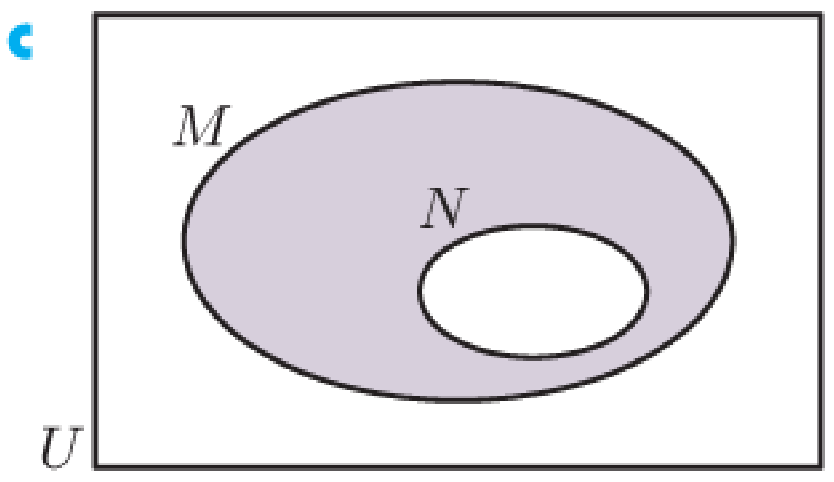
6 **a**



b

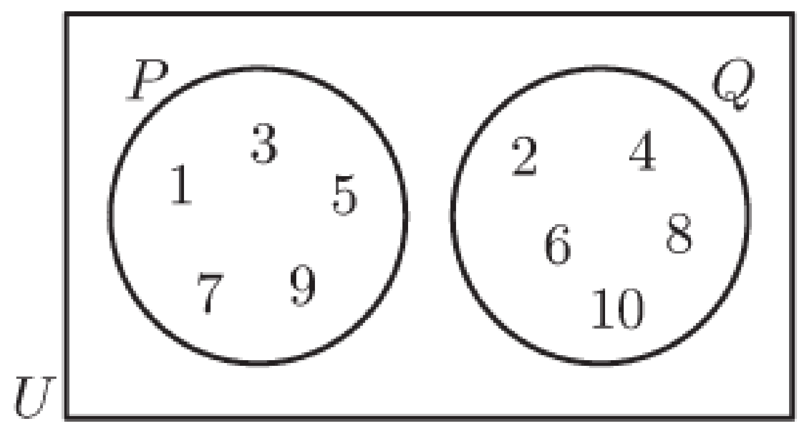


c

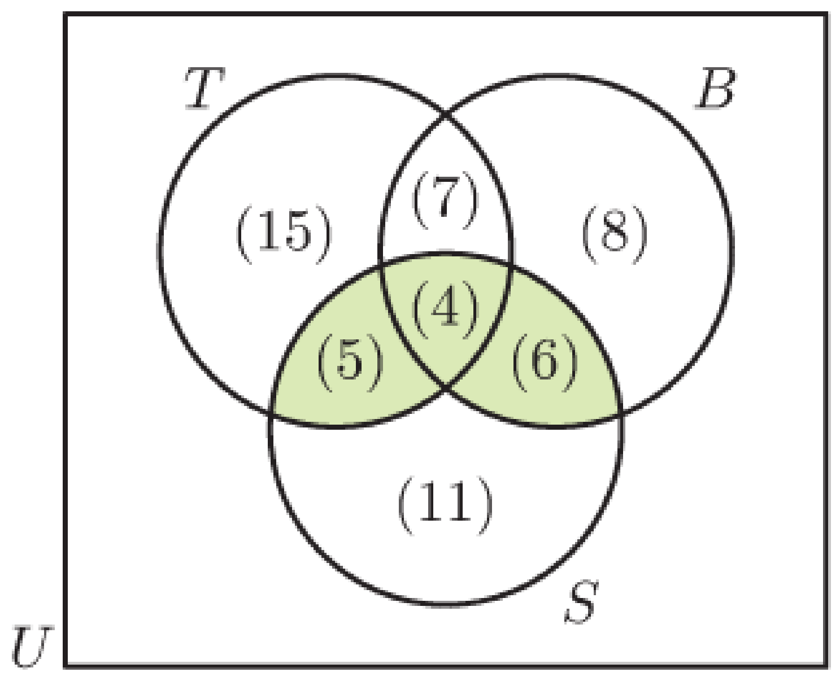


7 As $P \subseteq Q$, then all elements in P are also in Q.
An element which is not in Q must not be in P either.
$\therefore$ $P \cap Q' = \varnothing$, P and Q' are disjoint.

8 **a** $P = \{1, 3, 5, 7, 9\}$, $Q = \{2, 4, 6, 8, 10\}$
b P and Q are disjoint. **c**

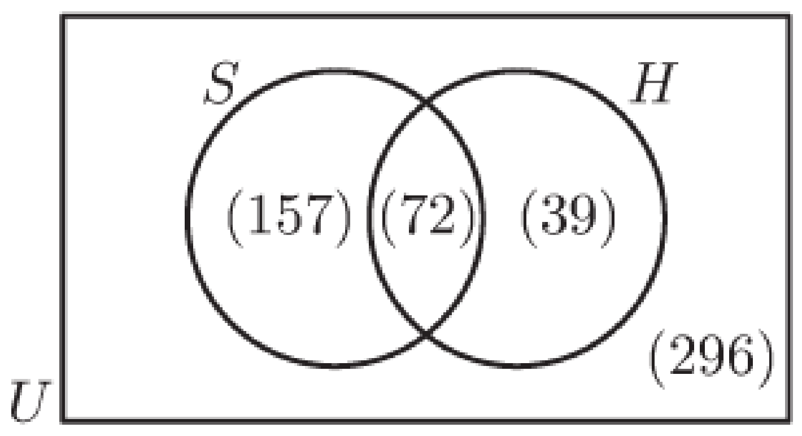


9 **a** 56 members
b **i** 8 members
ii 25 members
iii 5 members
c



d 11 members

10 **a**



b **i** 72 students
ii 39 students
iii 268 students

11 8 dishes

12 **a** 9 students **b** 7 students **c** 17 students

REVIEW SET 2B

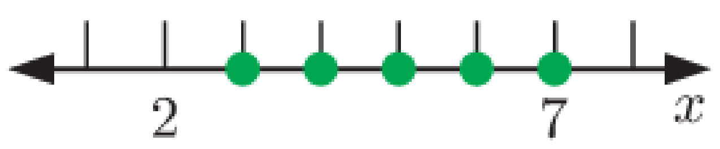
1 $\varnothing$, $\{1\}$, $\{3\}$, $\{5\}$, $\{1, 3\}$, $\{1, 5\}$, $\{3, 5\}$, $\{1, 3, 5\}$

2 **a** $S \cap T = \varnothing$ **b** $s + t$

3 **a** $\{x \in \mathbb{R} \mid 5 < x < 12\}$, infinite
b $\{x \in \mathbb{Z} \mid -4 \leqslant x < 7\}$, finite
c $\{x \in \mathbb{N} \mid x > 45\}$, infinite

4 **a** $x \in]2, 5]$ **b** $x \in [4, \infty[$
c $x \in]-\infty, -3] \cup [1, \infty[$

5 **a** $S = \{3, 4, 5, 6, 7\}$ **b**

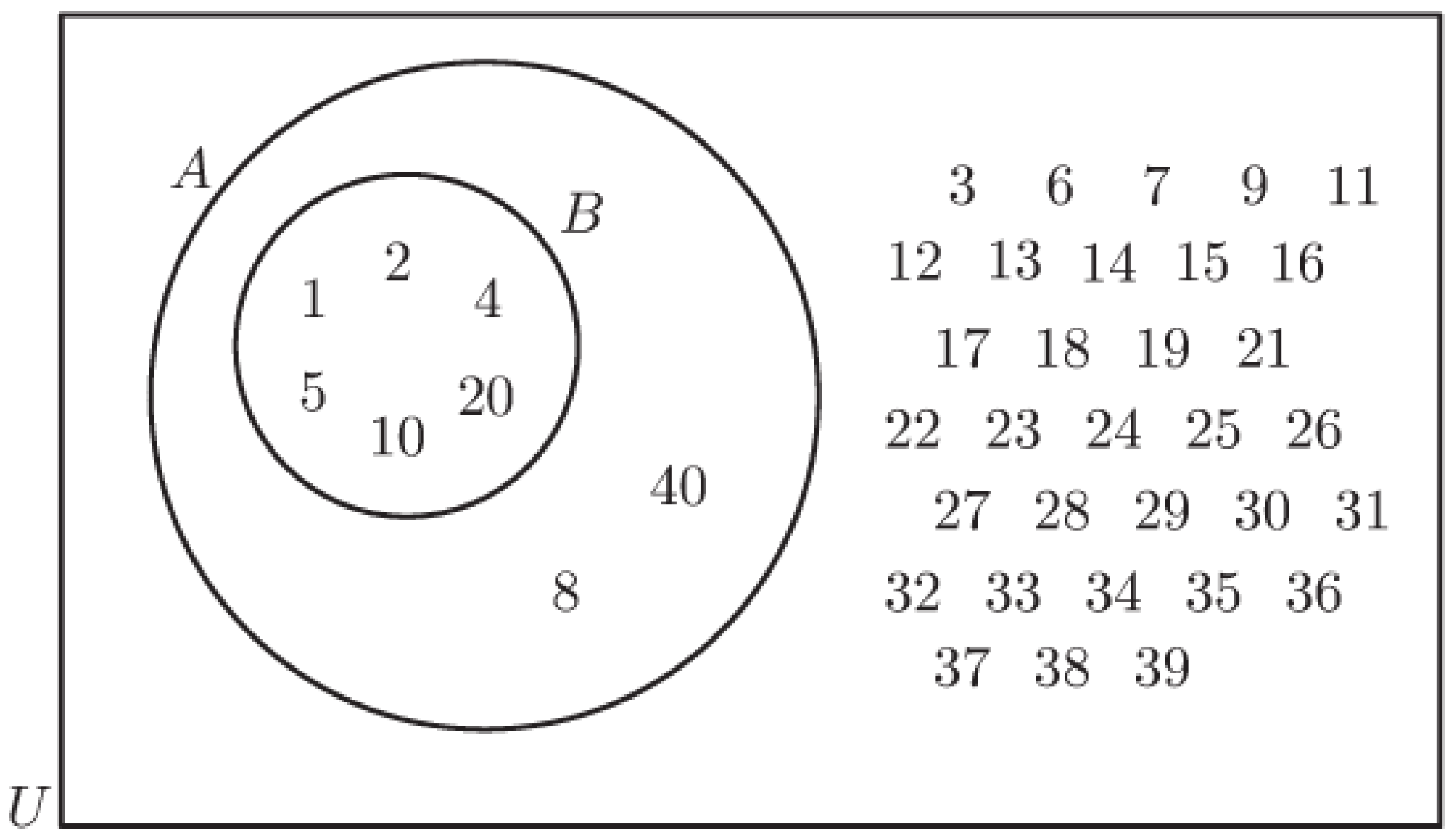


c 5

6 **a** $A \subseteq B$ **b** $A \subseteq B$ **c** $A \nsubseteq B$ **d** $A \subseteq B$

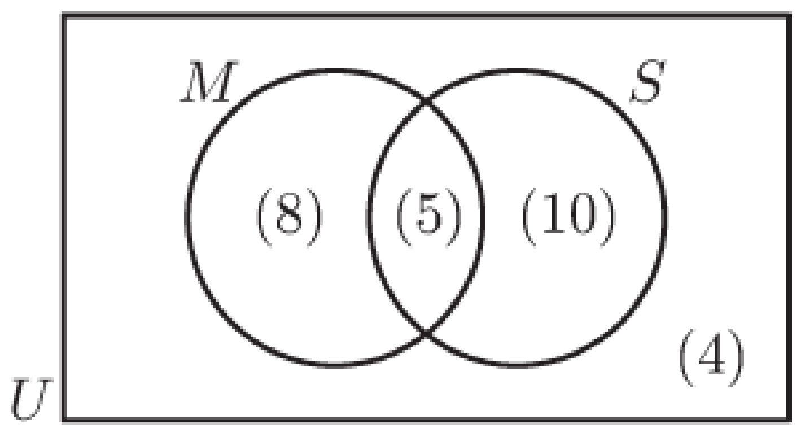
7 **a** $X' = \{\text{orange, yellow, green, blue}\}$
b $X' = \{-5, -3, -2, 0, 1, 2, 5\}$
c $X' = \{x \in \mathbb{Q} \mid x \geqslant -8\}$

8 **a** $A = \{1, 2, 4, 5, 8, 10, 20, 40\}$, $B = \{1, 2, 4, 5, 10, 20\}$
b $B \subset A$
c



9 **a** $P = \{3, 4, 5, 6, 7, 8, 9\}$ **b** 7 **c** finite
d **i** 2 and 15 are in Q, but not in P
ii $R = \{3, 6, 9\}$, all these elements are in P, but $R \neq P$
e **i** $\{9\}$ **ii** $\{9\}$ **iii** $\{2, 3, 6, 9, 15\}$

10 **a**



b **i** 27 members
ii 8 members
iii 14 members

11 4 students

12 **a** 1 delegate **b** 7 delegates **c** 15 delegates

EXERCISE 3A

1 **a** 11 **b** $\sqrt{15}$ **c** $\sqrt{30}$ **d** 42
e 45 **f** $\sqrt{90}$ **g** $-\sqrt{540}$ **h** $\sqrt{288}$
2 **a** $\sqrt{6}$ **b** $\sqrt{6}$ **c** 2 **d** $\frac{1}{2}$
e $\sqrt{\frac{1}{3}}$ **f** $\sqrt{5}$ **g** $\frac{1}{4}$ **h** $\frac{1}{4}$
3 **a** $2\sqrt{3}$ **b** $2\sqrt{5}$ **c** $3\sqrt{3}$ **d** $3\sqrt{6}$
e $5\sqrt{2}$ **f** $4\sqrt{5}$ **g** $4\sqrt{6}$ **h** $6\sqrt{3}$
4 **a** $5\sqrt{2}$ **b** $-\sqrt{2}$ **c** $2\sqrt{5}$ **d** $8\sqrt{5}$
e $-2\sqrt{5}$ **f** $9\sqrt{3}$ **g** $-3\sqrt{6}$ **h** $3\sqrt{2}$
5 **a** $2\sqrt{3}$ **b** $8\sqrt{2}$ **c** $5\sqrt{6}$ **d** $10\sqrt{3}$
e $3\sqrt{3}$ **f** $-\sqrt{2}$
6 **a** $3\sqrt{2}-2$ **b** $12-14\sqrt{6}$ **c** $-8+10\sqrt{2}$
d $-12\sqrt{2}+36$ **e** $22+9\sqrt{2}$ **f** $22+14\sqrt{7}$
g $-7-\sqrt{3}$ **h** $34-30\sqrt{2}$ **i** $30\sqrt{5}-47$
7 **a** $11+6\sqrt{2}$ **b** $39-12\sqrt{3}$ **c** $46+6\sqrt{5}$
d $89-28\sqrt{10}$ **e** 2 **f** -23
g 7 **h** 218 **i** $10+6\sqrt{3}$

EXERCISE 3B

1 **a** $\frac{\sqrt{3}}{3}$ **b** $\frac{11\sqrt{3}}{3}$ **c** $\frac{\sqrt{6}}{9}$ **d** $6\sqrt{2}$
e $\frac{\sqrt{6}}{2}$ **f** $\frac{\sqrt{2}}{8}$ **g** $3\sqrt{5}$ **h** $-\frac{3\sqrt{5}}{5}$
i $\frac{\sqrt{5}}{15}$ **j** $3\sqrt{7}$ **k** $\frac{2\sqrt{11}}{11}$ **l** $\frac{\sqrt{3}}{9}$
2 **a** $\dfrac{3-\sqrt{2}}{7}$ **b** $\dfrac{6+2\sqrt{2}}{7}$ **c** $2\sqrt{6}+2$
d $\dfrac{\sqrt{21}-2\sqrt{3}}{3}$ **e** $-3-2\sqrt{2}$ **f** $2\sqrt{2}+4$
g $-7-3\sqrt{5}$ **h** $\dfrac{-38+11\sqrt{10}}{6}$ **i** $\dfrac{7+\sqrt{5}}{11}$
j $\dfrac{28+\sqrt{2}}{23}$ **k** $\dfrac{17+7\sqrt{7}}{3}$ **l** $\dfrac{\sqrt{11}-1}{5}$
3 **a** $3-2\sqrt{2}$ **b** $\frac{14}{17}-\frac{1}{34}\sqrt{2}$ **c** $3-2\sqrt{2}$
d $\frac{11}{49}+\frac{6}{49}\sqrt{2}$ **e** $3-2\sqrt{2}$ **f** $-\frac{7}{41}-\frac{2}{41}\sqrt{2}$
g $\frac{7}{529}+\frac{17}{529}\sqrt{2}$ **h** $\frac{45}{343}+\frac{29}{343}\sqrt{2}$

EXERCISE 3C

1 **a** $2^1=2,\ 2^2=4,\ 2^3=8,\ 2^4=16,\ 2^5=32,\ 2^6=64$
b $3^1=3,\ 3^2=9,\ 3^3=27,\ 3^4=81,\ 3^5=243,$
$3^6=729$
c $4^1=4,\ 4^2=16,\ 4^3=64,\ 4^4=256,\ 4^5=1024,$
$4^6=4096$
2 **a** -1 **b** 1 **c** 1 **d** -1 **e** 1 **f** -1
g -1 **h** -32 **i** -32 **j** -64 **k** 625 **l** -625
3 **a** 16 384 **b** 2401 **c** -3125 **d** -3125
e 262 144 **f** 262 144 **g** $-262\,144$
h 902.436 039 6 **i** $-902.436\,039\,6$ **j** $-902.436\,039\,6$
4 **a** $0.\overline{1}$ **b** $0.02\overline{7}$ **c** $0.\overline{012\,345\,679}$ **d** 1
Notice that $a^{-n}=\dfrac{1}{a^n}$ and $a^0=1$.
5 3 **6** 7
7 **a** $21+23+25+27+29=125=5^3$
b $43+45+47+49+51+53+55=343=7^3$
c $133+135+137+139+141+143+145+147+149$
$+151+153+155=1728=12^3$

EXERCISE 3D

1 **a** k^6 **b** c^{8+m} **c** r^{11} **d** 7^5
e m^6 **f** x^{3a-2} **g** 7^{6d} **h** m^{3t}
i 11^{2xy} **j** 7^{6-n} **k** x^{6s} **l** 3^{13}
m j^{12x} **n** z^{7-4t} **o** 13^{5cd} **p** w^{7p-1}
q k^{5t-3} **r** x^{3m-n}
2 **a** $4b^3$ **b** $6w^5$ **c** $4p^2$ **d** $30c^{11}$ **e** d^4 **f** $3ab^2$
g $4n^3$ **h** t^7 **i** $20s^2t^4$ **j** k^{11} **k** $\dfrac{3xy^3}{2}$ **l** b^9
3 **a** 2^2 **b** 2^{-2} **c** 2^3 **d** 2^{-3} **e** 2^5 **f** 2^{-5}
g 2^1 **h** 2^{-1} **i** 2^6 **j** 2^{-6} **k** 2^7 **l** 2^{-7}
4 **a** 3^2 **b** 3^{-2} **c** 3^3 **d** 3^{-3} **e** 3^1 **f** 3^{-1}
g 3^4 **h** 3^{-4} **i** 3^0 **j** 3^5 **k** 3^{-5}
5 **a** 2^{1+a} **b** 2^{2+b} **c** 2^{3+t} **d** 2^{2x+2} **e** 2^{n-1}
f 2^{c-2} **g** 2^{2m} **h** 2^{1+n} **i** 2^1 **j** 2^{3x-1}
6 **a** 3^{2+p} **b** 3^{3a} **c** 3^{1+2n} **d** 3^{3+d} **e** 3^{2+3t}
f 3^{y-1} **g** 3^{1-y} **h** 3^{2-3t} **i** 3^{3a-1} **j** 3^3
7 **a** 2^5 **b** 5^6 **c** 2^{4p} **d** 5^{a+2}
e 2^{5n} **f** 2^{3m-4n} **g** 5^{2p-4} **h** 3^{2t+4}
i 2^{10-5r} **j** 3^{3-y} **k** 2^{2k} **l** 5^{4-3a}
8 **a** 1 **b** $\frac{1}{3}$ **c** $\frac{1}{49}$ **d** $\dfrac{1}{x^3}$ **e** $\frac{6}{5}$ (or $1\frac{1}{5}$)
f 1 **g** $\frac{4}{7}$ **h** 6 **i** $\frac{9}{16}$ **j** $\frac{5}{2}$ (or $2\frac{1}{2}$)
k $\frac{27}{125}$ **l** $\frac{151}{5}$ (or $30\frac{1}{5}$)
9 **a** 3^{-2} **b** 2^{-4} **c** 5^{-3} **d** $3^1\times5^{-1}$
e $2^2\times3^{-3}$ **f** $2^{c-3}\times3^{-2}$
g $3^{2k}\times2^{-1}\times5^{-1}$ **h** $2^p\times3^{p-1}\times5^{-2}$
10 **a** $4a^2$ **b** $9n^2$ **c** $125m^3$ **d** m^3n^3
e $\dfrac{a^3}{8}$ **f** $\dfrac{9}{m^2}$ **g** $\dfrac{p^4}{q^4}$ **h** $\dfrac{t^2}{25}$
11 **a** $4a^2b^2$ **b** $4a^2$ **c** $36b^4$ **d** $-8a^3$
e $-27m^6n^6$ **f** $16a^4b^{16}$ **g** 1 $(a\neq0,\ b\neq0)$
h $\dfrac{m^4}{81n^4}$ **i** $\dfrac{x^3y^3}{8}$ **j** $\dfrac{-8a^6}{b^6}$ **k** $\dfrac{16a^6}{b^2}$ **l** $\dfrac{9p^4}{q^6}$
12 **a** x^5+x^3 **b** $x^4-2x^3+3x^2$ **c** x^5-x
d $x^5-x^4+2x^3-2x^2$ **e** $x^6-2x^4+x^2$
f x^3-2x^2+x **g** x^2+x-1
h x^4+2x+x^{-2} **i** x^4-x^{-2}
13 **a** $\dfrac{a}{b^2}$ **b** $\dfrac{1}{a^2b^2}$ **c** $\dfrac{4a^2}{b^2}$ **d** $\dfrac{1}{25m^4}$
e $\dfrac{9b^2}{a^4}$ **f** $\dfrac{1}{27x^3y^{12}}$ **g** $\dfrac{a^2}{bc^2}$ **h** $\dfrac{a^2c^2}{b}$
i a^3 **j** $\dfrac{b^3}{a^2}$ **k** $\dfrac{2}{ad^2}$ **l** $12am^3$

14 **a** a^{-n} **b** $5a^{-m}$ **c** b^n **d** 2^{3-n}
e 3^{n-2} **f** $3a^{m-4}$ **g** $a^n b^m$ **h** a^{-2n-2}

15 **a** x^{-2} **b** $2x^{-1}$ **c** $x + x^{-1}$ **d** $x^2 - 2x^{-3}$
e $x^{-1} + 3x^{-2}$ **f** $4x^{-1} - 5x^{-3}$
g $7x - 4x^{-1} + 5x^{-2}$ **h** $3x^{-1} - 2x^{-2} + 5x^{-4}$

16 **a** $1 + 3x^{-1}$ **b** $3x^{-1} - 2$ **c** $5x^{-2} - x^{-1}$
d $x^{-2} + 2x^{-3}$ **e** $x + 5x^{-1}$ **f** $x + 1 - 2x^{-1}$
g $2x - 3 + 4x^{-1}$ **h** $x - 3x^{-1} + 5x^{-2}$
i $5x^{-1} - 1 - x$ **j** $8x^{-1} + 5 - 2x^2$
k $16x^{-2} - 3x^{-1} + x$ **l** $5x^2 - 3 + x^{-1} + 6x^{-2}$

17 **a** $4x + 2x^2$ **b** $5x^2 - 4x^3$ **c** $6x^3 + 3x^4$
d $x^3 + 3x$ **e** $x^4 + x^3 - 4x^2$ **f** $x^6 - 3x^4 + 6x^3$
g $x^5 - 6x^3 + 10x^2$ **h** $x^5 + 4x^3 + x^2$

EXERCISE 3E

1 **C** and **D**

2 **a** 2.59×10^2 **b** 2.59×10^5 **c** 2.59×10^9
d 2.59×10^0 **e** 2.59×10^{-1} **f** 2.59×10^{-4}
g 4.07×10^1 **h** 4.07×10^3 **i** 4.07×10^{-2}
j 4.07×10^5 **k** 4.07×10^8 **l** 4.07×10^{-5}

3 **a** 4.745×10^7 kg **b** 3×10^{-3} m
c 2.599×10^6 hands **d** 4.7×10^{-7} m

4 **a** 4000 **b** 380 000 **c** 86 **d** 43 300 000
e 0.004 **f** 0.000 038 **g** 0.86 **h** 0.000 000 433

5 **a** 7 400 000 000 people **b** 0.0112 kg
c 0.000 000 5 m **d** 7 300 000 kg

6 **a** $4.5 \times 10^7 = 45\,000\,000$ **b** $3.8 \times 10^{-4} = 0.000\,38$
c $2.1 \times 10^5 = 210\,000$ **d** $4 \times 10^{-3} = 0.004$
e $6.1 \times 10^3 = 6100$ **f** $1.6 \times 10^{-6} = 0.000\,001\,6$
g $3.9 \times 10^4 = 39\,000$ **h** $6.7 \times 10^{-2} = 0.067$

7 **a** 4.964×10^{13} **b** 4×10^{-8} **c** 3.43×10^{-10}
d 1.6416×10^{10} **e** $4.121\,64 \times 10^{-3}$ **f** $\approx 5.27 \times 10^{-18}$
g $\approx 1.36 \times 10^2$ **h** $\approx 2.63 \times 10^{-6}$ **i** 1.728×10^9

8 7.5×10^7 peanuts **9** 2.61×10^{-6} m

10 **a** 1.15×10^{10} m
b We have assumed that we will always be on the side of the planet that is closest to the next planet, at the time when the planets are closest. It could take a very long time for these ideal conditions to occur.

11 **a** **i** $\approx 1.80 \times 10^{10}$ m **ii** $\approx 2.59 \times 10^{13}$ m
b $\approx 9.46 \times 10^{15}$ m **c** $\approx 3.99 \times 10^{16}$ m
d **i** $\approx 9.27 \times 10^{21}$ m **ii** $\approx 5.46 \times 10^{22}$
iii $\approx 9.46 \times 10^{12}$ hours ($\approx$ 1.08 billion years)

12 **a** It allows us to write very small numbers without having to write and count lots of zeros.
b **i** $\approx$ 1839 times **ii** $\approx$ 1836 times **iii** $\approx$ 1.001 times
c 47 electrons, 60 neutrons **d** $\approx 2.18 \times 10^{21}$ electrons

REVIEW SET 3A

1 **a** $4\sqrt{5}$ **b** $-\sqrt{6}$ **c** $20\sqrt{3} - 15$
d $4 + 3\sqrt{2}$ **e** $86 - 60\sqrt{2}$ **f** 4

2 **a** -1 **b** 27 **c** $\frac{2}{3}$

3 **a** x^6 **b** 2^{-7} (or $\frac{1}{128}$) **c** $a^6 b^{18}$

4 **a** $\frac{1}{27}$ **b** $\dfrac{y}{x}$ **c** $\dfrac{b}{a}$ **5** **a** 3^3 **b** 3^{2t} **c** 2^{3-m}

6 **a** $\dfrac{5x}{y^2}$ **b** $\dfrac{1}{j^7}$ **c** $3g^3h^3$

7 **a** $\dfrac{t^3}{64s^3}$ **b** $1 \;\; (m \neq 0, \;\; n \neq 0)$ **c** $25p^6q^2$

8 **a** $x + 8x^{-1}$ **b** $4x^2 + x^3 + x^5$ **c** k^{-2x-6}

9 **a** a^6b^7 **b** $\dfrac{2}{3x}$ **c** $\dfrac{y^2}{5}$

10 **a** 460 000 000 000 **b** 1.9 **c** 0.0032

11 **a** 1.274×10^7 m **b** 1.2×10^{-4} m

12 313 sheets **13** 2.8×10^9 km

REVIEW SET 3B

1 **a** $-\sqrt{11}$ **b** $\sqrt{2}$ **c** $17 - 11\sqrt{3}$ **d** 28

2 **a** $\frac{2\sqrt{3}}{3}$ **b** $\frac{\sqrt{35}}{5}$ **c** $6 - 3\sqrt{3}$ **d** $\dfrac{4 - \sqrt{7}}{9}$

3 **a** m^4 **b** $1 \;\; (y \neq 0)$ **c** $\dfrac{w^2}{49z^2}$

4 **a** k^{x-2} **b** 11^{r-4} **c** 3^{2+b}

5 **a** ab^{-2} **b** jk^4l^{-a} **c** $x^{-3} - 2x^{-5}$

6 **a** 2^{-4} **b** 3^{k+4} **c** 5^{3a-b}

7 **a** $x - 5 + \dfrac{6}{x}$ **b** $x^6 + 2x^2 + \dfrac{1}{x^2}$ **c** $x^3 - 2x - \dfrac{3}{x}$

8 **a** $\dfrac{a^{18}}{64b^6}$ **b** $\dfrac{25}{d^8}$ **c** $2z^4$

9 **a** $\dfrac{1}{x^5}$ **b** $\dfrac{2}{a^2b^2}$ **c** $\dfrac{2a}{b^2}$

10 **a** 3^{3-2a} **b** 3^{6-8x}

11 **a** 143 000 km **b** 0.000 000 082 m

12 **a** $\approx 1.96 \times 10^{-5}$ s **b** ≈ 0.0110 s **13** 7500 sheets

EXERCISE 4A

1 **a** $x = \pm 4$ **b** $x = \pm\sqrt{7}$ **c** no real solutions
d $x = 0$ **e** $x = \pm\sqrt{5}$ **f** no real solutions

2 **a** $x = 3$ **b** $x = \pm 2$ **c** no real solutions
d $x = \sqrt[5]{-13}$ **e** $x = -2$ **f** $x = \sqrt[3]{7}$
g $x = \pm\sqrt[4]{6}$ **h** $x = \frac{2}{3}$ **i** $x = \pm\frac{1}{2}$
j $x = \sqrt[5]{\frac{1}{3}}$ **k** $x = \sqrt[3]{-6}$ **l** $x = \pm 3$

3 **a** $x = 7$ or -1 **b** no real solutions **c** $x = -4 \pm \sqrt{13}$
d $x = 7$ **e** $x = 4$ or -1 **f** $x = \dfrac{-1 \pm \sqrt{14}}{3}$
g $x = 0$ or $2\sqrt{2}$ **h** $x = \dfrac{\sqrt{3} \pm \sqrt{2}}{2}$ **i** $x = \dfrac{-1 \pm \sqrt{7}}{2}$

4 **a** $x = 1 + \sqrt[3]{17}$ **b** $x = -4$ **c** $x = 2 \pm \sqrt[4]{20}$
d no real solutions **e** $x = -4 + \sqrt[5]{-12}$ **f** $x = 0$ or $\frac{2}{3}$
g $x = \dfrac{3 \pm \sqrt[4]{15}}{2}$ **h** $x = -1$ **i** $x = 4 - \sqrt[3]{-22}$

5 **a** $x = 2$ **b** $x = 1 \pm \sqrt[4]{11}$ **c** $x = -2$

6 **a** $x = 6$ **b** $x = \pm 3$ **c** $x = -3$ **d** $x = \pm\frac{1}{7}$
e $x = \pm 2$ **f** $x = -\frac{1}{4}$ **g** no real solutions
h $x = \dfrac{5 + \sqrt[3]{5}}{2}$

EXERCISE 4B

1 **a** $x = 0$ **b** $a = 0$ or $b = 0$ **c** $x = 0$ or $y = 0$
d $a = 0$ or $b = 0$ or $c = 0$

2 **a** $x = 0$ or 5 **b** $x = 0$ or -3 **c** $x = -1$ or 3

d $x = 0$ or 7 **e** $x = -6$ or $\frac{3}{2}$ **f** $x = -\frac{1}{2}$ or $\frac{1}{2}$

3 **a** $x = 0$ or -5 **b** $x = 5$ **c** $x = \frac{1}{3}$ **d** $x = 5$ or $-\frac{2}{3}$

e $x = 0, -1$, or 2 **f** $x = -2, -4$, or $\frac{1}{2}$

4 **a** $a = 0,\ b \neq 0$ **b** $x = 0$ or $y = 0,\ z \neq 0$

c no solutions **d** $x = 0,\ y \neq 0$

EXERCISE 4C.1

1 **a** $x = 0$ or $-\frac{7}{4}$ **b** $x = 0$ or $-\frac{1}{3}$ **c** $x = 0$ or $\frac{7}{3}$

d $x = 0$ or $\frac{11}{2}$ **e** $x = 0$ or $\frac{8}{3}$ **f** $x = 0$ or $\frac{3}{2}$

2 **a** $x = 2$ or 3 **b** $x = 1$ **c** $x = -4$ or 2

d $x = -3$ or -4 **e** $x = -2$ or 4 **f** $x = 3$ or 7

g $x = 3$ **h** $x = -4$ or 3 **i** $x = -11$ or 3

j $x = -4$ or 1 **k** $x = -7$ or 5 **l** $x = -2$ or 5

3 **a** $x = \frac{2}{3}$ **b** $x = -\frac{1}{2}$ or 7 **c** $x = -\frac{2}{3}$ or 6

d $x = \frac{1}{3}$ or -2 **e** $x = \frac{3}{2}$ or 1 **f** $x = -\frac{2}{3}$ or -2

g $x = -\frac{2}{3}$ or 4 **h** $x = \frac{1}{2}$ or $-\frac{3}{2}$ **i** $x = -\frac{1}{4}$ or 3

j $x = -\frac{3}{4}$ or $\frac{5}{3}$ **k** $x = \frac{1}{7}$ or -1 **l** $x = -2$ or $\frac{28}{15}$

4 **a** $x = 2$ or 5 **b** $x = -3$ or 2 **c** $x = 0$ or $-\frac{3}{2}$

d $x = 1$ or 2 **e** $x = \frac{1}{2}$ or -1 **f** $x = 3$

g $x = 1$ or -2 **h** $x = 6$ or -4 **i** $x = 7$ or -5

j $x = 4$ or -2

EXERCISE 4C.2

1 **a** $x = 2 \pm \sqrt{3}$ **b** $x = -3 \pm \sqrt{7}$ **c** $x = 7 \pm \sqrt{3}$

d $x = 2 \pm \sqrt{7}$ **e** $x = -3 \pm \sqrt{2}$ **f** $x = 1 \pm \sqrt{7}$

g $x = -3 \pm \sqrt{11}$ **h** $x = 4 \pm \sqrt{6}$ **i** no real solutions

2 **a** $x = -1 \pm \frac{1}{\sqrt{2}}$ **b** $x = \frac{5}{2} \pm \frac{\sqrt{19}}{2}$ **c** $x = -2 \pm \sqrt{\frac{7}{3}}$

d $x = 1 \pm \sqrt{\frac{7}{3}}$ **e** $x = \frac{3}{2} \pm \sqrt{\frac{37}{20}}$ **f** $x = -\frac{1}{2} \pm \frac{\sqrt{6}}{2}$

3 **a** $x = \frac{2}{3} \pm \frac{\sqrt{10}}{3}$ **b** $x = -\frac{1}{10} \pm \frac{\sqrt{21}}{10}$ **c** $x = -\frac{5}{6} \pm \frac{\sqrt{13}}{6}$

4 $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

EXERCISE 4C.3

1 **a** $x = 2 \pm \sqrt{7}$ **b** $x = -3 \pm \sqrt{2}$ **c** $x = 2 \pm \sqrt{3}$

d $x = -2 \pm \sqrt{5}$ **e** $x = 2 \pm \sqrt{2}$ **f** $x = \dfrac{1 \pm \sqrt{7}}{2}$

g $x = \dfrac{5 \pm \sqrt{37}}{6}$ **h** $x = 2 \pm \sqrt{10}$ **i** $x = \dfrac{7 \pm \sqrt{33}}{4}$

2 **a** $x = -2 \pm 2\sqrt{2}$ **b** $x = \dfrac{-5 \pm \sqrt{57}}{8}$ **c** $x = \dfrac{5 \pm \sqrt{13}}{2}$

d $x = \dfrac{-4 \pm \sqrt{7}}{9}$ **e** $x = \dfrac{-7 \pm \sqrt{97}}{4}$ **f** $x = \dfrac{1 \pm \sqrt{145}}{8}$

g $x = \dfrac{1 \pm \sqrt{7}}{2}$ **h** $x = \dfrac{1 \pm \sqrt{5}}{2}$ **i** $x = \dfrac{3 \pm \sqrt{17}}{4}$

EXERCISE 4C.4

1 **a** $\Delta = 13$ **b** 2 distinct irrational roots **c** $x = \frac{7}{2} \pm \frac{\sqrt{13}}{2}$

2 **a** $\Delta = 0$ **b** 1 root (repeated) **c** $x = \frac{1}{2}$

3 **a** $x^2 = -5$, $\therefore$ no real roots **b** $\Delta = -20$

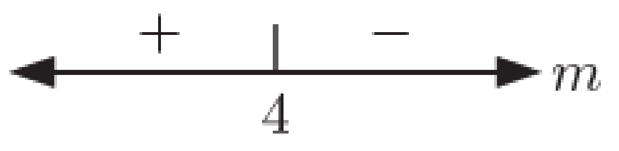
4 **a** 2 distinct irrational roots **b** 2 distinct rational roots

c 2 distinct rational roots **d** 2 distinct irrational roots

e no real roots **f** a repeated root

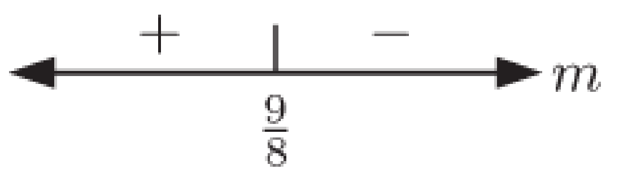
5 **a**, **c**, **d**, and **f**

6 **a** $\Delta = 16 - 4m$



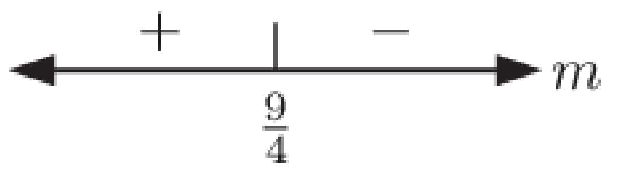
i $m = 4$ **ii** $m < 4$ **iii** $m > 4$

b $\Delta = 9 - 8m$



i $m = \frac{9}{8}$ **ii** $m < \frac{9}{8},\ m \neq 0$ **iii** $m > \frac{9}{8}$

c $\Delta = 9 - 4m$



i $m = \frac{9}{4}$ **ii** $m < \frac{9}{4},\ m \neq 0$ **iii** $m > \frac{9}{4}$

7 For $k = -8 + 4\sqrt{7}$, repeated root is $x = 1 - \frac{\sqrt{7}}{2}$.

For $k = -8 - 4\sqrt{7}$, repeated root is $x = 1 + \frac{\sqrt{7}}{2}$.

EXERCISE 4C.5

1 **a** **i** sum $= -4$, product $= -21$ **ii** roots are -7 and 3

b **i** sum $= 5$, product $= 5$ **ii** roots are $\frac{5}{2} \pm \frac{\sqrt{5}}{2}$

c **i** sum $= 3$, product $= \frac{5}{4}$ **ii** roots are $\frac{1}{2}$ and $\frac{5}{2}$

d **i** sum $= \frac{4}{3}$, product $= -\frac{2}{3}$ **ii** roots are $\frac{2}{3} \pm \frac{\sqrt{10}}{3}$

2 $k = -\frac{3}{5}$, roots are -1 and $\frac{1}{3}$

3 **a** $3\alpha = \dfrac{6}{a}$, $2\alpha^2 = \dfrac{a-2}{a}$

b $a = 4$, roots are $\frac{1}{2}$ and 1; or $a = -2$, roots are -1 and -2

4 $k = 4$, roots are $-\frac{1}{2}$ and $\frac{3}{2}$; or $k = 16$, roots are $-\frac{5}{4}$ and $\frac{3}{4}$

5 **a** $x^2 + 2x - 15 = 0$ **b** $x^2 - 4x + 1 = 0$

6 **a** $3x^2 - 2x - 4 = 0$ **b** $3x^2 + 4x - 16 = 0$

7 **a** $2x^2 - 13x + 17 = 0$

b $k(8x^2 - 10x - 1) = 0,\ k \in \mathbb{R},\ k \neq 0$

8 $k(x^2 + 7x - 44) = 0,\ k \in \mathbb{R},\ k \neq 0$

9 $k(4x^2 - 61x + 81),\ k \in \mathbb{R},\ k \neq 0$

10 $7x^2 - 48x + 64 = 0$

11 $k(8x^2 - 70x + 147) = 0,\ k \in \mathbb{R},\ k \neq 0$

EXERCISE 4D

1 **a** $x = -2$ or -7 **b** $x = 4$

c $x \approx 1.29$ or -1.54 **d** $x = 1.5$ or -2.5

e $x \approx 1.18$ or 2.82 **f** no real solutions

g $x \approx 0.360$ or 1.39 **h** $x \approx -5.99$ or 4.18

2 **a** $x = -3$ or -4 **b** $x \approx 1.85$ or -4.85

c $x \approx 0.847$ or -1.18 **d** no real solutions

3 **a** $x = -3, 0$, or 3 **b** $x \approx -1.13$ **c** $x = 3, 2$, or -4

d $x = 1$ **e** $x = 0.5, \approx 0.618$, or ≈ -1.62

f $x \approx 4.36, 0.406$, or -2.26

4 **a** no real solutions **b** $x \approx 1.34$ or -3.17

c $x = 1$ or -1 **d** no real solutions **e** $x \approx -2.27$ or 2.43

f $x \approx -3.36, -1.65, 0.192$, or 2.82

5 **a** $x = 0, \approx 1.73$, or -1.73 **b** $x \approx -0.811$

6 **a** $x^3 - 6x^2 + 2x - 6 = 0$ **b** $x \approx 5.83$

EXERCISE 4E

1 **a** $x = -2$ or 3 **b** $x = -2$ or 3

2 **a** $x \approx 3.21$ **b** $x \approx 0.387$ or -1.72 **c** $x \approx 2.46$

d $x \approx 5.17$ **e** $x \approx 1.52$ or 2.83 **f** $x \approx 3.56$ or -1.30

3 **a** $x \approx -1.59$ **b** $x \approx -0.861$, 1.24, or 16
c $x \approx -2.62$ **d** $x \approx -0.572$ or 0.821
e $x \approx -2.24$ or 2.34 **f** $x \approx -0.577$ or 0.577

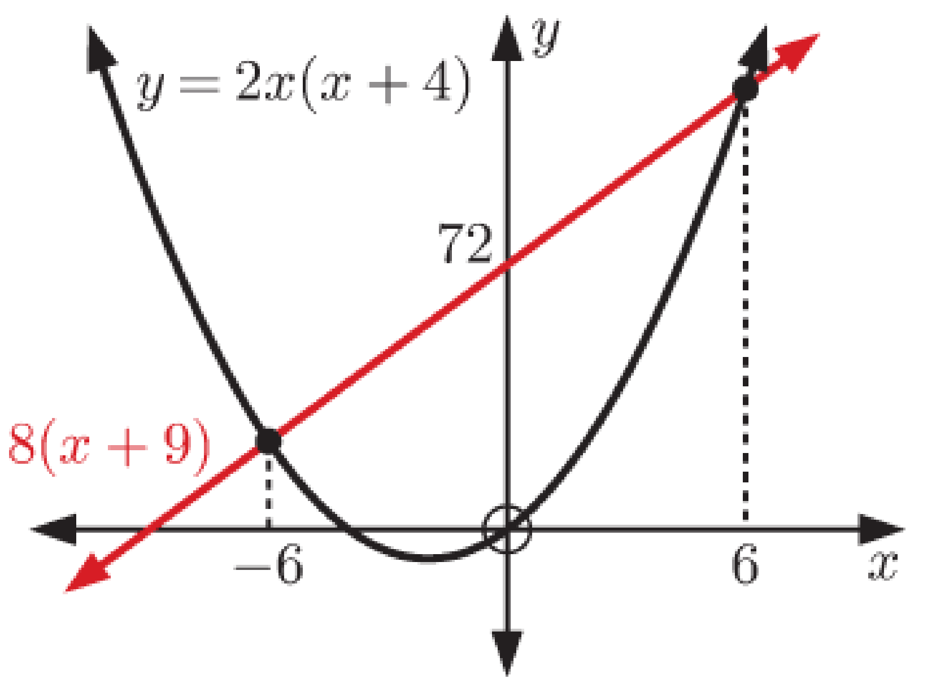
4 **a** **i** $x = 1$ or 5 **ii** $x = 3$ **iii** no real solutions
b **i** $k > -7$ **ii** $k = -7$ **iii** $k < -7$

REVIEW SET 4A

1 **a** $x = \pm\sqrt{19}$ **b** $x = -3$ or 7 **c** $x = 0$ or $2\sqrt{2}$
2 **a** no real solutions **b** $x = \frac{1}{3}$ **c** $x = 1 + \sqrt[5]{2}$
3 **a** $x = 0$ or -2 **b** $x = -3$ or $\frac{7}{2}$ **c** $x = -5, -1$, or 6
4 **a** $x = 0$ or $\frac{5}{3}$ **b** $x = 5$ or -1 **c** $x = -3$
d $x = 1$ or 3 **e** $x = \frac{1}{3}$ or -2 **f** $x = -6$ or 9
5 **a** $x = -5$ or 4 **b** $x = 2$ or 3 **c** $x = -\frac{5}{2}$ or 7
6 **a** $x = \dfrac{7 \pm \sqrt{41}}{2}$ **b** no real solutions **c** $x = \dfrac{-1 \pm \sqrt{37}}{6}$
7 **a** $\Delta = 49$, 2 distinct rational roots **b** $x = -\frac{1}{2}$ or $\frac{2}{3}$
8 $\Delta = -7$ $\therefore$ no real roots
9 **a** $m = \frac{9}{8}$ **b** $m < \frac{9}{8}$ **c** $m > \frac{9}{8}$
10 $k(2x^2 + 5x - 3) = 0,\ k \in \mathbb{R},\ k \neq 0$
11 $4x^2 + 3x - 2 = 0$ **12** $k = 3$, roots are $-\frac{1}{3}$ and 3
13 **a** $x = \frac{5}{2}$, 1, or -2 **b** $x = 2, \frac{1}{3}$, or 0
c $x = 4, 3$, or -5 **d** $x \approx 1.84$ or -6.92
14 **a** $x \approx 2.81$ **b** $x \approx 1.73$ **c** $x \approx -1.84$

REVIEW SET 4B

1 **a** $x = 0$ **b** $x = -\frac{5}{2}$ **c** $x = \sqrt{3} \pm 4$
2 **a** $x = \pm\frac{3}{2}$ **b** $x = \sqrt[5]{-18}$ **c** $x = \frac{1}{2}$ or $\frac{3}{2}$
3 **a** $p = 0,\ q \neq 0$ **b** $x = 0$ or $z = 0,\ y \neq 0$
c no solutions
4 **a** $x = 0$ or $\frac{5}{2}$ **b** $x = 0$ or 4 **c** $x = 1$ or 6
d $x = -2$ **e** $x = 6$ or -2 **f** $x = -\frac{5}{3}$ or 2
5 **a** $x^2 - 9 = 0,\quad x = \pm 3$
b $2x^2 + x - 3 = 0,\quad x = 1$ or $-\frac{3}{2}$
c $3x^2 - x - 2 = 0,\quad x = 1$ or $-\frac{2}{3}$
6 **a** $x = \dfrac{-5 \pm \sqrt{13}}{2}$ **b** $x = \dfrac{-11 \pm \sqrt{145}}{6}$
7 **a** $x = 3 \pm \sqrt{5}$ **b** $x = -2 \pm \frac{3}{\sqrt{2}}$ **c** $x = -\frac{3}{4}$ or 2
8 **a** $\Delta = 0$, 1 root (repeated)
b $\Delta = 41$, 2 distinct irrational roots
c $\Delta = -11$, no real roots
9 **a** **i** $x = \pm 6$
ii



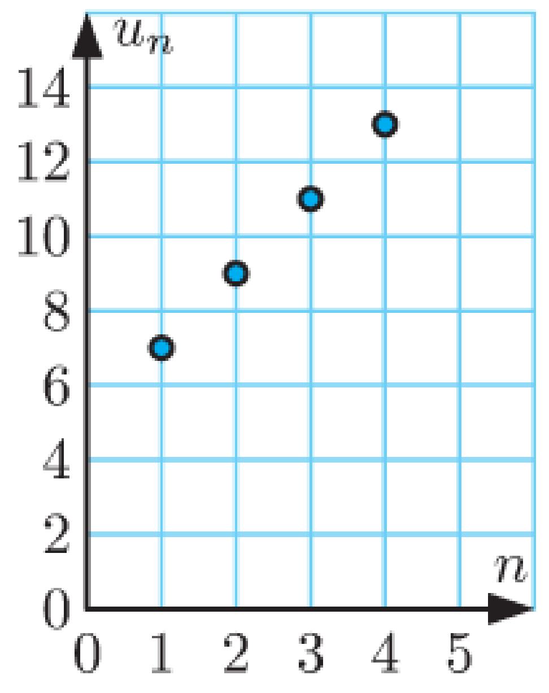
$x = -6$ or 6

c **i** $k > 0$ **ii** $k = 0$ **iii** $k < 0$
11 $k = 8\sqrt{2}$, roots are $-\sqrt{2}$ and $-3\sqrt{2}$; or
$k = -8\sqrt{2}$, roots are $\sqrt{2}$ and $3\sqrt{2}$

12 $k(64x^2 - 135x - 27) = 0,\ k \in \mathbb{R},\ k \neq 0$
13 **a** $x = 5, 0$, or -3 **b** $x \approx 4.93$, 0.814, or -1.74
c $x \approx 2.39$ or 0.449
14 **a** $x \approx 2.81$ **b** $x \approx 1.85$ **c** $x \approx -2.15$ or 3.58

EXERCISE 5A

1 **a** 4, 13, 22, 31 **b** 45, 39, 33, 27 **c** 2, 6, 18, 54
d 96, 48, 24, 12
2 **a** $u_2 = 3$ **b** $u_5 = 11$ **c** $u_{10} = 29$
3 **a** We start with 4 and add 3 each time.
b $u_1 = 4,\ u_4 = 13$ **c** $u_8 = 25$
4 **a** $u_1 = 7,\ u_2 = 9,$
$u_3 = 11,\ u_4 = 13$
b



5 **a** $u_1 = 1$ **b** $u_5 = 13$ **c** $u_{27} = 79$
6 **a** **B** **b** $u_{20} = 390$
7 **a** The sequence starts at 8, and each term is 8 more than the previous term. The next two terms are 40 and 48.
b The sequence starts at 2, and each term is 3 more than the previous term. The next two terms are 14 and 17.
c The sequence starts at 36, and each term is 5 less than the previous term. The next two terms are 16 and 11.
d The sequence starts at 96, and each term is 7 less than the previous term. The next two terms are 68 and 61.
e The sequence starts at 1, and each term is 4 times the previous term. The next two terms are 256 and 1024.
f The sequence starts at 2, and each term is 3 times the previous term. The next two terms are 162 and 486.
g The sequence starts at 480, and each term is half the previous term. The next two terms are 30 and 15.
h The sequence starts at 243, and each term is one third of the previous term. The next two terms are 3 and 1.
i The sequence starts at 50 000, and each term is one fifth of the previous term. The next two terms are 80 and 16.
8 **a** Each term is the square of the term number; 25, 36, 49.
b Each term is the cube of the term number; 125, 216, 343.
c Each term is $n(n+1)$ where n is the term number; 30, 42, 56.
9 **a** 79, 75 **b** 1280, 5120 **c** 625, 1296
d 13, 17 **e** 16, 22 **f** 6, 12
10 **a** 2, 4, 6, 8, 10 **b** -1, 1, 3, 5, 7
c 13, 15, 17, 19, 21 **d** 1, 5, 9, 13, 17
e 2, 4, 8, 16, 32 **f** 3, $\frac{3}{2}$, $\frac{3}{4}$, $\frac{3}{8}$, $\frac{3}{16}$
g -2, 4, -8, 16, -32 **h** 17, 11, 23, -1, 47

EXERCISE 5B.1

1 **a** **i** $u_1 = 19,\ d = 6$ **ii** $u_n = 6n + 13$ **iii** $u_{15} = 103$
b **i** $u_1 = 101,\ d = -4$ **ii** $u_n = 105 - 4n$
iii $u_{15} = 45$
c **i** $u_1 = 8,\ d = 1\frac{1}{2}$ **ii** $u_n = 1\frac{1}{2}n + 6\frac{1}{2}$ **iii** $u_{15} = 29$
d **i** $u_1 = 31,\ d = 5$ **ii** $u_n = 5n + 26$ **iii** $u_{15} = 101$
e **i** $u_1 = 5,\ d = -8$ **ii** $u_n = 13 - 8n$
iii $u_{15} = -107$

f **i** $u_1 = a,\ \ d = d$ **ii** $u_n = a + (n-1)d$
iii $u_{15} = a + 14d$

2 **a** $u_1 = 6,\ \ d = 11$ **b** $u_n = 11n - 5$
c $u_{50} = 545$ **d** yes, $u_{30} = 325$ **e** no

3 **a** $u_1 = 87,\ \ d = -4$ **b** $u_n = 91 - 4n$
c $u_{40} = -69$ **d** u_{97}

4 **b** $u_1 = 1,\ \ d = 3$ **c** $u_{57} = 169$ **d** $u_{150} = 448$

5 **b** $u_1 = 32,\ \ d = -\frac{7}{2}$ **c** $u_{75} = -227$ **d** $n \geqslant 68$

6 $u_{7692} = 100\,006$ **7** **b** $u_{200} = 1381$ **c** no

8 **a** $k = 17\frac{1}{2}$ **b** $k = 4$ **c** $k = 5$ **d** $k = \frac{3}{2}$
e $k = 7$ **f** $k = -4$ **g** $k = -2$ or 3 **h** $k = -1$ or 3

9 **a** $k = \frac{1}{2}$ or -2 **b** For $k = \frac{1}{2},\ \ d = -5$
For $k = -2,\ \ d = 15$

10 **a** $u_n = 6n - 1$ **b** $u_n = -\frac{3}{2}n + \frac{11}{2}$
c $u_n = -5n + 36$ **d** $u_n = -\frac{3}{2}n + \frac{1}{2}$

11 **b** $u_{30} = 815$ **12** $5, 6\frac{1}{4}, 7\frac{1}{2}, 8\frac{3}{4}, 10$

13 $-1, 3\frac{5}{7}, 8\frac{3}{7}, 13\frac{1}{7}, 17\frac{6}{7}, 22\frac{4}{7}, 27\frac{2}{7}, 32$

14 **a** $50, 48\frac{1}{2}, 47, 45\frac{1}{2}, 44$ **b** $u_{35} = -1$

15 **a** $k = 2$ **b** $u_n = \dfrac{3n-8}{2}$

16 **Hint:** Since $x > y > z > 0$, then $\dfrac{1}{x} < \dfrac{1}{y} < \dfrac{1}{z}$.

17 No, such a sequence would require $u_1 =$ some prime p, and $d \in \mathbb{Z}^+$. But then u_{p+1} must be a multiple of p.

18 **a** Month 1: 5 cars Month 4: 44 cars
Month 2: 18 cars Month 5: 57 cars
Month 3: 31 cars Month 6: 70 cars

b The total number of cars made increases by 13 each month. So, the common difference $d = 13$.

c 148 cars **d** 20 months

19 **a** Week 1: 2817 L Week 3: 2451 L
Week 2: 2634 L Week 4: 2268 L

b The amount in the tank decreases by the same amount (183 L) each week.

c in the 17th week

EXERCISE 5B.2

1 **a** 140.75 g **b** $u_n = 140.75n$

2 **a** 59.25 g **b** $u_n = 32 + 59.25n$ **c** $0 \leqslant n \leqslant 12$

3 **a** $u_n = 580 - 16n$ **b** $u_n = 9850 - \frac{7880}{29}n$

4 **a** 5.75 online friends per week **b** $u_n = 28.25 + 5.75n$

c No, the model is only intended to *estimate* the number of online friends. We can simply round to the nearest whole number.

d $143.25 \approx 143$ online friends

5 **a** $u_n = 1950 + 100n$

b **i** Catering is €100 per guest.
ii The venue hire is €1950 (with 0 guests).

c €10 450

EXERCISE 5C

1 **a** **i** $u_1 = 3,\ \ r = 2$ **ii** $u_n = 3 \times 2^{n-1}$ **iii** $u_9 = 768$

b **i** $u_1 = 2,\ \ r = 5$ **ii** $u_n = 2 \times 5^{n-1}$
iii $u_9 = 781\,250$

c **i** $u_1 = 512,\ \ r = \frac{1}{2}$ **ii** $u_n = 512 \times 2^{1-n}$
iii $u_9 = 2$

d **i** $u_1 = 1,\ \ r = 3$ **ii** $u_n = 3^{n-1}$ **iii** $u_9 = 6561$

e **i** $u_1 = 12,\ \ r = \frac{3}{2}$ **ii** $u_n = 12 \times (\frac{3}{2})^{n-1}$
iii $u_9 = \dfrac{3^9}{2^6} = \dfrac{19\,683}{64}$

f **i** $u_1 = \frac{1}{16},\ \ r = -2$ **ii** $u_n = \frac{1}{16}(-2)^{n-1}$
iii $u_9 = 16$

2 **a** $u_1 = 5,\ \ r = 2$ **b** $u_n = 5 \times 2^{n-1},\ \ u_{15} = 81\,920$

3 **a** $u_1 = 12,\ \ r = -\frac{1}{2}$
b $u_n = 12 \times (-\frac{1}{2})^{n-1},\ \ u_{13} = \frac{3}{1024}$

4 **a** $u_1 = 8,\ \ r = -\frac{3}{4}$ **b** $u_{10} \approx -0.601$

5 **a** $u_1 = 8,\ \ r = \frac{1}{\sqrt{2}}$ **b** **Hint:** $u_n = 2^3 \times (2^{-\frac{1}{2}})^{n-1}$

6 **a** $k = 6$ **b** $k = \frac{125}{2}$ **c** $k = \pm 14$ **d** $k = \pm 2$
e $k = \pm 36$ **f** $k = \pm 8$ **g** $k = 2$ **h** $k = -2$ or 4

7 **a** $k = -3$ or 4 **b** For $k = -3$, next term is $\frac{27}{2}$.
For $k = 4$, next term is 24.

8 **a** $u_n = 3 \times 2^{n-1}$ **b** $u_n = 32 \times (-\frac{1}{2})^{n-1}$
c $u_n = 3 \times (\pm\sqrt{2})^{n-1}$ **d** $u_n = 10 \times (\pm\sqrt{2})^{1-n}$

9 **a** $u_1 = 35\frac{5}{9},\ \ r = \frac{3}{2}$ **b** $u_{10} = 1366\frac{7}{8}$

10 **a** $u_9 = 13\,122$ **b** $u_{14} = 2916\sqrt{3} \approx 5050$
c $u_{18} = \frac{3}{32\,768} \approx 0.000\,091\,6$
d $u_{10} = -\frac{98\,415}{512} \approx -192$

11 2 primes
Consider $u_1 = 2,\ \ r = \frac{3}{2}$. The only other prime is $u_2 = 3$.

12 **a** $r = 2 \pm \sqrt{2}$
b For $r = 2 + \sqrt{2}$, geometric to arithmetic is $2 + 2\sqrt{2} : 1$.
For $r = 2 - \sqrt{2}$, geometric to arithmetic is $2 - 2\sqrt{2} : 1$.

EXERCISE 5D

1 **a** **i** $\approx$ 1553 ants **ii** $\approx$ 4823 ants **b** $\approx$ 12.2 weeks

2 **a** ≈ 278 **b** year 2057

3 **a** **i** $\approx$ 73 deer **ii** $\approx$ 167 deer **b** $\approx$ 30.5 years

4 **a** **i** ≈ 2860 **ii** $\approx 184\,000$ **b** $\approx$ 14.5 years

5 **a** $\approx$ 3.36 g **b** $\approx$ 10.2 more years

6 **a** €39 712.41 p.a. **b** €54 599.05 p.a.

EXERCISE 5E.1

1 £9367.58 **2** **a** €2233.58 **b** €233.58

3 $716.38 **4** **a** $20 977.42 **b** $23 077.89

5 **a** €37 305.85 **b** €7305.85

6 $11 222.90 **7** Bank A **8** £14 977

9 $11 478 **10** $22 054.85 **11** ¥3 000 340

EXERCISE 5E.2

1 **a** $8487.20 **b** $16 229.84 **c** $27 672.16

2 **a** $1218.99 **b** $1485.95 **c** $1811.36

3 $16 236.48

EXERCISE 5E.3

1 **a** $5567.55 **b** $5246.43

2 **a** €23 651.79 **b** €20 691.02

3 **a** $4782.47 **b** $782.47 **c** $3958.90

d The investment has not been effective. The real value of the investment after 6 years is less than what was originally invested.

4 **a** Real interest rate $= \frac{1.005}{1.001} \approx 1.003\,996 \approx 4\%$
b \$6602.66

5 $u_0\left(\dfrac{100+i}{100+r}\right)^{4y}$

EXERCISE 5E.4

1 €1280 **2** **a** €26 103.52 **b** €83 896.48
3 **a** ¥30 013 **b** ¥57 487 **4** 24.8%

EXERCISE 5E.5

1 74 614.60 pesos **2** \$6629.65
3 **a** \$9452.47 **b** \$12 482.59
4 **a** €6705.48 **b** €1705.48
5 **a** 2.82% p.a. **b** €4595.67 **6** \$1997.13
7 \$80 000 **8** \$108.69 **9** 2 years 9 months
10 13 years 3 months **11** 15 years **12** 14.5% p.a.
13 6.00% p.a. **14** 5.15% p.a. **15** 21.2% p.a.

EXERCISE 5F

1 **a** $S_3 = 18$ **b** $S_5 = 37$ **c** $S_{12} = 153$ **2** $u_5 = 7$

3 **a** **i** $S_n = \sum\limits_{k=1}^{n} (8k-5)$ **ii** $S_5 = 95$

b **i** $S_n = \sum\limits_{k=1}^{n} (47-5k)$ **ii** $S_5 = 160$

c **i** $S_n = \sum\limits_{k=1}^{n} 12(\frac{1}{2})^{k-1}$ **ii** $S_5 = 23\frac{1}{4}$

d **i** $S_n = \sum\limits_{k=1}^{n} 2(\frac{3}{2})^{k-1}$ **ii** $S_5 = 26\frac{3}{8}$

e **i** $S_n = \sum\limits_{k=1}^{n} \dfrac{1}{2^{k-1}}$ **ii** $S_5 = 1\frac{15}{16}$

f **i** $S_n = \sum\limits_{k=1}^{n} k^3$ **ii** $S_5 = 225$

4 **a** 24 **b** 27 **c** 10 **d** 25 **e** 168 **f** 310

5 $S_{20} = \sum\limits_{k=1}^{20} (3k-1) = 610$

7 **a**
$$\begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + \ldots. + & (n-1) & + & n \\ n & + & (n-1) & + & (n-2) & + \ldots. + & 2 & + & 1 \end{array}$$

b $S_n = \dfrac{n(n+1)}{2}$ **c** $a = 16,\ b = 3$

8 $S_n = \sum\limits_{k=1}^{n} (2k-1)$

9 **b**
$$\begin{aligned} (0+1)^3 &= 0^3 + 3(0)^2 + 3(0) + 1 \\ (1+1)^3 &= 1^3 + 3(1)^2 + 3(1) + 1 \\ (2+1)^3 &= 2^3 + 3(2)^2 + 3(2) + 1 \\ (3+1)^3 &= 3^3 + 3(3)^2 + 3(3) + 1 \\ &\ \vdots \\ (n+1)^3 &= n^3 + 3n^2 + 3n + 1 \end{aligned}$$

10 $\sum\limits_{k=1}^{n} (k+1)(k+2) = \dfrac{n(n^2+6n+11)}{3}$,
$\sum\limits_{k=1}^{10} (k+1)(k+2) = 570$

EXERCISE 5G

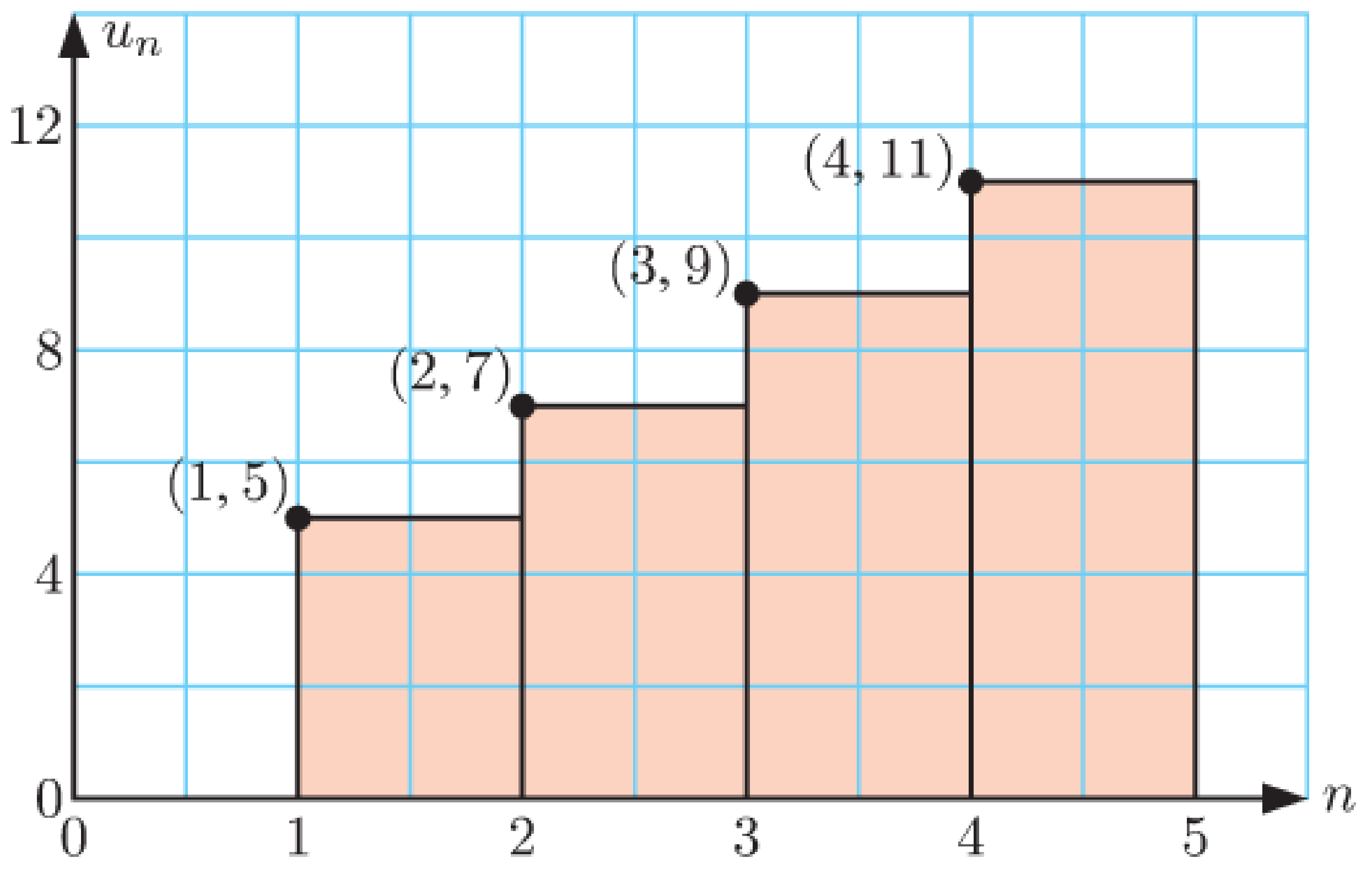
1 **a** 160 **b** 820 **c** $3087\frac{1}{2}$ **d** -1460
e -150 **f** -740
2 **a** 1749 **b** 184 **c** 2115 **d** $1410\frac{1}{2}$
3 **a** 160 **b** -630 **c** 135
4 $-115\frac{1}{2}$ **5** 18 layers
6 **a** **i** 38 laps **ii** 78 laps **b** 1470 laps
7 **a** \$450 **b** \$4125
8 **a** 65 seats **b** 1914 seats **c** 47 850 seats
9 **a** 14 025 **b** 71 071 **c** 3367 **d** 89 870
10 **a** $k = 5$ **b** $S_{25} = 2800$
11 $u_1 = 56,\ u_2 = 49$ **12** $S_{10} = 310$ **13** 8 terms
14 **a** $d = 3$ **b** $n = 11$ **15** 15 terms

16 **Hint:** $$\begin{aligned} S_n &= \frac{n}{2}(2u_1 + (n-1)d) \\ &= \frac{n}{2}(2\times 1 + (n-1)\times 1), \quad \text{and so on} \end{aligned}$$

17 **a** $u_n = 2n-1$
b $S_n = \dfrac{n}{2}(u_1 + u_n) = \dfrac{n}{2}(1 + 2n - 1) = n^2$

18 10, 4, -2 or -2, 4, 10 **19** $u_8 = 32$
20 2, 5, 8, 11, 14 or 14, 11, 8, 5, 2
21 **a** $u_1 = 7,\ u_2 = 10$ **b** $u_{20} = 64$
22 $S_{80} = -80$ **23** $u_1 = -\frac{7}{2},\ d = 2$

24 **a**



b S_n is the sum of the areas of the first n rectangles.

c **i** The left side of each rectangle increases in length by 2 units from the previous rectangle, $u_{n+1} = u_n + 2$.

ii The area of the $(n+1)$ th rectangle is u_{n+1}.
S_{n+1} is the sum of the areas of the first n rectangles and the $(n+1)$ th rectangle, $S_{n+1} = S_n + u_{n+1}$.

25 $15d$

EXERCISE 5H

1 **a** 6560 **b** 5115 **c** $\frac{3069}{128} \approx 24.0$
d $\approx 189\,134$ **e** $\frac{32\,769}{8192} \approx 4.00$ **f** ≈ 0.585

2 **a** $S_n = \dfrac{3+\sqrt{3}}{2}\left((\sqrt{3})^n - 1\right)$ **b** $S_n = 24\left(1 - (\frac{1}{2})^n\right)$
c $S_n = 1 - (0.1)^n$ **d** $S_n = \frac{40}{3}\left(1 - (-\frac{1}{2})^n\right)$

3 **a** 3069 **b** $\frac{4095}{1024} \approx 4.00$ **c** $-134\,217\,732$
4 **c** \$26 361.59
5 **a** The number of grains of wheat starts at 1, and each square has double the number of grains of the previous square.
c $u_n = 2^{n-1}$ **d** $(2^{64} - 1) \approx 1.84 \times 10^{19}$ grains of wheat
6 **a** \$5790 **b** $S_n = 100\,000((1.05)^n - 1)$ **c** $\approx$ \$40 710
7 £18 413.84

8 **a** $S_1 = \frac{1}{2}$, $S_2 = \frac{3}{4}$, $S_3 = \frac{7}{8}$, $S_4 = \frac{15}{16}$, $S_5 = \frac{31}{32}$

b $S_n = \dfrac{2^n - 1}{2^n}$ **c** $S_n = 1 - (\frac{1}{2})^n = \dfrac{2^n - 1}{2^n}$

d as $n \to \infty$, $S_n \to 1$

e As $n \to \infty$, the sum of the fractions approaches the area of a 1×1 unit square.

9 $u_4 = \frac{2}{3}$ or 54

10 Arithmetic: $u_{20} = 39$ or $7\frac{1}{3}$

Geometric: $u_{20} = 3^{19}$ or $\left(\frac{4}{3}\right)^{19}$

12 $n = 5$ **13** $n = 11$

14 **a** In 3 years she will earn \$183 000 under *Option B*, compared with \$126 100 under *Option A*.

b **i** $A_n = 40\,000 \times (1.05)^{n-1}$ **ii** $B_n = 59\,000 + 1000n$

c ≈ 13.1 years

e **i** graph 1 represents T_A, graph 2 represents T_B

ii P(22.3, 1 580 000) **iii** $0 \leqslant n \leqslant 22$

15 **a** $A_3 = \$8000 \times (1.03)^3 - (1.03)^2R - 1.03R - R$

b $A_8 = \$8000 \times (1.03)^8 - (1.03)^7R - (1.03)^6R$
$- (1.03)^5R - (1.03)^4R - (1.03)^3R - (1.03)^2R$
$- (1.03)R - R$
$= 0$

c $R = \$1139.65$

EXERCISE 5I

1 **a** It is geometric with $u_1 = \frac{3}{10}$ and $r = \frac{1}{10}$, and we are adding all the terms. Therefore it is an infinite geometric series.

b Using **a**, $S = \dfrac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{3}{9} = \frac{1}{3}$ $\therefore$ $0.\overline{3} = \frac{1}{3}$

2 **a** $0.\overline{4} = \frac{4}{9}$ **b** $0.\overline{16} = \frac{16}{99}$ **c** $0.\overline{312} = \frac{104}{333}$

4 **a** 54 **b** 14.175

5 **a** 1 **b** $4\frac{2}{7}$ **6** $u_1 = 9$, $r = \frac{2}{3}$

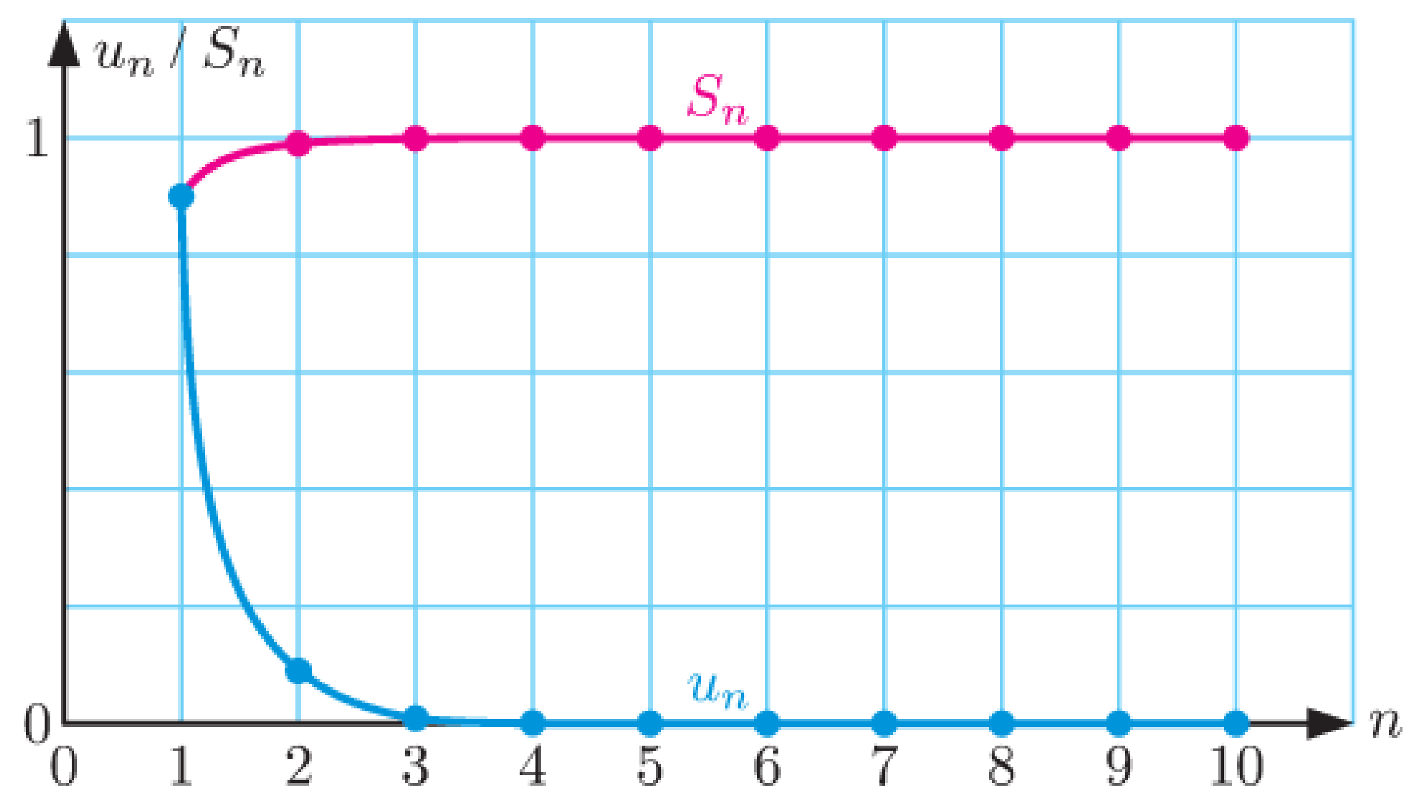
7 $u_1 = 8$, $r = \frac{1}{5}$ and $u_1 = 2$, $r = \frac{4}{5}$ **8** $S_5 = \frac{341}{16}$

9 **b** $S_n = 19 - 20(0.9)^n$ **c** 19 seconds **10** 70 cm

11 **a** $0.\overline{9} = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + ...$ which is geometric with $u_1 = \frac{9}{10}$ and $r = \frac{1}{10}$

$\therefore$ $0.\overline{9} = S = \dfrac{\frac{9}{10}}{1 - \frac{1}{10}} = 1$

c



12 $x = \frac{1}{2}$

13 $u_1 + u_2 + u_3 + u_4 + ...$ has common ratio r, $0 < r < 1$

a $u_1 - u_2 + u_3 - u_4 + ...$ has common ratio $-r$
$\therefore$ $-1 < -r < 1$ and so the series is convergent.
$\sqrt{u_1} + \sqrt{u_2} + \sqrt{u_3} + \sqrt{u_4} + ...$ has common ratio $\sqrt{r}$
$\therefore$ $-1 < \sqrt{r} < 1$ and so the series is convergent.

b $\frac{81}{8}$

REVIEW SET 5A

1 **a** $u_2 = 9$ **b** $u_6 = 19$ **c** $S_4 = 37$

2 $k = -\frac{11}{2}$ **3** **b** $u_1 = 6$, $r = \frac{1}{2}$ **c** $u_{16} \approx 0.000\,183$

4 $u_n = \frac{1}{6} \times 2^{n-1}$ or $-\frac{1}{6} \times (-2)^{n-1}$

5 23, 21, 19, 17, 15, 13, 11, 9

6 **a** ≈ 45.7 mL **b** $u_n \approx 45.7n$ **c** ≈ 594 mL

7 **a** $10\frac{4}{5}$ **b** $16 + 8\sqrt{2}$ **8** **a** 1272 **b** $302\frac{1}{2}$

9 **a** 2011: 630 000 sheets, 2012: 567 000 sheets

b $\approx 5\,630\,000$ sheets

10 **a** $1 + 4 + 9 + 16 + 25 + 36 + 49 = 140$

b $\frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \frac{7}{6} = \frac{99}{20}$

11 **a** $u_n = 3n + 1$ **12** **a** £15 425.20 **b** £15 453.77

13 \$4800 **14** **a** €6622.87 **b** €13 313.28

15 **a** \$24 076.91 **b** \$22 822.20

16 $u_n = 33 - 5n$, $S_n = \dfrac{n}{2}(61 - 5n)$

17 **a** 17 terms **b** $\frac{131\,071}{512} \approx 255.998$

18 $u_1 = 54$, $r = \frac{2}{3}$ or $u_1 = 150$, $r = -\frac{2}{5}$
$|\,r\,| < 1$ in both cases, so the series will converge.
For $u_1 = 54$, $r = \frac{2}{3}$, $S = 162$.
For $u_1 = 150$, $r = -\frac{2}{5}$, $S = 107\frac{1}{7}$.

19 **a** The number of cigarettes Tim smokes decreases by 5 each week, with 115 in the first week. The common difference $d = -5$.

b 24 weeks **c** 1380 cigarettes

20 **a** $\frac{5}{2}\sqrt{2}$

b **i** $S_{10} = 310 + 155\sqrt{2}$ **ii** $S = 320 + 160\sqrt{2}$

21 $u_1 = 3$ **22** $r = 4$

23 $x = 3$, $y = -1$, $z = \frac{1}{3}$ or $x = \frac{1}{3}$, $y = -1$, $z = 3$

24 **a** \$192 000

b **i** \$1000, \$1600, \$2200 **ii** \$189 600

c **i** \$500, \$600, \$720 **ii** \$196 242.12

d Option 3 **e** \$636.97

25 $x = \frac{3}{2}$ ($x = -\frac{6}{7}$ gives a divergent series)

REVIEW SET 5B

1 **a** $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}$ **b** 17, 22, 27, 32, 37

c $\frac{4}{3}, 1, \frac{4}{5}, \frac{2}{3}, \frac{4}{7}$

2 **b** $u_1 = 63$, $d = -5$ **c** $u_{37} = -117$ **d** $u_{54} = -202$

3 **a** $u_n = 73 - 6n$ **b** $u_{34} = -131$

4 **a** $S_{12} = 432$ **b** $S_{12} = \frac{12\,285}{256} \approx 48.0$

5 **a** $u_n = \frac{25}{6}n - \frac{265}{6}$

b Stacy makes $\approx$ £4.17 per customer. The setup fee for the stand $\approx$ £44.17.

c $\approx$ £105.83

6 $u_{12} = 10\,240$ **7** 27 metres

8 **a** \$8337.11 **b** \$8369.33 **c** \$8376.76

9 **a** $k = \pm\frac{2\sqrt{3}}{3}$ **b** For $k = \frac{2\sqrt{3}}{3}$, $r = \frac{\sqrt{3}}{6}$

For $k = -\frac{2\sqrt{3}}{3}$, $r = -\frac{\sqrt{3}}{6}$

10 **a** 35.5 km **b** 1183 km

11 **a** $\frac{1331}{2100} \approx 0.634$ **b** $\frac{98}{15} \approx 6.53$

12 $u_{11} = \frac{8}{19\,683} \approx 0.000\,406$ **13** 3.80% p.a.

14 182 months (15 years 2 months)

15 **a** \$10 069.82 **b** \$7887.74 **16** \$2174.63

17 **a** 70 **b** ≈ 241 **c** $\frac{64}{1875} \approx 0.0341$

18 **a** $u_n = \frac{3}{4} \times 2^{n-1}$ **b** $S_{15} = 24\,575\frac{1}{4}$

19 **a** ≈ 3470 iguanas **b** year 2029

20 **a** $0 < x < 1$ (we require $|2x - 1| < 1$) **b** $35\frac{5}{7}$

21 **a** The sequence is $2^{u_1}, 2^{u_1+d}, 2^{u_1+2d}, ...$
or $2^{u_1}, 2^d 2^{u_1}, (2^d)^2 2^{u_1}, ...$
which is geometric.

b $\frac{32}{7}$

22 **a** \$82 539.08

b

n (years)	0	1	2	3	4
V_n (\$)	100 000	106 000	112 360	119 101.60	126 247.70

c $V_n = 100\,000 \times (1.06)^n$ dollars

d $S_n = 6000n$ dollars

e

n (years)	0	1	2	3	4
T_n (\$)	100 000	112 000	124 360	137 101.60	150 247.70

f 19 years

23 $47\frac{6}{7}$ or $31\frac{1}{7}$ **24** $S_n = \dfrac{2 - 2^{\frac{1}{n+1}}}{2^{\frac{1}{n+1}} - 1}$

25 **a** $r = 4$

b **Hint:** If u_1 is the first term of the arithmetic sequence, show that $(u_1 + 7d) \times 4 = u_1 + 23d$.

EXERCISE 6A

1 **a** ≈ 57.2 mm **b** ≈ 33.5 cm **c** ≈ 40.5 m
d ≈ 138 cm

2 ≈ 41.4 cm **3** ≈ 68.5 mm

4 **a** ≈ 133 cm^2 **b** ≈ 9.62 m^2 **c** ≈ 58.5 cm^2
d ≈ 192 cm^2

5 ≈ 5.26 cm **6** ≈ 21.5 cm

7 **a** ≈ 191 m **b** ≈ 6.04 m s^{-1}

8 **a** $8\sqrt{2} \approx 11.3$ mm **b** $8\pi(1 + \sqrt{2}) \approx 60.7$ mm
c 128 mm^2

9 **c** $r = 0.98$ m, $\theta \approx 58.5$ **d** ≈ 1.29 m

EXERCISE 6B.1

1 **a** 5.7802 m^2 **b** ≈ 112 cm^2 **c** ≈ 14.9 cm^2

2 **a** 1440 cm^2 **b** ≈ 51.6 cm^2 **c** ≈ 181 m^2

3 **a** 23 814 cm^2

b

area = 2754 cm^2

area = 7446 cm^2

area = 2550 cm^2

area $\approx$ 5617 cm^2

c $\approx$ €540

4 **a** 26 940 cm^2 **b** ≈ 407 m^2 **5** ≈ 2310 cm^2

6 **a** $(10x^2 + 12x)$ cm^2 **b** $(1 + \sqrt{3})x^2$ cm^2

EXERCISE 6B.2

1 **a** ≈ 1005.3 cm^2 **b** ≈ 63.6 km^2 **c** ≈ 188.5 cm^2
d ≈ 549.8 m^2 **e** ≈ 1068.1 cm^2 **f** ≈ 84.8 cm^2

2 **a** ≈ 2210 cm^2 **b** ≈ 66.5 m^2 **c** $\approx 14\,800$ mm^2
d ≈ 12.1 cm^2

3 **a** $s \approx 5.39$ **b** ≈ 46.4 m^2 **c** $\approx$ \$835.24

4 **a** ≈ 50.3 m^2 **b** $\approx$ \$1166.16 **c** ≈ 150.8 m^2
d $\approx$ \$2789.73 **e** $\approx$ \$3960

5 ≈ 266 cm^2 **6** **a** $SA = 4\pi r^2$ **b** ≈ 5.40 m

7 **a** $SA = 3\pi r^2$ **b** **i** ≈ 4.50 cm **ii** ≈ 4.24 cm

8 **a** $SA = 6\pi x^2$ cm^2 **b** $SA = 3\pi r^2$ cm^2
c $SA = \pi x^2(1 + \sqrt{5})$ cm^2

9 **a** 4 cm **b** ≈ 25.1 cm **c** ≈ 84.1 mm

10 **a** ≈ 34.7 m^2 **b** ≈ 285.4 m^2 **c** ≈ 62.8 cm^2

11 $\approx 24\,600$ km **12** **a** $\dfrac{\theta\pi s}{180}$ **b** $\theta = \dfrac{360r}{s}$

EXERCISE 6C.1

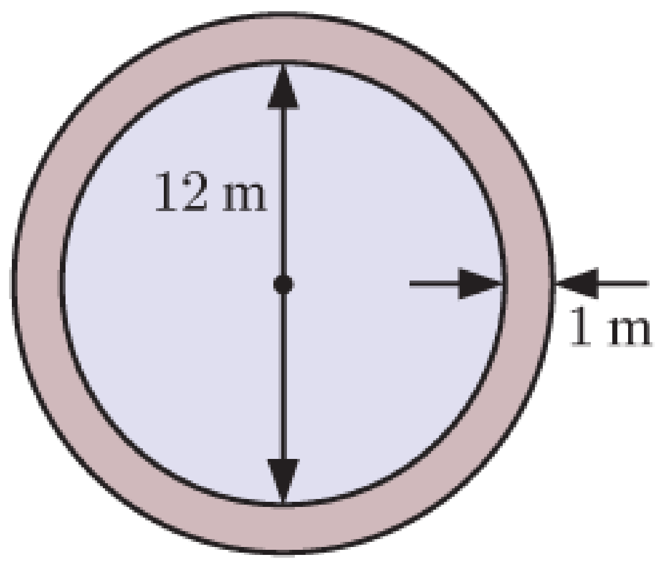
1 **a** 25.116 cm^3 **b** 373 cm^3 **c** 765.486 cm^3
d ≈ 2940 cm^3 **e** ≈ 3.13 m^3 **f** 1440 cm^3

2 **a** 648 000 000 mm^3 **b** ≈ 11.6 m^3 **c** 156 cm^3

3 **a** 0.5 m **b** 0.45 m **c** ≈ 0.373 m^3

4 **a** 7.176 m^3 **b** \$972

5 **a**



b ≈ 40.8 m^2
c ≈ 4.08 m^3

6 ≈ 81.1 tonnes

7 **a** 2 trailer loads **b** \$174.60
c **i** 2 loads **ii** \$95.90 **d** \$270.50

8 **a** 100 cm **b** 1 500 000 cm^3 (or 1.5 m^3) **c** 95 000 cm^2

9 **a** $\frac{8}{3} \approx 2.67$ cm **b** ≈ 3.24 cm **c** ≈ 1.74 cm

10 ≈ 12.7 cm

EXERCISE 6C.2

1 **a** ≈ 463 cm^3 **b** ≈ 4.60 cm^3 **c** ≈ 26.5 cm^3
d ≈ 1870 m^3 **e** ≈ 155 m^3 **f** ≈ 226 cm^3

2 **a** $\approx 29\,000$ m^3 **b** 480 m^3 **c** ≈ 497 cm^3

3 **a** ≈ 11.9 m^3 **b** 5.8 m **c** ≈ 1.36 m^3 more
e The hemispherical design, as it holds more concrete and is shorter.

4 **a** ≈ 4.46 cm **b** ≈ 2.60 m **c** ≈ 5.60 cm

6 **a** **i** ≈ 67.0 cm^3 **ii** ≈ 113 m^3
b $V = \frac{2}{3}\pi r^3$ This is half the volume of a sphere because when $h = r$, the cap is a hemisphere.

EXERCISE 6D

1 **a** 12.852 kL **b** ≈ 61.2 kL **c** ≈ 68.0 kL

2 **a** $\approx 12\,200$ cm^3 **b** ≈ 12.2 L **3** 594 425 kL

4 **a** ≈ 954 mL **b** 4.92 kL **c** 5155 tins **d** \$18 042.50

5 ≈ 0.553 m (or ≈ 55.3 cm)

6 **a** 1.32 m^3 **b** 1.32 kL **c** ≈ 10.5 cm **7** ≈ 7.8 cm

8 a ≈ 252 mL b i ≈ 189 mL ii 3.25 cm

9 35 truck loads

10 a $\approx 110\,000$ mm^3

b The external surface area and internal surface area of a container may be different.

c i $1\,870\,000$ mm^3 ii 1.87 L iii $\approx 502\,000$ mm^3

REVIEW SET 6A

1 a ≈ 18.3 cm b ≈ 38.3 cm c ≈ 91.6 cm^2

2 ≈ 10.4 cm

3 a ≈ 377.0 cm^2 b ≈ 339.8 cm^2 c ≈ 201.1 cm^2

4 a 71 m^2 b \$239.25

5 a ≈ 4.99 m^3 b 853 cm^3 c ≈ 0.452 m^3

6 ≈ 3.22 m^3 7 $\approx 82\,400$ cm^3 8 ≈ 1470 m^3

9 a 734.44 mL b ≈ 198 L 10 ≈ 68.4 mm

11 a height $= 3.3$ m $- 1.8$ m $- 0.8$ m $= 0.7$ m $= 70$ cm

b ≈ 1.06 m c ≈ 15.7 m^2

d **Hint:** Volume of silo
= volume of hemisphere + volume of cylinder
+ volume of cone

e ≈ 5.2 kL

REVIEW SET 6B

1 a $\theta^\circ \approx 76.6^\circ$ b ≈ 14.3 cm^2

2 a ≈ 29.1 cm b ≈ 25.1 cm^2

3 a ≈ 84.7 cm^2 b ≈ 7110 mm^2 c ≈ 8.99 m^2

4 ≈ 23.5 m^2 5 ≈ 434 cm^2

6 a ≈ 164 cm^3 b 120 m^3 c $\approx 10\,300$ mm^3

7 a 0.52 m^3 b 5.08 m^2 8 ≈ 5680 L 9 ≈ 1.03 m

10 a $\approx 6.08 \times 10^{18}$ m^2 b $\approx 1.41 \times 10^{27}$ m^3

11 a ≈ 56.5 cm^3 b 3 cm c ≈ 96.5 cm^2

EXERCISE 7A

1 a i $\frac{4}{5}$ ii $\frac{3}{5}$ iii $\frac{4}{3}$

b i $\frac{5}{8}$ ii $\frac{\sqrt{39}}{8}$ iii $\frac{5}{\sqrt{39}}$

c i $\frac{7}{\sqrt{65}}$ ii $\frac{4}{\sqrt{65}}$ iii $\frac{7}{4}$

d i $\frac{5}{\sqrt{61}}$ ii $\frac{6}{\sqrt{61}}$ iii $\frac{5}{6}$

2 a XY ≈ 4.9 cm, XZ ≈ 3.3 cm, YZ ≈ 5.9 cm

b i ≈ 0.83 ii ≈ 0.56 iii ≈ 1.48

3 a **Hint:** Base angles of an isosceles triangle are equal, and sum of all angles in a triangle is 180°.

b AB $= \sqrt{2} \approx 1.41$ m

c i $\frac{1}{\sqrt{2}} \approx 0.707$ ii $\frac{1}{\sqrt{2}} \approx 0.707$ iii 1

4 The OPP and ADJ sides will always be smaller than the HYP. So, the sine and cosine ratios will always be less than or equal to 1.

5 a i $\frac{a}{c}$ ii $\frac{b}{c}$ iii $\frac{a}{b}$ iv $\frac{b}{c}$ v $\frac{a}{c}$ vi $\frac{b}{a}$

b $A = 90^\circ - B$

c i $\sin\theta = \cos(90^\circ - \theta)$ ii $\cos\theta = \sin(90^\circ - \theta)$

iii $\tan\theta = \dfrac{1}{\tan(90^\circ - \theta)}$

6 a ≈ 7.50 m b ≈ 7.82 cm c ≈ 4.82 cm

d ≈ 5.17 m e ≈ 6.38 m f ≈ 4.82 cm

7 a $x \approx 3.98$ b i $y \approx 4.98$ ii $y \approx 4.98$

8 a $x \approx 2.87$, $y \approx 4.10$ b $x \approx 16.40$, $y \approx 18.25$

c $x \approx 10.77$, $y \approx 14.50$

9 a perimeter ≈ 23.2 cm, area ≈ 22.9 cm^2

b perimeter ≈ 17.0 cm, area ≈ 10.9 cm^2

10 ≈ 21.7 cm

EXERCISE 7B

1 a $\theta \approx 53.1^\circ$ b $\theta \approx 45.6^\circ$ c $\theta \approx 13.7^\circ$

d $\theta \approx 52.4^\circ$ e $\theta \approx 76.1^\circ$ f $\theta \approx 36.0^\circ$

2 a $\theta \approx 56.3^\circ$ b i $\phi \approx 33.7^\circ$ ii $\phi \approx 33.7^\circ$

3 a $\theta \approx 39.7^\circ$, $\phi \approx 50.3^\circ$ b $\alpha \approx 38.9^\circ$, $\beta \approx 51.1^\circ$

c $\theta \approx 61.5^\circ$, $\phi \approx 28.5^\circ$

4 a The triangle cannot be drawn with the given dimensions.

b The triangle cannot be drawn with the given dimensions.

c The result is not a triangle, but a straight line of length 9.3 m.

5 a $x \approx 2.65$, $\theta \approx 37.1^\circ$

b $x \approx 6.16$, $\theta \approx 50.3^\circ$, $y \approx 13.0$

6 $\approx 135^\circ$ 7 $\alpha \approx 6.92$

EXERCISE 7C

1 a $x \approx 4.13$ b $\alpha \approx 75.5^\circ$ c $\beta \approx 41.0^\circ$

d $x \approx 6.29$ e $\theta \approx 51.9^\circ$ f $x \approx 12.6$

2 $\approx 22.4^\circ$ 3 ≈ 11.8 cm

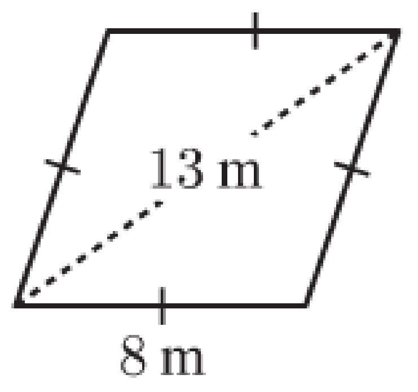
4 a ≈ 27.2 cm^2 b ≈ 153 m^2 5 $\approx 119^\circ$

6 ≈ 36.5 cm 7 a $x \approx 45.4$ b $x \approx 2.24$

8 a $x \approx 3.44$ b $\alpha \approx 51.5^\circ$

9 a ≈ 12.3 cm^2 b ≈ 14.3 cm^2

10 a



b ≈ 9.33 m

c $\approx 71.3^\circ$

11 a ≈ 2.59 cm b ≈ 8.46 cm

12 a $\theta \approx 36.9^\circ$ b $r \approx 11.3$ c $\alpha \approx 61.9^\circ$

13 ≈ 7.99 cm 14 $\approx 89.2^\circ$ 15 $\approx 47.2^\circ$ 16 ≈ 6.78 cm^2

EXERCISE 7D

1 ≈ 18.3 m 2 a ≈ 46.4 m b ≈ 259 m

3 $\approx 1.58^\circ$ 4 a $\approx 26.4^\circ$ b $\approx 26.4^\circ$

5 ≈ 142 m 6 $\theta \approx 12.6^\circ$ 7 ≈ 9.56 m

8 ≈ 46.7 m 9 $\beta \approx 129^\circ$ 10 ≈ 10.9 m

11 ≈ 104 m 12 ≈ 962 m 13 ≈ 3.17 km

14 ≈ 43.8 m 15 a ≈ 18.4 cm b $\approx 35.3^\circ$

16 a ≈ 10.8 cm b $\approx 36.5^\circ$ c ≈ 9.49 cm d $\approx 40.1^\circ$

17 a ≈ 82.4 cm b ≈ 77.7 L

18 a i 2 m ii ≈ 2.01 m b $\approx 6.84^\circ$

19 a ≈ 10.2 m b no 20 a ≈ 73.4 m b $\approx 16.2^\circ$

21 $\approx 67.0^\circ$

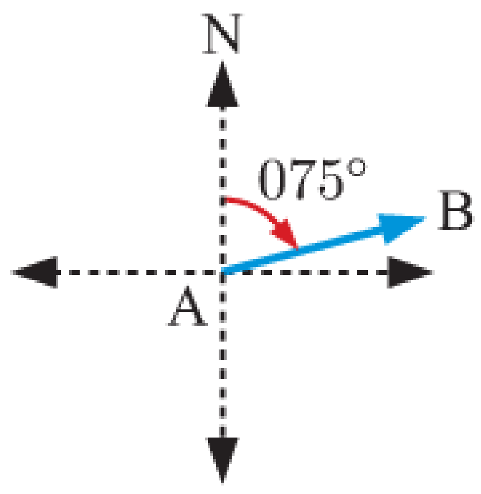
22 a ≈ 1.49 m^3 b ≈ 0.331 m^3 c ≈ 88.9 cm^3

23 a **Hint:** Consider

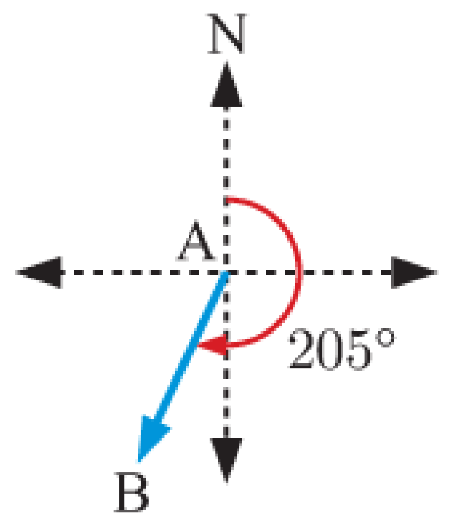
b ≈ 0.285 arc seconds

EXERCISE 7E

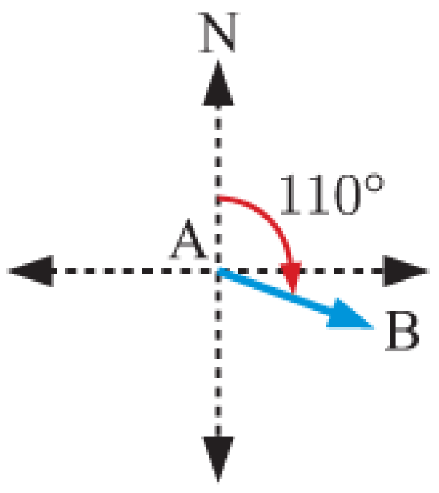
1 a



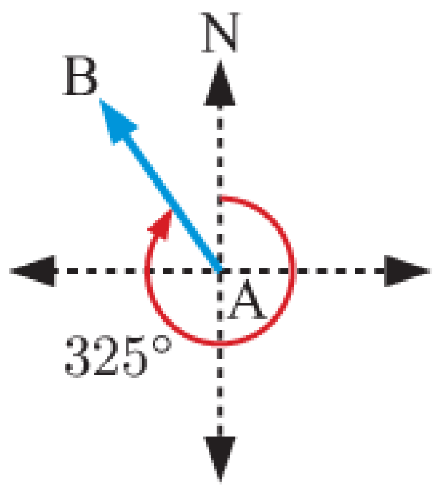
b



c



d

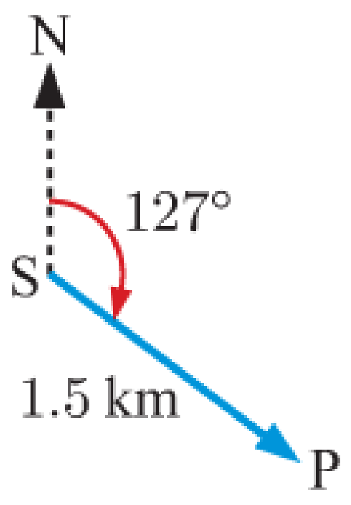


2 **a** 126° **b** 245° **c** 152° **d** 308°

3 **a** 072° **b** 252° **c** 162° **d** 342°
e 113° **f** 293°

4 ≈ 125° **5** **a** ≈ 224 m **b** ≈ 333° **c** ≈ 153°

6 **a**



b ≈ 1.20 km **c** ≈ 0.903 km

7 ≈ 2.41 km **8** ≈ 12.6 km

9 **a** ≈ 854 m **b** ≈ 203°

10 ≈ 73.3 km on the bearing ≈ 191°

11 ≈ 17.8 km on the bearing ≈ 162°

12 **a** ≈ 046.6° **b** ≈ 4.22 km

EXERCISE 7F

1 **a** **i** [EH] **ii** [EF] **iii** [EG] **iv** [FH]
b **i** [MR] **ii** [MN]

2 **a** **i** $\widehat{\text{AFE}}$ **ii** $\widehat{\text{BMF}}$ **iii** $\widehat{\text{ADE}}$ **iv** $\widehat{\text{BNF}}$
b **i** $\widehat{\text{BAM}}$ **ii** $\widehat{\text{BNM}}$ **iii** $\widehat{\text{EAN}}$

3 **a** **i** ≈ 36.9° **ii** ≈ 25.1° **iii** ≈ 56.3° **iv** ≈ 29.1°
b **i** ≈ 33.7° **ii** ≈ 33.7° **iii** ≈ 25.2° **iv** ≈ 30.8°
c **i** ≈ 59.0° **ii** ≈ 22.0° **iii** ≈ 22.6°
d **i** ≈ 64.9° **ii** ≈ 71.7°

4 ≈ 31.7°

REVIEW SET 7A

1 **a** 10 cm **b** $\frac{6}{10} = \frac{3}{5}$ **c** $\frac{8}{10} = \frac{4}{5}$ **d** $\frac{6}{8} = \frac{3}{4}$

2 **a** $x \approx 3.51$ **b** $x \approx 51.1$ **c** $x \approx 5.62$

3 ≈ 43.4 cm^2 **4** $\theta = 33°$, $x \approx 3.90$, $y \approx 7.15$

5 $\theta \approx 8.19°$ **6** ≈ 124°

7 **a** $x \approx 2.8$ **b** $x \approx 4.2$ **c** $x \approx 5.2$

8 ≈ 13.5 m **9** **a** 118° **b** 231° **c** 329°

10 13 km on the bearing ≈ 203° from the helipad.

11 ≈ 8.74° **12** ≈ 0.607 L **13** **a** ≈ 53.1° **b** ≈ 62.1°

REVIEW SET 7B

1 **a** AB ≈ 4.5 cm, AC ≈ 2.2 cm, BC ≈ 5.0 cm
b **i** ≈ 0.44 **ii** ≈ 0.90 **iii** ≈ 0.49

2 **a** $\theta \approx 34.8°$ **b** $\theta \approx 39.7°$ **c** $\theta \approx 36.0°$

3 AB ≈ 120 mm, AC ≈ 111 mm

4 $x \approx 25.7$, $\theta \approx 53.6°$, $\alpha \approx 36.4°$

5 **a** ≈ 200 cm **b** ≈ 1500 cm^2 **6** ≈ 2.54 cm

7 ≈ 204 m **8** **a** 90° **b** ≈ 33.9°

9 ≈ 3.91 km on the bearing ≈ 253° from his starting point.

10 ≈ 5.46 km **11** ≈ 485 m^3

12 **a** ≈ 14.4° **b** ≈ 18.9° **c** ≈ 21.8°

13 **a** **i** ≈ 27.6 cm **ii** ≈ 23.3 cm **b** ≈ 6010 cm^3

EXERCISE 8A

1 **a** $\frac{\pi}{2}$ **b** $\frac{\pi}{3}$ **c** $\frac{\pi}{6}$ **d** $\frac{\pi}{10}$ **e** $\frac{\pi}{20}$
f $\frac{3\pi}{4}$ **g** $\frac{5\pi}{4}$ **h** $\frac{3\pi}{2}$ **i** 2π **j** 4π
k $\frac{7\pi}{4}$ **l** 3π **m** $\frac{\pi}{5}$ **n** $\frac{4\pi}{9}$ **o** $\frac{23\pi}{18}$

2 **a** $\approx 0.641^c$ **b** $\approx 2.39^c$ **c** $\approx 5.55^c$ **d** $\approx 3.83^c$
e $\approx 6.92^c$

3 **a** 36° **b** 108° **c** 135° **d** 10° **e** 20°
f 140° **g** 18° **h** 27° **i** 210° **j** 22.5°

4 **a** ≈ 114.59° **b** ≈ 87.66° **c** ≈ 49.68°
d ≈ 182.14° **e** ≈ 301.78°

5 **a**

Degrees	0	45	90	135	180	225	270	315	360
Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

b

Deg.	0	30	60	90	120	150	180	210	240	270	300	330	360
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

EXERCISE 8B

1 **a** 7 cm **b** 12 cm **c** ≈ 13.0 m

2 **a** 6 cm^2 **b** 48 cm^2 **c** ≈ 8.21 cm^2

3 **a** arc length ≈ 49.5 cm, area ≈ 223 cm^2
b arc length ≈ 23.0 cm, area ≈ 56.8 cm^2

4 **a** $\approx 0.686^c$ **b** 0.6^c

5 **a** $\theta = 0.75^c$, area = 24 cm^2
b $\theta = 1.68^c$, area = 21 cm^2
c $\theta \approx 2.32^c$, area = 126.8 cm^2

6 **a** ≈ 3.15 m **b** ≈ 9.32 m^2

7 **a** ≈ 5.91 cm **b** ≈ 18.9 cm

8 **a** $\alpha \approx 0.3218^c$ **b** $\theta \approx 2.498^c$ **c** ≈ 387 m^2

9 **a** ≈ 11.7 cm **b** $r \approx 11.7$ **c** ≈ 37.7 cm **d** $\theta \approx 3.23^c$

10 ≈ 25.9 cm **11** **b** ≈ 2 h 24 min **12** ≈ 227 m^2

13 **a** $\alpha \approx 5.739$ **b** $\theta \approx 168.5$ **c** $\phi \approx 191.5$
d ≈ 71.62 cm

14 **a** 4 cm **b** **i** ≈ 2.16 cm^2 **ii** ≈ 29.3 cm^2

15 **a** **Hint:** Let the largest circle have radius r_1, and use a right angled triangle to show that $\sin\frac{\pi}{6} = \dfrac{r_1}{10 - r_1}$.
b $\frac{25\pi}{2}$ units^2 **c** $\frac{3}{4}$

EXERCISE 8C

1

θ (degrees)	0°	90°	180°	270°	360°	450°
θ (radians)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
sine	0	1	0	−1	0	1
cosine	1	0	−1	0	1	0
tangent	0	undef.	0	undef.	0	undef.

2 **a** **i** A(cos 26°, sin 26°), B(cos 146°, sin 146°), C(cos 199°, sin 199°)
ii A(0.899, 0.438), B(−0.829, 0.559), C(−0.946, −0.326)
b **i** A(cos 123°, sin 123°), B(cos 251°, sin 251°), C(cos(−35°), sin(−35°))
ii A(−0.545, 0.839), B(−0.326, −0.946), C(0.819, −0.574)

3 **a** **i** $\frac{1}{\sqrt{2}} \approx 0.707$ **ii** $\frac{\sqrt{3}}{2} \approx 0.866$

b

θ (degrees)	$30°$	$45°$	$60°$	$135°$	$150°$	$240°$	$315°$
θ (radians)	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{4\pi}{3}$	$\frac{7\pi}{4}$
sine	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$
cosine	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
tangent	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	$\sqrt{3}$	-1

4

Quadrant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	$0° < \theta < 90°$	$0 < \theta < \frac{\pi}{2}$	+ve	+ve	+ve
2	$90° < \theta < 180°$	$\frac{\pi}{2} < \theta < \pi$	−ve	+ve	−ve
3	$180° < \theta < 270°$	$\pi < \theta < \frac{3\pi}{2}$	−ve	−ve	+ve
4	$270° < \theta < 360°$	$\frac{3\pi}{2} < \theta < 2\pi$	+ve	−ve	−ve

5 **a** 1 and 4 **b** 2 and 3 **c** 3 **d** 2

6 **a** $\cos 400° = \cos(360 + 40)° = \cos 40°$

b $\sin \frac{5\pi}{7} = \sin\left(\frac{5\pi}{7} + 2\pi\right) = \sin \frac{19\pi}{7}$

c $\tan \frac{13\pi}{8} = \tan\left(\frac{13\pi}{8} - 3\pi\right) = \tan\left(-\frac{11\pi}{8}\right)$

7 **B** and **D** **8** **B** and **E**

9 **a** **i** ≈ 0.985 **ii** ≈ 0.985 **iii** ≈ 0.866 **iv** ≈ 0.866

v 0.5 **vi** 0.5 **vii** ≈ 0.707 **viii** ≈ 0.707

b $\sin(180° - \theta) = \sin\theta$ **c** $\sin(\pi - \theta) = \sin\theta$

d The points have the same y-coordinate.

e **i** $135°$ **ii** $129°$ **iii** $\frac{2\pi}{3}$ **iv** $\frac{5\pi}{6}$

10 **a** **i** ≈ 0.342 **ii** ≈ -0.342 **iii** 0.5

iv -0.5 **v** ≈ 0.906 **vi** ≈ -0.906

vii ≈ 0.174 **viii** ≈ -0.174

b $\cos(180° - \theta) = -\cos\theta$ **c** $\cos(\pi - \theta) = -\cos\theta$

d The x-coordinates of the points have the same magnitude but are opposite in sign.

e **i** $140°$ **ii** $161°$ **iii** $\frac{4\pi}{5}$ **iv** $\frac{3\pi}{5}$

11 $\tan(\pi - \theta) = -\tan\theta$

12 **a** ≈ 0.6820 **b** ≈ 0.8572 **c** ≈ -0.7986

d ≈ 0.9135 **e** ≈ 0.9063 **f** ≈ -0.6691

13 **a**

θ^c	$\sin\theta$	$\sin(-\theta)$	$\cos\theta$	$\cos(-\theta)$
0.75	≈ 0.682	≈ -0.682	≈ 0.732	≈ 0.732
1.772	≈ 0.980	≈ -0.980	≈ -0.200	≈ -0.200
3.414	≈ -0.269	≈ 0.269	≈ -0.963	≈ -0.963
6.25	≈ -0.0332	≈ 0.0332	≈ 0.999	≈ 0.999
−1.17	≈ -0.921	≈ 0.921	≈ 0.390	≈ 0.390

b $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$

c Q has coordinates $(\cos(-\theta), \sin(-\theta))$ or $(\cos\theta, -\sin\theta)$ (since it is the reflection of P in the x-axis)

$\therefore$ $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

d $\cos(2\pi - \theta) = \cos(-\theta) = \cos\theta$

$\sin(2\pi - \theta) = \sin(-\theta) = -\sin\theta$

e $\tan(2\pi - \theta) = -\tan\theta$

14 **a** The angle between [OP] and the positive x-axis is $\left(\frac{\pi}{2} - \theta\right)$.

$\therefore$ P is $\left(\cos\left(\frac{\pi}{2} - \theta\right), \sin\left(\frac{\pi}{2} - \theta\right)\right)$

b **i** In $\triangle$OXP, $\sin\theta = \frac{\text{XP}}{\text{OP}} = \frac{\text{XP}}{1}$

$\therefore$ $\text{XP} = \sin\theta$

ii In $\triangle$OXP, $\cos\theta = \frac{\text{OX}}{\text{OP}} = \frac{\text{OX}}{1}$

$\therefore$ $\text{OX} = \cos\theta$

c **i** $\cos\left(\frac{\pi}{2} - \theta\right) = \text{XP} = \sin\theta$

ii $\sin\left(\frac{\pi}{2} - \theta\right) = \text{OX} = \cos\theta$

d **i** $\cos \frac{\pi}{5} = \sin\left(\frac{\pi}{2} - \frac{\pi}{5}\right) = \sin \frac{3\pi}{10} \approx 0.809$

ii $\sin \frac{\pi}{8} = \cos\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \cos \frac{3\pi}{8} \approx 0.383$

e $\tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan\theta}$

EXERCISE 8D

1

	a	**b**	**c**	**d**	**e**
$\sin\theta$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$
$\cos\theta$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$
$\tan\theta$	1	-1	-1	0	1

2

	a	**b**	**c**	**d**	**e**
$\sin\beta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\cos\beta$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\tan\beta$	$\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$

3 **a** $\cos \frac{2\pi}{3} = -\frac{1}{2}$, $\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, $\tan \frac{2\pi}{3} = -\sqrt{3}$

b $\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$, $\tan\left(-\frac{\pi}{4}\right) = -1$

4 **a** $\cos \frac{\pi}{2} = 0$, $\sin \frac{\pi}{2} = 1$ **b** $\tan \frac{\pi}{2}$ is undefined

5 **a** $\frac{3}{4}$ **b** $\frac{1}{4}$ **c** $\frac{1}{4}$ **d** $-\frac{1}{4}$ **e** 1 **f** $\sqrt{2}$

g $\frac{1}{2}$ **h** $\frac{1}{2}$ **i** 2 **j** -1 **k** $-\sqrt{3}$ **l** $-\sqrt{3}$

6 **a** $\frac{\pi}{6}, \frac{5\pi}{6}$ **b** $\frac{\pi}{3}, \frac{2\pi}{3}$ **c** $\frac{\pi}{4}, \frac{7\pi}{4}$ **d** $\frac{2\pi}{3}, \frac{4\pi}{3}$

e $\frac{3\pi}{4}, \frac{5\pi}{4}$ **f** $\frac{4\pi}{3}, \frac{5\pi}{3}$

7 **a** $\frac{\pi}{4}, \frac{5\pi}{4}$ **b** $\frac{3\pi}{4}, \frac{7\pi}{4}$ **c** $\frac{\pi}{3}, \frac{4\pi}{3}$ **d** $0, \pi, 2\pi$

e $\frac{\pi}{6}, \frac{7\pi}{6}$ **f** $\frac{2\pi}{3}, \frac{5\pi}{3}$

8 **a** $\frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$ **b** $\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$ **c** $\frac{3\pi}{2}, \frac{7\pi}{2}$

9 **a** $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ **b** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$ **c** $\theta = \pi$

d $\theta = \frac{\pi}{2}$ **e** $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}$ **f** $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

g $\theta = 0, \pi, 2\pi$ **h** $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

i $\theta = \frac{5\pi}{6}, \frac{11\pi}{6}$ **j** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

10 **a** $\theta = k\pi$, $k \in \mathbb{Z}$ **b** $\theta = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

EXERCISE 8E

1 **a** $\cos\theta = \pm\frac{\sqrt{3}}{2}$ **b** $\cos\theta = \pm\frac{2\sqrt{2}}{3}$ **c** $\cos\theta = \pm 1$

d $\cos\theta = 0$

2 **a** $\sin\theta = \pm\frac{3}{5}$ **b** $\sin\theta = \pm\frac{\sqrt{7}}{4}$ **c** $\sin\theta = 0$

d $\sin\theta = \pm 1$

3 **a** $\sin\theta = \frac{\sqrt{5}}{3}$ **b** $\cos\theta = -\frac{\sqrt{21}}{5}$ **c** $\cos\theta = \frac{4}{5}$

d $\sin\theta = -\frac{12}{13}$

4 **a** $\tan\theta = -\frac{1}{2\sqrt{2}}$ **b** $\tan\theta = -2\sqrt{6}$ **c** $\tan\theta = \frac{1}{\sqrt{2}}$
d $\tan\theta = -\frac{\sqrt{7}}{3}$

5 **a** $\sin\theta = \frac{2}{\sqrt{13}}$, $\cos\theta = \frac{3}{\sqrt{13}}$ **b** $\sin\theta = \frac{4}{5}$, $\cos\theta = -\frac{3}{5}$
c $\sin\theta = -\sqrt{\frac{5}{14}}$, $\cos\theta = -\frac{3}{\sqrt{14}}$
d $\sin\theta = -\frac{12}{13}$, $\cos\theta = \frac{5}{13}$

6 $\sin\theta = \dfrac{-k}{\sqrt{k^2+1}}$, $\cos\theta = \dfrac{-1}{\sqrt{k^2+1}}$

EXERCISE 8F

1 **a** $\theta \approx 76.0°$ or $256°$ **b** $\theta \approx 33.9°$ or $326.1°$
c $\theta \approx 36.9°$ or $143.1°$ **d** $\theta = 90°$ or $270°$
e $\theta \approx 81.5°$ or $261.5°$ **f** $\theta \approx 83.2°$ or $276.8°$

2 **a** $\theta \approx 0.322$ or 3.46 **b** $\theta \approx 1.13$ or 5.16
c $\theta \approx 0.656$ or 2.49 **d** $\theta \approx 1.32$ or 4.97
e $\theta \approx 0.114$ or 3.26 **f** $\theta \approx 0.167$ or 2.97

3 **a** $\theta \approx 1.82$ or 4.46 **b** $\theta = 0, \pi$, or 2π
c $\theta \approx 1.88$ or 5.02 **d** $\theta \approx 3.58$ or 5.85
e $\theta \approx 0.876$ or 4.02 **f** $\theta \approx 0.674$ or 5.61
g $\theta \approx 0.0910$ or 3.05 **h** $\theta \approx 2.19$ or 4.10

4 **a** $\theta \approx -95.7°$ or $95.7°$ **b** $\theta \approx 53.1°$ or $126.9°$
c $\theta \approx -56.3°$ or $123.7°$ **d** $\theta \approx -36.9°$ or $36.9°$
e $\theta \approx -39.8°$ or $140.2°$ **f** $\theta \approx -140.5°$ or $-39.5°$

5 **a** $\theta \approx 1.27$ or 5.02
b For $\theta \approx 1.27$: $\sin\theta = \frac{\sqrt{91}}{10}$, $\tan\theta = \frac{\sqrt{91}}{3}$
For $\theta \approx 5.02$: $\sin\theta = -\frac{\sqrt{91}}{10}$, $\tan\theta = -\frac{\sqrt{91}}{3}$

EXERCISE 8G

1 **a** $y = \sqrt{3}x$ **b** $y = x$ **c** $y = -\frac{1}{\sqrt{3}}x$

2 **a** $y = \sqrt{3}x + 2$ **b** $y = -\sqrt{3}x$ **c** $y = \frac{1}{\sqrt{3}}x - 2$

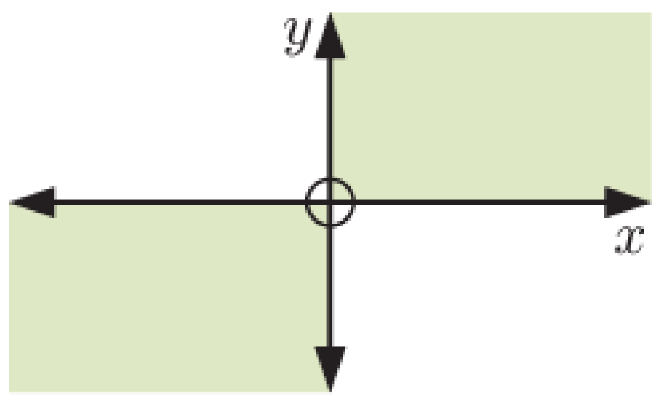
3 **a** $\theta \approx 1.25$ **b** $\theta \approx -0.983$ **c** $\theta \approx -0.381$

4 **a** $\theta \approx 23.2°$ **b** $\theta \approx 117°$ **c** $\theta \approx -11.3°$

REVIEW SET 8A

1 **a** $\frac{2\pi}{3}$ **b** $\frac{5\pi}{4}$ **c** $\frac{5\pi}{6}$ **d** 3π

2



3 **a** $(0.766, -0.643)$ **b** $(-0.956, 0.292)$
c $(0.778, 0.629)$ **d** $(0.866, -0.5)$

4 12 cm **5** **a** $\frac{\pi}{3}$ **b** 15° **c** 84°

6 **a** ≈ 0.358 **b** ≈ -0.035 **c** ≈ 0.259 **d** ≈ 1.072

7 **a** $\cos 360° = 1$, $\sin 360° = 0$
b $\cos(-\pi) = -1$, $\sin(-\pi) = 0$

8 **a** $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{2\pi}{3} = -\frac{1}{2}$, $\tan\frac{2\pi}{3} = -\sqrt{3}$
b $\sin\frac{8\pi}{3} = \frac{\sqrt{3}}{2}$, $\cos\frac{8\pi}{3} = -\frac{1}{2}$, $\tan\frac{8\pi}{3} = -\sqrt{3}$

9 **a** **i** 60° **ii** $\frac{\pi}{3}$ **b** $\frac{\pi}{3}$ cm **c** $\frac{\pi}{6}$ cm^2

10 $\tan x = \frac{1}{\sqrt{15}}$ **11** $\sin\theta = \pm\frac{\sqrt{7}}{4}$

12 **a** $\frac{\sqrt{3}}{2}$ **b** 0 **c** $\frac{1}{2}$ **13** **a** $\frac{2}{\sqrt{13}}$ **b** $-\frac{3}{\sqrt{13}}$

14 $\tan\theta = \frac{\sqrt{6}}{\sqrt{11}}$

16 **a** $\theta \approx 0.841$ or 5.44 **b** $\theta \approx 3.39$ or 6.03
c $\theta \approx 1.25$ or 4.39

17 **a** $y = \frac{1}{\sqrt{3}}x$ **b** $y = \sqrt{3}x + 3$

REVIEW SET 8B

1 **a** 72° **b** $\approx 83.65°$ **c** $\approx 24.92°$ **d** $\approx -302.01°$

2 ≈ 111 cm^2 **3** $\approx 103°$

4 radius ≈ 8.79 cm, area ≈ 81.0 cm^2 **5** 4.5 cm or 6 cm

6 **a** $\cos\frac{3\pi}{2} = 0$, $\sin\frac{3\pi}{2} = -1$
b $\cos\left(-\frac{\pi}{2}\right) = 0$, $\sin\left(-\frac{\pi}{2}\right) = -1$

7 **a** $\sin(\pi - p) = m$ **b** $\sin(p + 2\pi) = m$
c $\cos p = \sqrt{1 - m^2}$ **d** $\tan p = \dfrac{m}{\sqrt{1-m^2}}$

8 **a** 150°, 210° **b** 45°, 135° **c** 120°, 300°

9 **a** $\theta = \pi$ **b** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

10 **a** 133° **b** $\frac{14\pi}{15}$ **c** 174°

11 perimeter ≈ 34.1 cm, area ≈ 66.5 cm^2

13 **a** $\frac{\sqrt{7}}{4}$ **b** $-\frac{\sqrt{7}}{3}$ **c** $\frac{3}{4}$

14 **a** $2\frac{1}{2}$ **b** $1\frac{1}{2}$ **c** $-\frac{1}{2}$ **d** 3

17 **a** $\theta \approx 0.322$ **b** $\theta \approx 1.95$

EXERCISE 9A

1 **a** ≈ 28.9 cm^2 **b** ≈ 384 km^2 **c** 20 m^2

2 **a** ≈ 18.7 cm^2 **b** ≈ 28.3 cm^2 **c** ≈ 52.0 m^2

3 $x \approx 19.0$ **4** **a** ≈ 166 cm^2 **b** ≈ 1410 cm^2

5 ≈ 18.9 cm^2 **6** ≈ 137 cm^2

7 **a** ≈ 71.616 m^2 **b** ≈ 8.43 m

8 ≈ 374 cm^2 **9** ≈ 7.49 cm **10** ≈ 11.9 m

11 **a** $\approx 48.6°$ or $\approx 131.4°$ **b** $\approx 42.1°$ or $\approx 137.9°$

12 $\frac{1}{4}$ is not covered

13 **a** ≈ 36.2 cm^2 **b** ≈ 62.8 cm^2 **c** ≈ 40.4 mm^2
d ≈ 19.3 cm^2

14 ≈ 4.69 cm^2

EXERCISE 9B

1 **a** ≈ 28.8 cm **b** ≈ 3.38 km **c** ≈ 14.2 m

2 **a** $\theta \approx 82.8°$ **b** $\theta \approx 54.8°$ **c** $\theta \approx 98.2°$

3 $\mathrm{B\widehat{A}C} \approx 52.0°$, $\mathrm{A\widehat{B}C} \approx 59.3°$, $\mathrm{A\widehat{C}B} \approx 68.7°$

4 **a** $\approx 112°$ **b** ≈ 16.2 cm^2

5 **a** $\approx 40.3°$ **b** $\approx 107°$

6 **a** $\cos\theta = 0.65$ **b** $x \approx 3.81$

7 **a** $\theta \approx 75.2°$ **b** ≈ 6.30 m

8 **a** DB ≈ 4.09 m, BC ≈ 9.86 m
b $\mathrm{A\widehat{B}E} \approx 68.2°$, $\mathrm{D\widehat{B}C} \approx 57.5°$ **c** ≈ 17.0 m^2

9 **b** $x = 3 + \sqrt{22}$

10 **a** $x \approx 10.8$ **b** $x \approx 2.77$ **c** $x \approx 2.89$

11 $x \approx 1.41$ or 7.78 **12** BD ≈ 12.4 cm

13 $\theta \approx 71.6°$ **14** ≈ 6.40 cm

15 **a** $x = 2$ **b** $4\sqrt{6}$ cm^2 **16** $\approx 63°, 117°, 36°, 144°$

EXERCISE 9C.1

1 **a** $x \approx 28.4$ **b** $x \approx 13.4$ **c** $x \approx 3.79$
d $x \approx 10.3$ **e** $x \approx 4.49$ **f** $x \approx 7.07$

2 **a** $a \approx 21.3$ cm **b** $b \approx 76.9$ cm **c** $c \approx 5.09$ cm

3 **a** $B\widehat{A}C = 74°$, $AB \approx 7.99$ cm, $BC \approx 9.05$ cm
b $X\widehat{Z}Y = 108°$, $XZ \approx 13.5$ cm, $XY \approx 26.5$ cm

4 $x \approx 17.7$, $y \approx 33.1$ **5** $x = 11 + \frac{11}{2}\sqrt{2}$

EXERCISE 9C.2

1 **a** $x \approx 9.85$ **b** $x \approx 41.3$ **c** $x \approx 52.8$

2 $C \approx 62.1°$ or $C \approx 117.9°$

3 **a** $B\widehat{A}C \approx 30.9°$ **b** $A\widehat{B}C \approx 28.7°$
c $A\widehat{C}B \approx 30.1°$ **d** $B\widehat{A}C \approx 46.6°$
e $A\widehat{B}C \approx 55.5°$ or $124.5°$ **f** $A\widehat{C}B \approx 25.4°$ or $154.6°$

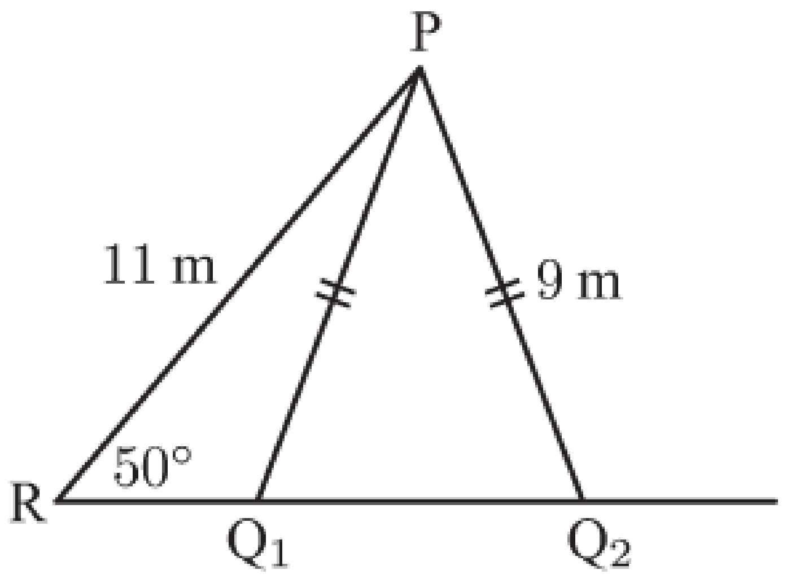
4 **a** We find that $\sin x \approx 1.04$ which has no solutions.
b The triangle cannot be drawn with the given dimensions.

5 **a** **i** $A\widehat{C}B \approx 22.9°$ **ii** $B\widehat{A}C \approx 127.1°$ **b** ≈ 25.1 cm^2

6 No, the angle opposite the 9.8 cm side has a sine of 1.05, which is impossible.

7 **a** $\approx 69.4°$ or $\approx 110.6°$

b



c For $P\widehat{Q}R \approx 69.4°$:
i $\approx 60.6°$
ii ≈ 43.1 m^2
iii ≈ 30.2 m

For $P\widehat{Q}R \approx 110.6°$:
i $\approx 19.4°$
ii ≈ 16.5 m^2
iii ≈ 23.9 m

EXERCISE 9D

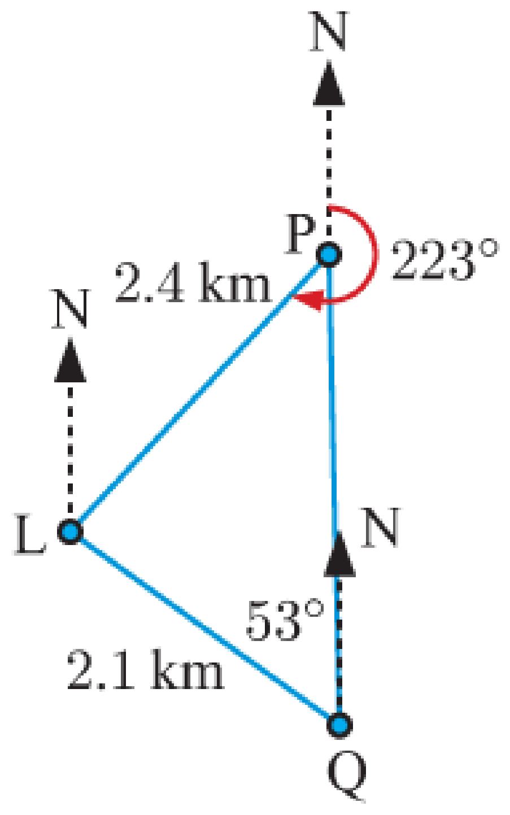
1 ≈ 17.7 m **2** ≈ 207 m **3** ≈ 10.1 km

4 $\approx 23.9°$ **5** ≈ 37.6 km

6 **a** ≈ 5.63 km **b** on the bearing $\approx 115°$
c **i** Esko **ii** ≈ 7.37 min (≈ 7 min 22 s) **d** $\approx 295°$

7 $\approx 9.38°$ **8** ≈ 69.1 m **9** **a** ≈ 38.0 m **b** ≈ 94.0 m

10 **a**

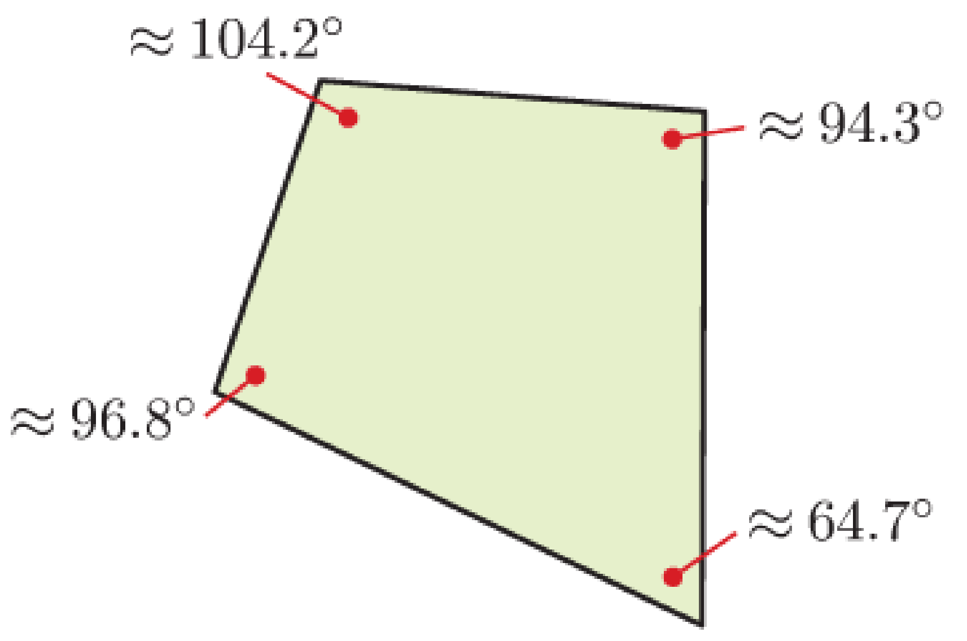


b ≈ 2.98 km
c $\approx 179°$

11 **a** $\approx 55.1°$ **b** $\approx 50.3°$ **12** $\approx 65.6°$ **13** ≈ 9.12 km

14 **a** ≈ 74.9 km^2 **b** ≈ 7490 ha **15** ≈ 85.0 mm

16



Area $\approx 13\,100$ m^2

17 ≈ 7400 m^2 **18** ≈ 2.52 km^2 **19** $\approx 32.2°$ and $\approx 87.8°$

20 ≈ 29.2 m **21** **a** ≈ 3.97 km **b** ≈ 1.13 km

22 **b** ≈ 1.467 cm

REVIEW SET 9A

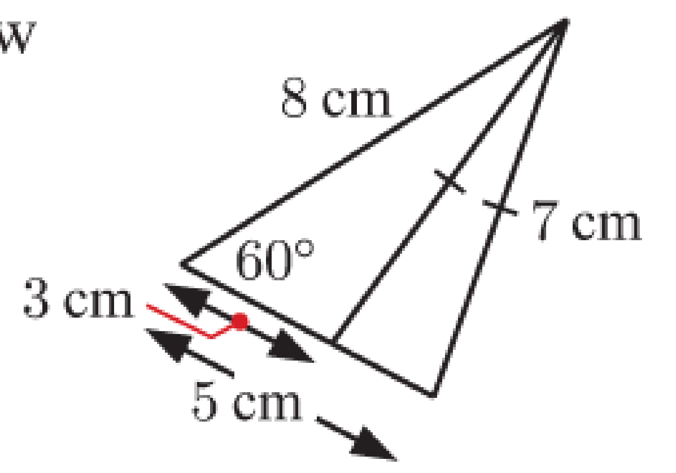
1 **a** ≈ 26.8 cm^2 **b** 14 km^2 **c** ≈ 33.0 m^2

2 ≈ 22.7 cm^2 **3** **a** ≈ 10.5 cm **b** ≈ 11.6 m

4 **a** $x \approx 9.24$ **b** $\theta \approx 59.2°$ **c** $x \approx 6.28$

5 ≈ 113 cm^2 **6** ≈ 51.6 cm^2

7 **a** $x = 3$ or 5 **b** Kady can draw 2 triangles:



8 **b** $x \approx 19.3$ **c** ≈ 43.3 cm

9 ≈ 7.21 cm, ≈ 11.2 cm, ≈ 12.5 cm

10 $\approx 74.4°$ **11** $x \approx 18.5$, $y \approx 13.8$ **12** 42 km

13 **a** $\approx 69.5°$ or $\approx 110.5°$
b For $A\widehat{B}C \approx 69.5°$, area ≈ 16.3 cm^2.
For $A\widehat{B}C \approx 110.5°$, area ≈ 8.09 cm^2.

14 ≈ 577 m

REVIEW SET 9B

1 **a** $x \approx 34.1$ **b** $x \approx 18.9$ **2** $\approx 47.5°$ or $132.5°$

3 **a** $\theta \approx 29.9°$ **b** $\theta \approx 103°$

4 **a** $AC \approx 12.6$ cm, $B\widehat{A}C \approx 48.6°$, $A\widehat{C}B \approx 57.4°$
b $P\widehat{R}Q = 51°$, $PQ \approx 7.83$ cm, $QR \approx 7.25$ cm
c $Y\widehat{X}Z \approx 78.3°$, $X\widehat{Y}Z \approx 55.5°$, $X\widehat{Z}Y \approx 46.2°$

5 **a** $x \approx 6.93$ **b** $x \approx 11.4$ **c** $x \approx 7.16$ **d** $x \approx 34.7$

6 ≈ 17.7 m **7** ≈ 7.32 m

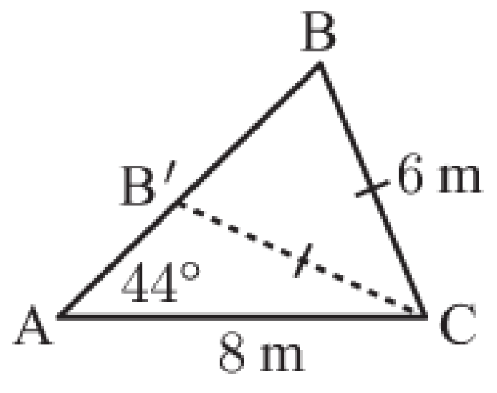
8 perimeter ≈ 578 m, area $\approx 15\,000$ m^2

9 ≈ 560 m on the bearing $\approx 079.7°$

10 $B\widehat{A}D \approx 90.5°$, $B\widehat{C}D \approx 94.3°$, $A\widehat{D}C \approx 70.2°$

11 $Q \approx 39.7°$ **12** **a** $\approx 10\,600$ m^2 **b** ≈ 1.06 ha

13 **a** The information given could give two triangles:



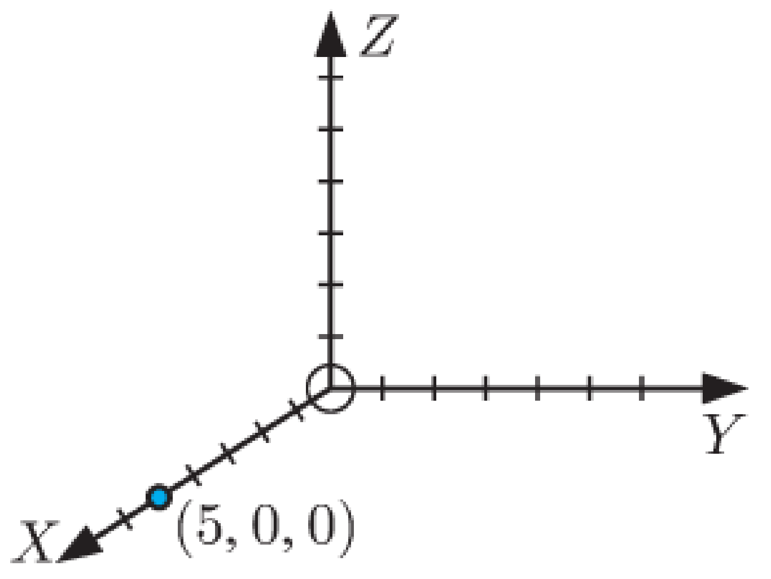
b ≈ 2.23 m^3

14 **a** **i** ≈ 20.8 mm **ii** ≈ 374 mm^2 **b** ≈ 2270 mm^3

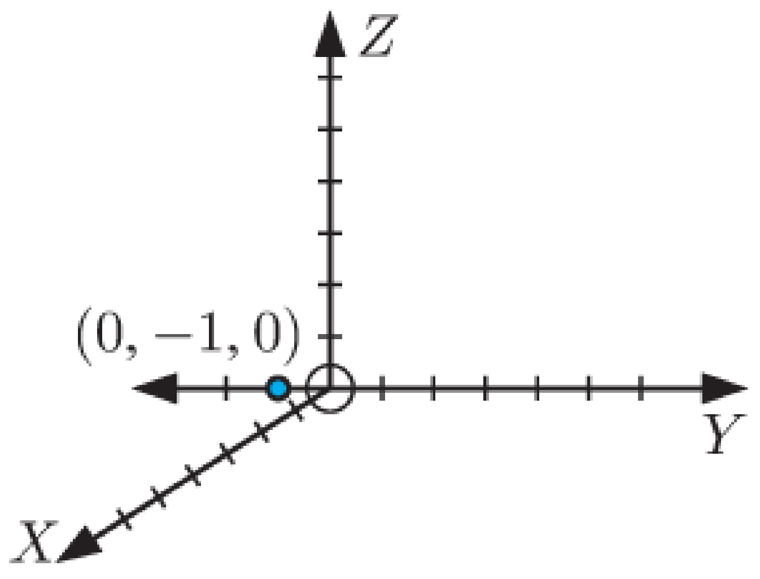
15 **a** **Hint:** Let $B\widehat{A}C = \theta$, so $A = \frac{1}{2}bc\sin\theta$.

EXERCISE 10A

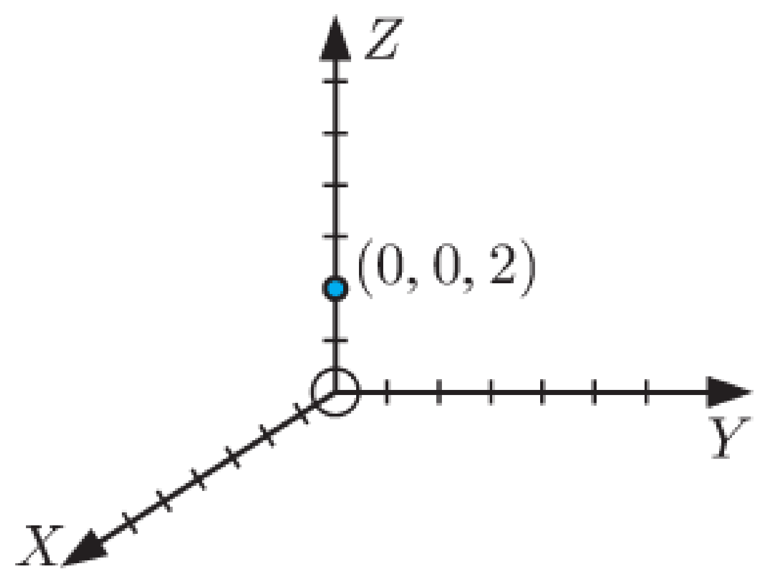
1 **a**



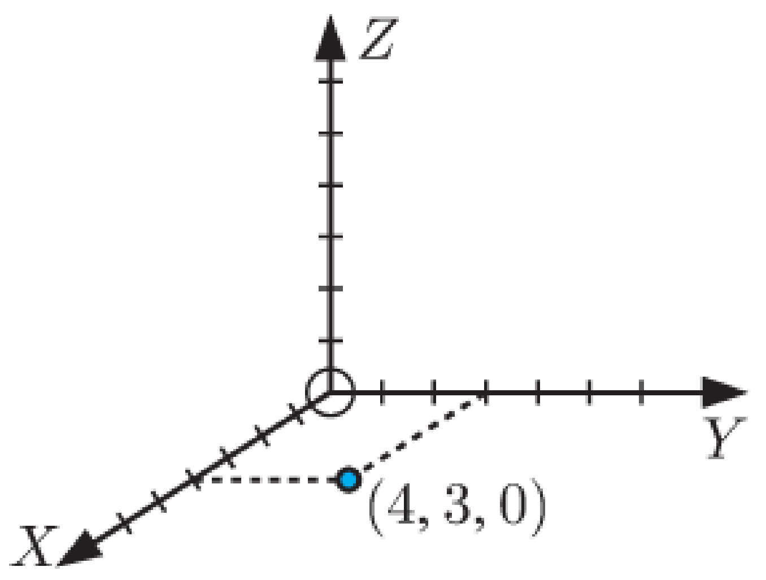
b



c



d



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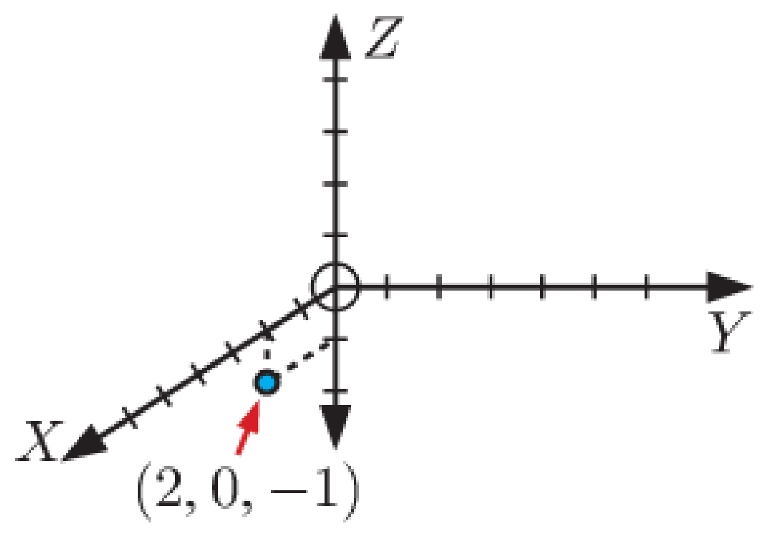

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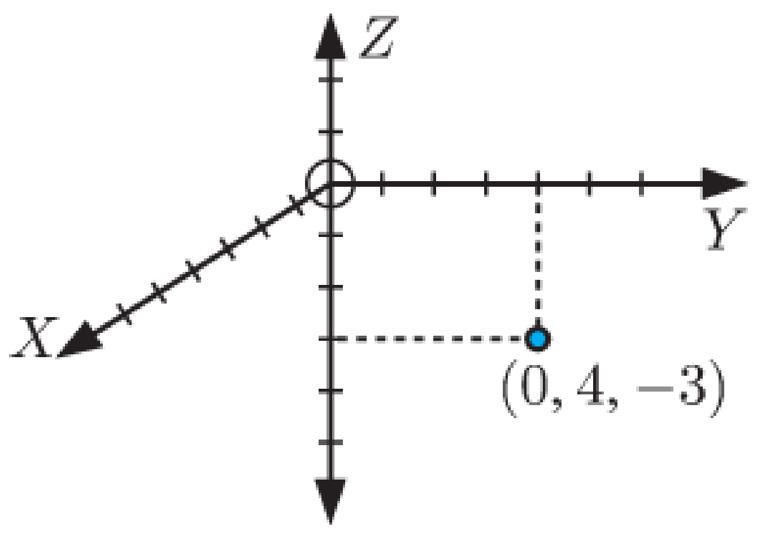

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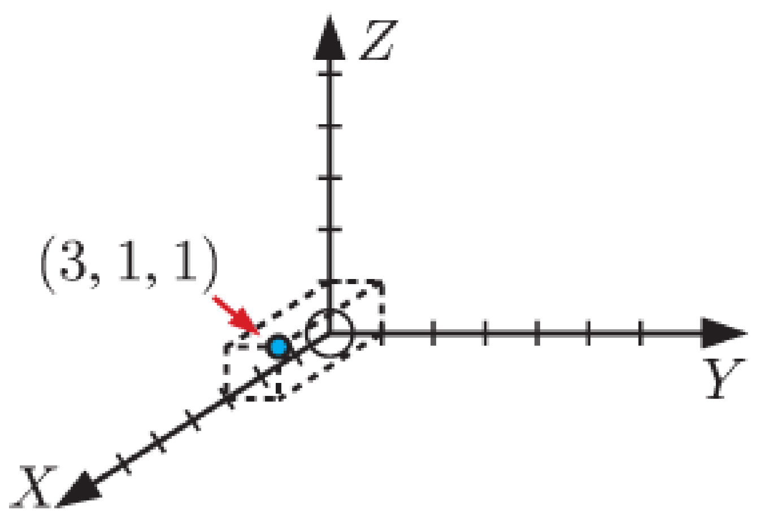

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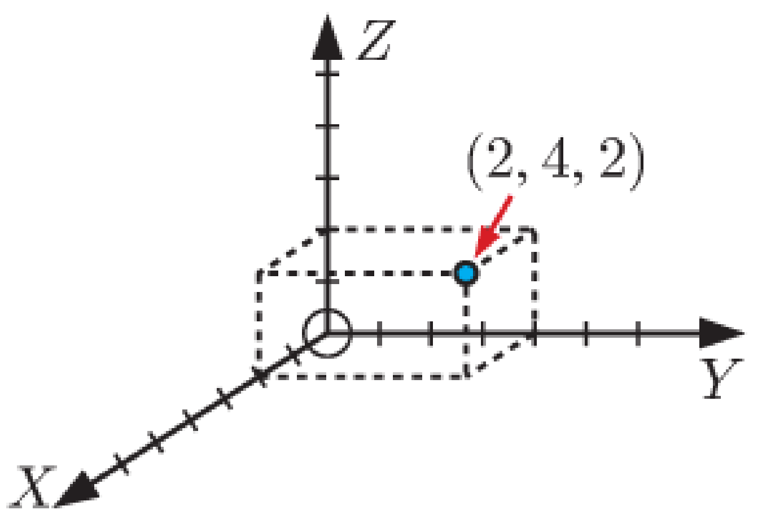

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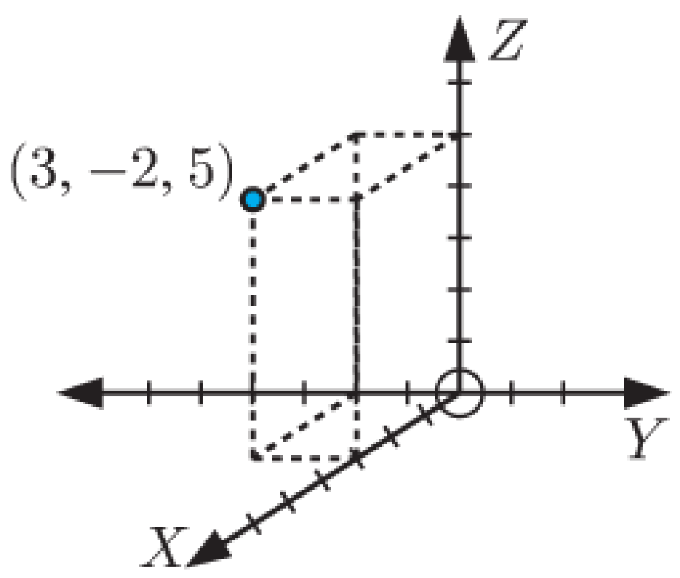

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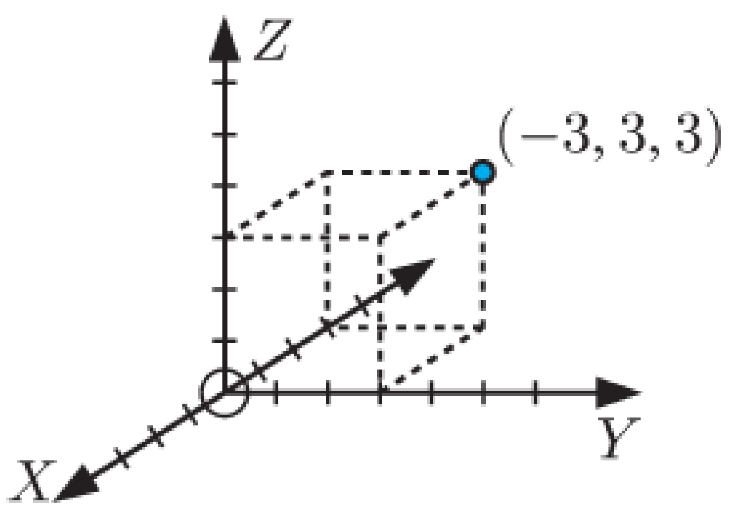

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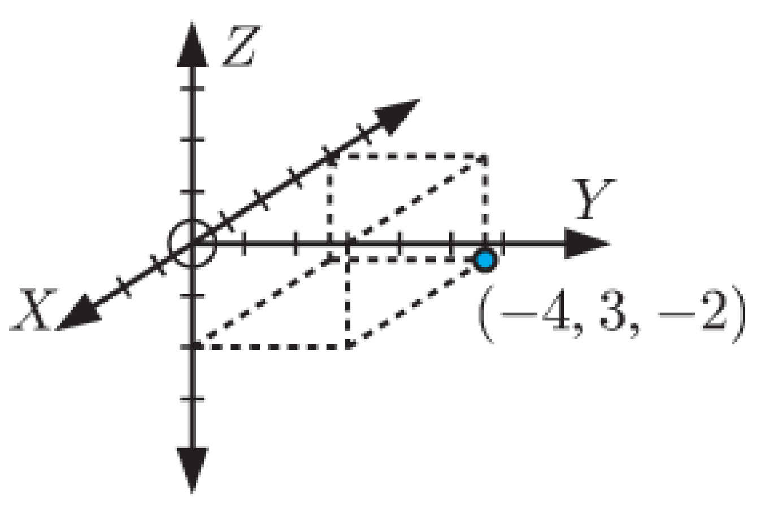

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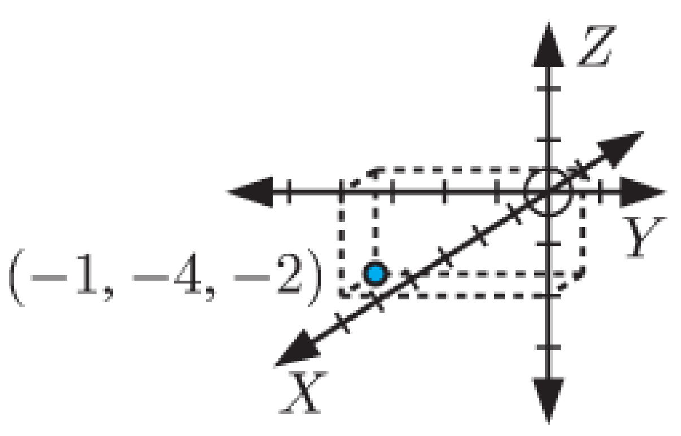


2 **a** **i** $2\sqrt{14}$ units **ii** $(3,\ -2,\ 1)$

b **i** $2\sqrt{5}$ units **ii** $(2,\ 1,\ -1)$

c **i** $2\sqrt{6}$ units **ii** $(3,\ -2,\ 1)$

d **i** $\sqrt{69}$ units **ii** $(-4,\ \frac{7}{2},\ 4)$

e **i** $5\sqrt{2}$ units **ii** $(\frac{3}{2},\ 3,\ \frac{1}{2})$

f **i** $\sqrt{83}$ units **ii** $(-\frac{3}{2},\ \frac{9}{2},\ -\frac{1}{2})$

3 **a** isosceles with $\text{AB} = \text{AC} = \sqrt{101}$ units **b** scalene

4 $\text{AB} = \sqrt{342}$ units, $\text{AC} = \sqrt{72}$ units, $\text{BC} = \sqrt{414}$ units

$\text{AB}^2 + \text{AC}^2 = (\sqrt{342})^2 + (\sqrt{72})^2 = 414 = \text{BC}^2$

$\therefore$ triangle ABC is right angled.

5 **a** $\text{M}(\frac{7}{2},\ -2,\ 3)$, $\text{N}(\frac{1}{2},\ -5,\ -1)$

b $\text{PR} = 2\sqrt{34}$ units, $\text{MN} = \sqrt{34}$ units $= \frac{1}{2}\,\text{PR}$

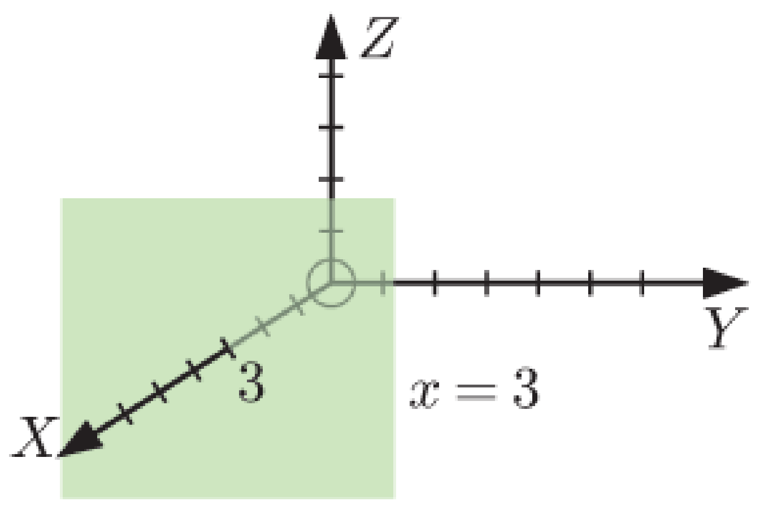
6 $\text{B}(-7,\ 11,\ 3)$ **7** $\text{P}(-2,\ 5,\ 0)$, $\text{Q}(4,\ -1,\ 3)$, $\text{R}(1,\ 7,\ 6)$

8 $k = 2 \pm \sqrt{23}$

9 **a** $x^2 + y^2 + z^2 = 9$, P lies on a sphere with centre $(0,\ 0,\ 0)$ and radius 3 units.

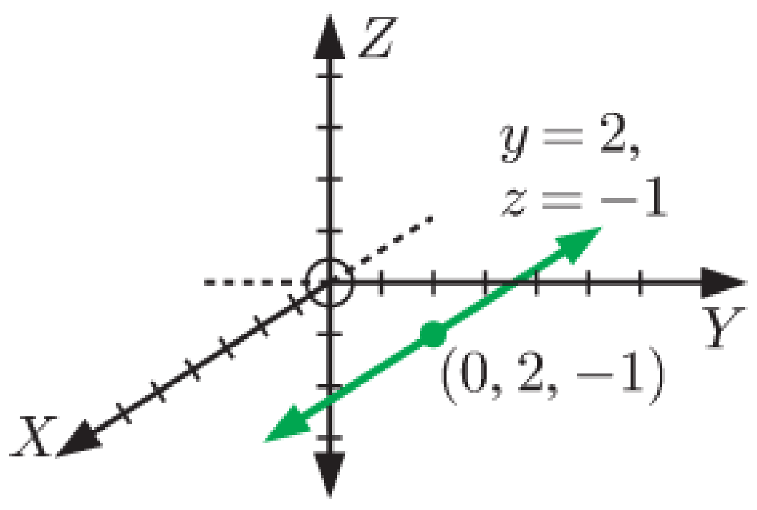
b $(x-2)^2 + (y-5)^2 + (z-4)^2 = 1$, P lies on a sphere with centre $(2,\ 5,\ 4)$ and radius 1 unit.

10 **a**



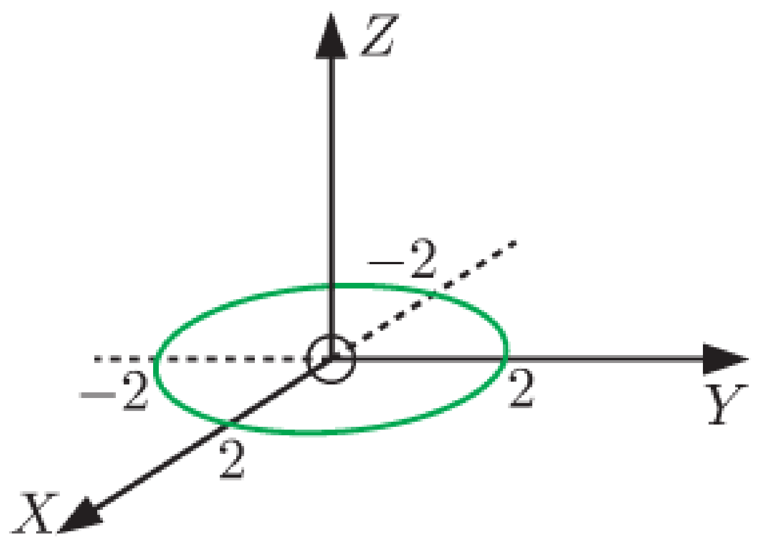
A plane parallel to the YOZ plane, passing through $(3,\ 0,\ 0)$.

b



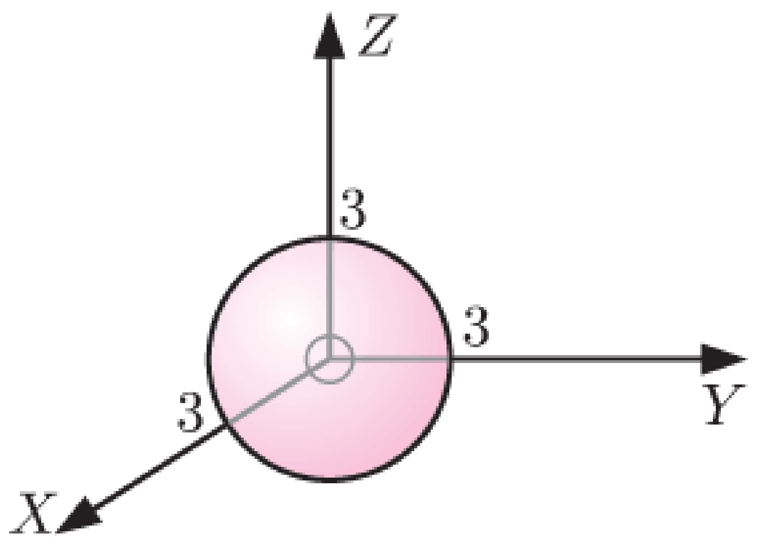
A line parallel to the X-axis, passing through $(0,\ 2,\ -1)$.

c



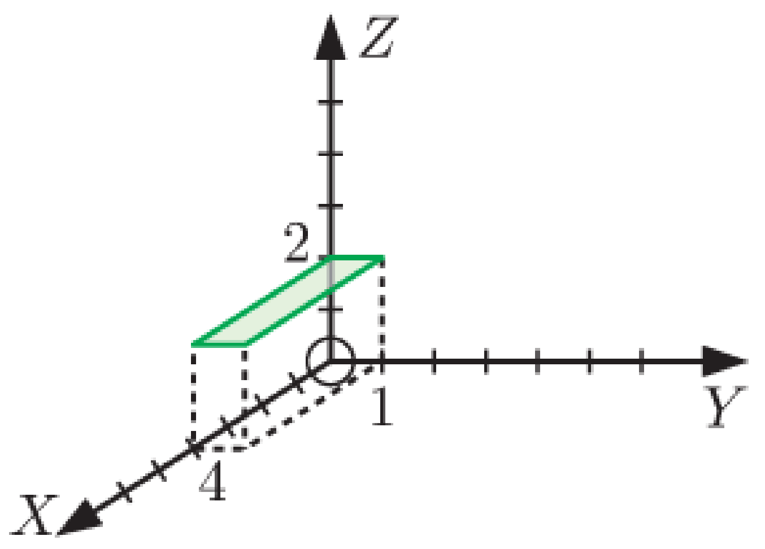
A circle in the XOY plane, centre $(0,\ 0,\ 0)$, radius 2 units.

d



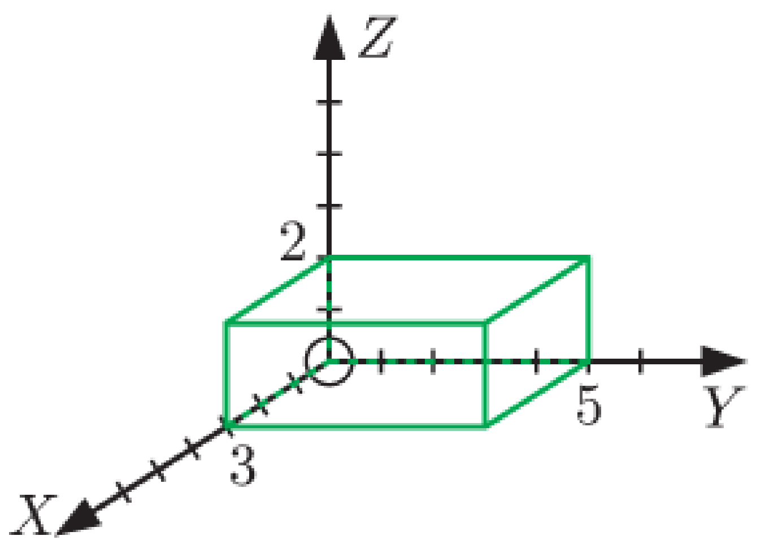
A sphere, centre $(0,\ 0,\ 0)$, radius 3 units.

e



A 4 by 1 rectangular plane 2 units above the XOY plane (as shown).

f



All points on and within a $3 \times 5 \times 2$ rectangular prism (as shown).

EXERCISE 10B

1 **a** 27 units^3 **b** 60 units^3 **c** 40 units^3

2 **a** $\text{D}(-7,\ 0,\ 3)$, $\text{E}(-7,\ 4,\ 0)$ **b** 42 units^3

c 5 units **d** 96 units^2

3 **a** The centre of the base is $(3,\ 3,\ 0)$ which is directly below the apex.

b 108 units^3

c **i** $\text{M}(6,\ 3,\ 0)$ **ii** $3\sqrt{10}$ units **iii** $36(1 + \sqrt{10})$ units^2

4 volume = 720 units^3, surface area = 564 units^2

5 **a** $\sqrt{41}$ units **b** 82π units^3 **c** $\sqrt{77}$ units

d ≈ 305 units^2

6 **a** $\sqrt{238}$ units **b** $\approx 15\,400$ units^3

7 **a** $(-3,\ 4,\ -3)$ **b** $\sqrt{38}$ units

c volume ≈ 981 units^3, surface area ≈ 478 units^2

8 **a** 4 units **b** $k = -3 \pm \sqrt{6}$ **c** ≈ 88.0 units^2

9 **a** $\text{D}(-4,\ -2,\ 2)$ **b** 16.8 units **c** ≈ 190 units^3

10 **a** **i** $\text{A}(10,\ 40,\ 0)$, $\text{B}(50,\ 160,\ 0)$, $\text{C}(110,\ 140,\ 0)$, $\text{D}(70,\ 20,\ 0)$

ii $(60,\ 90,\ 15)$

b $40\,000\ \text{m}^3$ **c** $\approx 8540\ \text{m}^2$

11 $2\,461\,200\ \text{m}^3$

EXERCISE 10C

1 **a** $\approx 50.2°$ **b** $\approx 48.1°$

2 **a** $\text{M}(3,\ 3,\ 0)$ **b** $\approx 25.4°$ **c** $\approx 50.2°$

3 **a** $\text{M}(4,\ 6,\ 0)$ **b** $\approx 66.5°$ **c** **i** $\approx 35.0°$ **ii** $\approx 44.1°$

4 **a** $\text{M}(2,\ 4,\ 0)$ **b** **i** $\approx 68.2°$ **ii** $\approx 60.5°$

5 **a** $\text{M}(-4,\ 3,\ 5)$ **b** **i** $\approx 30.3°$ **ii** $45°$ **c** $\approx 34.4°$

6 **a** **i** $3\sqrt{3}$ units **ii** 3 units **iii** $\sqrt{38}$ units **b** $\approx 29.1°$

7 **a** $\approx 67.3°$ **b** $\approx 22.3°$

8 **a** $\approx 53.6°$ **b** ≈ 10.8 units^2 **9** $k = -1$

10 **a** $\text{M}(\frac{5}{2},\ 3,\ 5)$ **b** **i** $\approx 66.8°$ **ii** $\approx 128°$ **iii** $\approx 76.0°$

11 **a** The bird is at $(30,\ 20,\ 10)$.

b **i** $10\sqrt{6}$ m ≈ 24.5 m **ii** $\approx 24.1°$

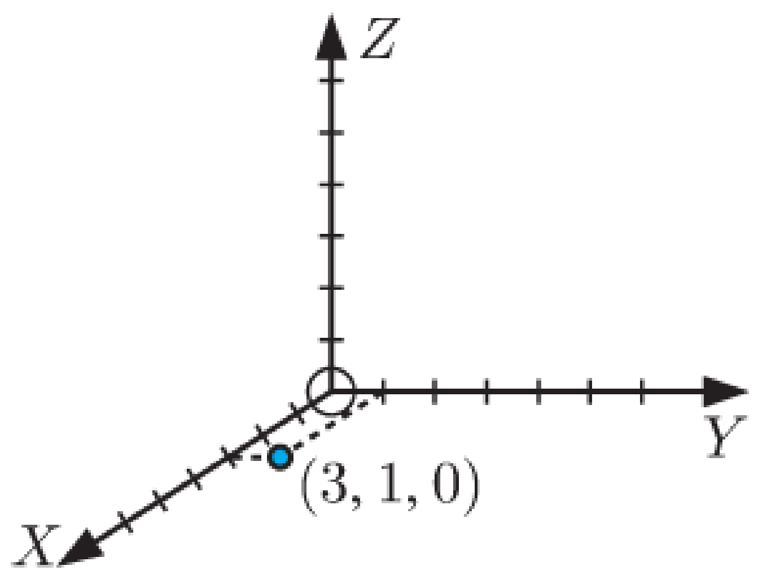
12 **a** $(-3,\ 4,\ \frac{1}{2})$ **b** $\frac{\sqrt{101}}{2} \approx 5.02$ km **c** $135°$ **d** $\approx 5.05°$

13 **a** $\frac{\sqrt{201}}{2} \approx 7.09$ km **b** Jack, Gabriel, Malina

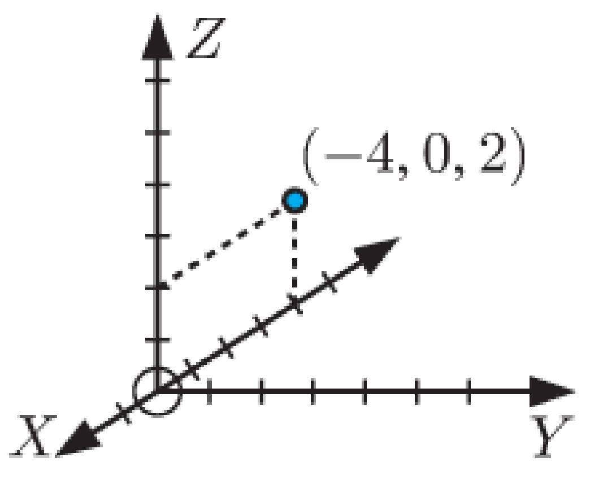
c **i** $\approx 1.71°$ **ii** $\approx 7.64°$

REVIEW SET 10A

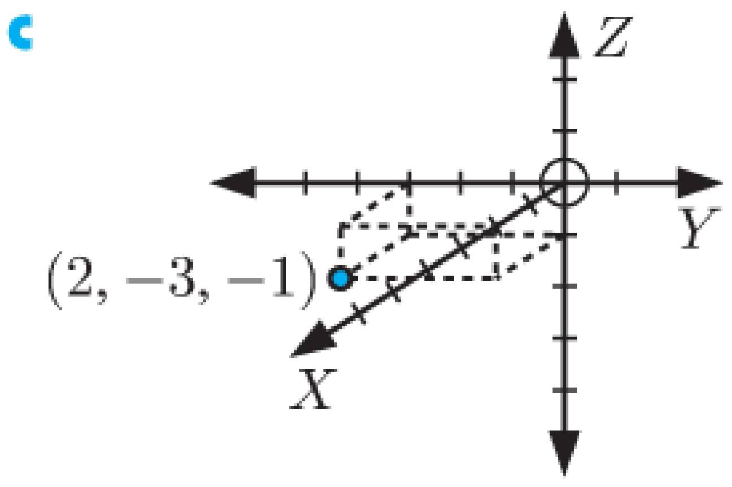
1 **a**



b



c



2 **a** **i** 6 units **ii** $(-1, -4, 1)$

b **i** $3\sqrt{10}$ units **ii** $(-\frac{5}{2}, -3, \frac{7}{2})$

3 isosceles with $AB = AC = \sqrt{41}$ units

4 **a** 128 units^3 **b** M(8, 4, 0) **c** $2\sqrt{13}$ units

d $32(2+\sqrt{13})$ units^2 ≈ 179 units^2 **e** $\approx 29.0°$

5 **a** $\sqrt{29}$ units

b volume ≈ 327 units^3, surface area ≈ 273 units^2

6 **a** M(5, 4, 3) **b** $\approx 64.9°$ **c** **i** $\approx 43.1°$ **ii** $\approx 25.1°$

7 $\approx 61.4°$ **8** **a** $k = 2$ **b** ≈ 68.9 units^2

9 **a** P(2, 7, −2.5), Q(8, 3, −2.9) **b** ≈ 7.22 m

c $\approx 3.17°$

REVIEW SET 10B

1 **a** **i** $\sqrt{41}$ units **ii** $(-2, 3, \frac{9}{2})$

b **i** $\sqrt{83}$ units **ii** $(-\frac{9}{2}, \frac{5}{2}, \frac{5}{2})$

2 **a** $PQ = \sqrt{14}$ units, $PR = \sqrt{45}$ units, $QR = \sqrt{59}$ units

$PQ^2 + PR^2 = (\sqrt{14})^2 + (\sqrt{45})^2 = 59 = QR^2$

$\therefore$ triangle PQR is right angled.

b $\approx 60.8°$

3 $k = 1 \pm \sqrt{30}$

4 **a** 96 units^3 **b** $2\sqrt{13}$ units

c $(104 + 16\sqrt{13}) \approx 162$ units^2

5 **a** $(-1, 0, -1)$ **b** $3\sqrt{5}$ units

c volume ≈ 1260 units^3, surface area ≈ 565 units^2

6 **a** $\approx 21.4°$ **b** ≈ 3.53 units^2

7 **a** M(6, 9, 5) **b** $\approx 71.6°$ **c** **i** $\approx 54.2°$ **ii** $\approx 36.7°$

8 **a** $H(2, -4, \frac{1}{5})$ **b** ≈ 4.48 km

c **i** $M(-4, 1, \frac{1}{2})$ **ii** ≈ 7.82 km **iii** $\approx 2.20°$

9 **a** R(0, 0, 3) **b** $\widehat{PRO} \approx 46.5°$, $\widehat{QRO} \approx 36.7°$

c $\widehat{PRQ} \approx 60.6°$

EXERCISE 11A

1 **a** ≈ 0.78 **b** ≈ 0.22

2 **a** ≈ 0.487 **b** ≈ 0.051 **c** ≈ 0.731

3 **a** 43 days **b** **i** ≈ 0.0465 **ii** ≈ 0.186 **iii** ≈ 0.465

4 **a** ≈ 0.0895 **b** ≈ 0.126

5 **a** ≈ 0.265 **b** ≈ 0.861 **c** ≈ 0.222

6 **a** ≈ 0.146 **b** ≈ 0.435 **c** ≈ 0.565

7 **a** **i** ≈ 0.171 **ii** ≈ 0.613 **b** ≈ 0.366 **c** ≈ 0.545

EXERCISE 11B

1 **a** 7510 **b** **i** ≈ 0.325 **ii** ≈ 0.653 **iii** ≈ 0.243

2 **a**

	Junior	*Middle*	*Senior*	*Total*
Sport	131	164	141	436
No sport	28	81	176	285
Total	159	245	317	721

b **i** $\frac{436}{721} \approx 0.605$ **ii** $\frac{131}{721} \approx 0.182$ **iii** $\frac{257}{721} \approx 0.356$

3 **a** **i** $\frac{743}{1235} \approx 0.602$ **ii** $\frac{148}{1235} \approx 0.120$ **iii** $\frac{1085}{1235} \approx 0.879$

iv $\frac{795}{1235} \approx 0.644$

b $\frac{52}{492} \approx 0.106$ **c** $\frac{518}{862} \approx 0.601$

EXERCISE 11C

1 **a** {A, B, C, D} **b** {1, 2, 3, 4, 5, 6, 7, 8}

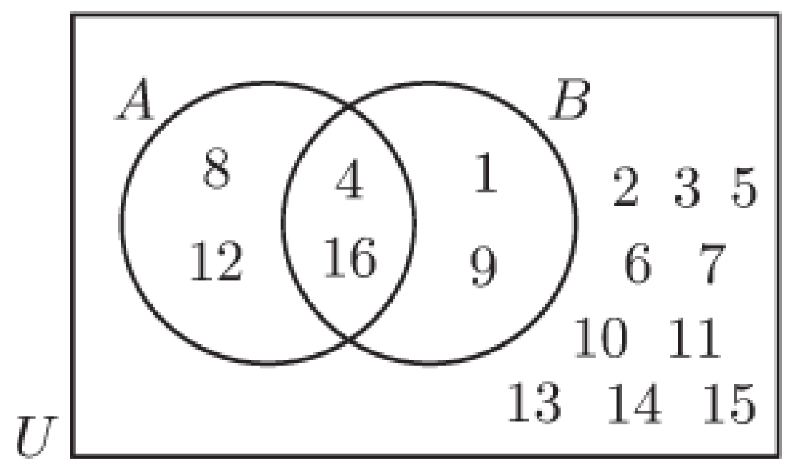
c {MM, MF, FM, FF}

2 **a** $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$

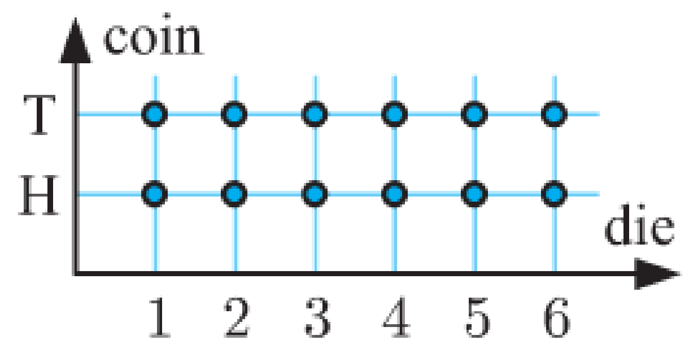
b **i** $A = \{4, 8, 12, 16\}$

ii $B = \{1, 4, 9, 16\}$

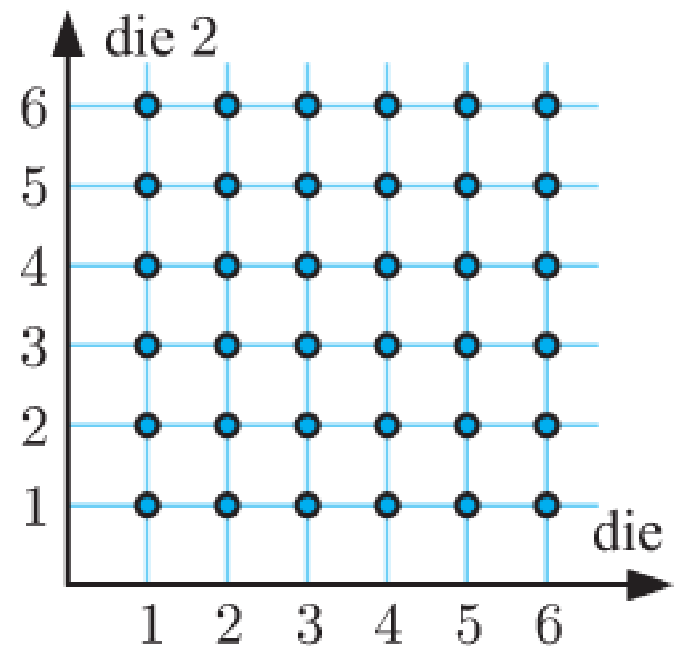
c



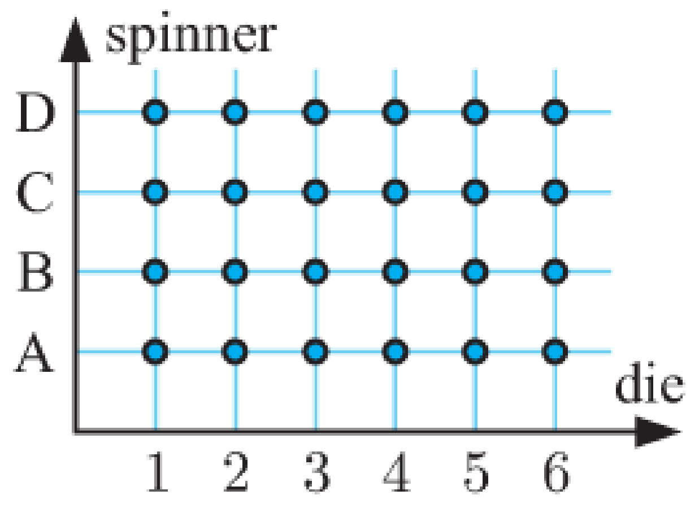
3 **a**



b



c



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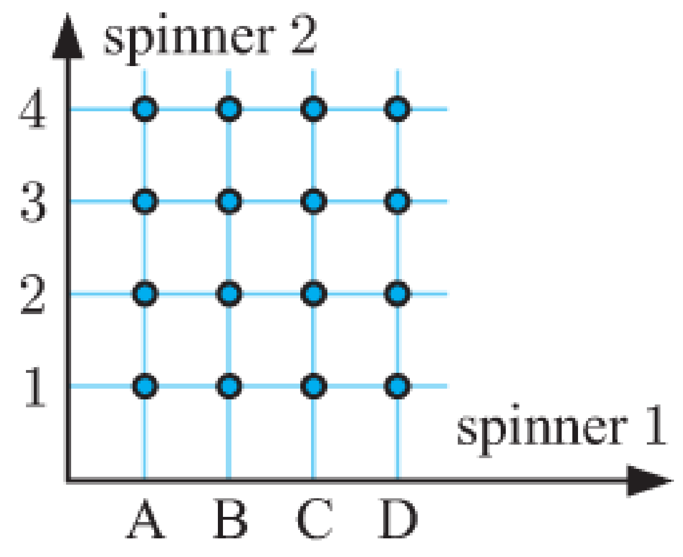
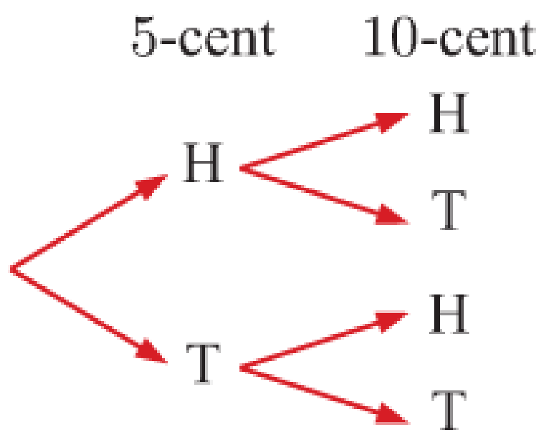
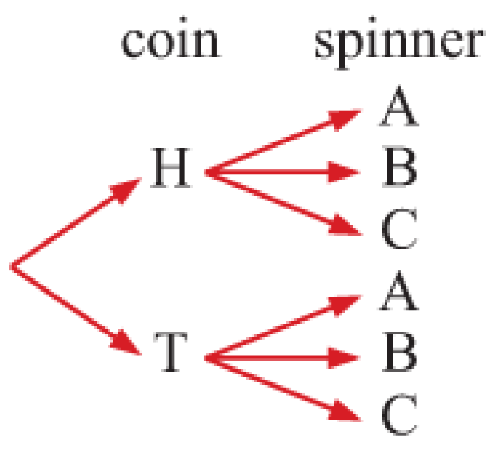


4 **a**



b



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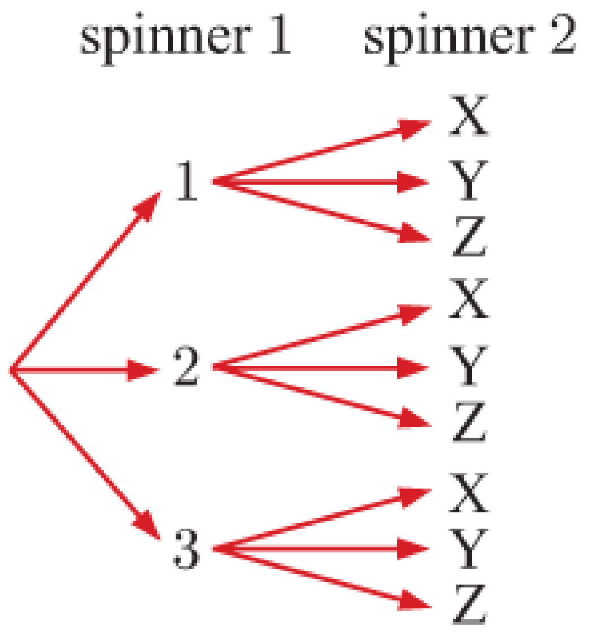

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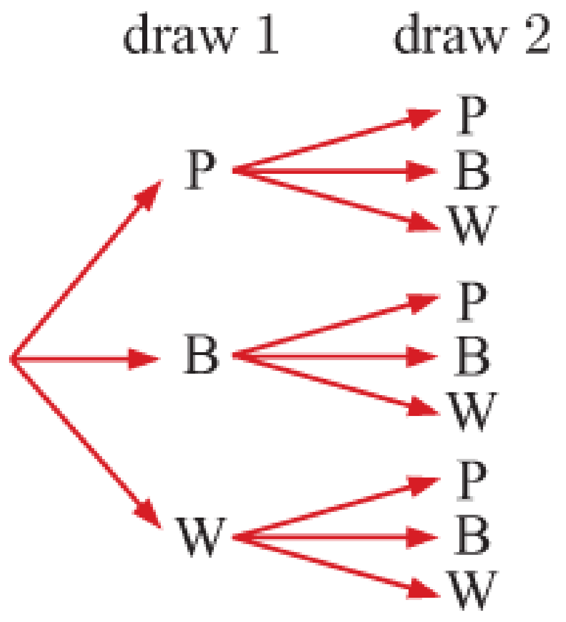

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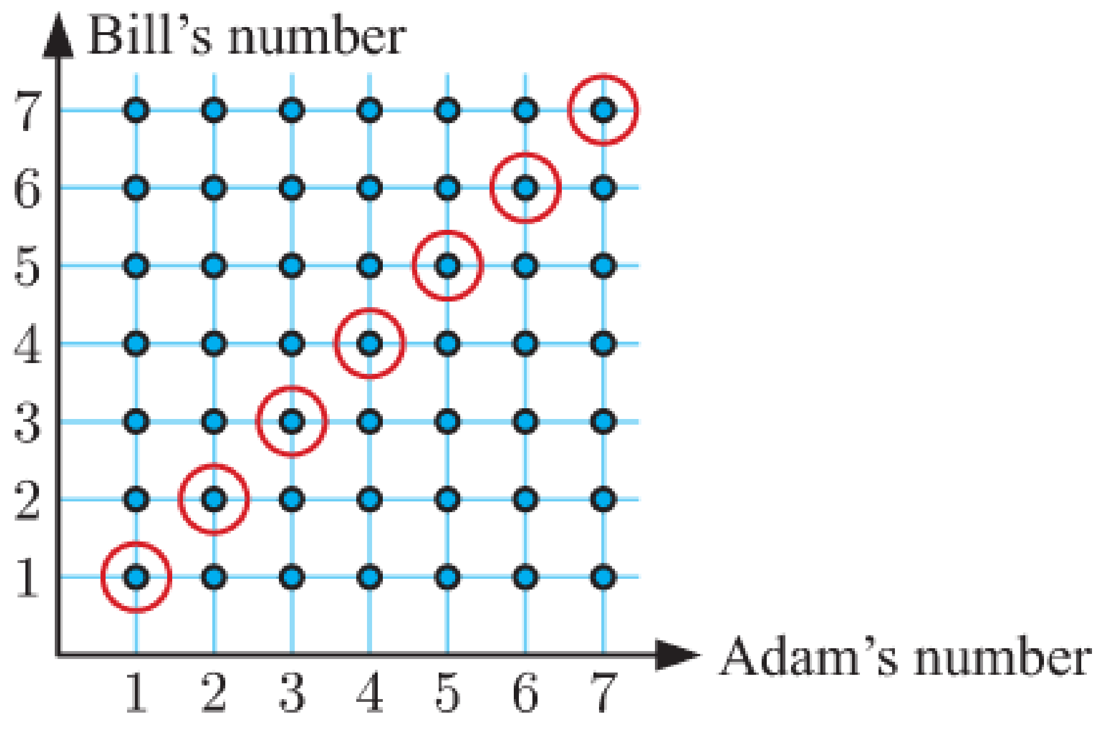

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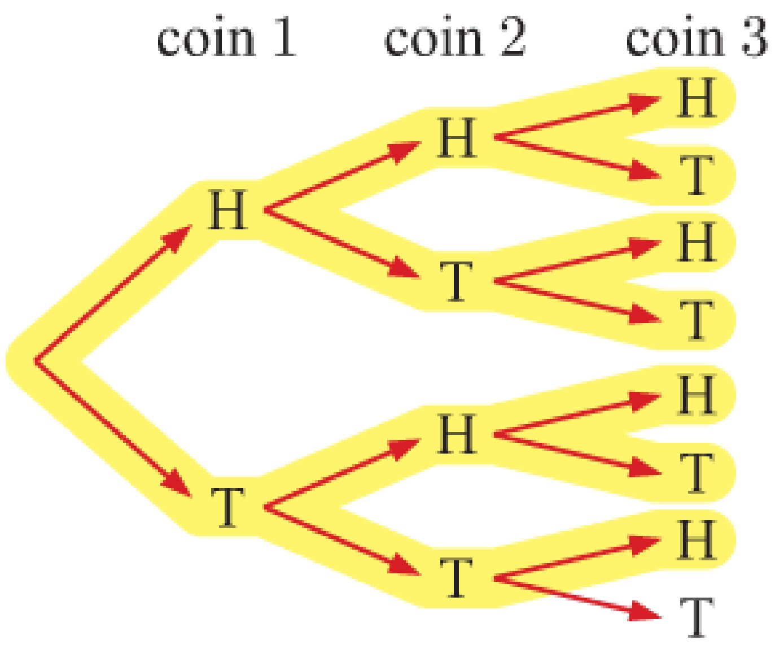


EXERCISE 11D

1 **a** $\frac{1}{5}$ **b** $\frac{1}{3}$ **c** $\frac{7}{15}$ **d** $\frac{4}{5}$ **e** $\frac{1}{5}$ **f** $\frac{8}{15}$

2 **a** $\frac{1}{2}$ **b** $\frac{2}{25}$ **c** $\frac{39}{200}$

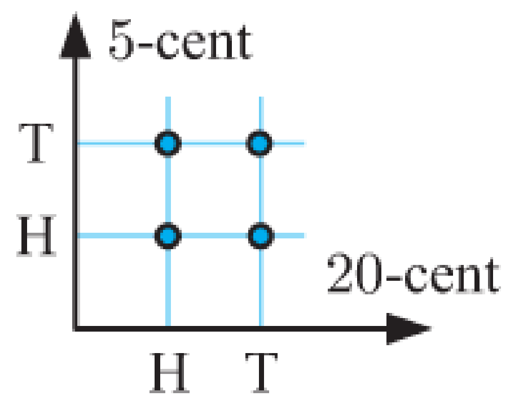
3 **a** $\frac{1}{4}$ **b** $\frac{1}{9}$ **c** $\frac{4}{9}$ **d** $\frac{1}{18}$ **e** $\frac{1}{6}$ **f** $\frac{13}{36}$
g $\frac{1}{12}$ **h** $\frac{1}{3}$

4 **a** $\frac{1}{7}$ **b** $\frac{2}{7}$ **c** $\frac{124}{1461}$ **d** $\frac{237}{1461}$
e $\frac{729}{1461}$ {remember leap years}

5 **a** {GGG, GGB, GBG, BGG, GBB, BGB, BBG, BBB}
b **i** $\frac{1}{8}$ **ii** $\frac{1}{8}$ **iii** $\frac{1}{8}$ **iv** $\frac{3}{8}$ **v** $\frac{1}{2}$ **vi** $\frac{7}{8}$

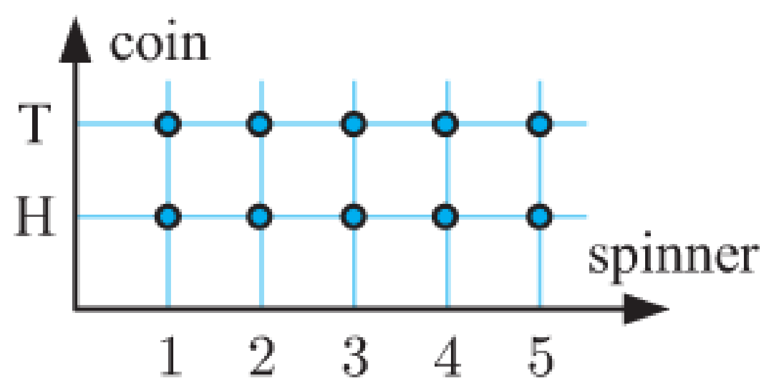
6 **a** {ABCD, ABDC, ACBD, ACDB, ADBC, ADCB, BACD, BADC, BCAD, BCDA, BDAC, BDCA, CABD, CADB, CBAD, CBDA, CDAB, CDBA, DABC, DACB, DBAC, DBCA, DCAB, DCBA}
b **i** $\frac{1}{2}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{2}$ **iv** $\frac{1}{2}$

7 **a**



b **i** $\frac{1}{4}$ **ii** $\frac{1}{4}$ **iii** $\frac{1}{2}$ **iv** $\frac{3}{4}$

8 **a**

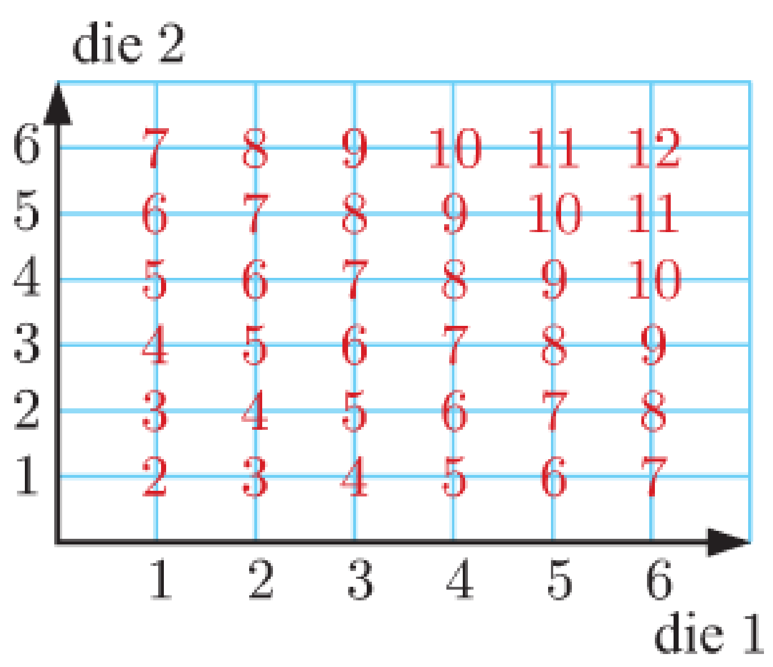


b **i** $\frac{1}{10}$ **ii** $\frac{3}{10}$ **iii** $\frac{2}{5}$ **iv** $\frac{3}{5}$

9 **a** $\frac{1}{36}$ **b** $\frac{1}{18}$ **c** $\frac{5}{9}$ **d** $\frac{11}{36}$ **e** $\frac{5}{18}$ **f** $\frac{25}{36}$

10 **a** Both grids show the sample space correctly, although **B** is more useful for calculating probabilities.
b $\frac{1}{6}$

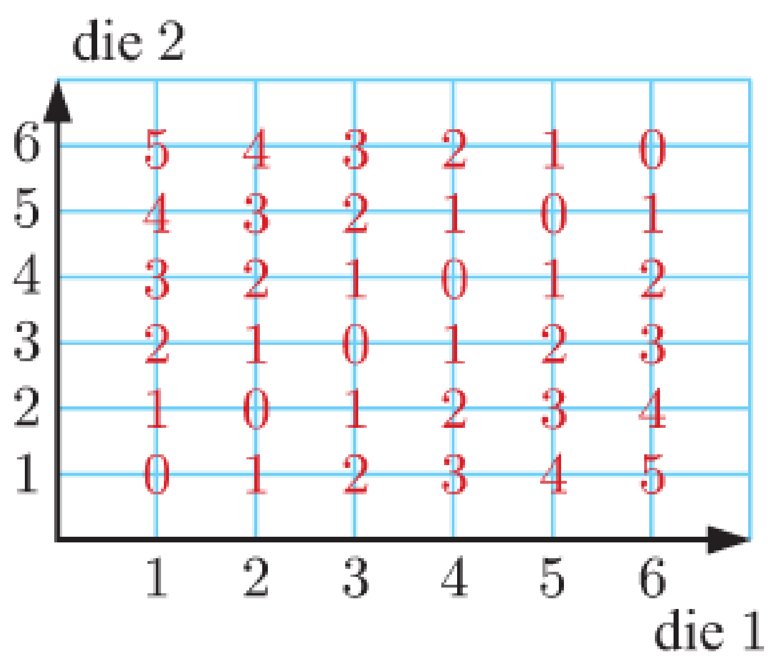
11 **a**



6	7	8	9	10	11	12
5	6	7	8	9	10	11
4	5	6	7	8	9	10
3	4	5	6	7	8	9
2	3	4	5	6	7	8
1	2	3	4	5	6	7
	1	2	3	4	5	6

b **i** $\frac{2}{36} = \frac{1}{18}$ **ii** $\frac{5}{36}$ **iii** $\frac{9}{36} = \frac{1}{4}$ **iv** $\frac{10}{36} = \frac{5}{18}$ **v** $\frac{10}{36} = \frac{5}{18}$ **vi** $\frac{26}{36} = \frac{13}{18}$

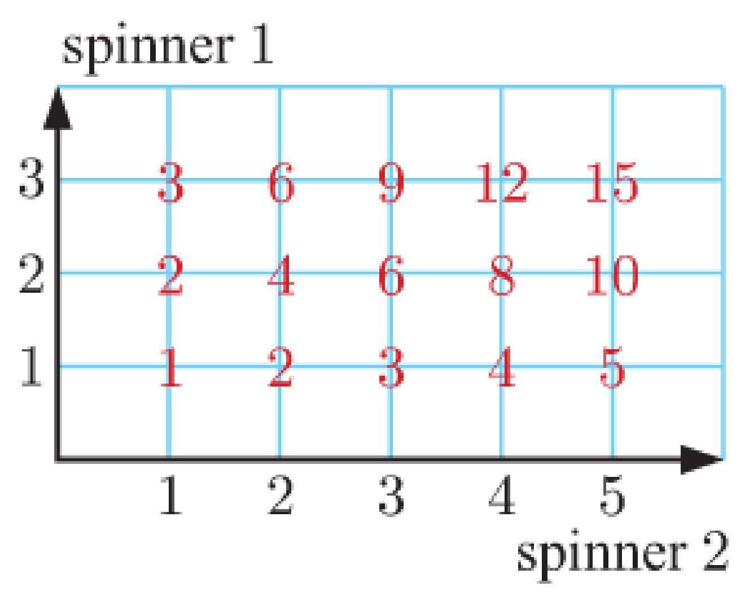
12 **a**



6	5	4	3	2	1	0
5	4	3	2	1	0	1
4	3	2	1	0	1	2
3	2	1	0	1	2	3
2	1	0	1	2	3	4
1	0	1	2	3	4	5
	1	2	3	4	5	6

b **i** $\frac{6}{36} = \frac{1}{6}$ **ii** $\frac{8}{36} = \frac{2}{9}$ **iii** $\frac{18}{36} = \frac{1}{2}$ **iv** $\frac{6}{36} = \frac{1}{6}$ **v** $\frac{24}{36} = \frac{2}{3}$

13 **a**



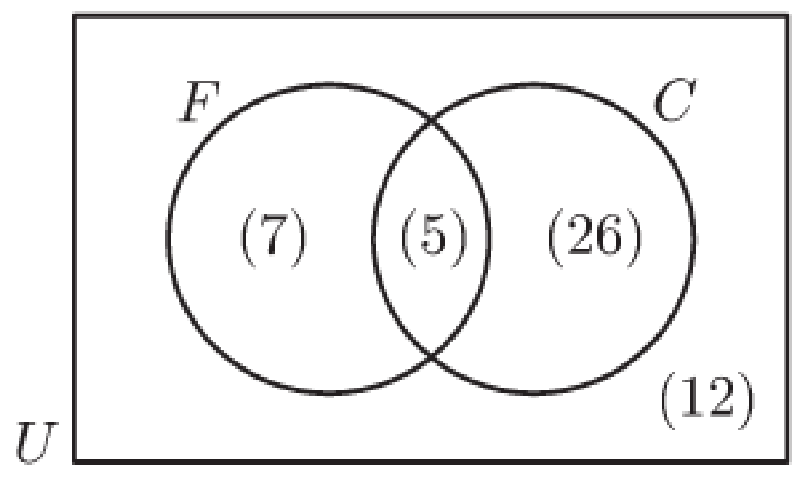
3	3	6	9	12	15
2	2	4	6	8	10
1	1	2	3	4	5
	1	2	3	4	5

b **i** $\frac{2}{15}$ **ii** $\frac{7}{15}$ **iii** $\frac{6}{15} = \frac{2}{5}$

14 **a** $\frac{3}{17}$ **b** $\frac{14}{17}$ **15** **a** $\frac{9}{65}$ **b** $\frac{4}{65}$ **c** $\frac{4}{5}$

16 **a** $\frac{17}{29}$ **b** $\frac{26}{29}$ **c** $\frac{5}{29}$ **17** **a** $\frac{37}{50}$ **b** $\frac{2}{5}$ **c** $\frac{17}{50}$

18 **a**



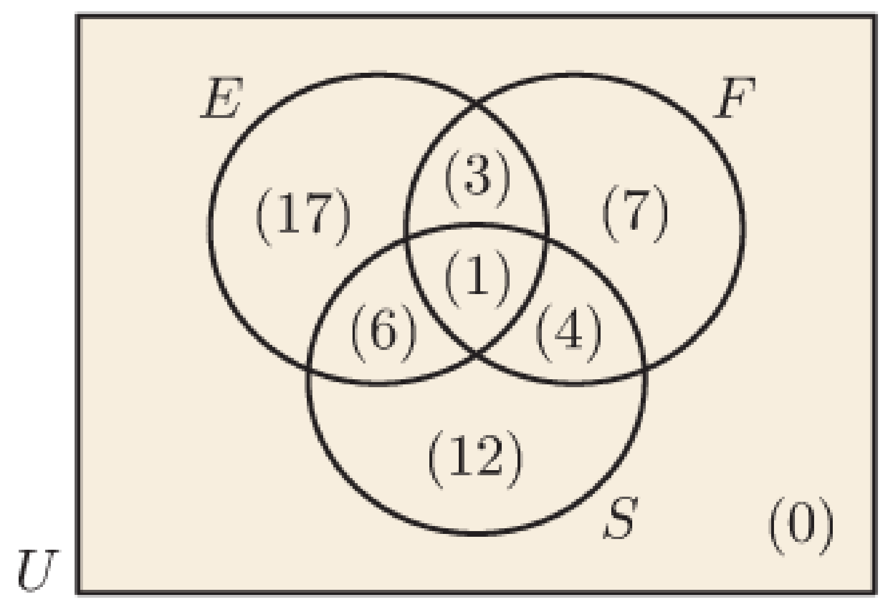
b **i** $\frac{19}{25}$ **ii** $\frac{13}{25}$ **iii** $\frac{6}{25}$

19 **a** $\frac{19}{40}$ **b** $\frac{1}{2}$ **c** $\frac{4}{5}$ **d** $\frac{5}{8}$ **e** $\frac{13}{40}$

20 **a** $\frac{7}{15}$ **b** $\frac{1}{15}$ **c** $\frac{2}{15}$

21 **a** $k = 5$
b **i** $\frac{7}{30}$ **ii** $\frac{11}{60}$ **iii** $\frac{7}{60}$ **iv** $\frac{53}{60}$ **v** $\frac{7}{60}$
vi $\frac{2}{15}$ **vii** $\frac{41}{60}$ **viii** $\frac{31}{60}$

22 **a**



b **i** $\frac{27}{50}$ **ii** $\frac{3}{10}$ **iii** $\frac{8}{25}$ **iv** $\frac{1}{5}$ **v** $\frac{2}{25}$

23 **a** $a = 3,\ b = 3$
b **i** $\frac{3}{10}$ **ii** $\frac{1}{10}$ **iii** $\frac{7}{40}$ **iv** $\frac{3}{8}$ **v** $\frac{5}{8}$

EXERCISE 11E

1 27 saves **2** $\approx$ 16 times **3** **a** $\frac{1}{4}$ **b** 50 occasions

4 15 days **5** 30 occasions

6 **a** **i** ≈ 0.55 **ii** ≈ 0.29 **iii** ≈ 0.16
b **i** $\approx$ 4125 people **ii** $\approx$ 2175 people
iii $\approx$ 1200 people

EXERCISE 11F

1 $\text{P}(A \cup B) = 0.55$ **2** $\text{P}(B) = 0.6$ **3** $\text{P}(X \cap Y) = 0.2$

4 **a** $\text{P}(A \cap B) = 0$ **b** A and B are mutually exclusive.

5 $\text{P}(A) = 0.35$

6 **a** yes
b **i** $\text{P}(A) = \frac{4}{15}$ **ii** $\text{P}(B) = \frac{7}{15}$ **iii** $\text{P}(A \cup B) = \frac{11}{15}$

7 **a** $\frac{11}{25}$ **b** $\frac{12}{25}$ **c** $\frac{8}{25}$ **d** $\frac{7}{25}$ **e** $\frac{4}{25}$ **f** $\frac{23}{25}$
g not possible **h** $\frac{11}{25}$ **i** not possible **j** $\frac{12}{25}$

8 $\text{P}(A \cup B) = 1$
Hint: Show $\text{P}(A' \cup B') = 2 - \text{P}(A \cup B)$

EXERCISE 11G

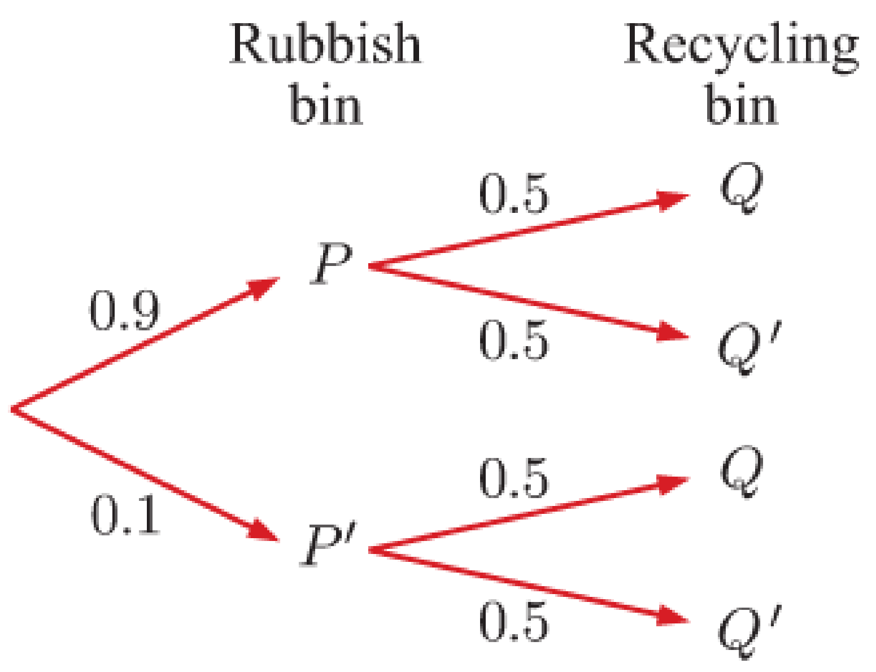
1 **a** $\frac{1}{24}$ **b** $\frac{1}{6}$ **2** **a** $\frac{1}{8}$ **b** $\frac{1}{8}$

3 **a** 0.0096 **b** 0.8096

4 **a** 0.56 **b** 0.06 **c** 0.14 **d** 0.24

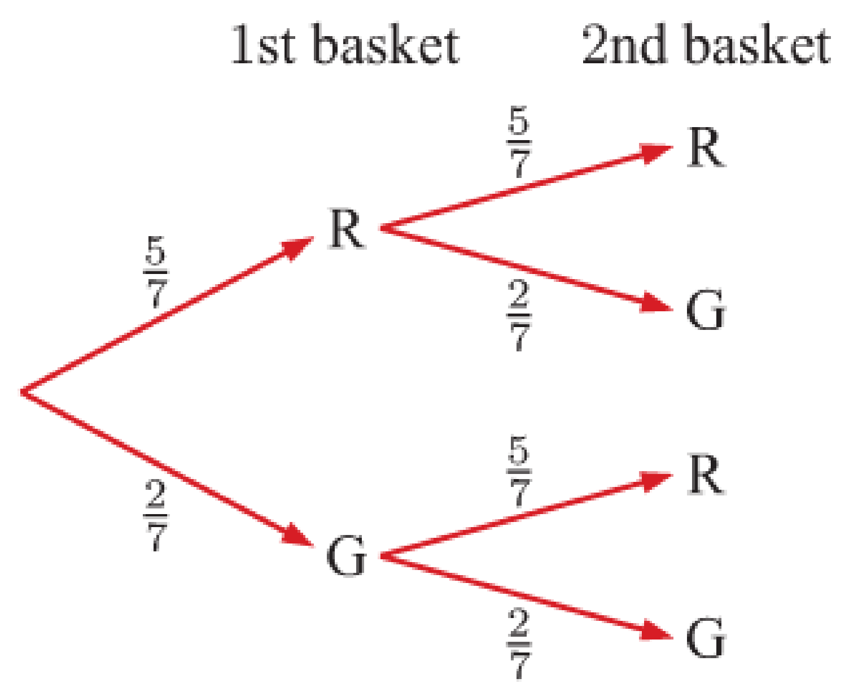
5 **a** $\frac{8}{125}$ **b** $\frac{12}{125}$ **c** $\frac{27}{125}$

6 **a**



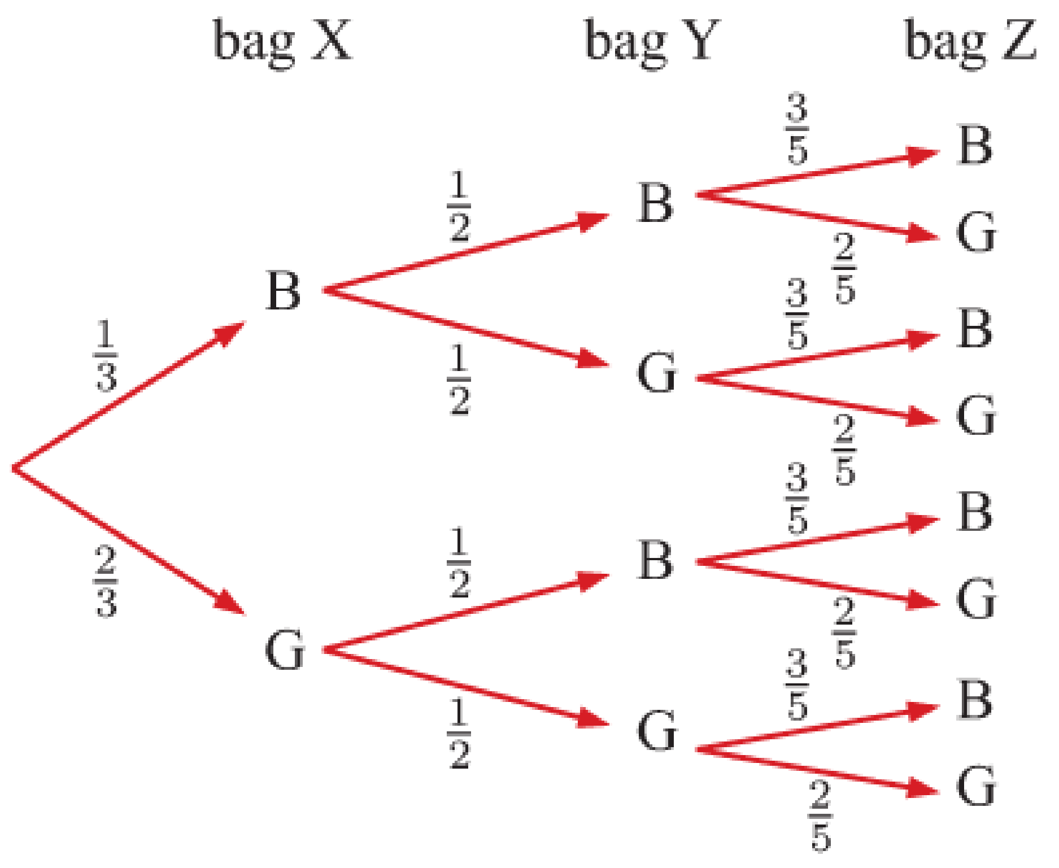
b **i** 0.45 **ii** 0.05

7 **a**



b **i** $\frac{25}{49}$ **ii** $\frac{20}{49}$

8 **a**



b **i** $\frac{1}{10}$ **ii** $\frac{1}{5}$ **iii** $\frac{13}{15}$

9 **a** $2p^2 - p^4$ **b** $p \approx 0.541$

10 Penny - Quentin - Penny
To win 2 matches in a row, Kane must win the middle match, so he should play against the weaker player in this match.

EXERCISE 11H

1 **a** $\frac{7}{15}$ **b** $\frac{7}{30}$ **2** **a** $\frac{2}{15}$ **b** $\frac{4}{15}$ **3** **a** $\frac{14}{55}$ **b** $\frac{1}{55}$

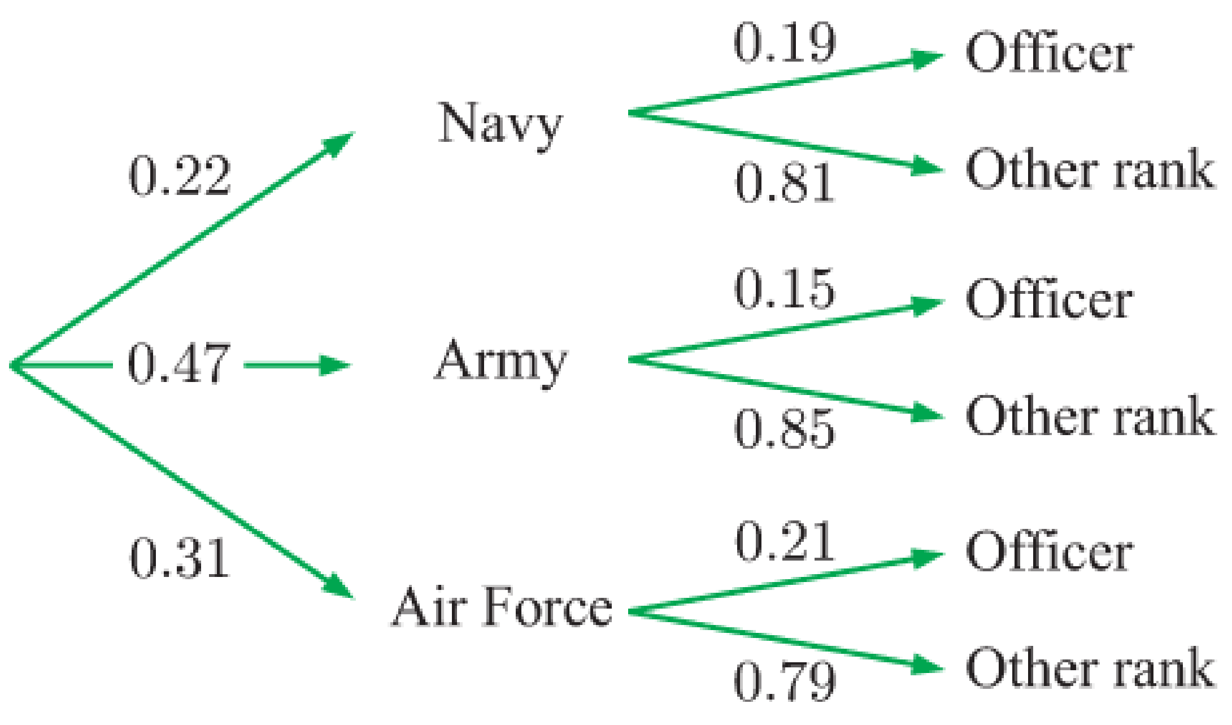
4 **a** $\frac{3}{100}$ **b** $\frac{3}{100} \times \frac{2}{99} \approx 0.000\,606$

c $\frac{3}{100} \times \frac{2}{99} \times \frac{1}{98} \approx 0.000\,006\,18$

d $\frac{97}{100} \times \frac{96}{99} \times \frac{95}{98} \approx 0.912$

5 **a** $\frac{4}{7}$ **b** $\frac{2}{7}$ **6** **a** $\frac{10}{21}$ **b** $\frac{1}{21}$

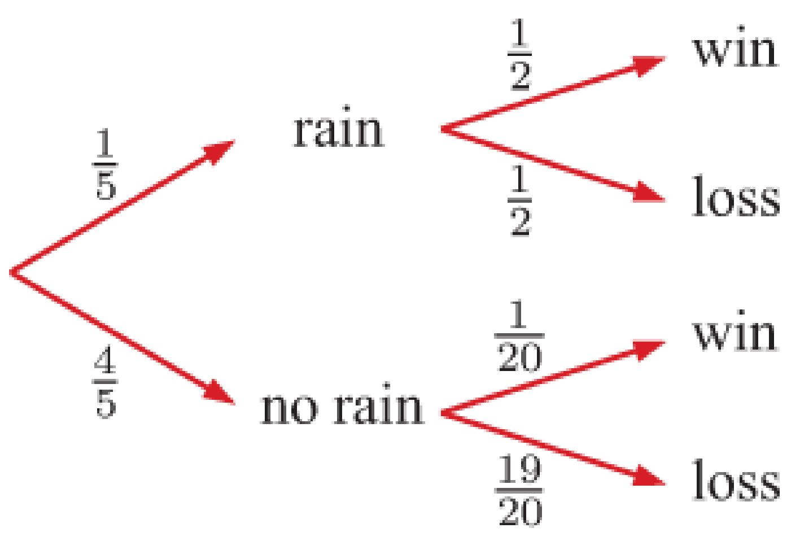
7 **a**



b **i** 0.1774 **ii** 0.9582 **iii** 0.8644

8 **a** 0.28 **b** 0.24

9 **a**



b $\frac{7}{50}$

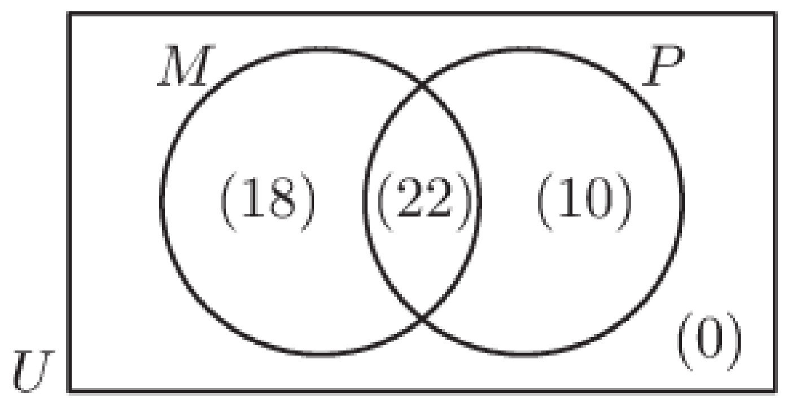
10 0.032 **11** $\frac{9}{38}$ **12** **a** $\frac{11}{30}$ **b** $\frac{19}{30}$

13 $\frac{187}{460} \approx 0.407$ **14** **a** $\frac{325}{833} \approx 0.390$ **b** $\frac{787}{833} \approx 0.945$

EXERCISE 11I

1 **a** $\frac{1}{4}$ **b** $\frac{1}{2}$ **c** 0 **2** $\frac{1}{2}$

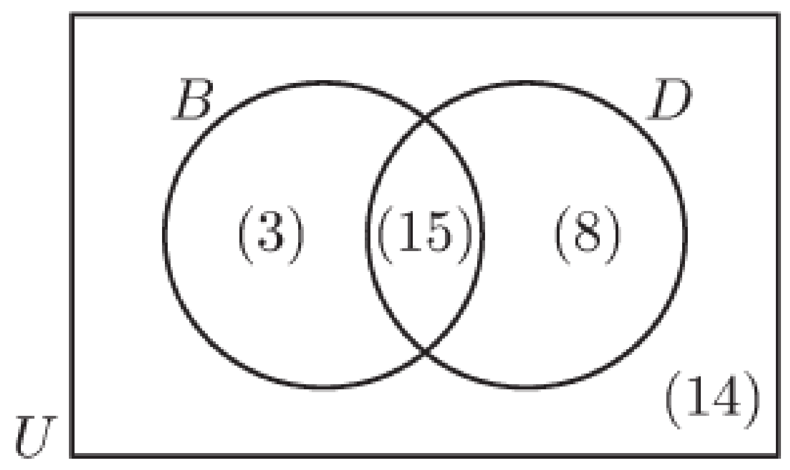
3 **a**



22 study both

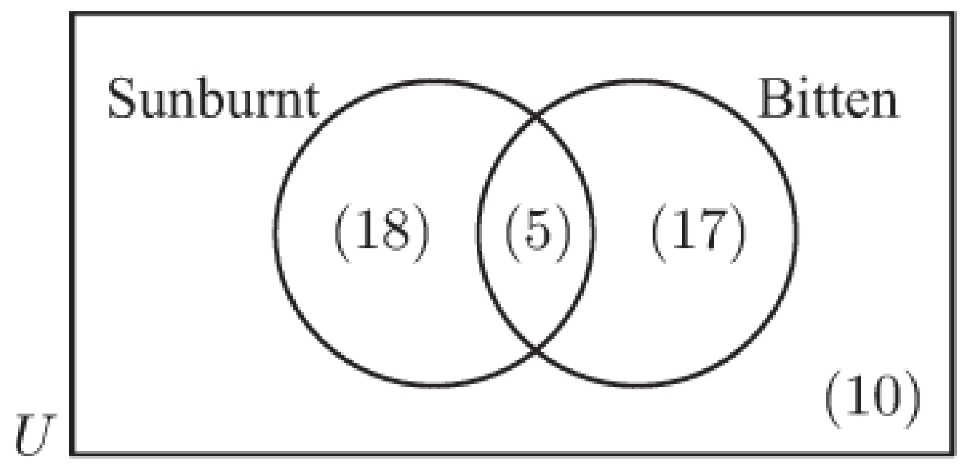
b **i** $\frac{9}{25}$ **ii** $\frac{11}{20}$

4 **a**



b **i** $\frac{3}{8}$ **ii** $\frac{15}{23}$

5 **a**



b **i** $\frac{14}{25}$ **ii** $\frac{4}{5}$ **iii** $\frac{5}{23}$ **iv** $\frac{9}{14}$

6 $\frac{7}{8}$

7 **a** $\frac{13}{20}$ **b** $\frac{7}{20}$ **c** $\frac{11}{50}$ **d** $\frac{7}{25}$ **e** $\frac{4}{7}$ **f** $\frac{1}{4}$

8 **a** $\frac{3}{5}$ **b** $\frac{2}{3}$ **9** **a** $\frac{23}{50}$ **b** $\frac{14}{23}$

10 **a** $\frac{10}{17}$ **b** $\frac{70}{163}$ **11** $\frac{7}{15}$

EXERCISE 11J

1 $\mathrm{P}(R \cap S) = 0.4 + 0.5 - 0.7 = 0.2$ and $\mathrm{P}(R) \times \mathrm{P}(S) = 0.2$
$\therefore$ R and S are independent events.

2 **a** **i** $\frac{7}{30}$ **ii** $\frac{7}{12}$ **iii** $\frac{7}{10}$

b No, as $\mathrm{P}(A \mid B) \neq \mathrm{P}(A)$.

3 **a** 0.35 **b** 0.85 **c** 0.15 **d** 0.15 **e** 0.5

4 **Hint:** Show $\mathrm{P}(A' \cap B') = \mathrm{P}(A')\,\mathrm{P}(B')$
using a Venn diagram and $\mathrm{P}(A \cap B)$.

5 $\mathrm{P}(B) = 0$ **6** 0.9

7 **a** $\mathrm{P}(D) = \frac{89}{400}$ **b** No, as $\mathrm{P}(D \mid C) \neq \mathrm{P}(D)$.

8 $\mathrm{P}(A \cup B) = 1$ or $\mathrm{P}(A \cap B) = 0$

EXERCISE 11K

1 **a** 0.0435 **b** ≈ 0.598 **2** **a** ≈ 0.773 **b** ≈ 0.556

3 $\frac{10}{13}$ **4** ≈ 0.424 **5** 0.0137 **6** $\frac{15}{83}$ **7** $\frac{99}{148}$

8 **a** $\frac{9}{19}$ **b** $\frac{10}{19}$ **10** **a** 0.95 **b** ≈ 0.306 **c** 0.6

11 **a** 0.104 **b** ≈ 0.267 **c** ≈ 0.0168

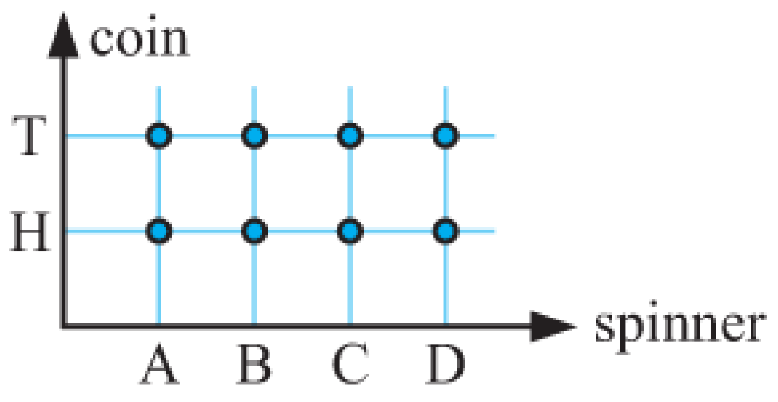
12 **a** $\mathrm{P}(L \mid T) = \frac{46}{205}$ **b** $\mathrm{P}(T \mid L) = \frac{46}{57}$

c Bayes' theorem tells us that $\mathrm{P}(L \mid T) = \mathrm{P}(T \mid L)\,\dfrac{\mathrm{P}(L)}{\mathrm{P}(T)}$.
Our answers to **a** and **b** differ since $\mathrm{P}(L) \neq \mathrm{P}(T)$.

REVIEW SET 11A

1 **a** ≈ 0.13 **b** ≈ 0.53

2 **a**



b **i** $\frac{3}{8}$ **ii** $\frac{1}{8}$ **iii** $\frac{5}{8}$

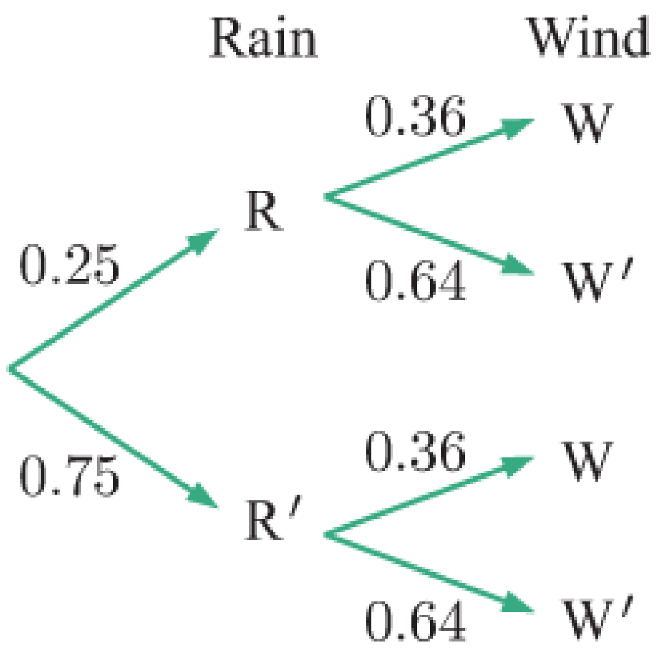
3 **a** Two events are independent if the occurrence of either event does not affect the probability that the other occurs. For A and B independent, $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$.

b Two events A and B are mutually exclusive if they have no common outcomes. $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$

4 0.496

5 **a** $\mathrm{P}(A \cup B) = x + 0.57$ **b** $x = 0.16$

6 **a**

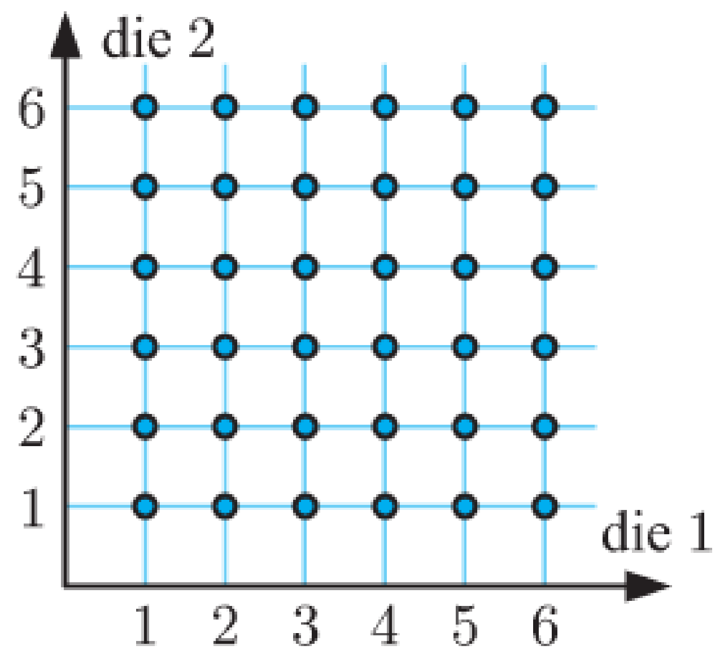


b **i** 0.09 **ii** 0.52

c It is assumed that the events are independent.

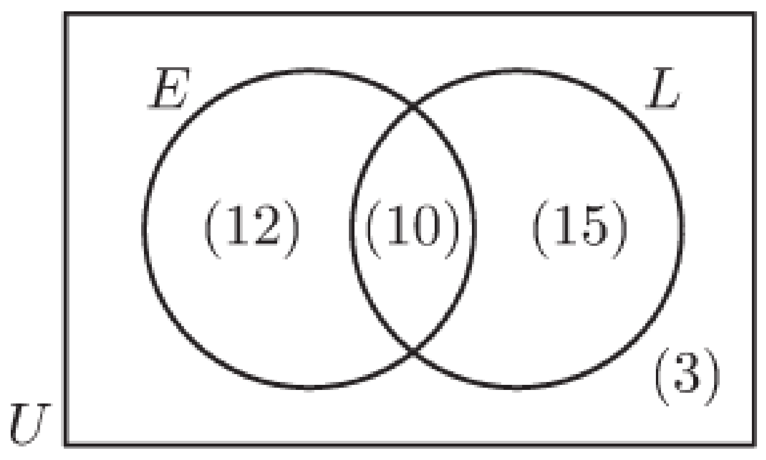
7 **a** 0 **b** 0.45 **c** 0.8

8 **a**



b **i** $\frac{2}{9}$ **ii** $\frac{5}{12}$

9 **a**



b **i** $\frac{1}{4}$ **ii** $\frac{37}{40}$ **iii** $\frac{2}{5}$

10 4350 seeds **11** **a** $\frac{25}{144}$ **b** $\frac{25}{72}$ **c** $\frac{7}{16}$ **d** $\frac{4}{9}$

12 ≈ 0.127

13 **a**

	Female	*Male*	*Total*
Smoker	20	40	60
Non-smoker	70	70	140
Total	90	110	200

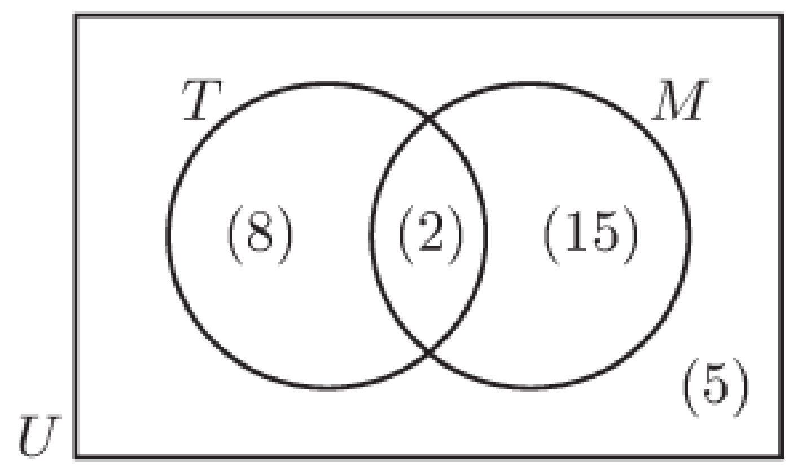
b **i** $\frac{7}{20}$ **ii** $\frac{1}{2}$

c **i** ≈ 0.121 **ii** ≈ 0.422

14 $\frac{69}{95}$ **15** **a** $\frac{1}{5}$ **b** $\mathrm{P}(B \mid A) \neq \mathrm{P}(B)$ **c** $\frac{2}{3}$ **16** $\frac{5}{324}$

REVIEW SET 11B

1 **a**



b **i** $\frac{1}{15}$ **ii** $\frac{2}{17}$

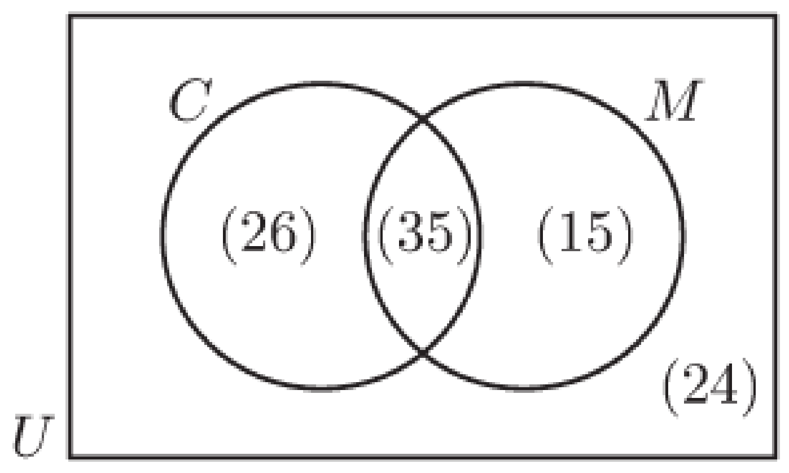
2 0.9975

3 **a** $\mathrm{P}(A \cap B) = 0.28$ which is not equal to 0.

$\therefore$ A and B are not mutually exclusive.

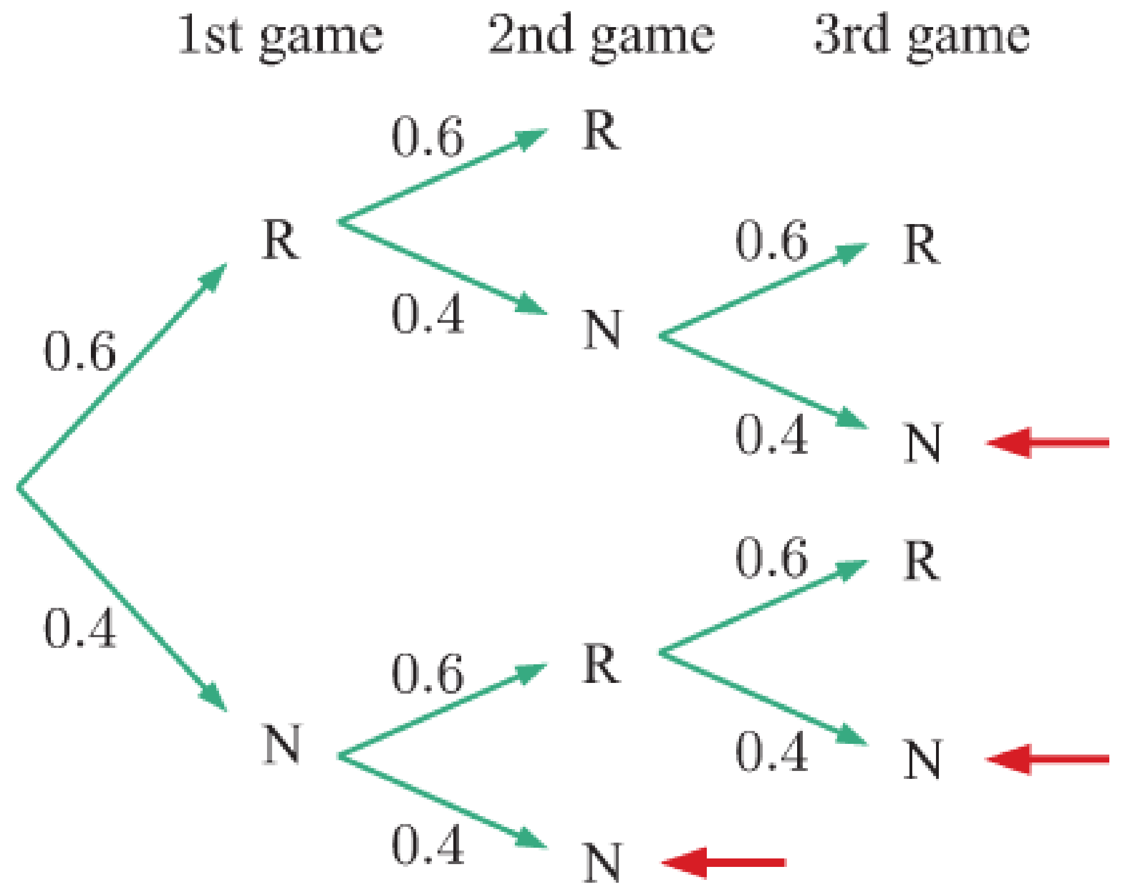
b $\mathrm{P}(A \cup B) = 0.82$

4 $\frac{5}{9}$ **5** **a**



b $\frac{35}{61}$

6 **a**



b P(N wins) $= 0.352$

7 **a** 0.93 **b** 0.8 **c** 0.2 **d** 0.65

8 **a** $\frac{4}{500} \times \frac{3}{499} \times \frac{2}{498} \approx 0.000\,000\,193$

b $1 - \frac{496}{500} \times \frac{495}{499} \times \frac{494}{498} \approx 0.0239$

9 **a** 0.2588 **b** ≈ 0.703

10 **a** $\mathrm{P}(B) = \frac{1}{3}$ **b** **i** $\frac{16}{21}$ **ii** $\frac{13}{21}$

11 $\frac{4}{9}$ **12** **a** ≈ 0.660 **b** $\frac{100}{7} \approx 14.3$ or 14 pieces

13 **a** $\frac{31}{70}$ **b** $\frac{21}{31}$ **14** $\frac{1}{2}$

15 **a** **i** 100 balloons **ii** 33 balloons

b **i** $\frac{19}{25}$ **ii** $\frac{37}{50}$

c **i** $\frac{17}{66}$ **ii** $\frac{2701}{4950}$ **iii** $\frac{25}{66}$ **iv** $\frac{29}{66}$

d **i** $\frac{1}{980}$ **ii** $\frac{17}{308}$

16 **b** ≈ 0.988 **c** **i** ≈ 0.547 **ii** ≈ 0.266 **d** females

e A 20 year old is expected to live much longer than 30 more years, so it is unlikely the insurance company will have to pay out the policy. A 50 year old however is expected to live for only another 26.45 years (males) or 31.59 years (females), so the insurance company may have to pay out the policy.

g For “third world” countries with poverty, lack of sanitation, and so on, the tables would show a significantly lower life expectancy.

EXERCISE 12A

1 This sample is too small to draw reliable conclusions from.

2
- The sample size is very small and may not be representative of the whole population.
- The sample was taken in a Toronto shopping mall. People living outside of the city are probably not represented.

3 **a** The sample is likely to under-represent full-time weekday working voters.

b The members of the golf club may not be representative of the whole electorate.

c Only people who catch the train in the morning such as full-time workers or students will be sampled.

d The voters in the street may not be representative of those in the whole electorate.

4 **a** The sample size is too small.

b With only 10 sheep being weighed, any errors in the measuring of weights will have more impact on the results.

5 **a** The whole population is being considered, not just a sample. There will be no sampling error as this is a census.

b measurement error

6 **a** Many of the workers may not return or even complete the survey.

b There may be more responses to the survey as many workers would feel that it is easier to complete a survey online rather than on paper and mailing it back. Responses would also be received more quickly however some workers may not have internet access and will therefore be unable to complete the survey.

7 **a** Yes; members with strong negative opinions regarding the management structure of the organisation are more likely to respond.

b No; the feedback from the survey is still valid. Although it might be biased, the feedback might bring certain issues to attention.

EXERCISE 12B

1 **Note:** Sample answers only - many answers are possible.

a 12, 6, 23, 10, 21, 25

b 11, 2, 10, 17, 24, 14, 25, 1, 21, 7

c 14, 24, 44, 34, 27, 1 **d** 166, 156, 129, 200, 452

2 **a** 17, 67, 117, 167, 217 **b** 1600 blocks of chocolate

3 **a** Select 5 random numbers between 1 and 365 inclusive. For example, 65, 276, 203, 165, and 20 represent 6th March, 3rd October, 22nd July, 14th June, and 20th January.

b Select a random number between 1 and 52 inclusive. Take the week starting on the Monday that lies in that week.

c Select a random number between 1 and 12 inclusive.

d Select 3 random numbers between 1 and 12 inclusive.

e Select a random number between 1 and 10 inclusive for the starting month.

f Select 4 random numbers between 1 and 52 inclusive. Choose the Wednesday that lies in that week.

4 **a** convenience sampling

b The people arriving first will spend more time at the show, and so are more likely to spend more than €20. Also, the sample size is relatively small.

c For example, systematic sample of every 10th person through the gate.

5 **a** systematic sampling **b** 14 days

c Only visitors who use the library on Mondays will be counted. Mondays may not be representative of all of the days.

6 **a** 160 members

b 20 tennis members, 15 lawn bowls members,
5 croquet members

7 1 departmental manager, 3 supervisors, 9 senior sales staff,
13 junior sales staff, 4 shelf packers

8 **a** It is easier for Mona to survey her own home room class, so this is a convenience sample.

b Mona's sample will not be representative of all of the classes in the school. Mona's survey may be influenced by her friends in her class.

c For example, a stratified sample of students from every class.

9 **a** Not all students selected for the sample will be comfortable discussing the topic.

b quota sample

10 **a** All students in Years 11 and 12 were asked, not just a sample.

b 0.48

c **i** Sample too small to be representative.

ii Sample too small to be representative.

iii Valid but unnecessarily large sample size.

iv Useful and valid technique.

v Useful and valid technique.

vi Useful and valid technique.

d **v** is simple random sampling, while **iii** and **iv** are systematic sampling, and **vi** is stratified or quota sampling.

EXERCISE 12C

Note: Sample answers only - many answers are possible.

1 **a** It does not allow for colours which are different from those given.

b What colour is your shirt?

c For example, one person might interpret a colour as blue whereas another person may interpret it as purple. A shirt may also be more than one colour which could lead to difficulties in interpreting the colour.

2 **a** The question could be interpreted as:

- "Do you have any medically diagnosed allergies?"
- "Do you have any life threatening allergies?"
- "Do you have any food allergies?"
- "Do you think you have any allergies?"

The question also does not specify if it is a structured yes/no type of question or if the respondent should list specific allergies.

b "Do you have any food or other type of allergies (medically diagnosed or otherwise), and if so, what are they?"

3 **a** The question could be interpreted as:

- "Do you have any animals in your household?"
- "Do you have any animals in your care at home or elsewhere?"

The question does not specify if it is a structured yes/no type of question or if it includes livestock or only domestic animals.

b "Do you have any domesticated animals in your household (not including livestock), and if so, what are they?"

4 **a** The journalist's question is misleading as it only mentions the proposed cuts to education, not the proposal to move those funds to health. This may produce a measurement error as the respondents are unlikely to give their views about the whole proposal.

b For example, "What are your views about the Government's proposal to move funding from education to health?".

5 **a** Many respondents would not be comfortable giving their address to someone they do not know.

b The question could be more specific, asking for the general area or suburb only. For example:

- "Which suburb do you live in?"
- "Which state do you live in?"

Giving a reason why this information is needed will also improve the response rate.

6 **a** **i** The question contains a double negative which could confuse respondents. The word "infectious" suggests that *not* immunising is undesirable behaviour, thus the question is biased.

ii "Have you been immunised against meningococcal disease?"

b **i** The question asks for two things:

- whether climate change is a major issue
- the respondent's political opinion on climate change.

It is not clear whether the respondent's response will reflect their general or political opinion on the issue.
The phrase "thrown around by politicians" is also rather emotive, thus the question is biased.

ii "Do you believe that climate change is an important issue?"

c **i** The question uses a positive fact about fair trade cocoa to try to persuade the respondent into answering "yes". So the question is biased.
It is also very long and takes a long time to get to the point.

ii "Do you believe that fair trade certified chocolate should be more expensive than uncertified chocolate?"

EXERCISE 12D

1 **a** discrete; 0, 1, 2, 3, ...

b categorical; red, yellow, orange, green

c continuous; 0 - 15 minutes

d continuous; 0 - 25 m

e categorical; Ford, BMW, Renault **f** discrete; 1, 2, 3, ...

g categorical; Australia, Hawaii, Dubai

h discrete; 0.0 - 10.0 **i** continuous; 0 - 4 L

j continuous; 0 - 80 hours **k** continuous; −20°C - 35°C

l categorical; cereal, toast, fruit, rice, eggs

m discrete; 0, 1, 2, ...

2 *Name*: categorical, *Age*: continuous, *Height*: continuous, *Country*: categorical, *Wins*: discrete, *Speed*: continuous, *Ranking*: discrete, *Prize money*: discrete

EXERCISE 12E

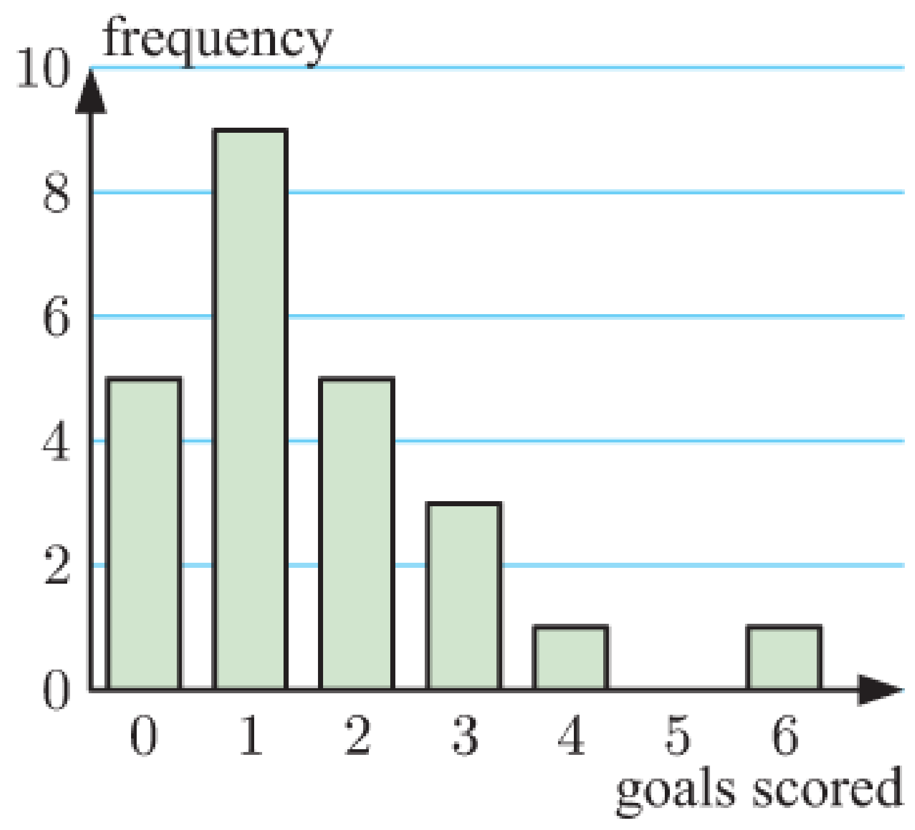
1 **a** the number of goals scored in a game

b variable is counted, not measured

c

Goals scored	*Tally*	*Frequency*	*Rel. Frequency*
0	𝍸	5	≈ 0.208
1	𝍸 ||||	9	0.375
2	𝍸	5	≈ 0.208
3	|||	3	0.125
4	|	1	≈ 0.042
5		0	0
6	|	1	≈ 0.042
	Total	24	

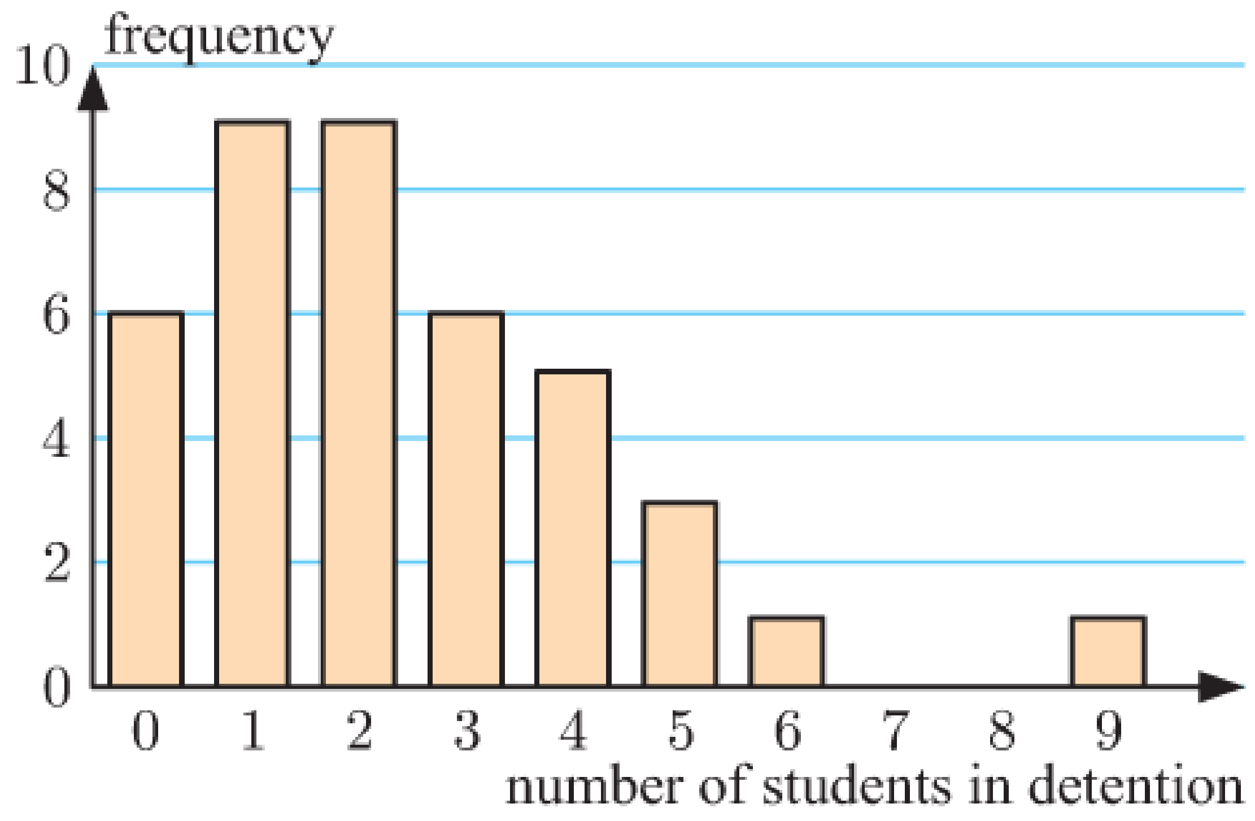
d



e 1 goal

f positively skewed, one outlier (6 goals) **g** ≈ 20.8%

2 **a**



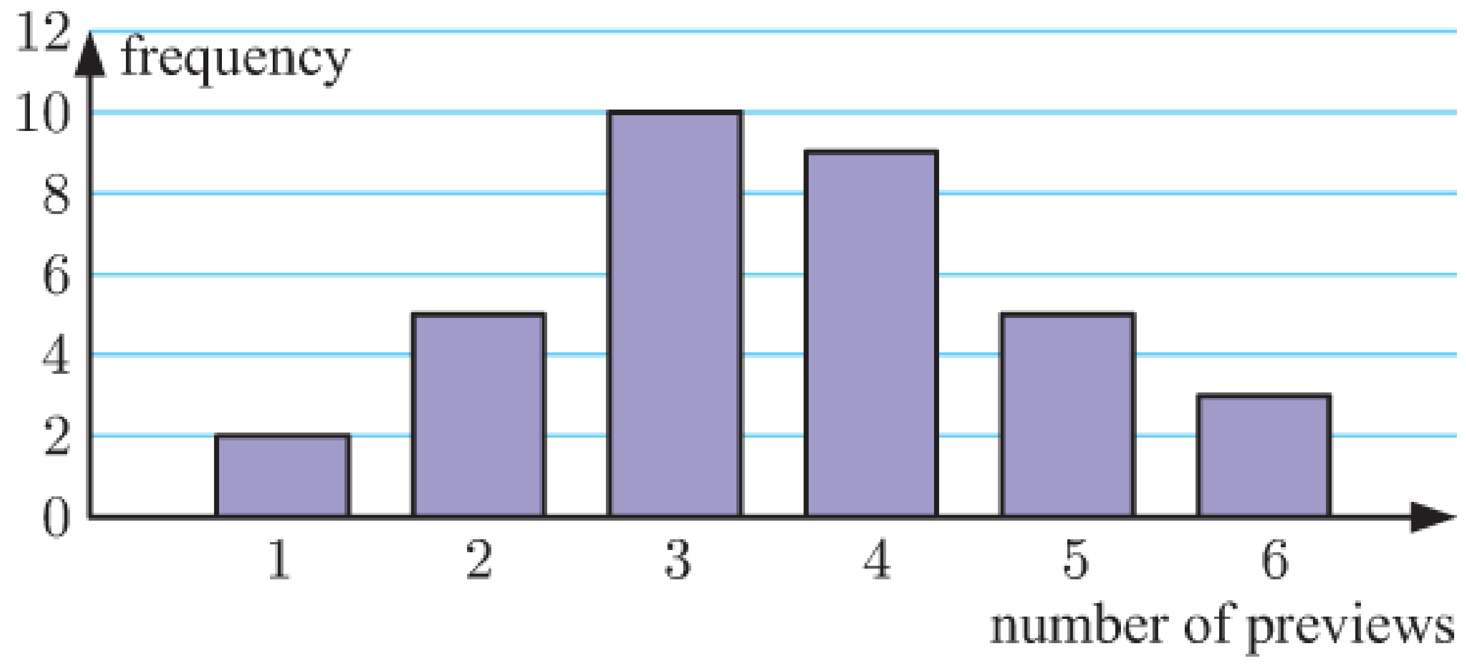
b 1 and 2 **c** positively skewed, one outlier (9 students)

d $12\frac{1}{2}\%$

3 **a**

Number of previews	*Tally*	*Frequency*
1	||	2
2	𝍸	5
3	𝍸 𝍸	10
4	𝍸 ||||	9
5	𝍸	5
6	|||	3
	Total	34

b



c 3 previews **d** symmetrical, no outliers **e** ≈ 79.4%

4 **a** 45 people **b** 1 time **c** 8 people **d** 20%

e positively skewed, no outliers

EXERCISE 12F

1 **a** 37 businesses **b** 40 - 49 employees

c negatively skewed **d** ≈ 37.8%

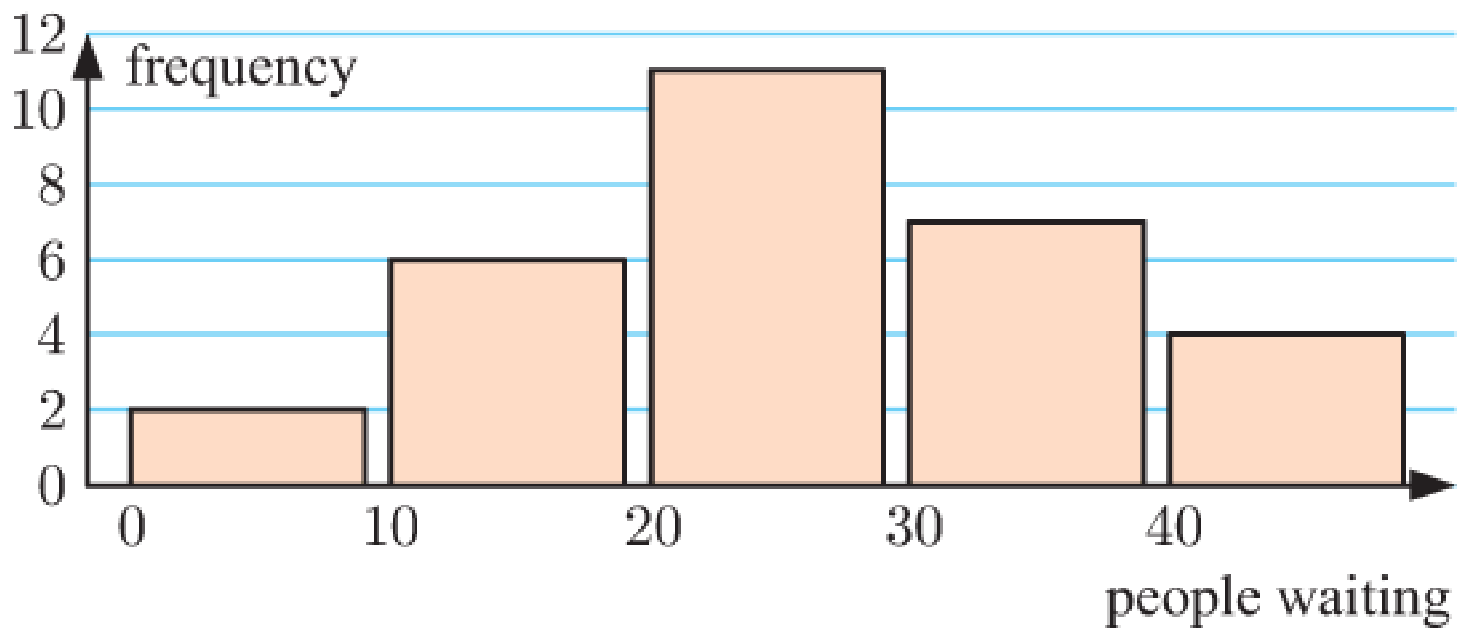
e No, only that it was in the interval 50 - 59 employees.

2 **a**

People waiting	*Tally*	*Frequency*	*Rel. Freq.*
0 - 9	||	2	≈ 0.067
10 - 19	𝍸 |	6	0.200
20 - 29	𝍸 𝍸 |	11	≈ 0.367
30 - 39	𝍸 ||	7	≈ 0.233
40 - 49	||||	4	≈ 0.133
	Total	30	

b 2 days **c** ≈ 36.7%

d

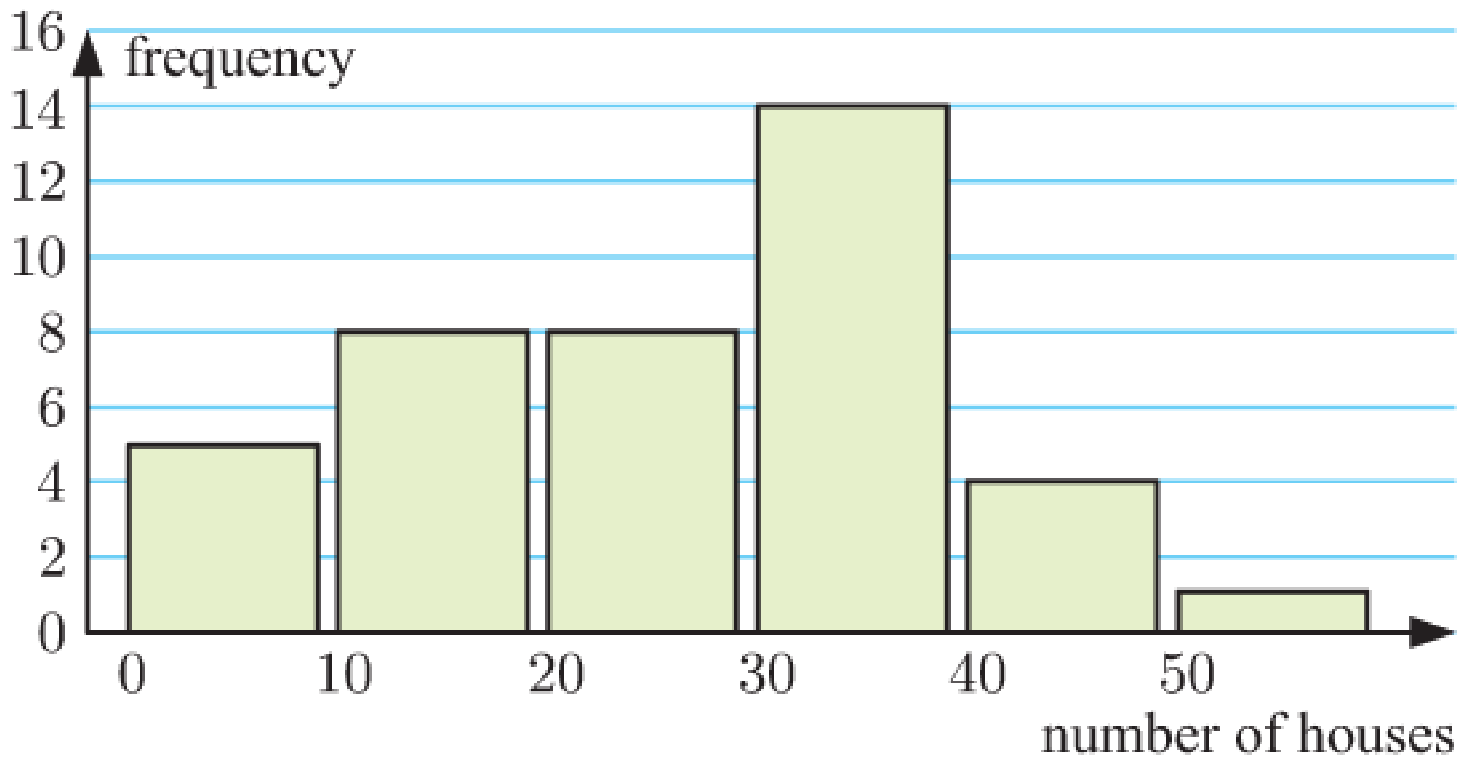


e 20 - 29 people

3 **a**

Number of houses	*Tally*	*Frequency*
0 - 9	𝍸	5
10 - 19	𝍸 |||	8
20 - 29	𝍸 |||	8
30 - 39	𝍸 𝍸 ||||	14
40 - 49	||||	4
50 - 59	|	1
	Total	40

b



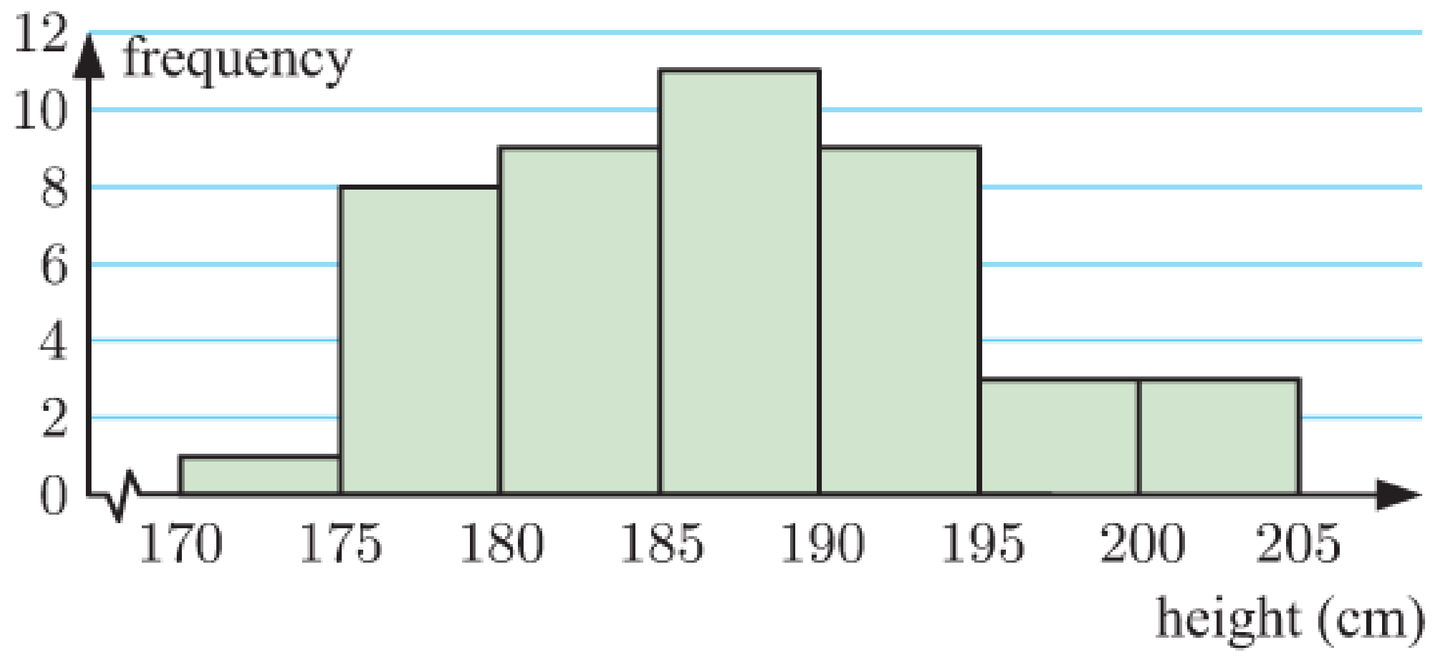
c 30 - 39 houses **d** 67.5%

EXERCISE 12G

1 **a** Height is measured on a continuous scale.

b

Heights of a volleyball squad



c $185 \leqslant H < 190$ cm. This is the class of values that appears most often.

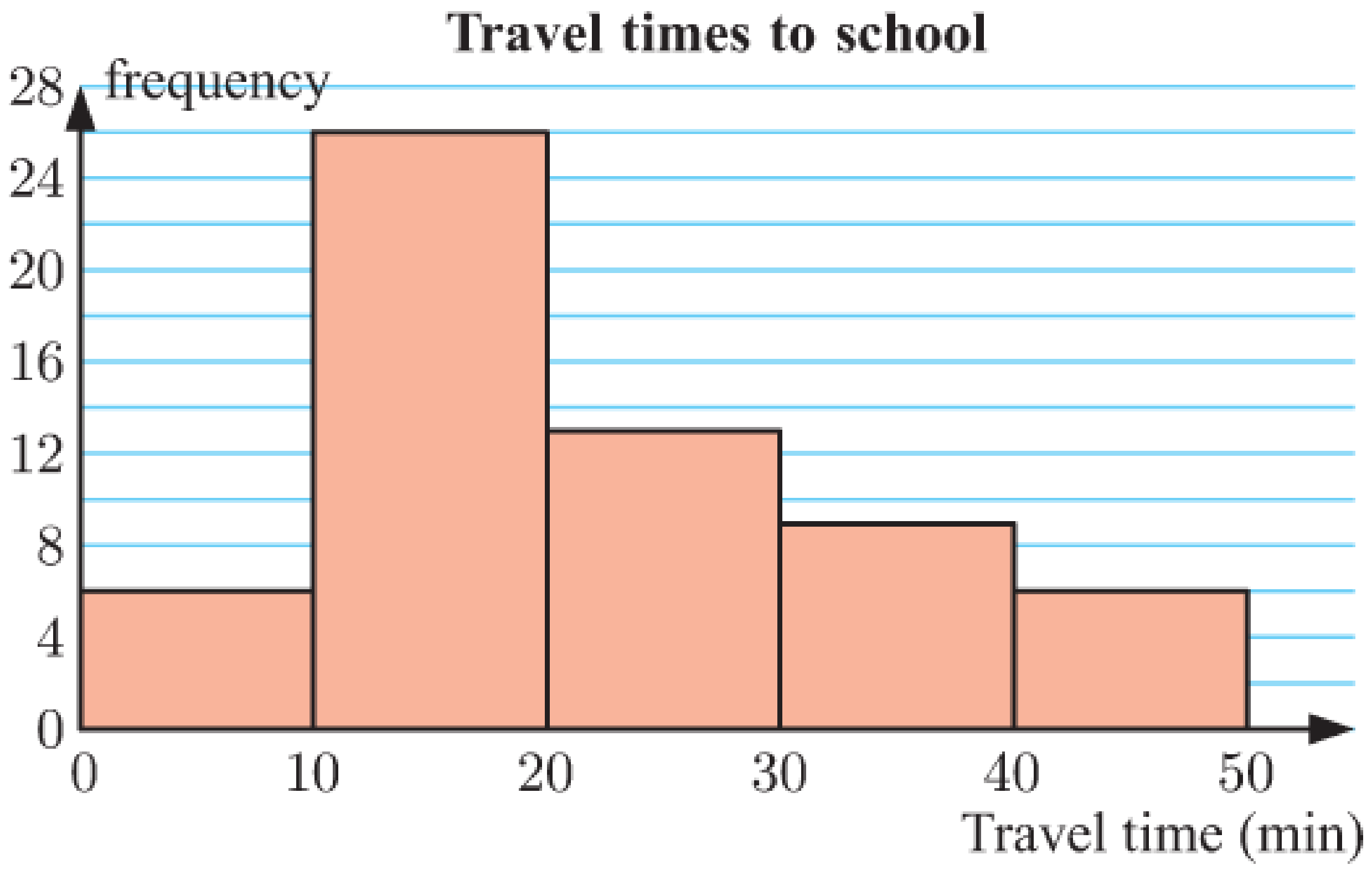
d slightly positively skewed

2 **a** continuous

b

Travel time (min)	*Tally*	*Frequency*
$0 \leqslant t < 10$	𝍸 |	6
$10 \leqslant t < 20$	𝍸 𝍸 𝍸 𝍸 𝍸 |	26
$20 \leqslant t < 30$	𝍸 𝍸 |||	13
$30 \leqslant t < 40$	𝍸 ||||	9
$40 \leqslant t < 50$	𝍸 |	6
	Total	60

c

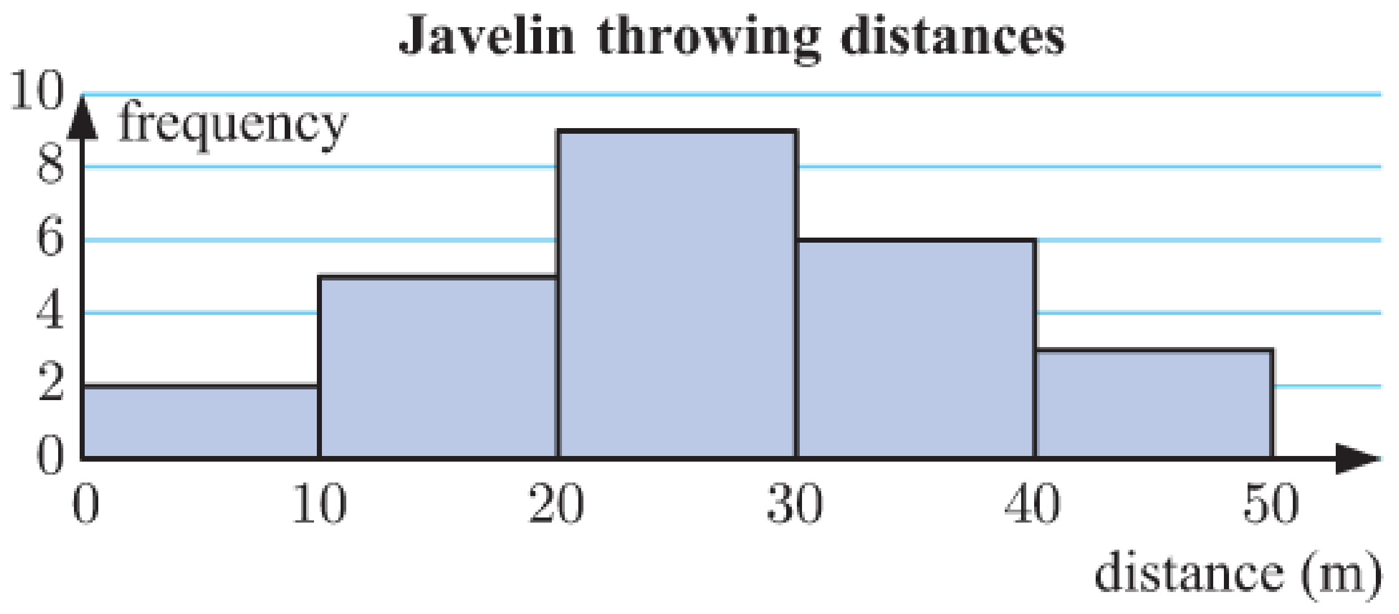


d positively skewed **e** $10 \leqslant t < 20$ minutes

3 **a, b**

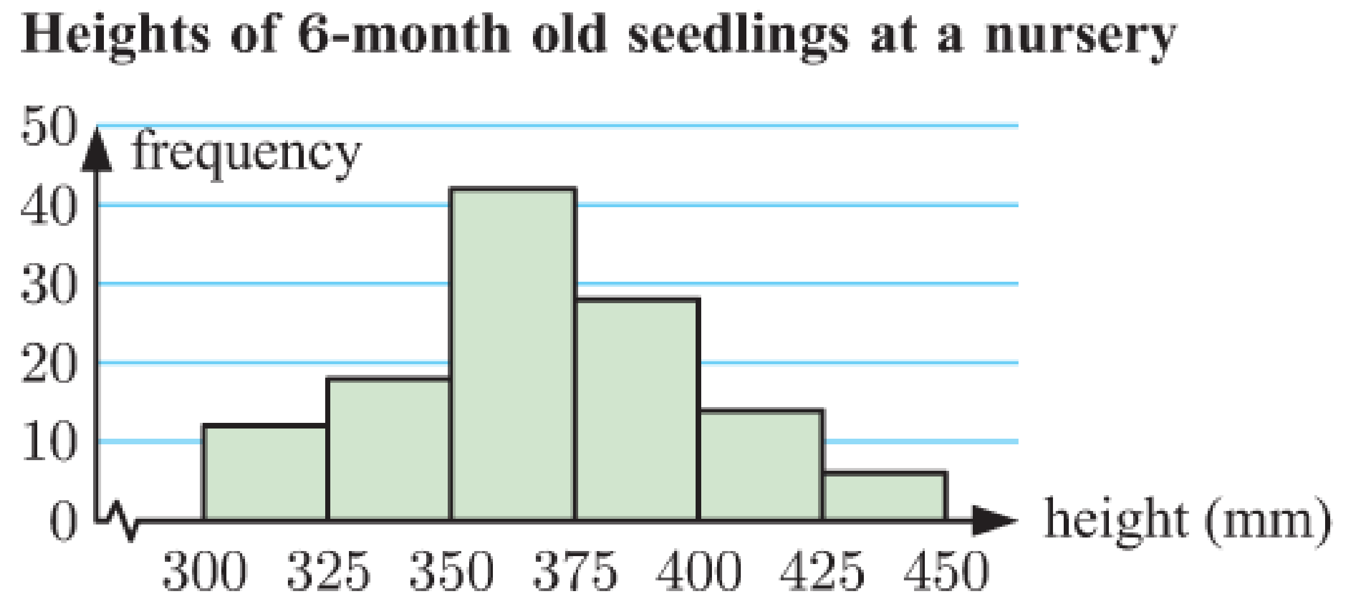
Distance (m)	*Tally*	*Frequency*
$0 \leqslant d < 10$	||	2
$10 \leqslant d < 20$	𝍸	5
$20 \leqslant d < 30$	𝍸 ||||	9
$30 \leqslant d < 40$	𝍸 |	6
$40 \leqslant d < 50$	|||	3
	Total	25

c



d $20 \leqslant d < 30$ m **e** 36%

4 **a**



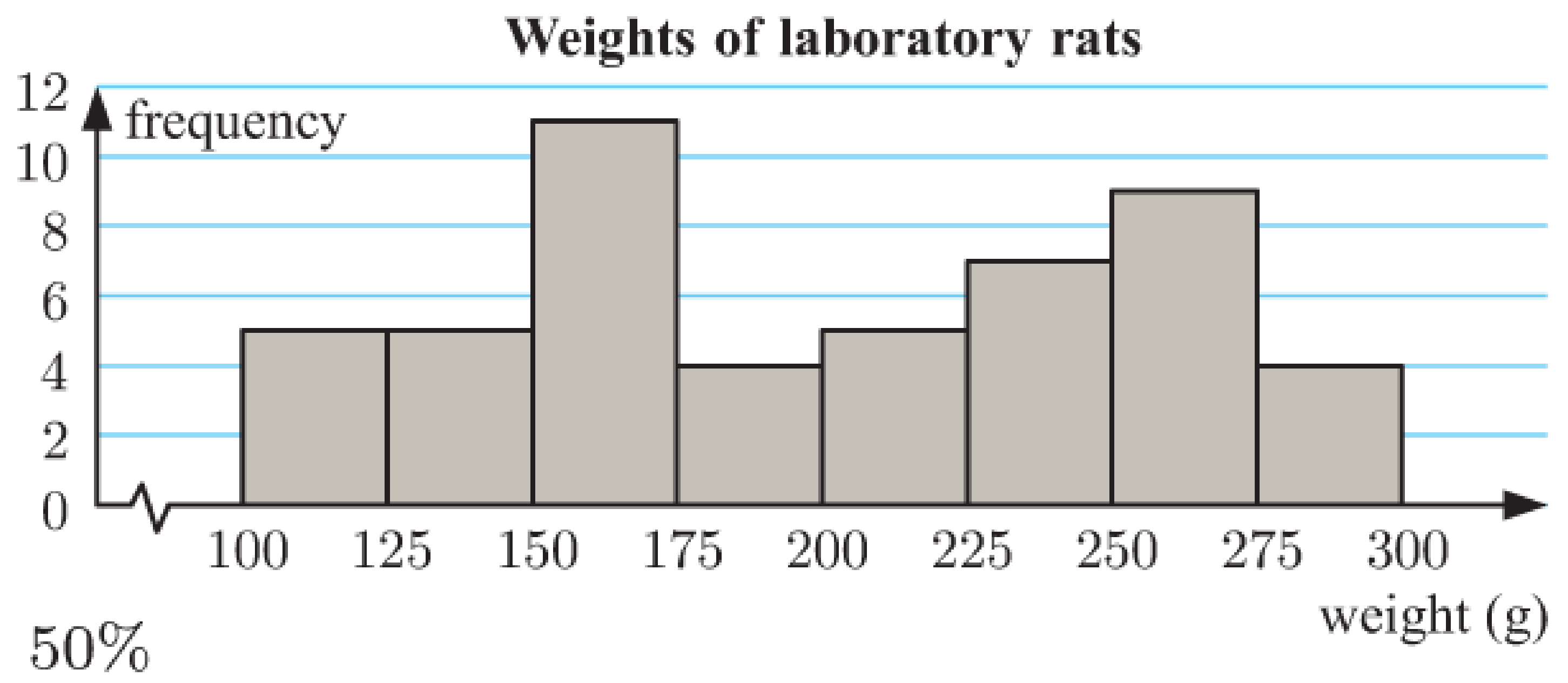
b 20 seedlings **c** ≈ 58.3%

d **i** ≈ 1218 seedlings **ii** ≈ 512 seedlings

5 **a, b**

Weight (g)	*Tally*	*Frequency*
$100 \leqslant w < 125$	𝍸	5
$125 \leqslant w < 150$	𝍸	5
$150 \leqslant w < 175$	𝍸 𝍸 |	11
$175 \leqslant w < 200$	||||	4
$200 \leqslant w < 225$	𝍸	5
$225 \leqslant w < 250$	𝍸 ||	7
$250 \leqslant w < 275$	𝍸 ||||	9
$275 \leqslant w < 300$	||||	4
	Total	50

c



d 50%

REVIEW SET 12A

1 **a** Students studying Italian may have an Italian background so surveying these students may produce a biased result.

b For example, Andrew could survey a randomly selected group of students as they entered the school grounds one morning.

2 **a** It would be too time consuming and expensive.

b

Age range	< 18	18 - 39	40 - 54	55 - 70	> 70
Sample size	50	82	123	69	26

3 **a** discrete **b** continuous **c** categorical
d categorical **e** categorical **f** continuous
g continuous **h** discrete **i** discrete

4 **a** convenience sampling

b Yes, the sample will be biased as people are more likely to be drinking on a Saturday night. It is sensible for this sample to be biased since drink-driving is illegal.

5 **a** The question could be interpreted as:
- "Do you consider yourself to be healthy?"
- "Are you not currently suffering from any health conditions?"
- "Do you eat a balanced diet and exercise regularly?"
- "Do you take any medication for any health conditions?"

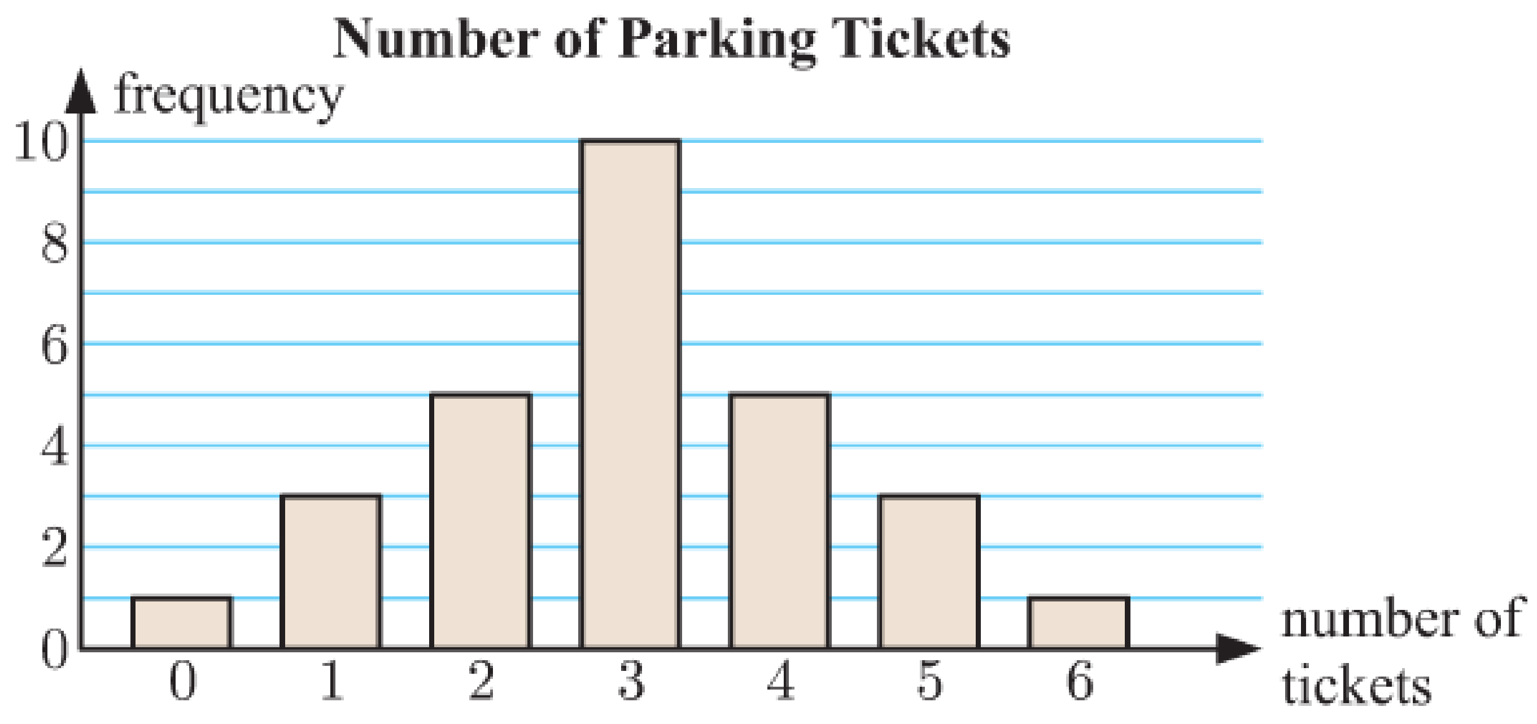
b "Do you eat a balanced diet and exercise regularly?"

6 **a** discrete **b** 1 round **c** positively skewed

7 **a**

Number of tickets	*Tally*	*Frequency*
0	|	1
1	|||	3
2	𝍸	5
3	𝍸 𝍸	10
4	𝍸	5
5	|||	3
6	|	1

b



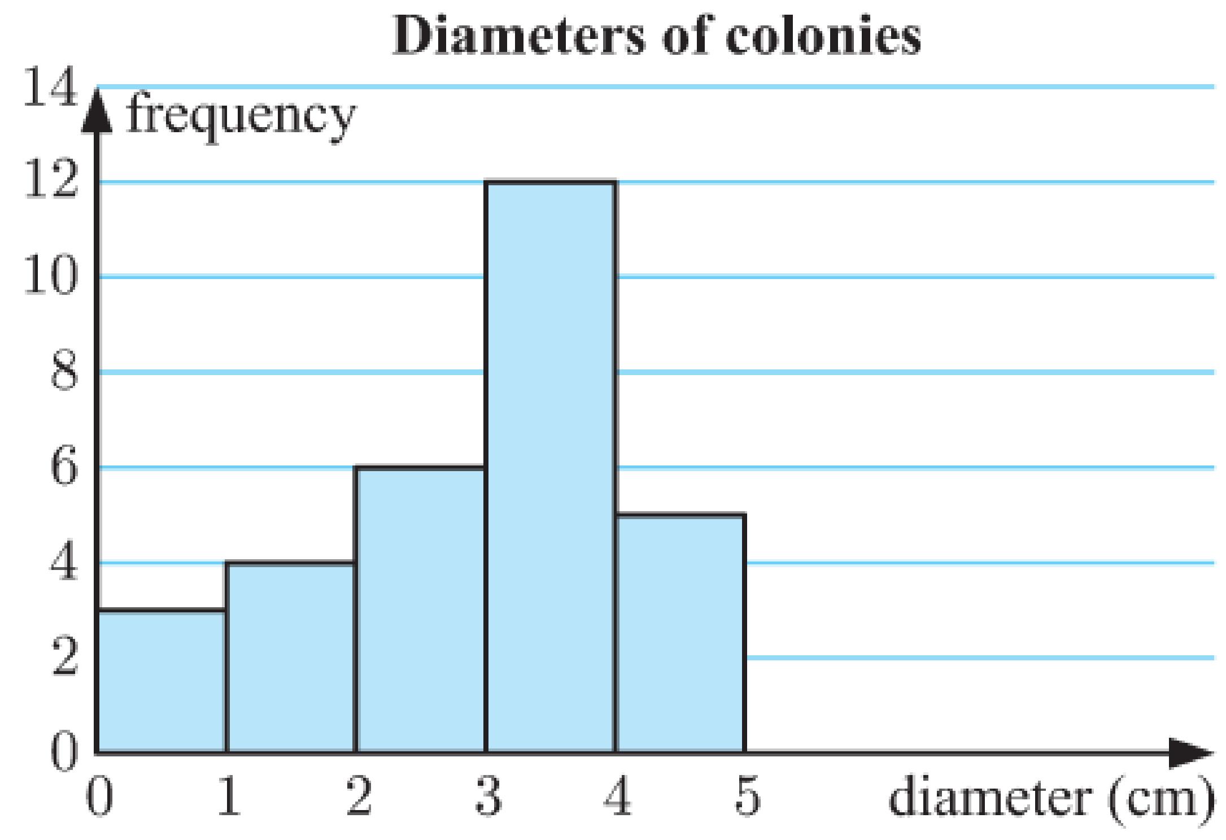
c The data is symmetric with no outliers.

8 **a** continuous

b

Diameter (d cm)	*Tally*	*Frequency*
$0 \leqslant d < 1$	|||	3
$1 \leqslant d < 2$	||||	4
$2 \leqslant d < 3$	𝍸 |	6
$3 \leqslant d < 4$	𝍸 𝍸 ||	12
$4 \leqslant d < 5$	𝍸	5
	Total	30

c



d $3 \leqslant d < 4$ cm **e** slightly negatively skewed

REVIEW SET 12B

1 **a** discrete **b** continuous **c** discrete

2 **a** systematic sampling

b A house will be visited if the last digit in its number is equal to the random number chosen by the promoter, with the random number 10 corresponding to the digit 0. Each house therefore has a 1 in 10 chance of being visited.

c Once the first house number has been chosen, the remaining houses chosen must all have the same second digit in their house number, that is, they are not randomly chosen. For example, it is impossible for two consecutively numbered houses to be selected for the sample.

3 **a** Petra's teacher colleagues are quite likely to ignore the emailed questionnaire as emails are easy to ignore.

b It is likely that the teachers who have responded will have strong opinions either for or against the general student behaviour. These responses may therefore not be representative of all teachers' views.

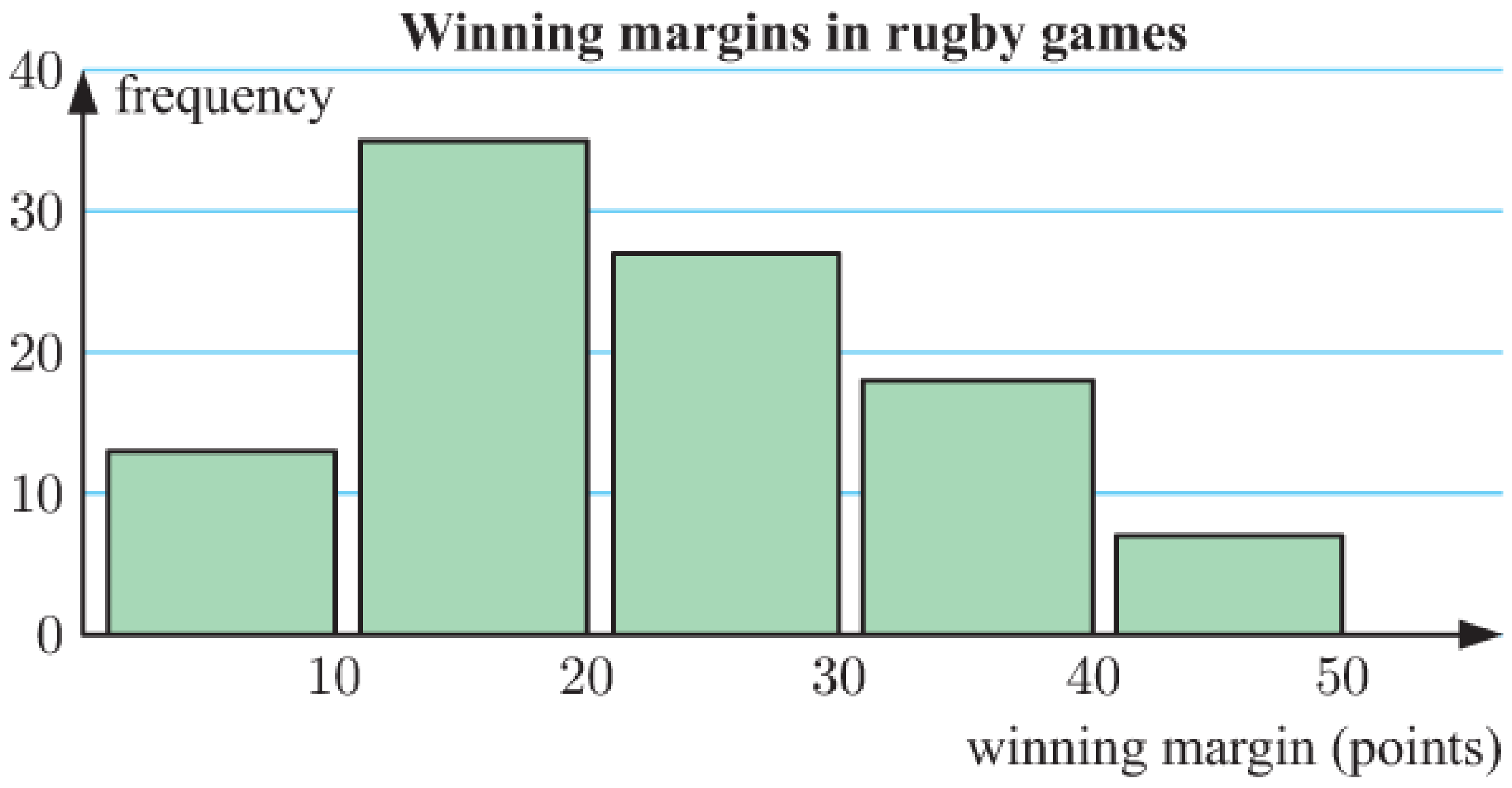
4 Did you learn about our services via:

- friends/family
- the internet
- newspaper
- television
- elsewhere?

5 **a** The tone is not neutral and it is a structured question. The only responses possible are yes or no.

b How would you describe your general behaviour when you were a child?

6



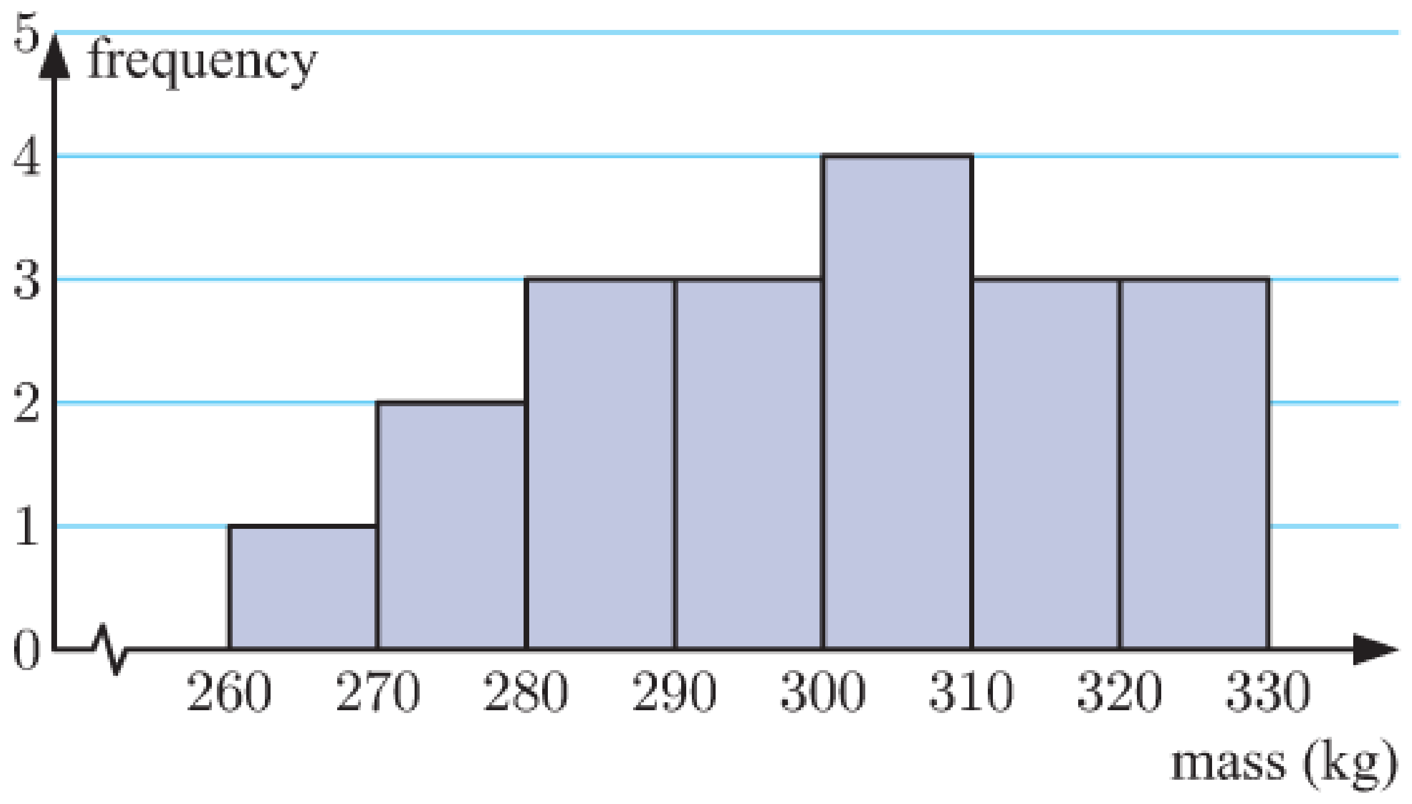
7 **a** Mass is measured on a continuous scale.

b

Mass (m kg)	*Frequency*
$260 \leqslant m < 270$	1
$270 \leqslant m < 280$	2
$280 \leqslant m < 290$	3
$290 \leqslant m < 300$	3
$300 \leqslant m < 310$	4
$310 \leqslant m < 320$	3
$320 \leqslant m < 330$	3

c $300 \leqslant m < 310$ kg

d



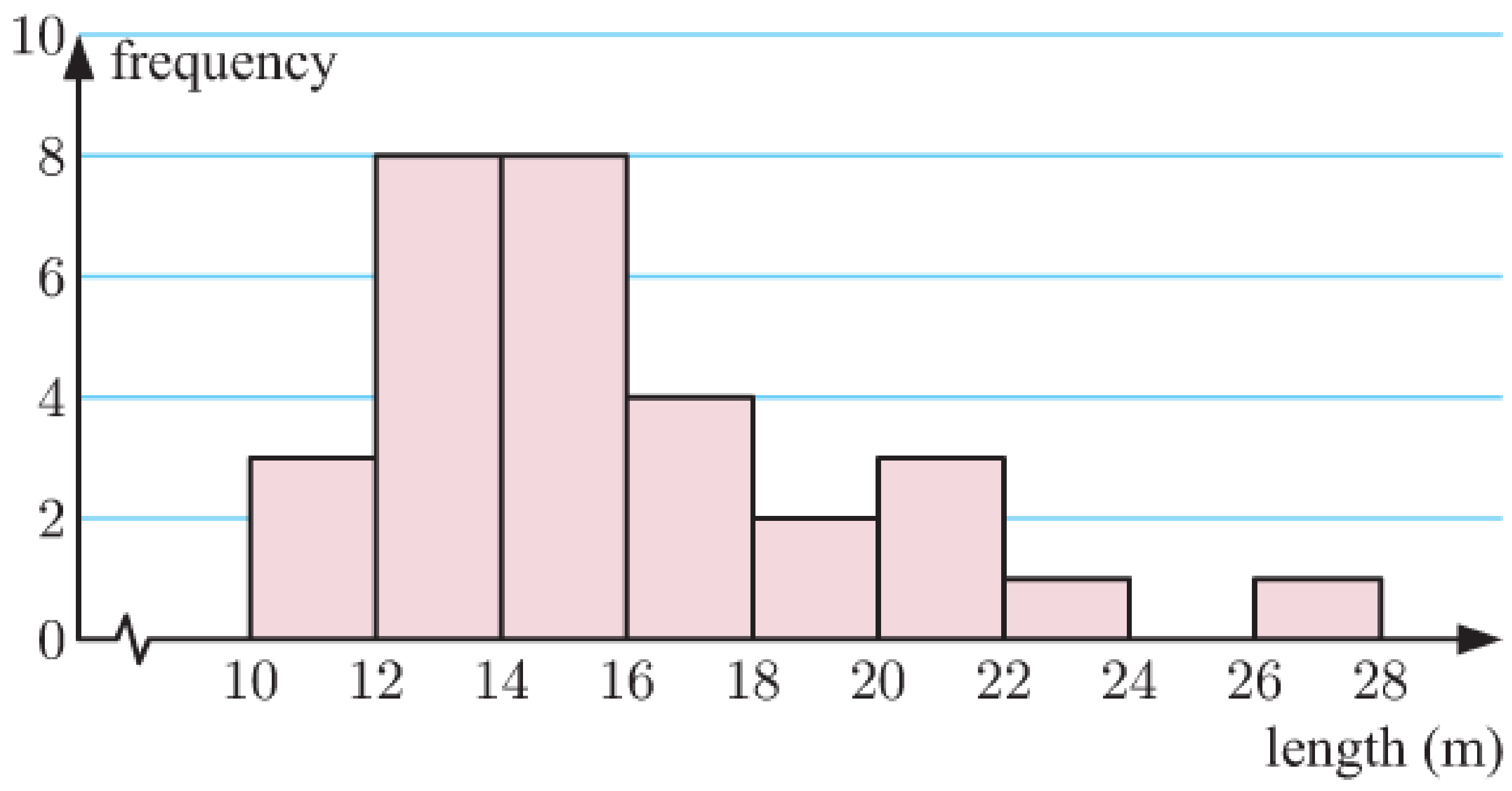
e slightly negatively skewed

8 **a** continuous

b

Length (l m)	*Frequency*
$10 \leqslant l < 12$	3
$12 \leqslant l < 14$	8
$14 \leqslant l < 16$	8
$16 \leqslant l < 18$	4
$18 \leqslant l < 20$	2
$20 \leqslant l < 22$	3
$22 \leqslant l < 24$	1
$24 \leqslant l < 26$	0
$26 \leqslant l < 28$	1

c



d positively skewed, one outlier (27.4 m)

EXERCISE 13A

1 **a** 1 cup **b** 2 cups **c** 1.8 cups

2 **a** **i** ≈ 5.61 **ii** 6 **iii** 6
b **i** ≈ 16.3 **ii** 17 **iii** 18
c **i** ≈ 24.8 **ii** 24.9 **iii** 23.5

3 9 **4** Ruth

5 **a** *data set A*: ≈ 6.46, *data set B*: ≈ 6.85

b *data set A*: 7, *data set B*: 7

c Data sets A and B differ only by their last value. This affects the mean, but not the median.

6 **a** **i** *motichoor ladoo*: ≈ 67.1, *malai jamun*: ≈ 53.6
ii *motichoor ladoo*: 69, *malai jamun*: 52

b The mean and median were much higher for the motichoor ladoo, so the motichoor ladoo were more popular.

7 **a** *Bus*: mean $= 39.7$, median $= 40.5$
Tram: mean ≈ 49.1, median $= 49$

b The tram data has a higher mean and median, but since there are more bus trips per day and more people travel by bus in total, the bus is more popular.

8 **a** 44 points **b** 44 points

c **i** Decrease, since 25 is lower than the mean of 44 for the first four matches.
ii 40.2 points

9 €185 604 **10** 116 **11** 17.25 goals per game
12 $x = 15$ **13** $a = 5$ **14** 37 marks
15 ≈ 14.8 **16** 6 and 12

EXERCISE 13B

1 **a** mean = \$363 770, median = \$347 200
The mean has been affected by the extreme values (the two values greater than \$400 000).
b **i** the mean **ii** the median

2 **a** mode = \$33 000, mean = \$39 300, median = \$33 500
b The mode is the lowest value in the data set.
c No, it is too close to the lower end of the distribution.

3 **a** mean ≈ 3.19 mm, median = 0 mm, mode = 0 mm
b The median is not the most suitable measure of centre as the data is positively skewed.
c The mode is the lowest value.
d 42 mm and 21 mm **e** no

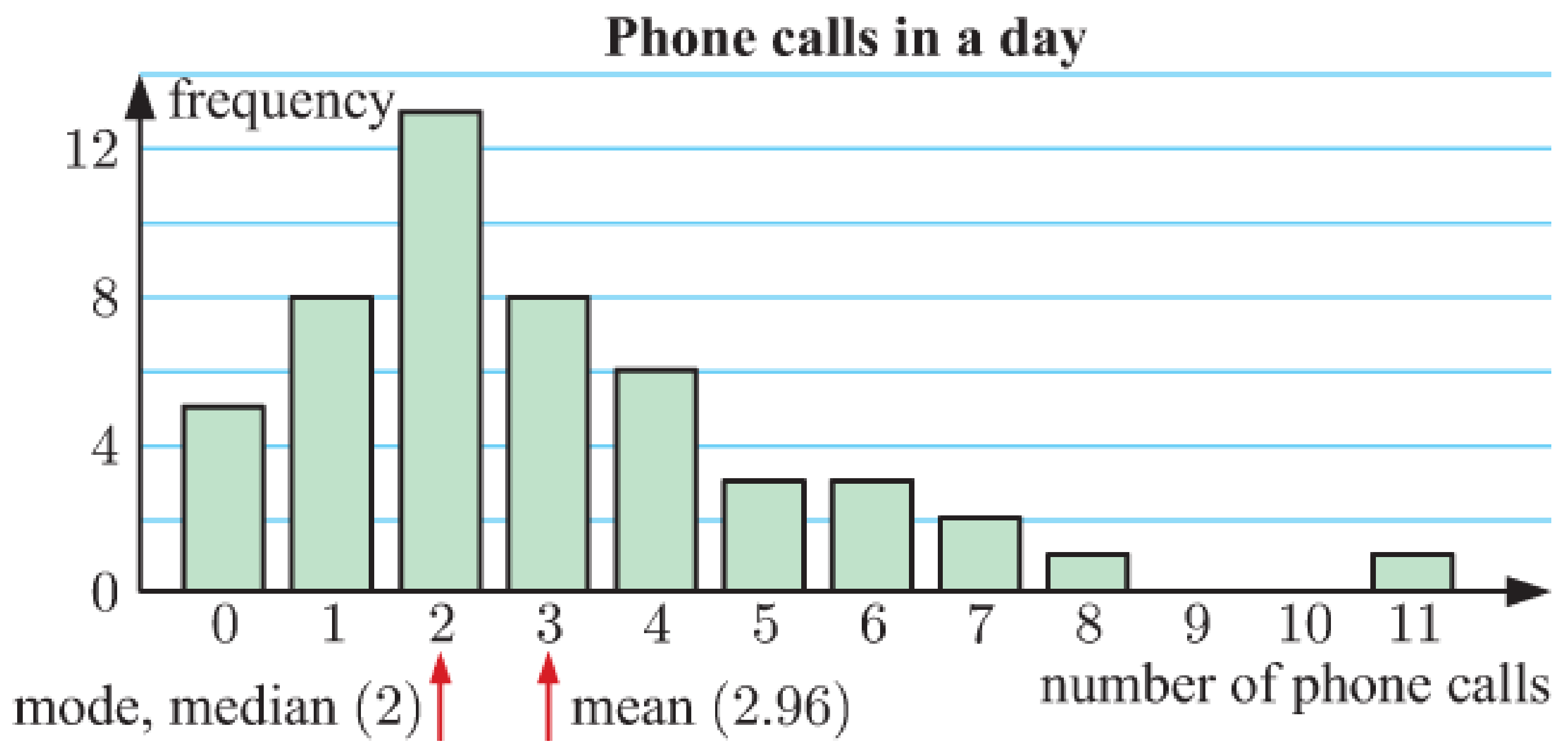
4 **a** mean ≈ 2.03, median = 2, mode = 1 and 2
b Yes, as Esmé can then offer a "family package" to match the most common number of children per family.
c 2 children, since this is one of the modes; it is also the median, and very close to the mean.

EXERCISE 13C

1 **a** 1 person **b** 2 people **c** ≈ 2.03 people

2 **a** **i** 2.96 phone calls **ii** 2 phone calls
iii 2 phone calls

b



c positively skewed
d The mean takes into account the larger numbers of phone calls.
e the mean

3 **a** **i** ≈ 2.61 children **ii** 2 children **iii** 2 children
b This school has more children per family than the average British family.
c positively skewed
d The values at the higher end increase the mean more than the median and the mode.

4 **a**

Pocket money (€)	*Frequency*
1	4
2	9
3	2
4	6
5	8

b 29 children
c **i** ≈ €3.17
ii €3
iii €2
d the mode

5 10.1 cm

6 **a** **i** \$63 000 **ii** \$56 000 **iii** \$66 600 **b** the mean

7 **a** $x = 5$ **b** 75%

EXERCISE 13D

1 **a** 40 phone calls **b** ≈ 15 minutes **2** ≈ 31.7

3 **a** 70 service stations **b** ≈ 411 000 litres (≈ 411 kL)
c ≈ 5870 L
d $6000 < P \leqslant 7000$ L. This is the most frequently occurring amount of petrol sales at a service station in one day.

4 **a**

Runs scored	*Tally*	*Frequency*
0 - 9	𝍸 𝍸 |	11
10 - 19	𝍸 |||	8
20 - 29	𝍸 |||	8
30 - 39	||	2
	Total	29

b ≈ 14.8 runs
c ≈ 14.9 runs; the estimate in **b** was very accurate.

5 **a** $p = 24$ **b** ≈ 3.37 minutes **c** ≈ 15.3%

6 **a** 125 people **b** ≈ 119 marks **c** $\frac{3}{25}$ **d** 28%

EXERCISE 13E

1 **a** **i** 13 **ii** $Q_1 = 9$, $Q_3 = 18$ **iii** 16 **iv** 9
b **i** 18.5 **ii** $Q_1 = 13$, $Q_3 = 23$ **iii** 19 **iv** 10
c **i** 26.5 **ii** $Q_1 = 20$, $Q_3 = 35$ **iii** 28 **iv** 15
d **i** 37 **ii** $Q_1 = 28$, $Q_3 = 52$ **iii** 49 **iv** 24

2 **a** *Jane*: mean = \$35.50, median = \$35.50
Ashley: mean = \$30.75, median = \$26.00
b *Jane*: range = \$18, IQR = \$9
Ashley: range = \$40, IQR = \$14
c Jane **d** Ashley

3 **a** range = 60, IQR = 8.5 **b** '67' is an outlier.
c range = 18, IQR = 8 **d** the range

4 **a** *Derrick*: range = 240 minutes, IQR = 30 minutes
Gareth: range = 170 minutes, IQR = 120 minutes
b **i** Gareth's **ii** Derrick's
c The IQR is most appropriate as it is less affected by outliers.

5 **a** g **b** **i** $m - a$ **ii** $\left(\frac{j+k}{2}\right) - \left(\frac{c+d}{2}\right)$

6

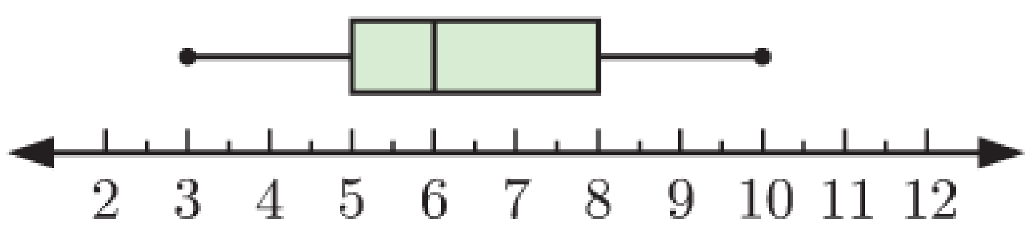
	Measure	median	mode	range	interquartile range
a	Value	11	9	13	6
b		18	14	26	12

EXERCISE 13F

1 **a** 35 points **b** 78 points **c** 13 points **d** 53 points
e 26 points **f** 65 points **g** 27 points

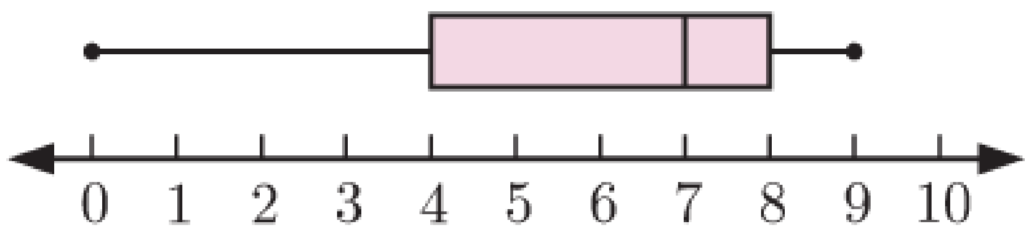
2 **a** **i** 98, 25 marks **ii** 70 marks **iii** 85 marks
iv 55, 85 marks
b range = 73, IQR = 30

3 **a** **i** min = 3, $Q_1 = 5$, med = 6, $Q_3 = 8$, max = 10
ii



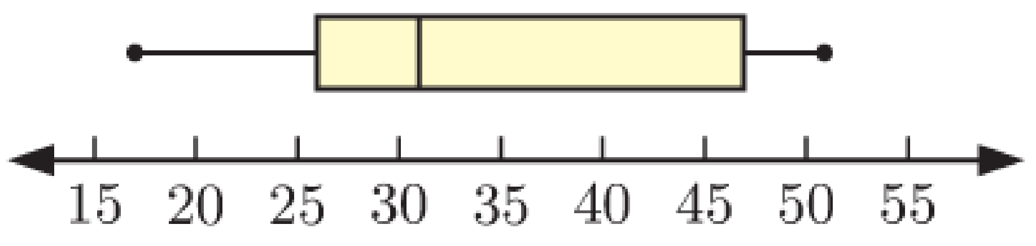
iii range = 7, IQR = 3

b **i** min = 0, $Q_1 = 4$, med = 7, $Q_3 = 8$, max = 9
ii



iii range = 9, IQR = 4

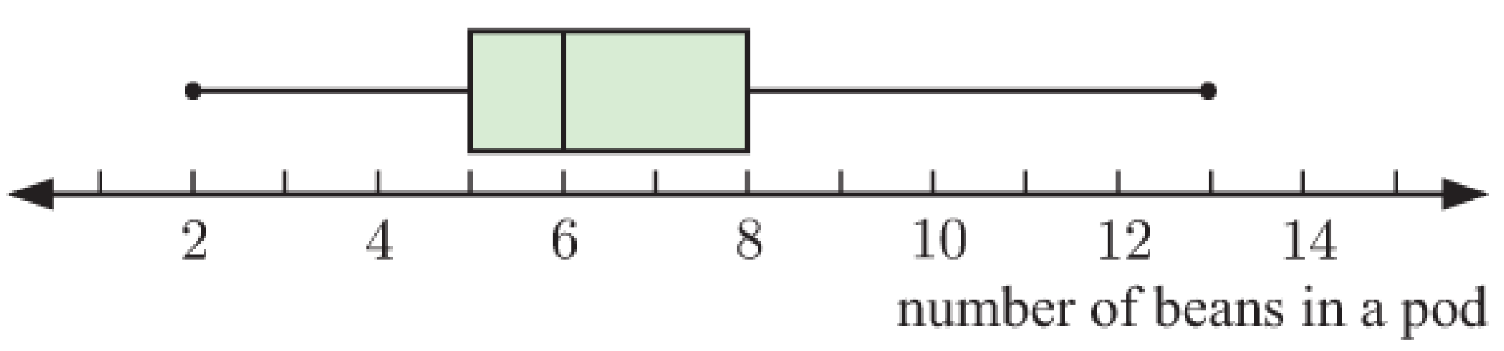
c **i** min = 17, $Q_1 = 26$, med = 31, $Q_3 = 47$, max = 51
ii



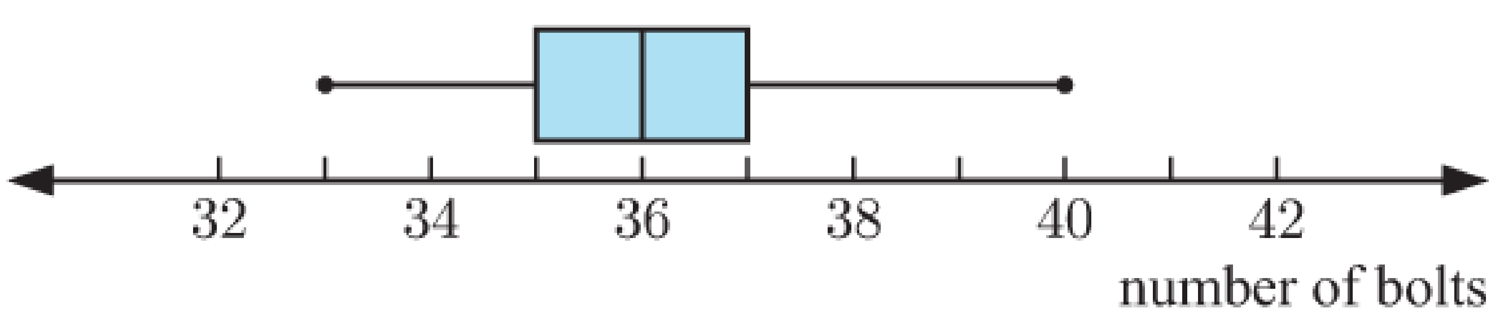
iii range = 34, IQR = 21

4 **a** median = 6, $Q_1 = 5$, $Q_3 = 8$ **b** IQR = 3

c



5 **a**

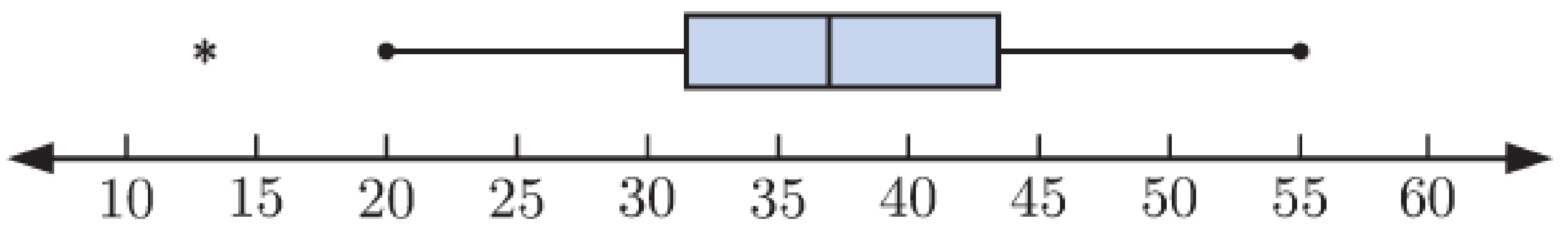


b range = 7, IQR = 2

EXERCISE 13G

1 **a** 12 **b** lower = 13.5, upper = 61.5 **c** 13

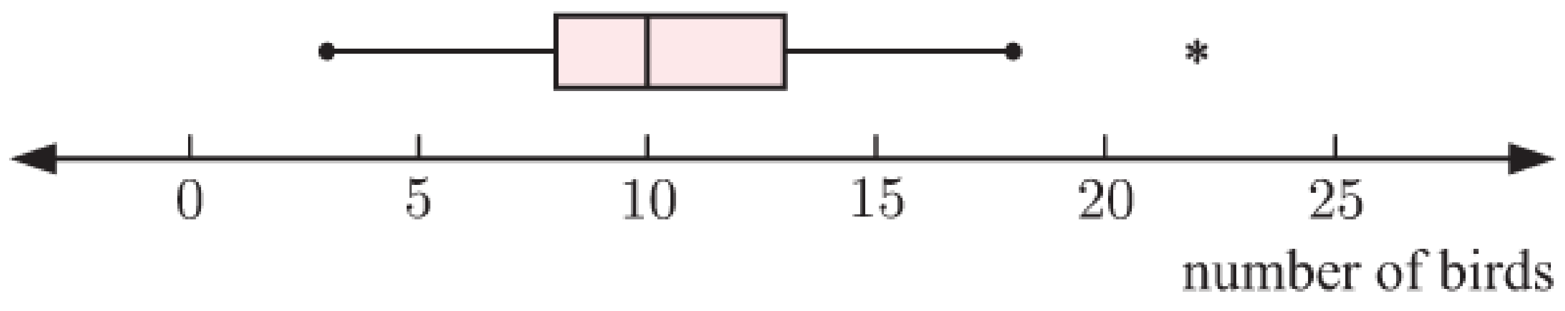
d



2 **a** median = 10, $Q_1 = 8$, $Q_3 = 13$ **b** IQR = 5

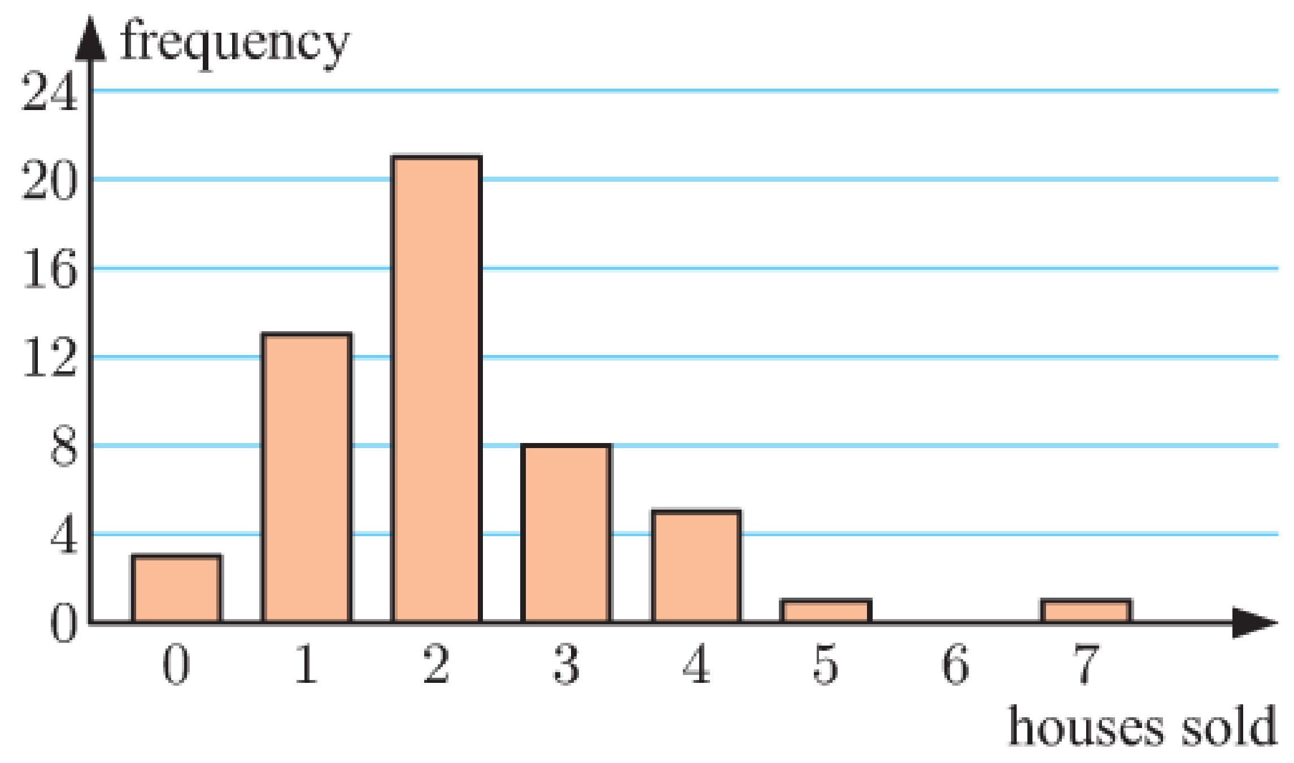
c lower = 0.5, upper = 20.5, 22 is an outlier

d



3 **a** **A** **b** **D** **c** **C** **d** **B**

4 **a** **Houses sold by a real estate agent**

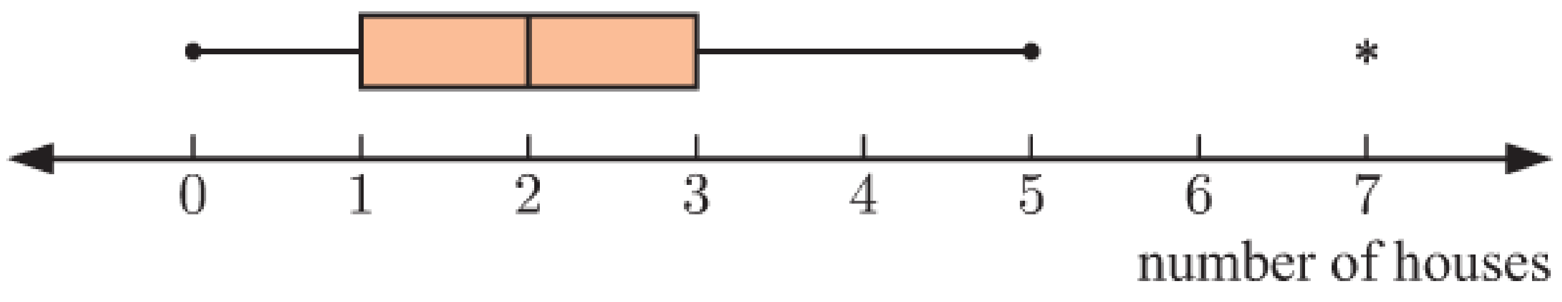


b 7 houses appears to be an outlier.

c lower boundary = −2, upper boundary = 6

7 houses is an outlier

d



EXERCISE 13H

1 **a**

Statistic	*Year 9*	*Year 12*
minimum	6	36
Q_1	30	60
median	45	84
Q_3	60	96
maximum	72	105

b **i** Year 9: 66 min

Year 12: 69 min

ii Year 9: 30 min

Year 12: 36 min

c **i** cannot tell **ii** true, since Year 9 $Q_1 <$ Year 12 min

2 **a** Friday: min = €20, Q_1 = €50, med = €70,
Q_3 = €100, max = €180

Saturday: min = €40, Q_1 = €80, med = €100,
Q_3 = €140, max = €200

b **i** Friday: €160, Saturday: €160

ii Friday: €50, Saturday: €60

3 **a** **i** class 1 (96%) **ii** class 1 (37%) **iii** class 1

b 18% **c** 55% **d** **i** 25% **ii** 50%

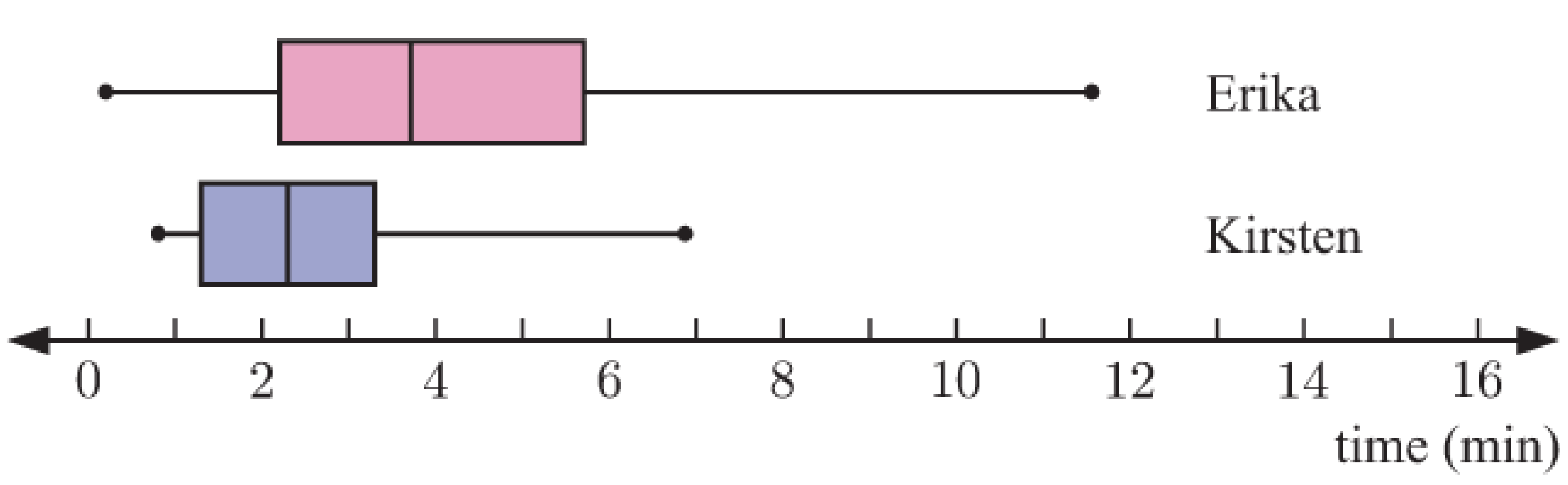
e **i** slightly positively skewed **ii** negatively skewed

f .... class 2, ... class 1

4 **a** Kirsten: min = 0.8 min, Q_1 = 1.3 min, med = 2.3 min,
Q_3 = 3.3 min, max = 6.9 min

Erika: min = 0.2 min, Q_1 = 2.2 min, med = 3.7 min,
Q_3 = 5.7 min, max = 11.5 min

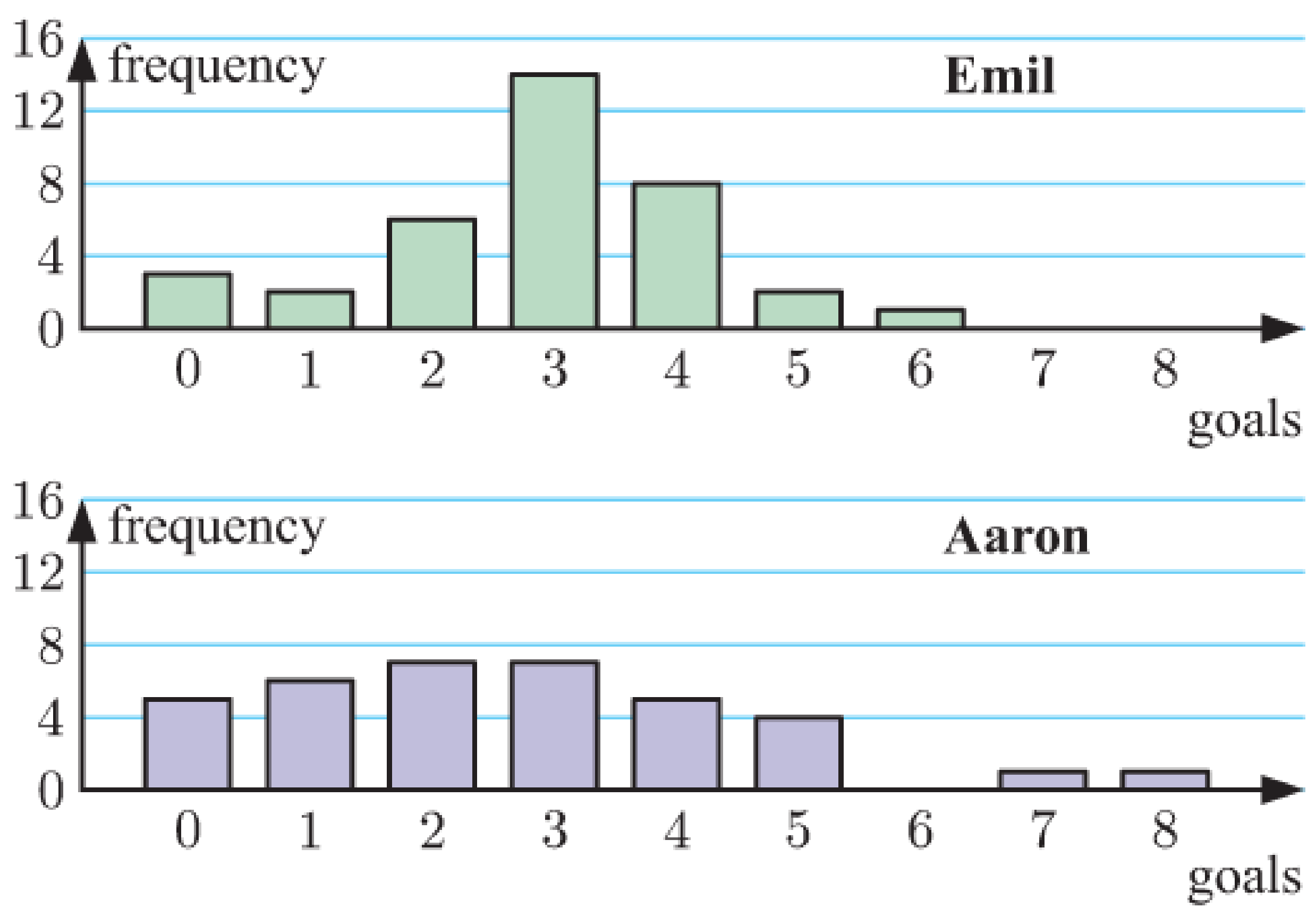
b **Phone call duration**



c Both are positively skewed (Erika's more so than Kirsten's). Erika's phone calls were more varied in duration.

5 **a** discrete

c



d Emil: approximately symmetrical

Aaron: positively skewed

e Emil: mean ≈ 2.89, median = 3, mode = 3

Aaron: mean ≈ 2.67, median = 2.5, mode = 2, 3

Emil's mean and median are slightly higher than Aaron's.

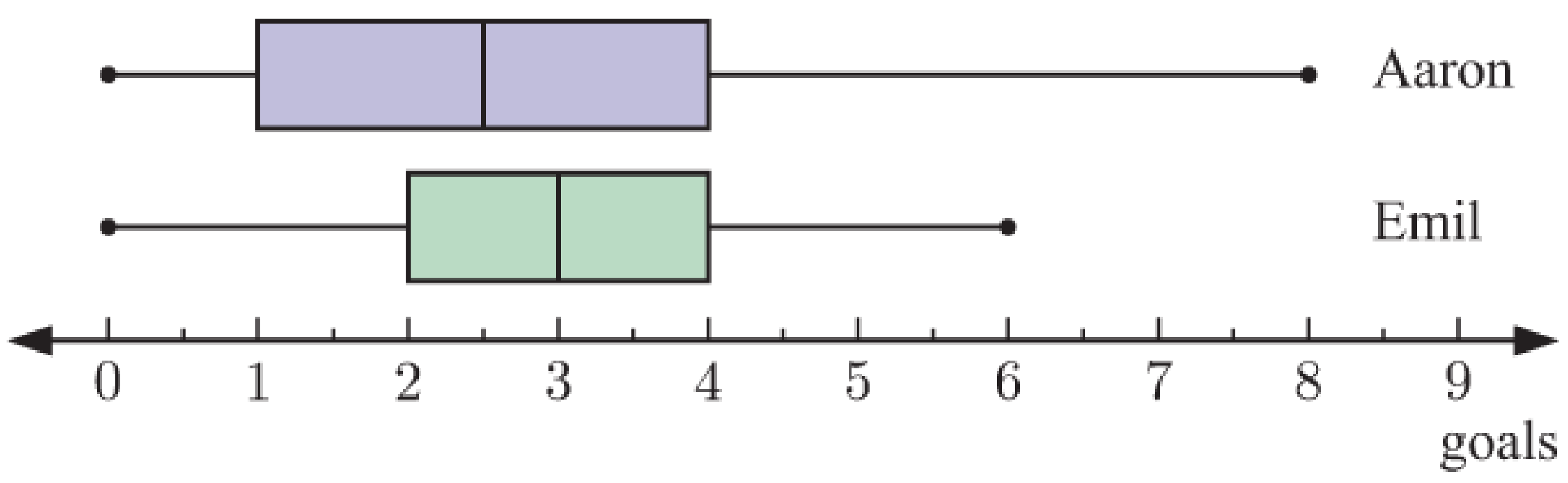
Emil has a clear mode of 3, whereas Aaron has two modes (2 and 3).

f Emil: range = 6, IQR = 2

Aaron: range = 8, IQR = 3

Emil's data set demonstrates less variability than Aaron's.

g



h Emil is more consistent with his scoring (in terms of goals) than Aaron.

6 **a** continuous (the data is measured)

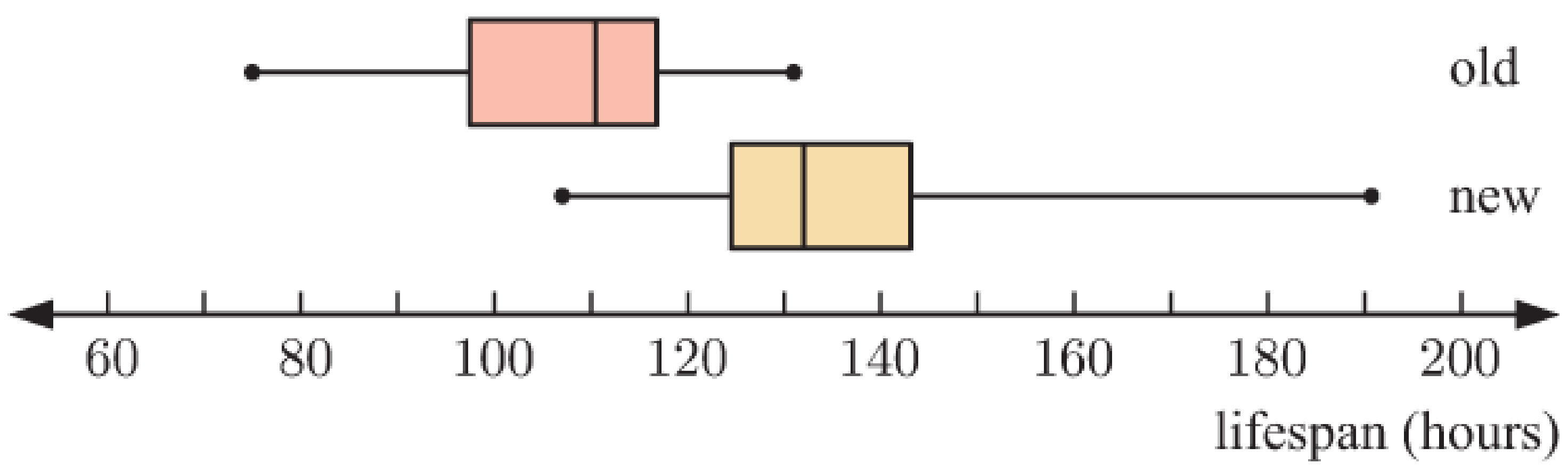
b Old type: mean = 107 hours, median = 110.5 hours,
range = 56 hours, IQR = 19 hours

New type: mean = 134 hours, median = 132 hours,
range = 84 hours, IQR = 18.5 hours

The "new" type of light globe has a higher mean and median than the "old" type.

The IQR is relatively unchanged going from "old" to "new", however, the range of the "new" type is greater, suggesting greater variability.

c



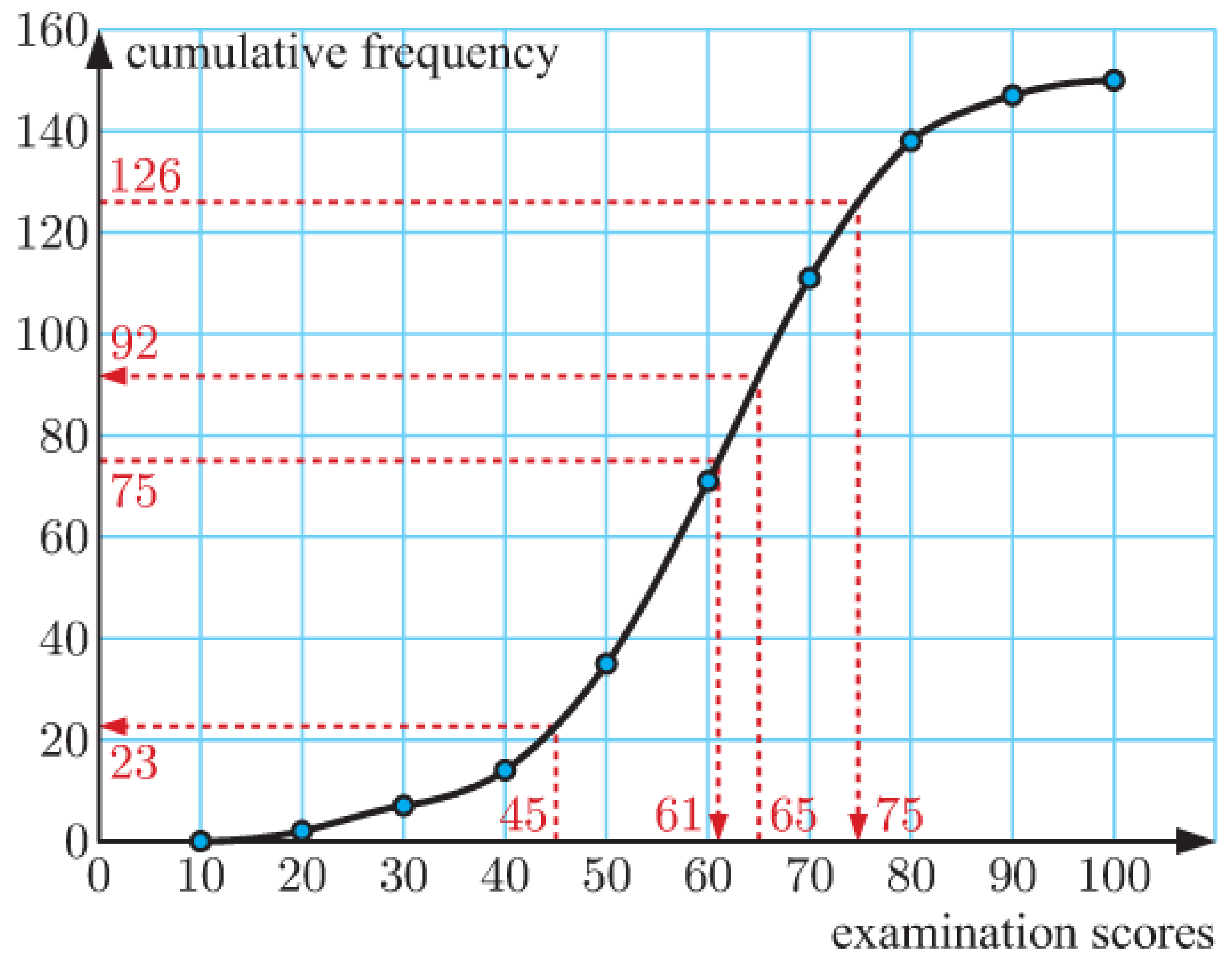
d Old type: negatively skewed

New type: positively skewed

e The "new" type of light globes do last longer than the "old" type. From **c**, both the mean and median for the "new" type are close to 20% greater than that of the "old" type. The manufacturer's claim appears to be valid.

EXERCISE 13I

1 **a**



b ≈ 61 marks **c** ≈ 92 students **d** 76 students

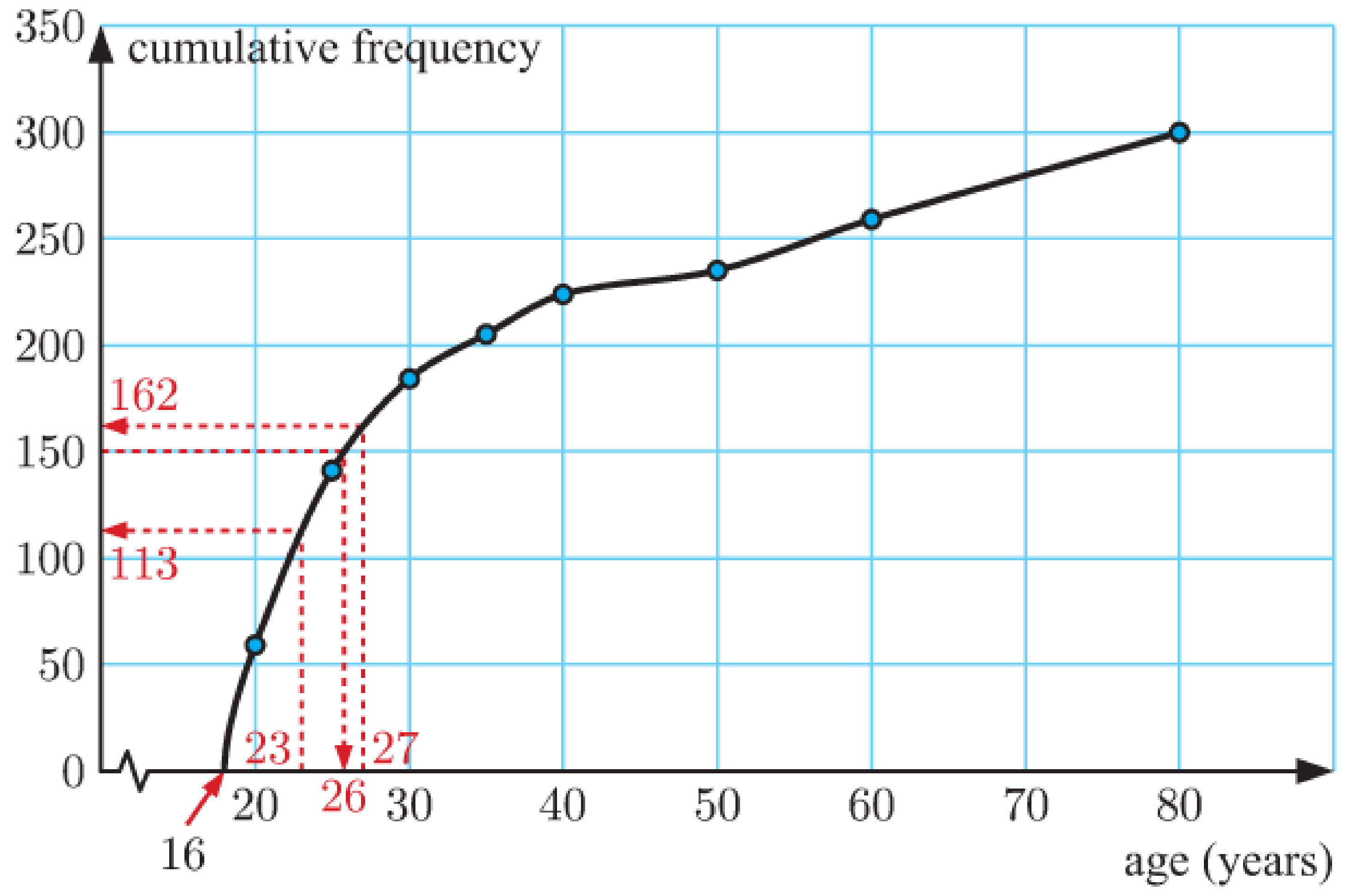
e ≈ 23 students **f** ≈ 75 marks

2 **a** ≈ 9 seedlings **b** $\approx 28.3\%$ **c** ≈ 7.1 cm

d ≈ 2.4 cm

e 10 cm, which means that 90% of the seedlings are shorter than 10 cm.

3 **a**

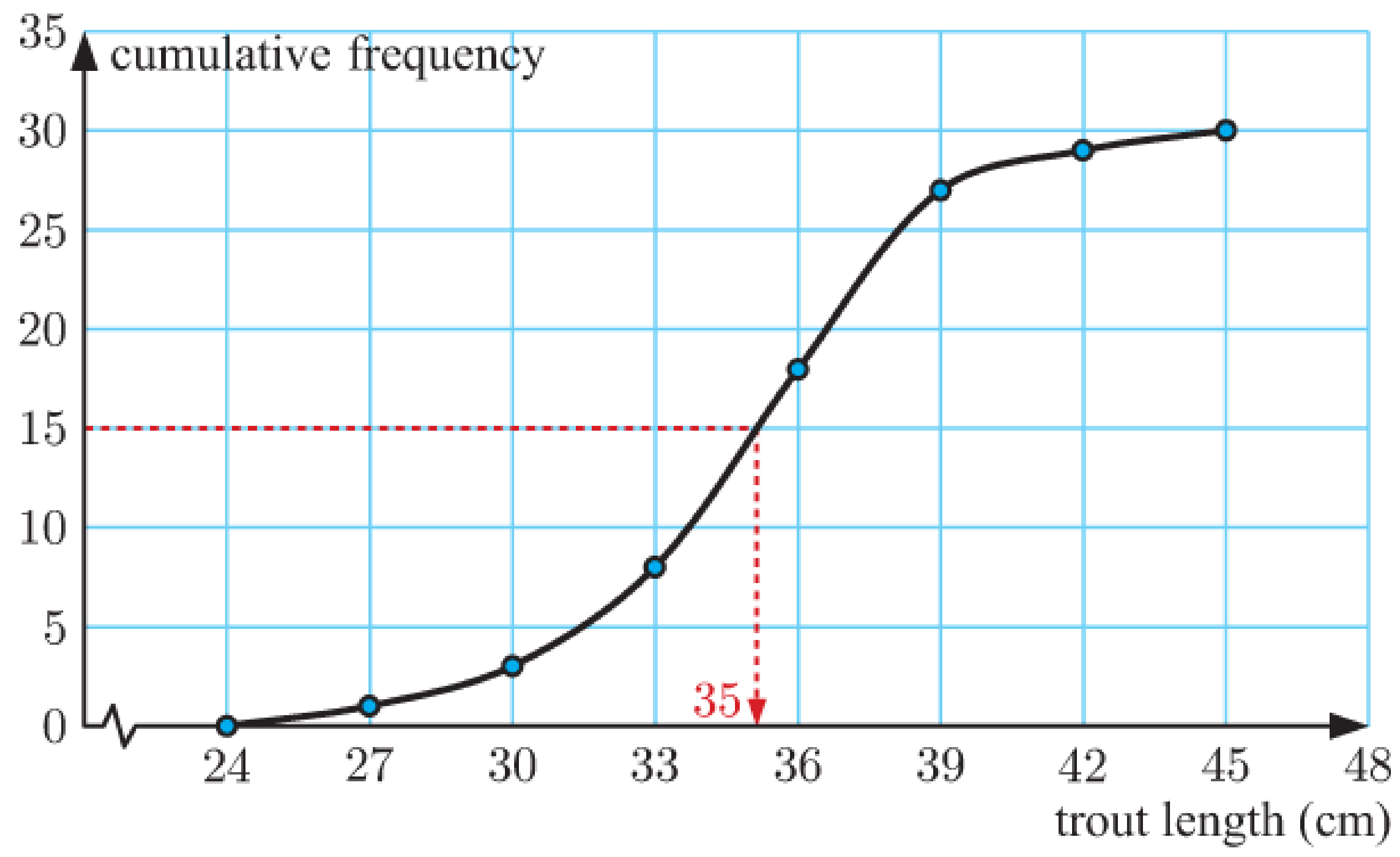


b ≈ 26 years **c** $\approx 37.7\%$ **d** **i** ≈ 0.54 **ii** ≈ 0.04

4 **a**

Length (cm)	*Frequency*	*Cumulative frequency*
$24 \leqslant x < 27$	1	1
$27 \leqslant x < 30$	2	3
$30 \leqslant x < 33$	5	8
$33 \leqslant x < 36$	10	18
$36 \leqslant x < 39$	9	27
$39 \leqslant x < 42$	2	29
$42 \leqslant x < 45$	1	30

b



c median ≈ 35 cm

d median $= 34.5$ cm; the median found from the graph is a good approximation.

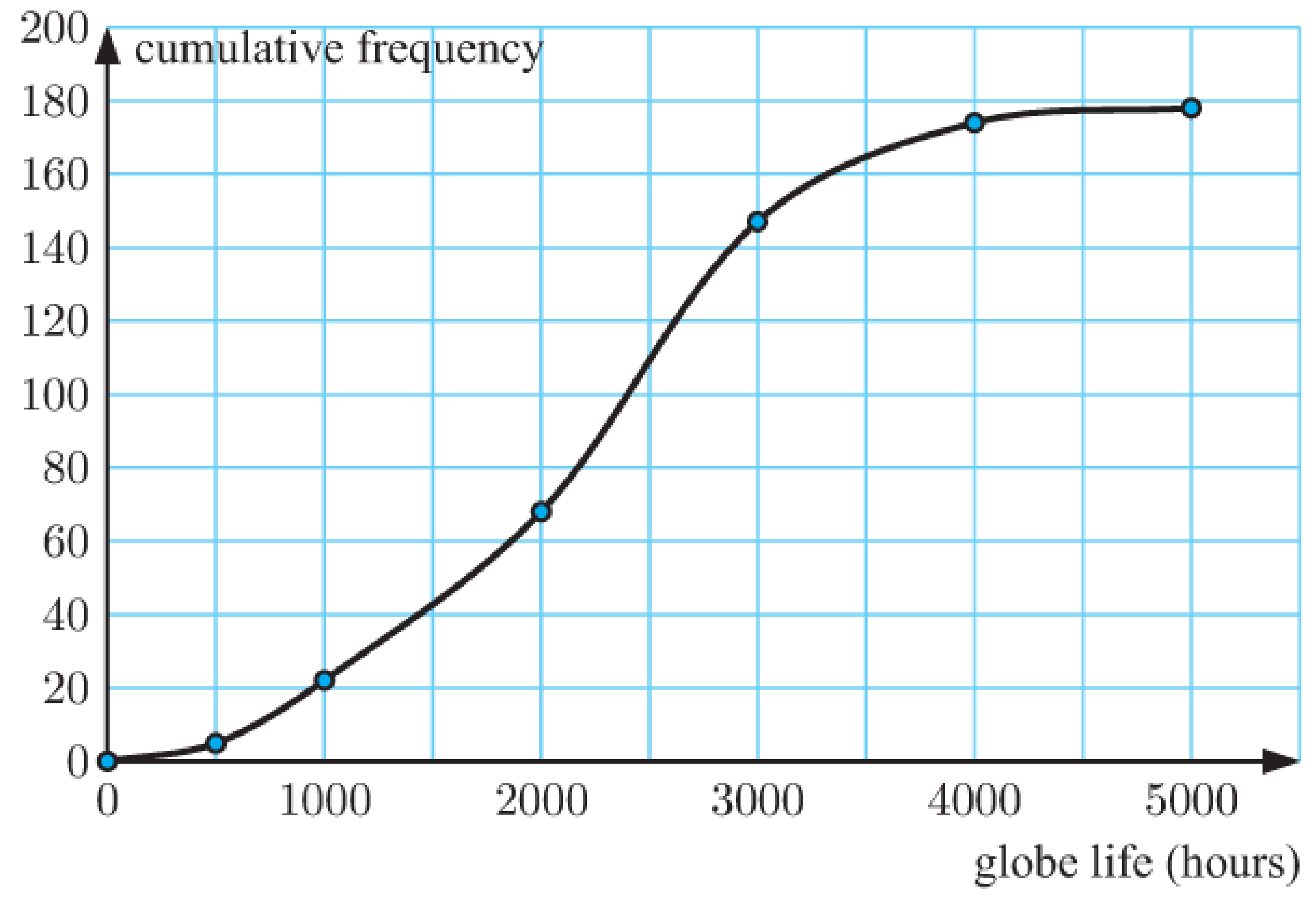
5 **a** ≈ 27 min **b** ≈ 29 min **c** ≈ 31.3 min

d ≈ 4.3 min **e** ≈ 28.2 min

f

Time (t min)	$21 \leqslant t < 24$	$24 \leqslant t < 27$	$27 \leqslant t < 30$	$30 \leqslant t < 33$	$33 \leqslant t < 36$
Number of competitors	5	15	30	20	10

6 **a**



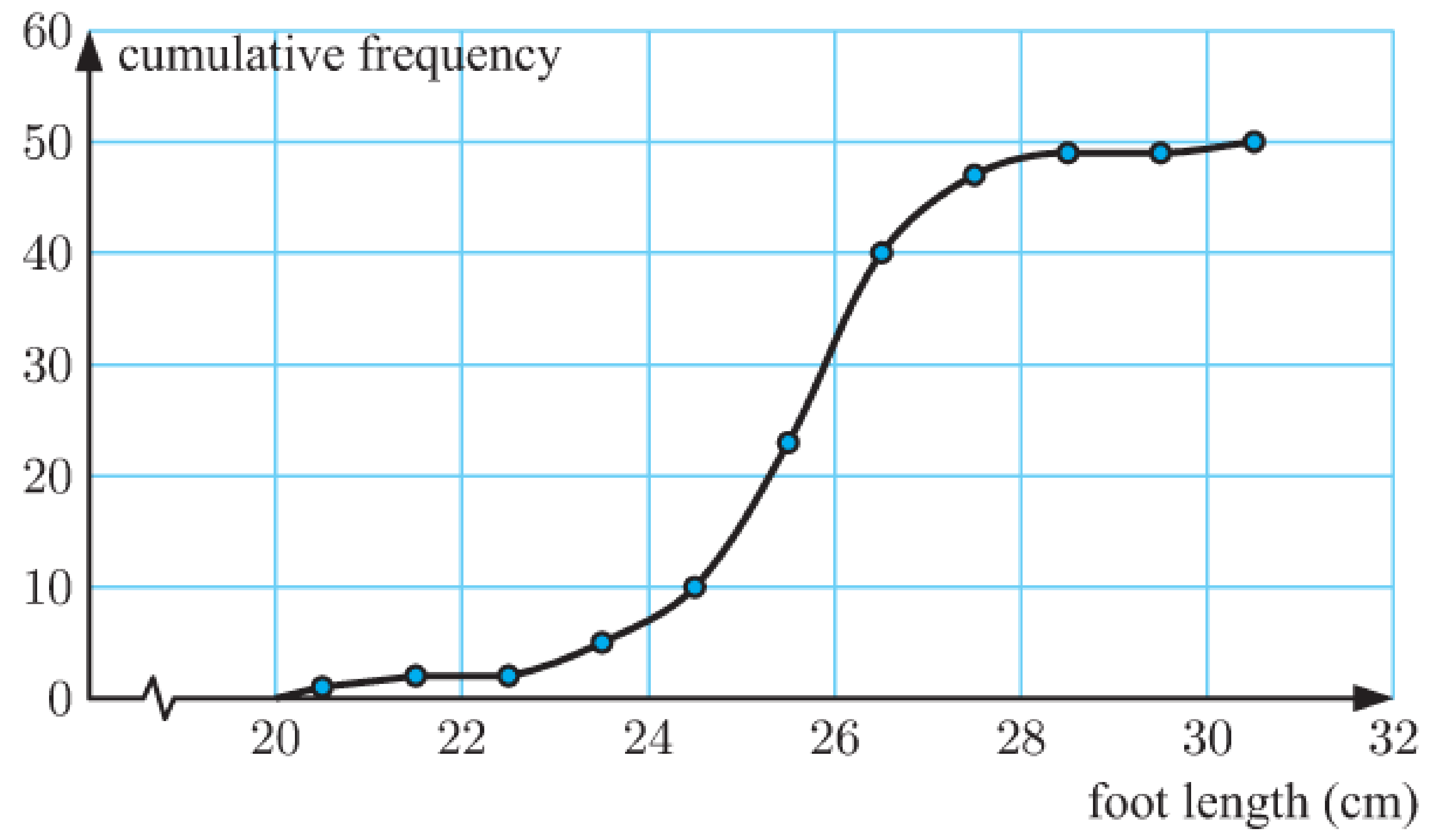
b ≈ 2280 hours **c** $\approx 71\%$ **d** ≈ 67

7 **a** $19.5 \leqslant l < 20.5$ cm

b

Foot length (cm)	*Frequency*	*Cumulative frequency*
$19.5 \leqslant l < 20.5$	1	1
$20.5 \leqslant l < 21.5$	1	2
$21.5 \leqslant l < 22.5$	0	2
$22.5 \leqslant l < 23.5$	3	5
$23.5 \leqslant l < 24.5$	5	10
$24.5 \leqslant l < 25.5$	13	23
$25.5 \leqslant l < 26.5$	17	40
$26.5 \leqslant l < 27.5$	7	47
$27.5 \leqslant l < 28.5$	2	49
$28.5 \leqslant l < 29.5$	0	49
$29.5 \leqslant l < 30.5$	1	50

c



d **i** ≈ 25.2 cm **ii** ≈ 18 people

EXERCISE 13J

1 **a** *Data set A*: mean $= \dfrac{10+7+5+8+10}{5} = 8$

Data set B: mean $= \dfrac{4+12+11+14+1+6}{6} = 8$

b Data set B appears to have a greater spread than data set A, as data set B has more values that are a long way from the mean, such as 1 and 14.

c *Data set A*: $\sigma^2 = 3.6$, $\sigma \approx 1.90$
Data set B: $\sigma^2 \approx 21.7$, $\sigma \approx 4.65$

2 **a** The data is positively skewed. **b** $\sigma \approx 1.59$
c $\sigma^2 \approx 2.54$

3 **a** $\mu = 24.25$, $\sigma \approx 3.07$ **b** $\mu = 28.25$, $\sigma \approx 3.07$
c If each data value is increased or decreased by the same amount, then the mean will also be increased or decreased by that amount, however the population standard deviation will be unchanged.

4 $\sigma \approx 2.64$, $s \approx 2.71$

5 **a** *Danny*: ≈ 3.21 hours; *Jennifer*: 2 hours
b Danny
c *Danny*: $\sigma \approx 0.700$ hours, $s \approx 0.726$ hours;
Jennifer: $\sigma \approx 0.423$ hours, $s \approx 0.439$ hours
d Jennifer

6 **a**

			Standard deviation		
	Mean $\bar{x}$	*Median*	σ	s	*Range*
Boys	32.02	31.05	≈ 4.52	≈ 4.77	13.8
Girls	34.77	35.85	≈ 3.76	≈ 3.96	11.7

b **i** boys **ii** boys
c Tyson could increase his sample size.

7 **a** *Rockets*: mean $= 5.7$, range $= 11$
Bullets: mean $= 5.7$, range $= 11$
b We suspect the Rockets, since they twice scored zero runs.
Rockets: $\sigma = 3.9$, $s \approx 4.11$ $\leftarrow$ greater variability
Bullets: $\sigma \approx 3.29$, $s \approx 3.47$
c standard deviation

8 **a** **i** *Museum*: ≈ 934 visitors; *Art gallery*: ≈ 1230 visitors
ii *Museum*: $\sigma \approx 208$ visitors, $s \approx 211$ visitors;
Art gallery: $\sigma \approx 84.6$ visitors, $s \approx 86.0$ visitors
b the museum
c **i** '0' is an outlier.
ii This outlier corresponded to Christmas Day, so the museum was probably closed which meant there were no visitors on that day.
iii Yes, although the outlier is not an error, it is not a true reflection of a visitor count for a particular day.
iv *Museum*: mean ≈ 965 visitors, $\sigma \approx 121$ visitors, $s \approx 123$ visitors
v The outlier had greatly increased the population standard deviation.

9 $s_A > s_B$ does not imply that $\sigma_A > \sigma_B$.
Hint: Find a counter example.

10 $p = 6$, $q = 9$ **11** $a = 8$, $b = 6$ **12** **b** $\mu = \pm 8.7$

13 $\sigma \approx 0.775$ **14** $\mu = 14.48$ years, $\sigma \approx 1.75$ years

15 **a** Data set A **b** *Data set A*: 8, *Data set B*: 8
c *Data set A*: 2, *Data set B*: ≈ 1.06
Data set A does have a wider spread.
d The standard deviation takes all of the data values into account, not just two.

16 **a** The female students' marks are in the range 16 to 20 whereas the male students' marks are in the range 12 to 19.
i the females **ii** the males
b *Females*: $\mu \approx 17.5$, $\sigma \approx 1.02$
Males: $\mu \approx 15.5$, $\sigma \approx 1.65$

17 The results for the mean will differ by 1, but the results for the standard deviation will be the same. Jess' question is worded so that the respondent will not include themselves.

18 **a** $\bar{x} \approx 48.3$ cm **b** $\sigma \approx 2.66$ cm, $s \approx 2.70$ cm

19 **a** $\bar{x} \approx 17.45$ **b** $\sigma \approx 7.87$, $s \approx 7.91$

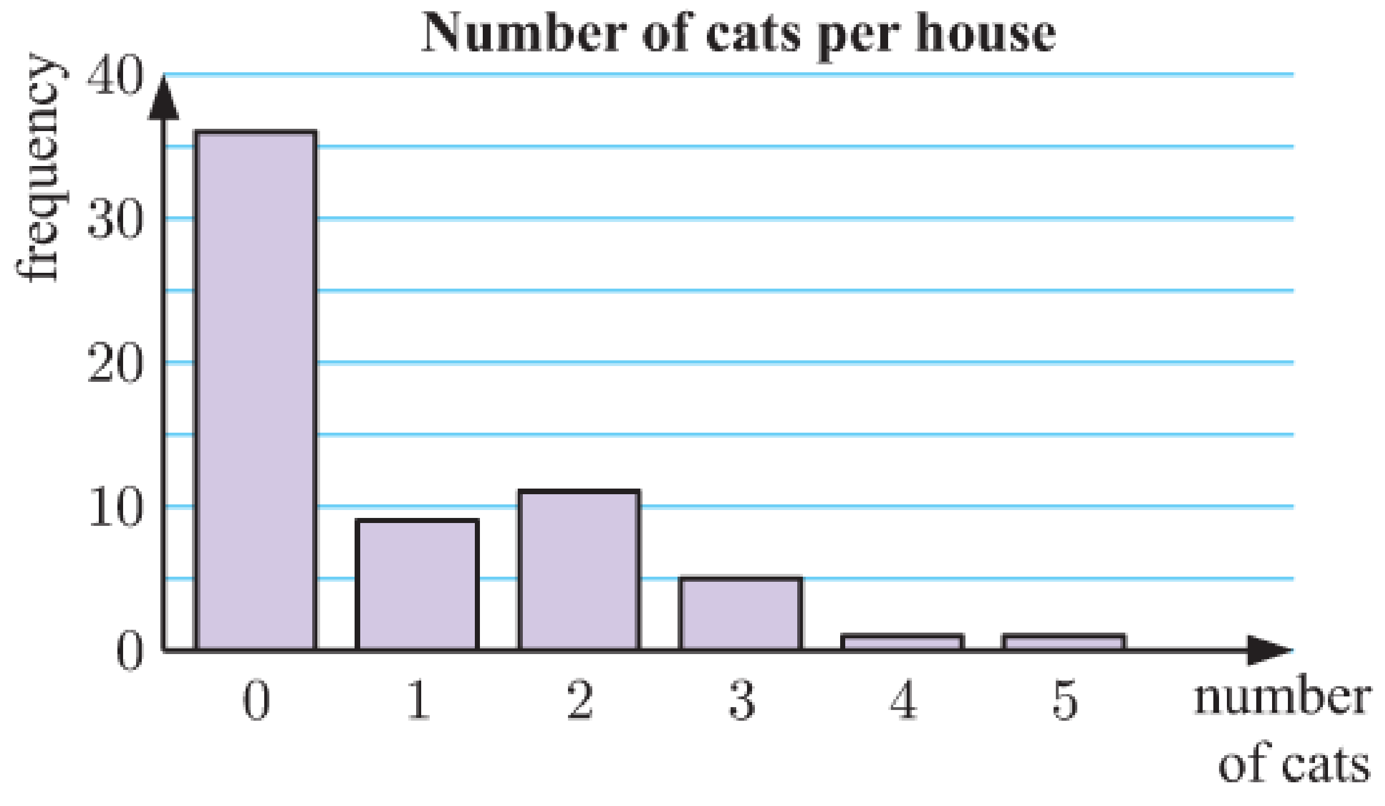
20 **a** $\bar{x} \approx \$780.60$ **b** $\sigma \approx \$31.74$, $s \approx \$31.82$

21 **a** $\bar{x} = 40.35$ hours, $\sigma \approx 4.23$ hours, $s \approx 4.28$ hours
b $\bar{x} = 40.6$ hours, $\sigma \approx 4.10$ hours, $s \approx 4.15$ hours
The mean increases slightly; the standard deviation decreases slightly. These are good approximations.

REVIEW SET 13A

1 **a** **i** ≈ 4.67 **ii** 5 **b** **i** 3.99 **ii** 3.9

2 **a**



b positively skewed
c **i** 0 cats **ii** ≈ 0.873 cats **iii** 0 cats
d The mean, as it suggests that some people have cats. (The mode and median are both 0.)

3 **a**

Distribution	*Girls*	*Boys*
median	36 s	34.5 s
mean	36 s	34.45 s
modal class	34.5 - 35.5 s	34.5 - 35.5 s

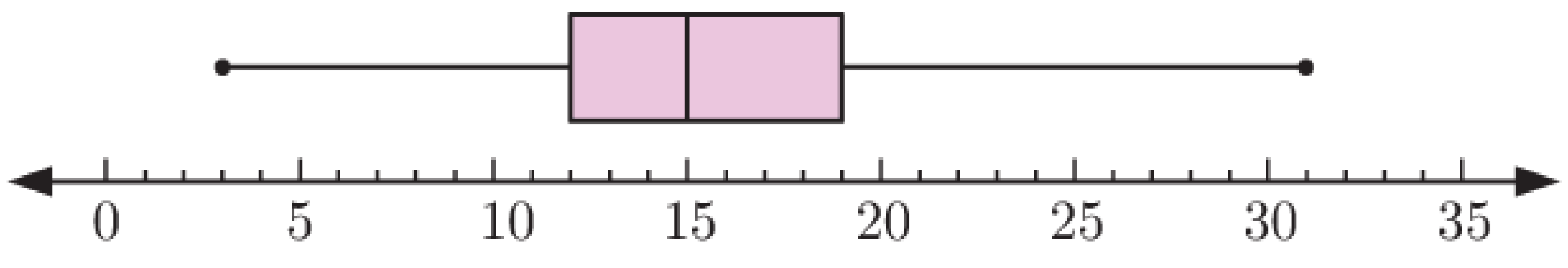
b The girls' distribution is positively skewed and the boys' distribution is approximately symmetrical. The median and mean swim times for boys are both about 1.5 seconds lower than for girls. Despite this, the distributions have the same modal class because of the skewness in the girls' distribution. The analysis supports the conjecture that boys generally swim faster than girls with less spread of times.

4 $a = 8$, $b = 6$

5 **b** $k + 3$

6 **a** We do not know each individual data value, only the intervals they fall in, so we cannot calculate the mean winning margin exactly.
b ≈ 22.6 points

7 **a** min $= 3$, $Q_1 = 12$, med $= 15$, $Q_3 = 19$, max $= 31$
b range $= 28$, IQR $= 7$
c



8 **a** 101.5 **b** 7.5 **c** 100.2 **d** ≈ 7.59

9 **a** A: min $= 11$ s, $Q_1 = 11.6$ s, med $= 12$ s, $Q_3 = 12.6$ s, max $= 13$ s
B: min $= 11.2$ s, $Q_1 = 12$ s, med $= 12.6$ s, $Q_3 = 13.2$ s, max $= 13.8$ s
b A: range $= 2.0$ s, IQR $= 1.0$ s
B: range $= 2.6$ s, IQR $= 1.2$ s
c **i** A, the median time is lower.
ii B, the range and IQR are higher.

10 **a** ≈ 58.5 s **b** ≈ 6 s **c** ≈ 53 s

11 **a** ≈ 88 students

b $m \approx 24$

c

Time (t min)	*Frequency*
$5 \leqslant t < 10$	20
$10 \leqslant t < 15$	40
$15 \leqslant t < 20$	48
$20 \leqslant t < 25$	42
$25 \leqslant t < 30$	28
$30 \leqslant t < 35$	17
$35 \leqslant t < 40$	5

12 **a** $\sigma^2 \approx 63.0$, $\sigma \approx 7.94$ **b** $\sigma^2 \approx 0.969$, $\sigma \approx 0.984$

13 **a** $\overline{x} \approx 49.6$ matches, $\sigma \approx 1.60$ matches, $s \approx 1.60$ matches

b The claim is not justified, but a larger sample is needed.

14 **a** $\overline{x} \approx 33.6$ L **b** $\sigma \approx 7.63$ L, $s \approx 7.66$ L

15 **a** No, extreme values have less effect on the standard deviation of a larger population.

b **i** mean **ii** standard deviation

c A low standard deviation means that the weight of biscuits in each packet is, on average, close to 250 g.

REVIEW SET 13B

1 **a**

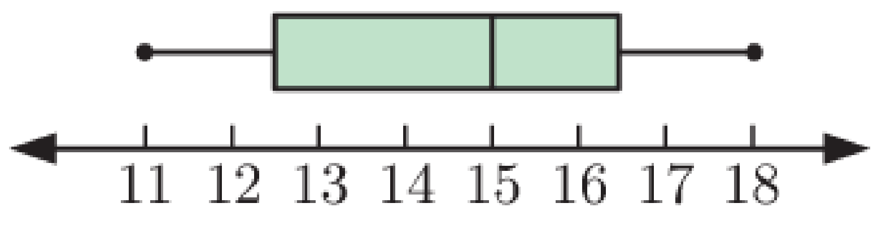
	mean (seconds)	median (seconds)
Week 1	≈ 16.0	16.3
Week 2	≈ 15.1	15.1
Week 3	≈ 14.4	14.3
Week 4	14.0	14.0

b Yes, Heike's mean and median times have gradually decreased each week which indicates that her speed has improved over the 4 week period.

2 **a** 5 **b** 3.52 **c** 3.5 **3** **a** $x = 7$ **b** 6

4 $p = 7$, $q = 9$ (or $p = 9$, $q = 7$) **5** ≈ 414 patrons

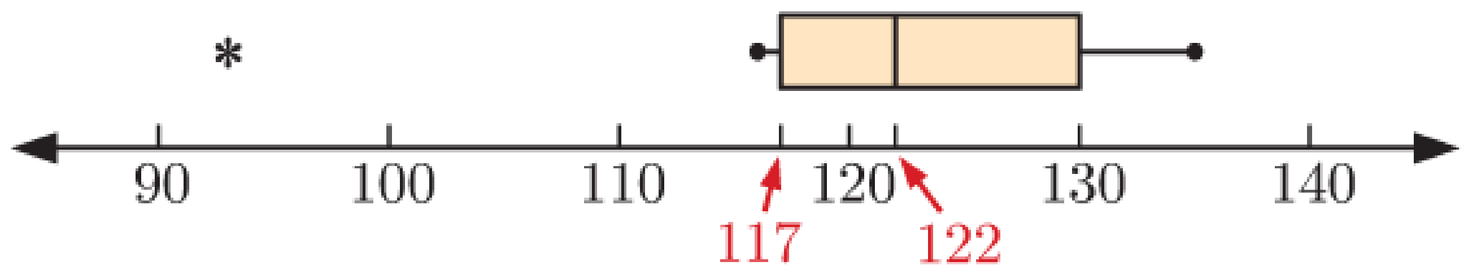
6



7 **a** $\sigma \approx 11.7$, $s \approx 12.4$ **b** $Q_1 = 117$, $Q_3 = 130$

c yes, 93

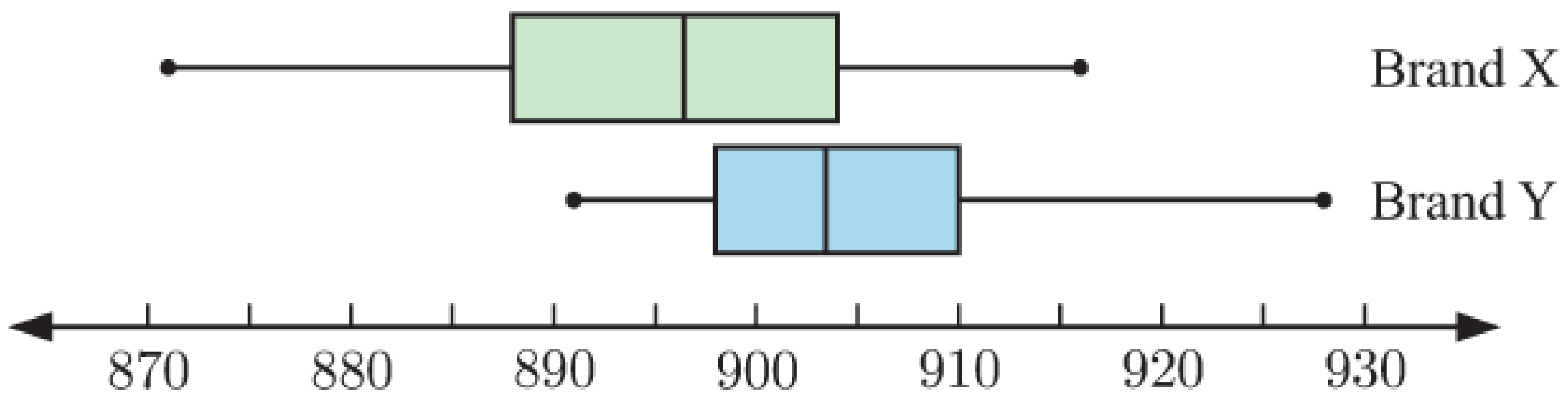
d



8 **a**

	Brand X	*Brand Y*
min	871	891
Q_1	888	898
median	896.5	903.5
Q_3	904	910
max	916	928
IQR	16	12

b



c **i** Brand Y, as the median is higher.

ii Brand Y, as the IQR is lower, so less variation.

9 **a** $p = 12$, $m = 6$

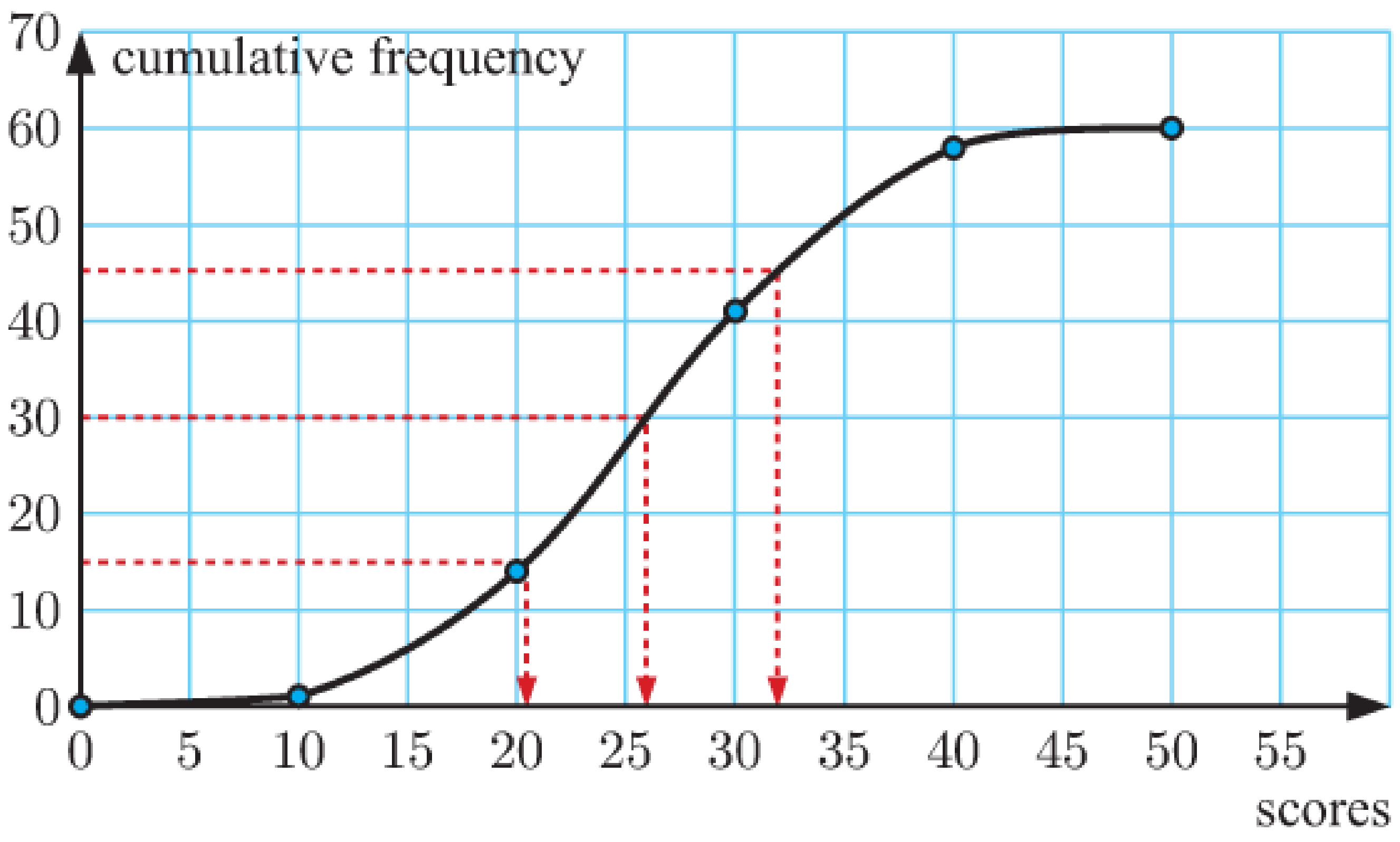
b

Measure	*Value*
mode	9
median	9
range	4

c $\overline{x} = \frac{254}{30} = \frac{127}{15}$

10 **a** ≈ 77 days **b** ≈ 12 days

11 **a**

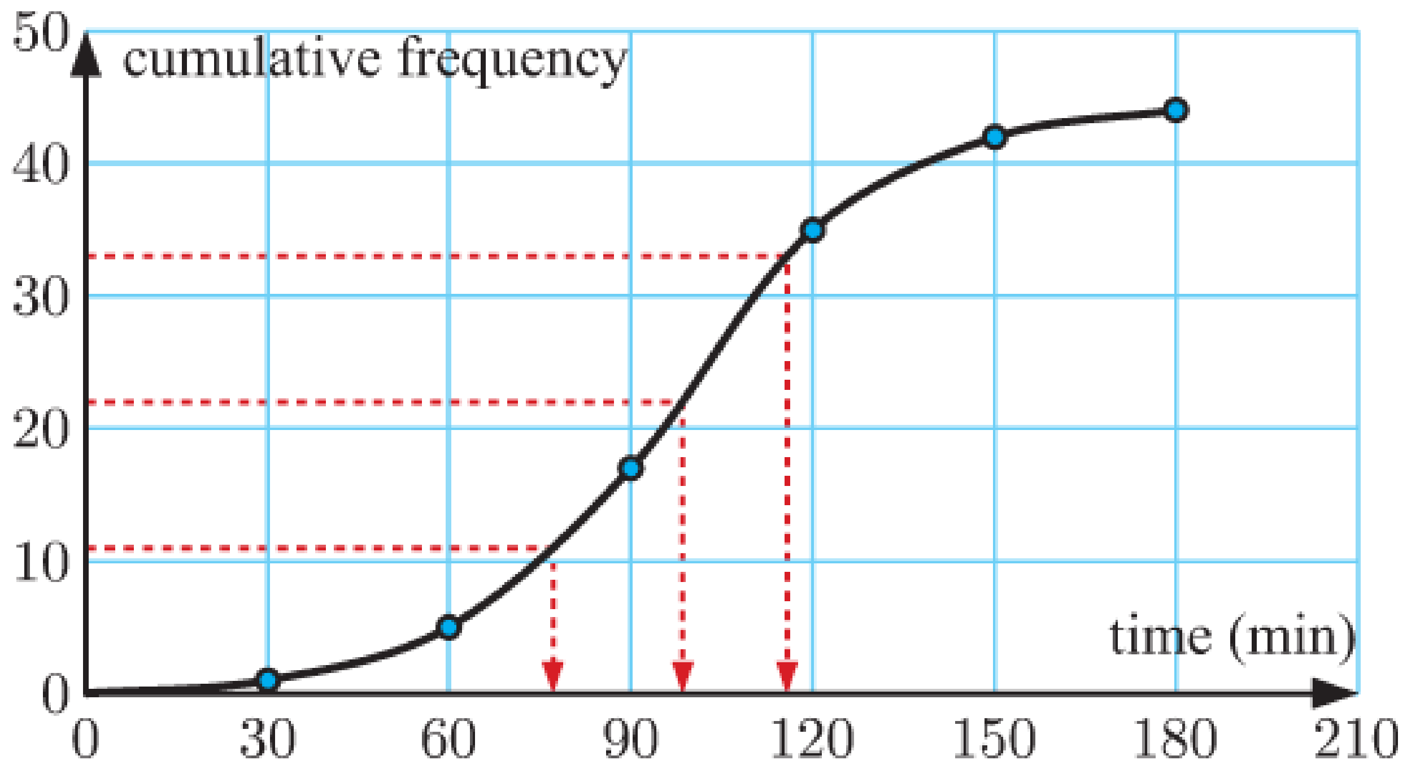


b **i** median ≈ 26 **ii** IQR ≈ 11.5

iii $\overline{x} \approx 26.0$ **iv** $\sigma \approx 8.31$

12 **a** 44 players **b** $90 \leqslant t < 120$ min

c



d **i** ≈ 98.6 min **ii** ≈ 96.8 min **iii** no

e ".... between 77.2 and 115.7 minutes."

13 **a** $\overline{x} \approx$ €207.02 **b** $\sigma =$ €38.80, $s \approx$ €38.89

14 **a** Kevin: $\overline{x} = 41.2$ min; Felicity: $\overline{x} = 39.5$ min

b Kevin: $\sigma \approx 7.61$ min, $s \approx 7.81$ min;
Felicity: $\sigma \approx 9.22$ min, $s \approx 9.46$ min

c Felicity **d** Kevin

15 10 data values

EXERCISE 14A

1 **a** $y = x^2 - 3x + 1$

x	-2	-1	0	1	2
y	11	5	1	-1	-1

b $y = x^2 + 2x - 5$

x	-2	-1	0	1	2
y	-5	-6	-5	-2	3

c $y = 2x^2 - x + 3$

x	-4	-2	0	2	4
y	39	13	3	9	31

d $y = -3x^2 + 2x + 4$

x	-4	-2	0	2	4
y	-52	-12	4	-4	-36

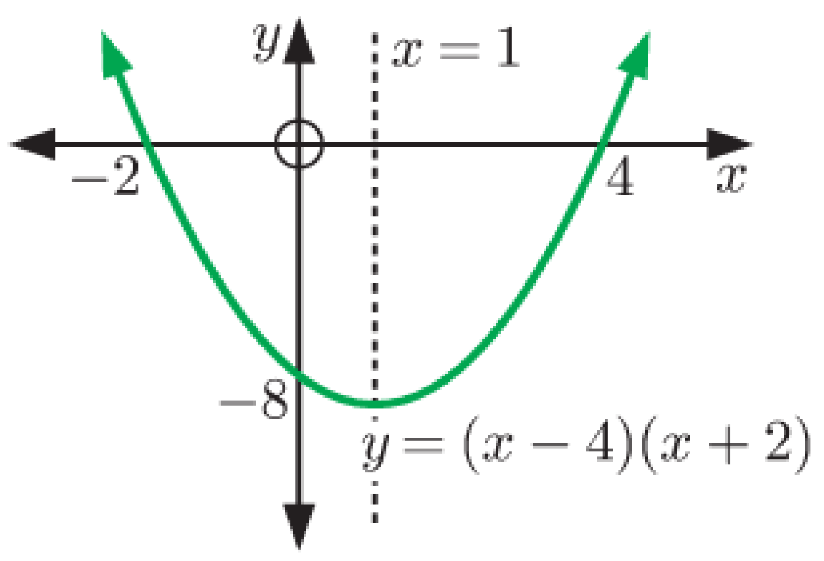
2 **a** no **b** yes **c** yes **d** yes **e** no **f** yes

3 **a** $x = -1$ or -2 **b** $x = 2$ **c** $x = 1$ or 5

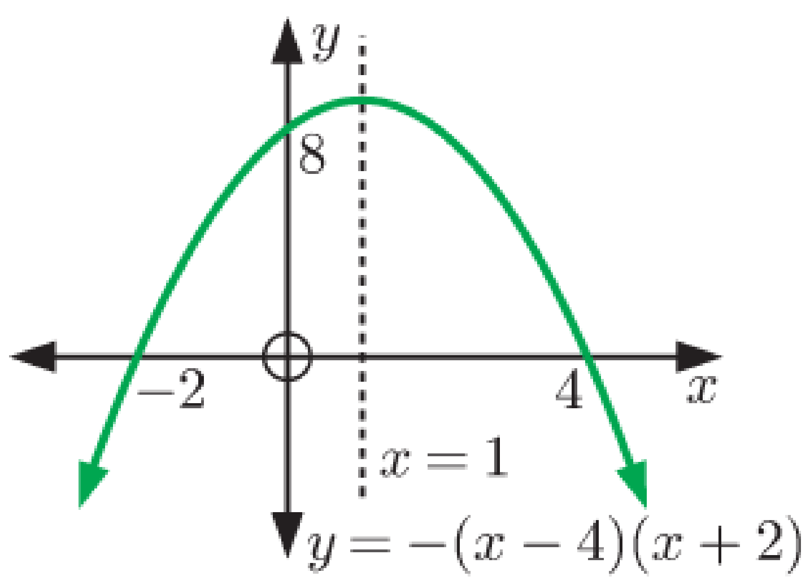
d $x = -3$ or $\frac{1}{2}$ **e** $x = -6$ or 1 **f** no real solutions

EXERCISE 14B.1

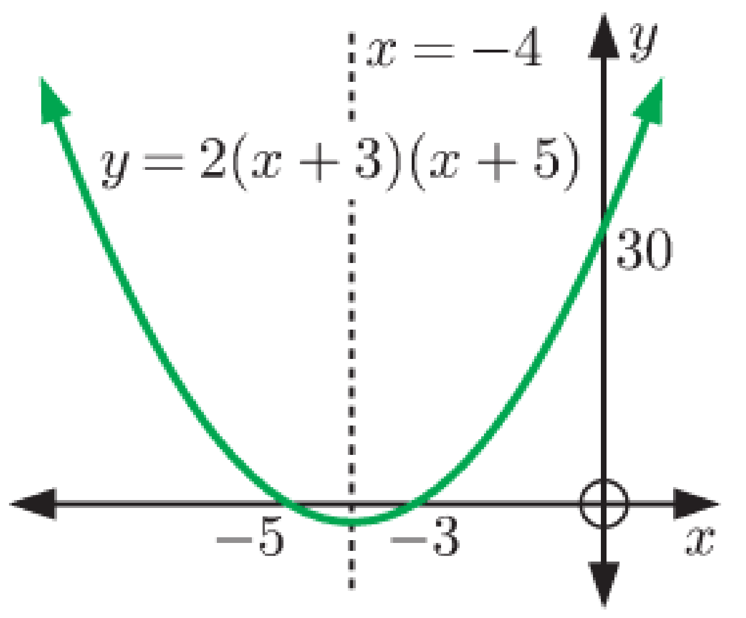
1 **a**



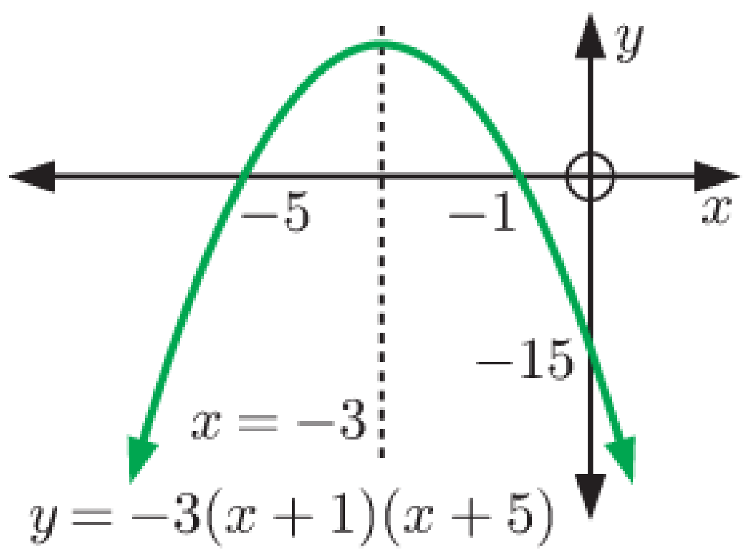
b



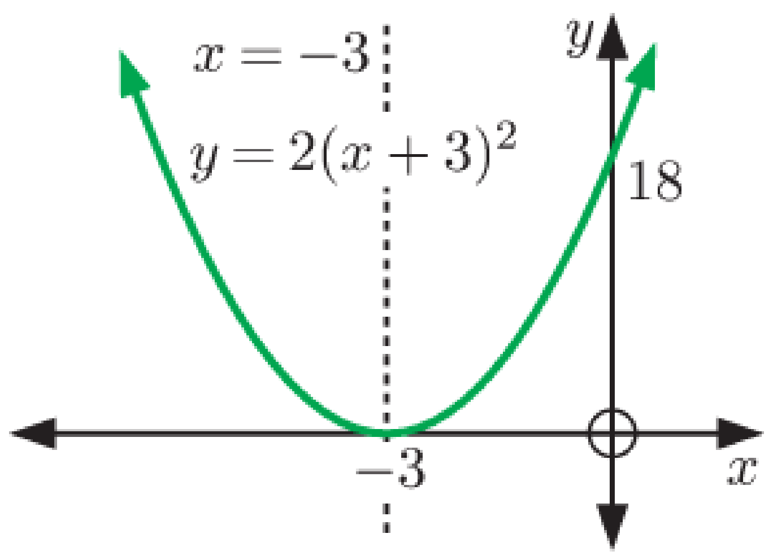
c



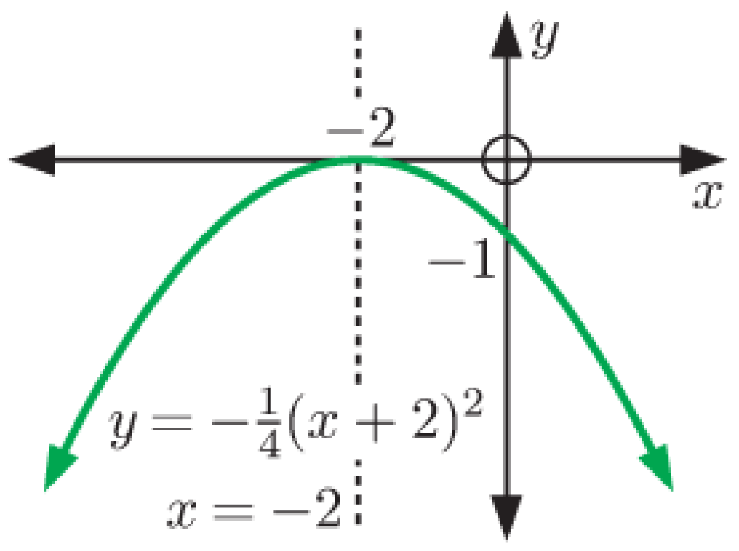
d



e

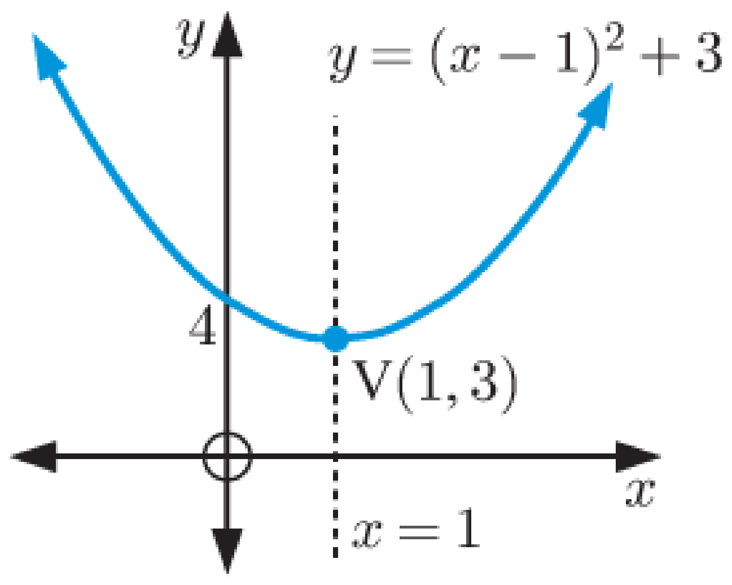


f



2 **a** **C** **b** **E** **c** **B** **d** **F** **e** **G** **f** **H**
g **I** **h** **A** **i** **D**

3 **a**



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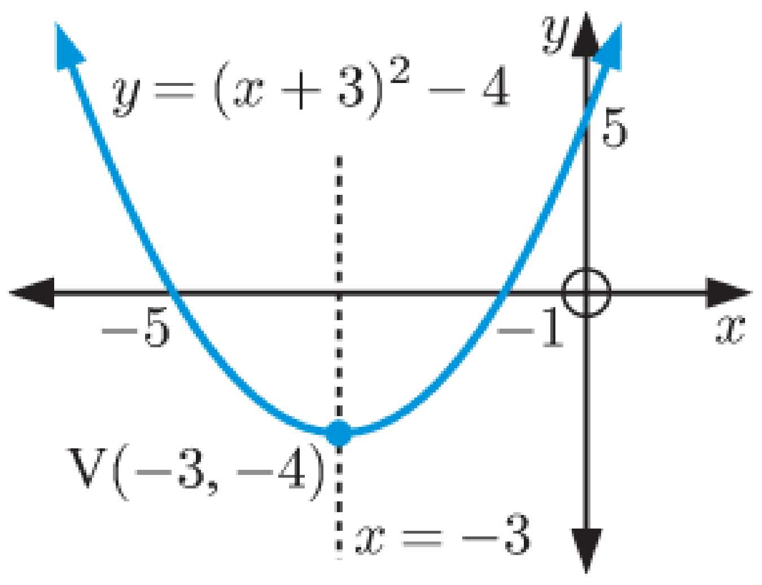

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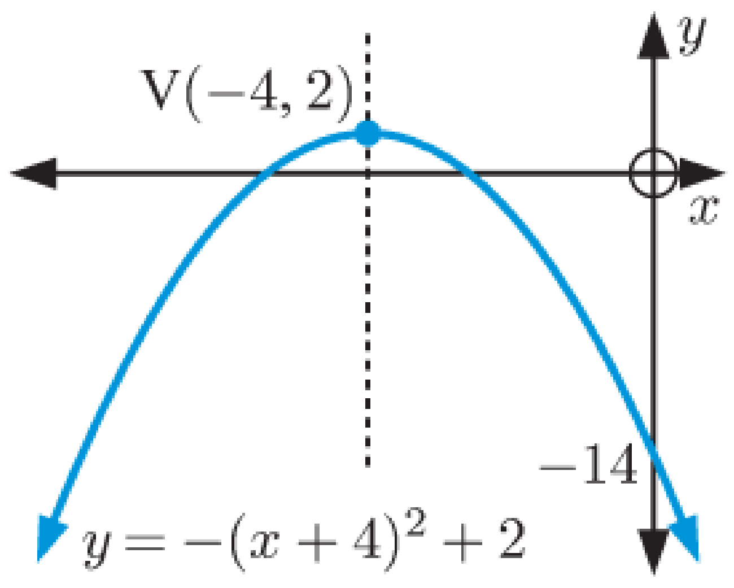

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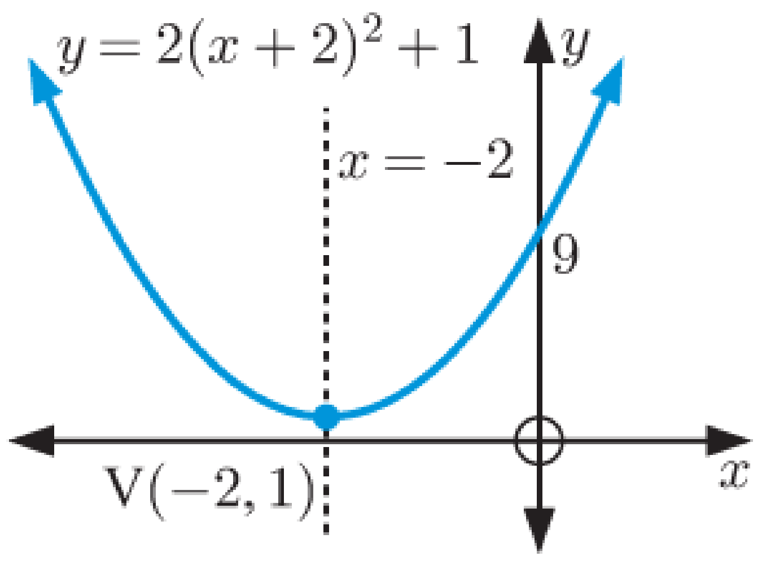

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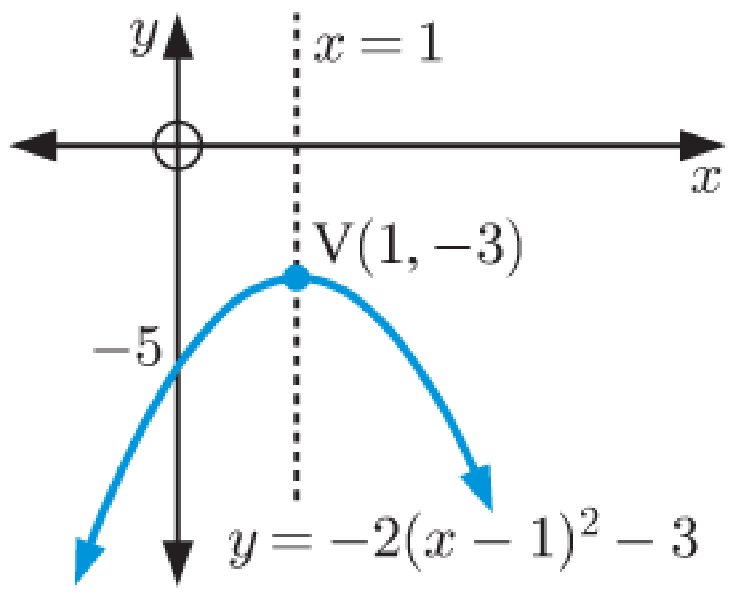

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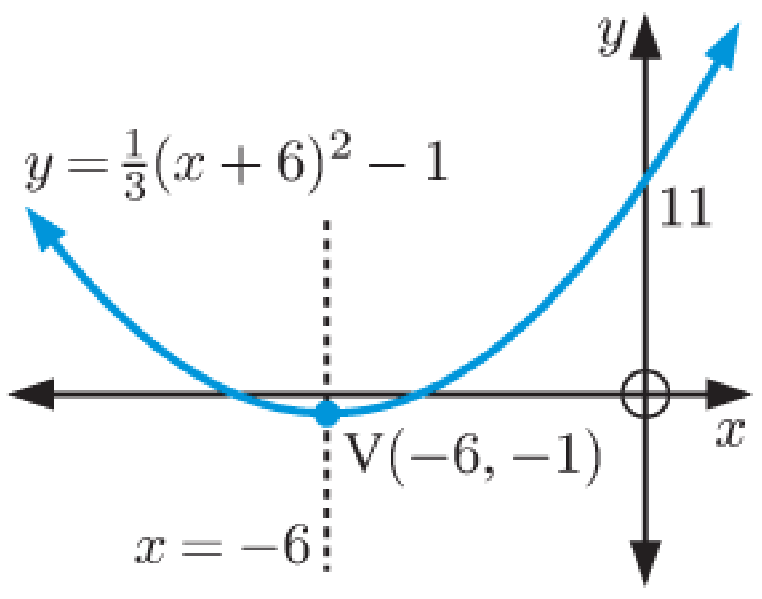
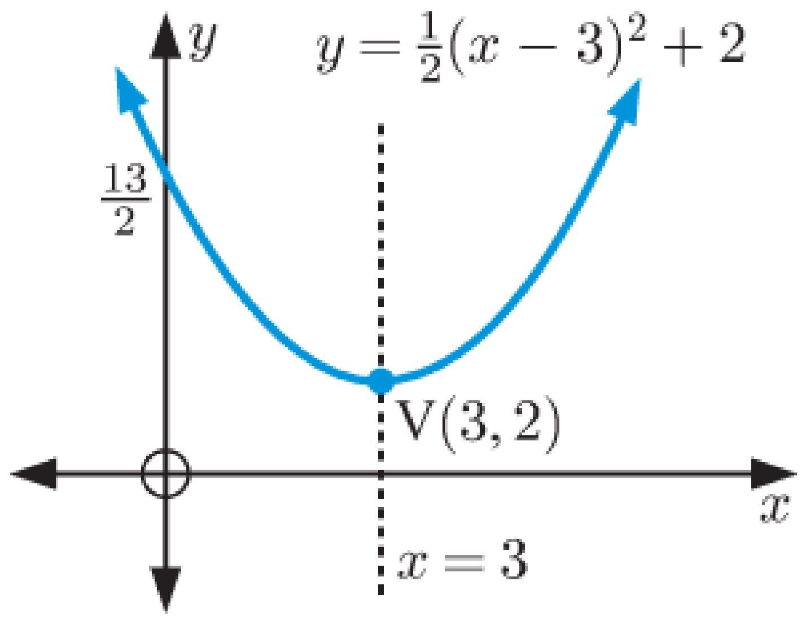
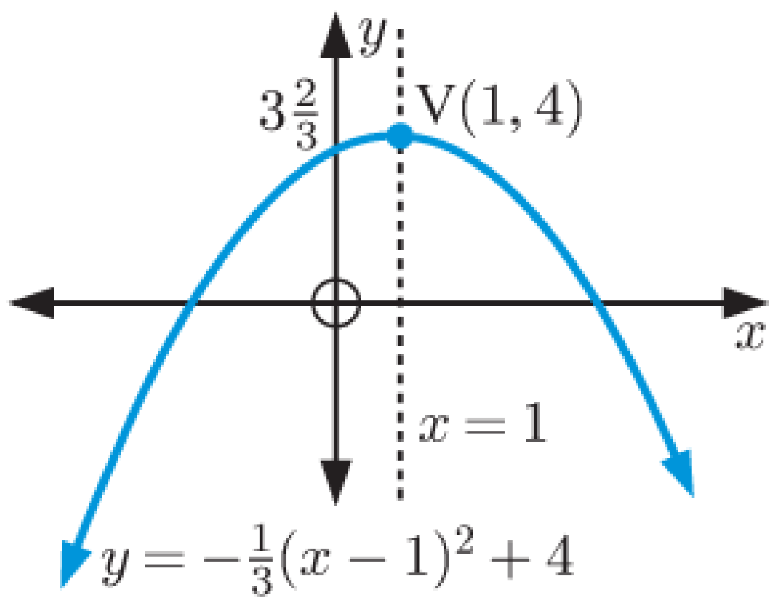


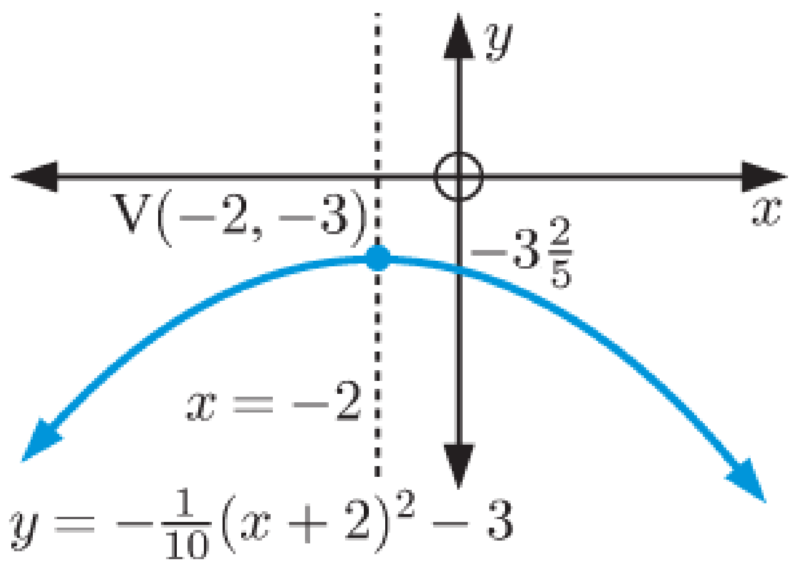
g



h

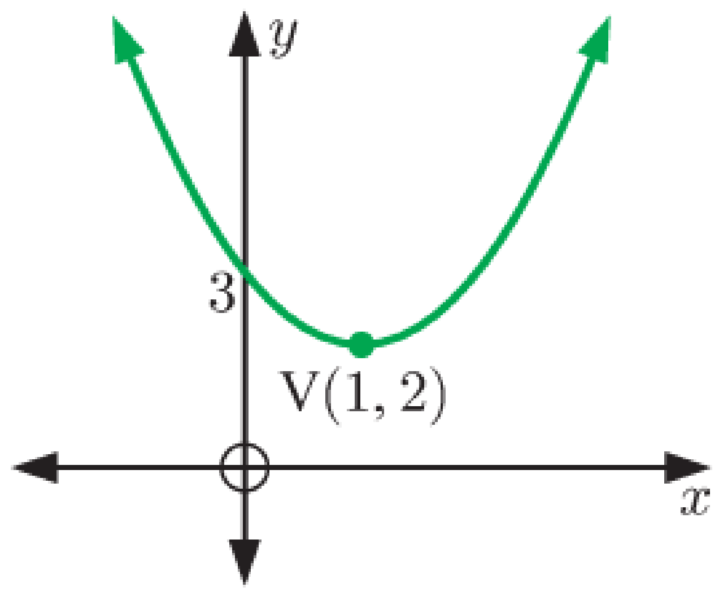


i

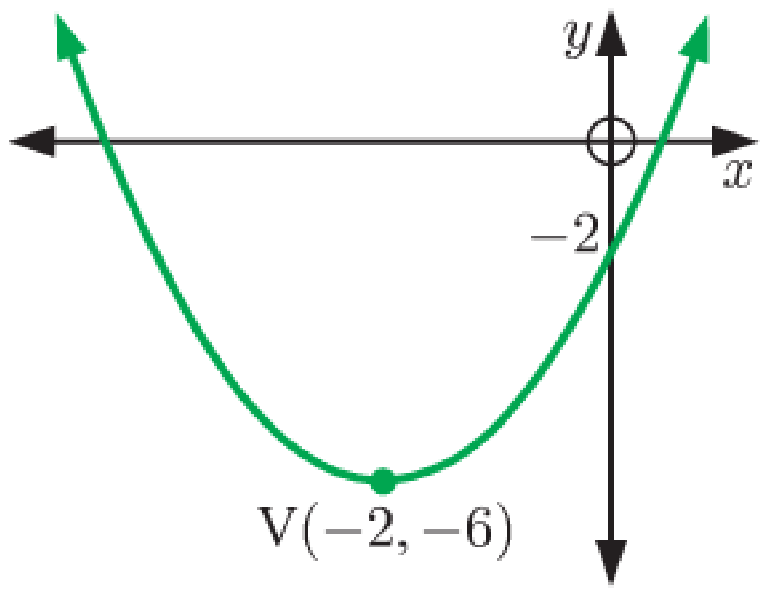


EXERCISE 14B.2

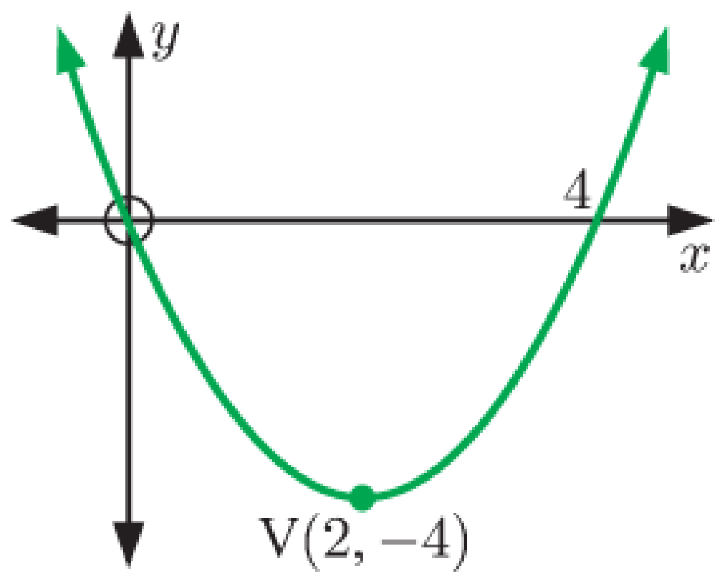
1 **a** $y=(x-1)^2+2$



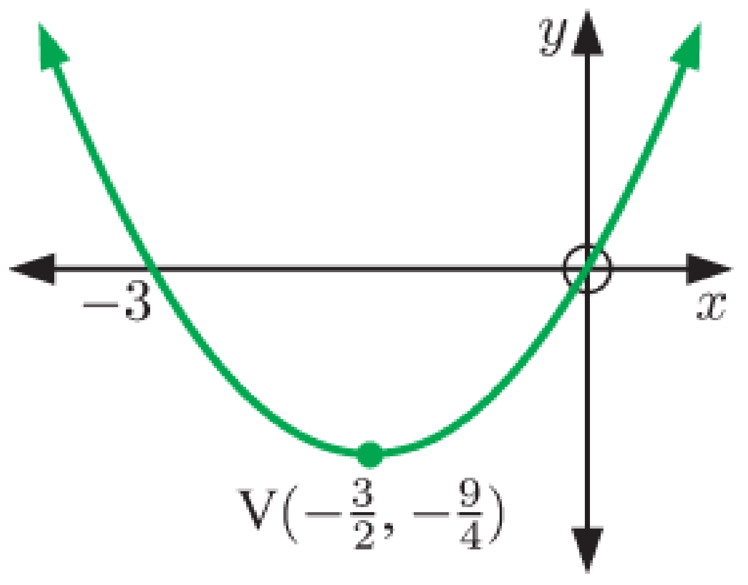
b $y=(x+2)^2-6$



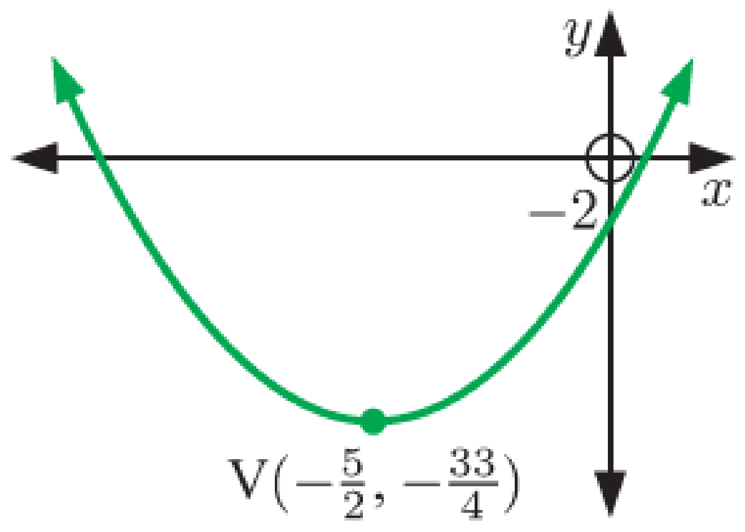
c $y=(x-2)^2-4$



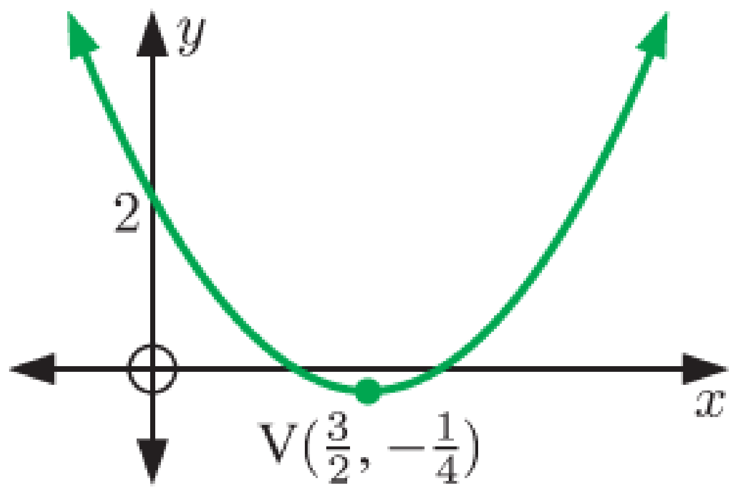
d $y=\left(x+\frac{3}{2}\right)^2-\frac{9}{4}$



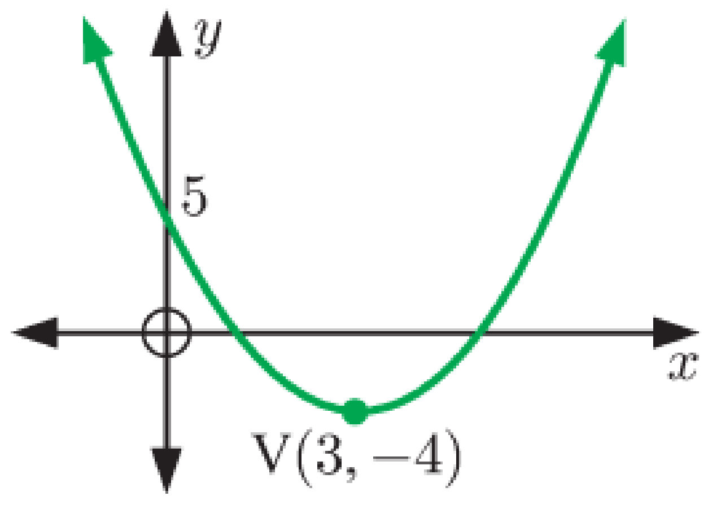
e $y=\left(x+\frac{5}{2}\right)^2-\frac{33}{4}$



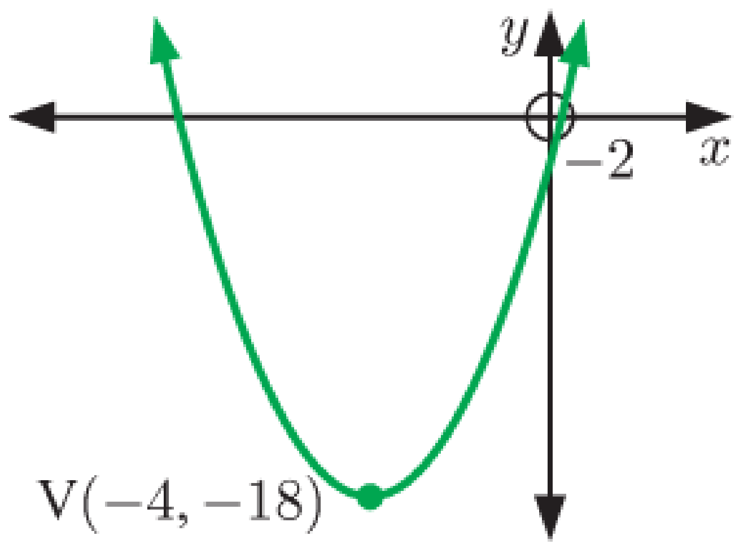
f $y=\left(x-\frac{3}{2}\right)^2-\frac{1}{4}$



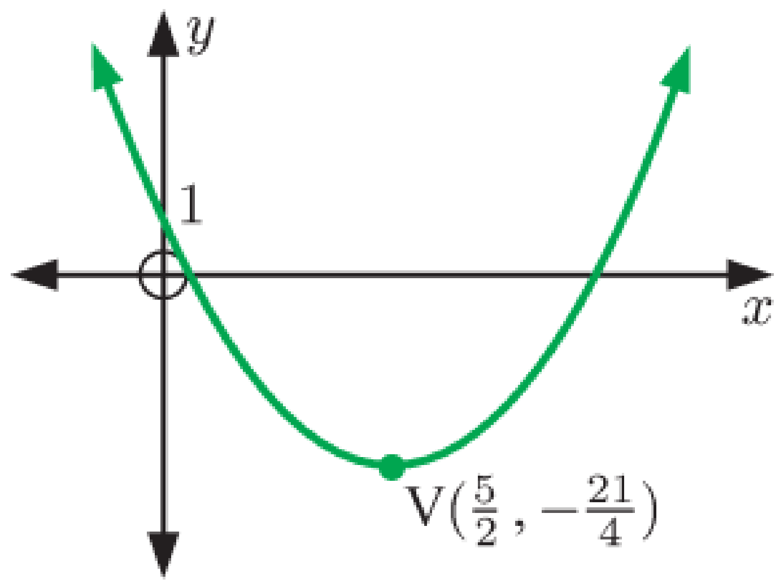
g $y=(x-3)^2-4$



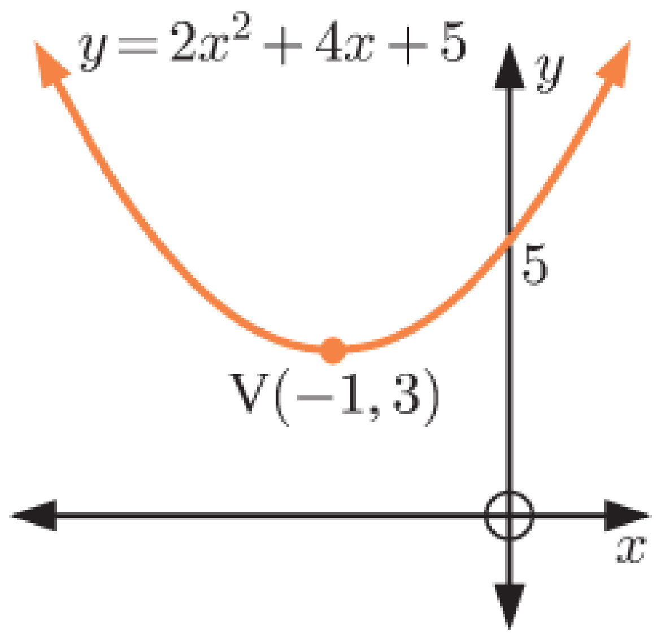
h $y=(x+4)^2-18$



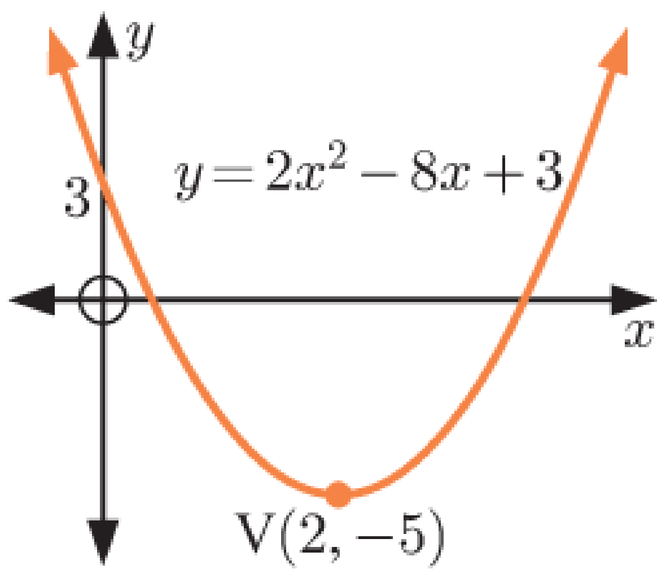
i $y=\left(x-\frac{5}{2}\right)^2-\frac{21}{4}$



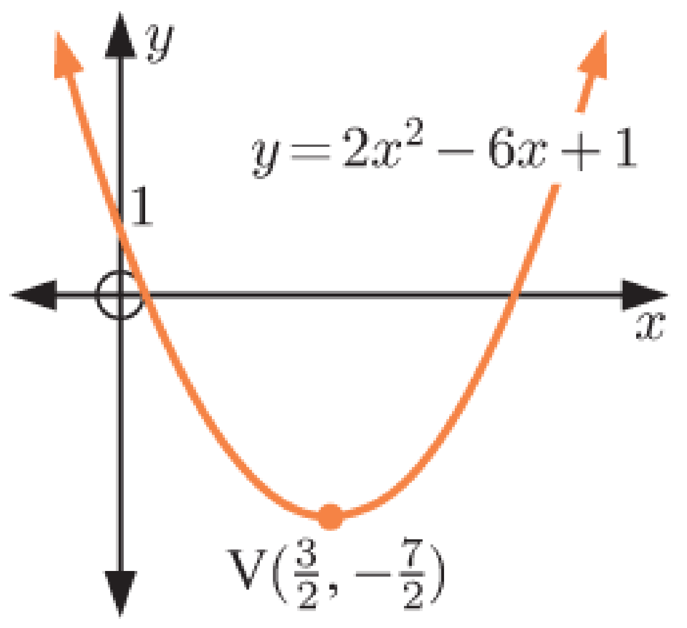
2 **a** **i** $y=2(x+1)^2+3$
ii $(-1, 3)$ **iii** 5
iv



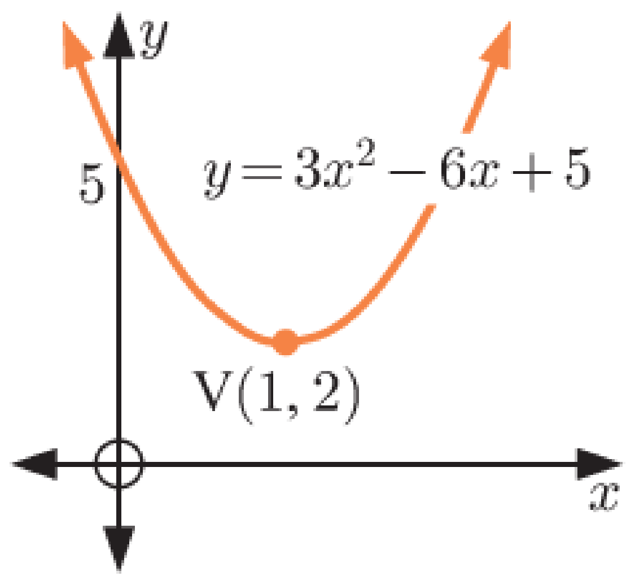
b **i** $y=2(x-2)^2-5$
ii $(2, -5)$ **iii** 3
iv



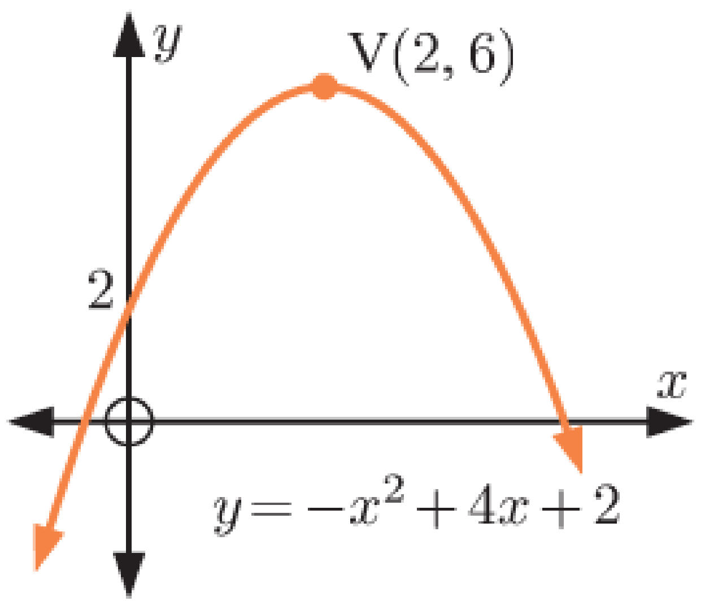
c **i** $y = 2(x-\frac{3}{2})^2 - \frac{7}{2}$
ii $(\frac{3}{2}, -\frac{7}{2})$ **iii** 1
iv



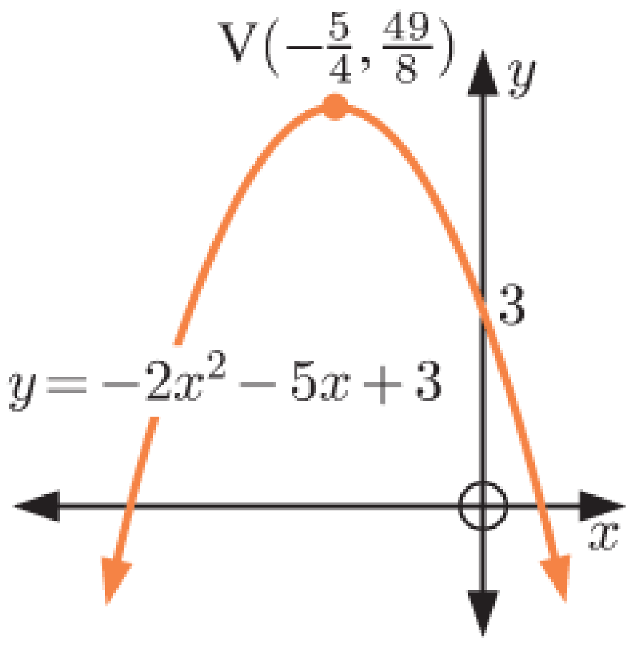
d **i** $y = 3(x-1)^2 + 2$
ii (1, 2) **iii** 5
iv



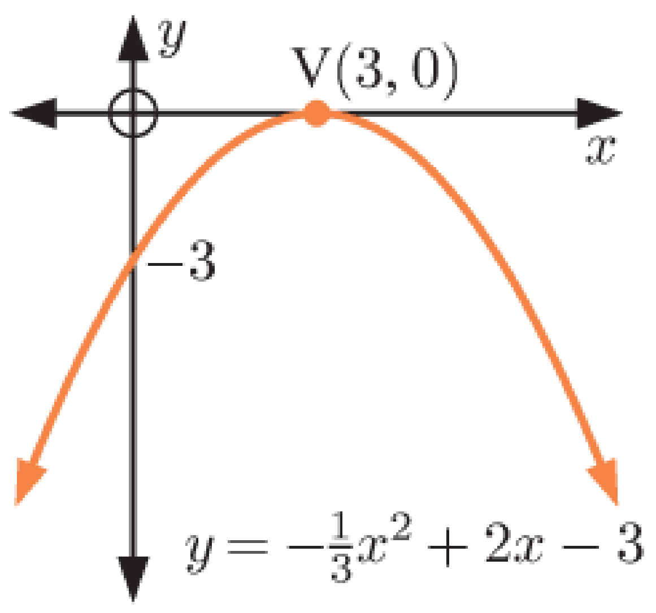
e **i** $y = -(x-2)^2 + 6$
ii (2, 6) **iii** 2
iv



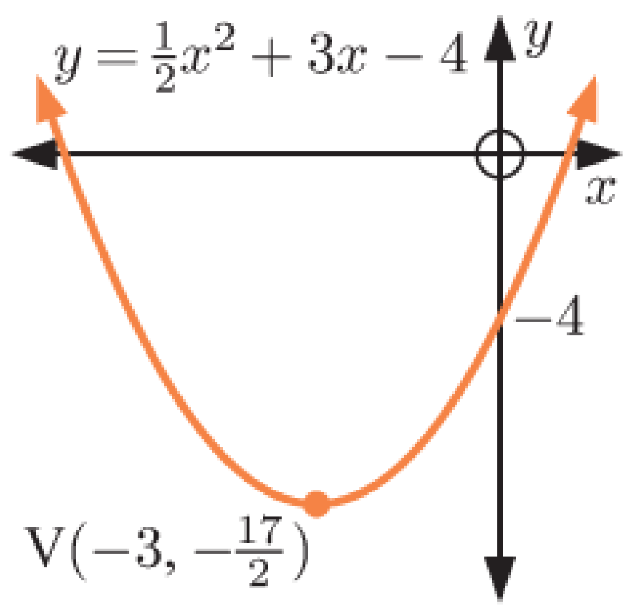
f **i** $y = -2(x+\frac{5}{4})^2 + \frac{49}{8}$
ii $(-\frac{5}{4}, \frac{49}{8})$ **iii** 3
iv



g **i** $y = -\frac{1}{3}(x-3)^2$
ii (3, 0) **iii** −3
iv



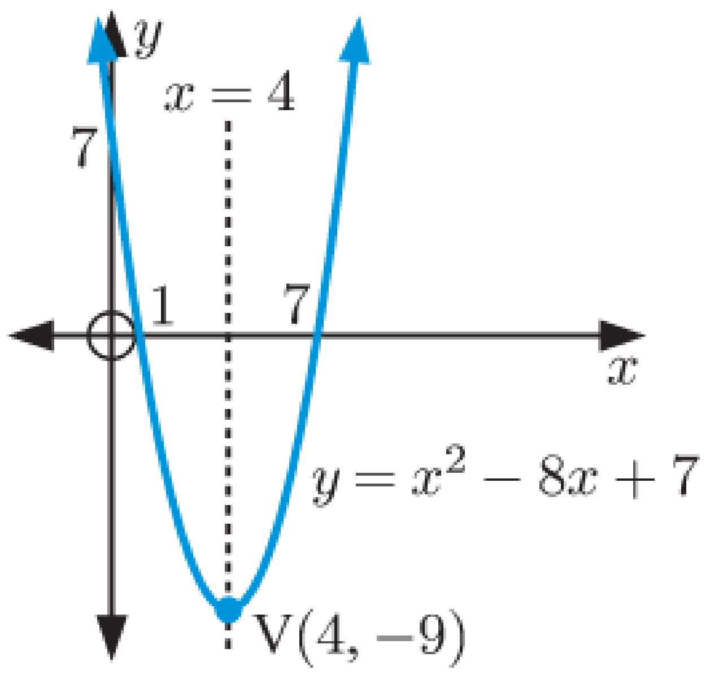
h **i** $y = \frac{1}{2}(x+3)^2 - \frac{17}{2}$
ii $(-3, -\frac{17}{2})$ **iii** −4
iv



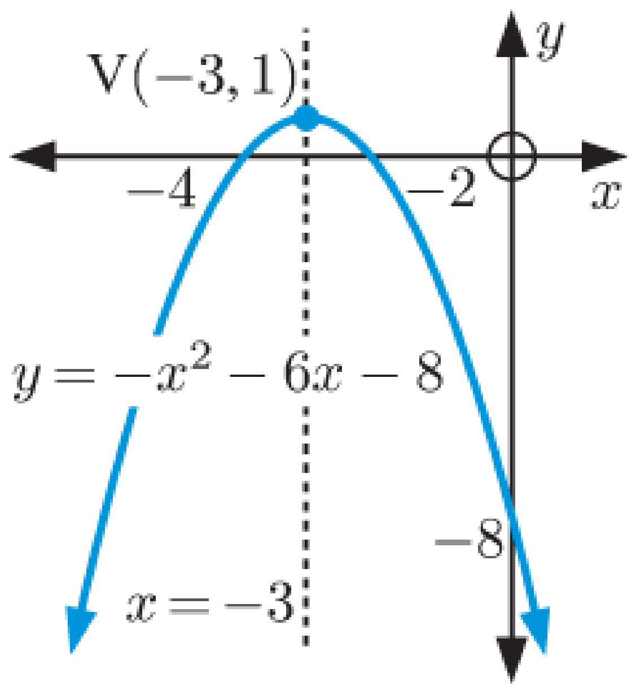
EXERCISE 14B.3

1 **a** **i** (2, −2) **ii** minimum turning point
b **i** (−1, −4) **ii** minimum turning point
c **i** (0, 4) **ii** minimum turning point
d **i** (0, 1) **ii** maximum turning point
e **i** (−2, −15) **ii** minimum turning point
f **i** (−2, −5) **ii** maximum turning point
g **i** $(-\frac{3}{2}, -\frac{11}{2})$ **ii** minimum turning point
h **i** $(\frac{5}{2}, -\frac{19}{2})$ **ii** minimum turning point
i **i** $(1, -\frac{9}{2})$ **ii** maximum turning point
j **i** (14, −43) **ii** minimum turning point

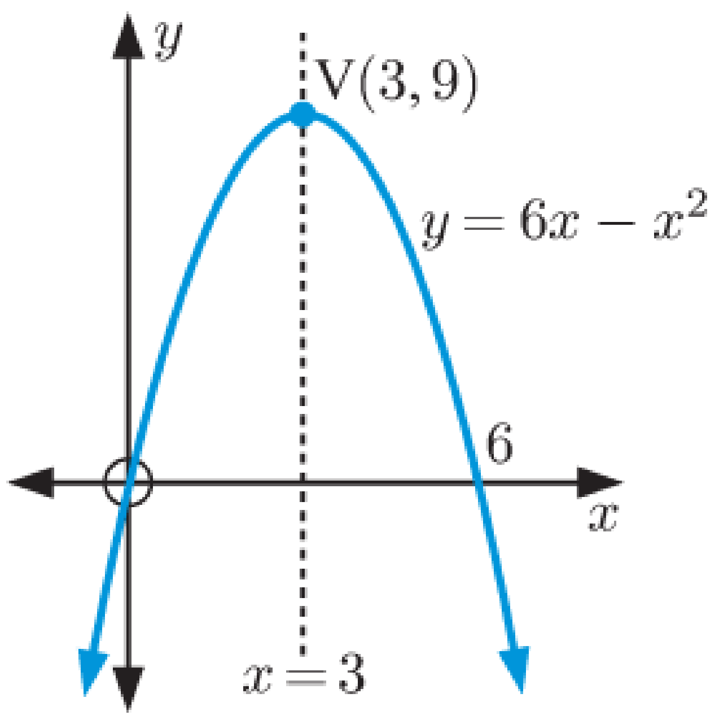
2 **a** **i** $x = 4$
ii (4, −9)
iii x-intercepts 1, 7,
y-intercept 7
iv



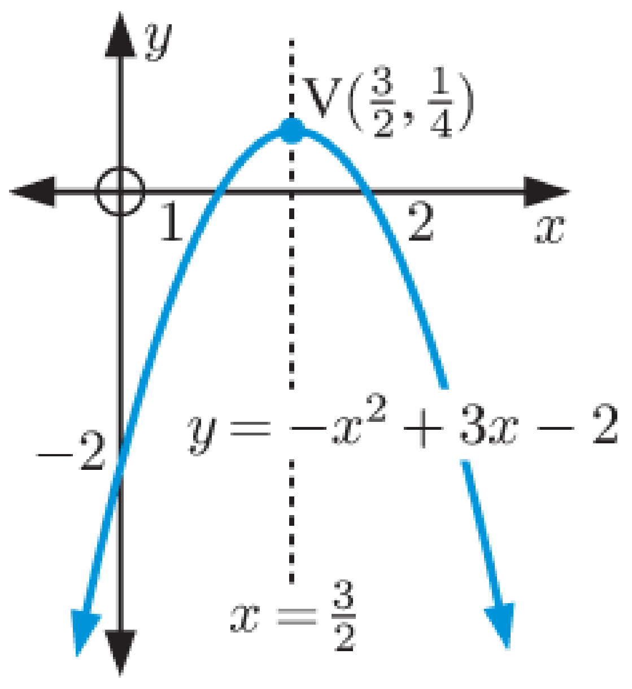
b **i** $x = -3$
ii (−3, 1)
iii x-int. −2, −4,
y-intercept −8
iv



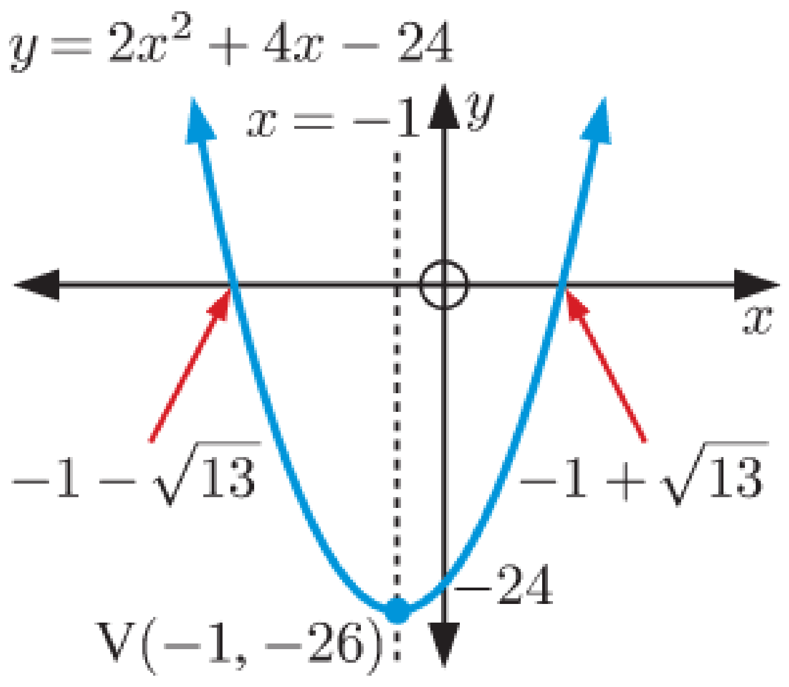
c **i** $x = 3$ **ii** (3, 9)
iii x-intercepts 0, 6,
y-intercept 0
iv



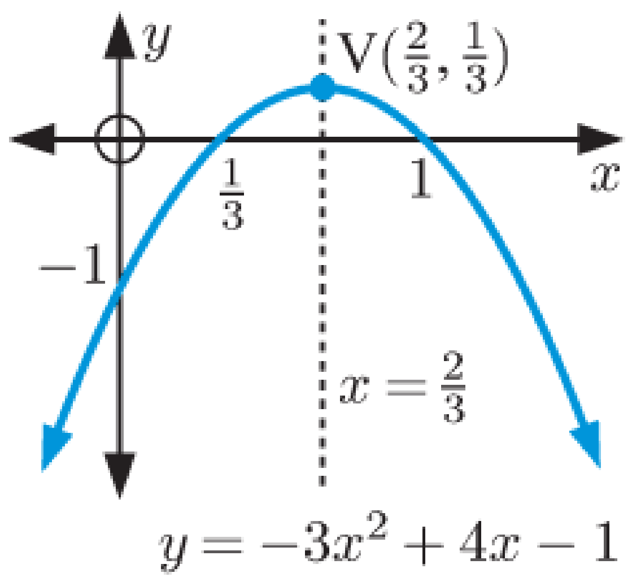
d **i** $x = \frac{3}{2}$ **ii** $(\frac{3}{2}, \frac{1}{4})$
iii x-intercepts 1, 2,
y-intercept −2
iv



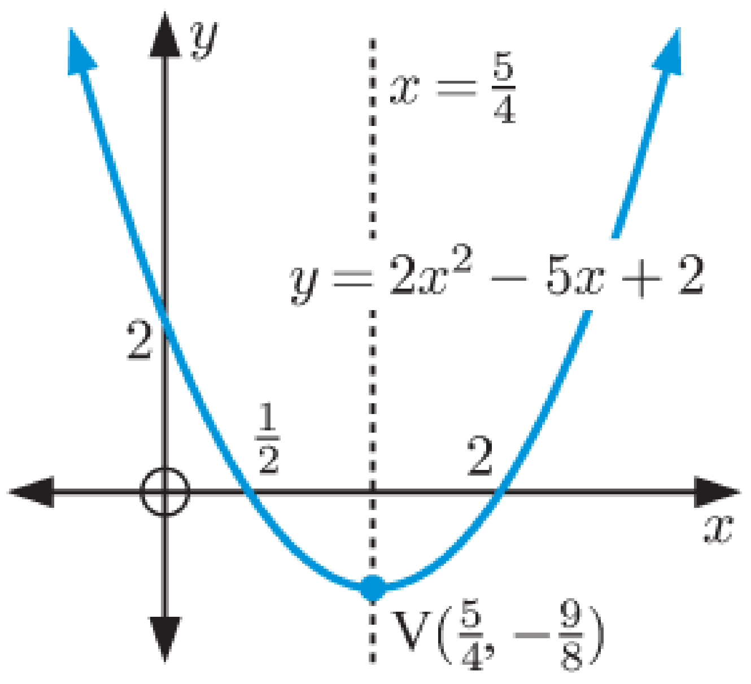
e **i** $x = -1$
ii (−1, −26)
iii x-int. $-1 \pm \sqrt{13}$,
y-intercept −24
iv



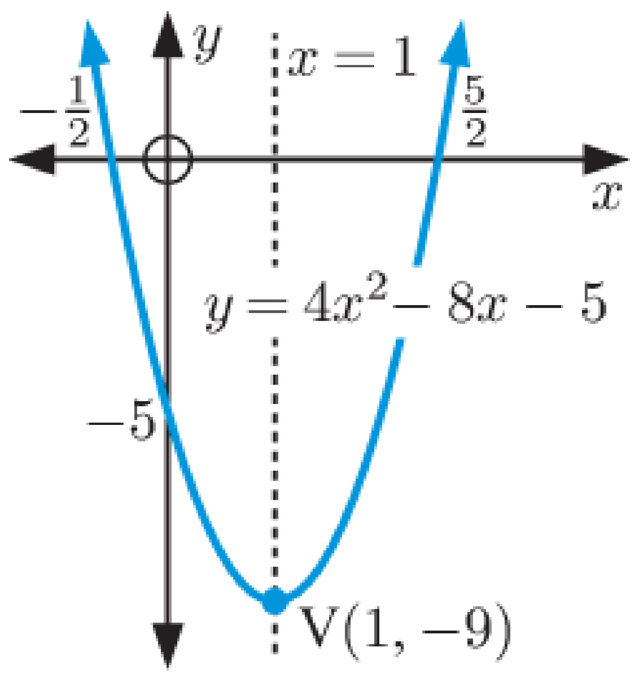
f **i** $x = \frac{2}{3}$
ii $(\frac{2}{3}, \frac{1}{3})$
iii x-intercepts $\frac{1}{3}$, 1,
y-intercept −1
iv



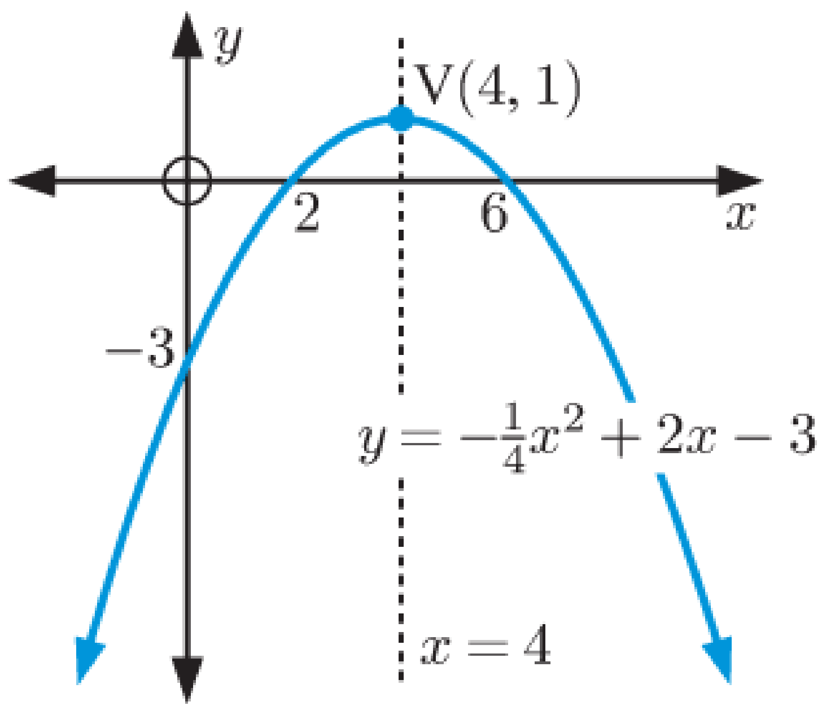
g **i** $x = \frac{5}{4}$
ii $(\frac{5}{4}, -\frac{9}{8})$
iii x-intercepts $\frac{1}{2}$, 2,
y-intercept 2
iv



h **i** $x = 1$
ii (1, −9)
iii x-intercepts $-\frac{1}{2}$, $\frac{5}{2}$,
y-intercept −5
iv



i **i** $x = 4$
ii (4, 1)
iii x-intercepts 2, 6,
y-intercept −3
iv



3 **Hint:** $y = ax^2 + bx + c$ has vertex with x-coordinate $-\dfrac{b}{2a}$
and y-coordinate $a\left(-\dfrac{b}{2a}\right)^2 + b\left(-\dfrac{b}{2a}\right) + c$.

EXERCISE 14C

1 **a** $\Delta = 9$ which is > 0, graph cuts x-axis twice; is concave up.
b $\Delta = 12$ which is > 0, graph cuts x-axis twice; is concave up.
c $\Delta = -12$ which is < 0, graph lies entirely below the x-axis; is concave down, negative definite.
d $\Delta = 57$ which is > 0, graph cuts x-axis twice; is concave up.

e $\Delta = 0$, graph touches x-axis; is concave up.

f $\Delta = 17$ which is > 0, graph cuts x-axis twice; is concave down.

g $\Delta = 121$ which is > 0, graph cuts x-axis twice; is concave up.

h $\Delta = 25$ which is > 0, graph cuts x-axis twice; is concave down.

i $\Delta = 0$, graph touches x-axis; is concave up.

2 **a** concave up

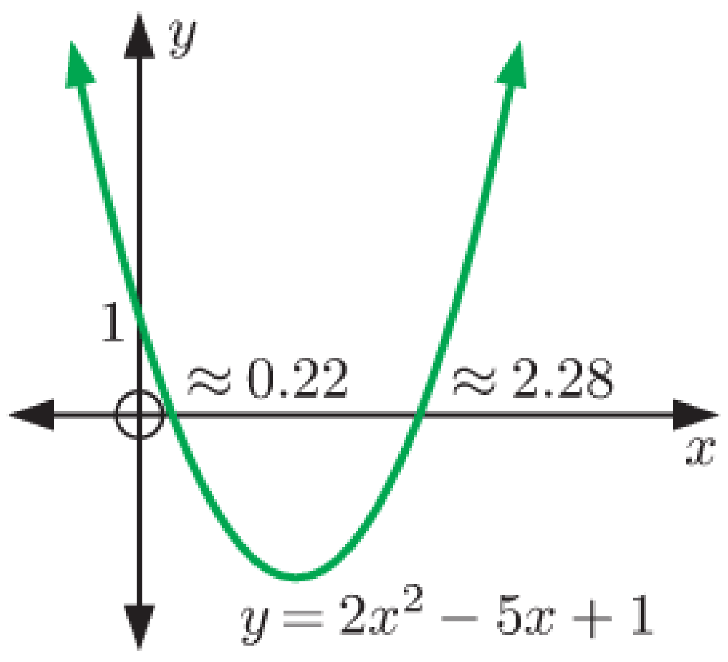
b $\Delta = 17$ which is > 0

$\therefore$ cuts x-axis twice

c x-intercepts ≈ 0.22 and 2.28

d y-intercept is 1

e



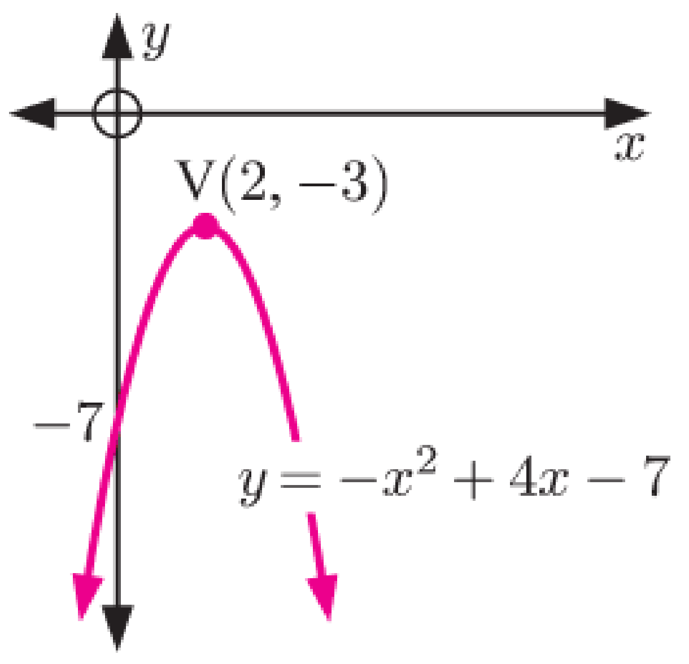
3 **a** $\Delta = -12$ which is < 0

$\therefore$ does not cut x-axis

b negative definite, since $a < 0$ and $\Delta < 0$

c vertex is $(2, -3)$, y-intercept is -7

d



4 **a** $a = 2$ which is > 0 and $\Delta = -40$ which is < 0

$\therefore$ positive definite.

b $a = -2$ which is < 0 and $\Delta = -23$ which is < 0

$\therefore$ negative definite.

c $a = 1$ which is > 0 and $\Delta = -15$ which is < 0

$\therefore$ positive definite so $x^2 - 3x + 6 > 0$ for all x.

d $a = -1$ which is < 0 and $\Delta = -8$ which is < 0

$\therefore$ negative definite so $4x - x^2 - 6 < 0$ for all x.

5

Constant	a	b	c	d	e	f	Δ_1	Δ_2
Sign	$+$	$-$	$+$	$-$	$+$	0	$-$	$+$

6 **a** **i** $k < \frac{9}{4}$ **ii** $k = \frac{9}{4}$ **iii** $k > \frac{9}{4}$

b **i** $k < 4$ **ii** $k = 4$ **iii** $k > 4$

c **i** $k > -\frac{4}{3}$ **ii** $k = -\frac{4}{3}$ **iii** $k < -\frac{4}{3}$

7 $a = 3$ which is > 0 and $\Delta = k^2 + 12$ which is always > 0 $\{$as $k^2 \geqslant 0$ for all $k\}$ $\therefore$ cannot be positive definite.

8 $k = -2$, the graph touches the x-axis in this case.

EXERCISE 14D

1 **a** $y = 2(x-1)(x-2)$ **b** $y = 3(x-2)^2$

c $y = (x-1)(x-3)$ **d** $y = -(x-3)(x+1)$

e $y = -3(x-1)^2$ **f** $y = -2(x+2)(x-3)$

2 **a** $y = \frac{3}{2}(x-2)(x-4)$ **b** $y = -\frac{1}{2}(x+4)(x-2)$

c $y = -\frac{4}{3}(x+3)^2$

3 **a** $y = 3x^2 - 18x + 15$ **b** $y = -4x^2 + 6x + 4$

c $y = -x^2 + 6x - 9$ **d** $y = 4x^2 + 16x + 16$

e $y = \frac{3}{2}x^2 - 6x + \frac{9}{2}$ **f** $y = -\frac{1}{3}x^2 + \frac{2}{3}x + 5$

4 **a** $y = -(x-2)^2 + 4$ **b** $y = 2(x-2)^2 - 1$

c $y = \frac{1}{3}(x+3)^2 - 4$ **d** $y = -2(x-3)^2 + 8$

e $y = \frac{2}{3}(x-4)^2 - 6$ **f** $y = -\frac{5}{9}(x+2)^2 + 5$

g $y = -2(x-2)^2 + 3$ **h** $y = \frac{3}{2}(x+4)^2 + 3$

i $y = 2(x - \frac{1}{2})^2 - \frac{3}{2}$

5 $y = 3$

EXERCISE 14E

1 **a** $(1, 7)$ and $(2, 8)$ **b** $(4, 5)$ and $(-3, -9)$

c $(3, 0)$ (touching) **d** graphs do not meet

2 **a** $(0.586, 5.59)$ and $(3.41, 8.41)$

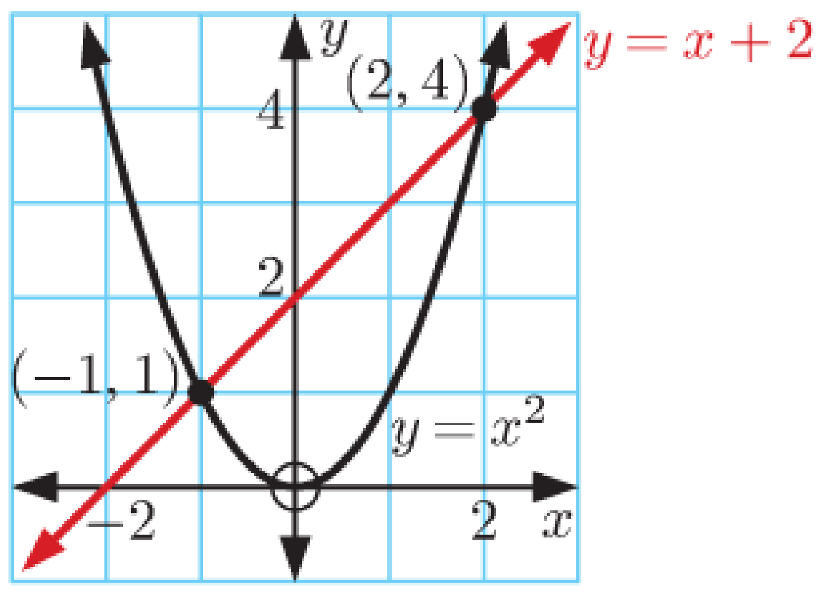
b $(3, -4)$ (touching) **c** graphs do not meet

d $(-2.56, -18.8)$ and $(1.56, 1.81)$

3 **a** $(-1, 1)$ and $(2, 4)$

c $x < -1$ or $x > 2$

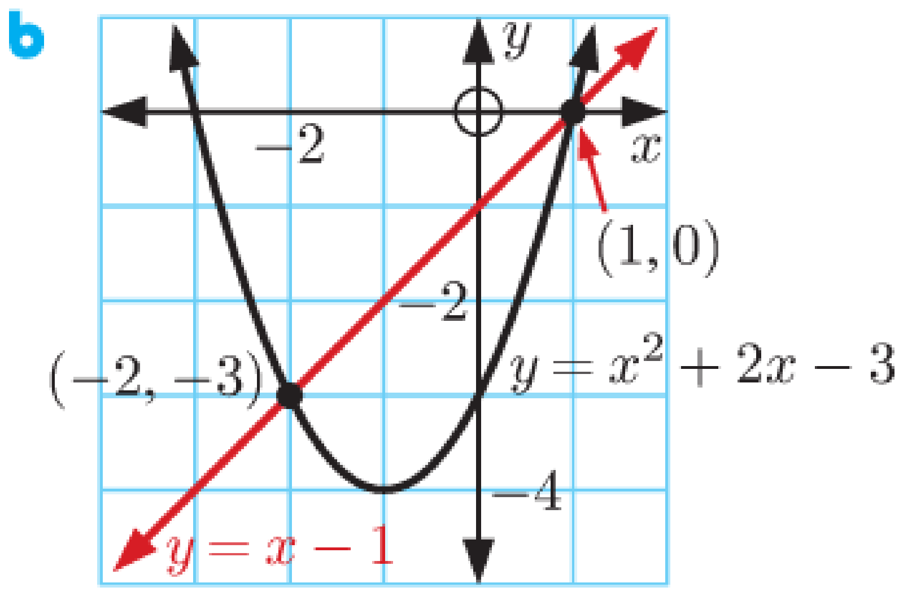
b



4 **a** $(-2, -3)$ and $(1, 0)$

c $x < -2$ or $x > 1$

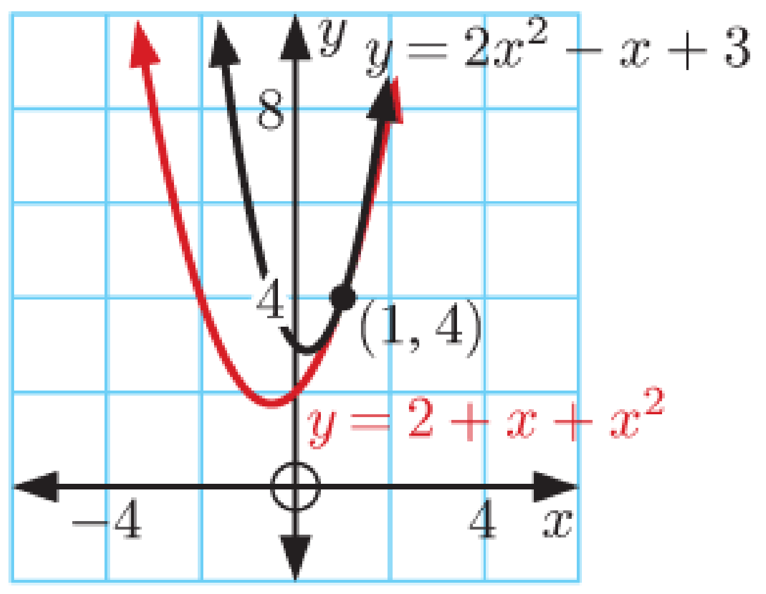
b



5 **a** $(1, 4)$

c $x \in \mathbb{R}$, $x \neq 1$

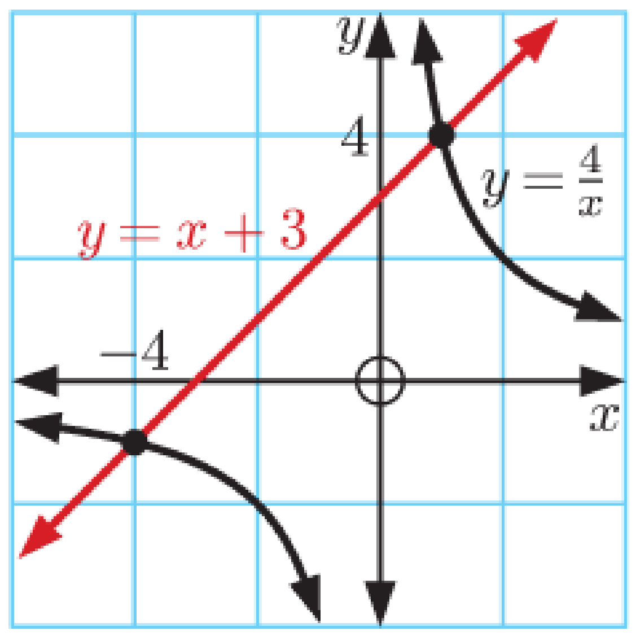
b



6 **a** $x = -4$ or 1

c $x < -4$ or $0 < x < 1$

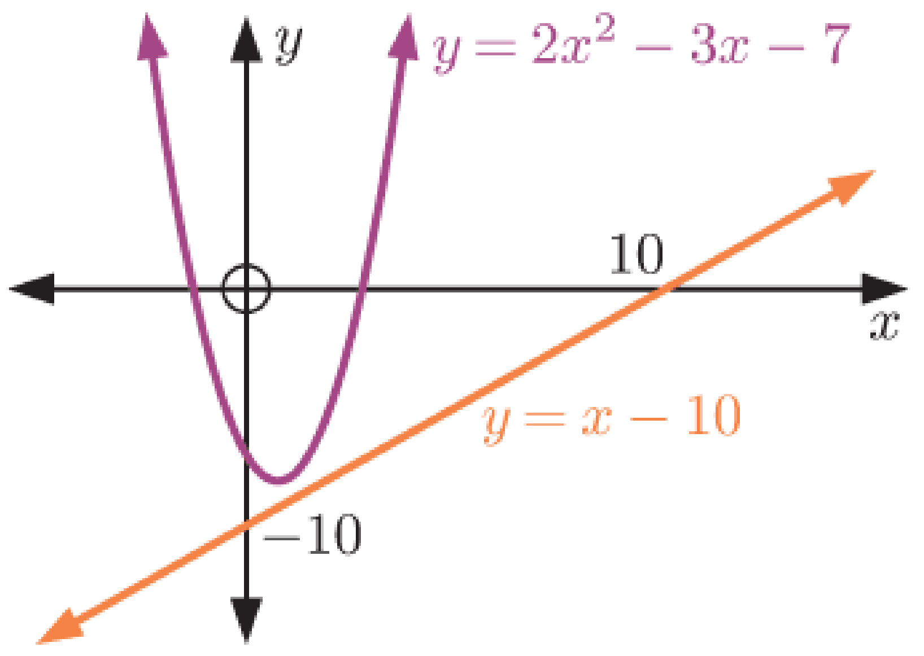
b



7 $c = -9$ **8** $m = 0$ or -8 **9** -1 or 11

10 **a** $c < -9$

b example: $c = -10$



12 **a** $c > -2$ **b** $c = -2$ **c** $c < -2$

13 **Hint:** A straight line through $(0, 3)$ will have an equation of the form $y = mx + 3$.

14 $b = 8$, $c = -14$ **15** **a** $c = a^2$, $m \in \mathbb{R}$ **b** $m = 2a$

EXERCISE 14F

1 7 and -5 or -7 and 5 **2** 5 or $\frac{1}{5}$ **3** 14

4 18 and 20 or -18 and -20

5 15 and 17 or -15 and -17 **6** 15 sides **7** ≈ 3.48 cm

8 **b** 6 cm by 6 cm by 7 cm **9** ≈ 11.2 cm square

10 no **12** ≈ 61.8 km h^{-1} **13** 32 elderly citizens

14 **a** $y = -\frac{8}{9}x^2 + 8$
b No, as the tunnel is only 4.44 m high when it is the same width as the truck.

15 **a** $h = -5(t-2)^2 + 80$ **b** 75 m **c** 6 seconds

EXERCISE 14G

1 **a** min. -1, when $x = 1$ **b** max. 8, when $x = -1$
c max. $8\frac{1}{3}$, when $x = \frac{1}{3}$ **d** min. $-1\frac{1}{8}$, when $x = -\frac{1}{4}$
e min. $4\frac{15}{16}$, when $x = \frac{1}{8}$ **f** max. $6\frac{1}{8}$, when $x = \frac{7}{4}$

2 **a** 40 refrigerators **b** €4000

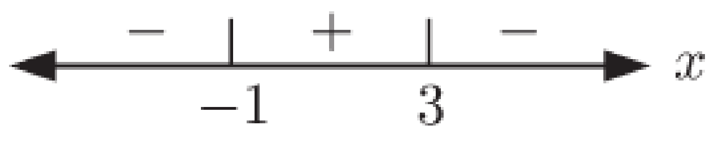
4 500 m by 250 m

5 **a** $41\frac{2}{3}$ m by $41\frac{2}{3}$ m **b** 50 m by $31\frac{1}{4}$ m

6 **b** $3\frac{1}{8}$ units **7** **a** $y = 6 - \frac{3}{4}x$ **b** 3 cm by 4 cm

8 $m = \dfrac{\sum_{i=1}^{n} a_i b_i}{\sum_{i=1}^{n} {a_i}^2}$ **9** $y = x^4 - 2(a^2+b^2)x^2 + (a^2-b^2)^2$
least value $= -4a^2b^2$

EXERCISE 14H.1

1 **a**


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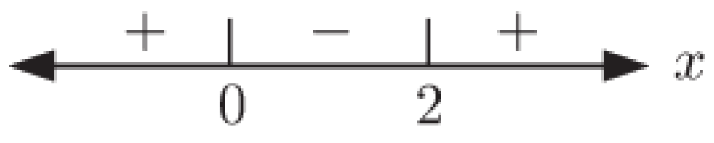

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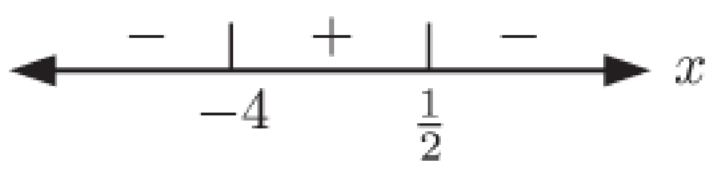

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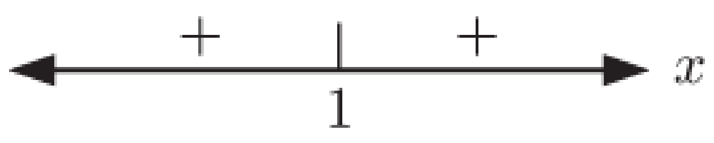

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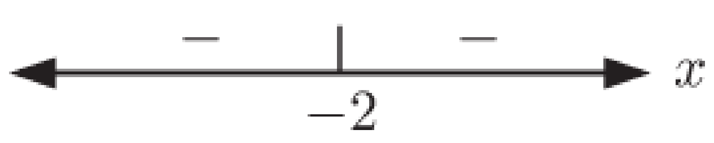

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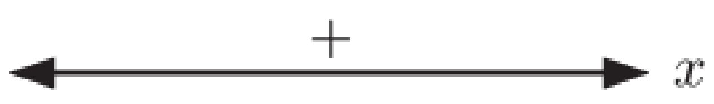

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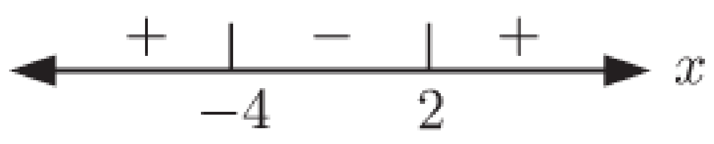

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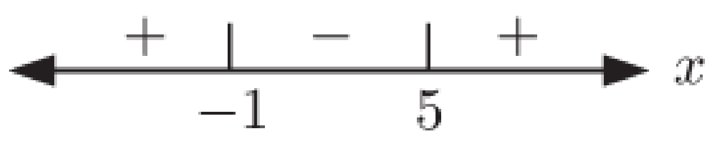

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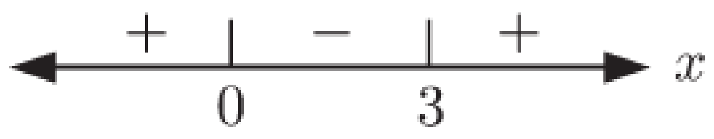

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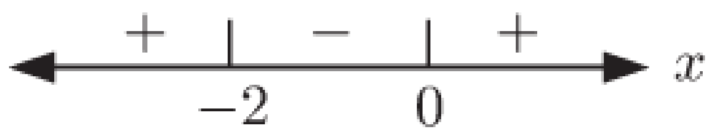

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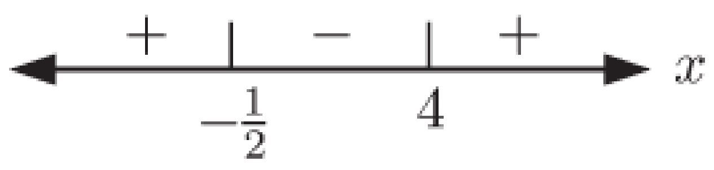

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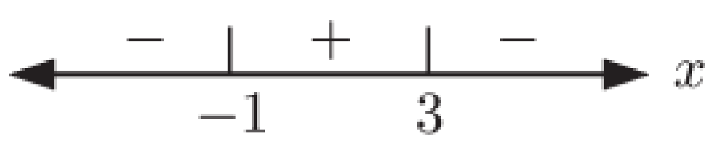

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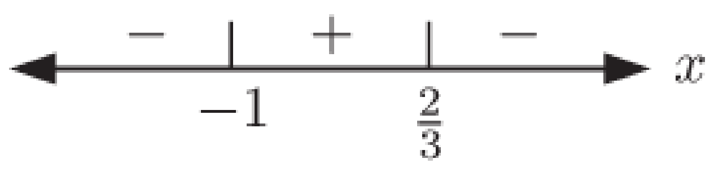

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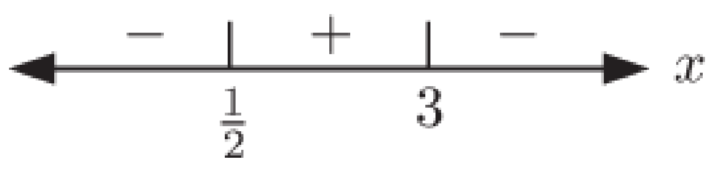

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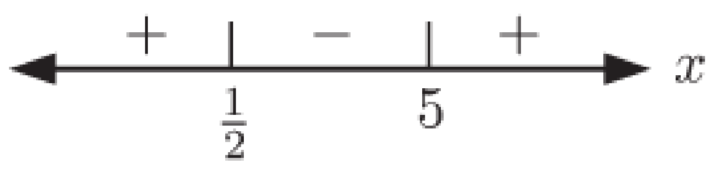

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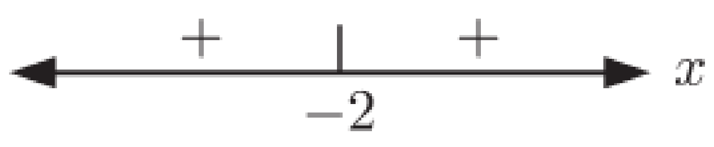

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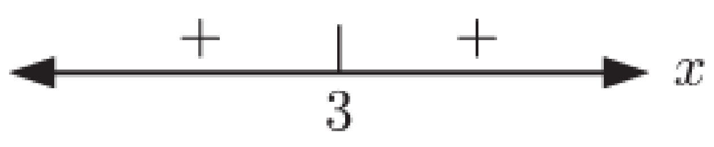

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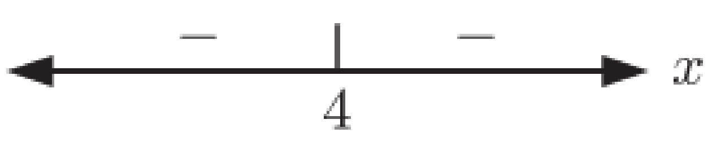

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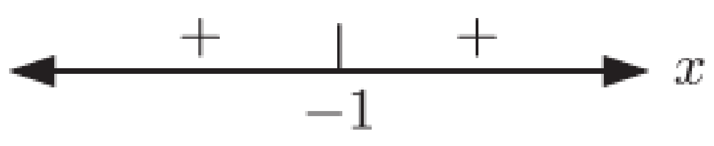

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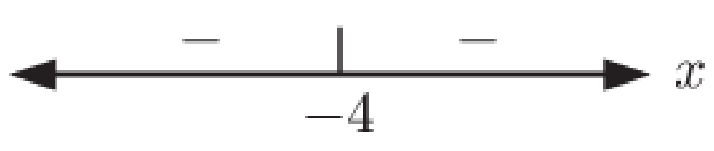

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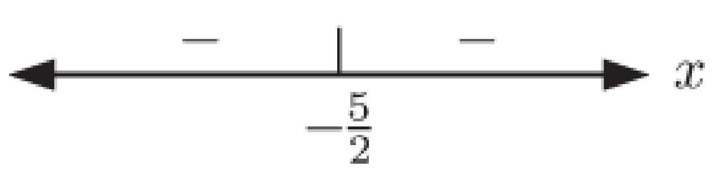

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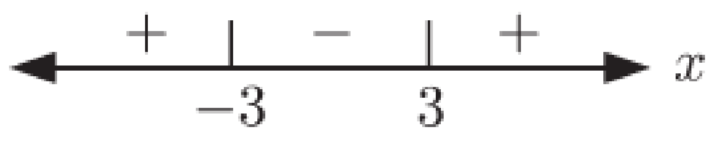

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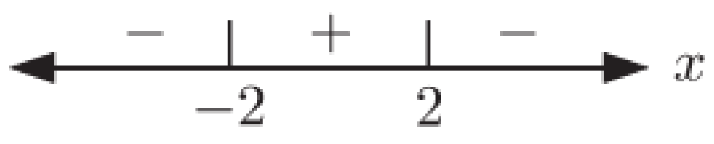

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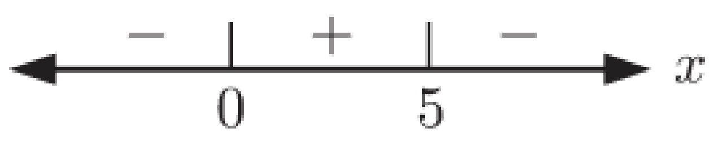

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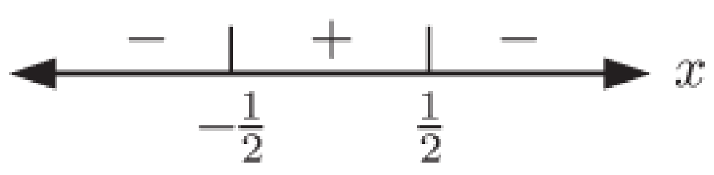

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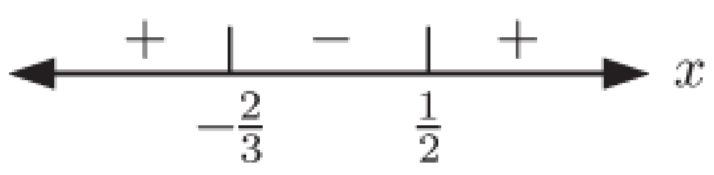

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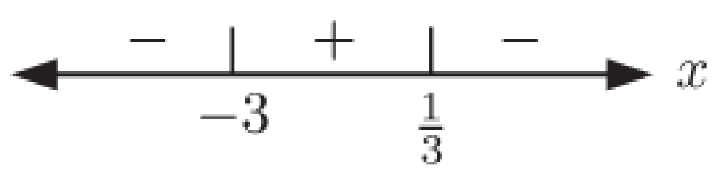

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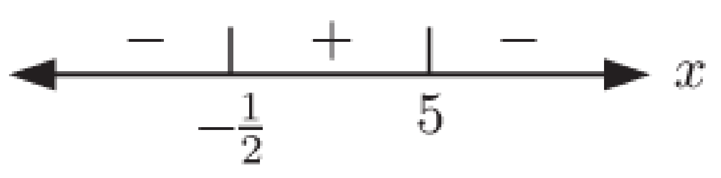

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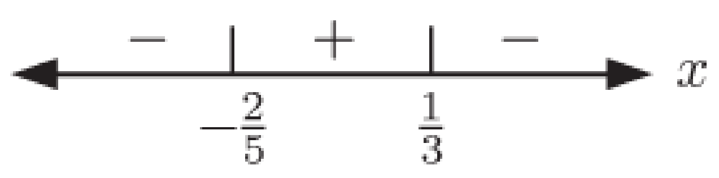

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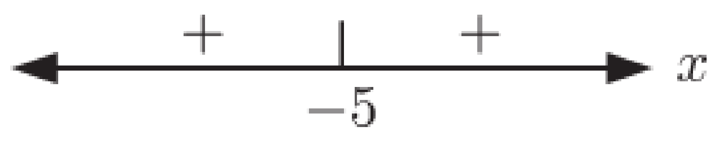
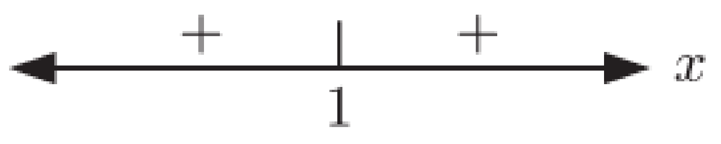


b


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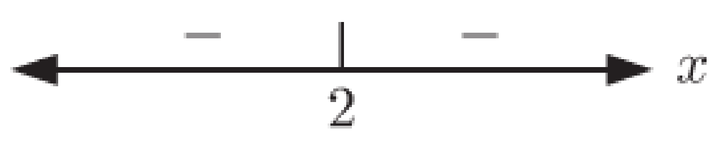

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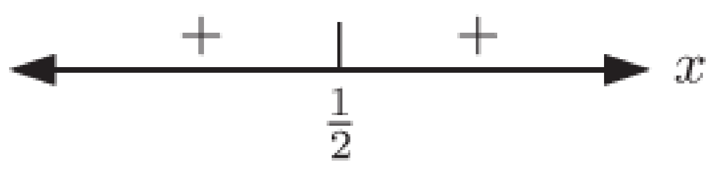
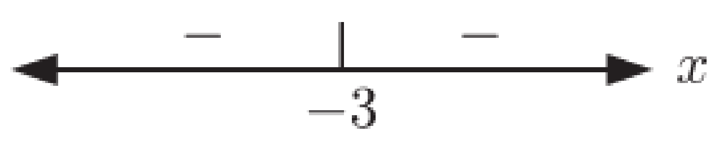
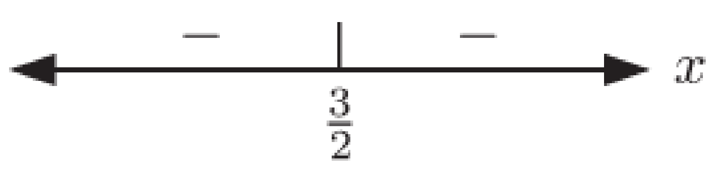


e


f


EXERCISE 14H.2

1 **a** $-5 \leqslant x \leqslant 2$ **b** $-3 \leqslant x \leqslant 2$ **c** no solutions
d all $x \in \mathbb{R}$ **e** $-\frac{1}{2} < x < 3$ **f** $-\frac{3}{2} < x < 4$

2 **a** $x \leqslant 0$ or $x \geqslant 1$ **b** $-\frac{2}{3} < x < 0$ **c** $x \neq -2$
d $-5 \leqslant x \leqslant 3$ **e** $x < -2$ or $x > 6$ **f** $-4 < x < 1$

3 **a** $x \leqslant 0$ or $x \geqslant 3$ **b** $-2 < x < 2$
c $x \leqslant -\sqrt{2}$ or $x \geqslant \sqrt{2}$ **d** $-3 \leqslant x \leqslant 7$
e $x < 5$ or $x > 6$ **f** $x < -6$ or $x > 7$
g $x \leqslant -1$ or $x \geqslant \frac{3}{2}$ **h** no solutions
i $-\frac{3}{2} < x < \frac{1}{3}$ **j** $x < -\frac{4}{3}$ or $x > 4$
k $x \neq 1$ **l** $\frac{1}{3} \leqslant x \leqslant \frac{1}{2}$ **m** $x < -\frac{1}{6}$ or $x > 1$
n $x \leqslant -\frac{1}{4}$ or $x \geqslant \frac{2}{3}$ **o** $x < \frac{3}{2}$ or $x > 3$

4 **a** **i** $k < -8$ or $k > 0$ **ii** $k = -8$ or 0
iii $-8 < k < 0$
b **i** $-1 < k < 1,\ k \neq 0$ **ii** $k = -1$ or 1
iii $k < -1$ or $k > 1$
c **i** $k < -6$ or $k > 2$ **ii** $k = -6$ or $k = 2$
iii $-6 < k < 2$

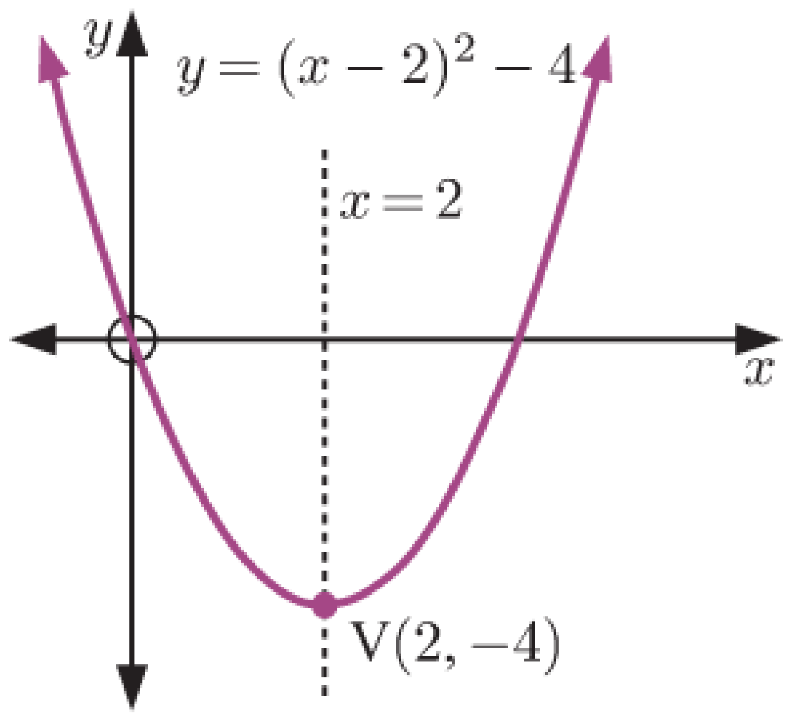
5 **a** **i** $k < -2$ or $k > 6$ **ii** $k = -2$ or $k = 6$
iii $-2 < k < 6$
b **i** $k < -\frac{13}{9}$ or $k > 3$ **ii** $k = -\frac{13}{9}$ or $k = 3$
iii $-\frac{13}{9} < k < 3$
c **i** $-\frac{4}{3} < k < 0,\ k \neq -1$ **ii** $k = -\frac{4}{3}$ or $k = 0$
iii $k < -\frac{4}{3}$ or $k > 0$

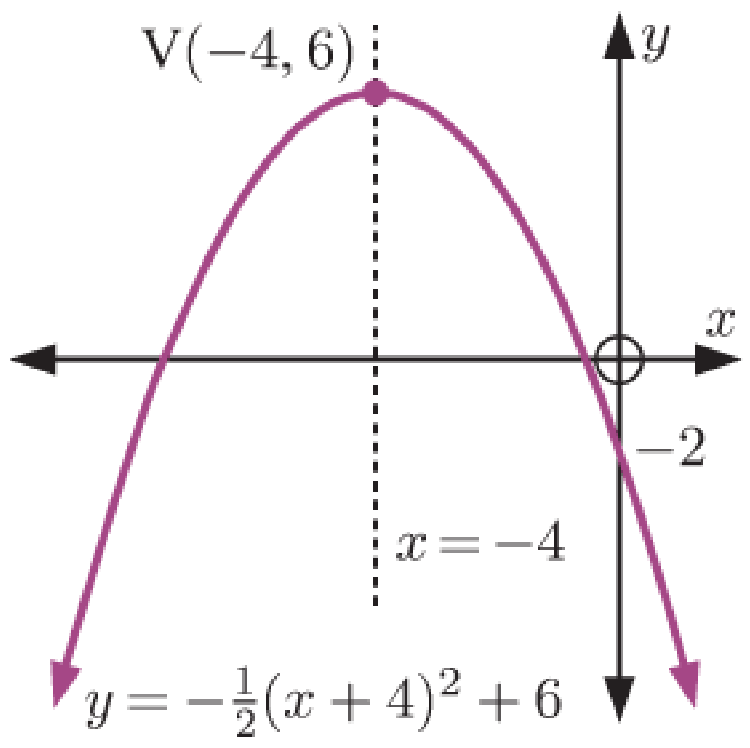
6 **a** $m > 3$ **b** $m < -1$

7 **a** $m < -1$ or $m > 7$ **b** $m = -1$ or $m = 7$
c $-1 < m < 7$

8 **a** $a < 6 - 2\sqrt{10}$ or $a > 6 + 2\sqrt{10}$ **b** $a = 6 \pm 2\sqrt{10}$
c $6 - 2\sqrt{10} < a < 6 + 2\sqrt{10}$

REVIEW SET 14A

1 **a**


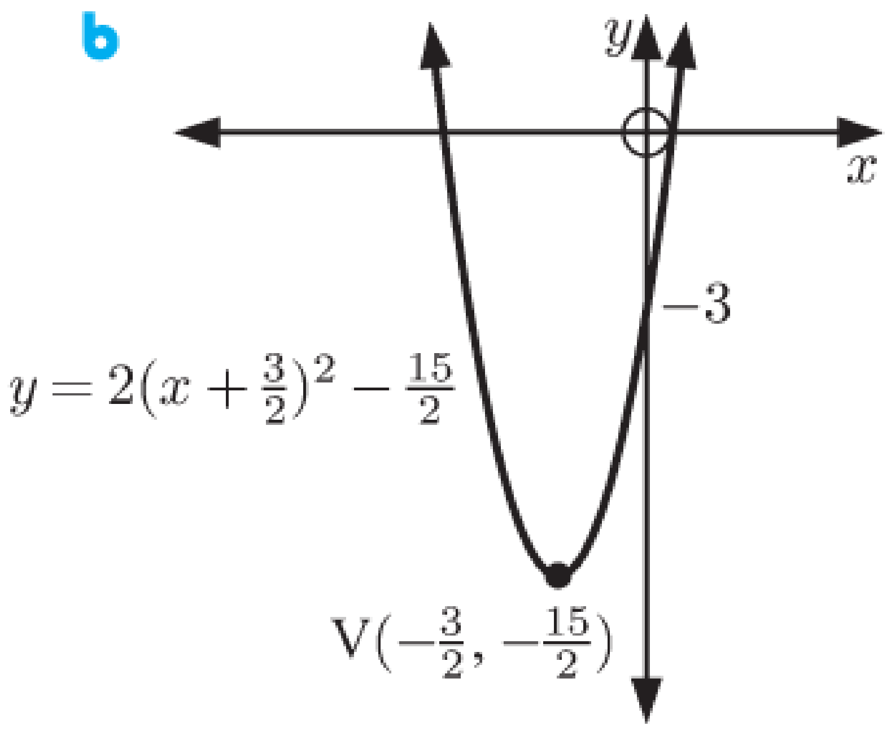
b


2 $(4,\ 4)$ and $(-3,\ 18)$ **3** $k < -3\frac{1}{8}$

4 **a** $m = \frac{9}{8}$ **b** $m < \frac{9}{8}$ **c** $m > \frac{9}{8}$ **5** $\frac{6}{5}$ or $\frac{5}{6}$

6 **Hint:** Let the line have equation $y = mx + 10$.

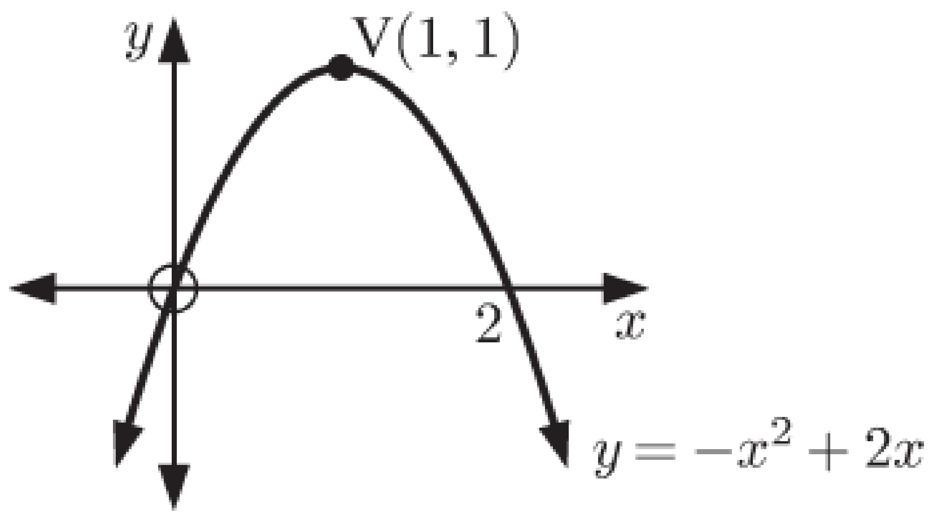
7 **a** $y = 2(x+\frac{3}{2})^2 - \frac{15}{2}$ **b**



8 **a** $y = \frac{20}{9}(x-2)^2 - 20$ **b** $y = -\frac{2}{7}(x-1)(x-7)$

c $y = \frac{2}{9}(x+3)^2$

9



10 $\frac{1}{2}$

11 **a** **i** $\Delta > 0$ **ii** $a < 0$

b **i** $A(-m,\ 0)$, $B(-n,\ 0)$ **ii** $x = \dfrac{-m-n}{2}$

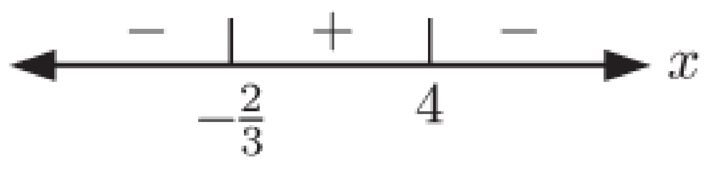
13 $y = -4x^2 + 4x + 24$ **14** $k = \frac{3}{2}$

15 **a** $c = 8$ **b** $3a + b = -3$, $a - b = -5$

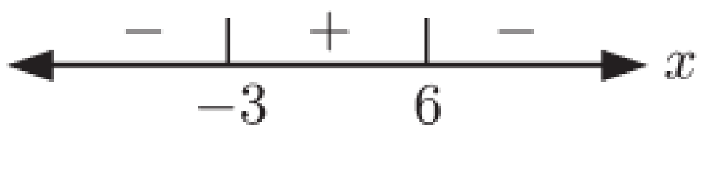
c $a = -2$, $b = 3$, $y = -2x^2 + 3x + 8$

16 $m = -5$ or 19 **17** 21 m

18 **a**



b



19 **a** $x < -2$ or $x > 3$ **b** $-1 \leqslant x \leqslant 5$

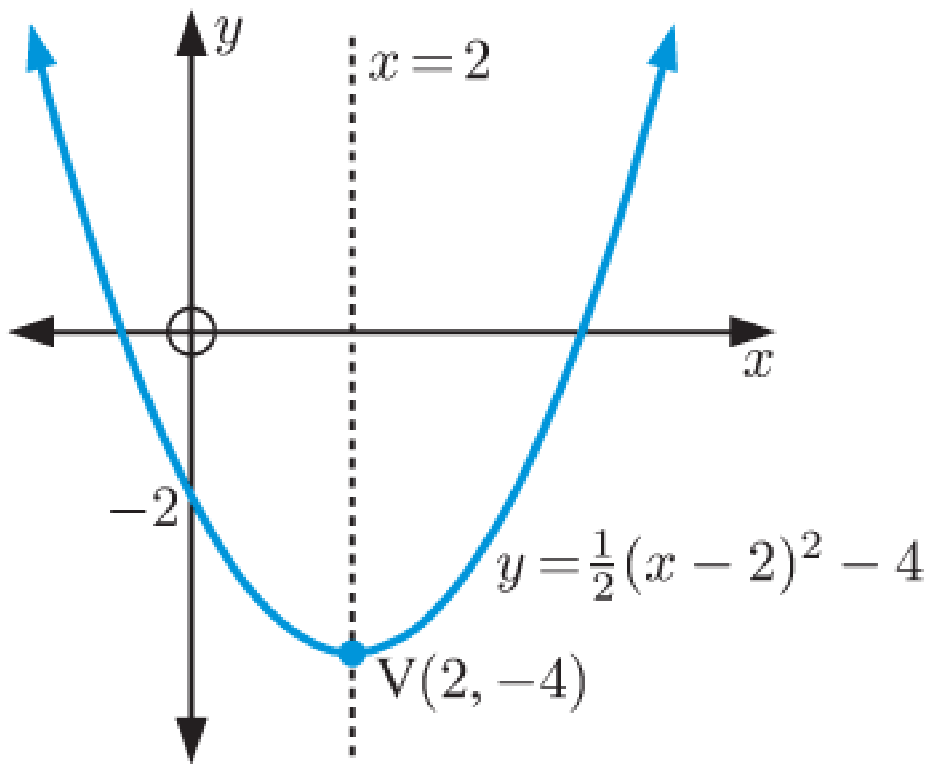
c $x < -\frac{5}{2}$ or $x > 2$

20 **a** $k < 6 - 2\sqrt{5}$ or $k > 6 + 2\sqrt{5}$ **b** $k = 6 \pm 2\sqrt{5}$

c $6 - 2\sqrt{5} < k < 6 + 2\sqrt{5}$

REVIEW SET 14B

1 **a** $x = 2$
b $(2,\ -4)$
c -2
d



2 $x = \frac{4}{3}$, $V(1\frac{1}{3},\ 12\frac{1}{3})$

3 **a** $\Delta = 65$, the graph cuts the x-axis twice

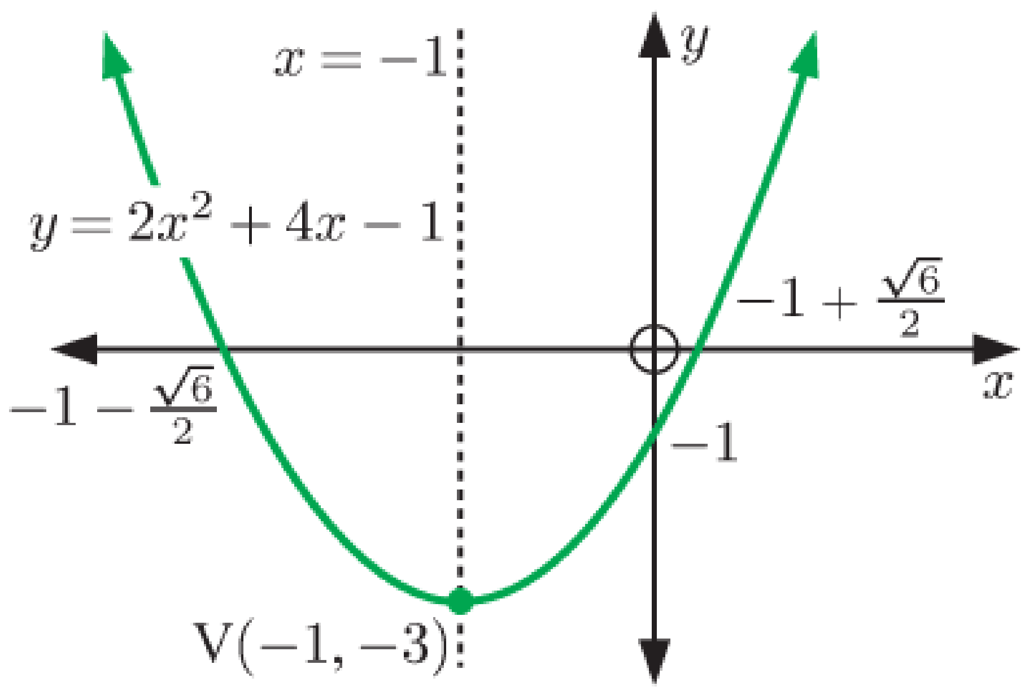
b $\Delta = 97$, the graph cuts the x-axis twice

4 $y = -6(x-2)^2 + 25$

5 **a** $y = -\frac{2}{5}(x+5)(x-1)$ **b** $(-2,\ 3\frac{3}{5})$, $x = -2$

6 **a** $x = -1$
b $(-1,\ -3)$
c x-int. $-1 \pm \frac{\sqrt{6}}{2}$
y-intercept -1

d



7 **a** $y = 2x^2 - 12x + 18$ **b** $y = -\frac{1}{2}x^2 + \frac{1}{2}x + 3$

c $y = x^2 + 7x - 3$ **d** $y = -2x^2 + 12x - 3$

8 **a** $c > -6$

b For example, when $c = -2$, points of intersection are $(-1,\ -5)$ and $(3,\ 7)$.

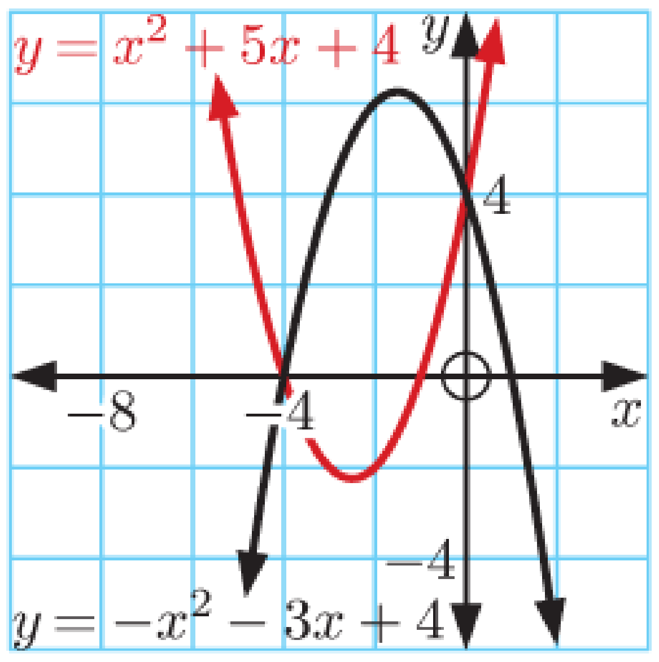
9 **a** minimum is $5\frac{2}{3}$ when $x = -\frac{2}{3}$

b maximum is $5\frac{1}{8}$ when $x = -\frac{5}{4}$

10 **a** $y = 3x^2 - 3x - 18$ **b** -18 **c** $(\frac{1}{2},\ -18\frac{3}{4})$

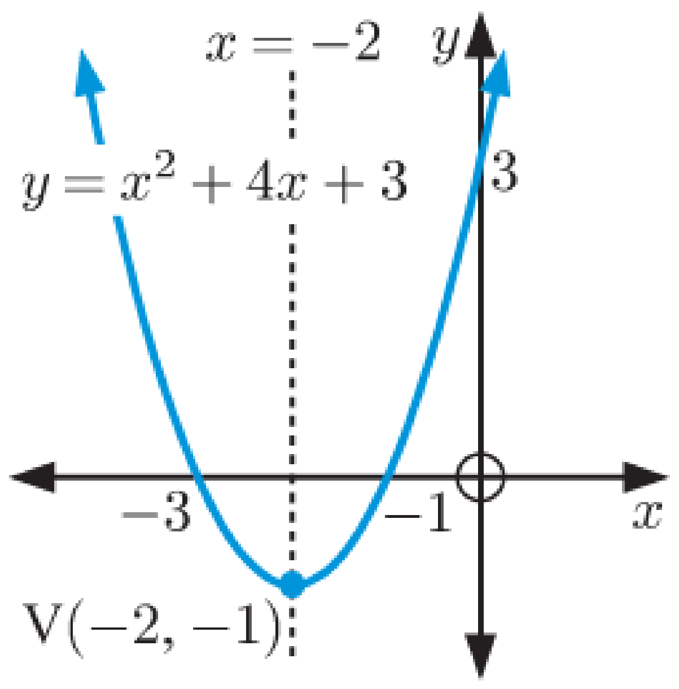
11 **a** $m = -2$, $n = 4$ **b** $k = 7$ **12** ≈ 13.5 cm square

13 **a** $x = -4$ or 0 **b**

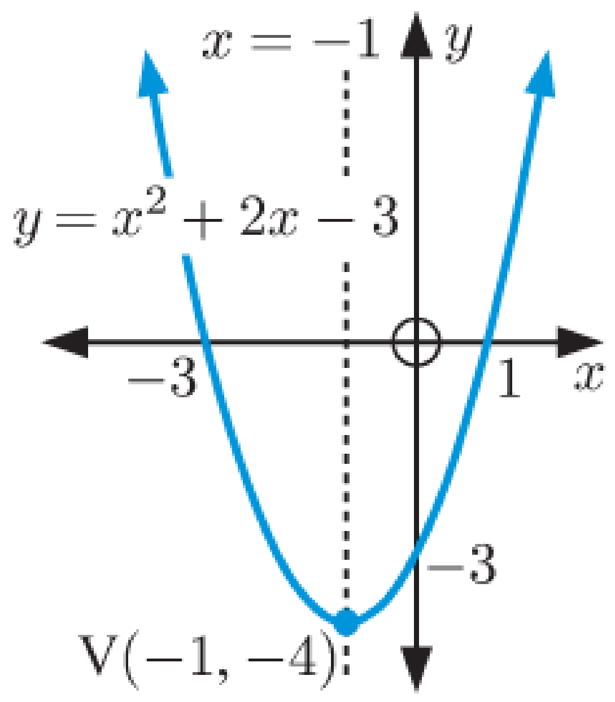


c $x < -4$ or $x > 0$

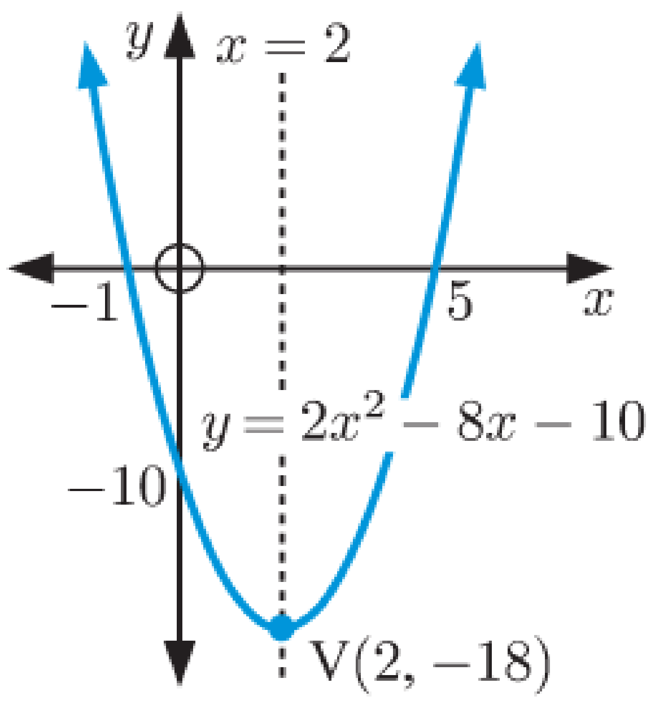
14 **a** **i** $y = (x+2)^2 - 1$
ii $y = (x+3)(x+1)$
iii



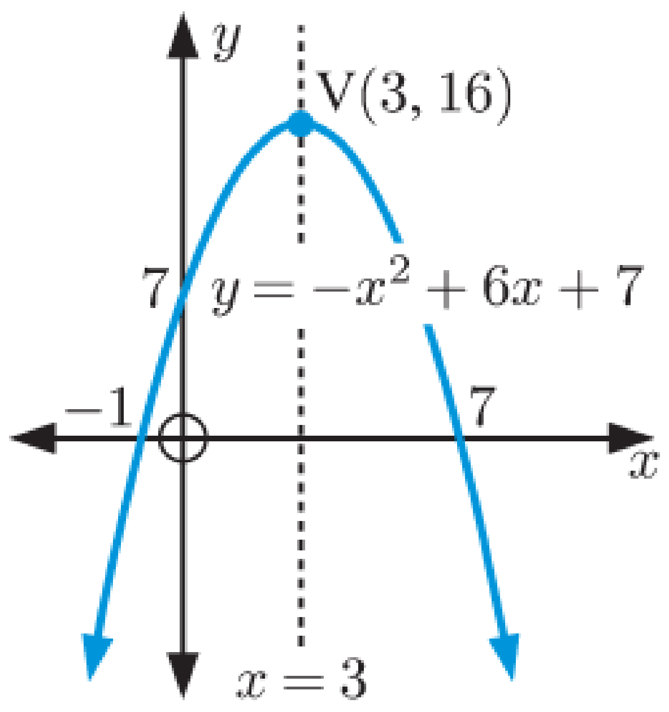
b **i** $y = (x+1)^2 - 4$
ii $y = (x+3)(x-1)$
iii



c **i** $y = 2(x-2)^2 - 18$
ii $y = 2(x-5)(x+1)$
iii



d **i** $y = -(x-3)^2 + 16$
ii $y = -(x-7)(x+1)$
iii

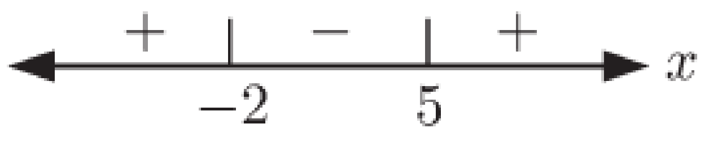


15 **a** $k = \pm 12$ **b** $(0,\ 4)$

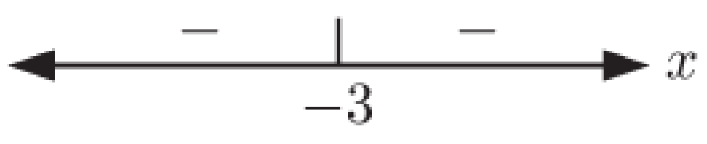
16 **b** $37\frac{1}{2}$ m by $33\frac{1}{3}$ m **c** $1250\ \text{m}^2$

17 **b** \$60, revenue is \$2400 per day

18 **a**



b



19 **a** $0 < x < \frac{3}{4}$ **b** $x \leqslant -1$ or $x \geqslant \frac{5}{2}$

c $x \leqslant \frac{1}{3}$ or $x \geqslant \frac{3}{2}$

20 **a** $-\frac{25}{2} < m < \frac{1}{2},\ \ m \neq 0$ **b** $m = -\frac{25}{2}$ or $m = \frac{1}{2}$
c $m < -\frac{25}{2}$ or $m > \frac{1}{2}$

EXERCISE 15A

1 **a** Is a function, since for every value of x there is only one corresponding value of y.
b Is not a function. When $x = 2,\ y = 1$ or 0.

2 **a** Is a function, since for any value of x there is at most one value of y.
b Is a function, since for any value of x there is at most one value of y.
c Is not a function. If $x^2 + y^2 = 9$, then $y = \pm\sqrt{9 - x^2}$. So, for example, for $x = 2,\ y = \pm\sqrt{5}$.

3 **a** function **b** not a function **c** function
d not a function

4 Not a function as a 2 year old child could pay \$0 or \$20.

5 No, because a vertical line (the y-axis) would cut the relation more than once.

6 No. A vertical line is not a function. It will not pass the "vertical line" test.

7 **a** $y^2 = x$ is a relation but not a function.
$y = x^2$ is a function (and a relation).
$y^2 = x$ has a horizontal axis of symmetry (the x-axis).
$y = x^2$ has a vertical axis of symmetry (the y-axis).
Both $y^2 = x$ and $y = x^2$ have vertex $(0, 0)$.
$y^2 = x$ is a rotation of $y = x^2$ clockwise through 90° about the origin *or* $y^2 = x$ is a reflection of $y = x^2$ in the line $y = x$.
b **i** The part of $y^2 = x$ in the first quadrant.
ii $y = \sqrt{x}$ is a function as any vertical line cuts the graph at most once.

8 **a** Both curves are functions since any vertical line will cut each curve at most once.
b $y = \sqrt[3]{x}$

EXERCISE 15B

1 **a** 2 **b** 2 **c** -16 **d** -68 **e** $\frac{17}{4}$

2 **a** -3 **b** 3 **c** 3 **d** -3 **e** $\frac{15}{2}$

3 **a** **i** $-\frac{7}{2}$ **ii** $-\frac{3}{4}$ **iii** $-\frac{4}{9}$ **b** $x = 4$ **c** $x = \frac{9}{5}$

4 **a** $7 - 3a$ **b** $7 + 3a$ **c** $-3a - 2$ **d** $7 - 6a$
e $1 - 3x$ **f** $7 - 3x - 3h$

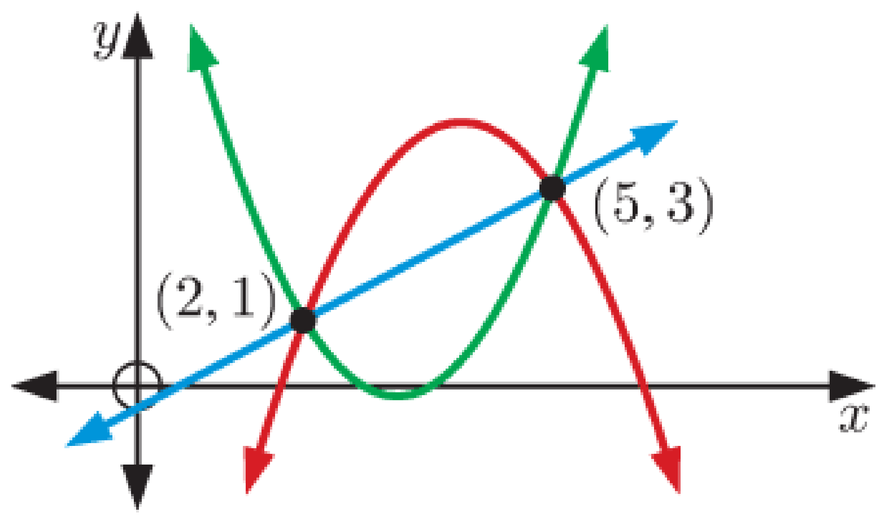
5 **a** $2x^2 + 19x + 43$ **b** $2x^2 - 11x + 13$
c $2x^2 - 3x - 1$ **d** $2x^4 + 3x^2 - 1$
e $18x^2 + 9x - 1$ **f** $2x^2 + (4h + 3)x + 2h^2 + 3h - 1$

6 **a** $9x^2$ **b** $\dfrac{x^2}{4}$ **c** $3x^2$ **d** $2x^2 - 4x + 7$

7 **a** $-\dfrac{1}{x}$ **b** $\dfrac{2}{x}$ **c** $\dfrac{2 + 3x}{x}$ **d** $\dfrac{2x + 1}{x - 1}$

8 f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x.

9 **Note:** Other answers are possible.



10 $f(x) = -2x + 5$

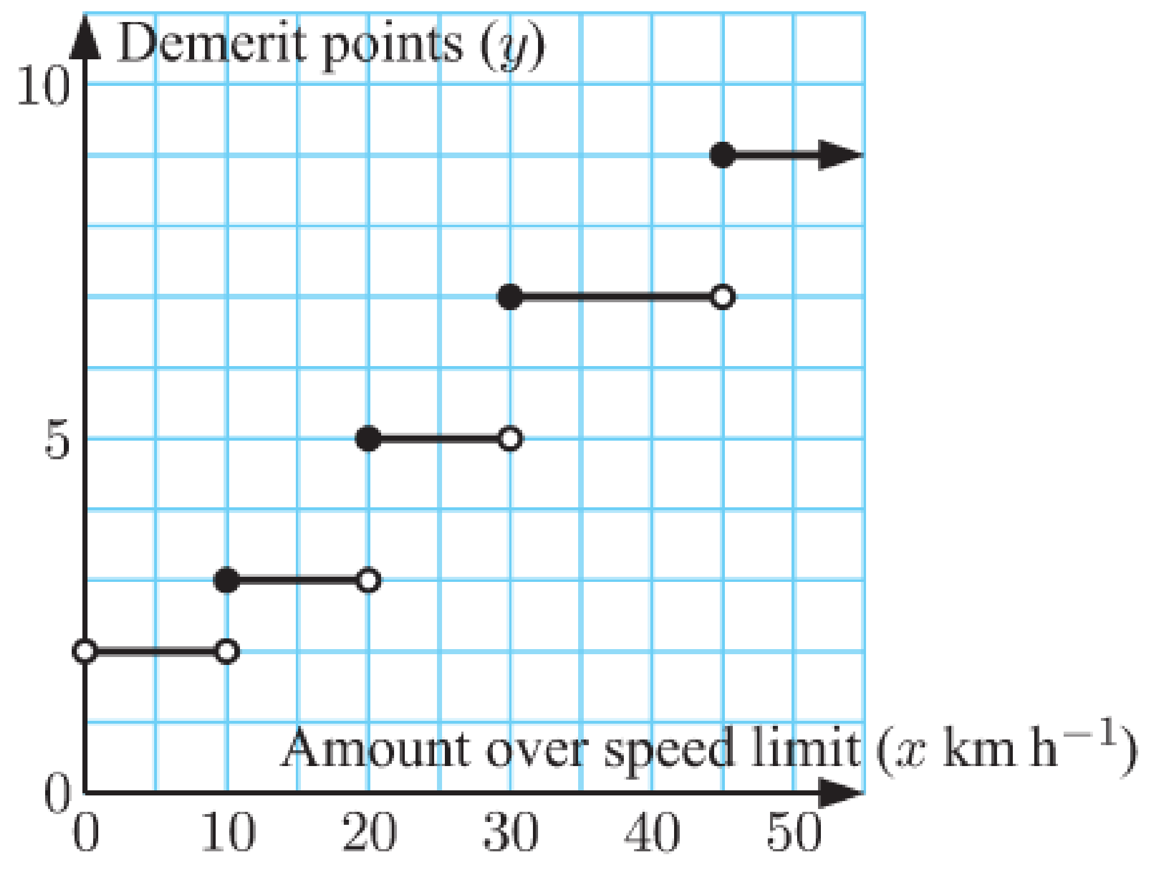
11 **a** $H(30) = 800$. After 30 minutes the balloon is 800 m high.
b $t = 20$ or 70. After 20 minutes and after 70 minutes the balloon is 600 m high.
c $0 \leqslant t \leqslant 80$ **d** 0 m to 900 m

12 $a = 3,\ b = -2$ **13** $a = 3,\ b = -1,\ c = -4$

14 **a** $V(4) = 5400$; $V(4)$ is the value of the photocopier in pounds after 4 years.
b $t = 6$. After 6 years the value of the photocopier is £3600.
c £9000 **d** $0 \leqslant t \leqslant 10$

EXERCISE 15C

1 **a**



b Yes, since for every value of x, there is at most one value of y.
c Domain is $\{x \mid x > 0\}$, Range is $\{2, 3, 5, 7, 9\}$

2 **a** At any moment in time there can be only one temperature, so the graph is a function.
b Domain is $\{t \mid 0 \leqslant t \leqslant 30\}$, Range is $\{T \mid 15 \leqslant T \leqslant 25\}$

3 **a** Domain is $\{x \mid -1 < x \leqslant 5\}$, Range is $\{y \mid 1 < y \leqslant 3\}$
b Domain is $\{x \mid x \neq 2\}$, Range is $\{y \mid y \neq -1\}$
c Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid 0 < y \leqslant 2\}$
d Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \leqslant \frac{25}{4}\}$
e Domain is $\{x \mid x \geqslant -4\}$, Range is $\{y \mid y \geqslant -3\}$
f Domain is $\{x \mid x \neq \pm 2\}$,
Range is $\{y \mid y \leqslant -1$ or $y > 0\}$

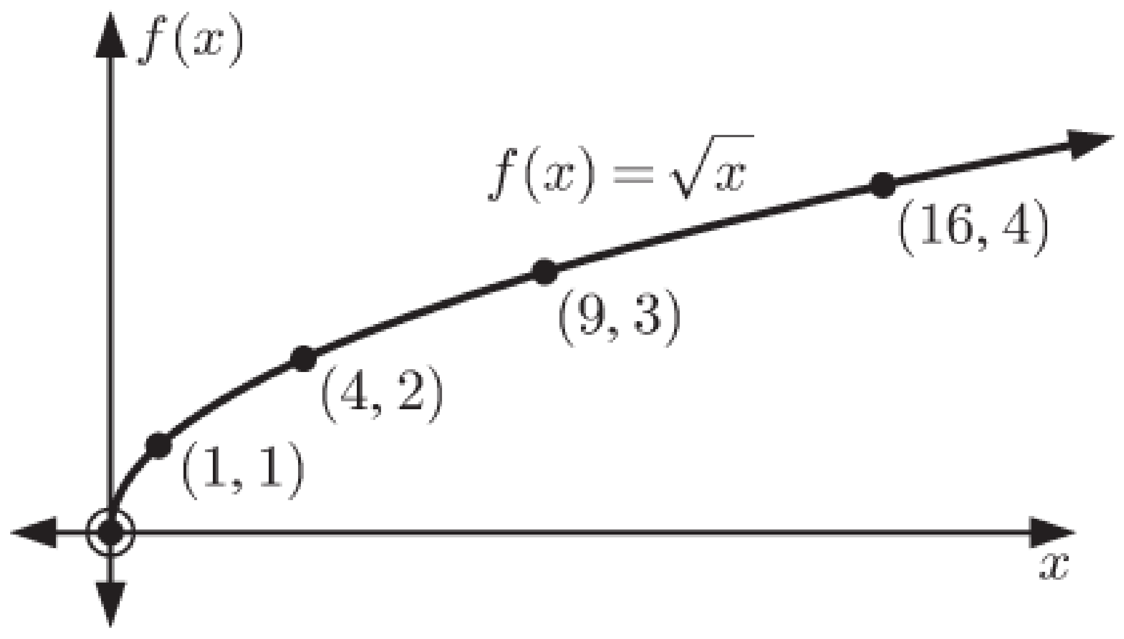
4 **a** true **b** false **c** true **d** true

5 **a** $\{y \mid y \geqslant 0\}$ **b** $\{y \mid y \leqslant 0\}$ **c** $\{y \mid y \geqslant 2\}$
d $\{y \mid y \leqslant 0\}$ **e** $\{y \mid y \leqslant 1\}$ **f** $\{y \mid y \geqslant 3\}$
g $\{y \mid y \geqslant -\frac{9}{4}\}$ **h** $\{y \mid y \leqslant 9\}$ **i** $\{y \mid y \leqslant \frac{25}{12}\}$

6 **a** $\{x \mid x \geqslant 0\}$ **b**

x	0	1	4	9	16
$f(x)$	0	1	2	3	4

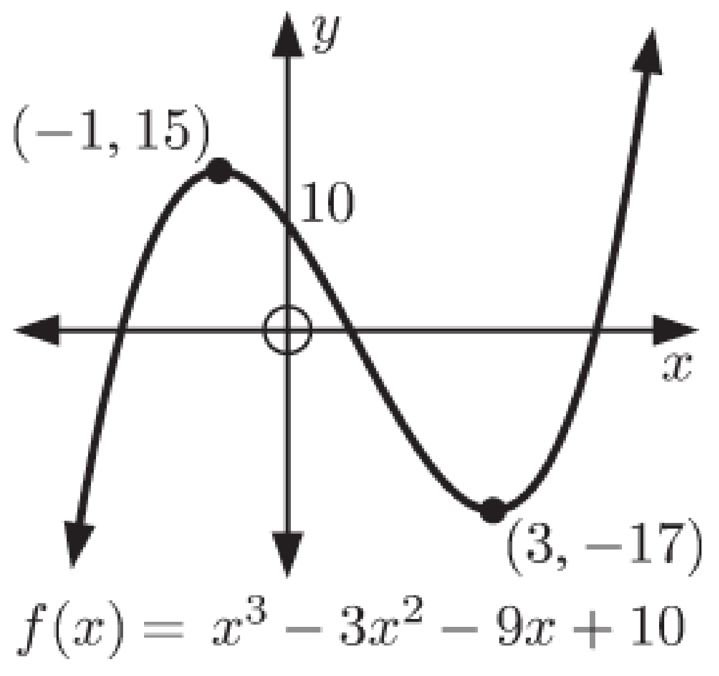
c



d $\{y \mid y \geqslant 0\}$

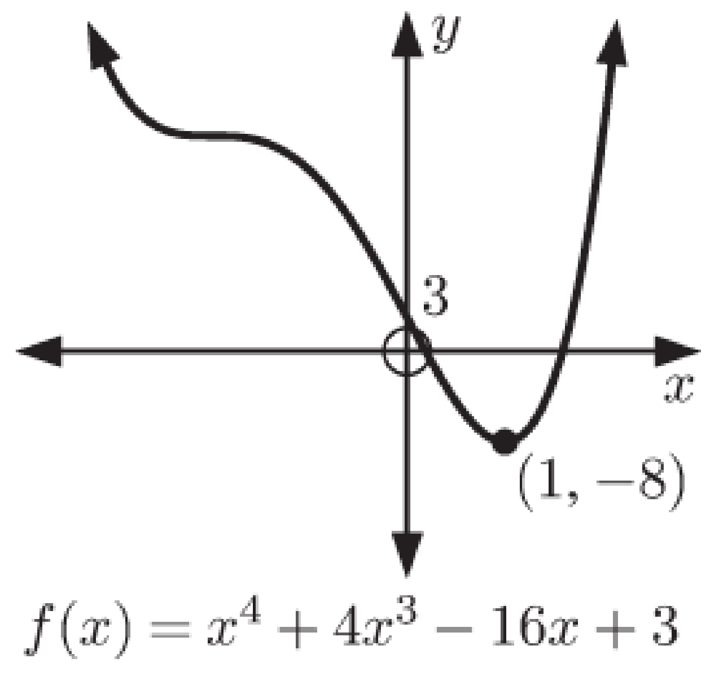
7 **a** Domain is $\{x \mid x \geqslant -6\}$, Range is $\{y \mid y \geqslant 0\}$
b Domain is $\{x \mid x \neq 0\}$, Range is $\{y \mid y > 0\}$
c Domain is $\{x \mid x \neq -1\}$, Range is $\{y \mid y \neq 0\}$
d Domain is $\{x \mid x > 0\}$, Range is $\{y \mid y < 0\}$
e Domain is $\{x \mid x \neq 3\}$, Range is $\{y \mid y \neq 0\}$
f Domain is $\{x \mid x \leqslant 4\}$, Range is $\{y \mid y \geqslant 0\}$

8 **a**



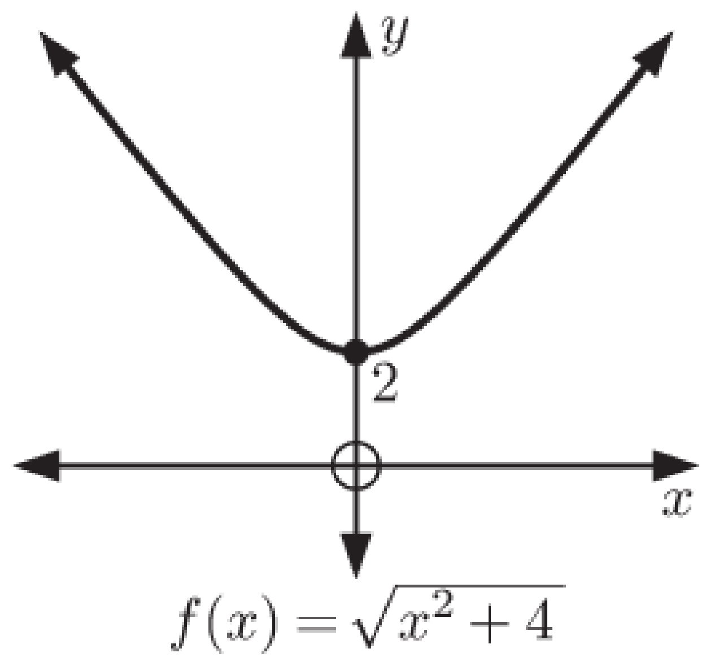
Domain is $\{x \mid x \in \mathbb{R}\}$,
Range is $\{y \mid y \in \mathbb{R}\}$

b



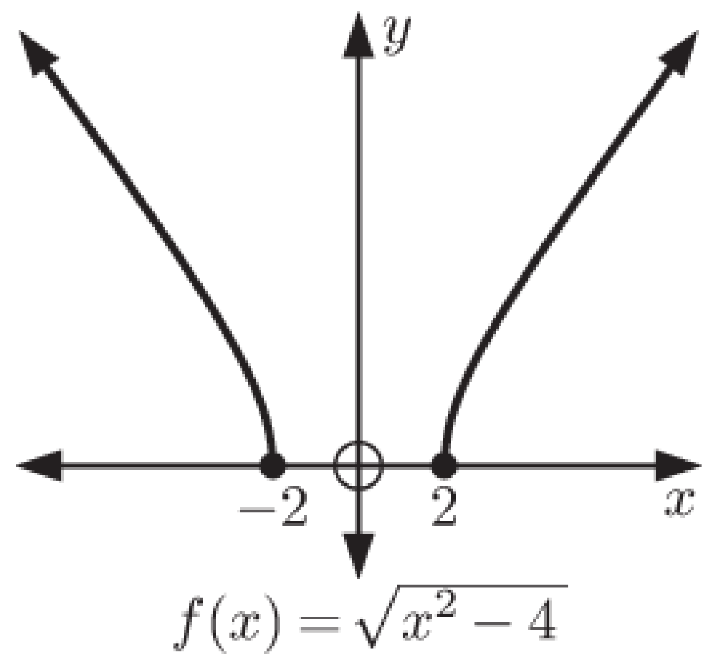
Domain is $\{x \mid x \in \mathbb{R}\}$,
Range is $\{y \mid y \geqslant -8\}$

c



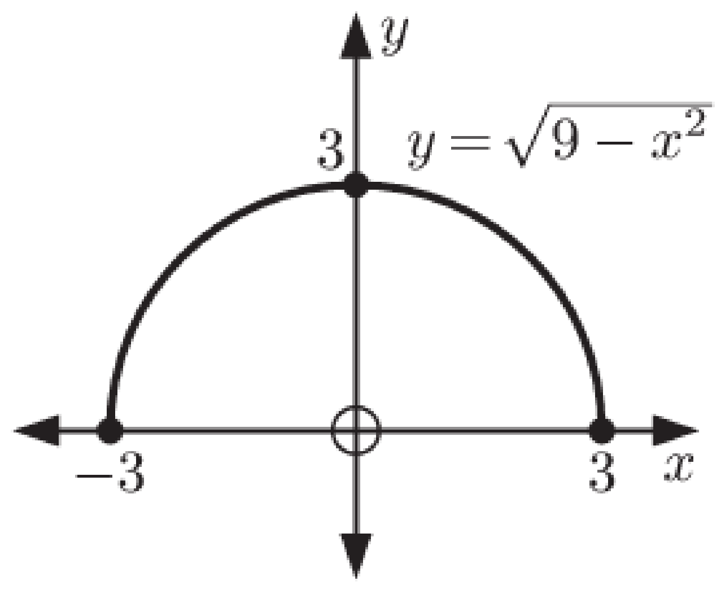
Domain is $\{x \mid x \in \mathbb{R}\}$,
Range is $\{y \mid y \geqslant 2\}$

d



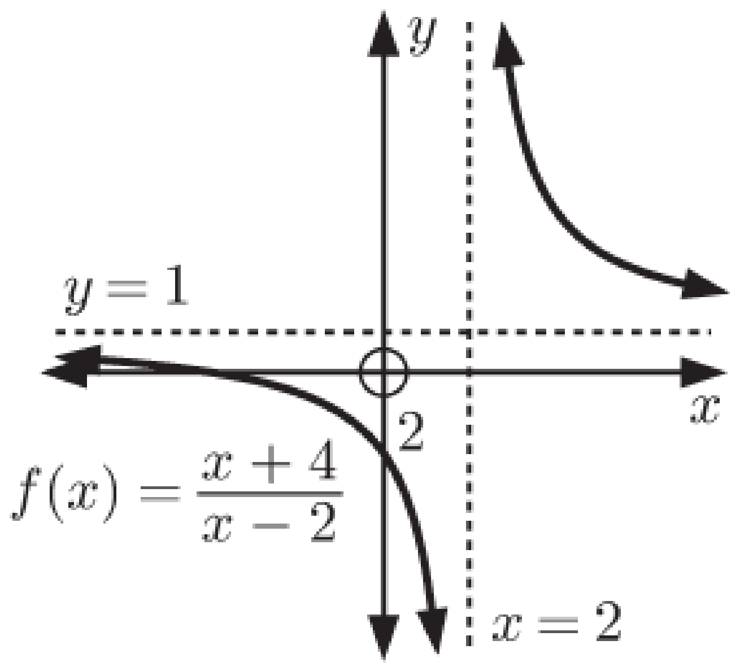
Domain is $\{x \mid x \leqslant -2$
or $x \geqslant 2\}$,
Range is $\{y \mid y \geqslant 0\}$

e



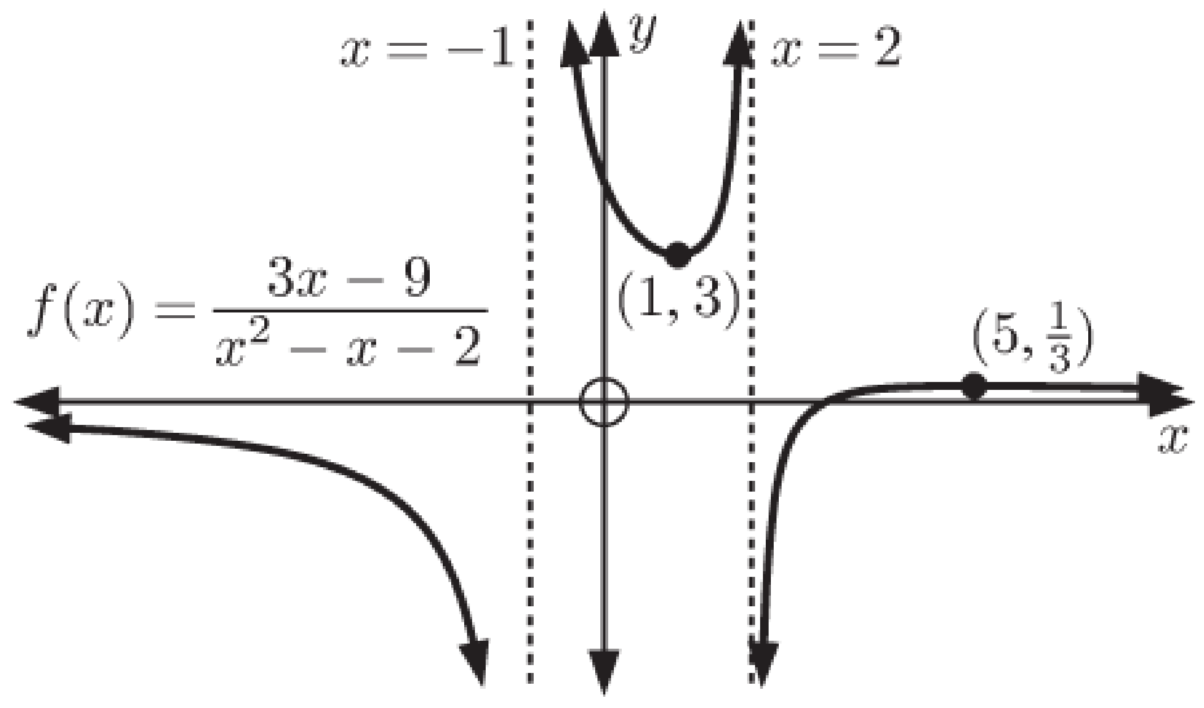
Domain is
$\{x \mid -3 \leqslant x \leqslant 3\}$,
Range is $\{y \mid 0 \leqslant y \leqslant 3\}$

f



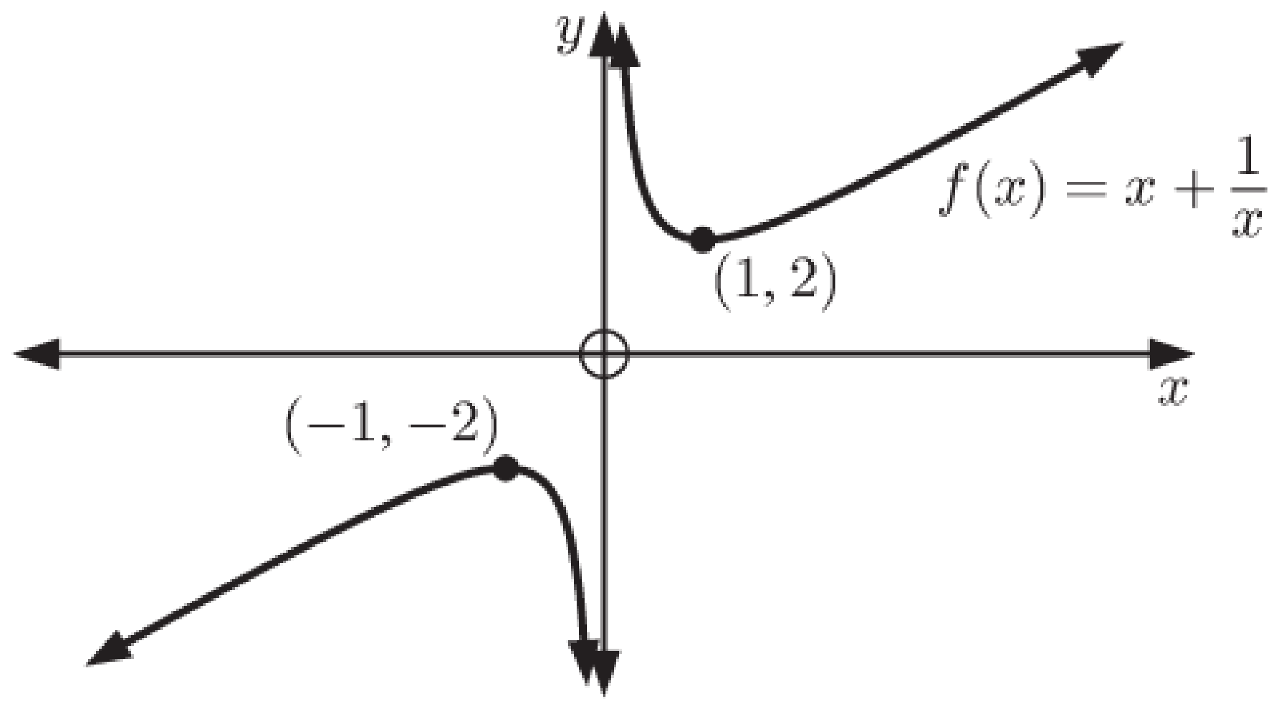
Domain is $\{x \mid x \neq 2\}$,
Range is $\{y \mid y \neq 1\}$

g



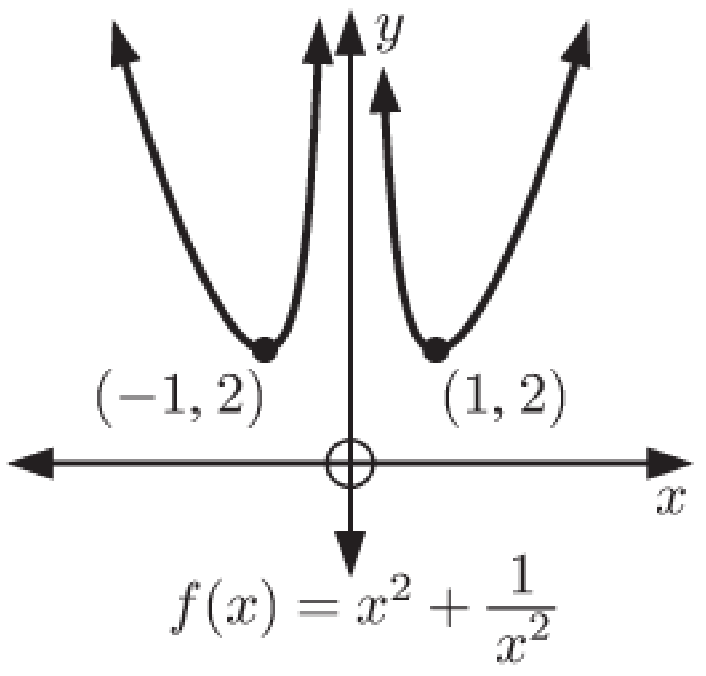
Domain is $\{x \mid x \neq -1 \text{ or } 2\}$,
Range is $\{y \mid y \leqslant \frac{1}{3} \text{ or } y \geqslant 3\}$

h



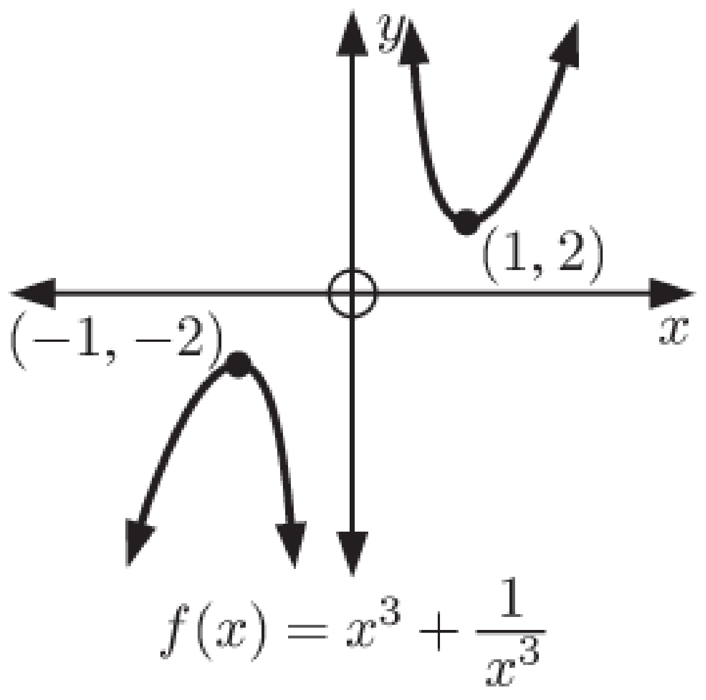
Domain is $\{x \mid x \neq 0\}$,
Range is $\{y \mid y \leqslant -2 \text{ or } y \geqslant 2\}$

i



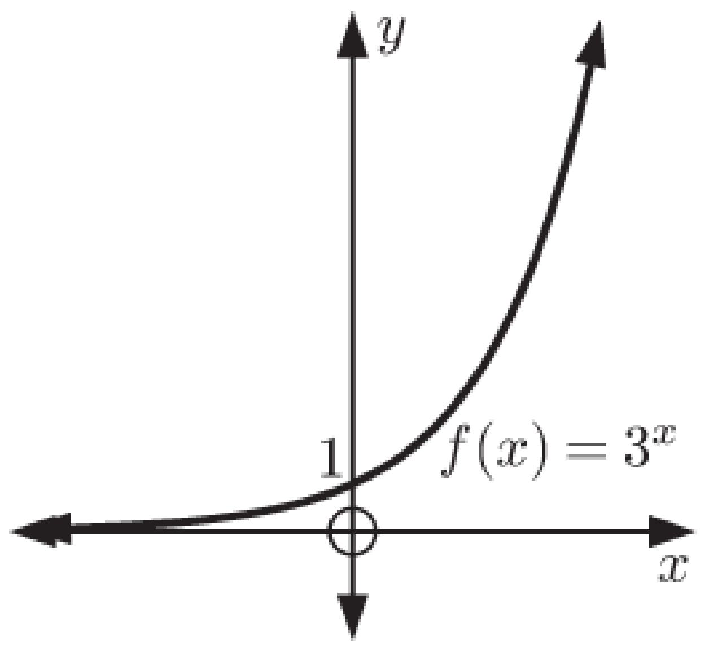
Domain is $\{x \mid x \neq 0\}$,
Range is $\{y \mid y \geqslant 2\}$

j



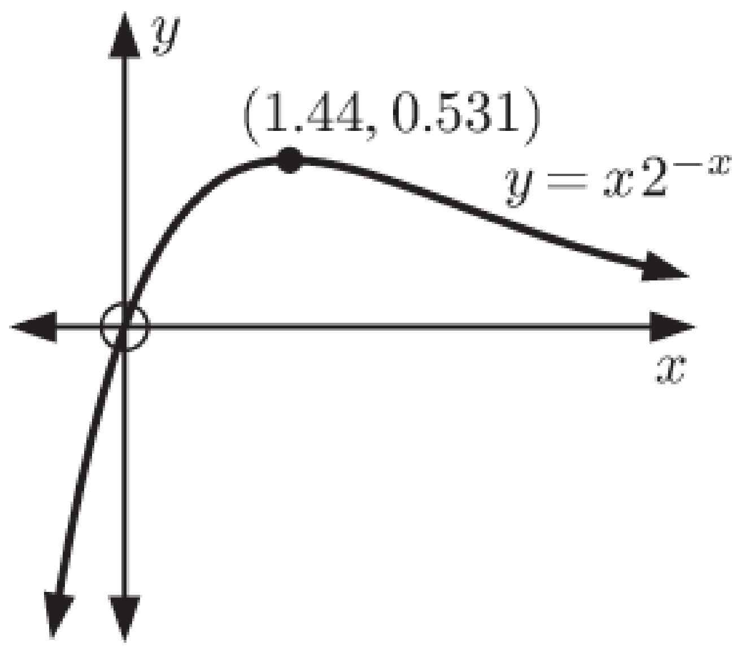
Domain is $\{x \mid x \neq 0\}$,
Range is $\{y \mid y \leqslant -2$
or $y \geqslant 2\}$

k



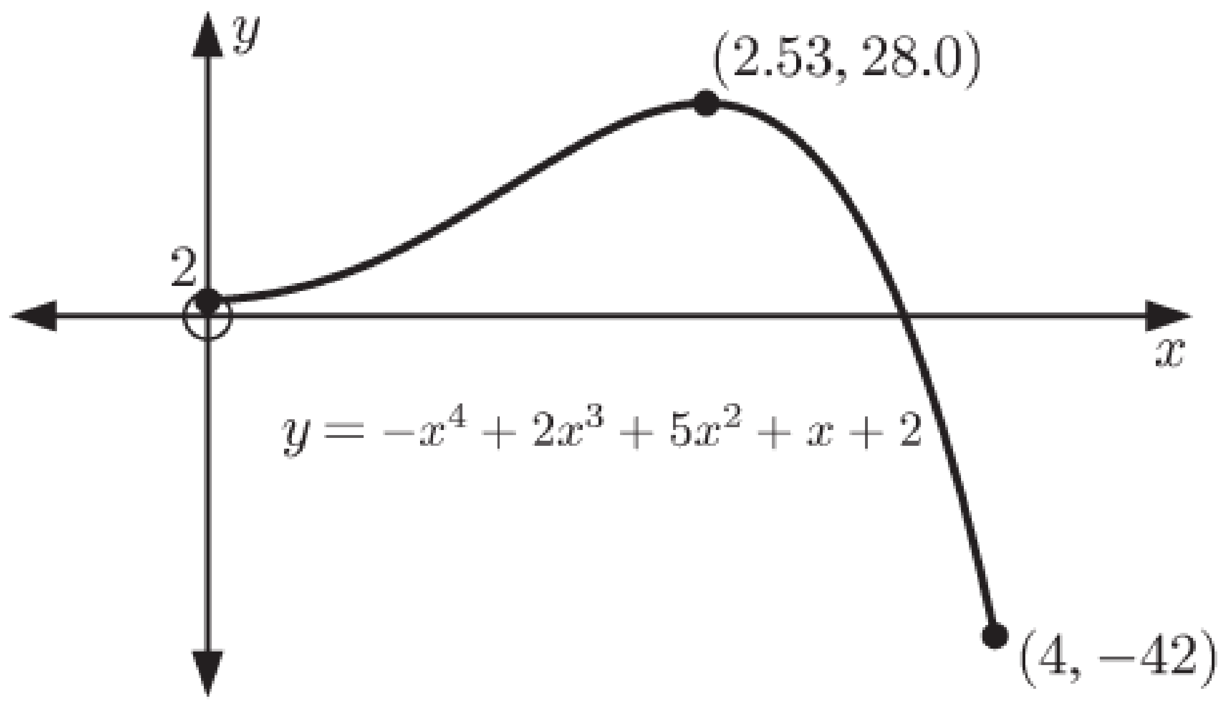
Domain is $\{x \mid x \in \mathbb{R}\}$,
Range is $\{y \mid y > 0\}$

l



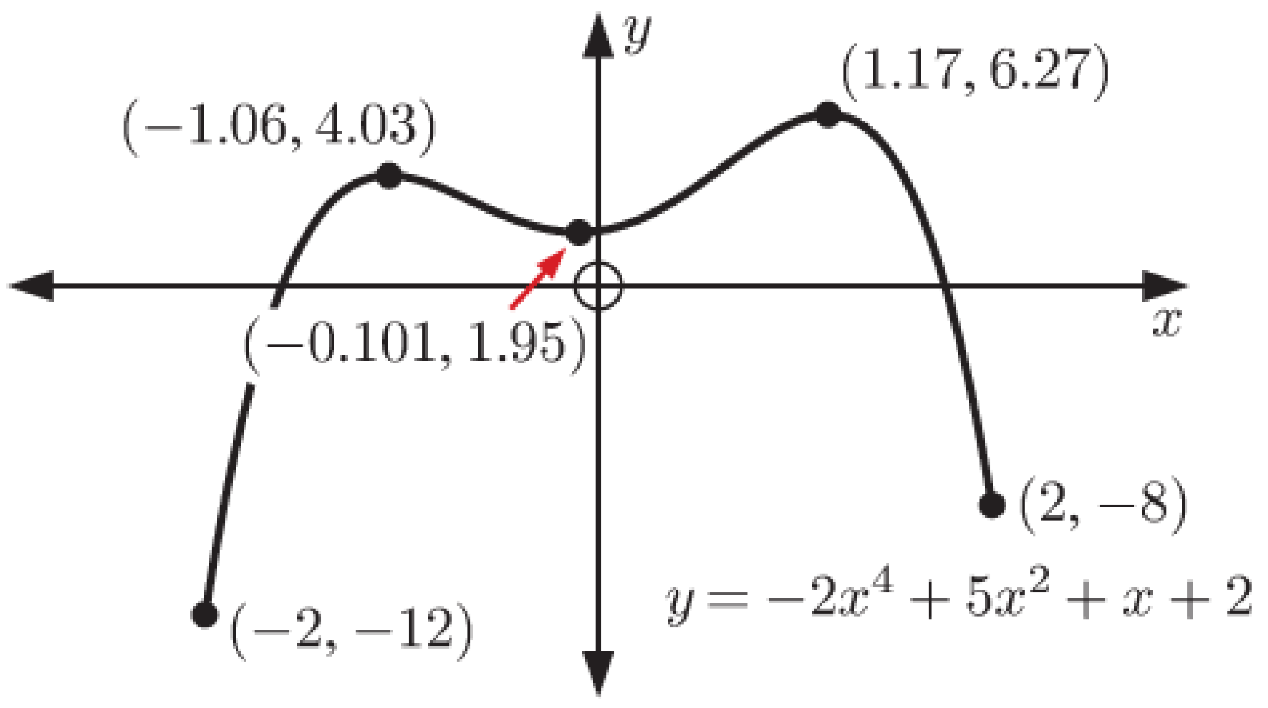
Domain is $\{x \mid x \in \mathbb{R}\}$,
Range is $\{y \mid y \leqslant 0.531\}$

9 **a**



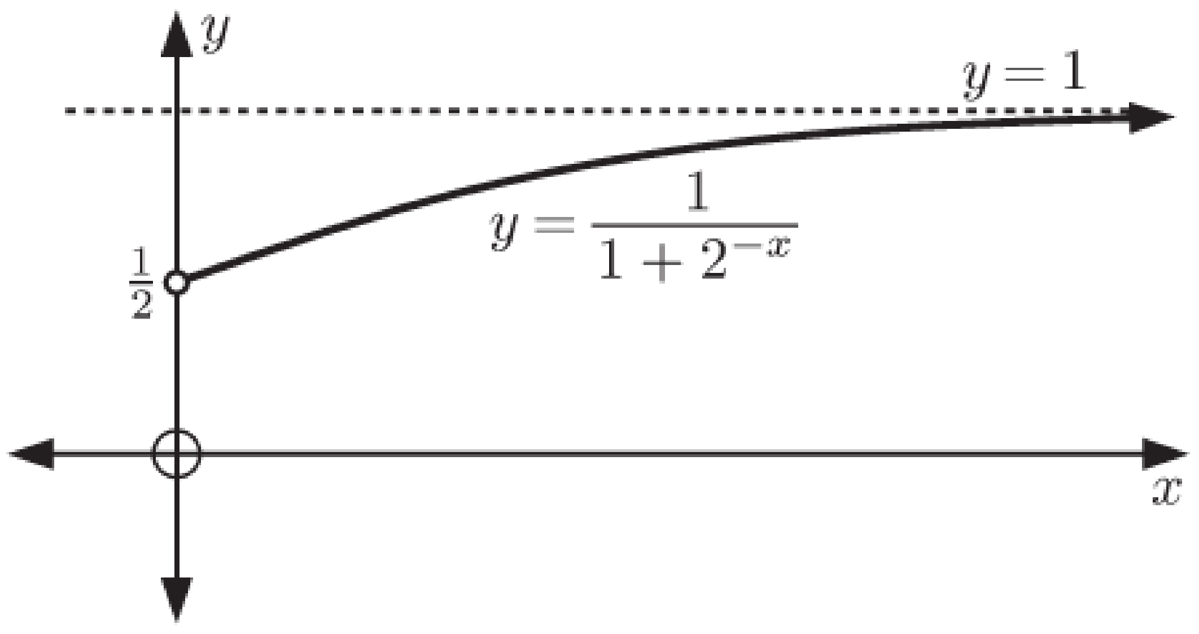
Range is $\{y \mid -42 \leqslant y \leqslant 28.0\}$

b



Range is $\{y \mid -12 \leqslant y \leqslant 6.27\}$

c



Range is $\{y \mid \frac{1}{2} < y < 1\}$

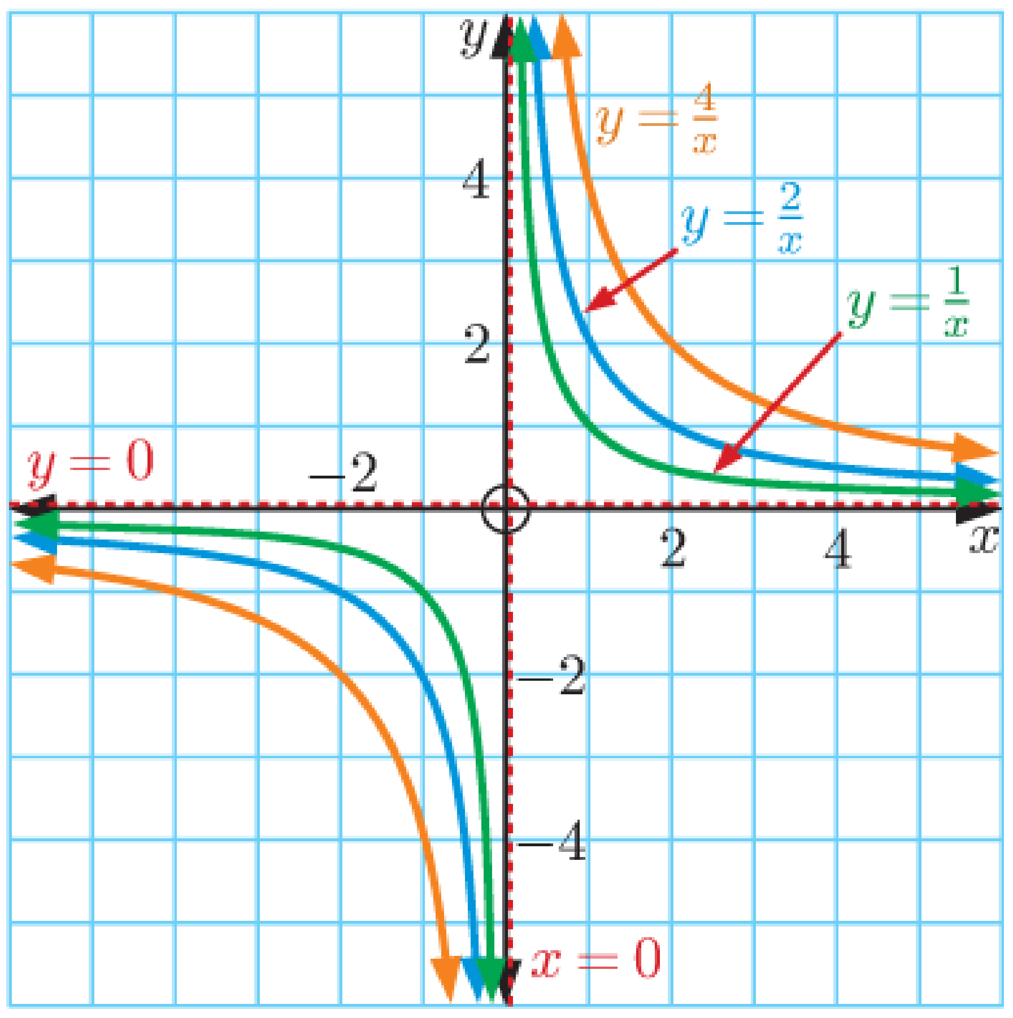
10 **a** $k \geqslant \frac{25}{4}$ **b** Range is $\{y \mid y \geqslant \sqrt{k - \frac{25}{4}}\}$

11 **a** Domain is $\{x \mid -2 \leqslant x \leqslant 2\}$
Range is $\{y \mid -2 \leqslant y \leqslant 2\}$

b Domain is $\{-2, -1, 0, 1, 2\}$
Range is $\{-2, -\sqrt{3}, 0, \sqrt{3}, 2\}$

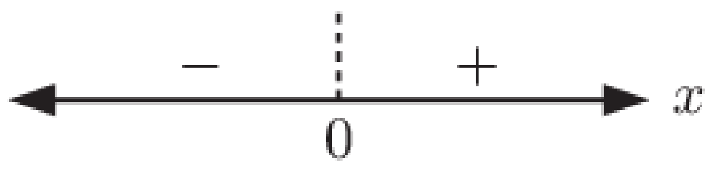
EXERCISE 15D.1

1 **a**

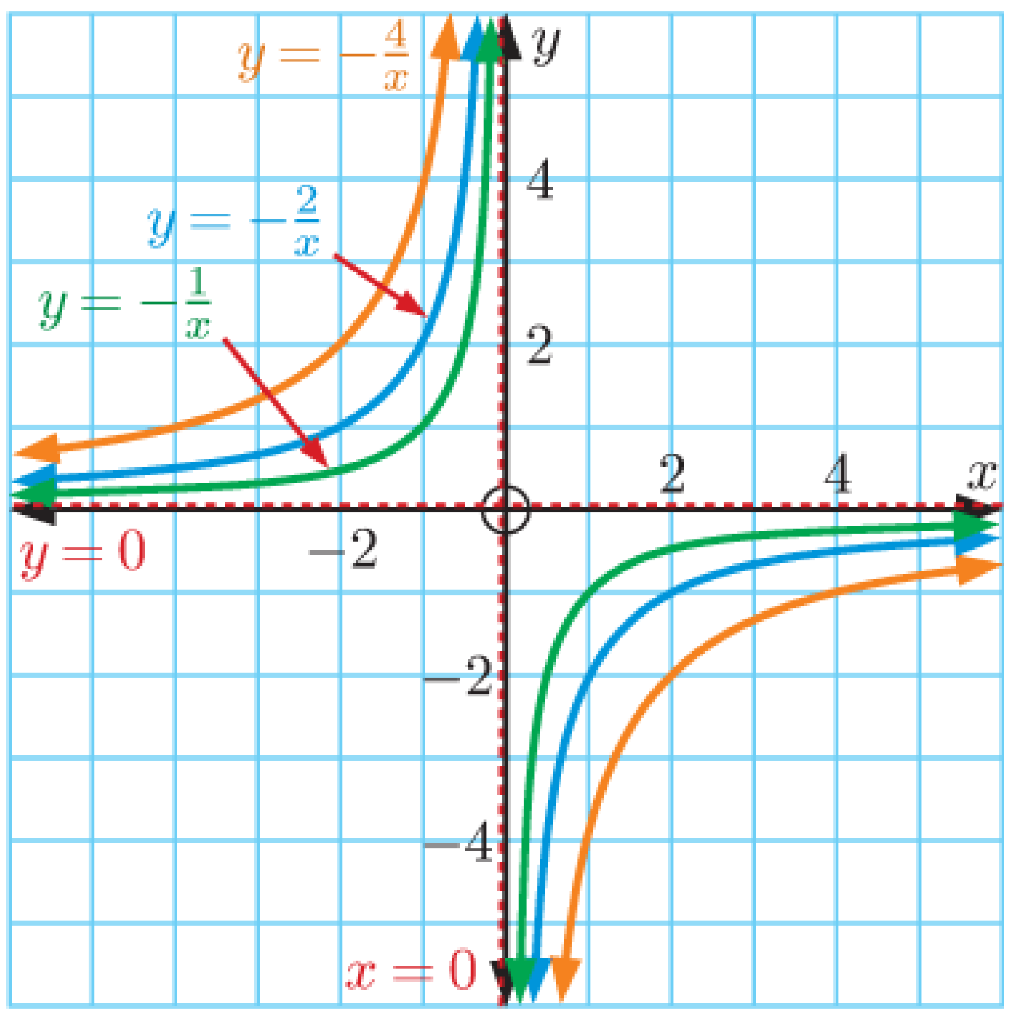


b **i** As k becomes larger the graphs move further from the origin.

ii quadrants 1 and 3 **iii**

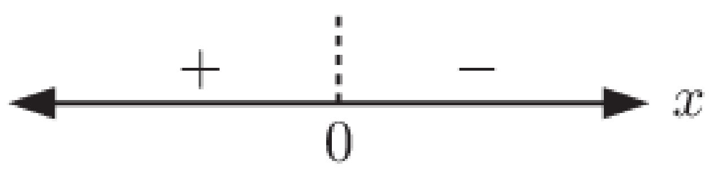


2 **a**



b **i** As $|\,k\,|$ becomes larger, the graphs move further from the origin.

ii quadrants 2 and 4 **iii**



3 **a** $\{x \mid x \neq 0\}$ **b** $\{y \mid y \neq 0\}$ **c** $x = 0$ **d** $y = 0$

4 **a** $y = \dfrac{6}{x}$ **b** $y = \dfrac{15}{x}$ **c** $y = -\dfrac{36}{x}$

EXERCISE 15D.2

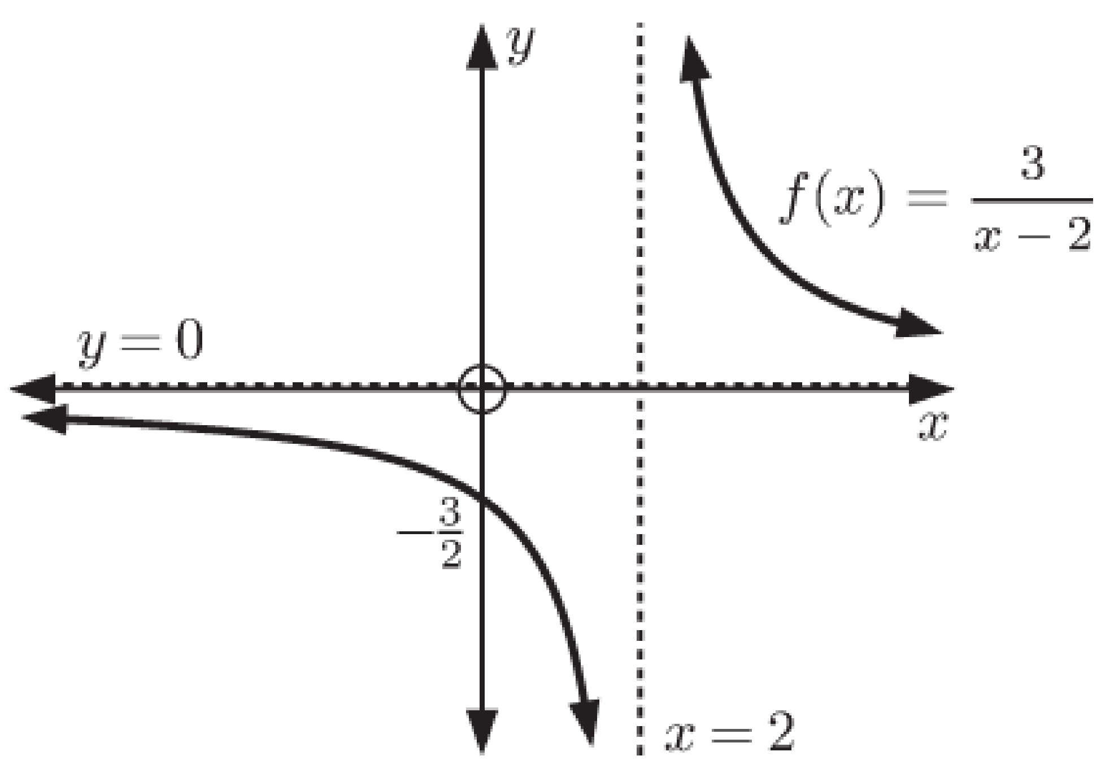
1 **a** **i** vertical asymptote $x = 2$, horizontal asymptote $y = 0$

ii Domain is $\{x \mid x \neq 2\}$, Range is $\{y \mid y \neq 0\}$

iii no x-intercept, y-intercept $-\frac{3}{2}$

iv as $x \to 2^-$, $f(x) \to -\infty$
as $x \to 2^+$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to 0^-$
as $x \to \infty$, $f(x) \to 0^+$

v



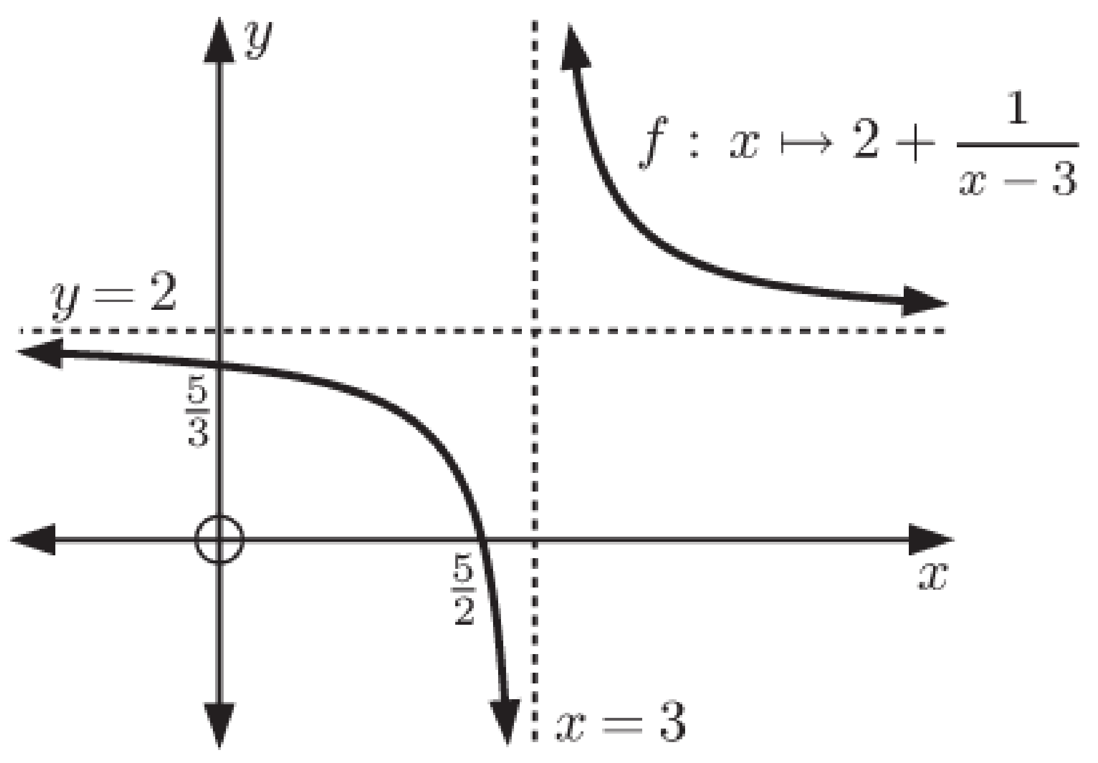
b **i** vertical asymptote $x = 3$, horizontal asymptote $y = 2$

ii Domain is $\{x \mid x \neq 3\}$, Range is $\{y \mid y \neq 2\}$

iii x-intercept $\frac{5}{2}$, y-intercept $\frac{5}{3}$

iv as $x \to 3^-$, $f(x) \to -\infty$
as $x \to 3^+$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to 2^-$
as $x \to \infty$, $f(x) \to 2^+$

v



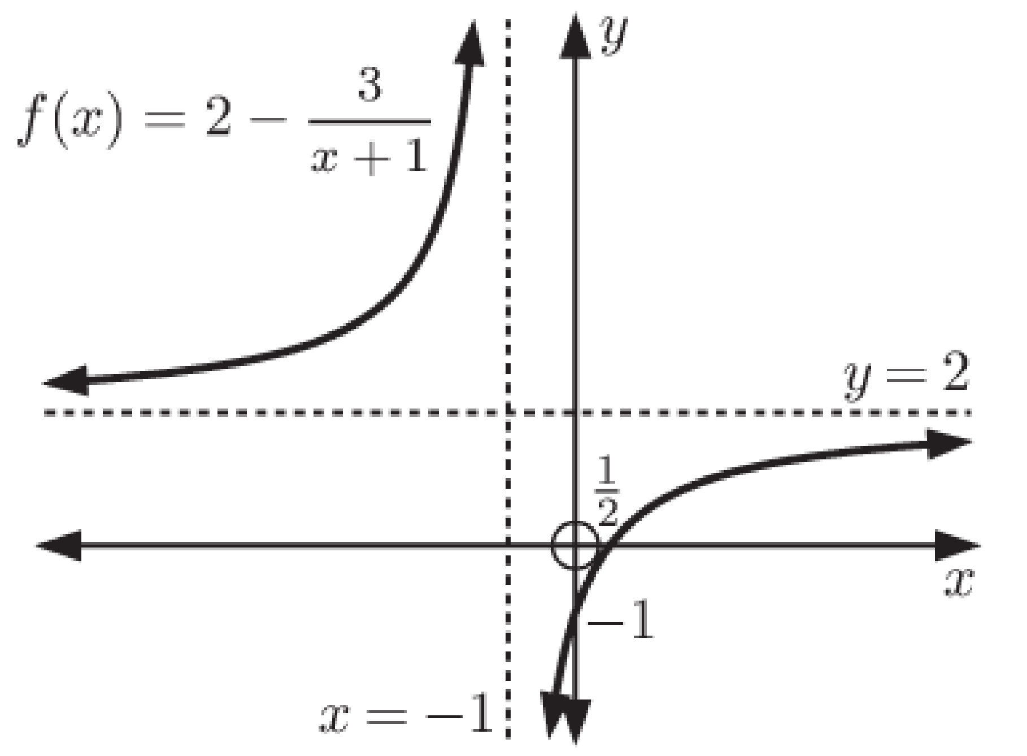
c **i** vertical asymptote $x = -1$, horizontal asymptote $y = 2$

ii Domain is $\{x \mid x \neq -1\}$, Range is $\{y \mid y \neq 2\}$

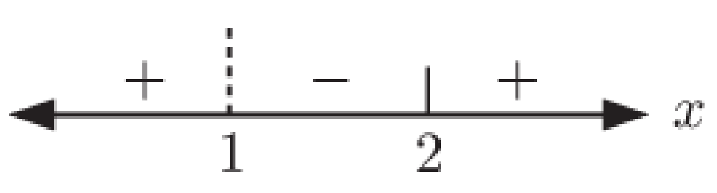
iii x-intercept $\frac{1}{2}$, y-intercept -1

iv as $x \to -1^-$, $f(x) \to \infty$
as $x \to -1^+$, $f(x) \to -\infty$
as $x \to -\infty$, $f(x) \to 2^+$
as $x \to \infty$, $f(x) \to 2^-$

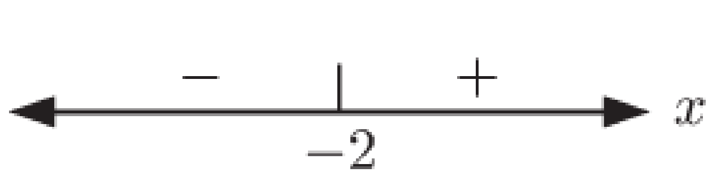
v



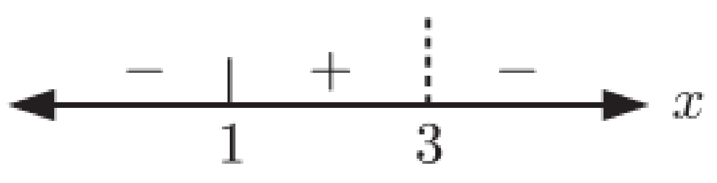
2 **a**



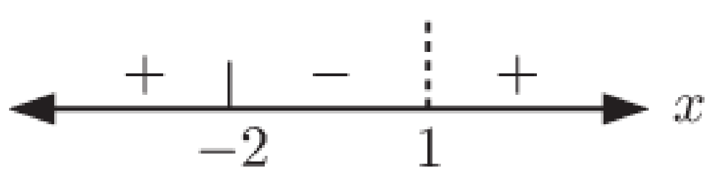
b



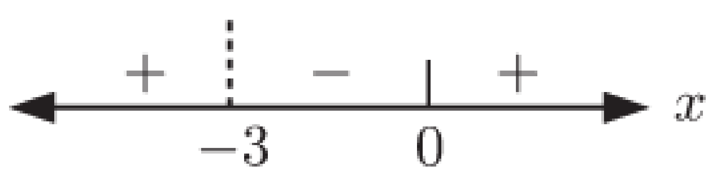
c



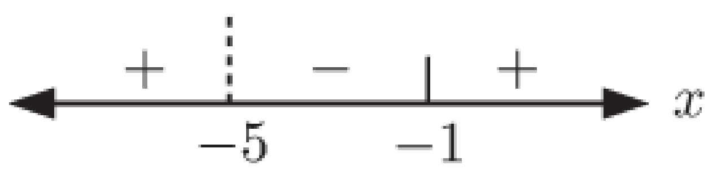
3 **a**



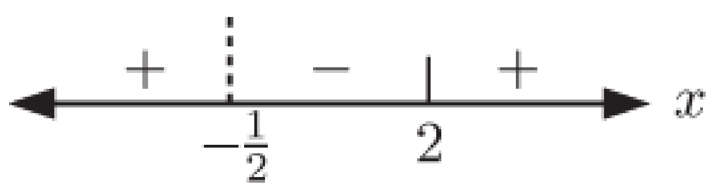
b



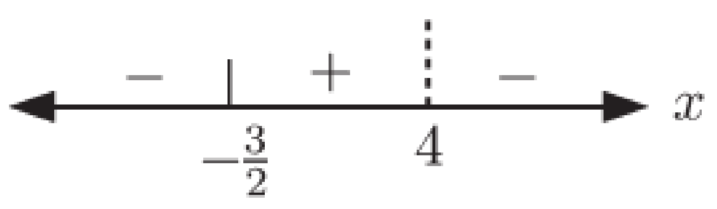
c



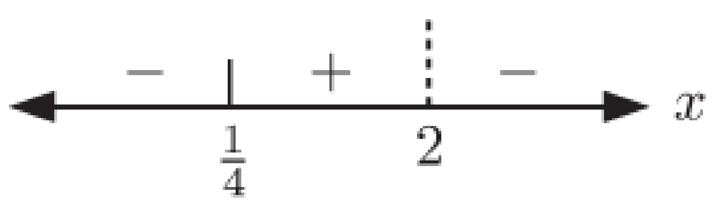
d



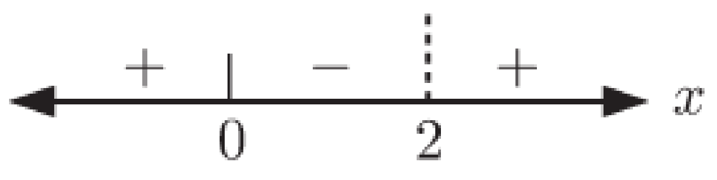
e



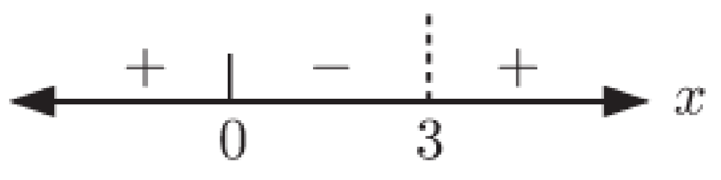
f



g



h

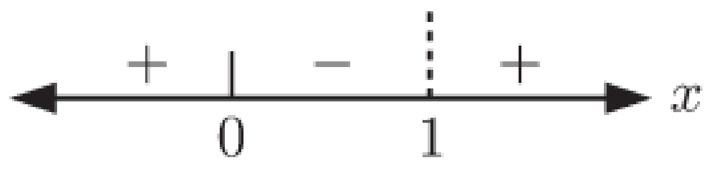


4 **a** **i** vertical asymptote is $x = 1$

ii x-intercept 0, y-intercept 0

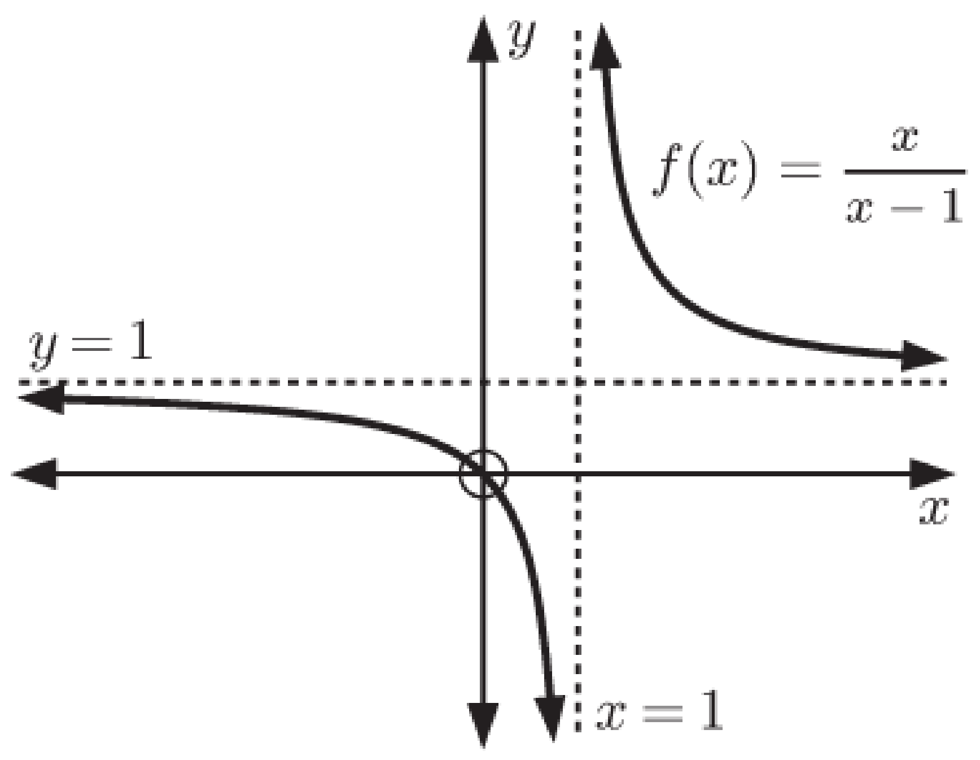
iii $f(x) = 1 + \dfrac{1}{x-1}$, horizontal asymptote is $y = 1$

iv



v as $x \to 1^-$, $f(x) \to -\infty$
as $x \to 1^+$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to 1^-$
as $x \to \infty$, $f(x) \to 1^+$

vi

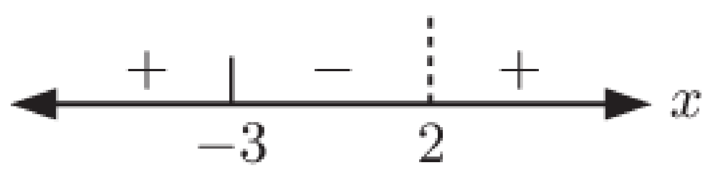


b **i** vertical asymptote is $x = 2$

ii x-intercept -3, y-intercept $-\frac{3}{2}$

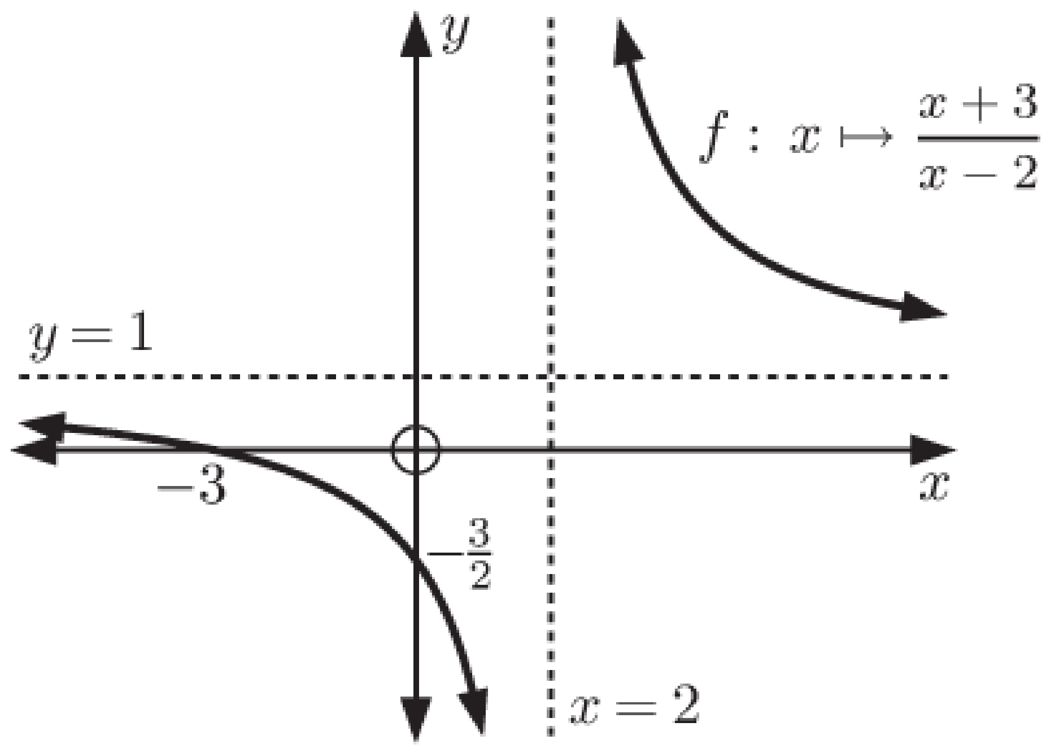
iii $f(x) = 1 + \dfrac{5}{x-2}$, horizontal asymptote is $y = 1$

iv



v as $x \to 2^-$, $f(x) \to -\infty$
as $x \to 2^+$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to 1^-$
as $x \to \infty$, $f(x) \to 1^+$

vi

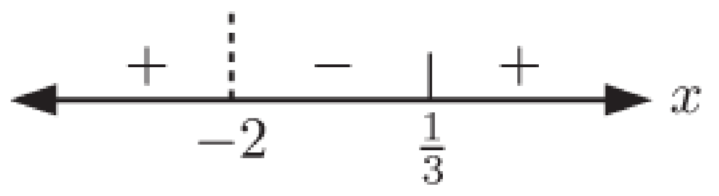


c **i** vertical asymptote is $x = -2$

ii x-intercept $\frac{1}{3}$, y-intercept $-\frac{1}{2}$

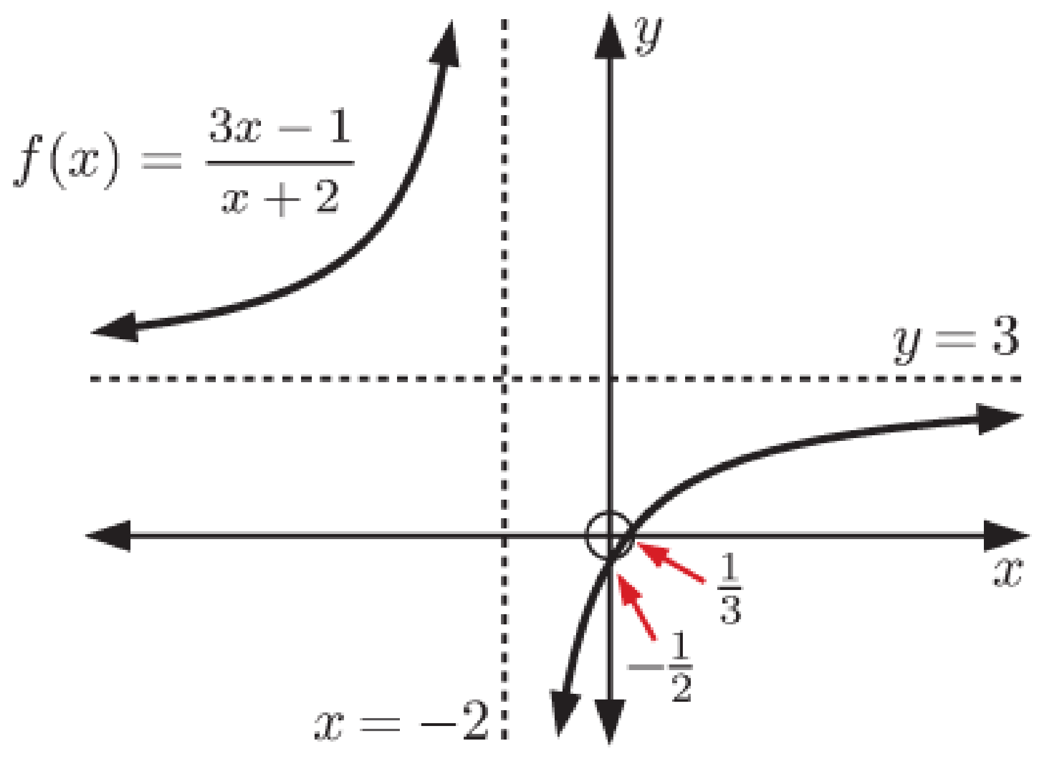
iii $f(x) = 3 - \dfrac{7}{x+2}$, horizontal asymptote is $y = 3$

iv



v as $x \to -2^-$, $f(x) \to \infty$
as $x \to -2^+$, $f(x) \to -\infty$
as $x \to -\infty$, $f(x) \to 3^+$
as $x \to \infty$, $f(x) \to 3^-$

vi

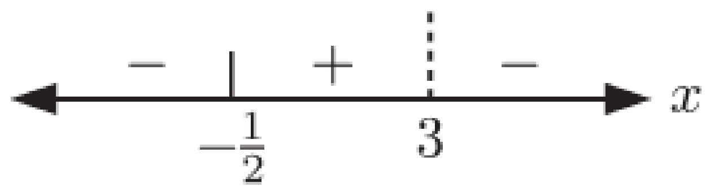


d **i** vertical asymptote is $x = 3$

ii x-intercept $-\frac{1}{2}$, y-intercept $\frac{1}{3}$

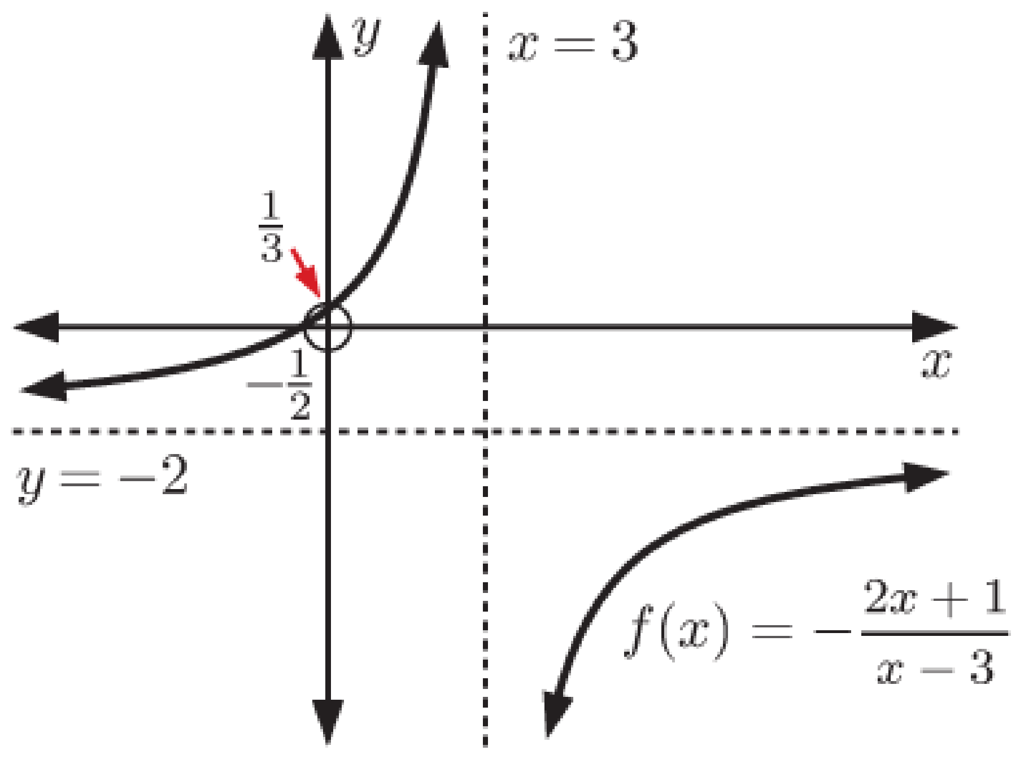
iii $f(x) = -2 - \dfrac{7}{x-3}$, horizontal asymptote is $y = -2$

iv



v as $x \to 3^-$, $f(x) \to \infty$
as $x \to 3^+$, $f(x) \to -\infty$
as $x \to -\infty$, $f(x) \to -2^+$
as $x \to \infty$, $f(x) \to -2^-$

vi

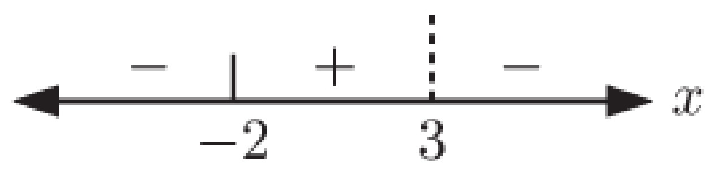


e **i** vertical asymptote is $x = 3$

ii x-intercept -2, y-intercept $\frac{4}{3}$

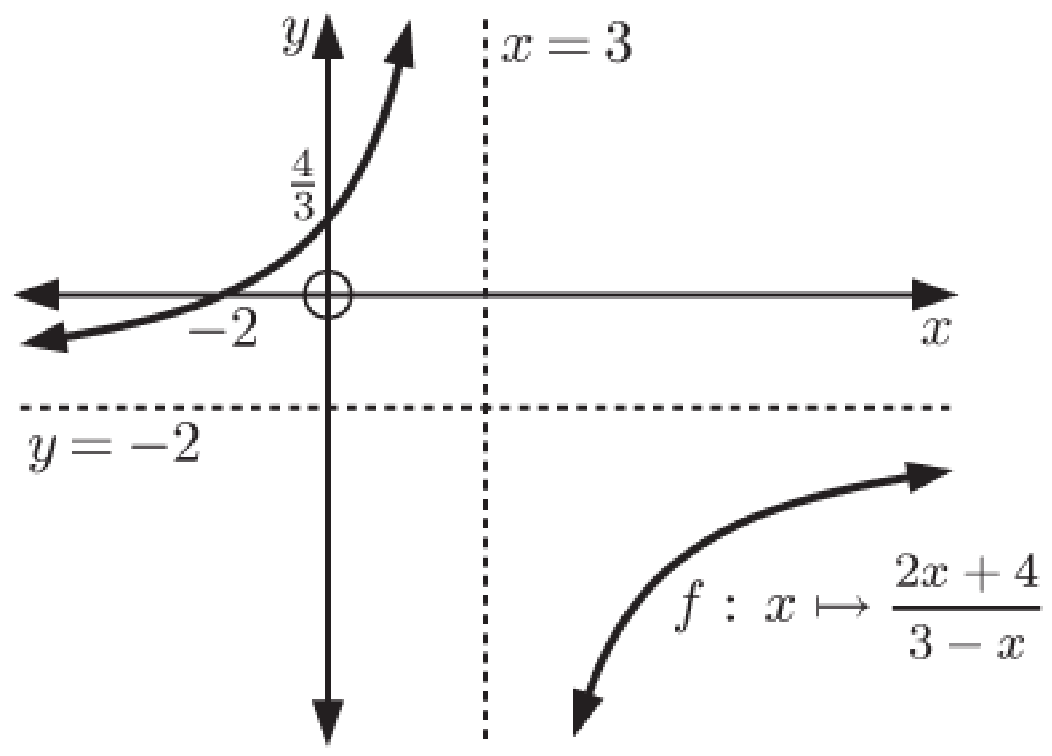
iii $f(x) = -2 + \dfrac{10}{3-x}$, horizontal asymptote is $y = -2$

iv



v as $x \to 3^-$, $f(x) \to \infty$
as $x \to 3^+$, $f(x) \to -\infty$
as $x \to -\infty$, $f(x) \to -2^+$
as $x \to \infty$, $f(x) \to -2^-$

vi

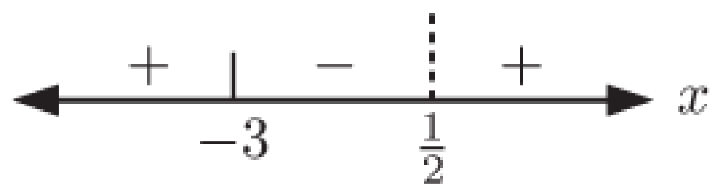


f **i** vertical asymptote is $x = \frac{1}{2}$

ii x-intercept -3, y-intercept -3

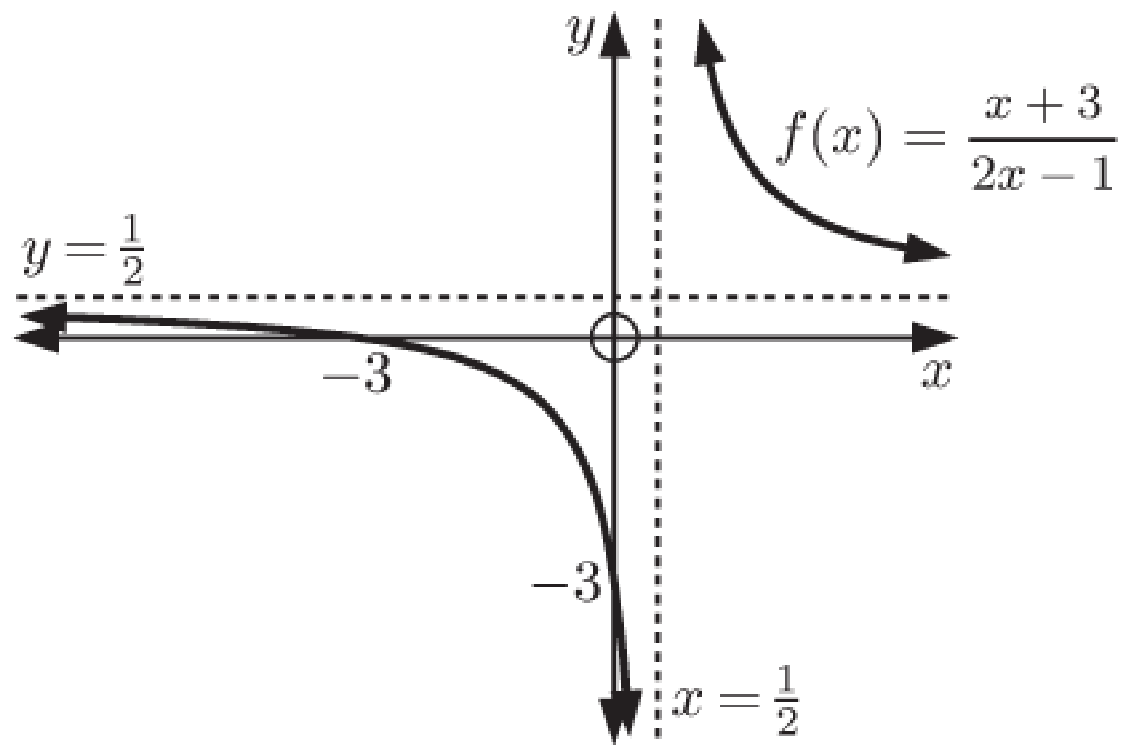
iii $f(x) = \frac{1}{2} + \dfrac{7}{4x-2}$, horizontal asymptote is $y = \frac{1}{2}$

iv

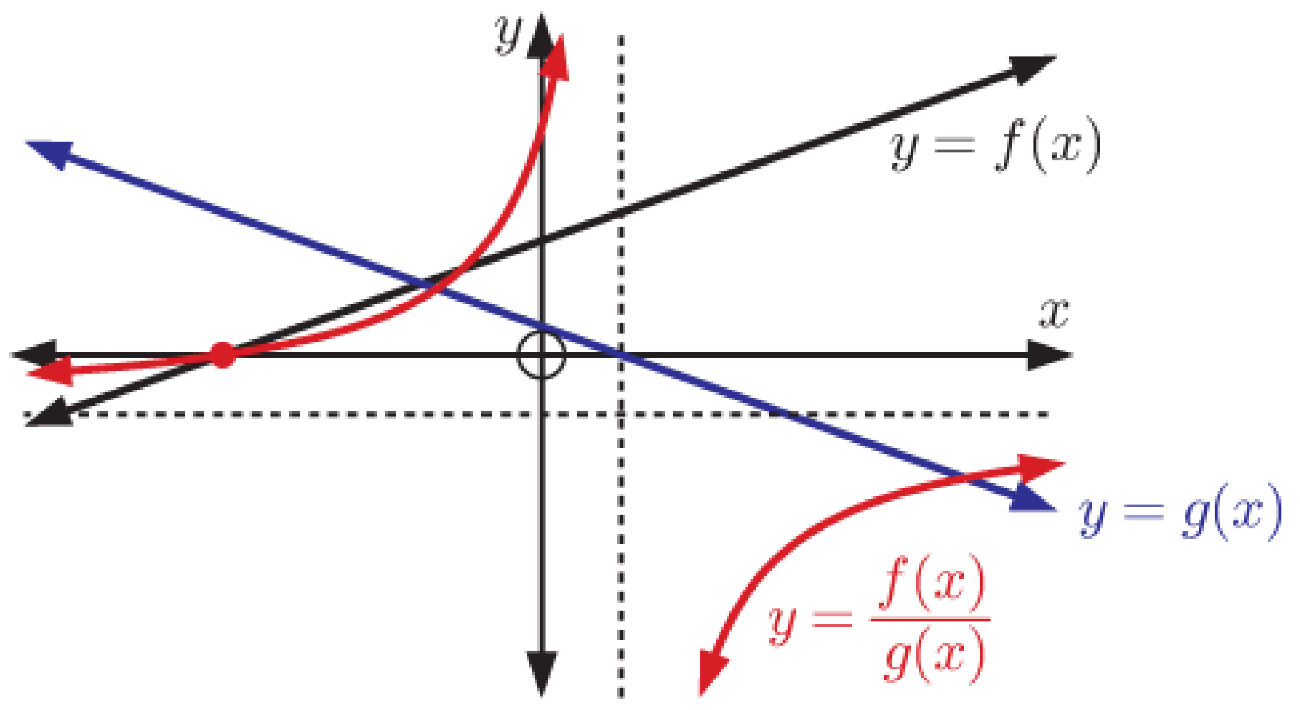


v as $x \to \frac{1}{2}^-$, $f(x) \to -\infty$
as $x \to \frac{1}{2}^+$, $f(x) \to \infty$
as $x \to -\infty$, $f(x) \to \frac{1}{2}^-$
as $x \to \infty$, $f(x) \to \frac{1}{2}^+$

vi



5



6 **a** Domain is $\{x \mid x \neq -\frac{d}{c}\}$

b vertical asymptote is $x = -\frac{d}{c}$

c x-intercept is $-\frac{b}{a}$, y-intercept is $\frac{b}{d}$

d $\dfrac{ax+b}{cx+d} = \dfrac{\frac{a}{c}(cx+d) - \frac{ad}{c} + b}{cx+d}$ and so on

As $x \to \infty$, $\dfrac{b - \frac{ad}{c}}{cx+d} \to 0$.

$\therefore$ the horizontal asymptote is $y = \frac{a}{c}$.

EXERCISE 15E

1 **a** $-2 - 2x^2$ **b** $1 + 4x^2$ **c** -10 **d** -4

2 **a** $-4x^2 - 16x - 13$ **b** $10 - 2x^2$ **c** 14 **d** $-\frac{73}{16}$

3 **a** $25x - 42$ **b** $\sqrt{8}$ **c** -7 **d** 2

4 **a** **i** $x^2 - 6x + 10$ **ii** $2 - x^2$ **b** $x = \pm\frac{1}{\sqrt{2}}$

5 **a** $(f \circ g)(x) = 9 - \sqrt{x^2 + 4}$

Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \leqslant 7\}$

b 53

c $(f \circ f)(x) = 9 - \sqrt{9 - \sqrt{x}}$

Domain is $\{x \mid 0 \leqslant x \leqslant 81\}$, Range is $\{y \mid 6 \leqslant y \leqslant 9\}$

6 **a** $-6x - 9$ **b** $x = -1$

7 **a** **i** $1 - 9x^2$ **ii** $1 + 6x - 3x^2$ **b** $x = -\frac{1}{9}$

8 **a** $(f \circ g)(x) = \dfrac{1}{x - 3}$

Domain is $\{x \mid x \neq 3\}$, Range is $\{y \mid y \neq 0\}$

b $(f \circ g)(x) = -\dfrac{1}{x^2 + 3x + 2}$

Domain is $\{x \mid x \neq -1, \ x \neq -2\}$

Range is $\{y \mid y \geqslant 4, \ y < 0\}$

9 **a** $f \circ g = \{(2, 7), (5, 2), (7, 5), (9, 9)\}$

b $g \circ f = \{(0, 2), (1, 0), (2, 1), (3, 3)\}$

10 **a** $(f \circ g)(x) = \dfrac{4x - 2}{3x - 1}$, Domain is $\{x \mid x \neq \frac{1}{3} \text{ or } 1\}$

b $(g \circ f)(x) = 2x + 5$, Domain is $\{x \mid x \neq -2\}$

c $(g \circ g)(x) = x$, Domain is $\{x \mid x \neq 1\}$

11 **a** Let $x = 0$, $\therefore$ $b = d$ and so

$ax + b = cx + b$

$\therefore$ $ax = cx$ for all x

Let $x = 1$, $\therefore$ $a = c$

b $(f \circ g)(x) = [2a]x + [2b + 3] = 1x + 0$ for all x

$\therefore$ $2a = 1$ and $2b + 3 = 0$

c Yes, $\{(g \circ f)(x) = [2a]x + [3a + b]\}$

12 **a** $(f \circ g)(x) = \sqrt{1 - x^2}$

b Domain is $\{x \mid -1 \leqslant x \leqslant 1\}$, Range is $\{y \mid 0 \leqslant y \leqslant 1\}$

c $(g \circ f)(x) = 1 - x$

d Domain is $\{x \mid x \leqslant 1\}$, Range is $\{y \mid y \geqslant 0\}$

13 **a** $R_g \cap D_f \neq \varnothing$

b Domain is $\{x \mid x \in D_g, \ g(x) \in D_f\}$

14 **a** $V \circ D = 6800 - 400t$

This is the value of Mila's car t years after purchase.

b 4400; the value of Mila's car 6 years after purchase is $4400.

15 **a** **i** $T \circ S$ **ii** $S \circ T$ **b** €715

16 **a** $V = 2000 - 20t$

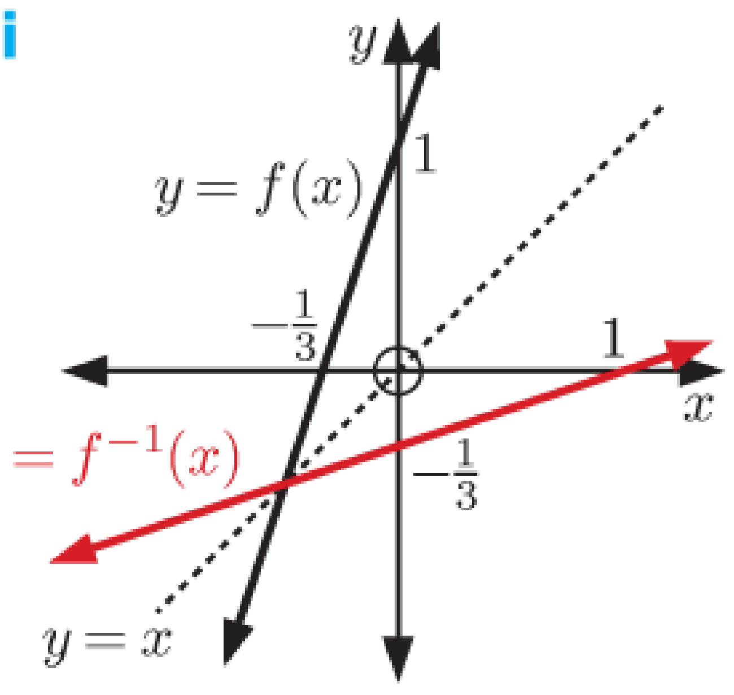
c $H \circ V = \sqrt[3]{\dfrac{24\,000 - 240t}{\pi}}$

This is the height of the solution after t minutes.

d $(H \circ V)(30) \approx 17.5$; the height of the solution after 30 minutes is about 17.5 cm.

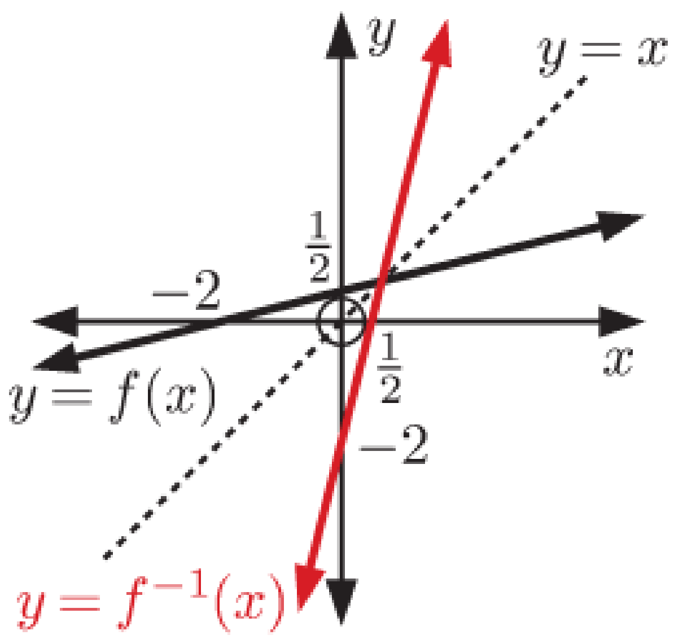
EXERCISE 15F

1 **a** **i**



ii, iii $f^{-1}(x) = \dfrac{x - 1}{3}$

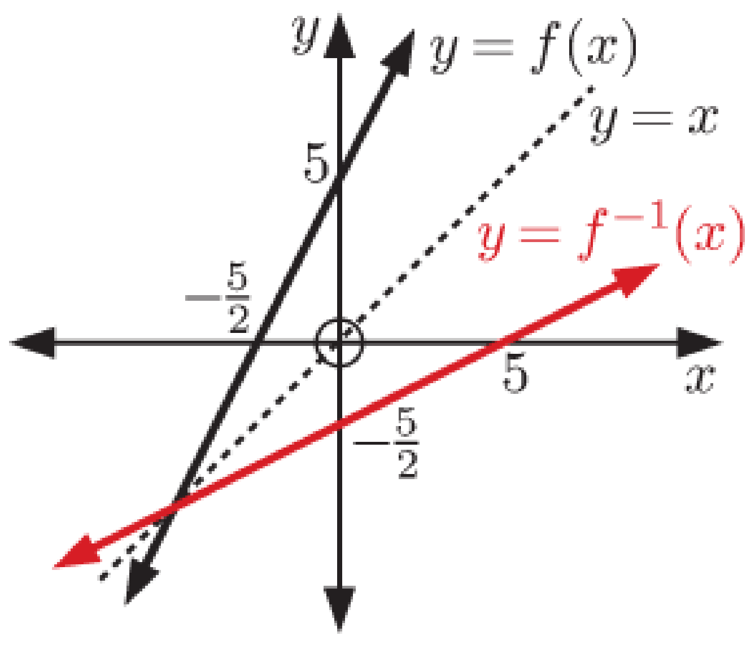
b **i**



ii, iii $f^{-1}(x) = 4x - 2$

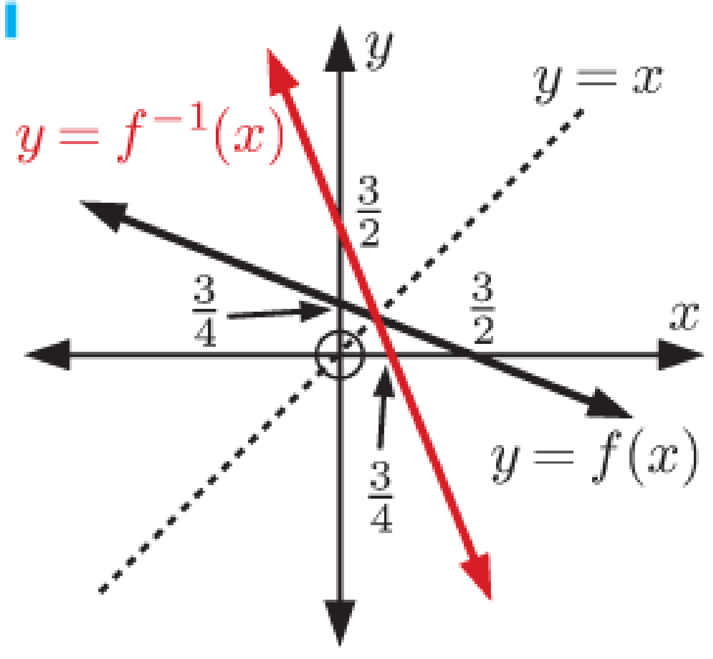
2 **a** **i** $f^{-1}(x) = \dfrac{x - 5}{2}$

ii

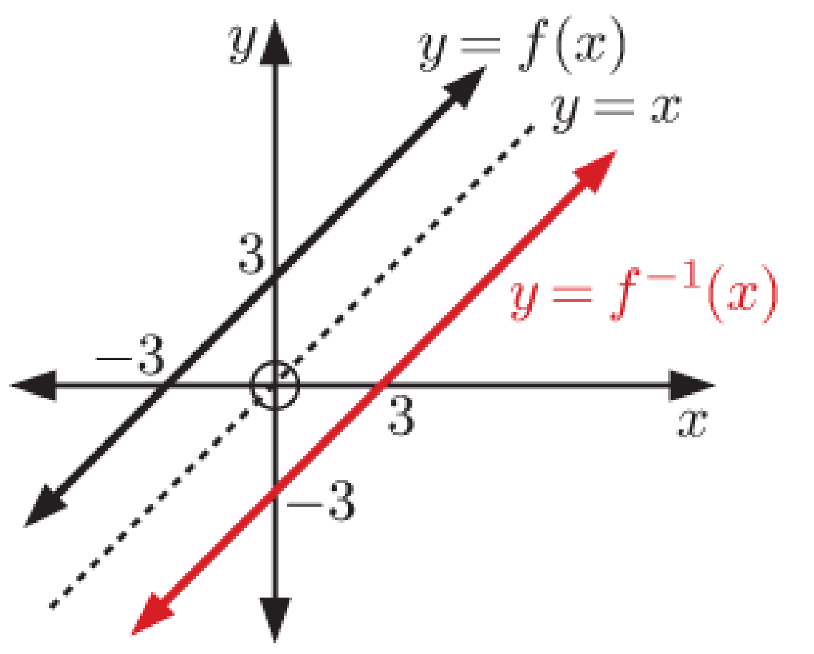


b **i** $f^{-1}(x) = -2x + \frac{3}{2}$

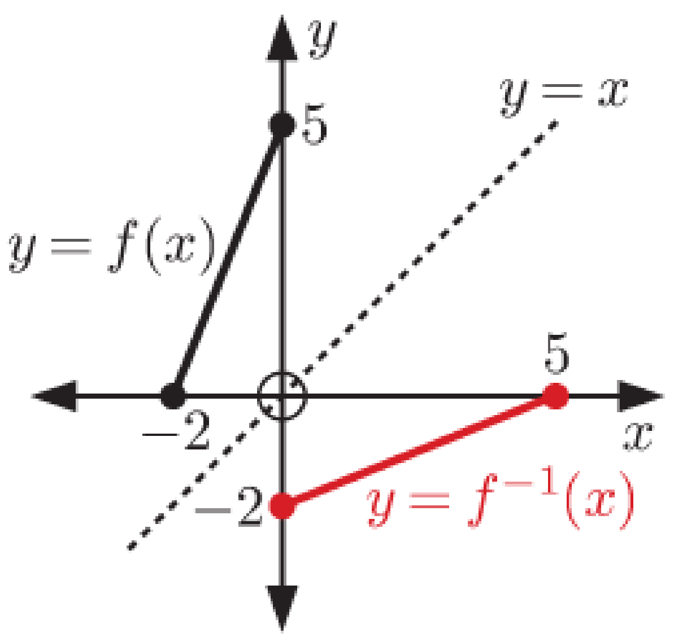
ii



c **i** $f^{-1}(x) = x - 3$ **ii**



3 **a**



f:

Domain is $\{x \mid -2 \leqslant x \leqslant 0\}$

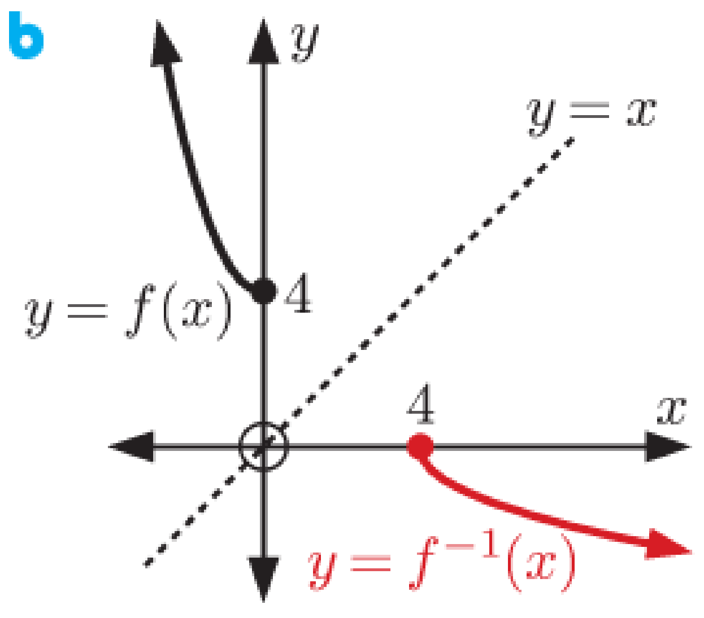
Range is $\{y \mid 0 \leqslant y \leqslant 5\}$

f^{-1}:

Domain is $\{x \mid 0 \leqslant x \leqslant 5\}$

Range is $\{y \mid -2 \leqslant y \leqslant 0\}$

b



f:

Domain is $\{x \mid x \leqslant 0\}$

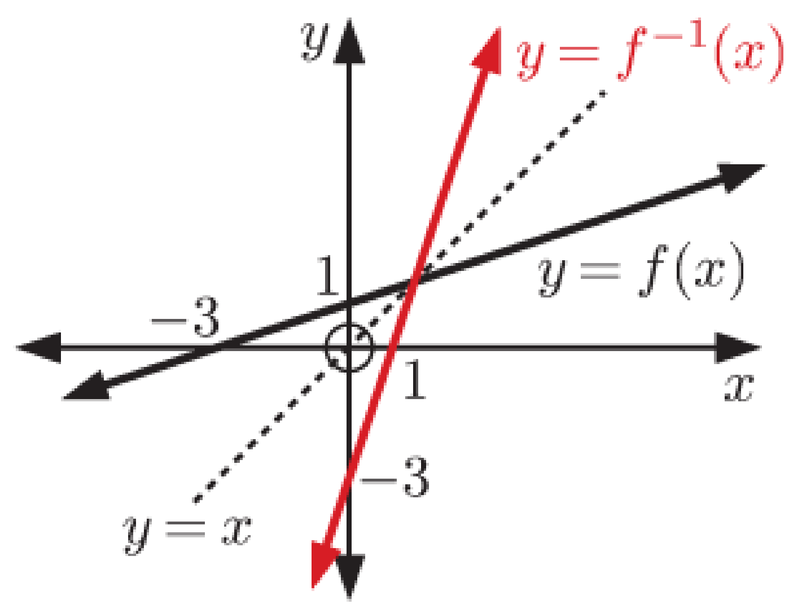
Range is $\{y \mid y \geqslant 4\}$

f^{-1}:

Domain is $\{x \mid x \geqslant 4\}$

Range is $\{y \mid y \leqslant 0\}$

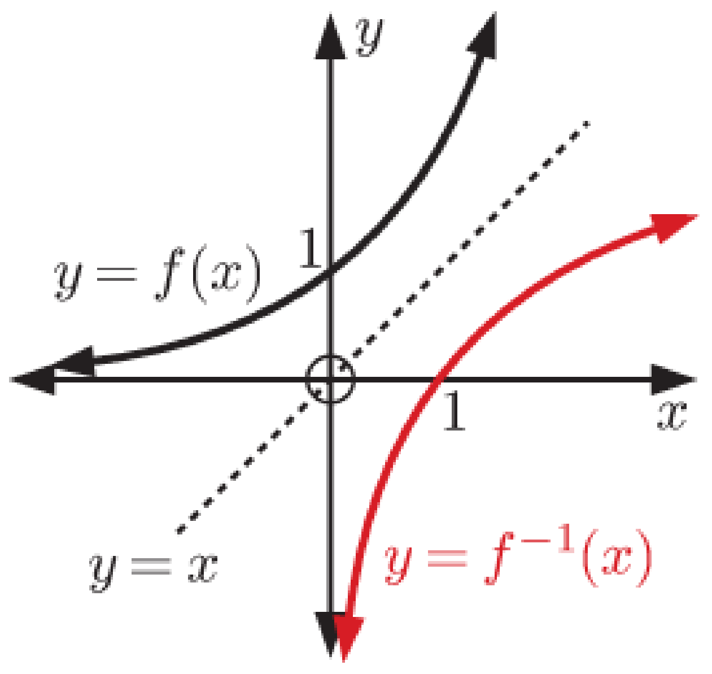
c



f:
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

f^{-1}:
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

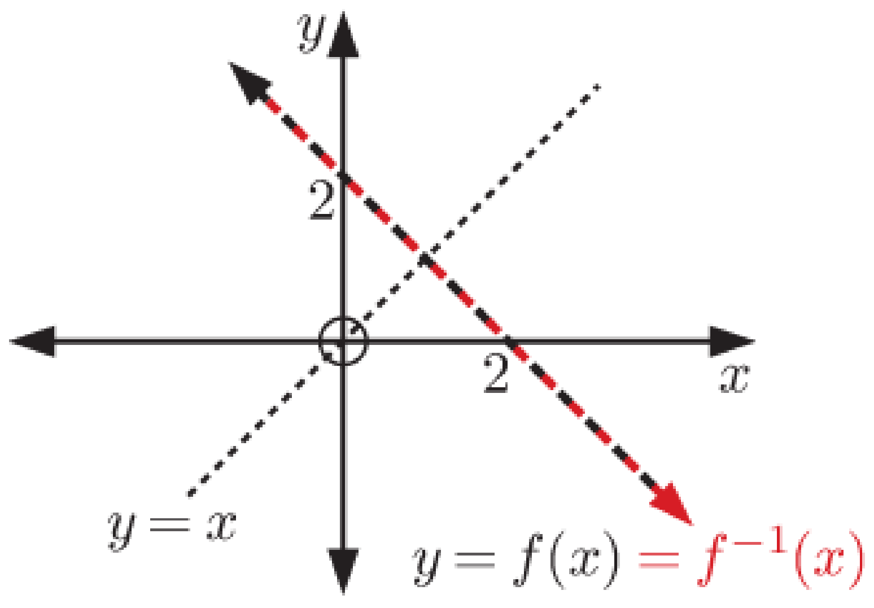
d



f:
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y > 0\}$

f^{-1}:
Domain is $\{x \mid x > 0\}$
Range is $\{y \mid y \in \mathbb{R}\}$

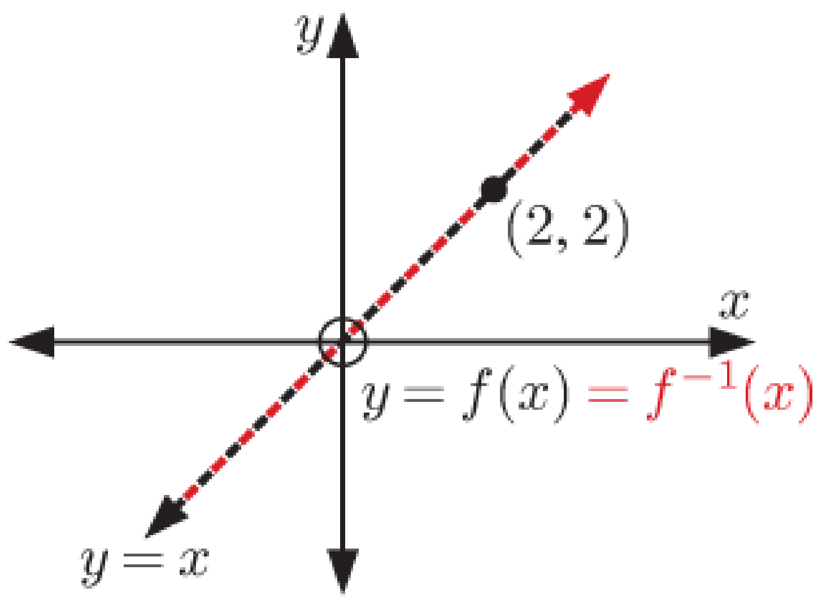
e



f:
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

f^{-1}:
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

f



f:
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

f^{-1}:
Domain is $\{x \mid x \in \mathbb{R}\}$
Range is $\{y \mid y \in \mathbb{R}\}$

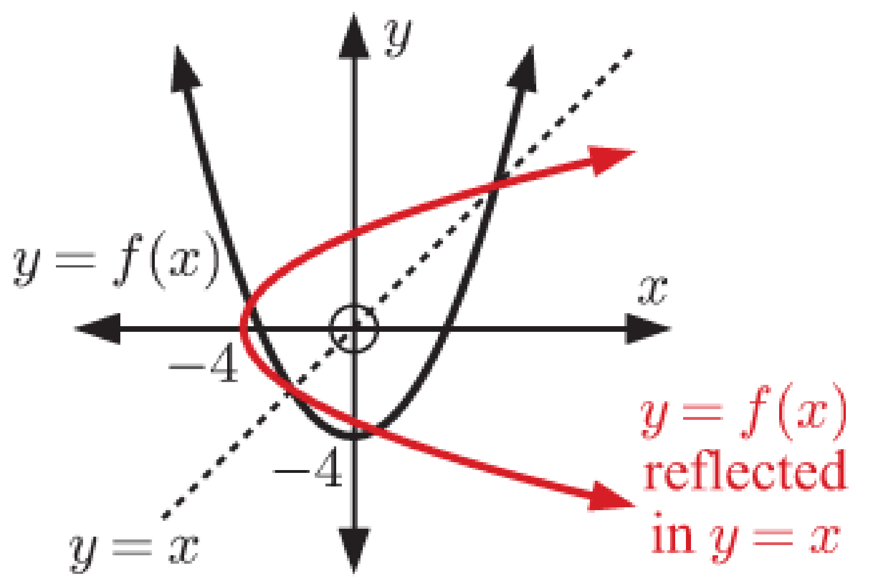
4 $(f^{-1})^{-1}(x) = 2x - 5 = f(x)$

5 **a** $\{(2, 1), (4, 2), (5, 3)\}$ **b** inverse does not exist
c $\{(1, 2), (0, -1), (2, 0), (3, 1)\}$
d $\{(-1, -1), (0, 0), (1, 1)\}$

6 $f(x) = x$ and $f(x) = -x + c, \ c \in \mathbb{R}$

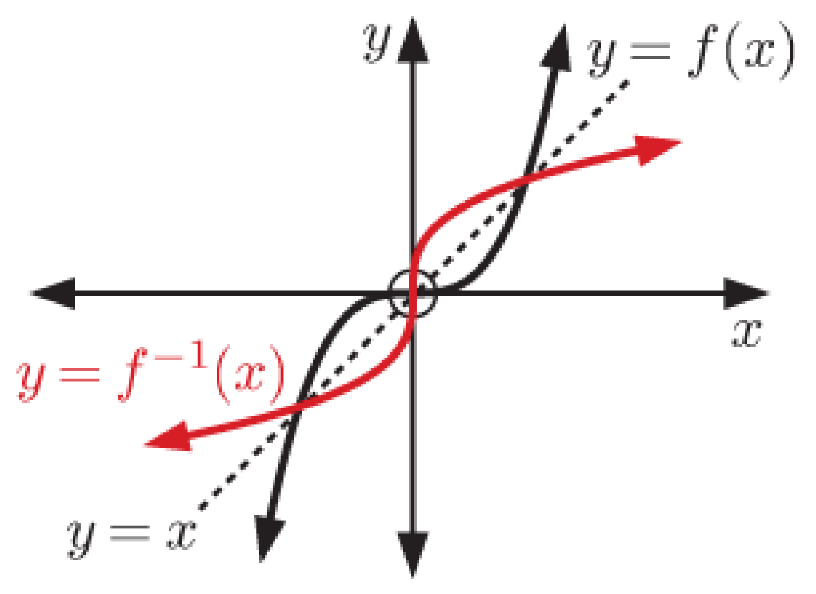
7 Range is $\{y \mid -2 \leqslant y < 3\}$

8 **a**



b no **c** yes, it is $f^{-1} : x \mapsto \sqrt{x+4}$

9



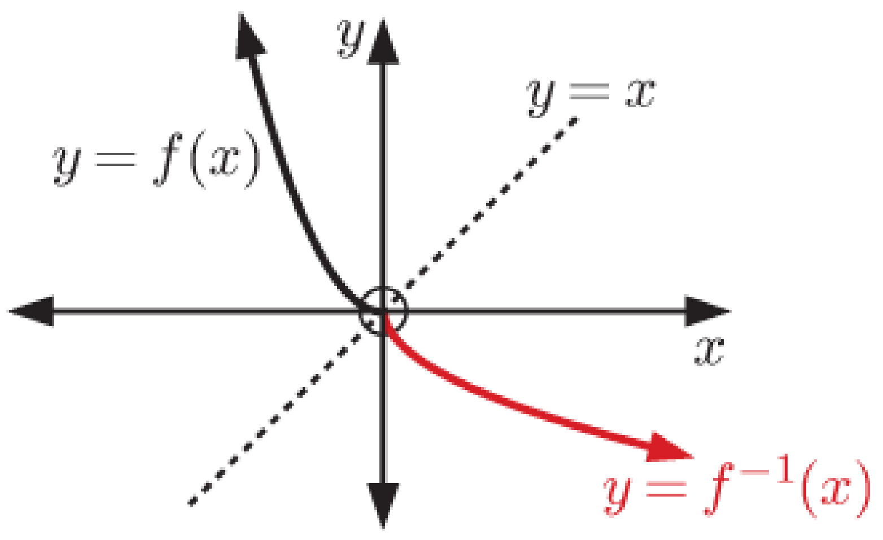
10 f is $y = \dfrac{1}{x}, \ x \neq 0$ $\therefore$ f^{-1} is $x = \dfrac{1}{y}$

$\therefore \ y = \dfrac{1}{x}$

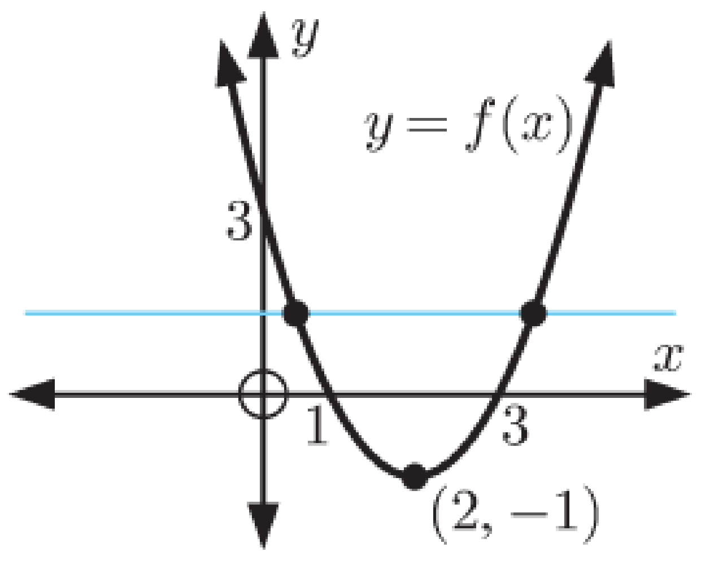
$\therefore \ f = f^{-1}$
$\therefore \ f$ is self-inverse.

11 **a** The inverse function must also be a function and must therefore satisfy the vertical line test, which it can only do if the original function satisfies the horizontal line test.
b **i** is the only one.
c **i** Domain is $\{x \mid x \geqslant 1\}$ or $\{x \mid x \leqslant 1\}$
ii Domain is $\{x \mid x \geqslant 1\}$ or $\{x \mid x \leqslant -2\}$

12 **a** $f^{-1}(x) = -\sqrt{x}$ **b**



13 **a**

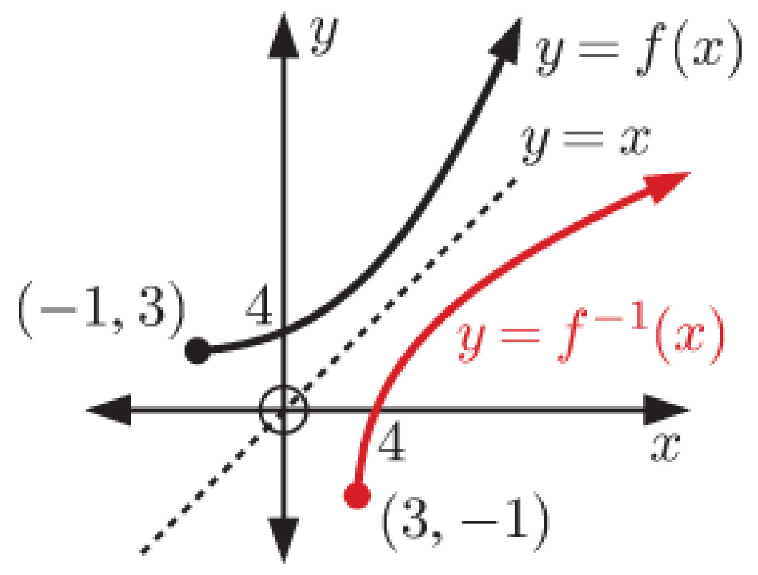


Every vertical line cuts the graph once. So, it is a function.
A horizontal line above the vertex cuts the graph **twice**. So, it does not have an inverse.

b For $x \geqslant 2$, all horizontal lines cut the graph at most once.
$\therefore \ g(x)$ has an inverse.
Hint: Inverse is $x = y^2 - 4y + 3$ for $y \geqslant 2$
c g: Domain is $\{x \mid x \geqslant 2\}$, Range is $\{y \mid y \geqslant -1\}$
g^{-1}: Domain is $\{x \mid x \geqslant -1\}$, Range is $\{y \mid y \geqslant 2\}$
d **Hint:** Find $gg^{-1}(x)$ and $g^{-1}g(x)$ and show that they both equal x.

14 **a** $f^{-1}(x) = \sqrt{x-3} - 1, \ x \geqslant 3$

b



c f:
Domain is $\{x \mid x \geqslant -1\}$
Range is $\{y \mid y \geqslant 3\}$
f^{-1}:
Domain is $\{x \mid x \geqslant 3\}$
Range is $\{y \mid y \geqslant -1\}$

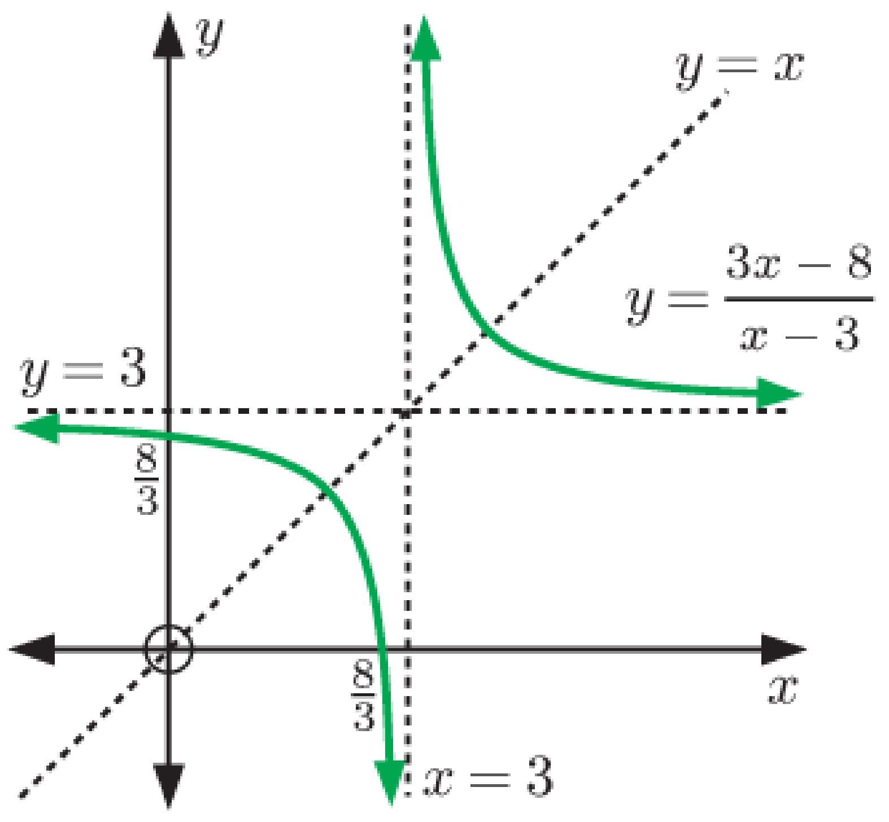
15 **a** $f^{-1}(x) = 3 - \sqrt{13 - x}$
b f: Domain is $\{x \mid x \leqslant 3\}$, Range is $\{y \mid y \leqslant 13\}$
f^{-1}: Domain is $\{x \mid x \leqslant 13\}$, Range is $\{y \mid y \leqslant 3\}$

16 **a** $k = \frac{5}{2}$
b **i** $f^{-1}(x) = \dfrac{5 - \sqrt{2x + 13}}{2}$
ii Domain is $\{x \mid x \geqslant -\frac{13}{2}\}$, Range is $\{y \mid y \leqslant \frac{5}{2}\}$

17 **a** $g^{-1}(x) = 8 - 2x$ **b** $x = 10$
c $f^{-1}(-3) - g^{-1}(6) = -4 - (-4) = 0$ **d** $x = 3$

18 **a** $8x - 6$ **b** $k = 10$
c $(f^{-1} \circ g^{-1})(x) = \dfrac{x+3}{8}$ and $(g \circ f)^{-1}(x) = \dfrac{x+3}{8}$

19 **a**



$y = \dfrac{3x - 8}{x - 3}$ is symmetrical about $y = x$
$\therefore \ f$ is a self-inverse function.

b $f^{-1}(x) = \dfrac{3x-8}{x-3}$ and $f(x) = \dfrac{3x-8}{x-3}$

$\therefore\ f = f^{-1}$ $\therefore$ f is a self-inverse function

20 $d = -a$ **21** **a** $\text{B}(f(x), x)$

22 **a** Domain is $\{x \mid x \geqslant 0\}$

b No, as $f(x)$ does not pass the horizontal line test.

c **i** $g^{-1}(x) = (3 - \sqrt{x+8})^2$

ii g: Domain is $\{x \mid 0 \leqslant x \leqslant 9\}$

Range is $\{y \mid -8 \leqslant y \leqslant 1\}$

g^{-1}: Domain is $\{x \mid -8 \leqslant x \leqslant 1\}$

Range is $\{y \mid 0 \leqslant y \leqslant 9\}$

d **i** $h^{-1}(x) = (3 + \sqrt{x+8})^2$

ii h: Domain is $\{x \mid x \geqslant 9\}$

Range is $\{y \mid y \geqslant -8\}$

h^{-1}: Domain is $\{x \mid x \geqslant -8\}$

Range is $\{y \mid y \geqslant 9\}$

iii $x = -8$

REVIEW SET 15A

1 **a** **i** Domain is $\{x \mid x \in \mathbb{R}\}$ **ii** Range is $\{y \mid y > -4\}$

iii yes, it is a function

b **i** Domain is $\{x \mid x \in \mathbb{R}\}$

ii Range is $\{y \mid y \leqslant -1$ or $y \geqslant 1\}$

iii no, not a function

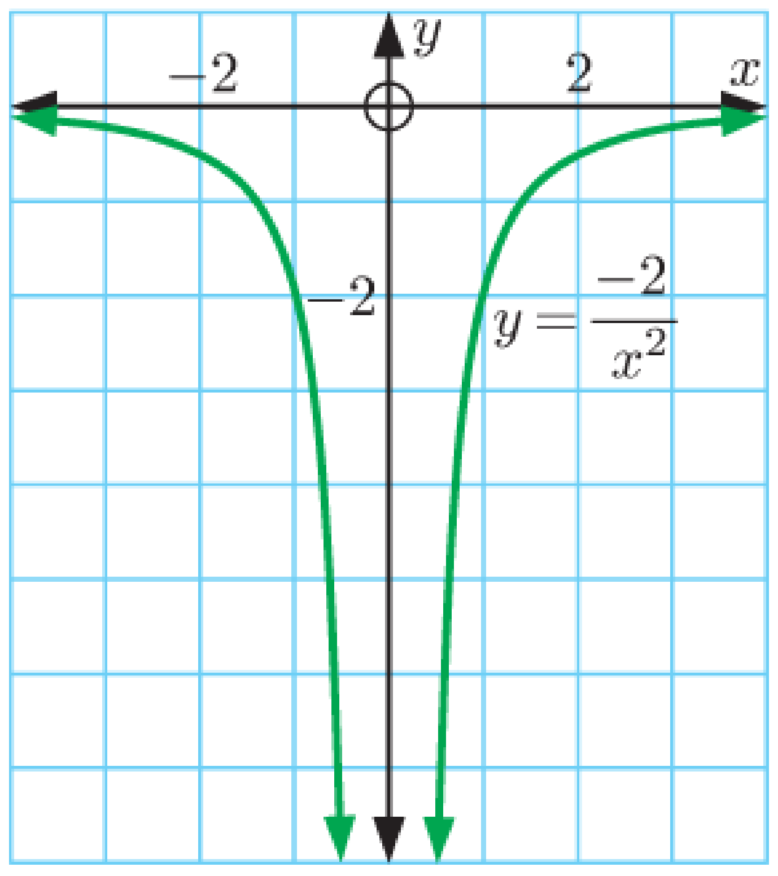
c **i** Domain is $\{x \mid x \in \mathbb{R}\}$

ii Range is $\{y \mid -5 \leqslant y \leqslant 5\}$ **iii** yes, it is a function

2 **a** 0 **b** -15 **c** $-\frac{5}{4}$ **3** $a = -6,\ b = 13$

4 **a** $x = 0$

b



c Domain is $\{x \mid x \neq 0\}$

Range is $\{y \mid y < 0\}$

5 **a** $f(-3) = (-3)^2 = 9,\quad g(-\frac{4}{3}) = 1 - 6(-\frac{4}{3}) = 9$

b $x = -4$

6 **a** Domain is $\{x \mid x \geqslant -4\}$, Range is $\{y \mid y \geqslant 0\}$

b Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \leqslant 1\}$

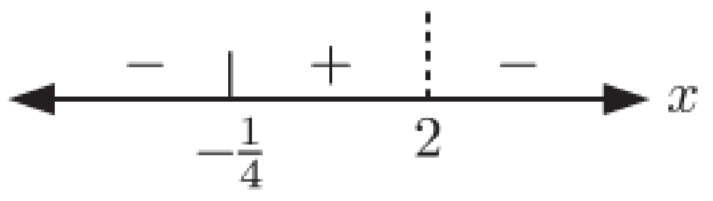
c Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \geqslant -\frac{1}{8}\}$

7 **a** $y = -\dfrac{20}{x}$ **b** $y = \dfrac{60}{x}$

8 **a** vertical asymptote $x = 2$, horizontal asymptote $y = -4$

b Domain is $\{x \mid x \neq 2\}$, Range is $\{y \mid y \neq -4\}$

c

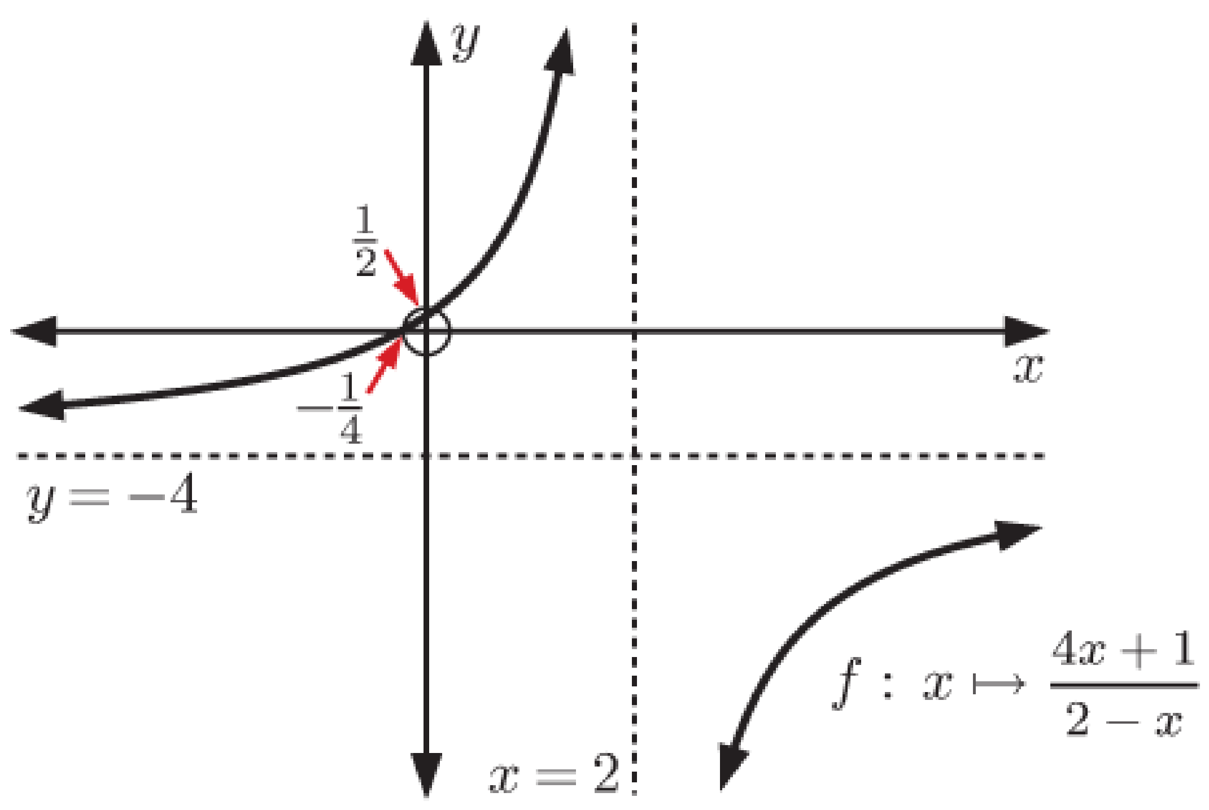


as $x \to 2^-$, $f(x) \to \infty$ as $x \to -\infty$, $f(x) \to -4^+$

as $x \to 2^+$, $f(x) \to -\infty$ as $x \to \infty$, $f(x) \to -4^-$

d x-intercept $-\frac{1}{4}$, y-intercept $\frac{1}{2}$

e



9 **a** $6x - 3$ **b** $x = 1$

10 **a** $1 - 2\sqrt{x}$ **b** $\sqrt{1-2x}$ **c** 3

11 **a** $(f \circ g)(x) = \sqrt{x^2 - 1}$

Domain is $\{x \mid x \leqslant -1$ or $x \geqslant 1\}$

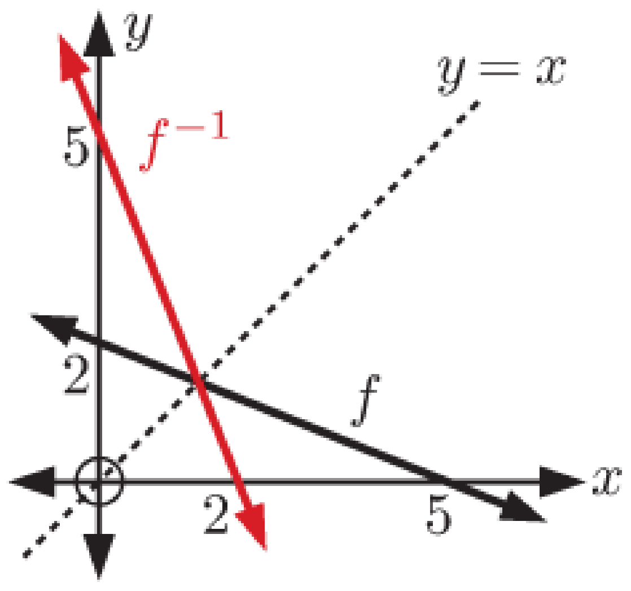
Range is $\{y \mid y \geqslant 0\}$

b $(g \circ f)(x) = x - 1$

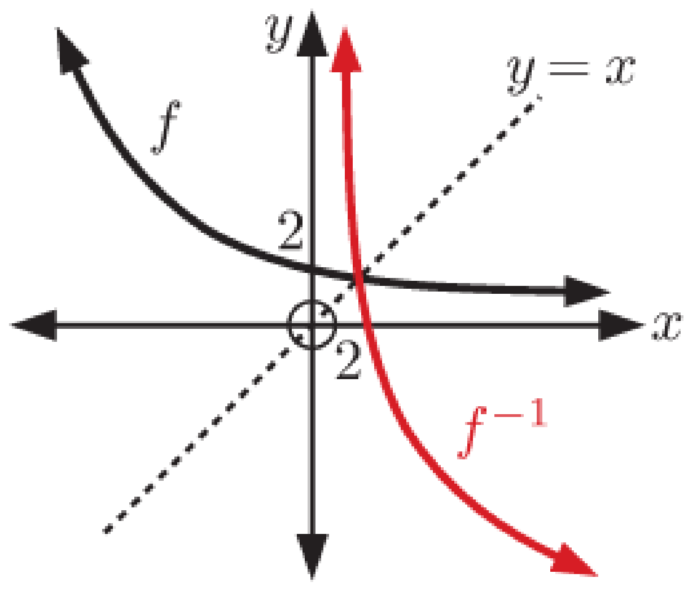
Domain is $\{x \mid x \geqslant -2\}$, Range is $\{y \mid y \geqslant -3\}$

12 $a = 1,\ b = -1$

13 **a**

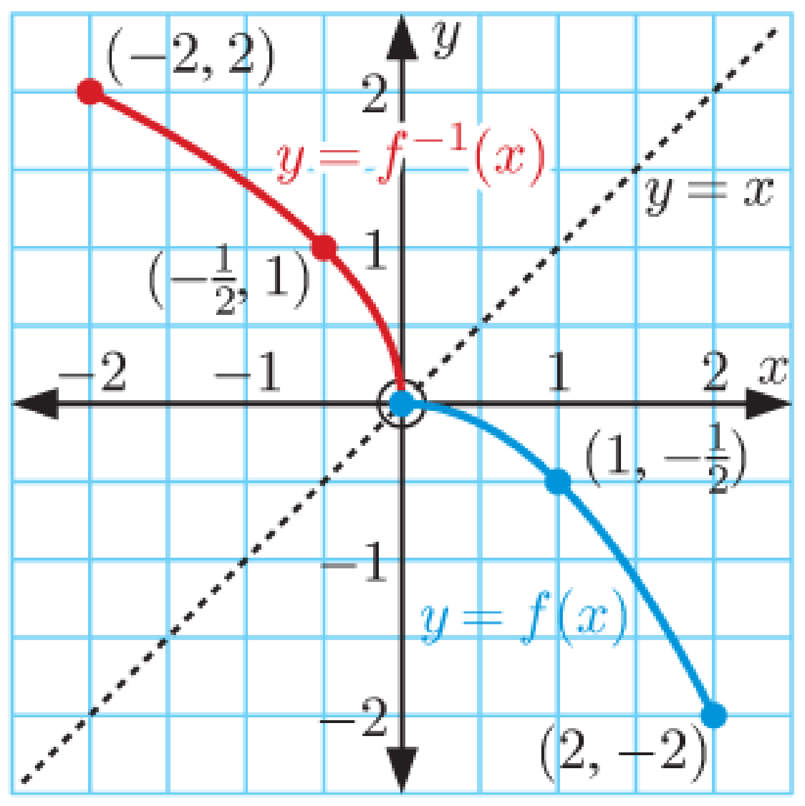


b



14 **a** $f^{-1}(x) = \dfrac{x-2}{4}$ **b** $f^{-1}(x) = \dfrac{3-4x}{5}$

15 **a**

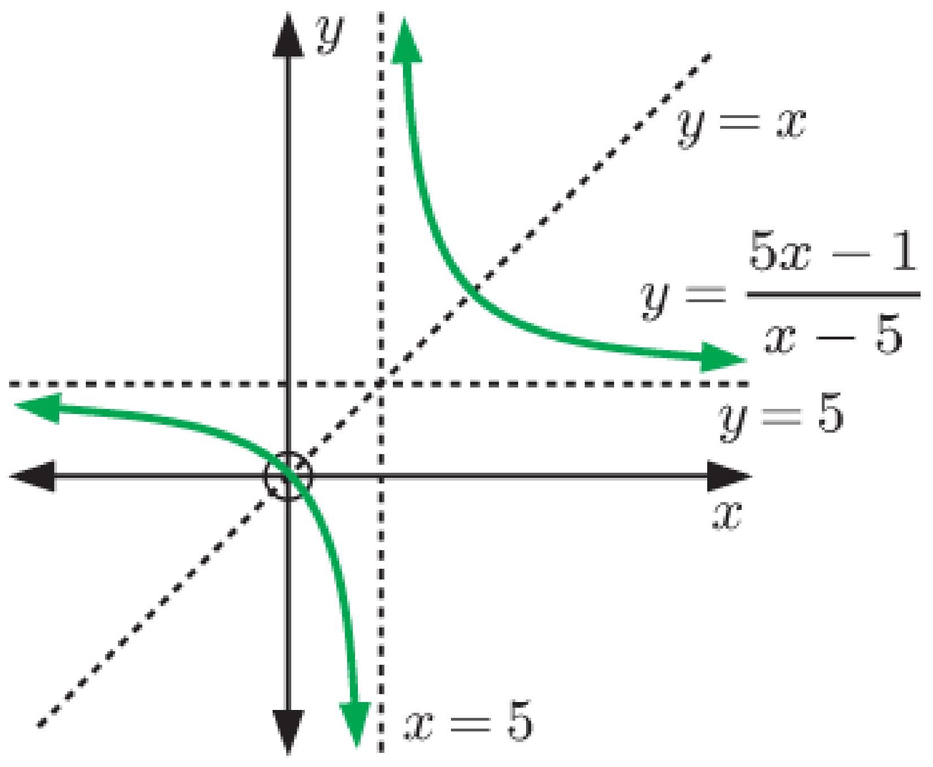


b Range is $\{y \mid 0 \leqslant y \leqslant 2\}$

c **i** $x = \sqrt{3}$ **ii** $x = -\frac{1}{2}$

16 $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x) = x - 2$

17 **a**



$y = \dfrac{5x-1}{x-5}$ is symmetrical about $y = x$

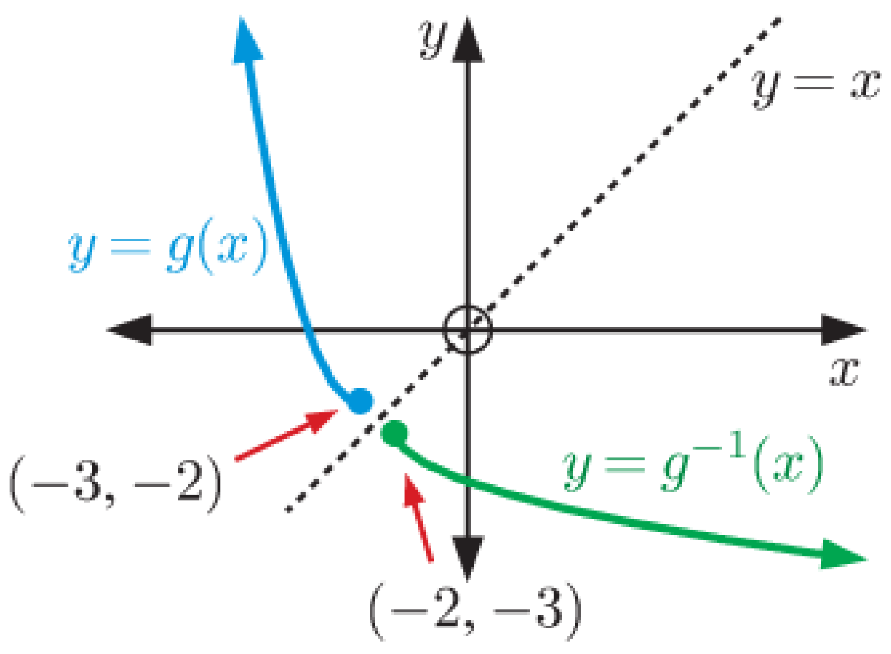
$\therefore$ f is a self-inverse function.

b $f^{-1}(x) = \dfrac{5x-1}{x-5}$ and $f(x) = \dfrac{5x-1}{x-5}$

$\therefore$ $f = f^{-1}$ $\therefore$ f is a self-inverse function.

18 **a** -4 **b** 1

19 **a, d**



b Any horizontal line cuts the graph at most once.

c $g^{-1}(x) = -3 - \sqrt{x+2},\ x \geqslant -2$

e Range of g is $\{y \mid y \geqslant -2\}$

f Domain of g^{-1} is $\{x \mid x \geqslant -2\}$,
Range of g^{-1} is $\{y \mid y \leqslant -3\}$

REVIEW SET 15B

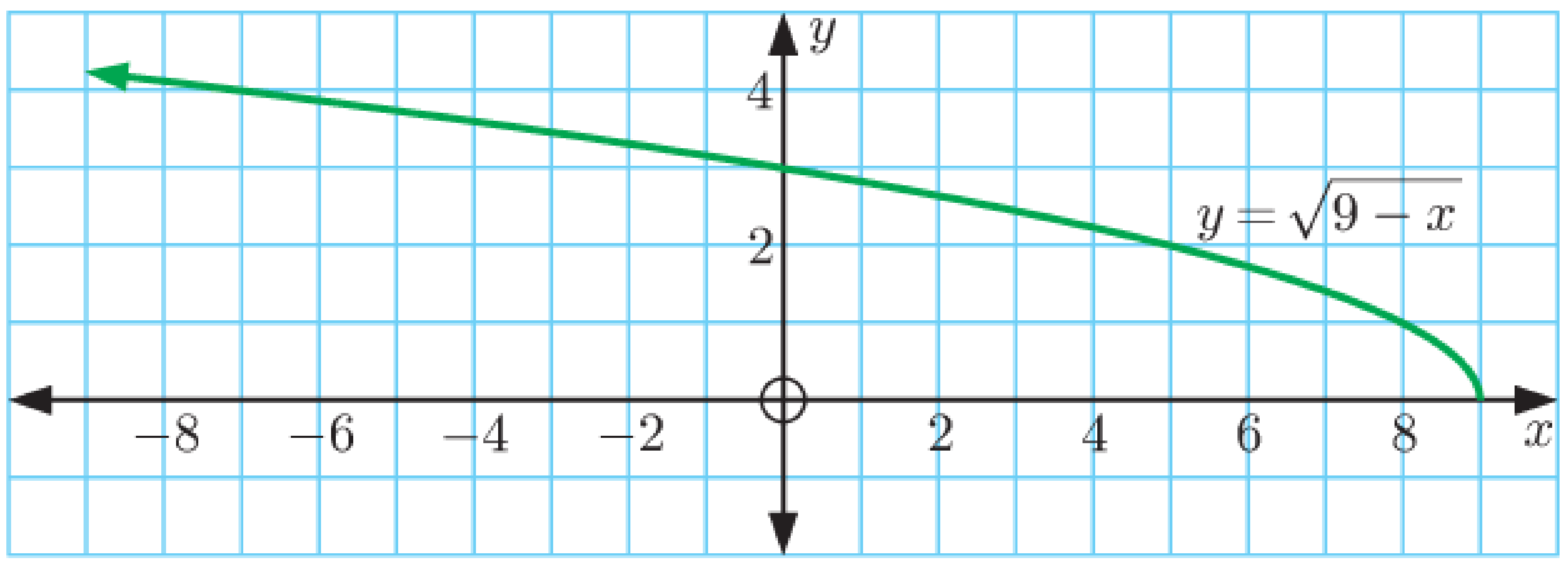
1 **a** Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y \geqslant -4\}$
b Domain is $\{x \mid x \geqslant -2\}$, Range is $\{y \mid 1 \leqslant y < 3\}$
c Domain is $\{x \mid x \in \mathbb{R}\}$, Range is $\{y \mid y = -1,\ 1,\text{ or } 2\}$

2 **a** $x^2 - x - 2$ **b** $16x^2 - 12x$

3 **a** is a function **b** is not a function

4 **a** Domain is $\{x \mid x \neq \frac{1}{2}\}$, Range is $\{y \mid y \neq 10\}$
b Domain is $\{x \mid x \geqslant -7\}$, Range is $\{y \mid y \geqslant 0\}$

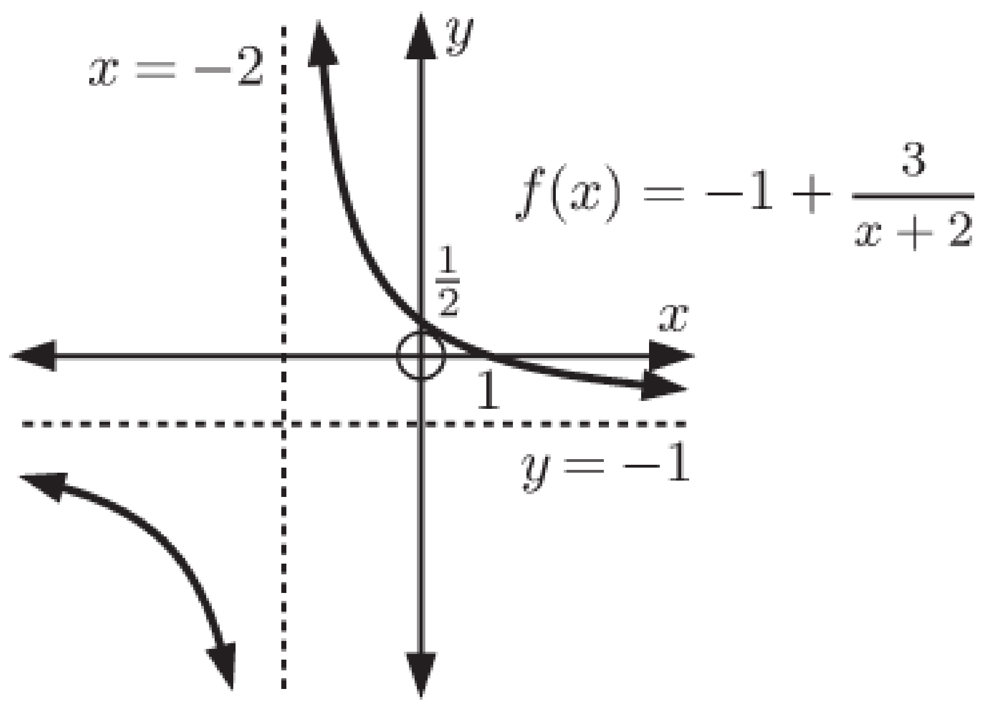
5 **a**



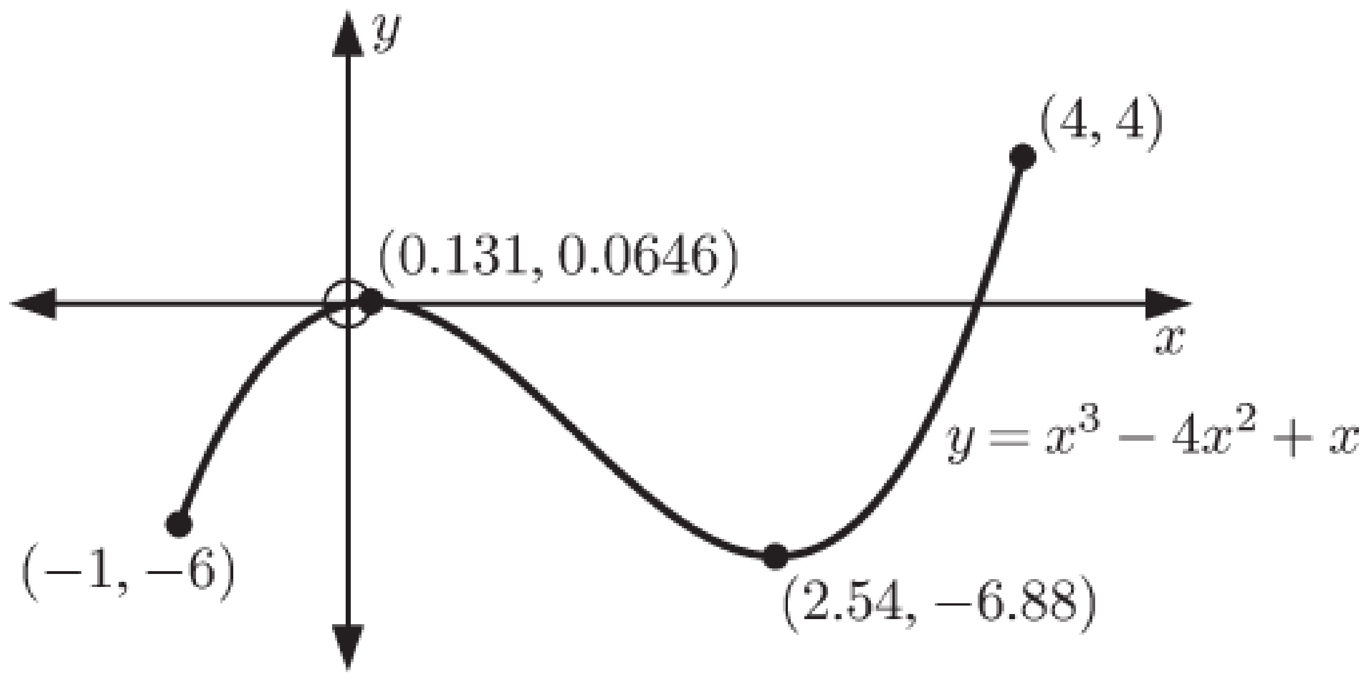
b It is a function.
c Domain is $\{x \mid x \leqslant 9\}$, Range is $\{y \mid y \geqslant 0\}$

6 $a = -2$ **7** $a = 1,\ b = -6,\ c = 5$

8 **a** vertical asymptote is $x = -2$,
horizontal asymptote is $y = -1$
b Domain is $\{x \mid x \neq -2\}$, Range is $\{y \mid y \neq -1\}$
c x-intercept is 1, y-intercept is $\frac{1}{2}$
d as $x \to -2^-$, $f(x) \to -\infty$ as $x \to -\infty$, $f(x) \to -1^-$
as $x \to -2^+$, $f(x) \to \infty$ as $x \to \infty$, $f(x) \to -1^+$
e



9



Range is $\{y \mid -6.88 \leqslant y \leqslant 4\}$

10 **a** $-4x^2 + 4x + 2$ **b** $5 - 2x^2$ **c** 2

11 $(f \circ g)(x) = \dfrac{1}{(x^2 - 4x + 3)^2}$

Domain is $\{x \mid x \neq 3,\ x \neq 1\}$, Range is $\{y \mid y > 0\}$

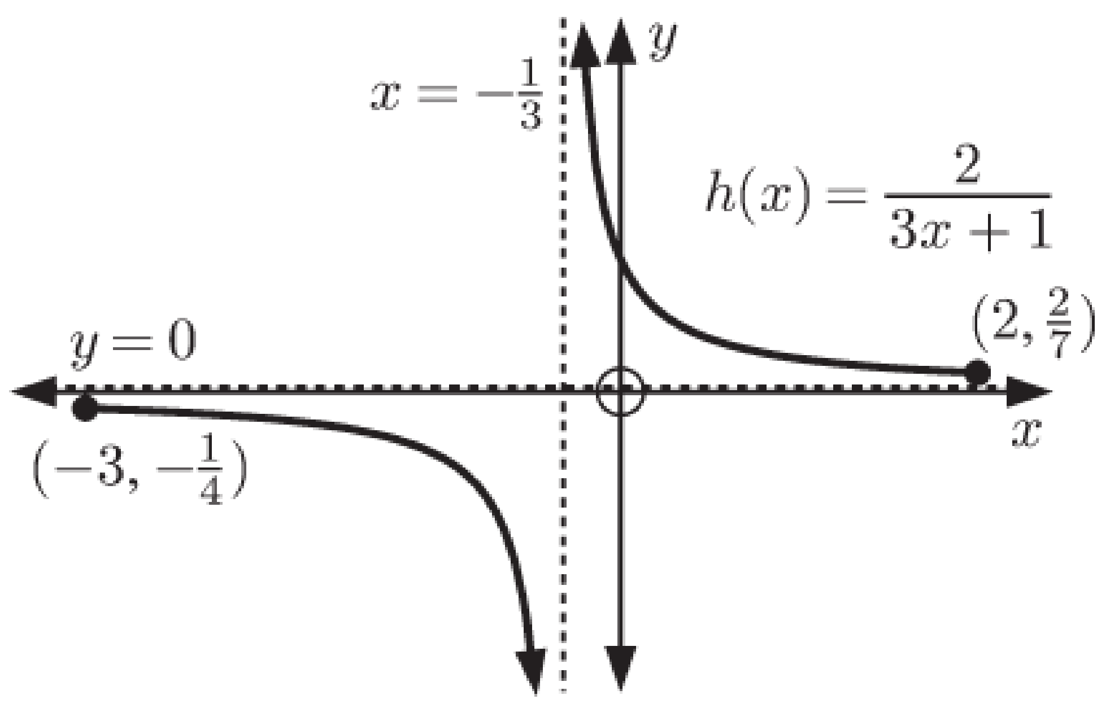
12 **a** **i** $6x^2 - 3x + 5$ **ii** $18x^2 + 57x + 45$
b $x = -\frac{5}{11}$

13 **a** $D \circ S = 4.9t^2$
This is the distance travelled by the object after t seconds.
b $(D \circ S)(5) = 122.5$ m
The object has travelled 122.5 m after 5 seconds.

14 $f^{-1}(x) = \dfrac{2x + 29}{5}$

15 $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x) = \dfrac{4x + 6}{15}$

16 **a** $(g \circ f)(x) = \dfrac{2}{3x + 1}$ **b** $x = -\frac{1}{2}$
c **i** vertical asymptote $x = -\frac{1}{3}$,
horizontal asymptote $y = 0$
ii



iii Range is $\{y \mid y \leqslant -\frac{1}{4}\ \text{ or }\ y \geqslant \frac{2}{7}\}$

17 16

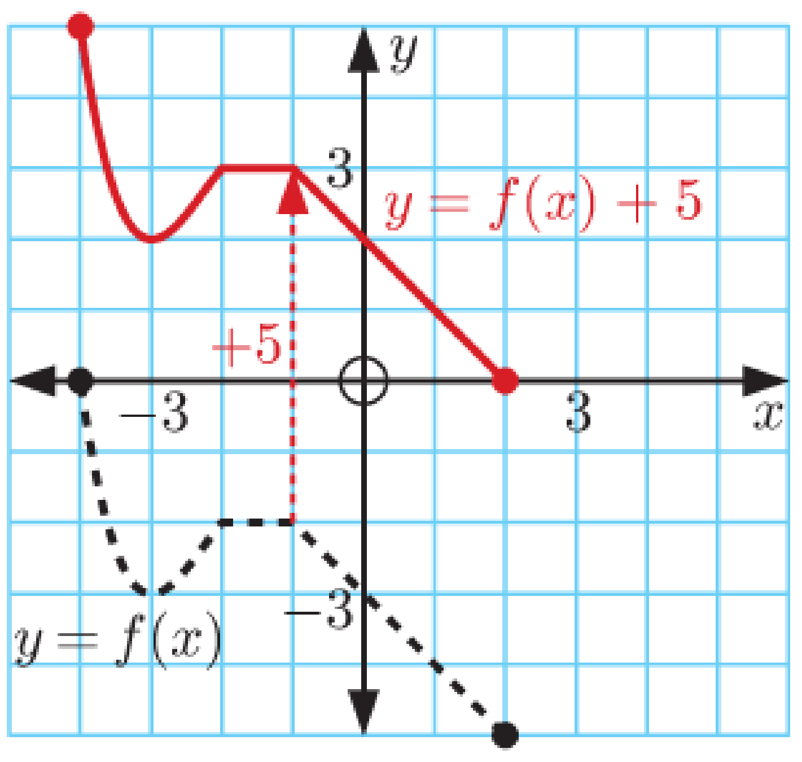
18 **a** $a = 2,\ b = -1$
b Domain is $\{x \mid x \neq 2\}$, Range is $\{y \mid y \neq -1\}$

19 **a** $\dfrac{3x}{x - 2}$ **b** $\dfrac{2x + 1}{x - 1}$

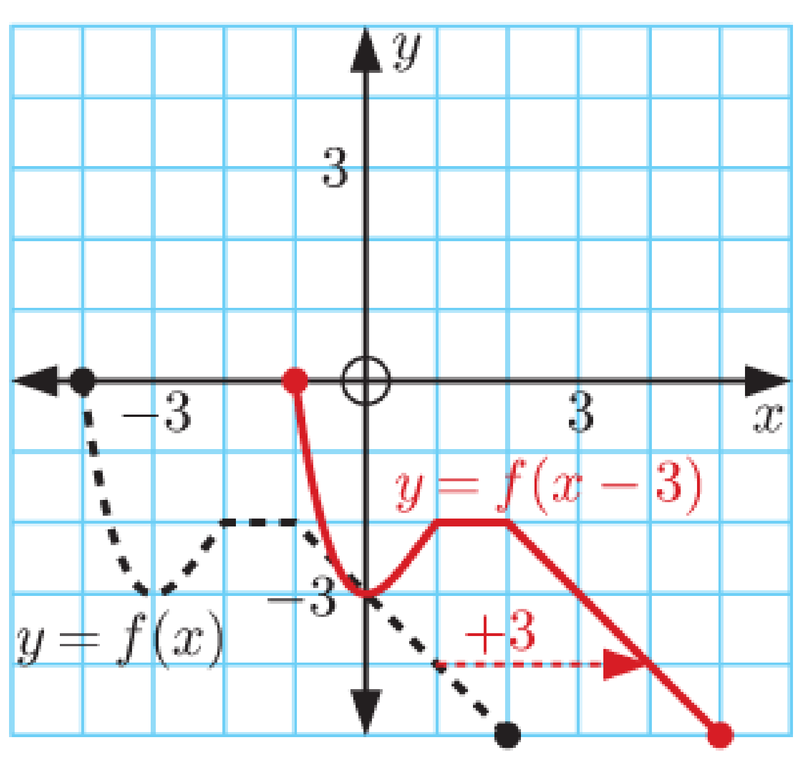
20 **a** $f^{-1}(x) = \sqrt{4 - \sqrt{x + 13}}$
Domain is $\{x \mid -13 \leqslant x \leqslant 3\}$,
Range is $\{y \mid 0 \leqslant y \leqslant 2\}$
b $g^{-1}(x) = \sqrt{4 + \sqrt{x + 13}}$
Domain is $\{x \mid x \geqslant -13\}$, Range is $\{y \mid y \geqslant 2\}$
c $h^{-1}(x) = -\sqrt{4 - \sqrt{x + 13}}$
Domain is $\{x \mid -13 \leqslant x \leqslant 3\}$,
Range is $\{y \mid -2 \leqslant y \leqslant 0\}$
d $j^{-1}(x) = -\sqrt{4 + \sqrt{x + 13}}$
Domain is $\{x \mid x \geqslant -13\}$, Range is $\{y \mid y \leqslant -2\}$

EXERCISE 16A

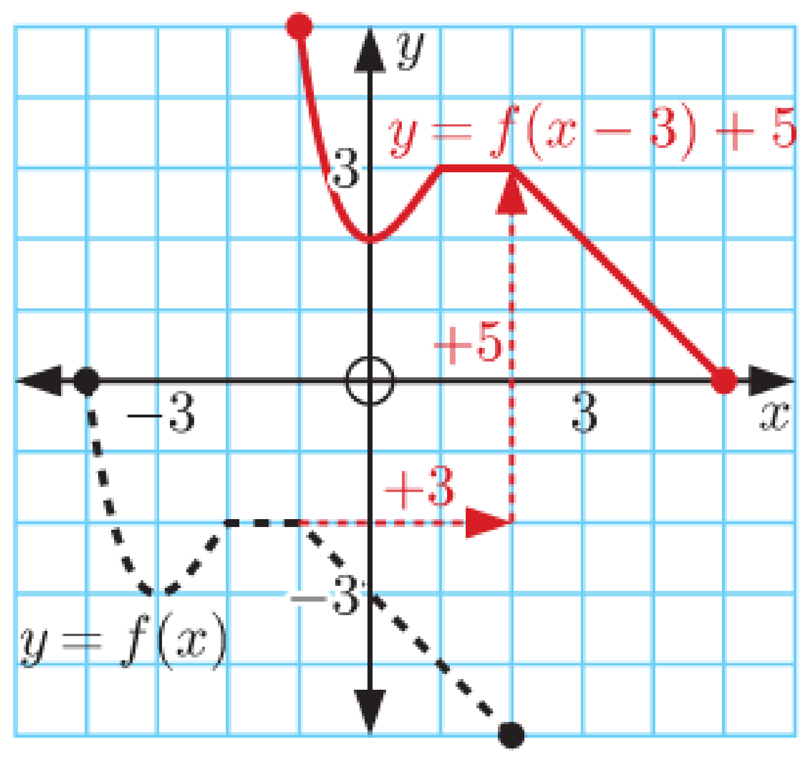
1 **a**



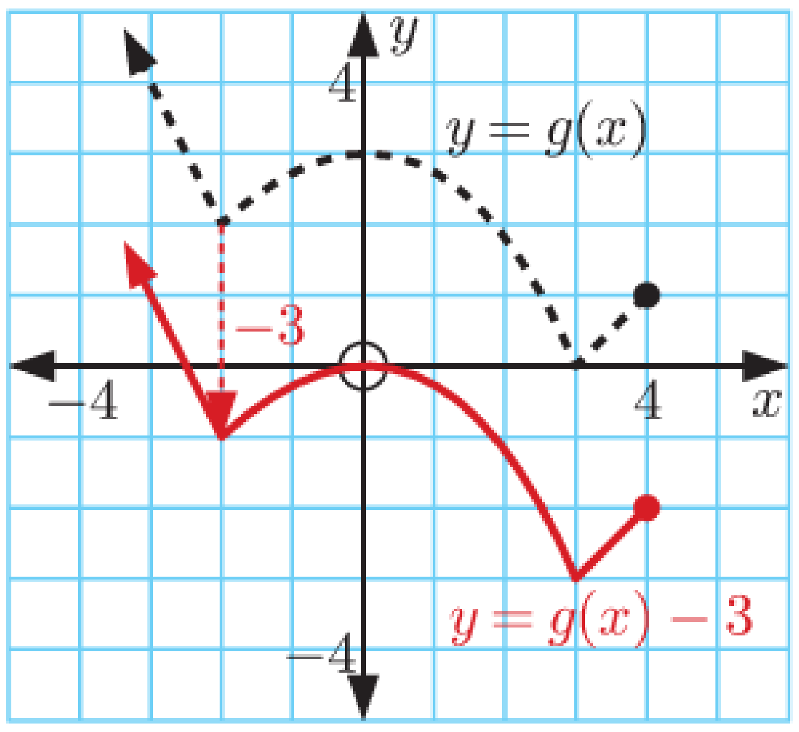
b



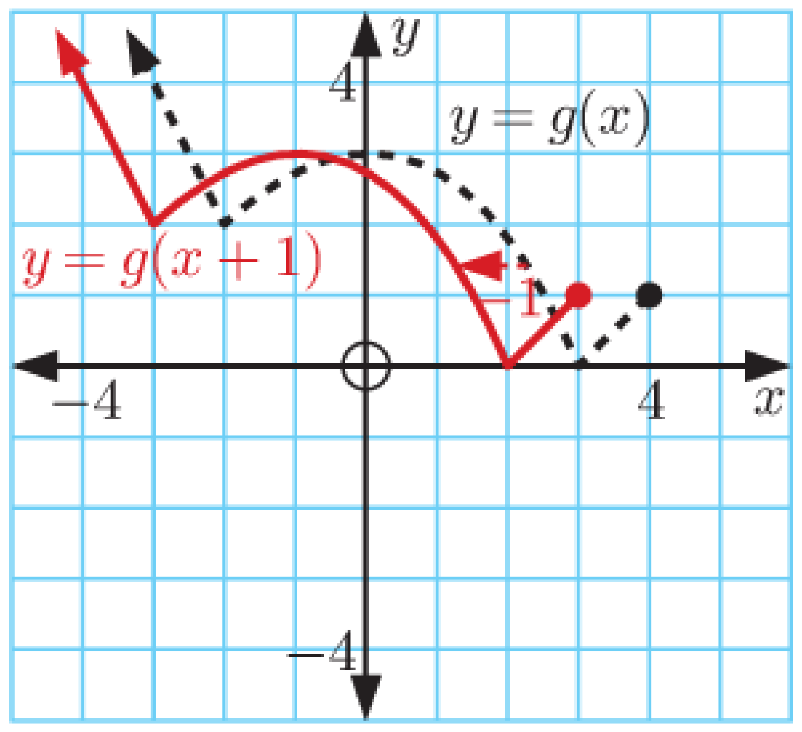
c



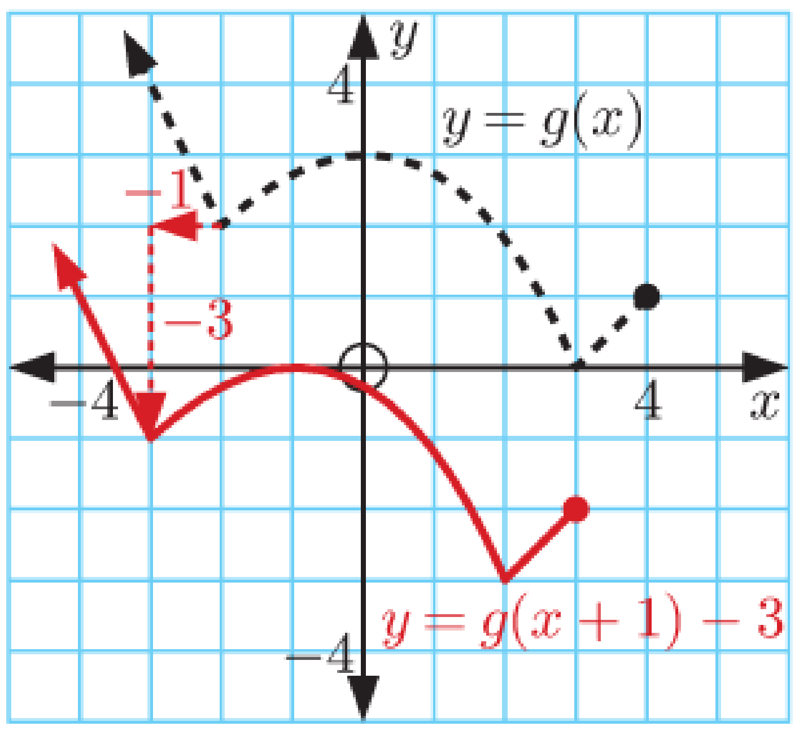
2 **a**



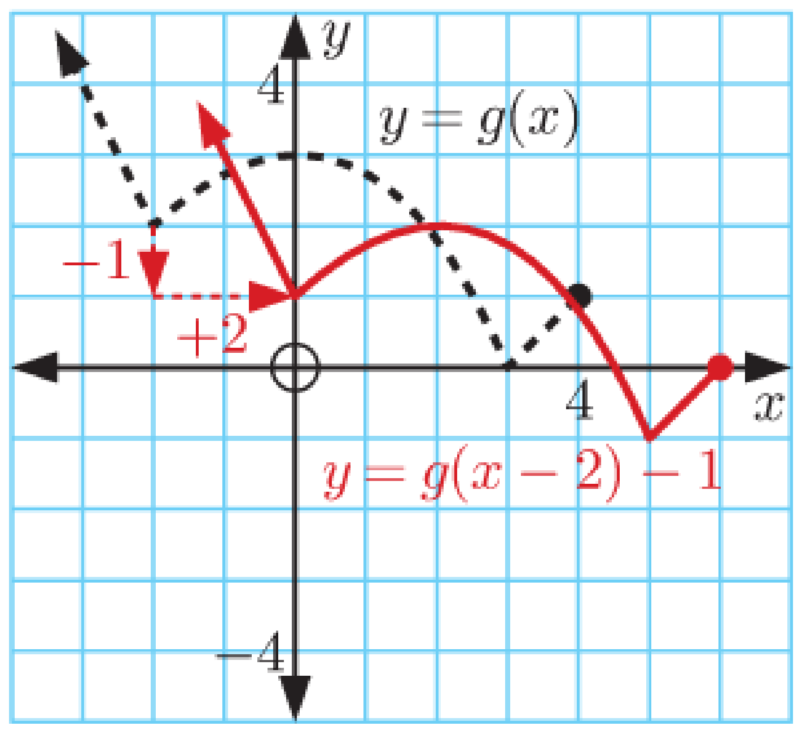
b



c



d

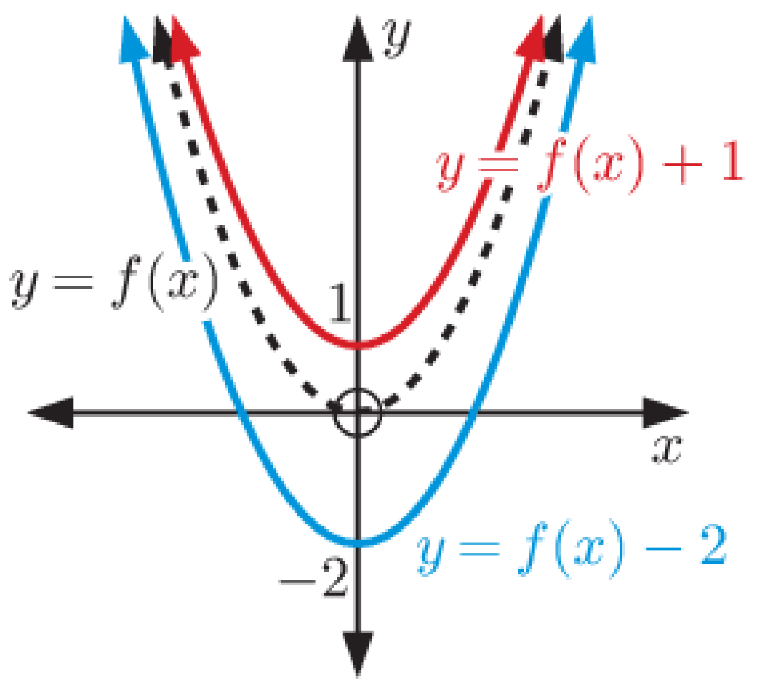


3 **a** $g(x) = f(x - 4)$ **b** $g(x) = f(x + 1) + 3$

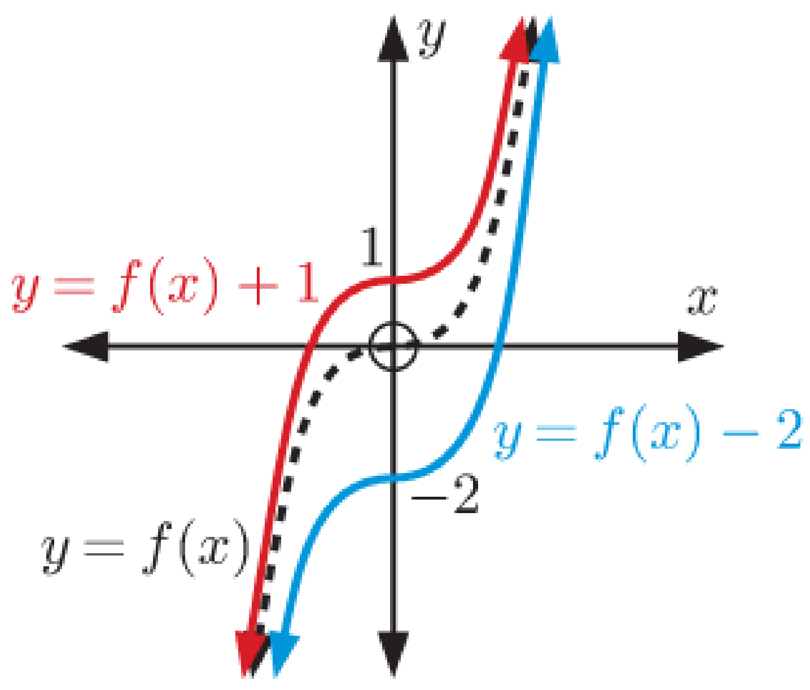
4 **a** $g(x) = 2x - 1$ **b** $g(x) = 3x + 2$

c $g(x) = -x^2 + 5x - 4$ **d** $g(x) = x^2 - 6x + 4$

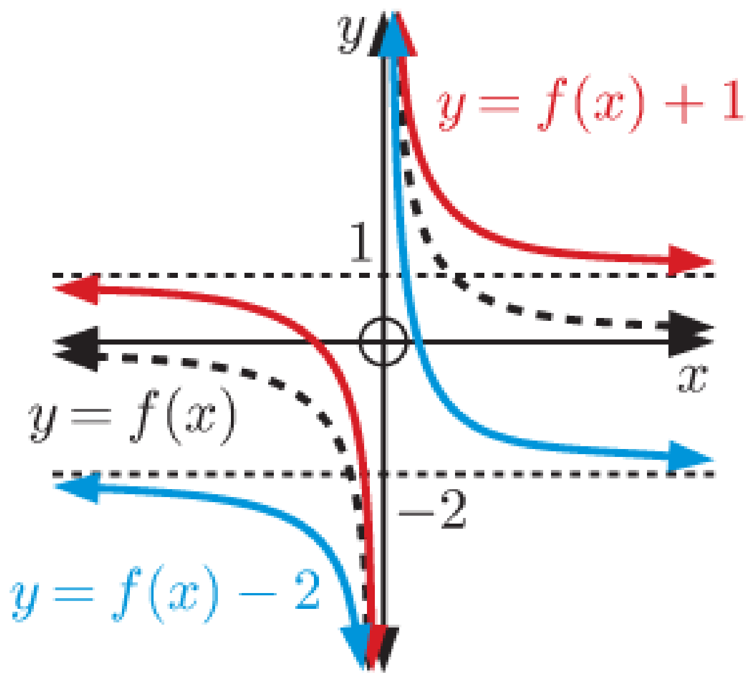
5 **a**



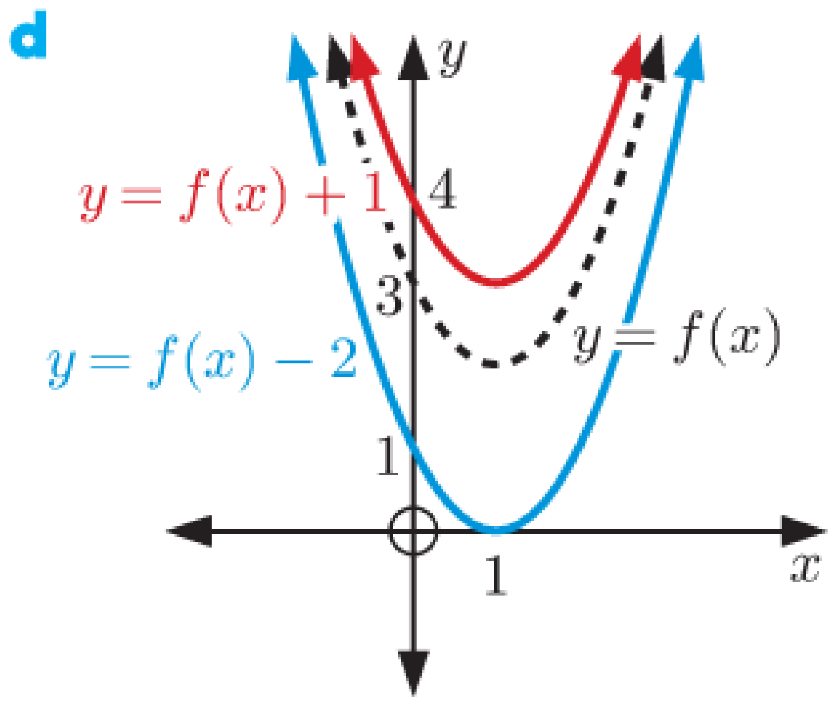
b



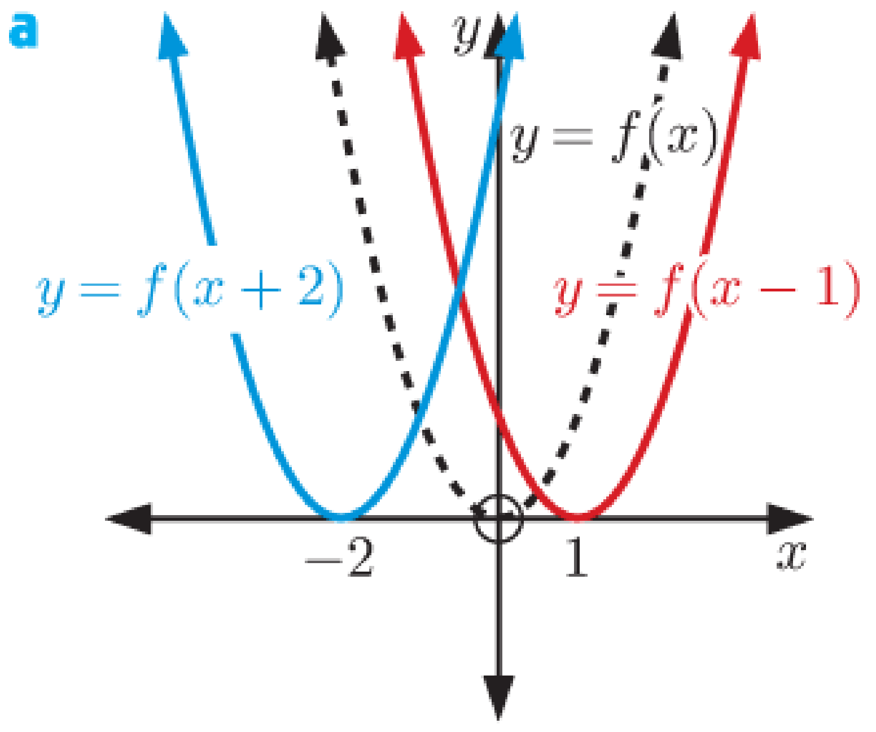
c



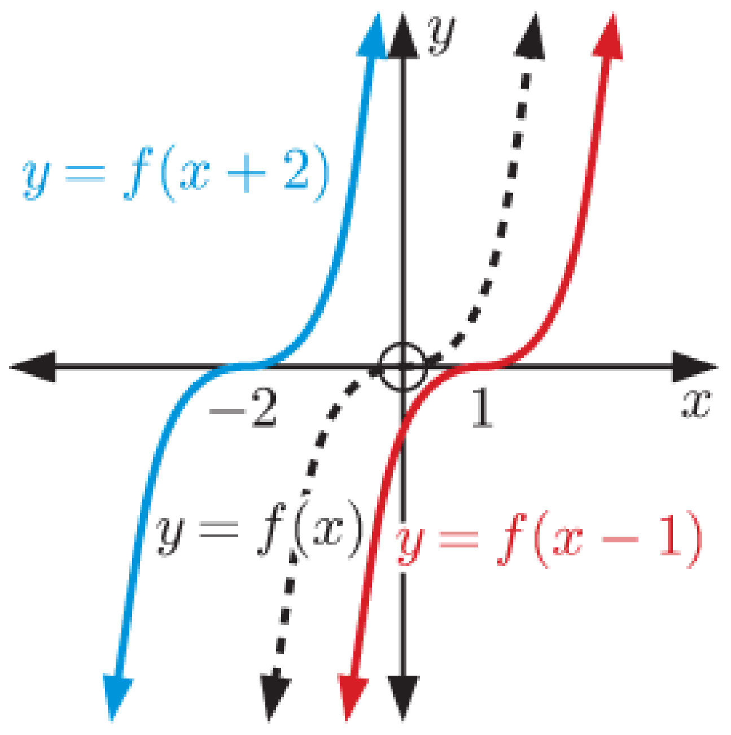
d



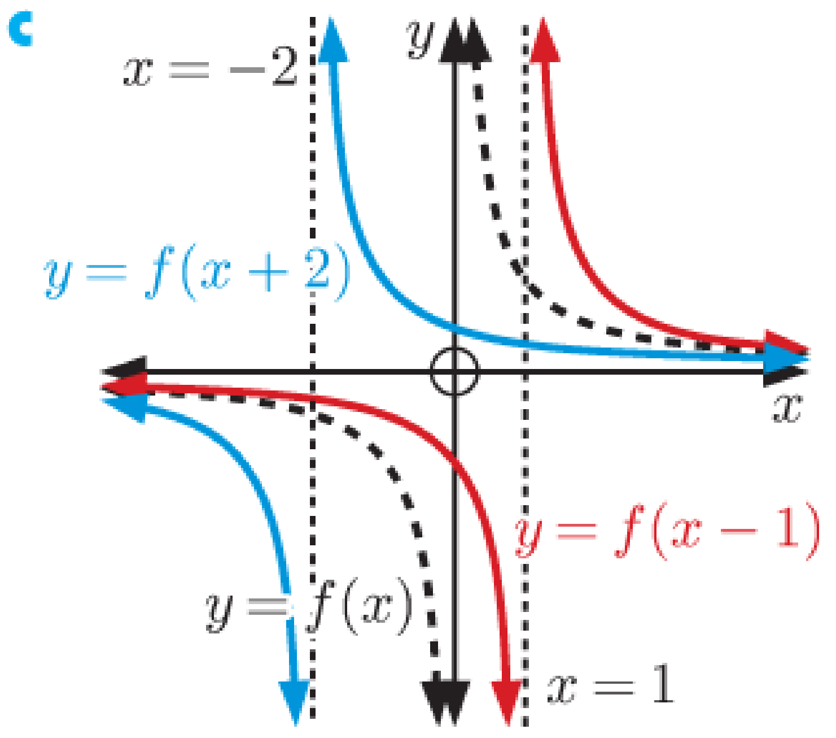
6 **a**



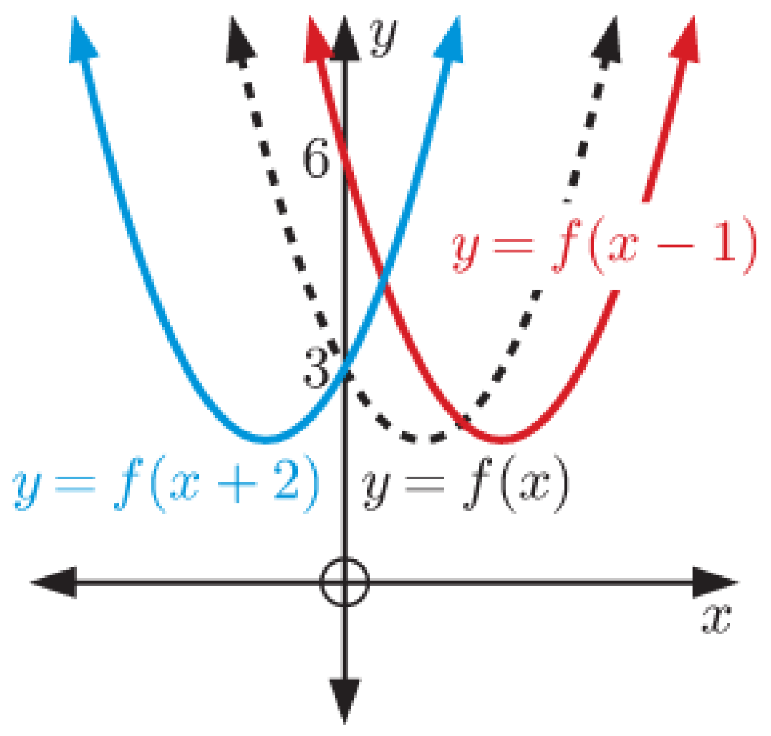
b



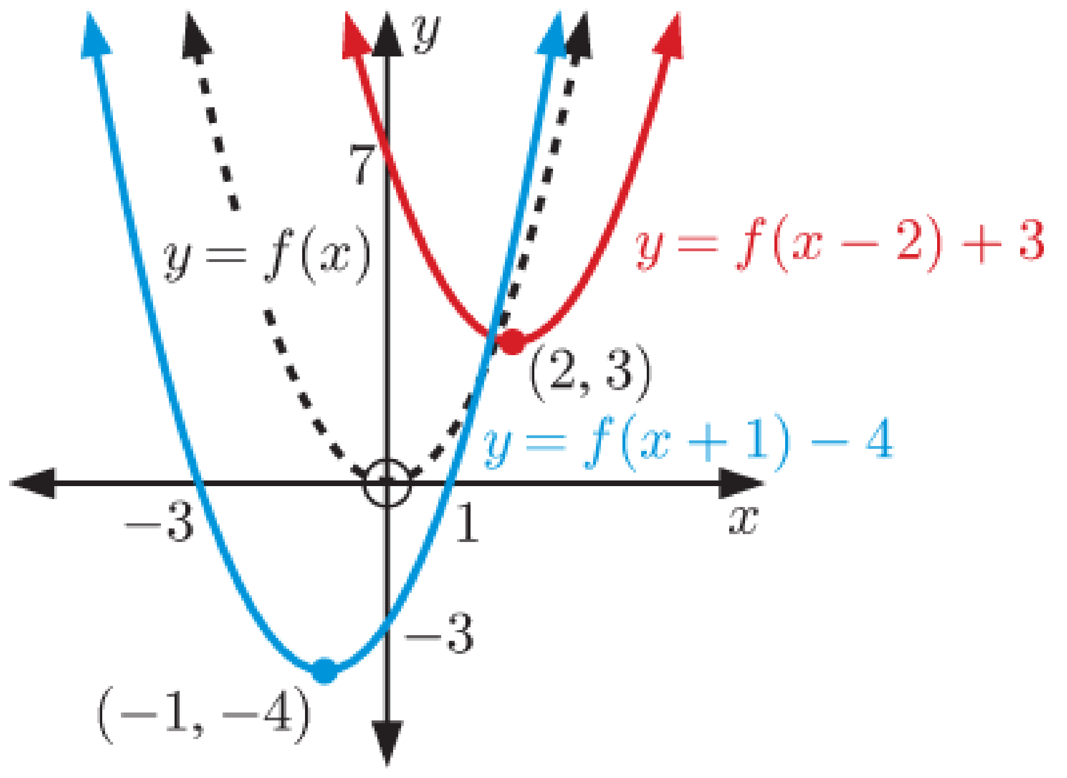
c



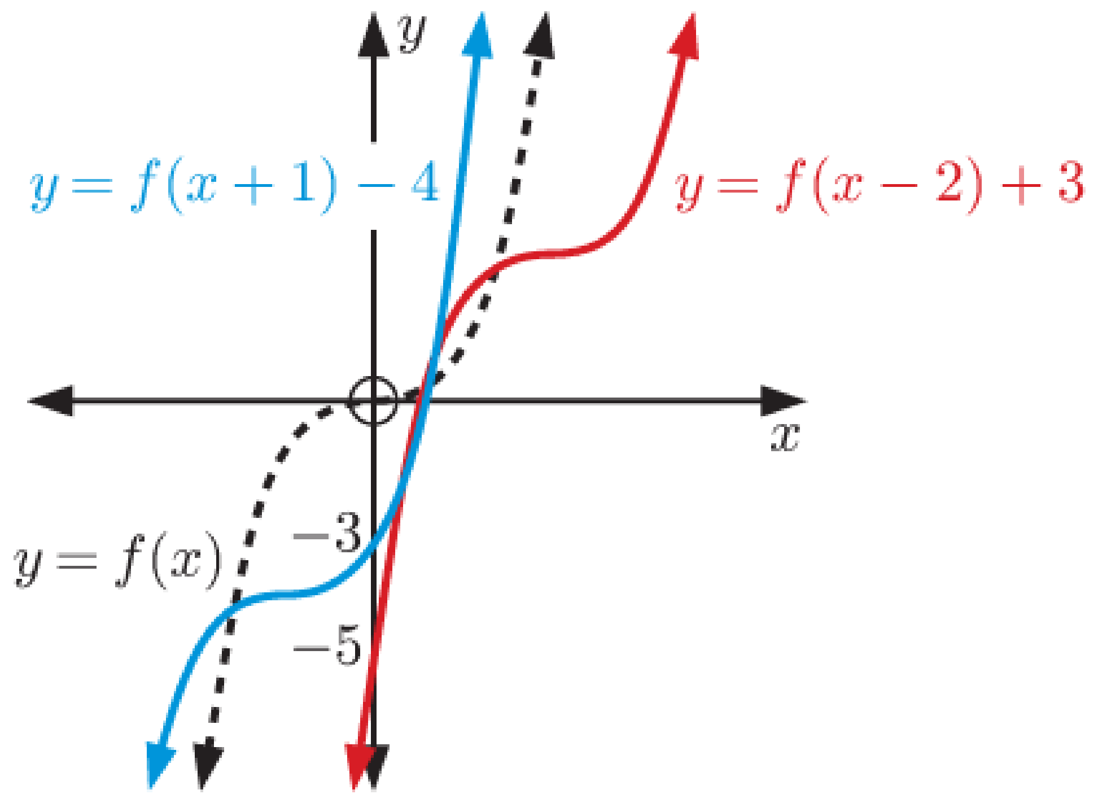
d



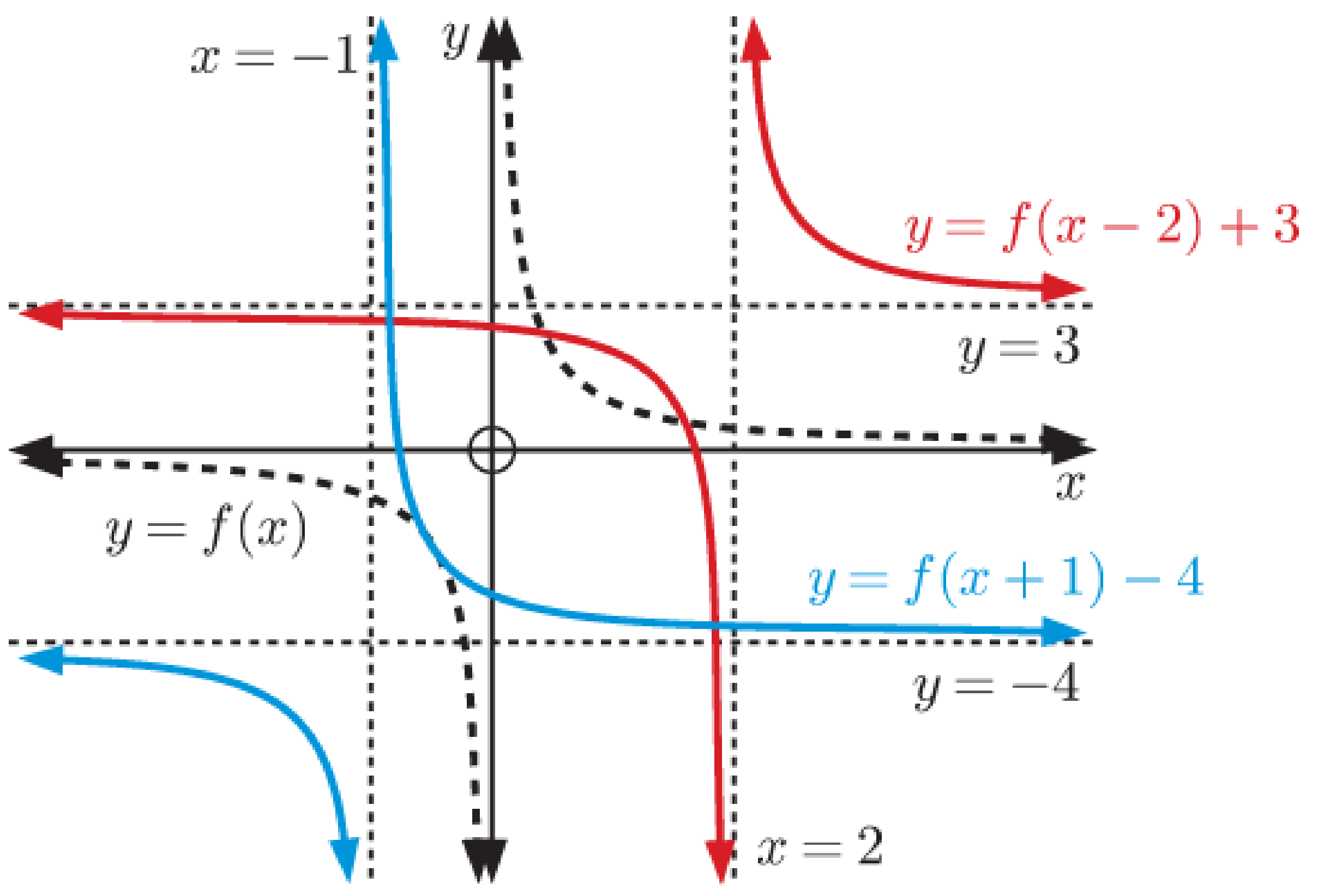
7 **a**



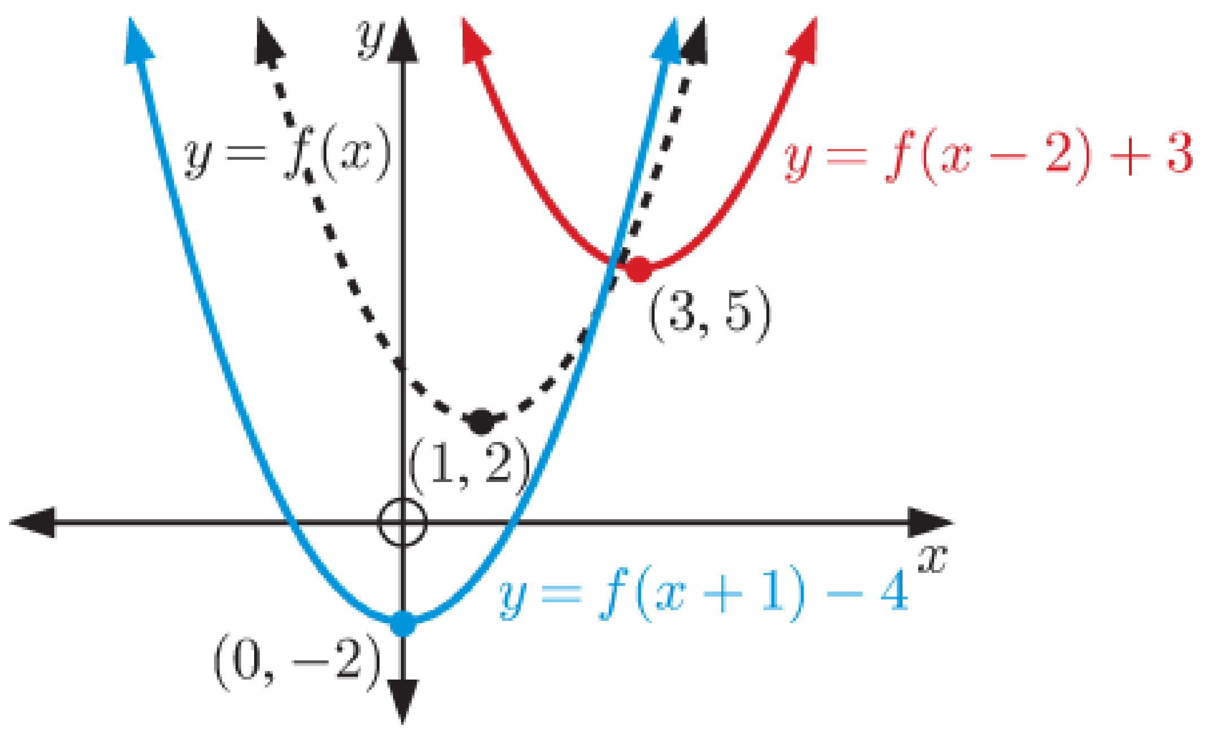
b



c



d



8 $(1, -9)$

9 **a** y-intercept is -1 **b** x-intercepts are -2 and 5

c inconclusive

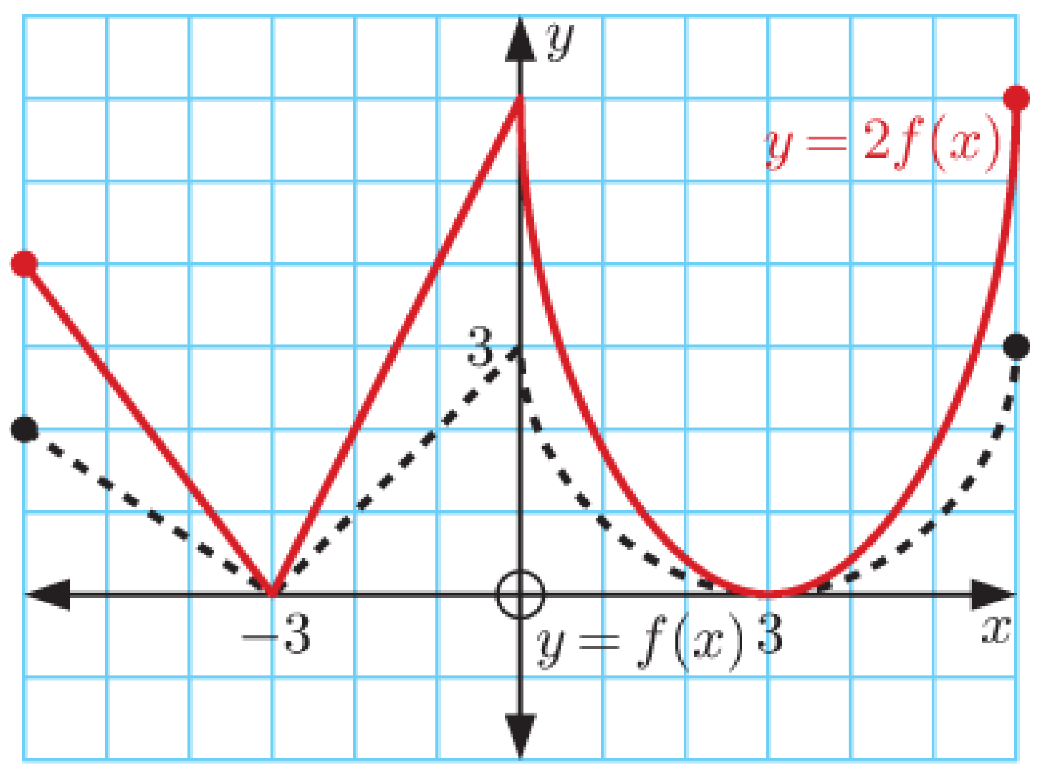
10 $g(x) = x^2 - 8x + 12$ **11** $g(x) = \dfrac{7x + 15}{x + 2}$

12 **a** **i** $(3, 2)$ **ii** $(0, 11)$ **iii** $(5, 6)$

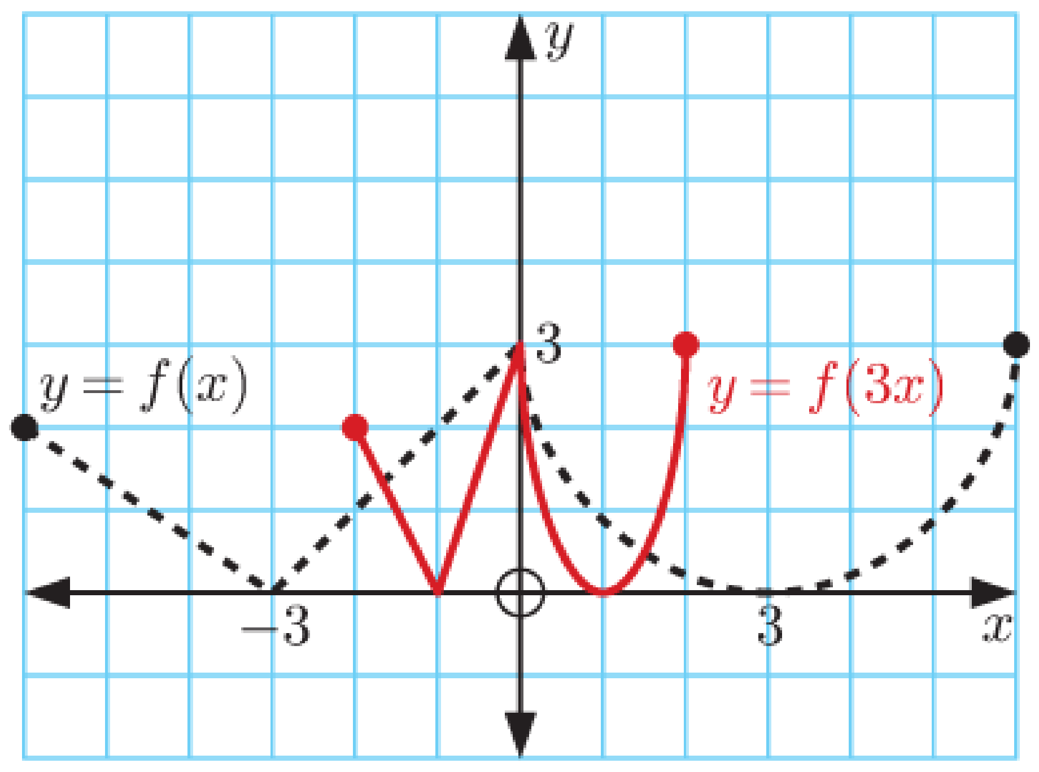
b **i** $(-2, 4)$ **ii** $(-5, 25)$ **iii** $\left(-1\frac{1}{2}, 2\frac{1}{4}\right)$

EXERCISE 16B

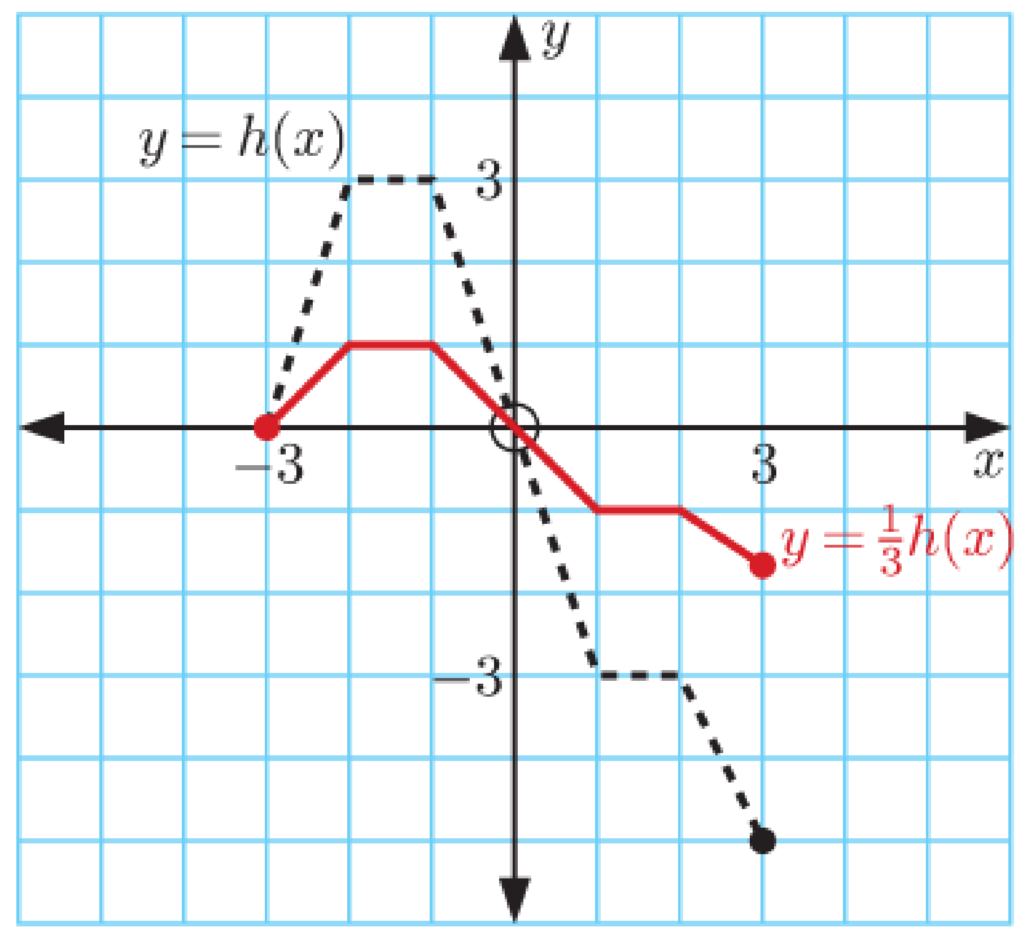
1 **a**



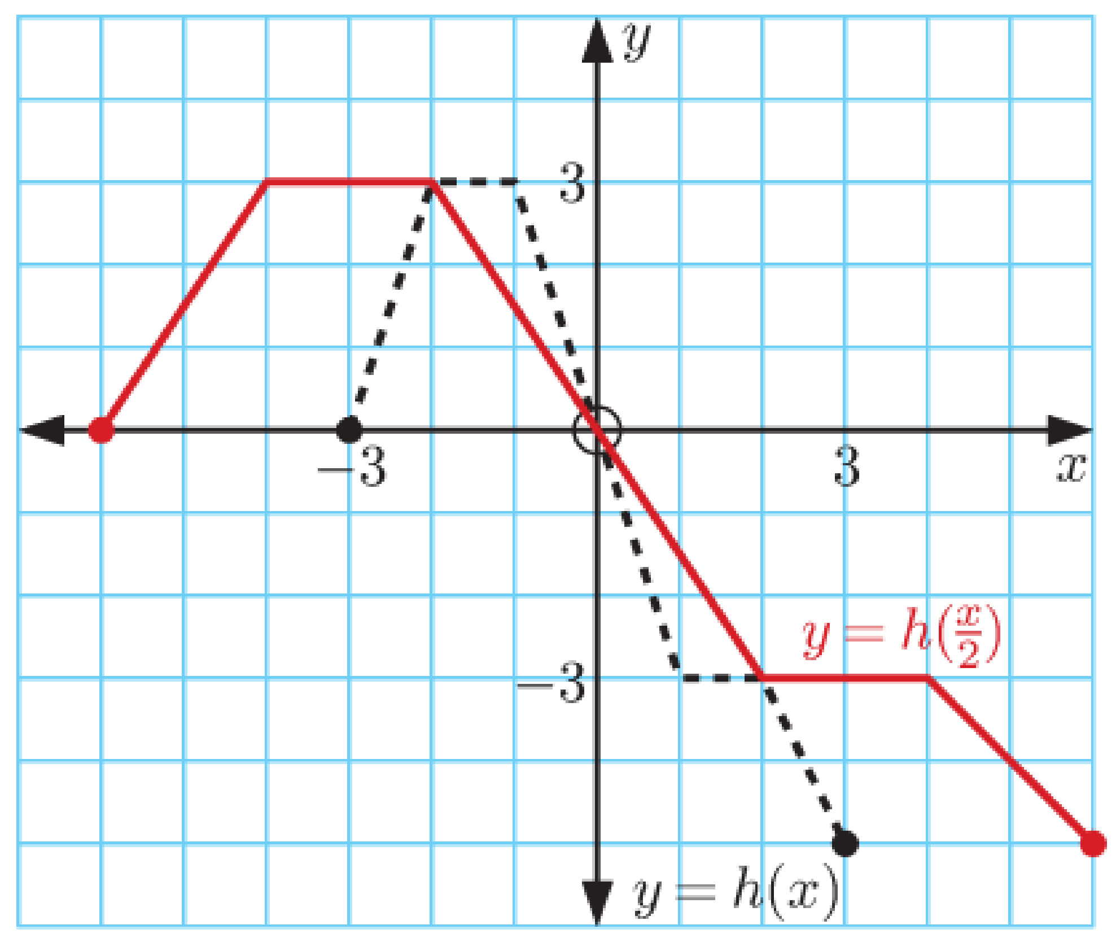
b



2 **a**

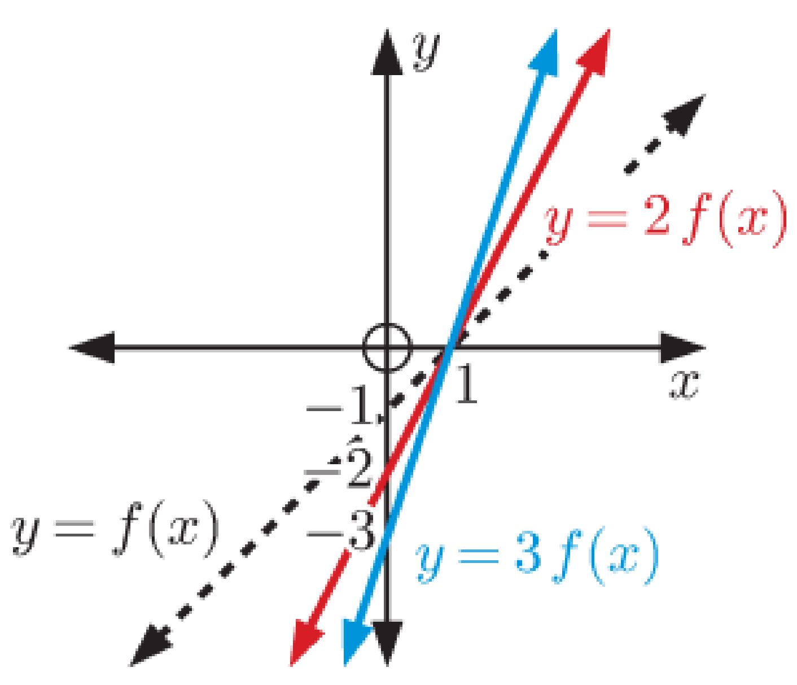


b

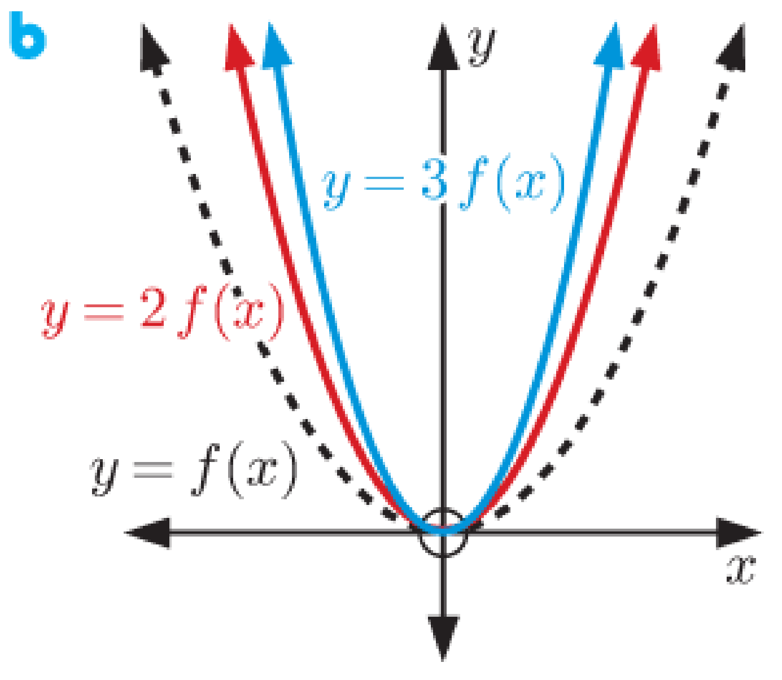


3 **a** $g(x) = 2f(x)$ **b** $g(x) = f\left(\dfrac{x}{3}\right)$ **4** *cm*

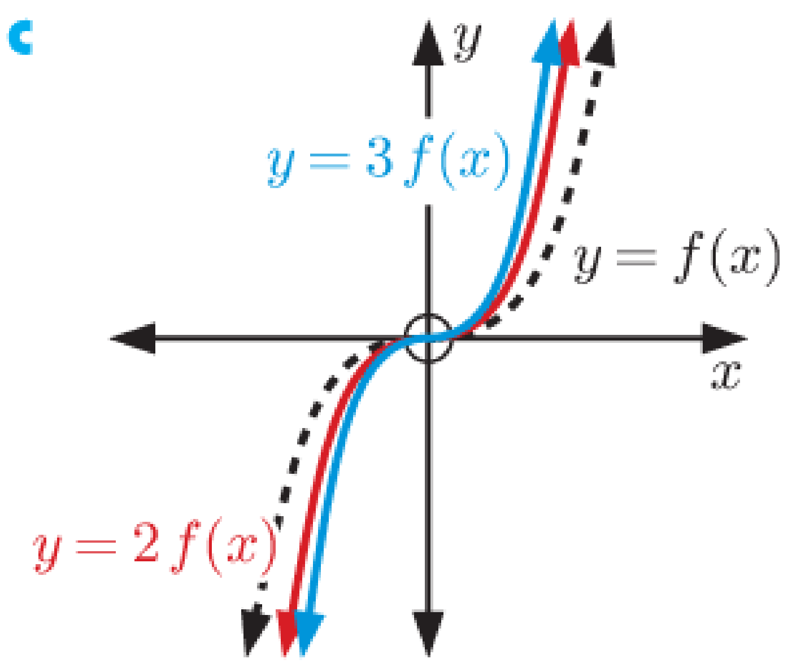
5 **a**



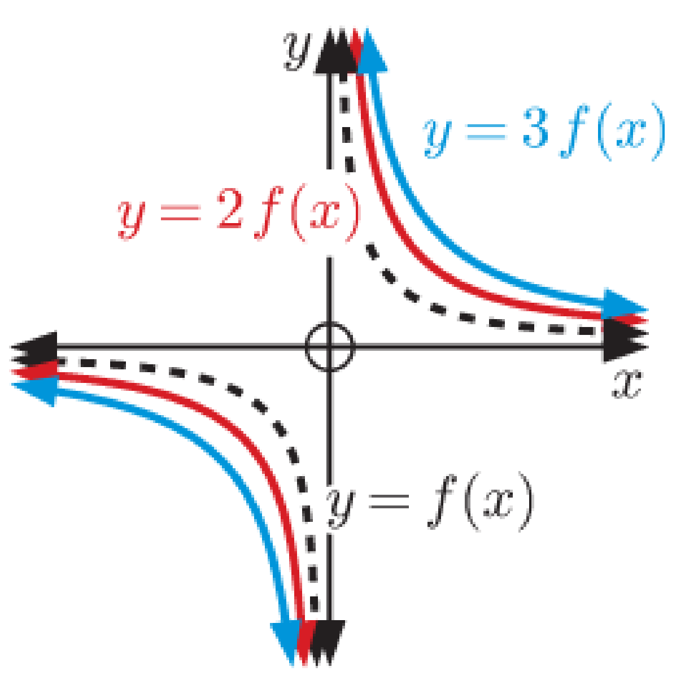
b



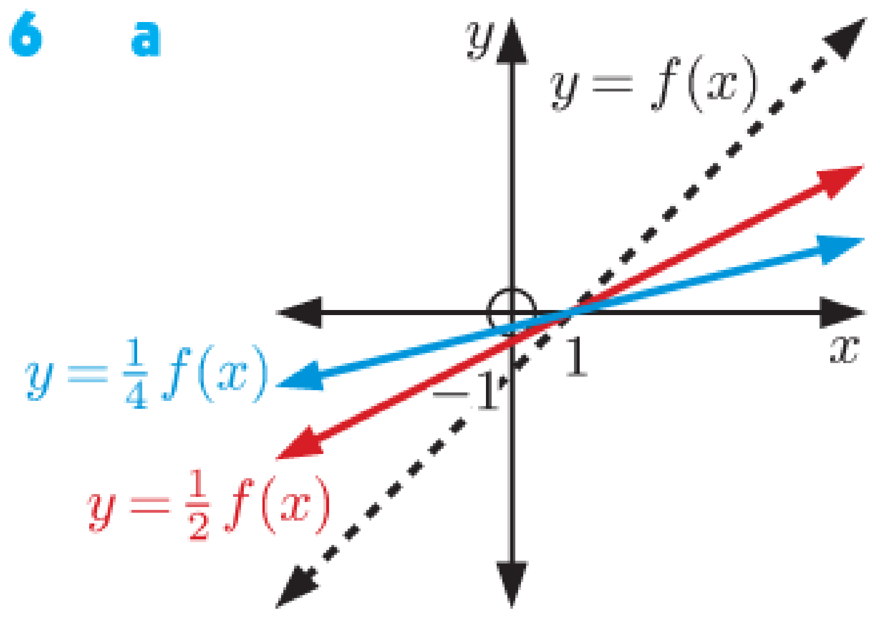
c



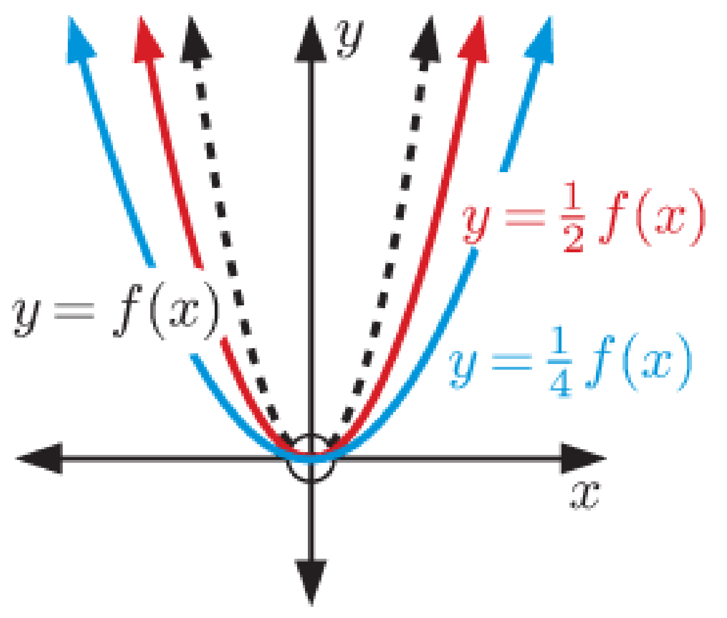
d



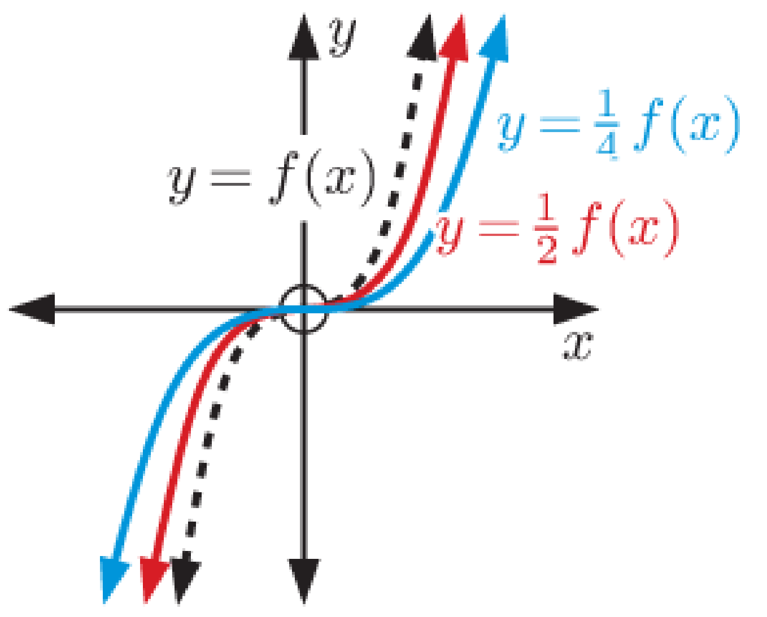
6 **a**



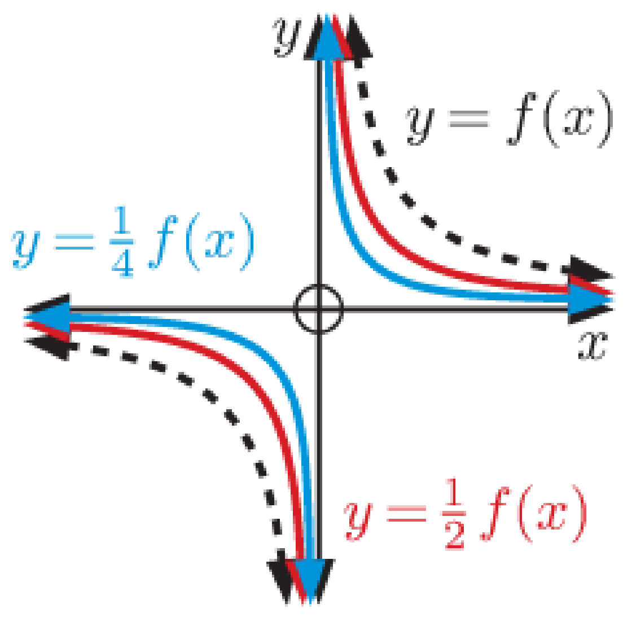
b



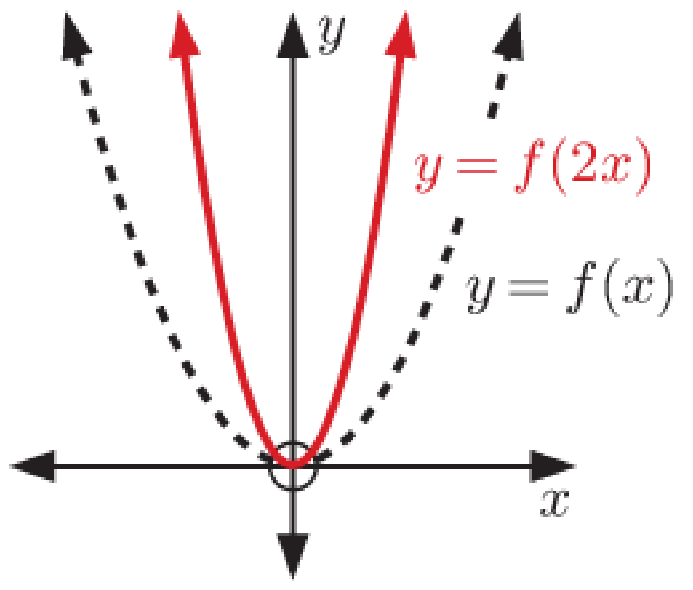
c



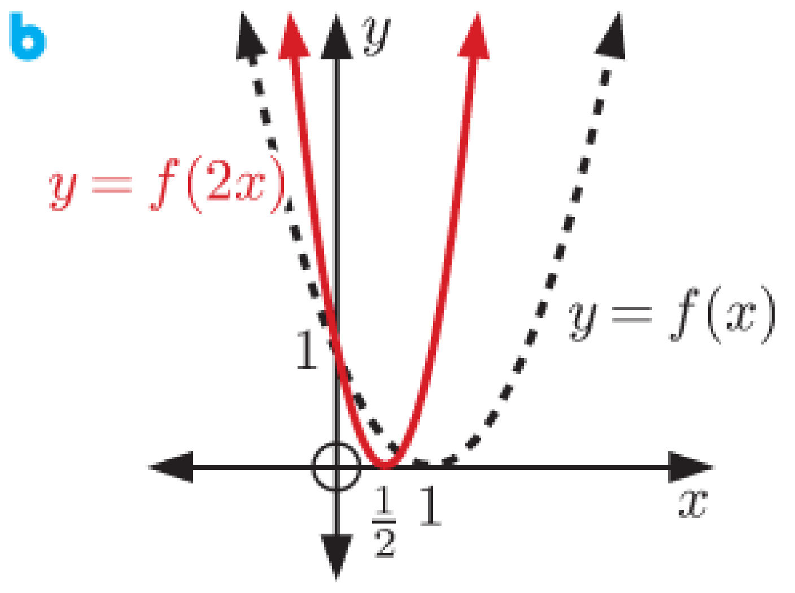
d



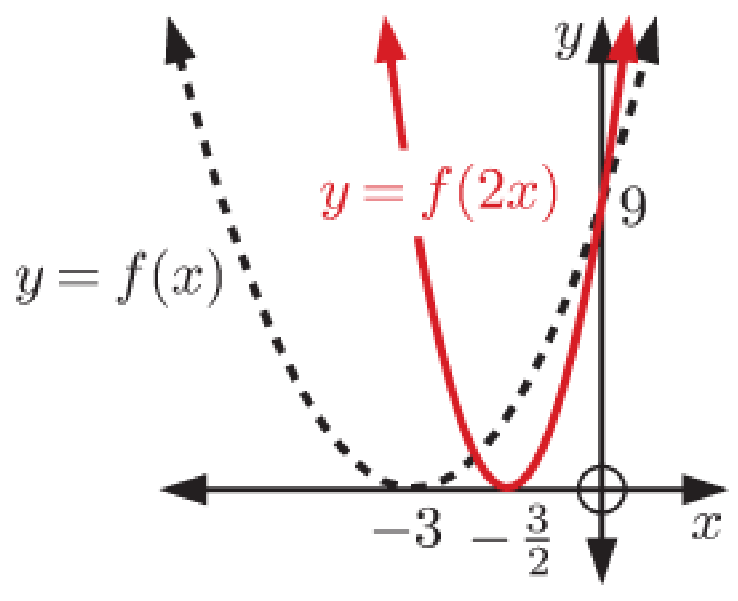
7 **a**



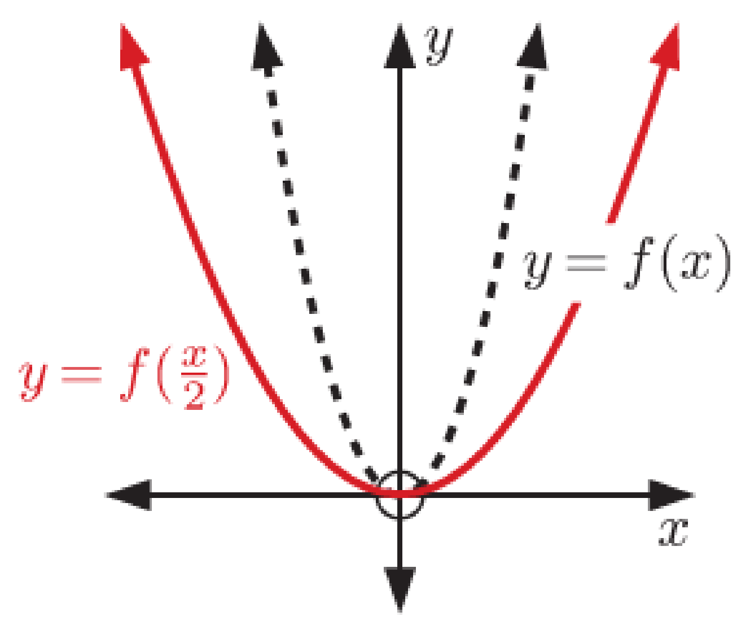
b



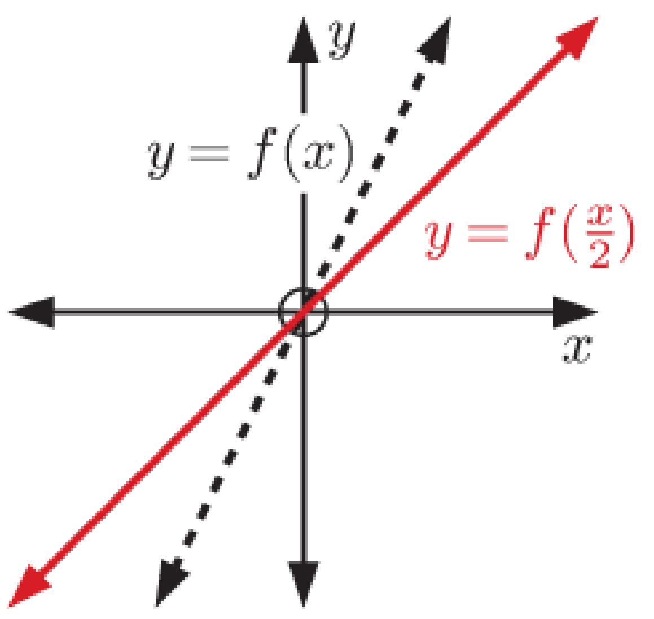
c



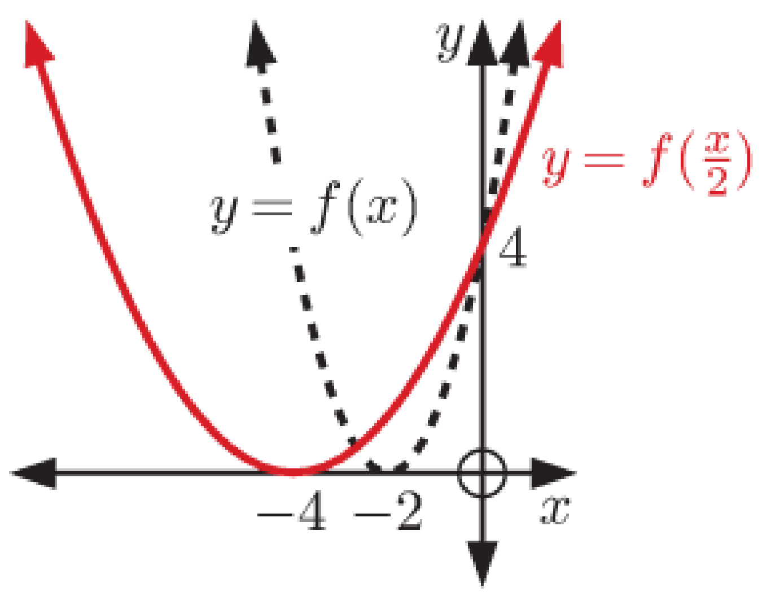
8 **a**



b



c

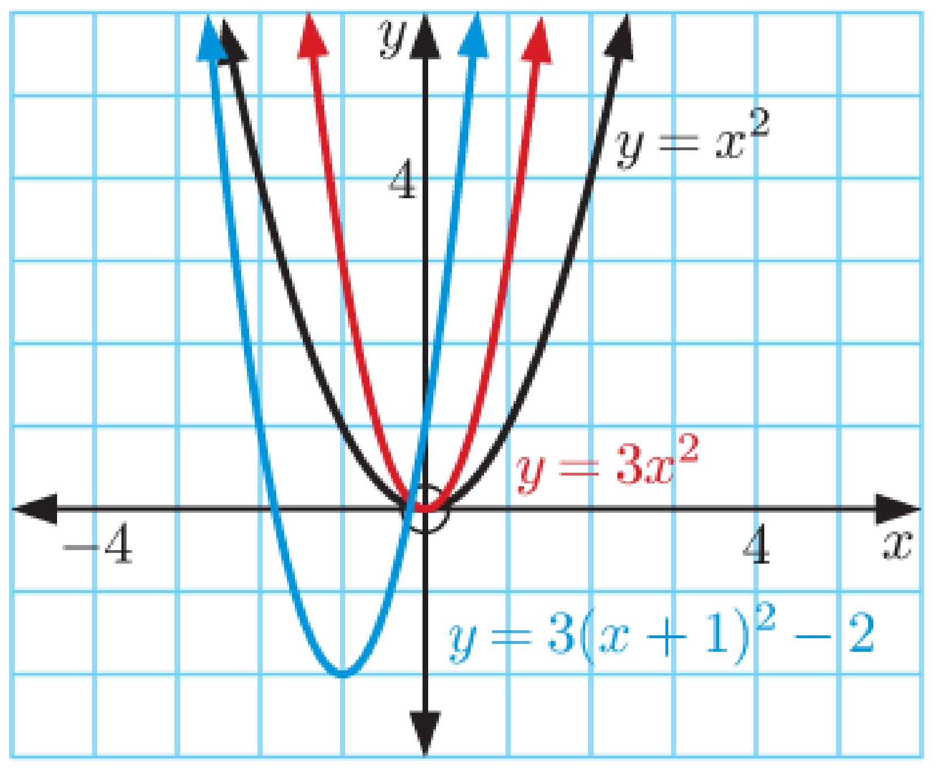


9 **a** (2, 25) **b** (−25, −15)

10 **a** $g(x) = 2x^2 + 4$ **b** $g(x) = 5 - x$

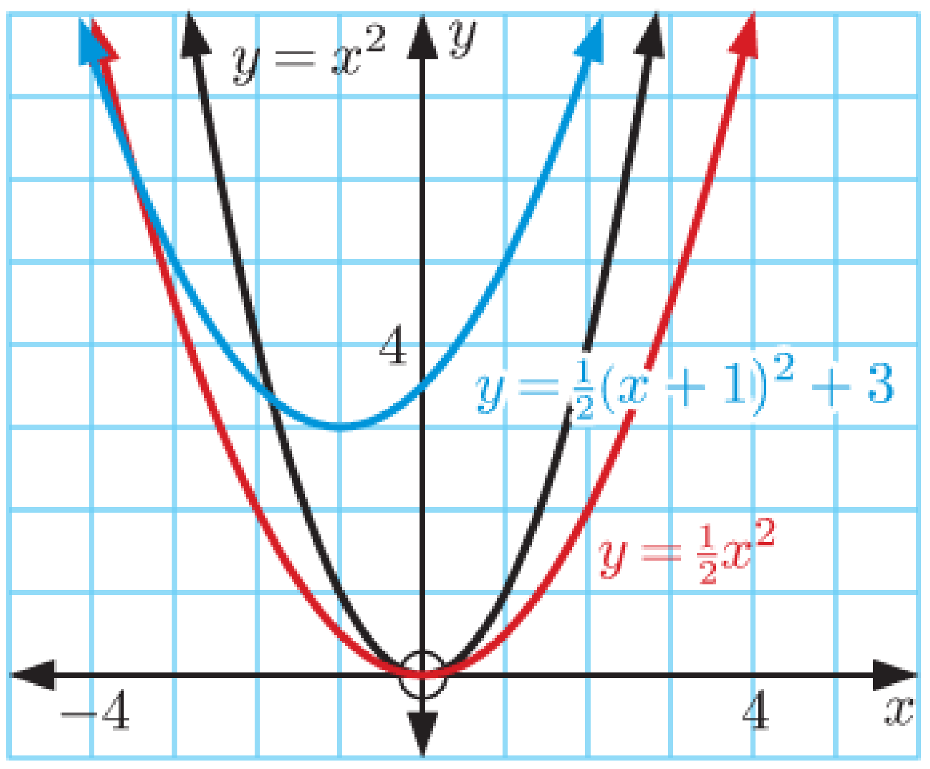
c $g(x) = \frac{1}{4}x^3 + 2x^2 - \frac{1}{2}$ **d** $g(x) = 8x^2 + 2x - 3$

11



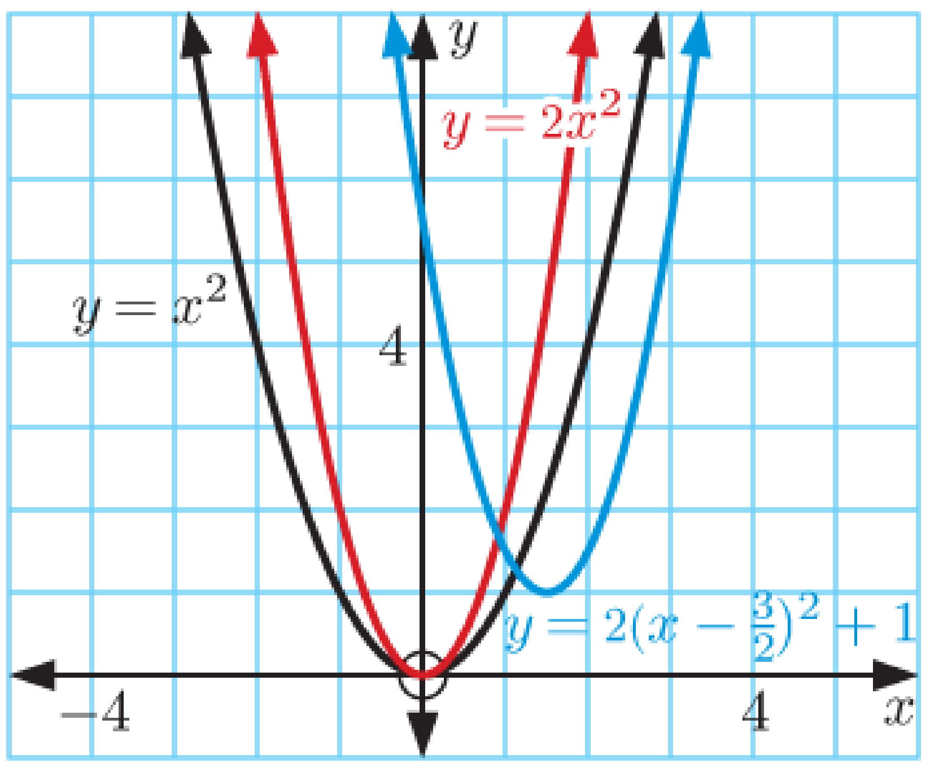
$y = x^2$ is transformed to $y = 3(x+1)^2 - 2$ by vertically stretching with scale factor 3 and then translating through $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

12



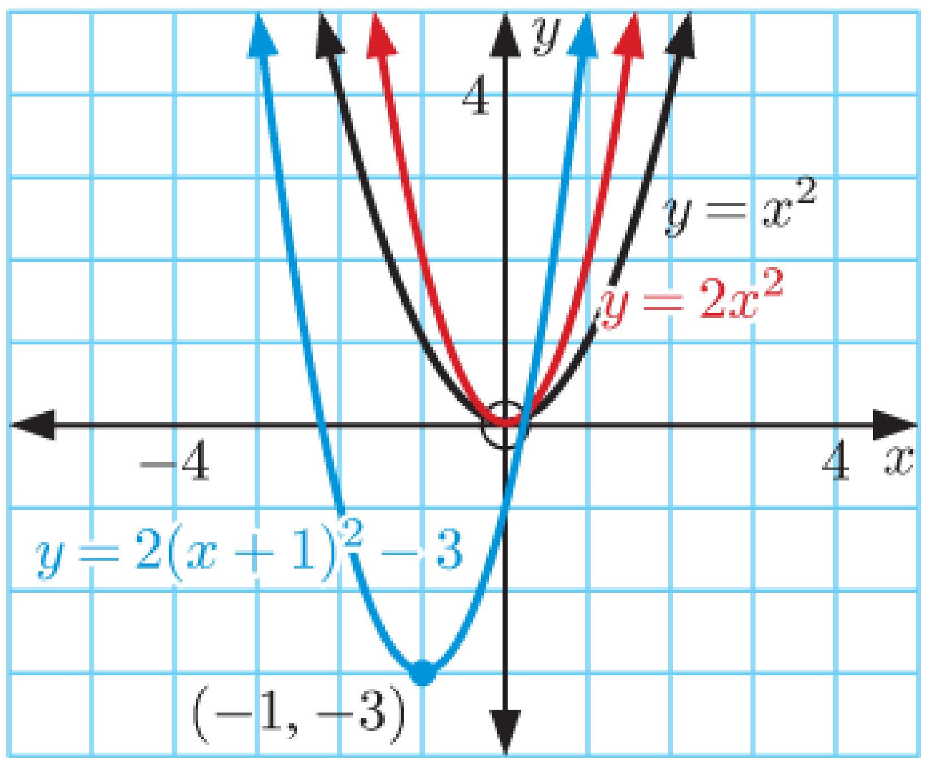
$y = x^2$ is transformed to $y = \frac{1}{2}(x+1)^2 + 3$ by vertically stretching with scale factor $\frac{1}{2}$ and then translating through $\begin{pmatrix}-1\\3\end{pmatrix}$.

13



$y = x^2$ is transformed to $y = 2(x - \frac{3}{2})^2 + 1$ by vertically stretching with scale factor 2 and then translating through $\begin{pmatrix}\frac{3}{2}\\1\end{pmatrix}$.

14



$y = x^2$ is transformed to $y = 2(x+1)^2 - 3$ by vertically stretching with scale factor 2 and then translating through $\begin{pmatrix}-1\\-3\end{pmatrix}$.

15 **a** Horizontally stretching with scale factor $\frac{1}{2}$, then vertically stretching with scale factor 3.

b **i** $(\frac{3}{2}, -15)$ **ii** $(\frac{1}{2}, 6)$ **iii** $(-1, 3)$

c **i** $(4, \frac{1}{3})$ **ii** $(-6, \frac{2}{3})$ **iii** $(-14, 1)$

16 $a = 5$, $b = \sqrt{10}$

17 **a** $y = \dfrac{1}{2x}$ **b** $y = \dfrac{3}{x}$ **c** $y = \dfrac{1}{x+3}$

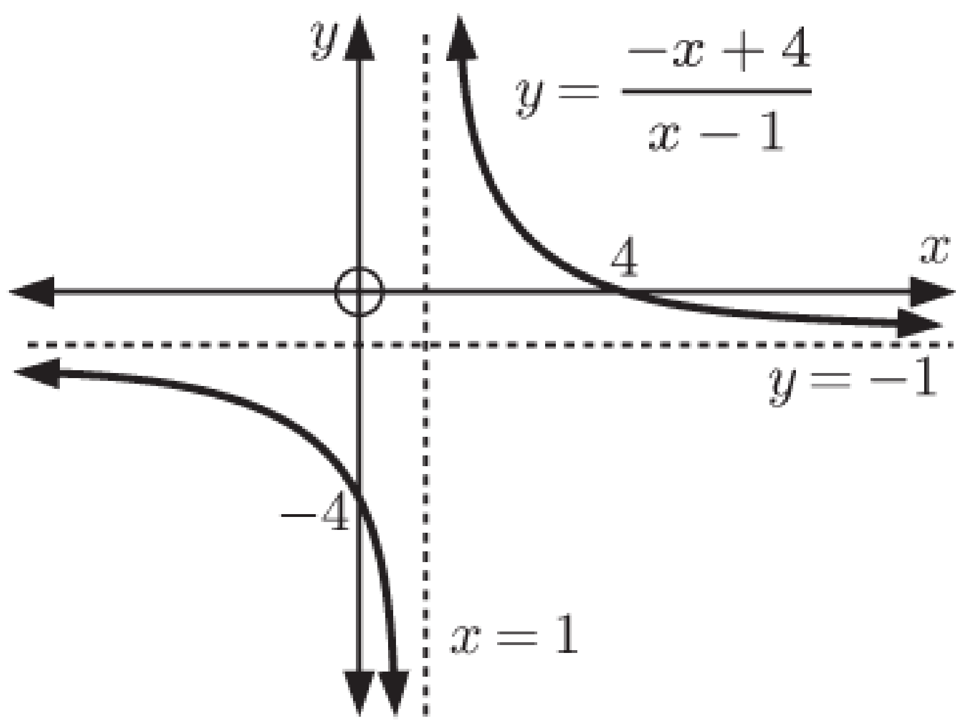
d $y = 4 + \dfrac{1}{x} = \dfrac{4x+1}{x}$

18 **a** $g(x) = \dfrac{3}{x-1} - 1 = \dfrac{-x+4}{x-1}$

b vertical asymptote $x = 1$, horizontal asymptote $y = -1$

c Domain is $\{x \mid x \neq 1\}$, Range is $\{y \mid y \neq -1\}$

d

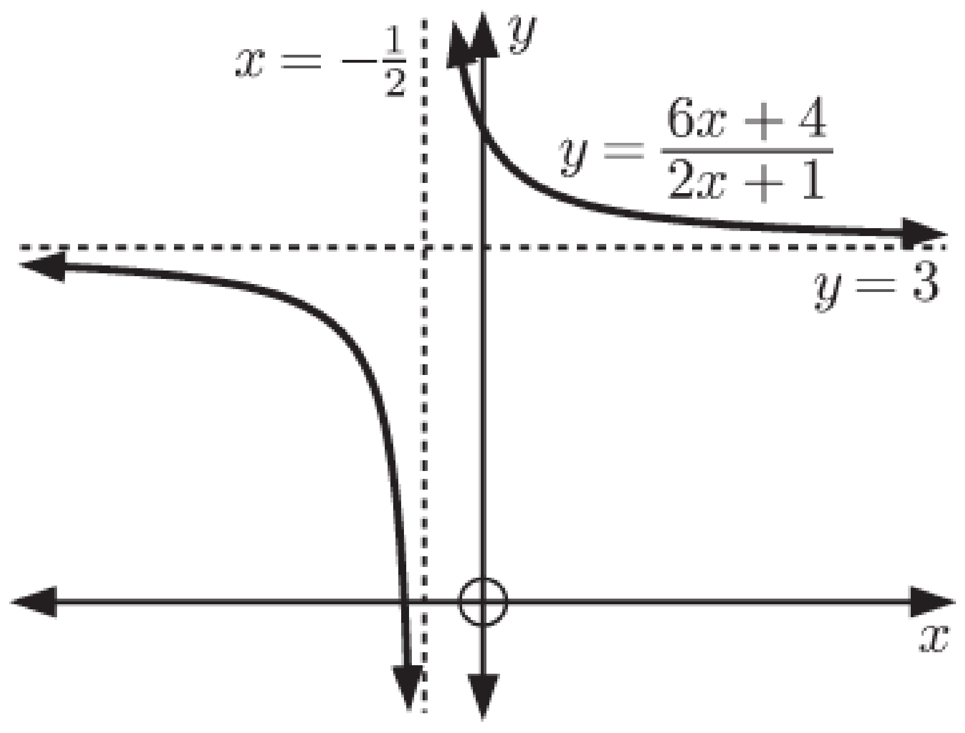


19 **a** $g(x) = \dfrac{6x+4}{2x+1}$

b vertical asymptote $x = -\frac{1}{2}$, horizontal asymptote $y = 3$

c Domain is $\{x \mid x \neq -\frac{1}{2}\}$, Range is $\{y \mid y \neq 3\}$

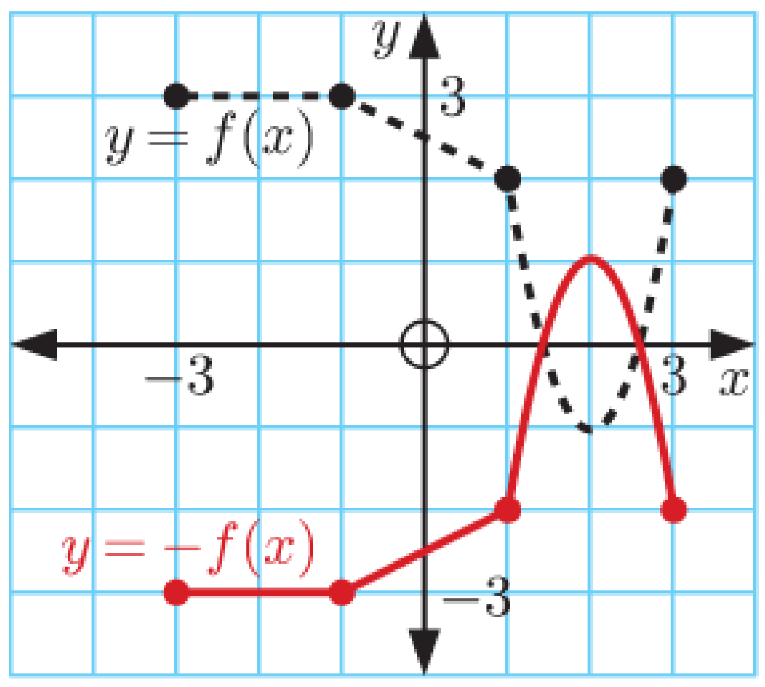
d



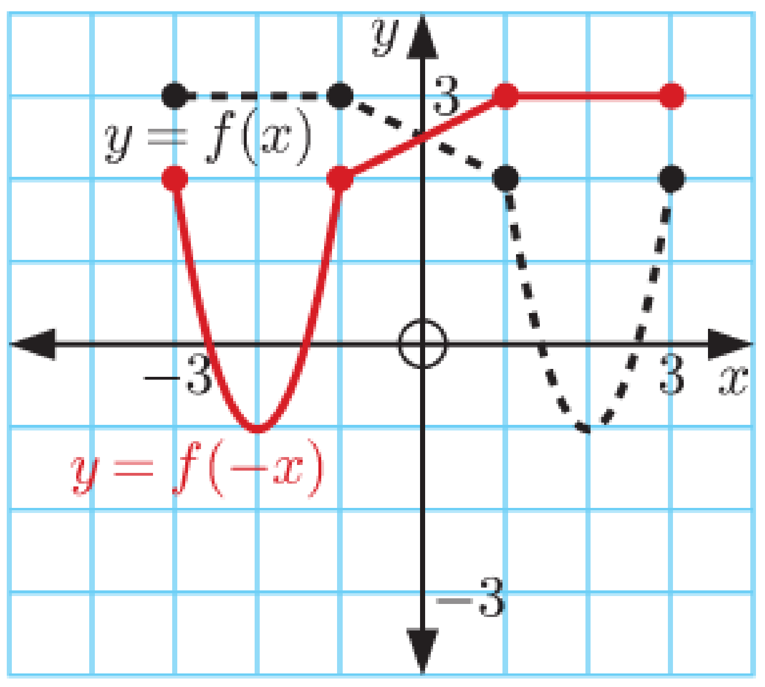
20
- A vertical stretch with scale factor 4 followed by a translation through $\begin{pmatrix}3\\33\end{pmatrix}$, *or*
- a translation through $\begin{pmatrix}3\\8\frac{1}{4}\end{pmatrix}$ followed by a vertical stretch with scale factor 4.

EXERCISE 16C

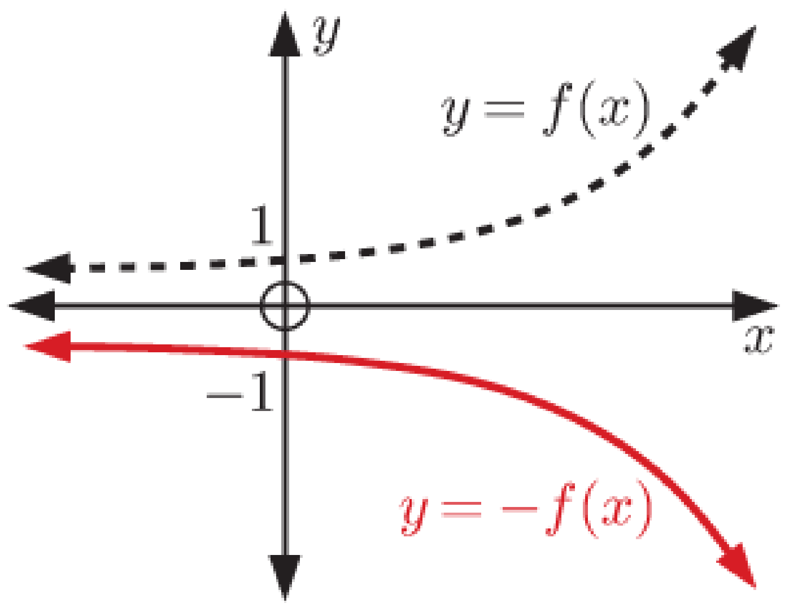
1 **a**



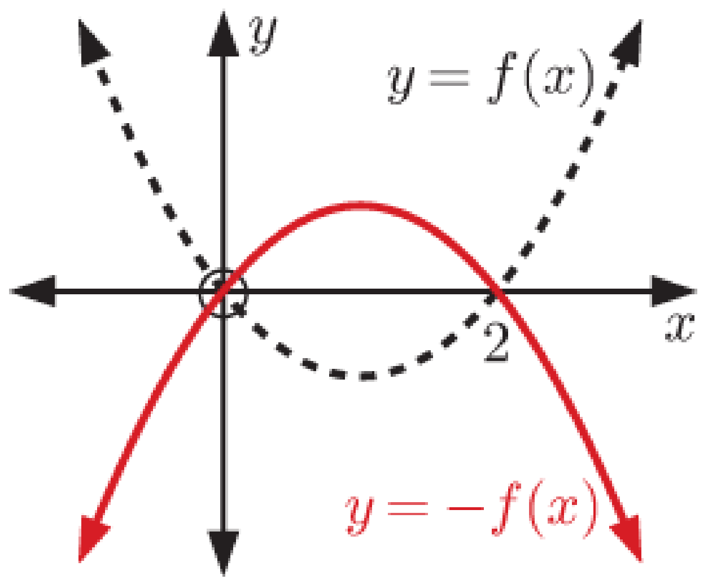
b



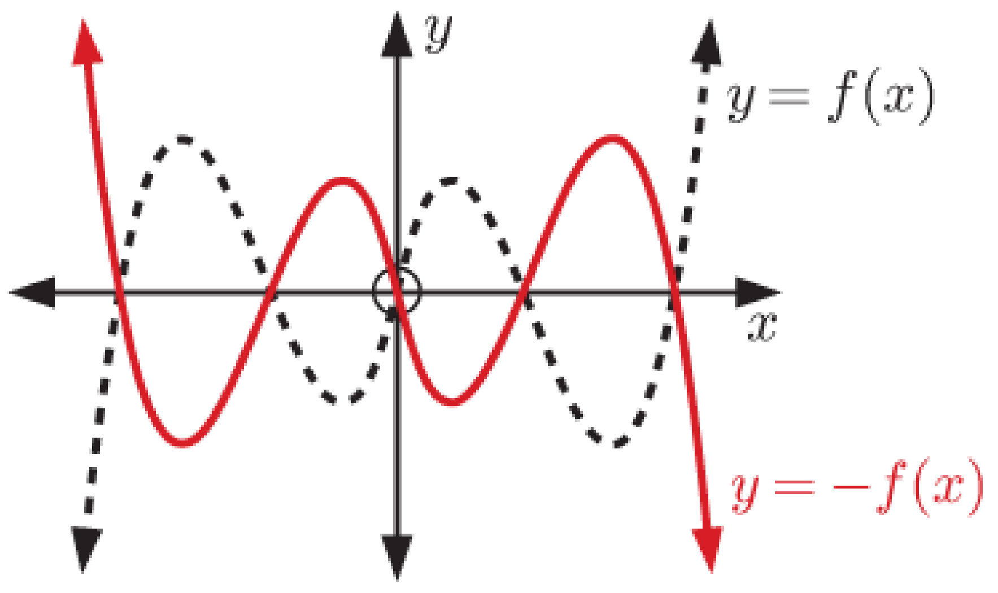
2 **a**



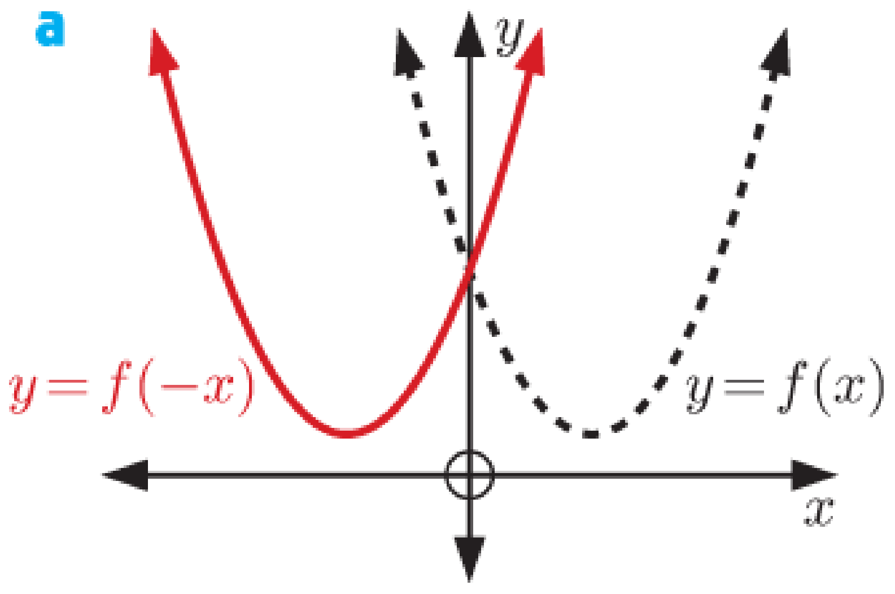
b



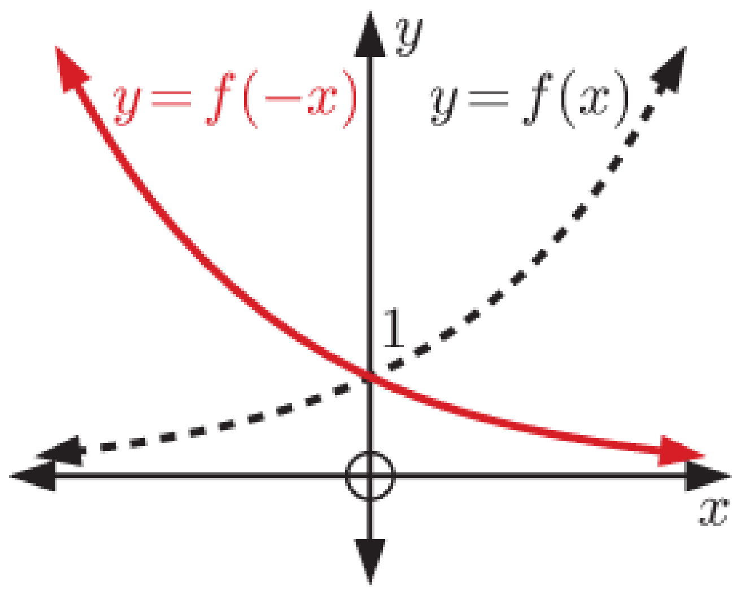
c



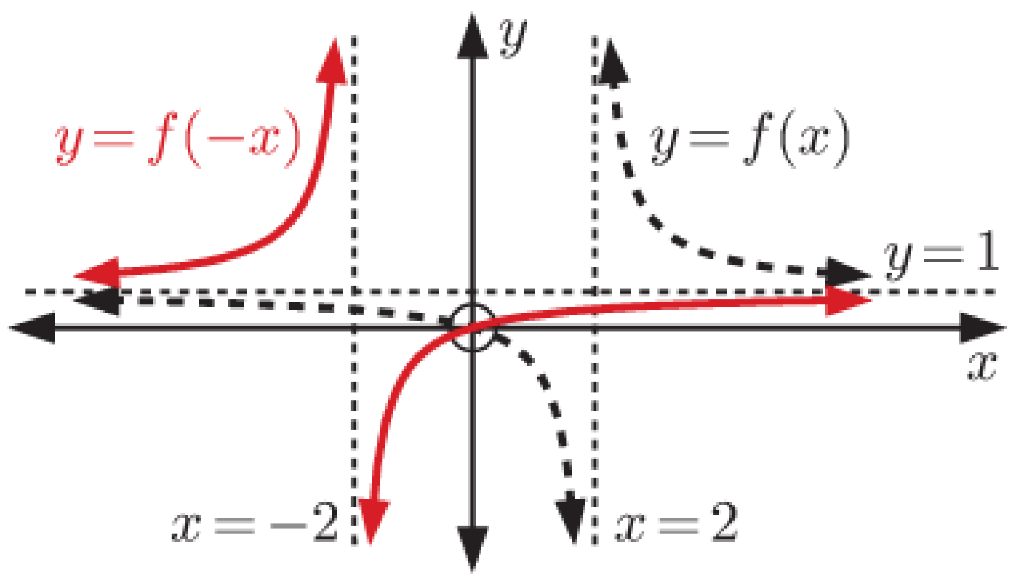
3 **a**



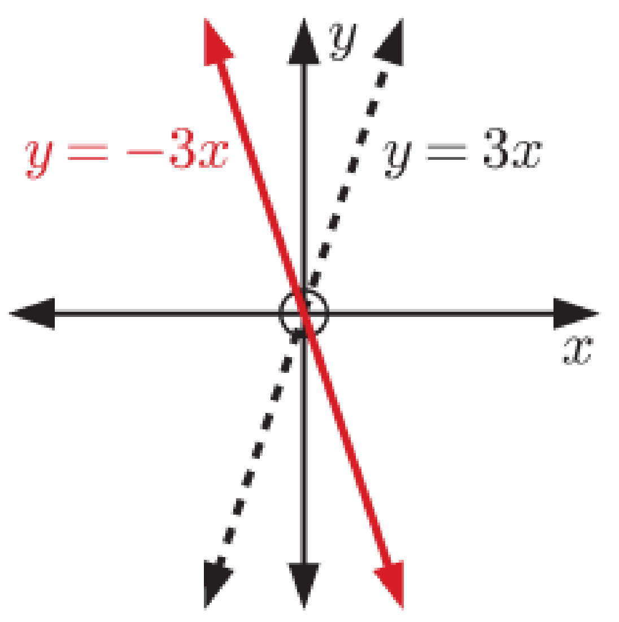
b



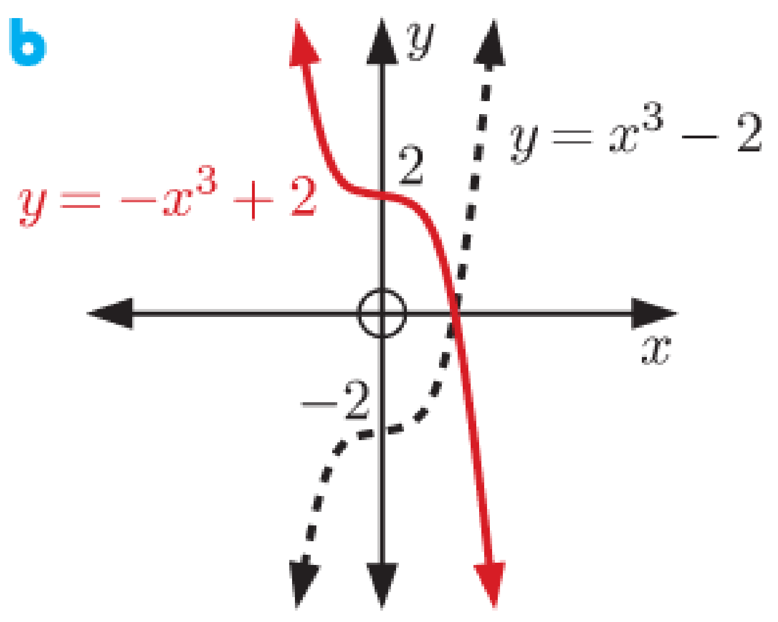
c



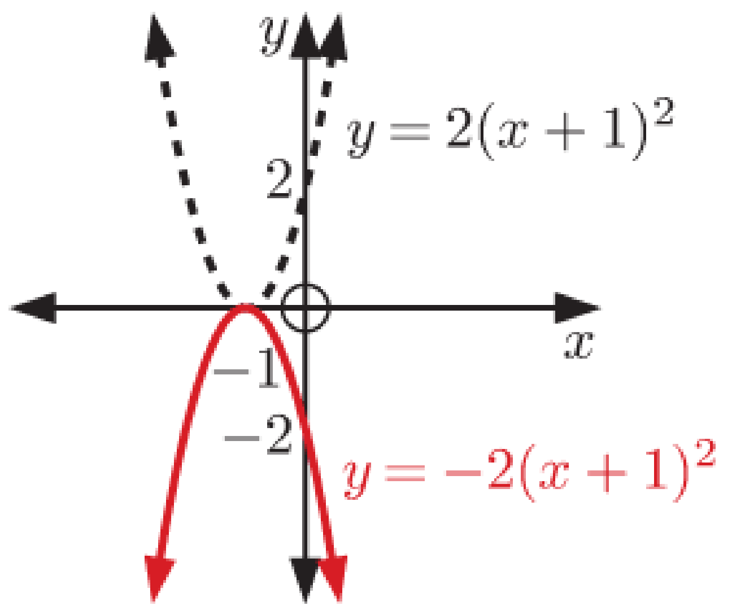
4 **a**



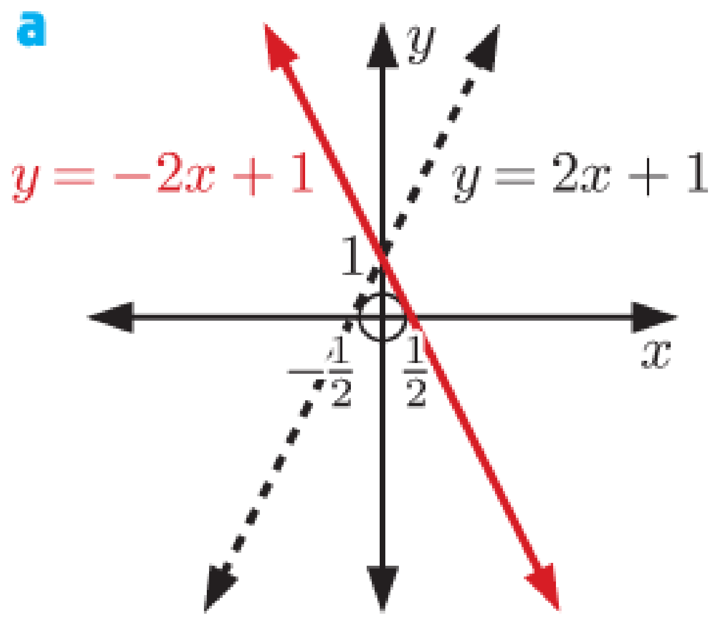
b



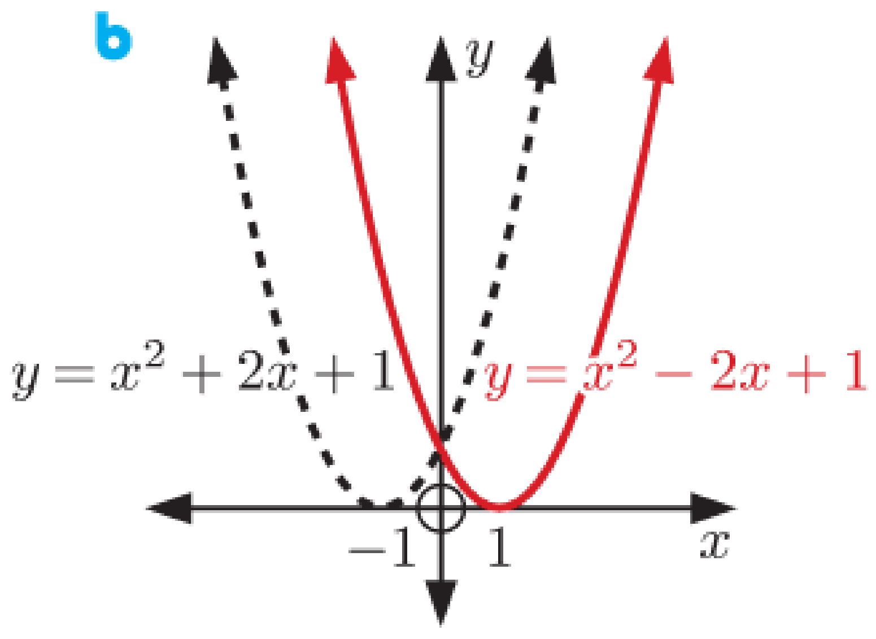
c



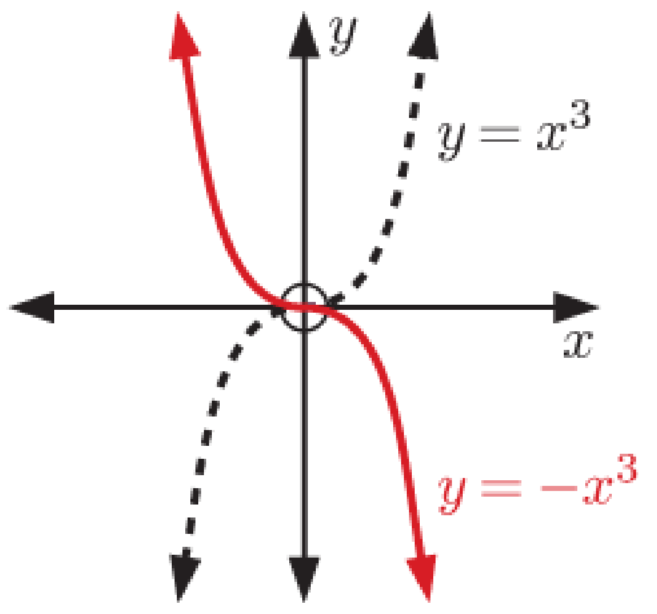
5 **a**



b



c



6 **a** $g(x)=-5x-7$ **b** $g(x)=2^{-x}$

c $g(x)=-2x^2-1$

d $g(x)=x^4+2x^3-3x^2-5x-7$

7 **a** **i** $(3, 0)$ **ii** $(2, 1)$ **iii** $(-3, -2)$

b **i** $(7, 1)$ **ii** $(-5, 0)$ **iii** $(-3, 2)$

8 **a** **i** $(-2, -1)$ **ii** $(0, 3)$ **iii** $(1, 2)$

b **i** $(-5, -4)$ **ii** $(0, 3)$ **iii** $(-2, 3)$

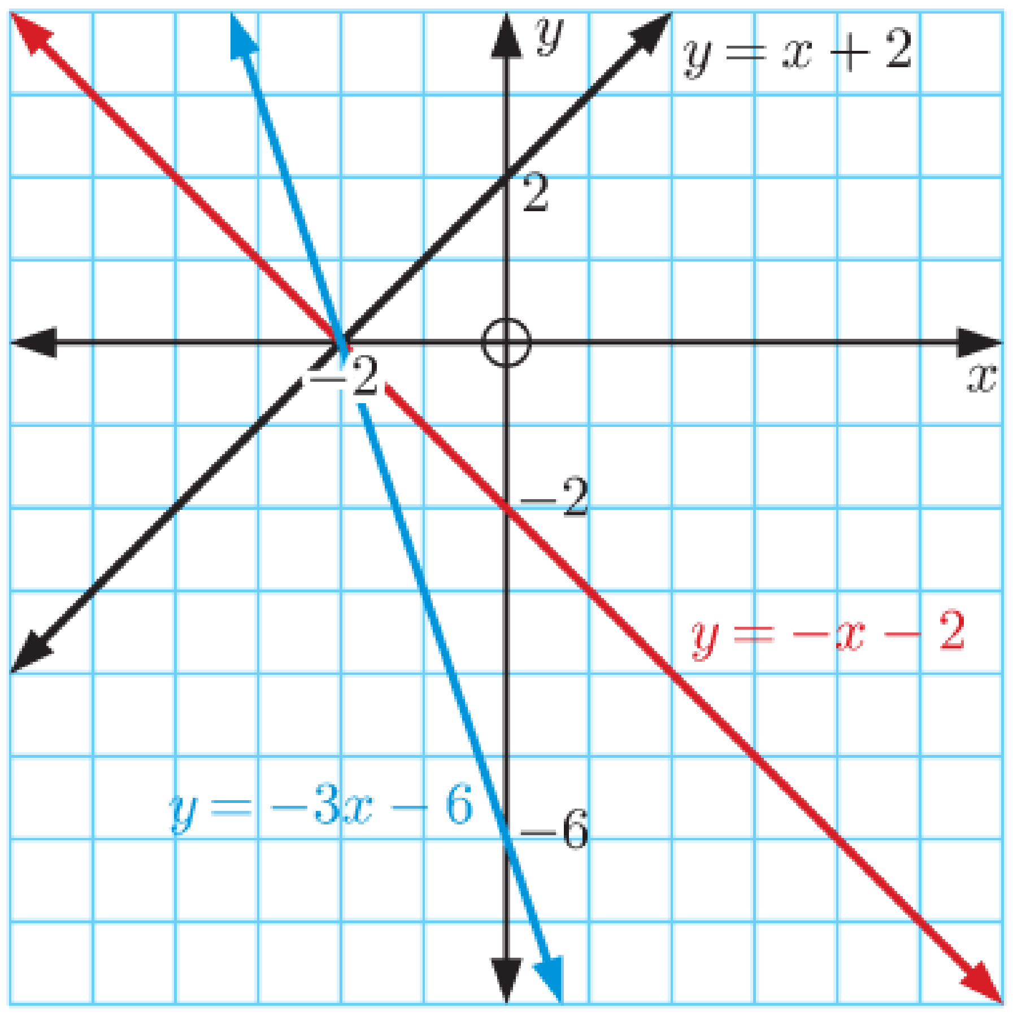
9 **a** A reflection in the y-axis and a reflection in the x-axis.

b $(-3, 7)$ **c** $(5, 1)$

10 **a** A reflection in the x-axis.

b A vertical stretch with scale factor 3.

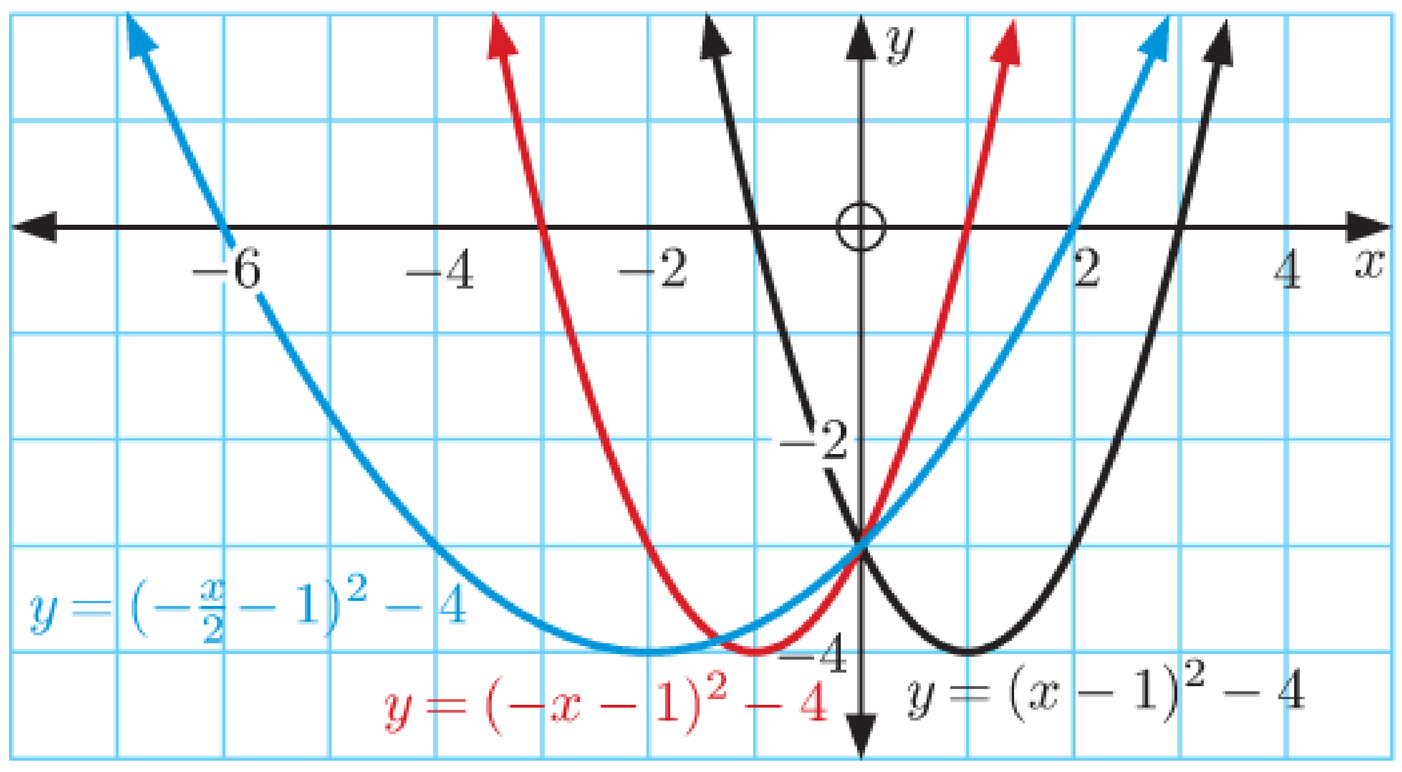
c



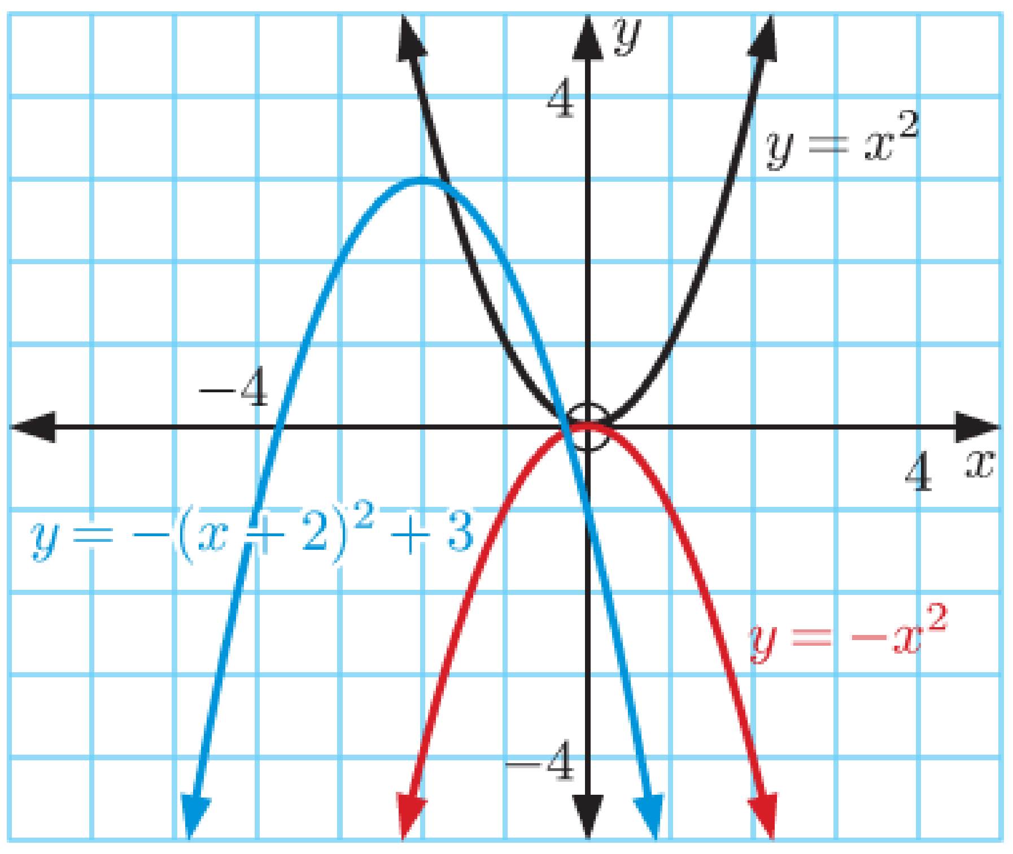
11 **a** A reflection in the y-axis.

b A horizontal stretch with scale factor 2.

c

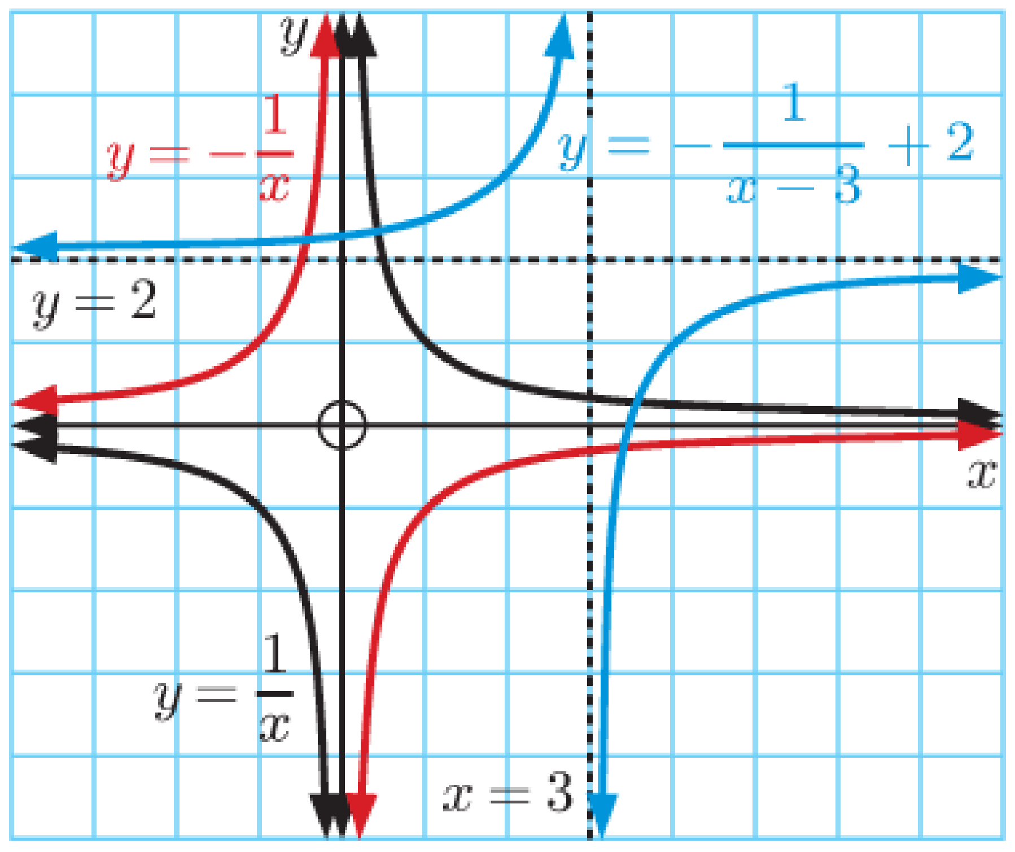


12



$y=x^2$ is transformed to $y=-(x+2)^2+3$ by reflecting in the x-axis and then translating through $\begin{pmatrix}-2\\3\end{pmatrix}$.

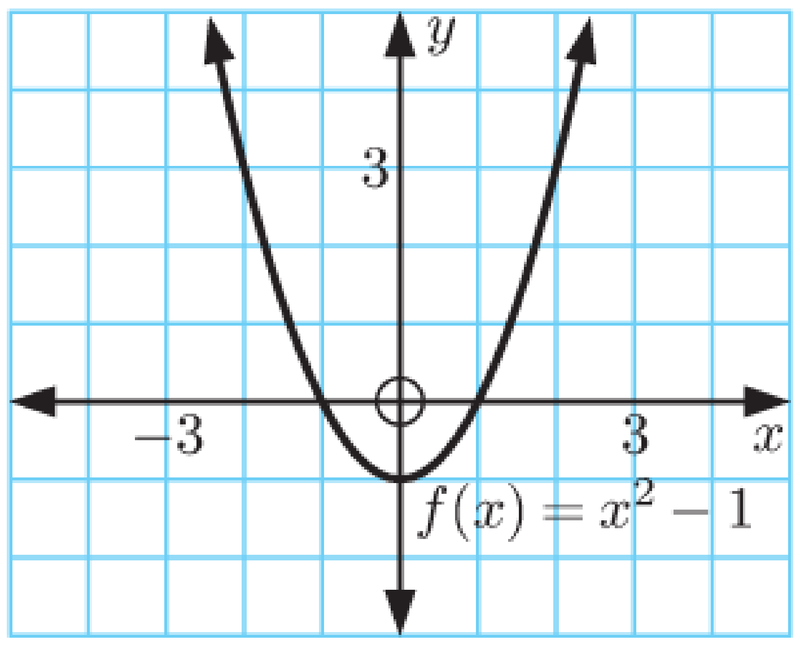
13



$y=\frac{1}{x}$ is transformed to $y=-\frac{1}{x-3}+2$ by reflecting in the x-axis and then translating through $\begin{pmatrix}3\\2\end{pmatrix}$.

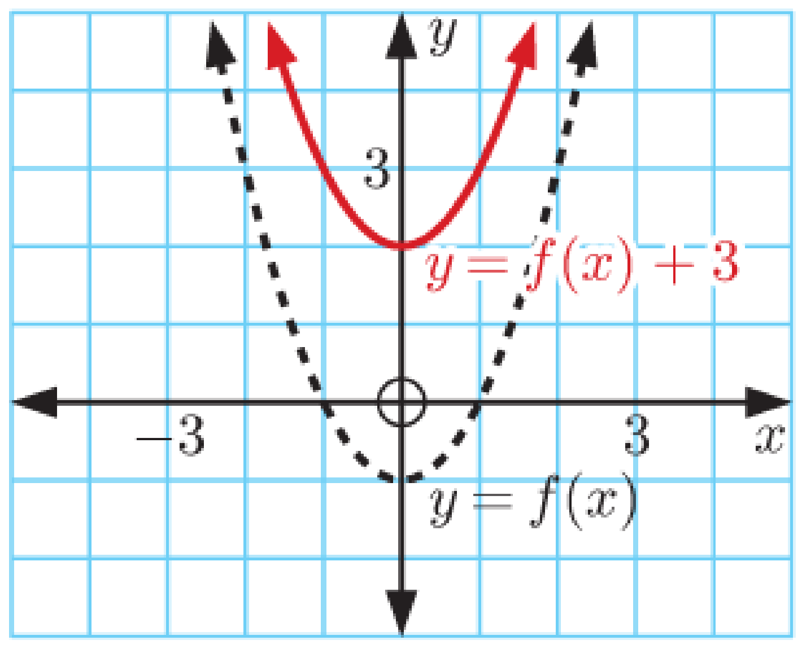
EXERCISE 16D

1 **a**



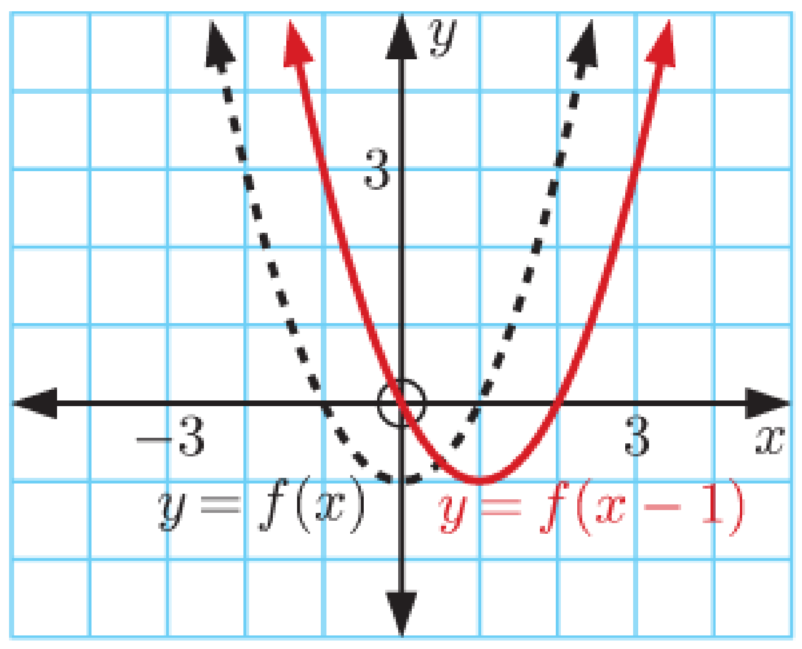
x-intercepts are ±1, y-intercept is -1

b **i**



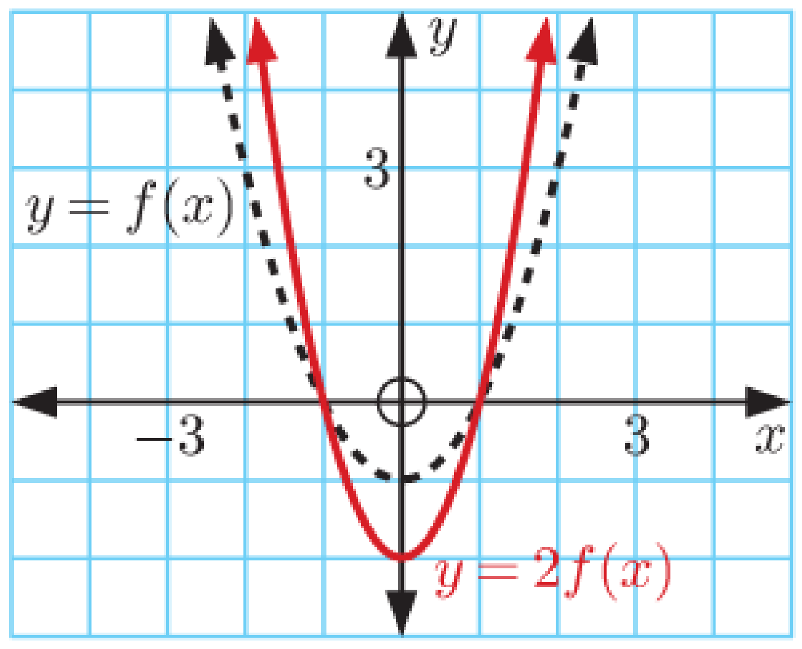
$y=f(x)$ has been translated 3 units upwards.

ii



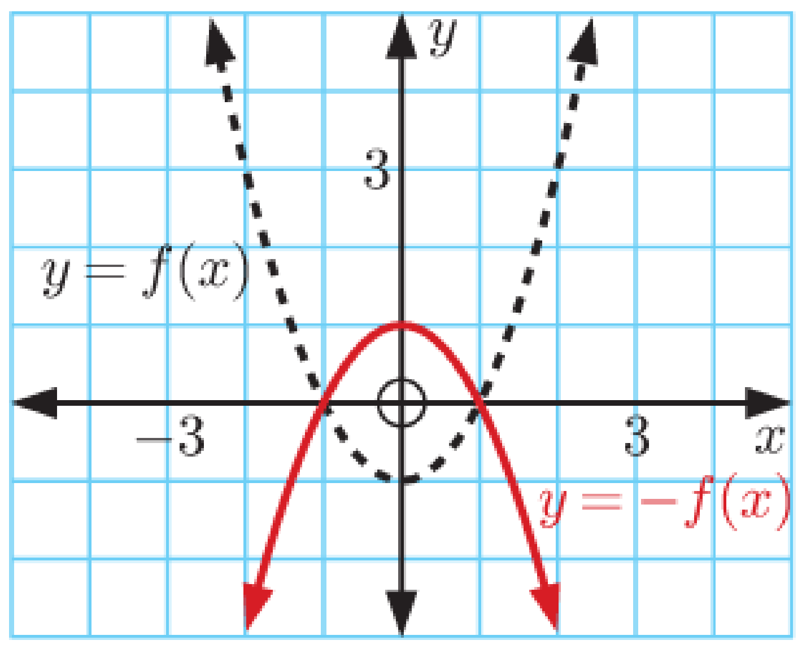
$y=f(x)$ has been translated 1 unit to the right.

iii



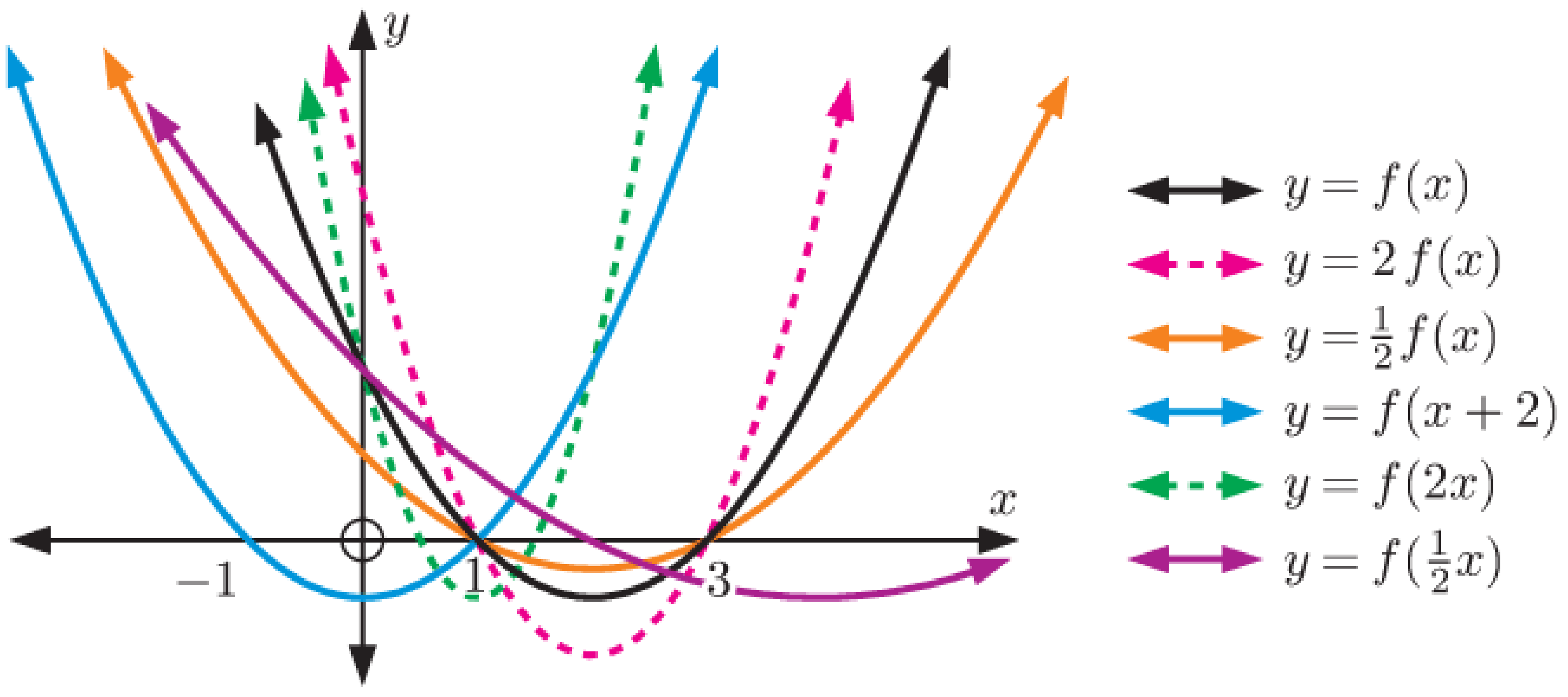
$y=f(x)$ has been vertically stretched with scale factor 2.

iv



$y=f(x)$ has been reflected in the x-axis.

2



3 **a** **i** A vertical translation through $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

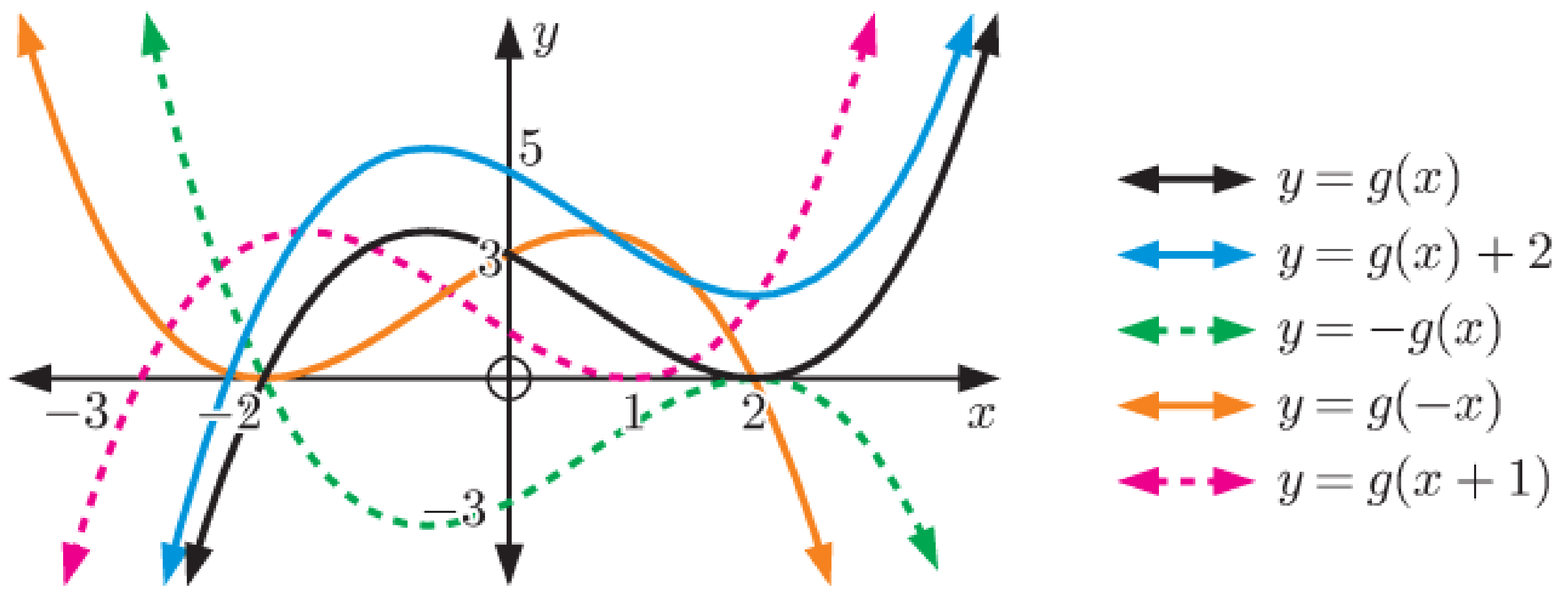
ii $g(x) = f(x) - 2$

b **i** A vertical stretch with scale factor $\frac{1}{2}$.

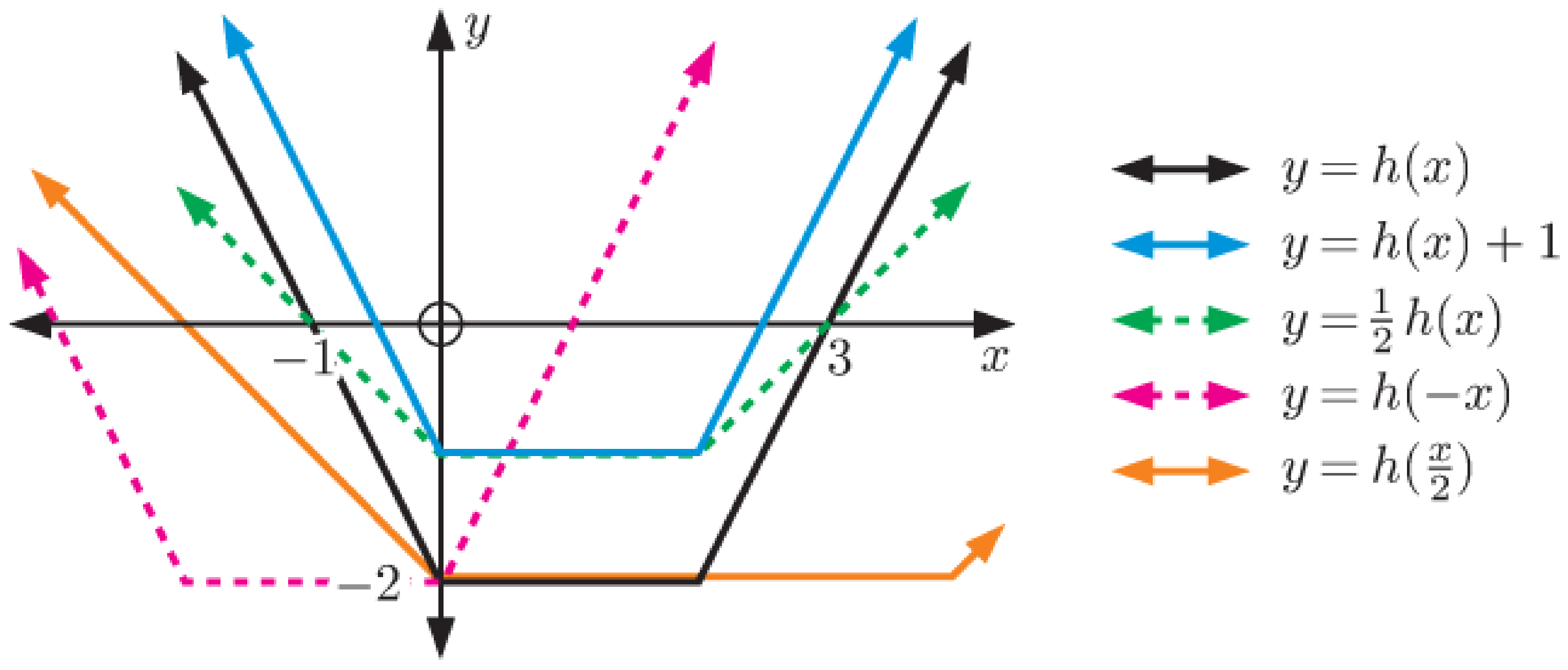
ii $g(x) = \frac{1}{2}f(x)$

c **i** A reflection in the y-axis. **ii** $g(x) = f(-x)$

4

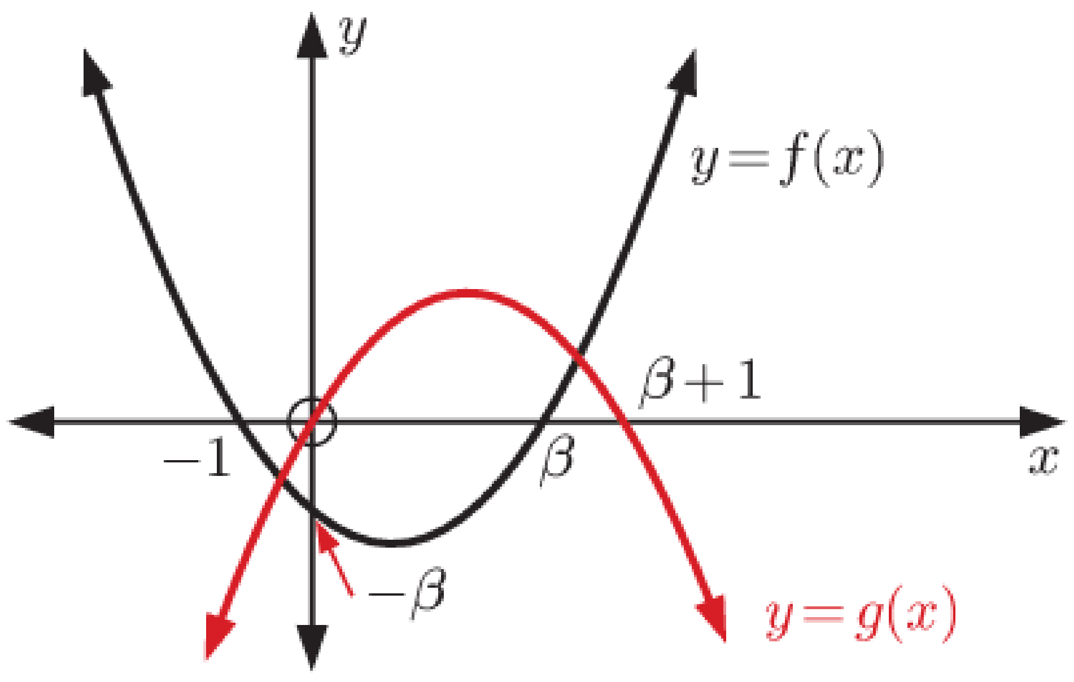


5



6 **a** x-intercepts are -1 and β, y-intercept is $-\beta$

b, c



7 **a** $f(-x-4) - 1$ **b** $f(-x+4) - 1$

c $\frac{1}{2}f(x+2) + \frac{1}{2}$ **d** $\frac{1}{2}f(x+2) + 1$

e $f(\frac{1}{4}x - 3) - 5$ **f** $f\left(\dfrac{x-3}{4}\right) - 5$

8 **a** A reflection in the x-axis, then a translation through $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

b A horizontal stretch with scale factor 2, then a translation through $\begin{pmatrix} 0 \\ -7 \end{pmatrix}$.

c A translation through $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then a horizontal stretch with scale factor $\frac{1}{3}$.

d A vertical stretch with scale factor 2, a translation through $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, then a horizontal stretch with scale factor 4.

e A vertical stretch with scale factor 2, a horizontal stretch with scale factor $\frac{1}{3}$, then a translation through $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$.

f A reflection in the x-axis, a vertical stretch with scale factor 4, a horizontal stretch with scale factor 2, then a translation through $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$.

9 **a** Domain is $\{x \mid x \geqslant -3\}$, Range is $\{y \mid -3 \leqslant y < 4\}$

b Domain is $\{x \mid x \geqslant \frac{1}{3}\}$, Range is $\{y \mid -10 < y \leqslant 4\}$

c Domain is $\{x \mid x \geqslant 3\}$, Range is $\{y \mid \frac{10}{3} \leqslant y < \frac{17}{3}\}$

10 **a** $5\sqrt{2-x} + 15$

Domain is $\{x \mid x \leqslant 2\}$, Range is $\{y \mid y \geqslant 15\}$

b $5\sqrt{2-x} + 3$

Domain is $\{x \mid x \leqslant 2\}$, Range is $\{y \mid y \geqslant 3\}$

c $5\sqrt{-x-2} + 3$

Domain is $\{x \mid x \leqslant -2\}$, Range is $\{y \mid y \geqslant 3\}$

11 **a** The vertical stretch has scale factor $|\,a\,|$. The reflection in the x-axis occurs if $a < 0$. Each point is then moved h units right and k units up.

b The function has shape ∪ if $a > 0$ and ∩ if $a < 0$.

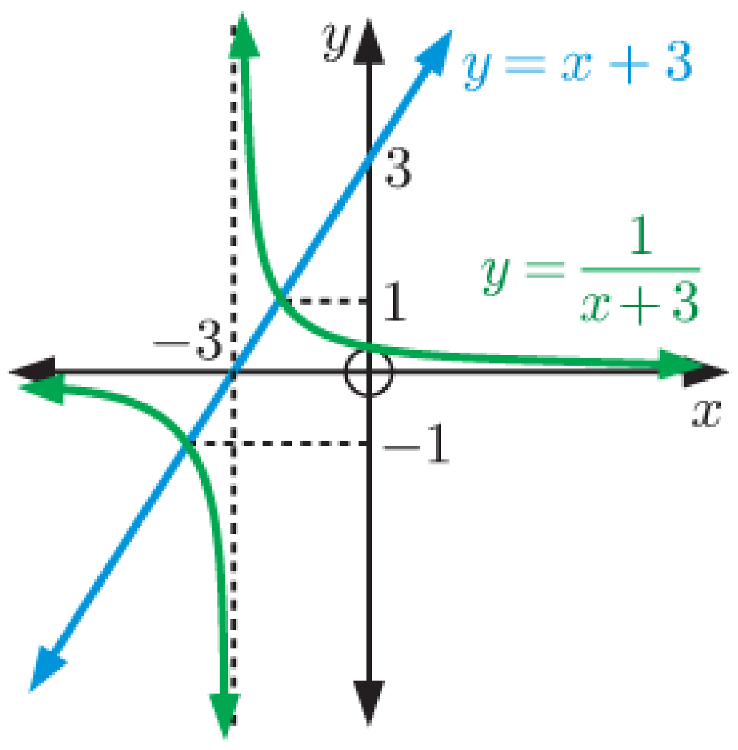
The function has vertex (h, k), and y-intercept $ah^2 + k$.

12 **a** $5 + \dfrac{-4}{2x+3}$

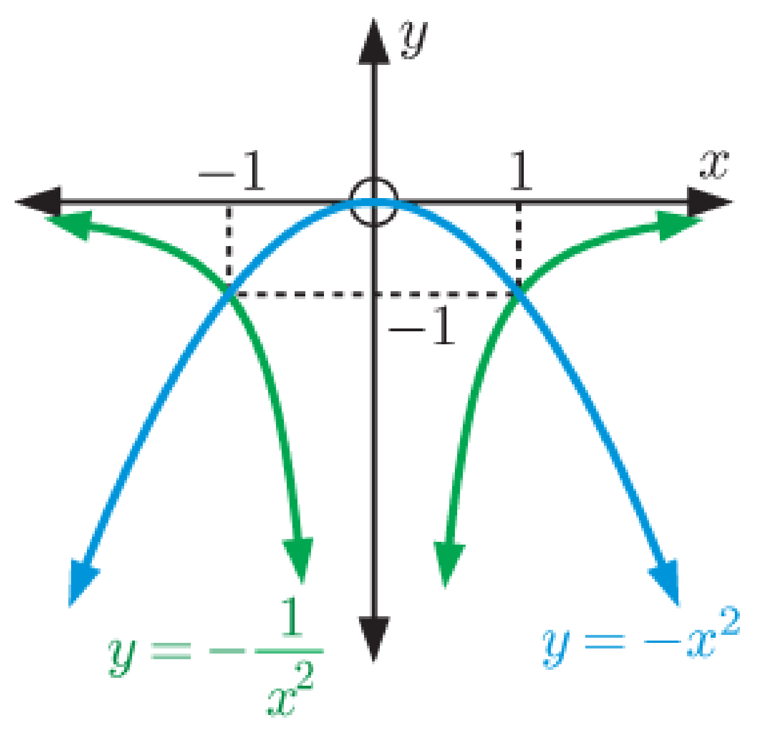
b A reflection in the x-axis, a vertical stretch with scale factor 4, a translation through $\begin{pmatrix} -3 \\ 5 \end{pmatrix}$, then a horizontal stretch with scale factor $\frac{1}{2}$.

EXERCISE 16E

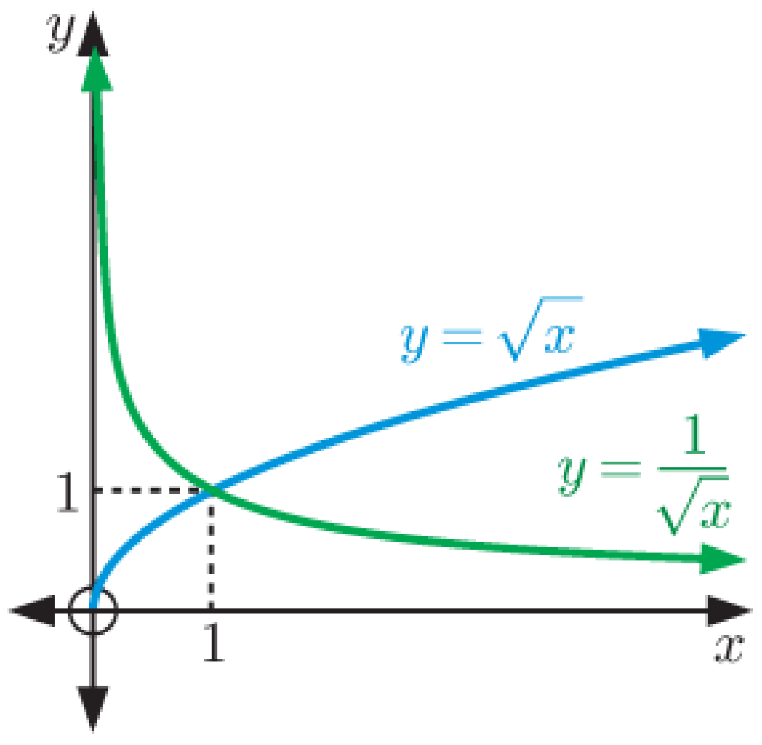
1 **a**



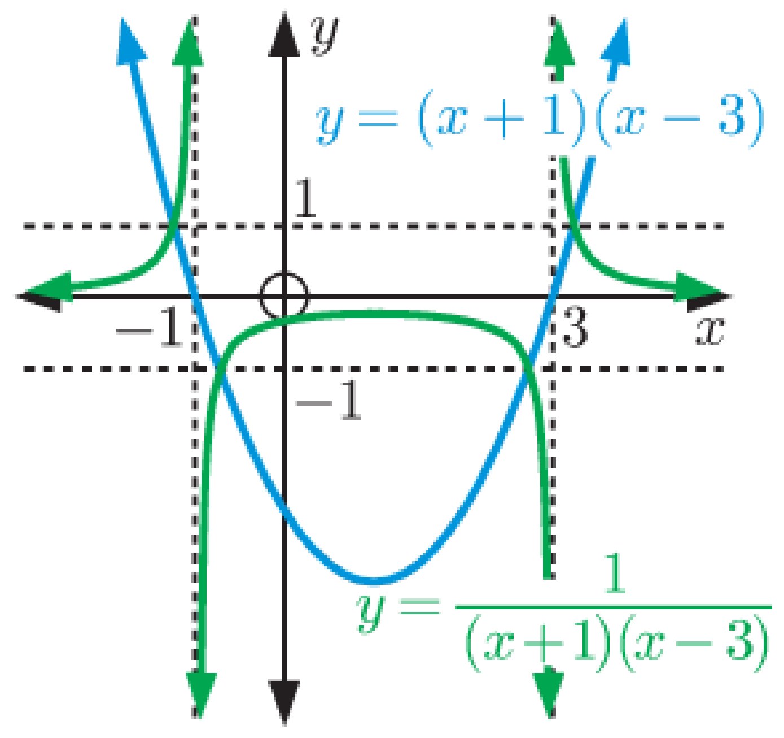
b



c



d



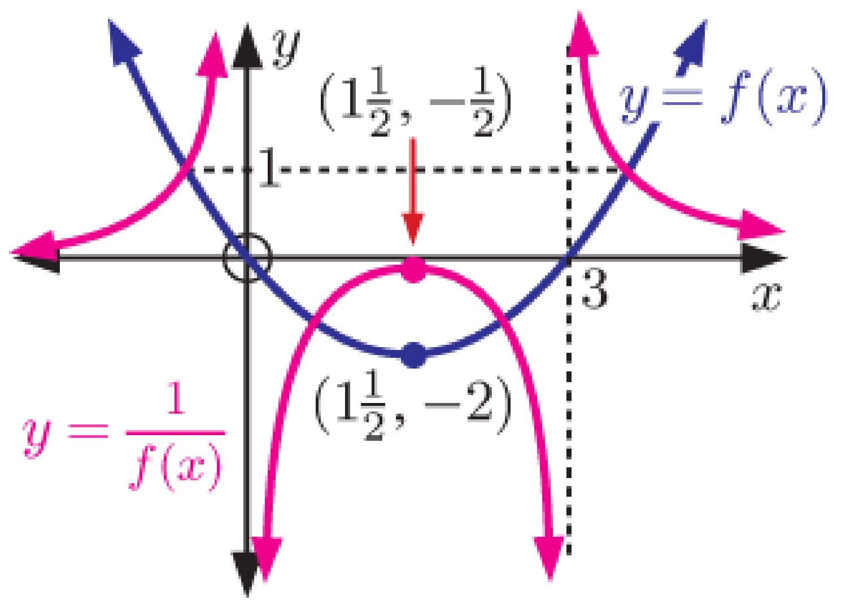
2 **a** invariant points are $(-2, 1)$ and $(-4, -1)$

b invariant points are $(-1, -1)$ and $(1, -1)$

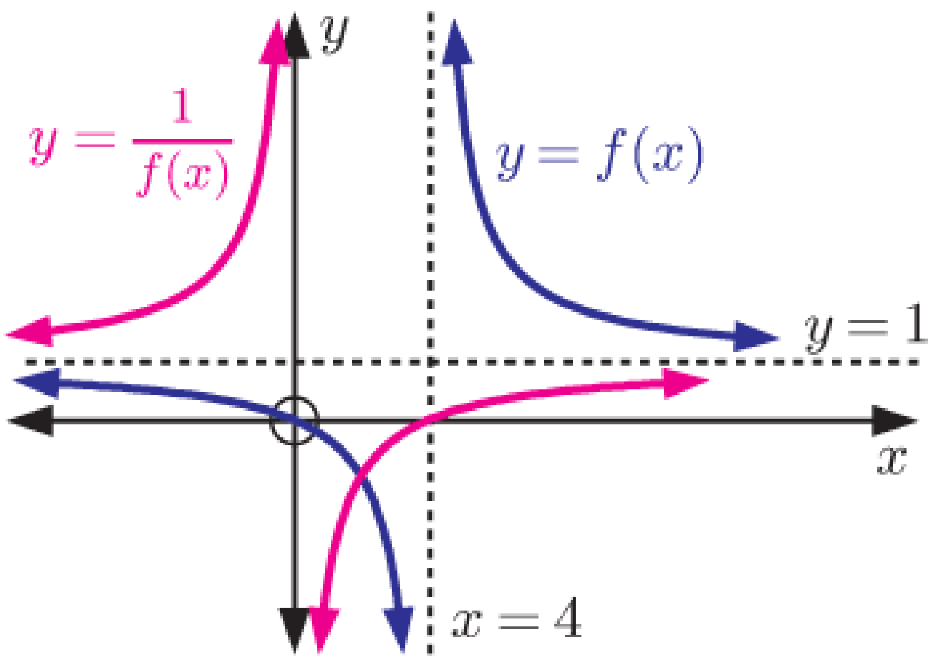
c invariant point is $(1, 1)$

d invariant points are $(\approx -1.24, 1)$, $(\approx -0.732, -1)$, $(\approx 2.73, -1)$, and $(\approx 3.24, 1)$

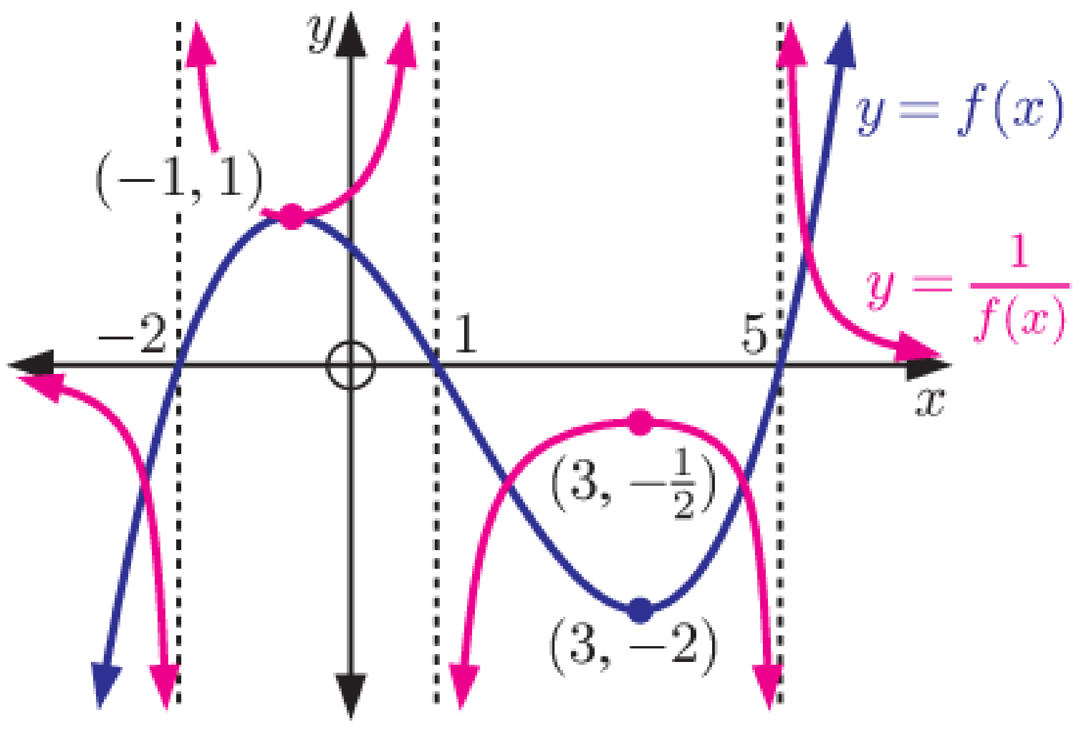
3 **a**



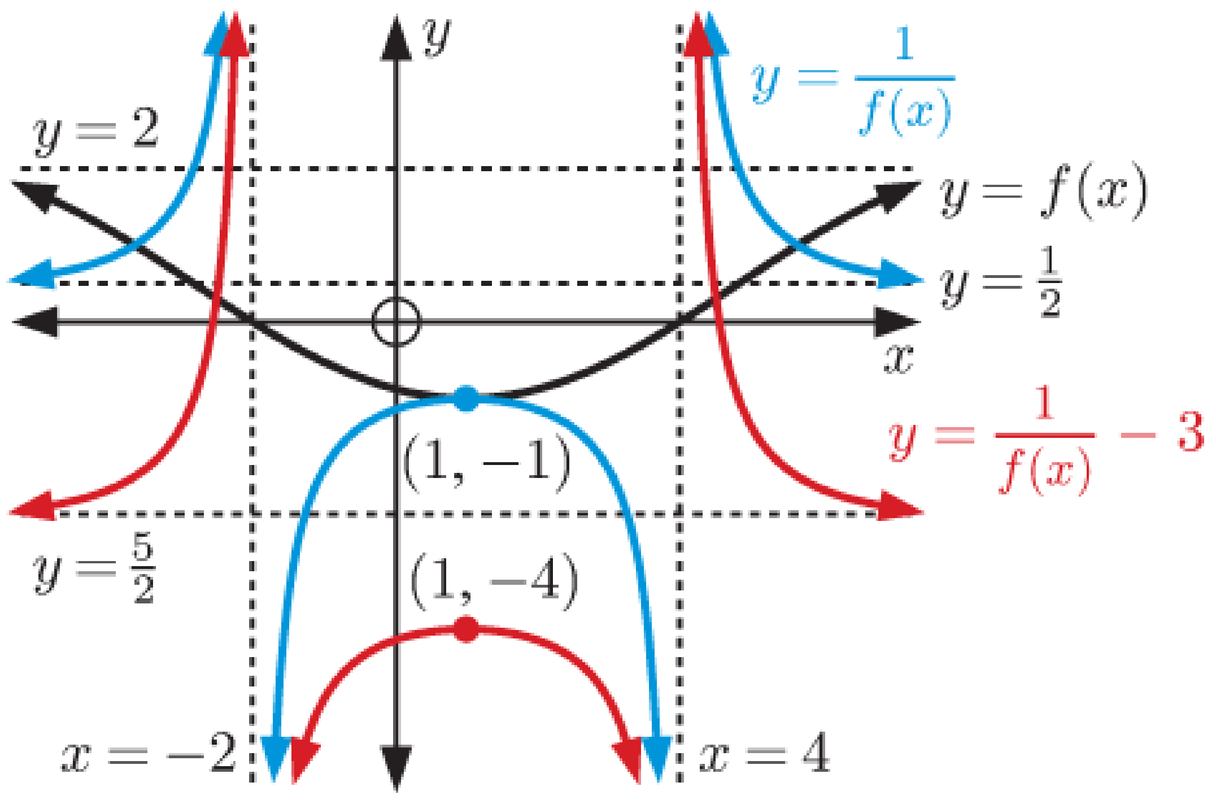
b



c

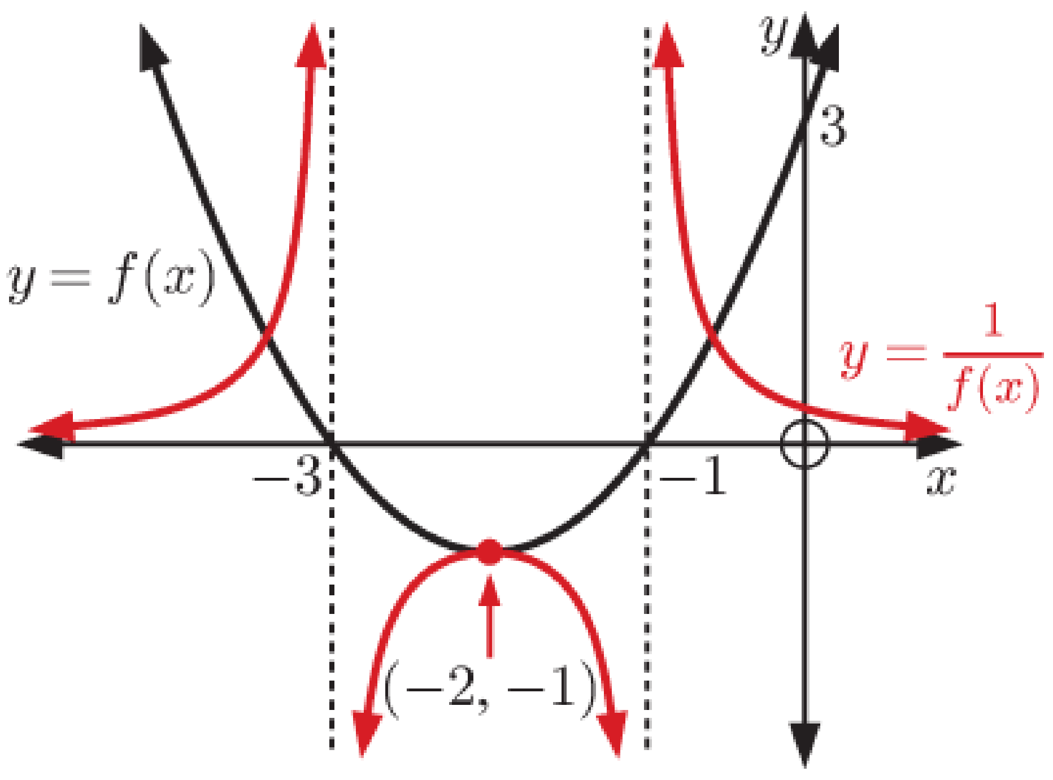


4



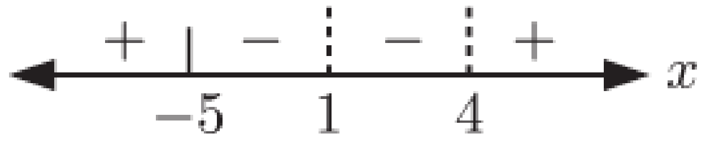
5 **a** x-intercepts -3 and -1, y-intercept 3, vertex $(-2, -1)$

b



c $x=-\frac{9}{2}$ or $\frac{1}{2}$

6

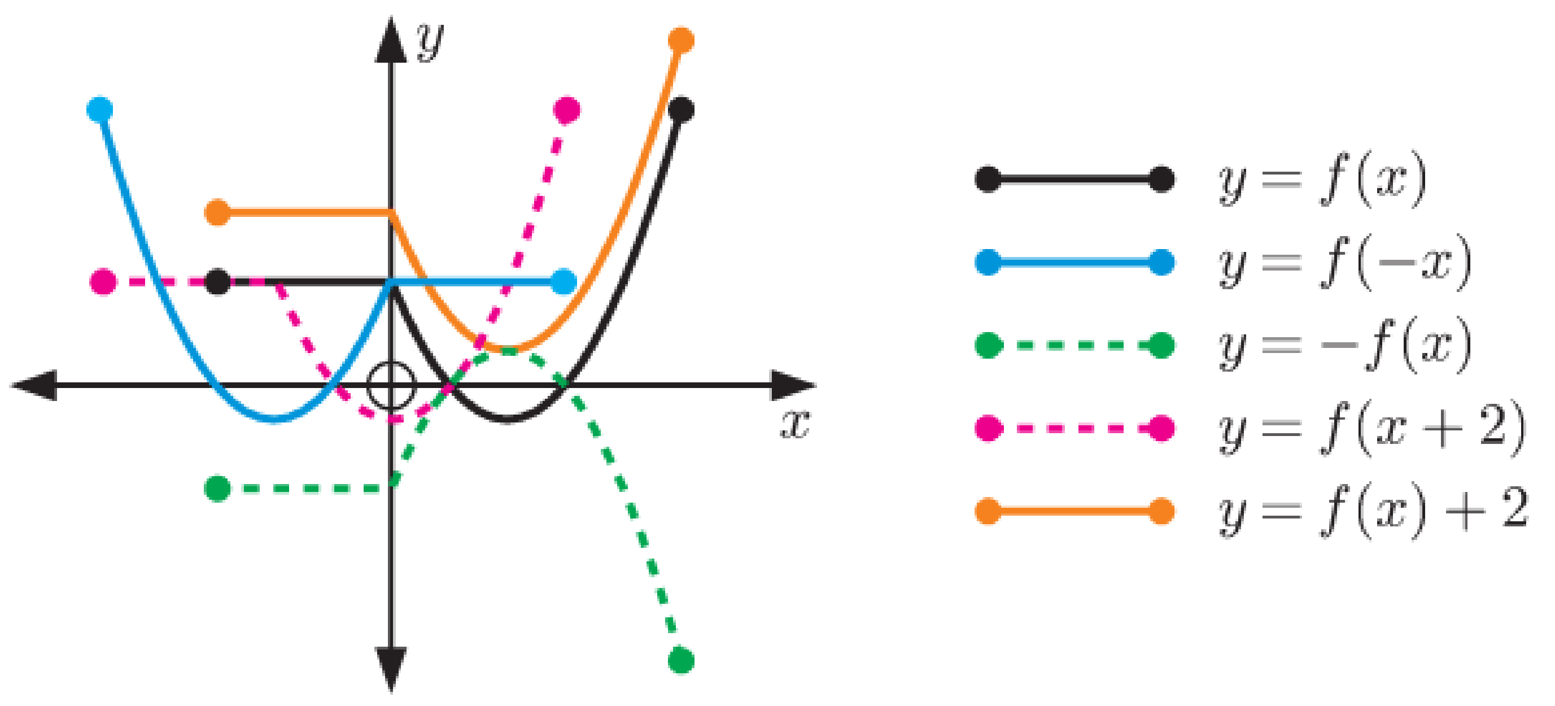


7 **a** Domain is $\{x \mid -1 \leqslant x \leqslant 6\}$, Range is $\{y \mid \frac{1}{5} < y \leqslant \frac{1}{2}\}$

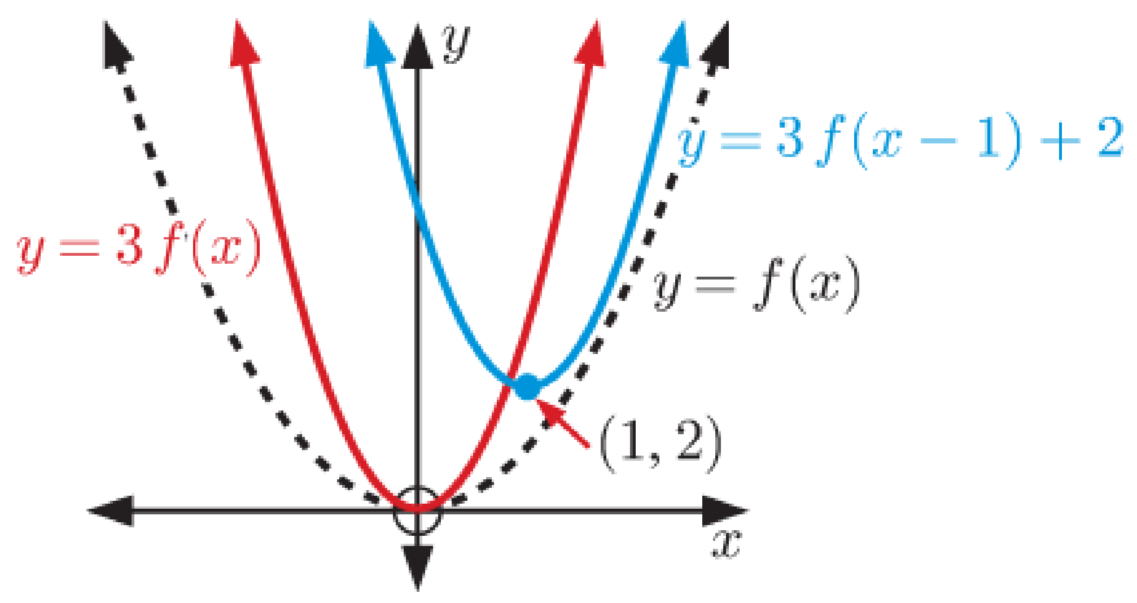
b Range is $\{y \mid y \leqslant -\frac{1}{3}$ or $y \geqslant \frac{1}{3}\}$, cannot say about the domain.

REVIEW SET 16A

1



2

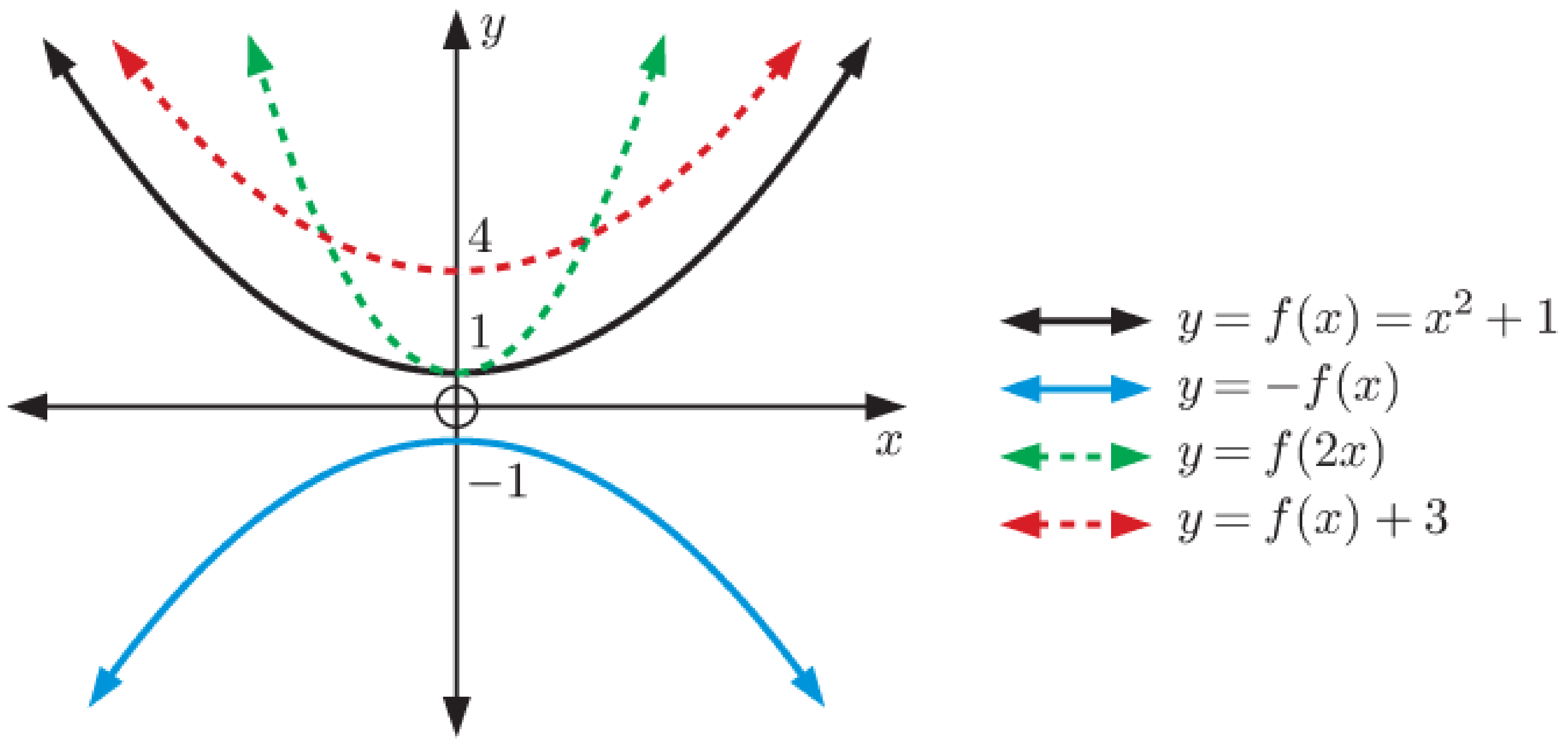


3 **a** $g(x)=4x-10$ **b** $g(x)=5x^2+30$

c $g(x)=-3x-5$ **d** $g(x)=\frac{2}{9}x^2-\frac{1}{3}x+4$

e $g(x)=-x^3$

4



5 $g(x)$ is the result of transforming $f(x)$ 3 units to the left and 4 units down.

$\therefore$ domain of $g(x)$ is $\{x \mid -5 \leqslant x \leqslant 0\}$
range of $g(x)$ is $\{y \mid -5 \leqslant y \leqslant 3\}$.

6 **a** $g(x)=(x-1)^2+8$

b **i** $\{y \mid y \geqslant 4\}$ **ii** $\{y \mid y \geqslant 8\}$

8 $g(x)=3x^2+5x+9$

9 **a** $-f(x+2)+3$ **b** $2f(x-4)-2$

10 **a** (0, 4) **b** (0, 6) **c** $(\frac{1}{2}, 3)$

11 **a** x-intercepts -9 and -3

b x-intercepts -5 and 1, y-intercept -9

c x-intercepts -10 and 2, y-intercept -3

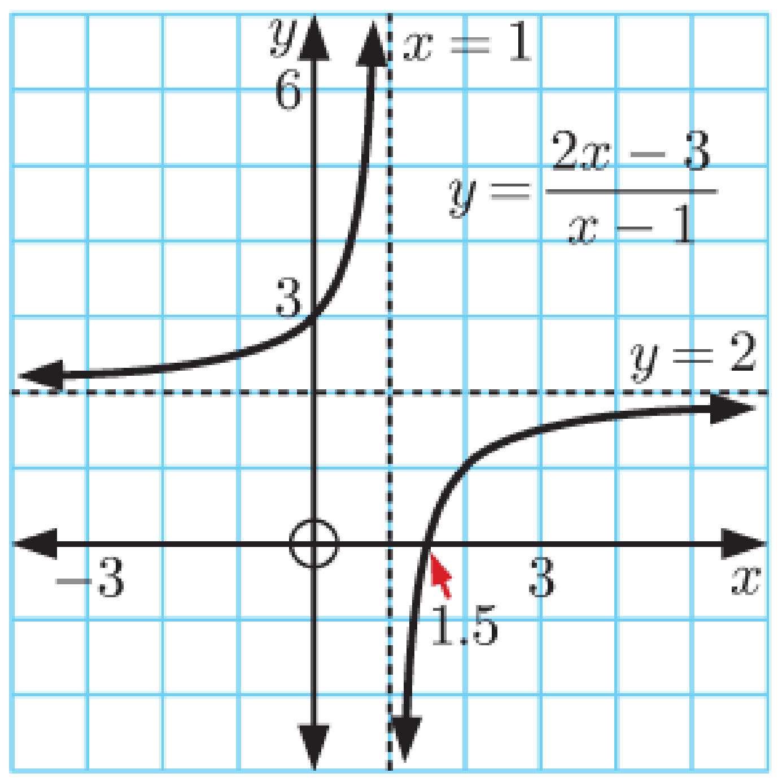
d x-intercepts -5 and 1, y-intercept 3

12 **a** $g(x)=\dfrac{2x-3}{x-1}$

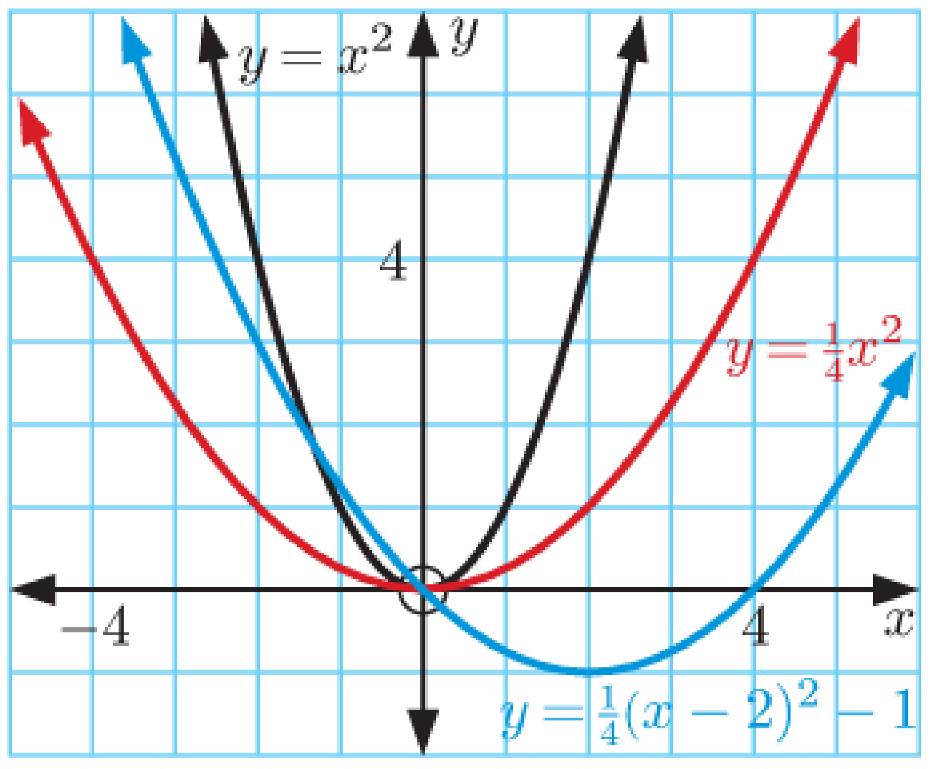
b vertical asymptote $x=1$, horizontal asymptote $y=2$

c Domain is $\{x \mid x \neq 1\}$
Range is $\{y \mid y \neq 2\}$

d

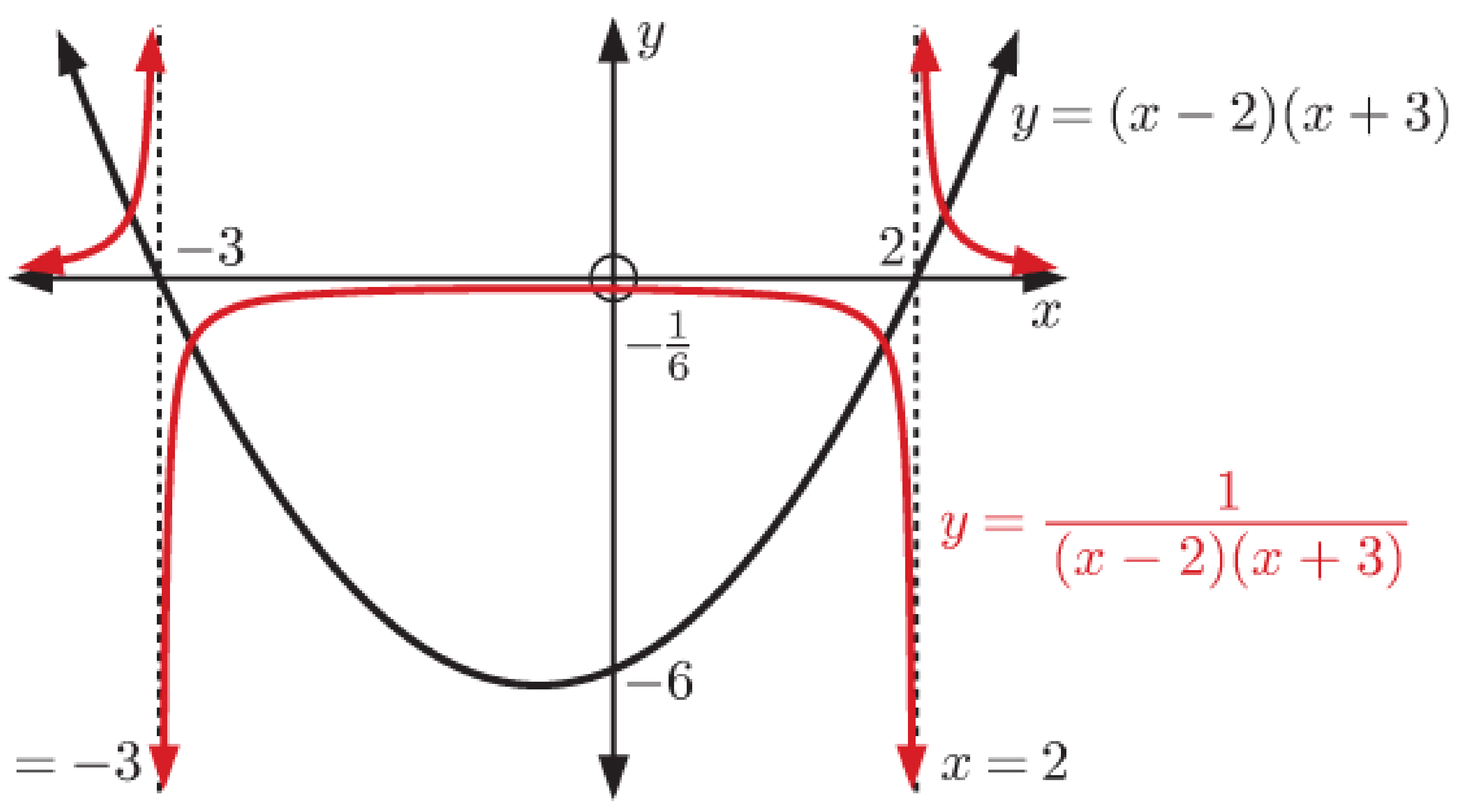


13



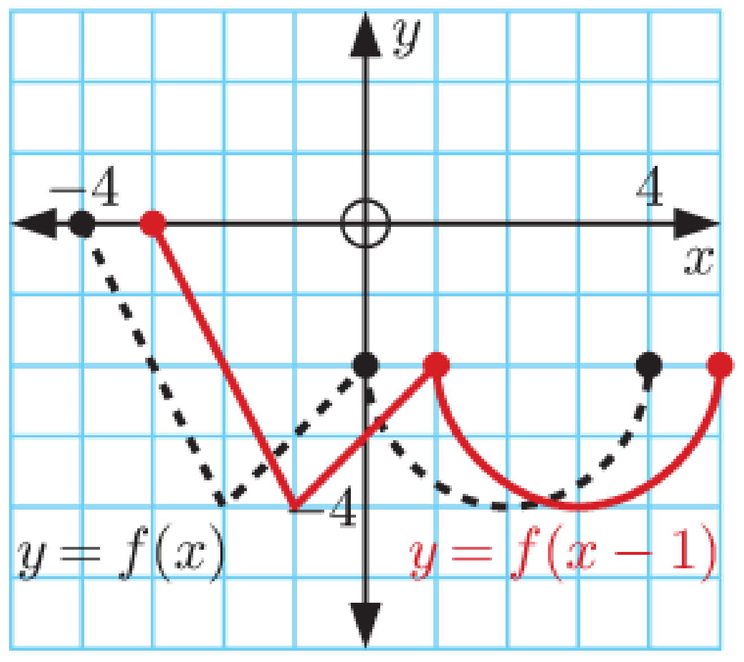
$y=x^2$ is transformed to $y=\frac{1}{4}(x-2)^2-1$ by vertically stretching with scale factor $\frac{1}{4}$ and then translating through $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

14

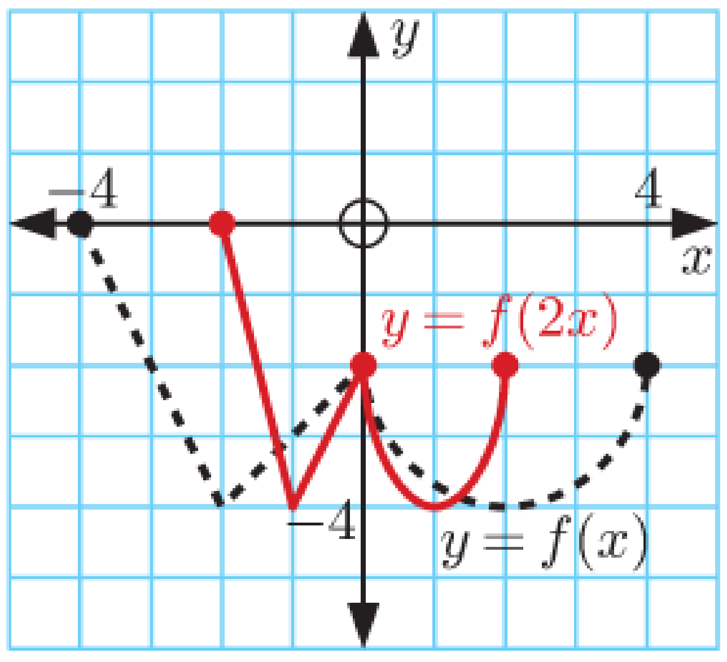


REVIEW SET 16B

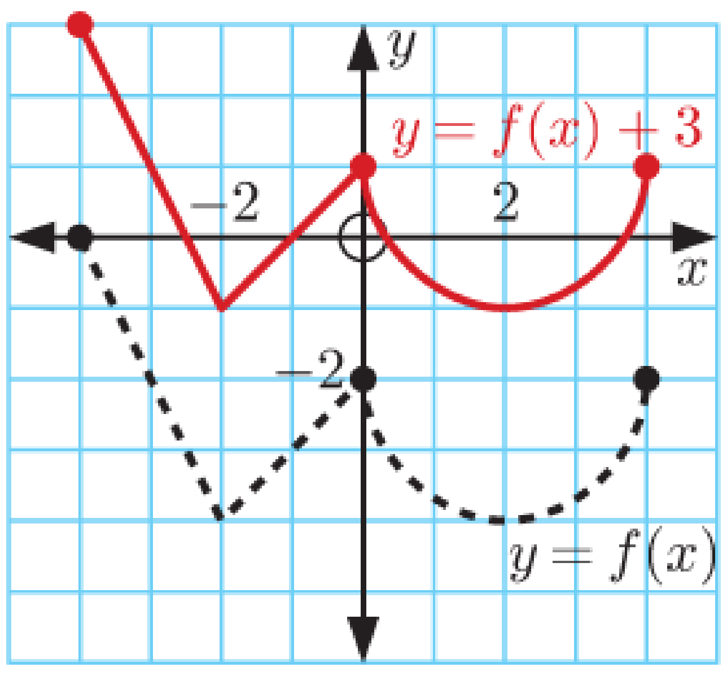
1 **a**



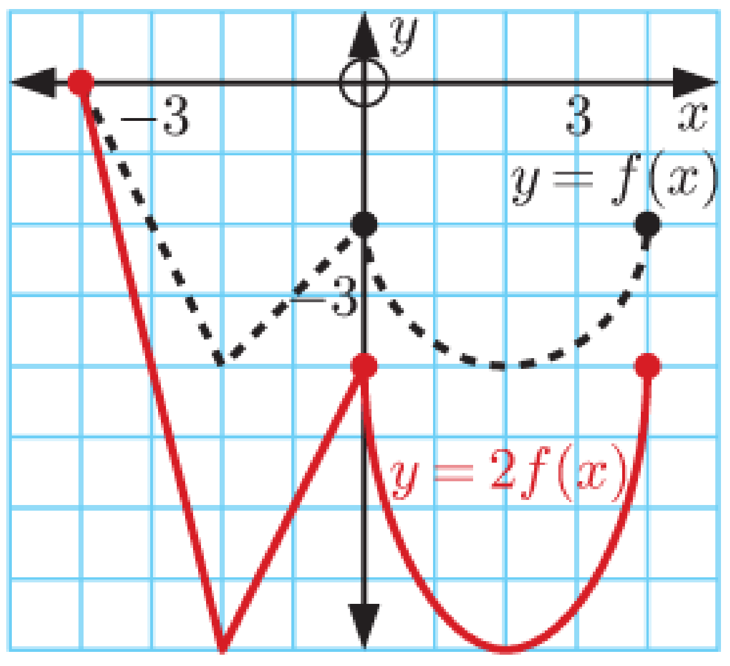
b



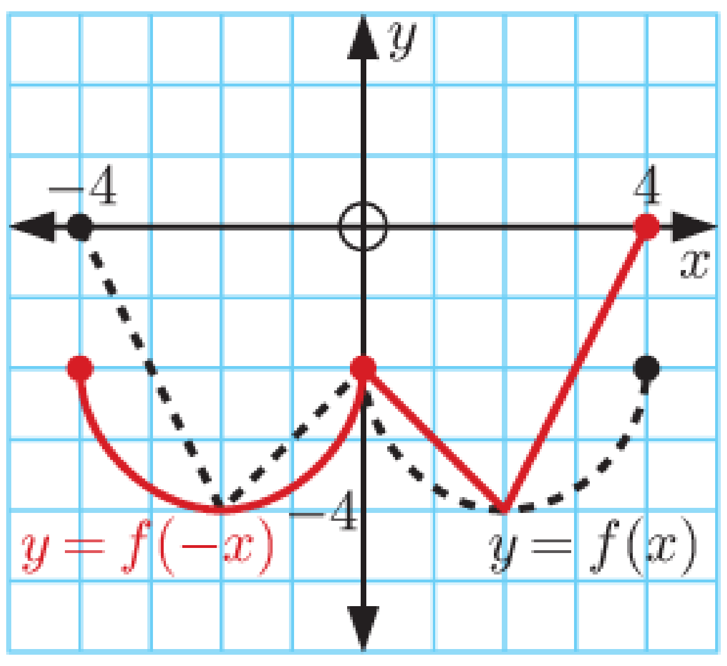
c



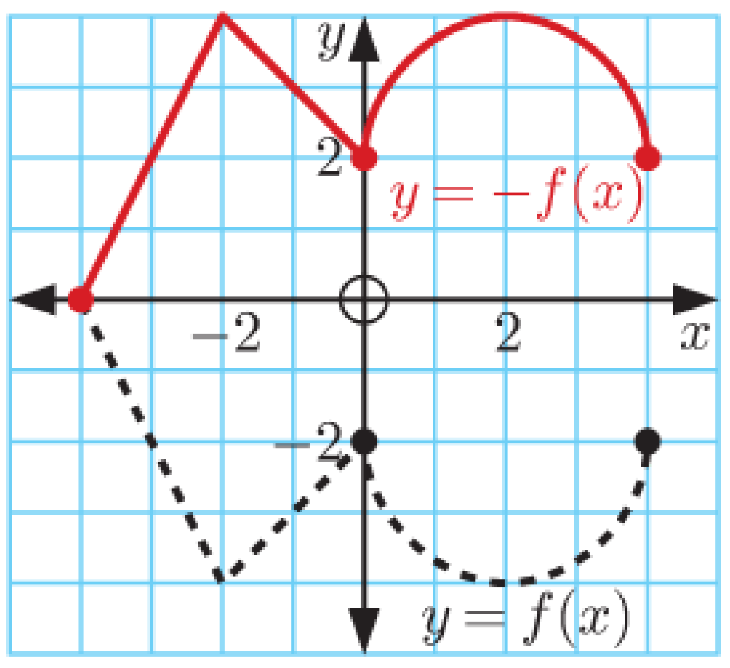
d



e



f

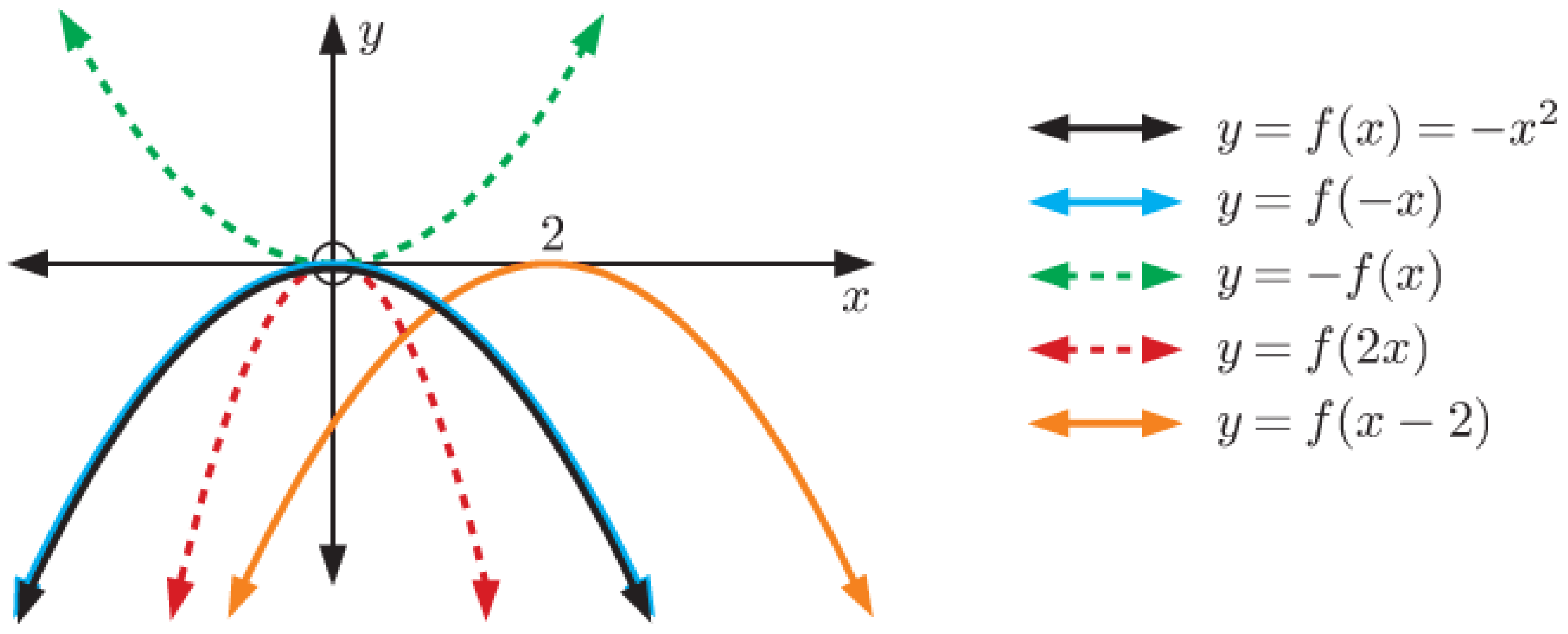


2 **a** $g(x) = 3x - x^2$ **b** $g(x) = 16 - x$

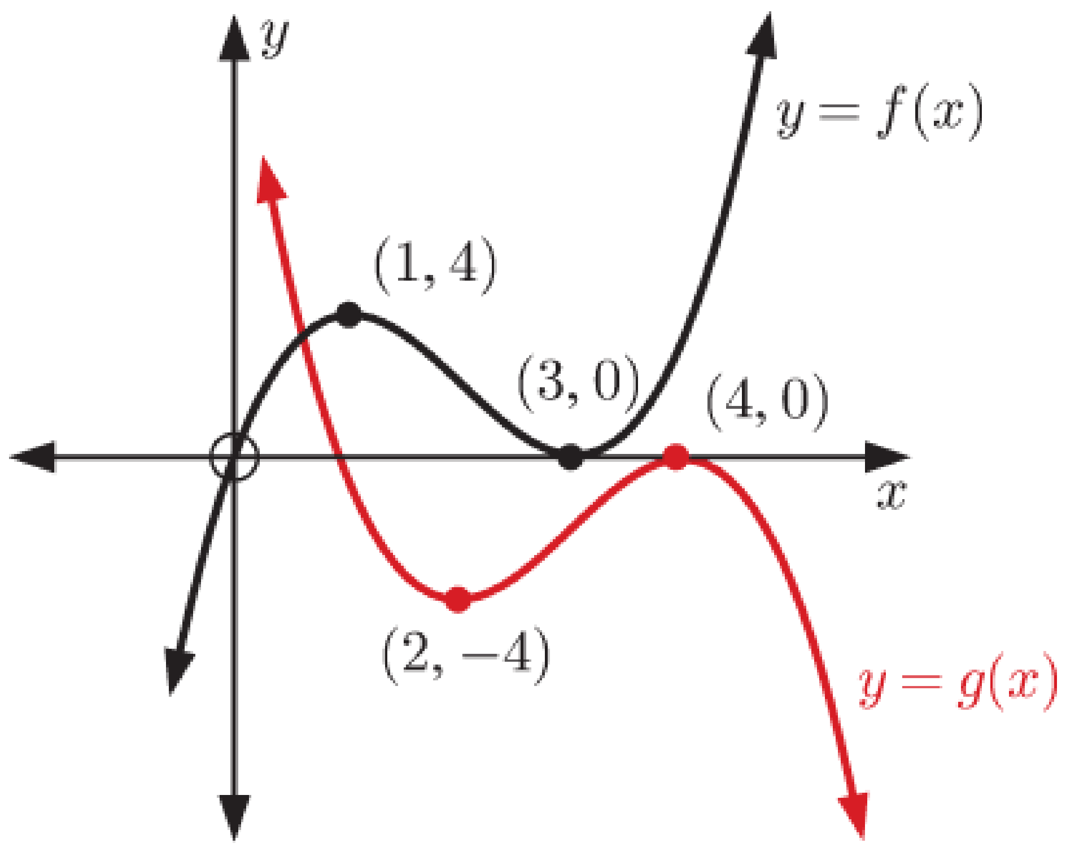
c $g(x) = \frac{1}{12}x + 2$

3 $g(x) = -x^2 - 6x - 7$

4



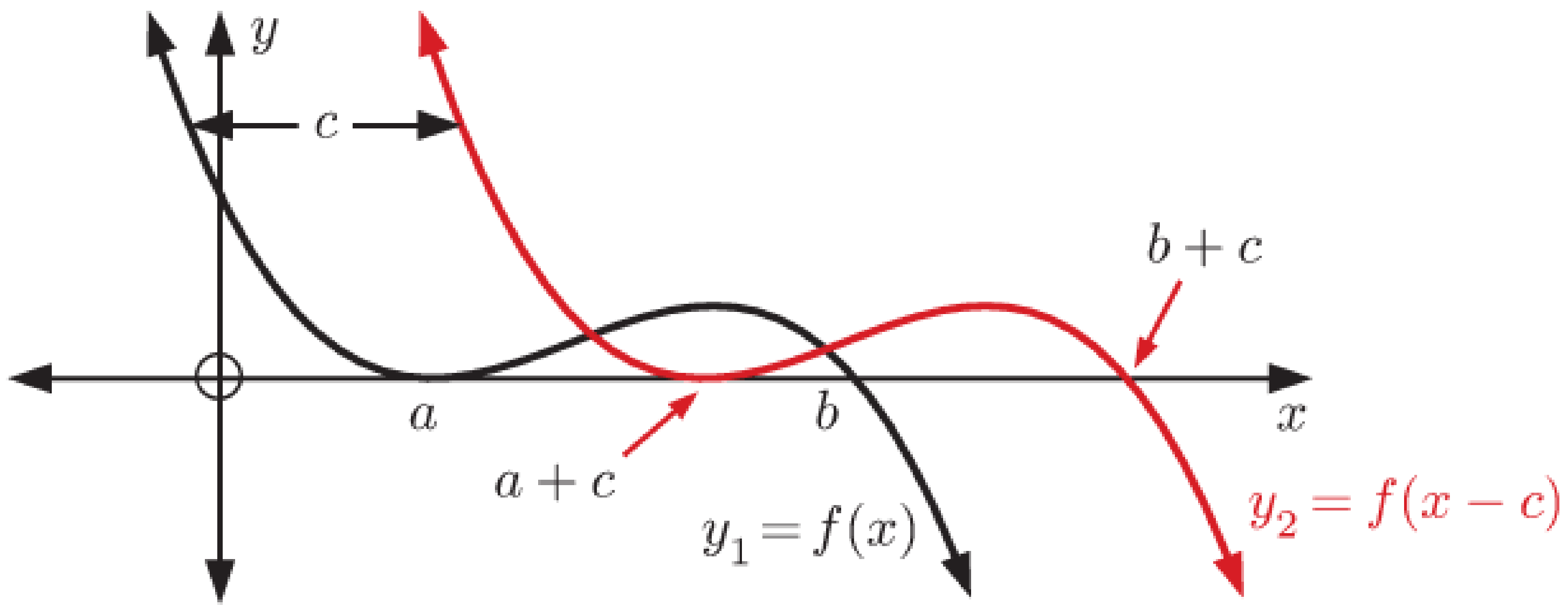
5 **a**



b $(2, -4)$ and $(4, 0)$

6 $y = -2x^2 + 5x - 3$

7



8 A reflection in the x-axis, then a translation through $\begin{pmatrix}\frac{5}{2}\\ -\frac{7}{2}\end{pmatrix}$.

9 $(1, 6)$

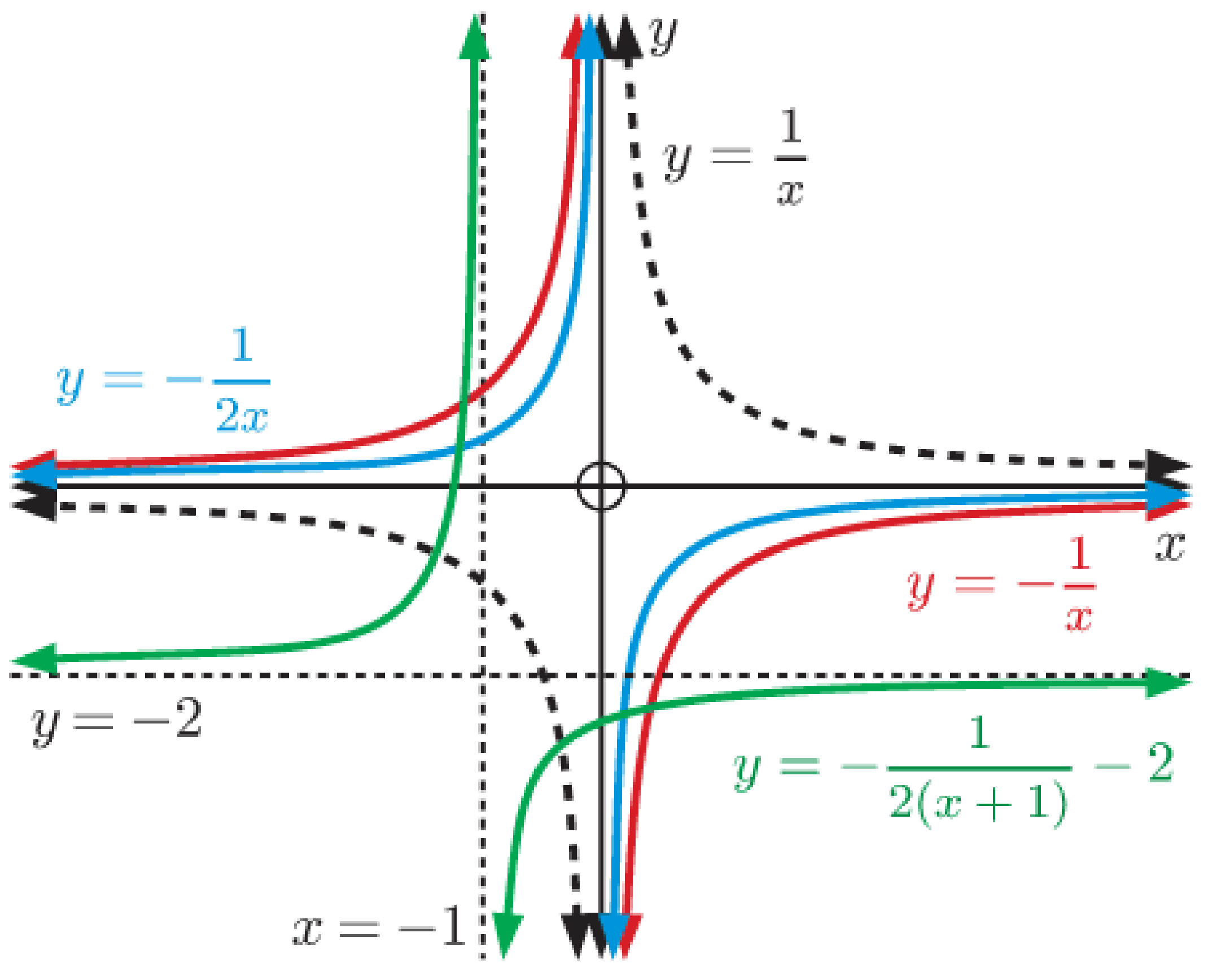
10 **a** A vertical stretch with scale factor 2, then a translation through $\begin{pmatrix}-1\\ 3\end{pmatrix}$.

b A reflection in the x-axis, a horizontal stretch with scale factor $\frac{3}{2}$, then a translation through $\begin{pmatrix}0\\ -6\end{pmatrix}$.

c A vertical stretch with scale factor $\frac{1}{3}$, a reflection in the y-axis, then a translation through $\begin{pmatrix}2\\ 0\end{pmatrix}$.

11 $b = 8$, $c = -20$

12 **a**

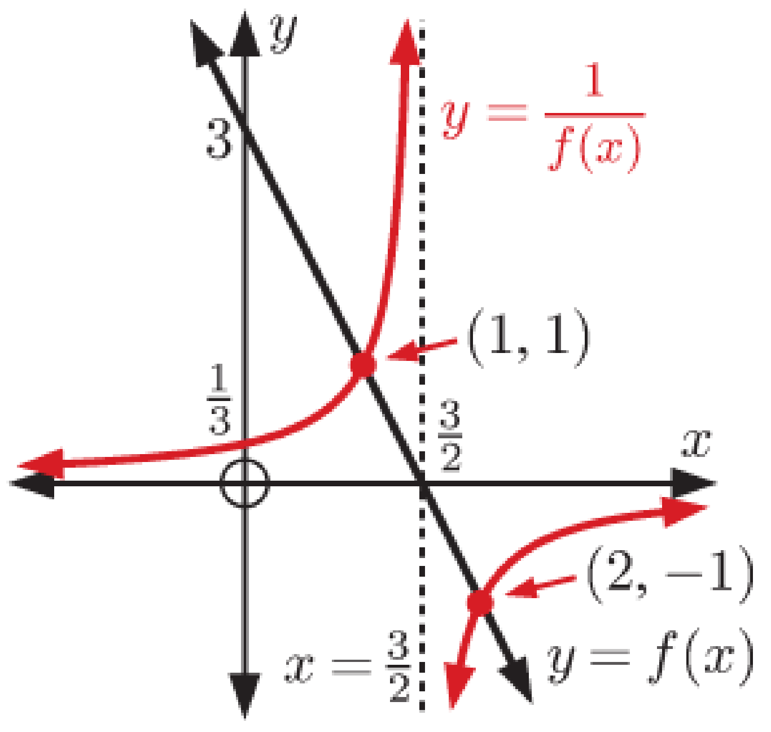


b A reflection in the x-axis, a vertical stretch with scale factor $\frac{1}{2}$, then a translation through $\begin{pmatrix}-1\\ -2\end{pmatrix}$.

c $y = \dfrac{-4x - 5}{2x + 2}$

Domain is $\{x \mid x \neq -1\}$, Range is $\{y \mid y \neq -2\}$

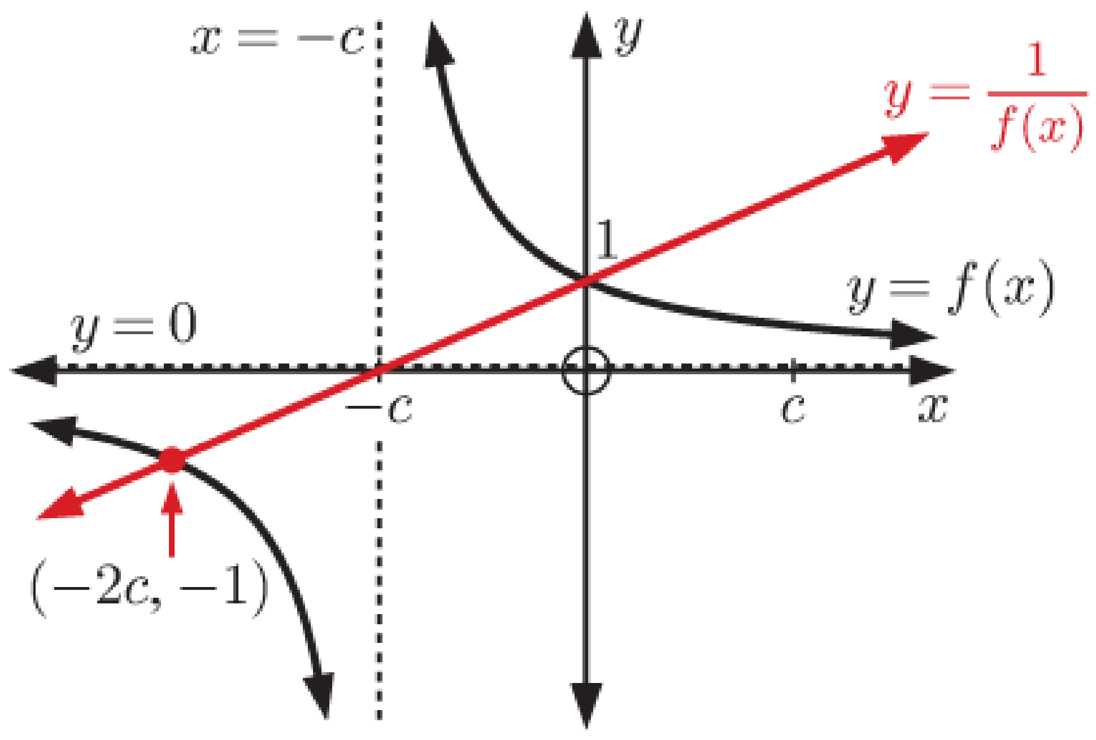
13 **a, d**



b $(1, 1)$ and $(2, -1)$

c y-intercept $\frac{1}{3}$ and vertical asymptote $x = \frac{3}{2}$

14

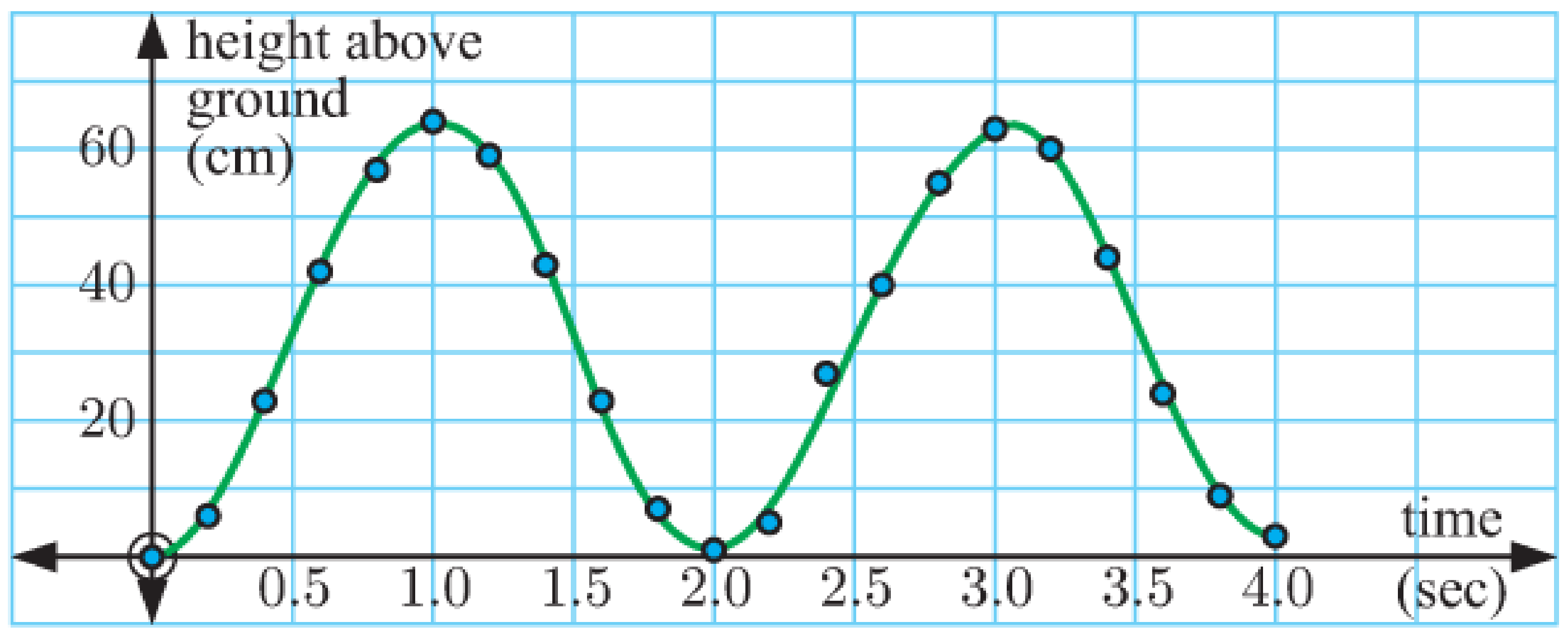


EXERCISE 17A

1 **a** periodic **b** periodic **c** periodic **d** not periodic

e periodic **f** periodic **g** not periodic **h** not periodic

2 **a**



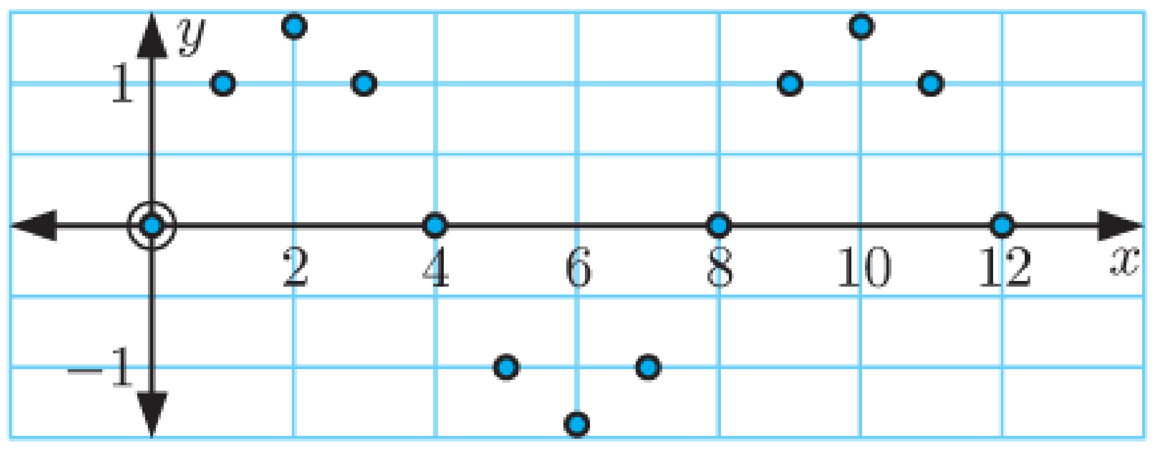
b A curve can be fitted to the data.

c The data is periodic.

i $y = 32$ (approximately) **ii** ≈ 64 cm

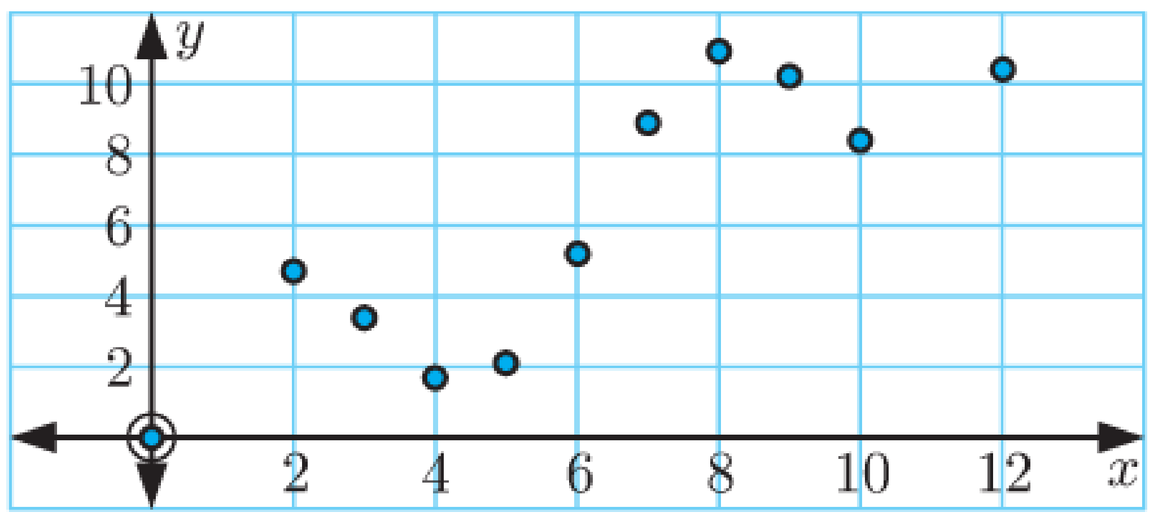
iii ≈ 2 seconds **iv** ≈ 32 cm

3 **a**



Data exhibits periodic behaviour.

b



Not enough information to say data is periodic.

EXERCISE 17B

1 **a** 0

b **i** $\theta = 0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi$ **ii** $\theta = \frac{3\pi}{2},\ \frac{7\pi}{2}$

iii $\theta = \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{13\pi}{6},\ \frac{17\pi}{6}$ **iv** $\theta = \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{7\pi}{3},\ \frac{8\pi}{3}$

c **i** $0 < \theta < \pi,\quad 2\pi < \theta < 3\pi$ **d** $\{y \mid -1 \leqslant y \leqslant 1\}$

ii $\pi < \theta < 2\pi,\quad 3\pi < \theta < 4\pi$

2 **a** 1

b **i** $\theta = \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2},\ \frac{7\pi}{2}$ **ii** $\theta = 0,\ 2\pi,\ 4\pi$

iii $\theta = \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{8\pi}{3},\ \frac{10\pi}{3}$ **iv** $\theta = \frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{11\pi}{4},\ \frac{13\pi}{4}$

c **i** $0 \leqslant \theta < \frac{\pi}{2},\quad \frac{3\pi}{2} < \theta < \frac{5\pi}{2},\quad \frac{7\pi}{2} < \theta \leqslant 4\pi$

ii $\frac{\pi}{2} < \theta < \frac{3\pi}{2},\quad \frac{5\pi}{2} < \theta < \frac{7\pi}{2}$

d $\{y \mid -1 \leqslant y \leqslant 1\}$

EXERCISE 17C

1 **a** vertical translation 1 unit downwards

b horizontal translation $\frac{\pi}{4}$ units to the right

c vertical stretch, scale factor 2

d horizontal stretch, scale factor $\frac{1}{4}$

e horizontal stretch, scale factor 4

f translation $\frac{\pi}{3}$ units right and 2 units upwards

2 **a** vertical stretch, scale factor $\frac{1}{2}$ **b** reflection in the x-axis

c translation $\frac{\pi}{6}$ units left and 2 units downwards

3 **a** $\frac{2\pi}{5}$ **b** $\frac{10\pi}{3}$ **c** 2 **d** $\frac{2\pi}{3}$ **e** 6π **f** 100

4 **a** $b = \frac{2}{5}$ **b** $b = 3$ **c** $b = \frac{1}{6}$ **d** $b = \frac{\pi}{2}$ **e** $b = \frac{\pi}{50}$

5 **a** maximum 4, minimum −4 **b** maximum 8, minimum 2

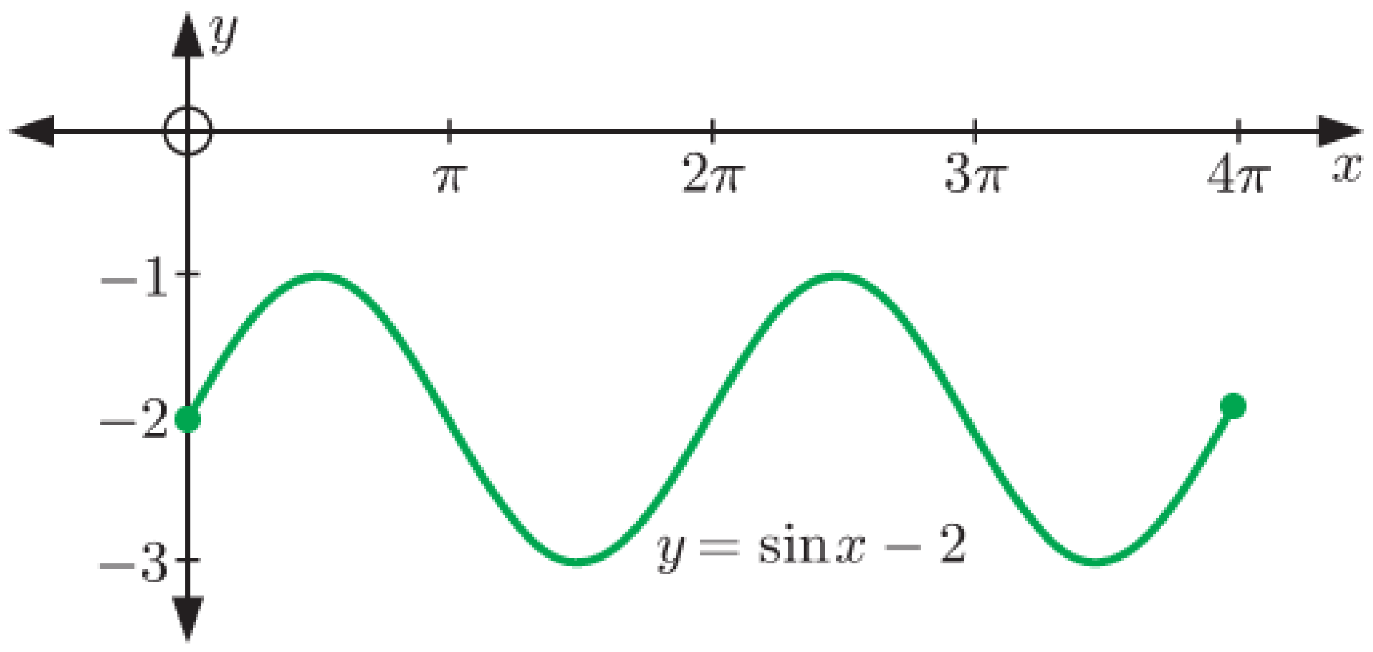
c maximum −2, minimum −6

6 **a** 4 **b** $\frac{2\pi}{3}$ **c** $\{y \mid -2 \leqslant y \leqslant 6\}$

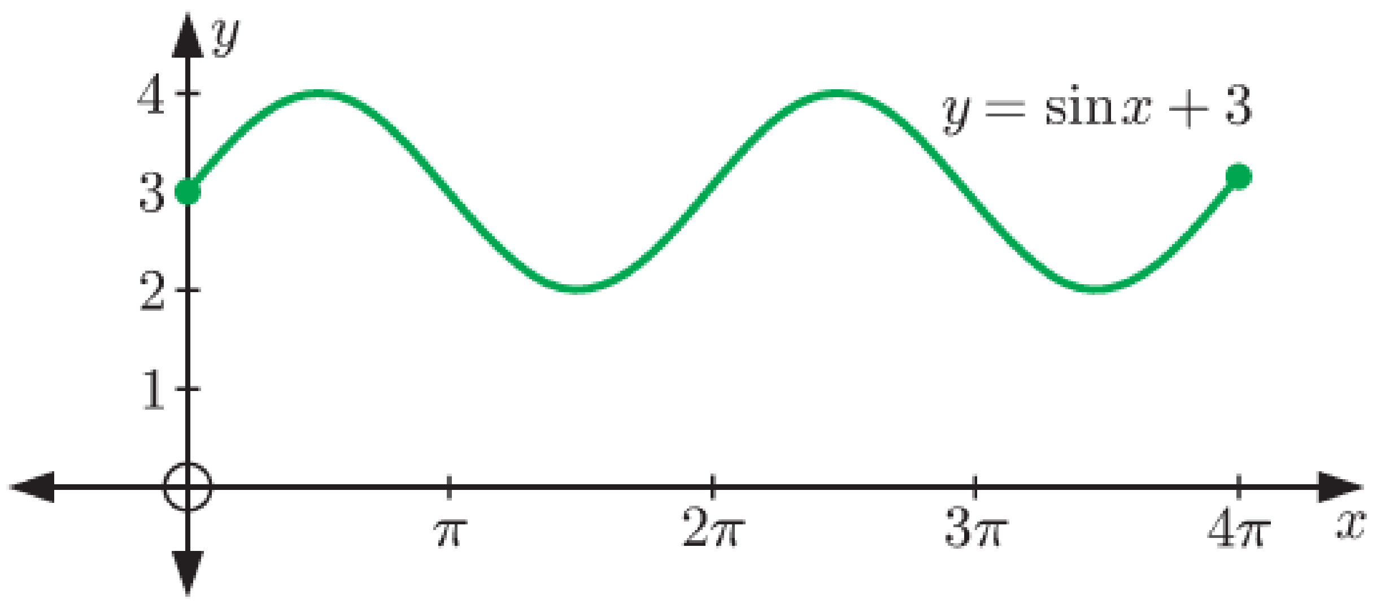
7 $|\,a\,| =$ amplitude, $b = \dfrac{2\pi}{\text{period}}$, $c =$ horizontal translation,

$d =$ vertical translation

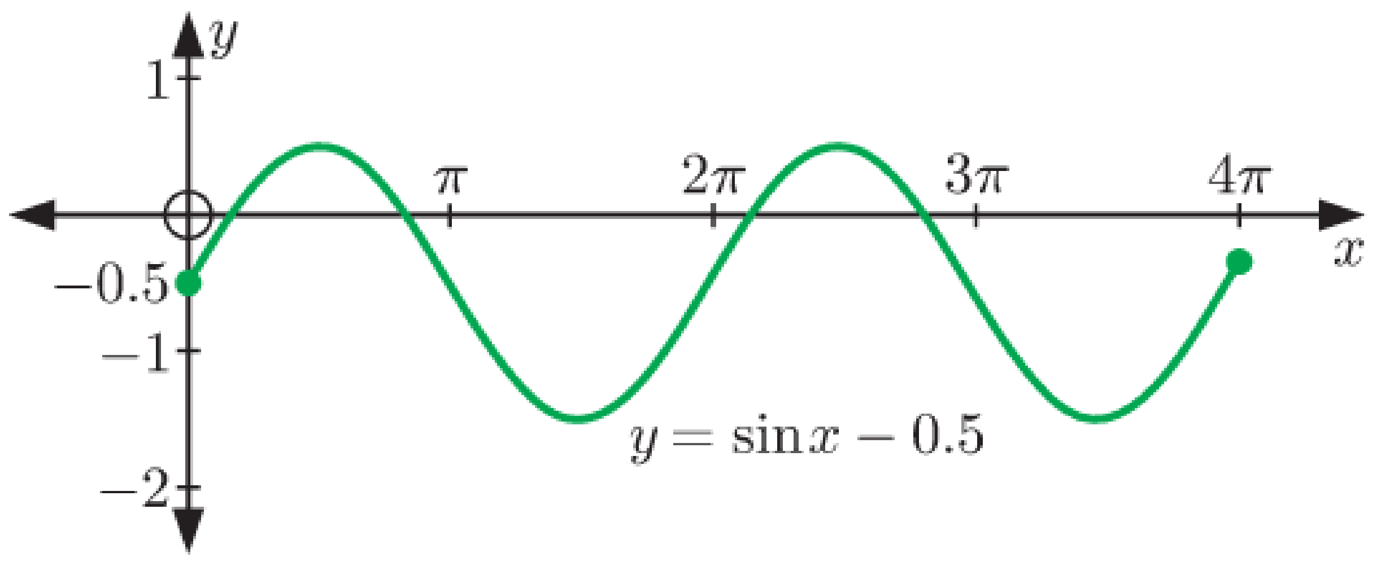
8 **a**



b



c



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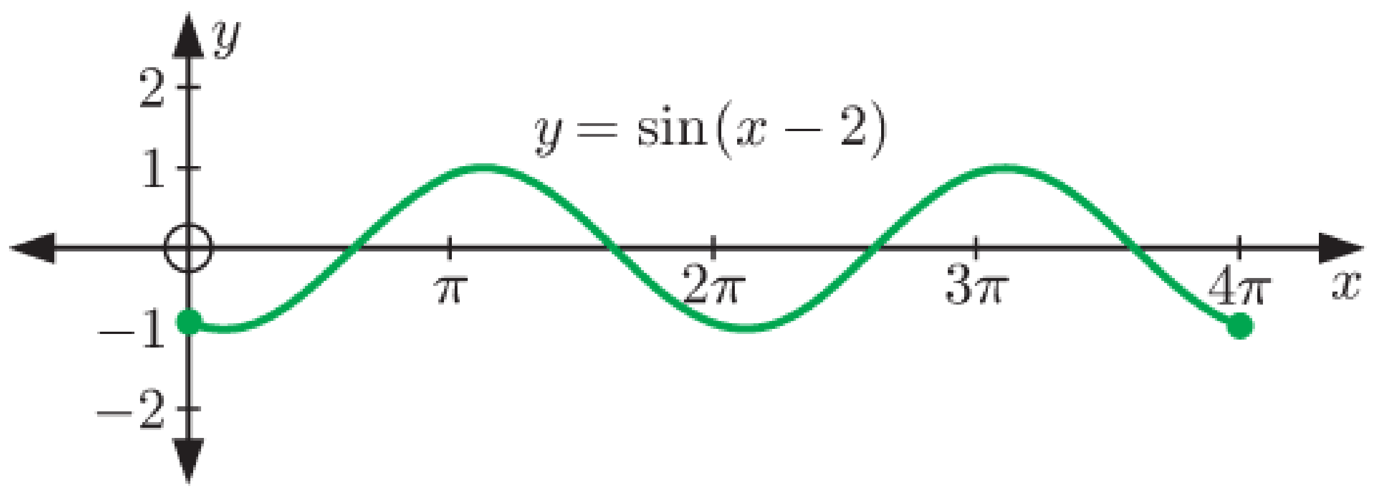

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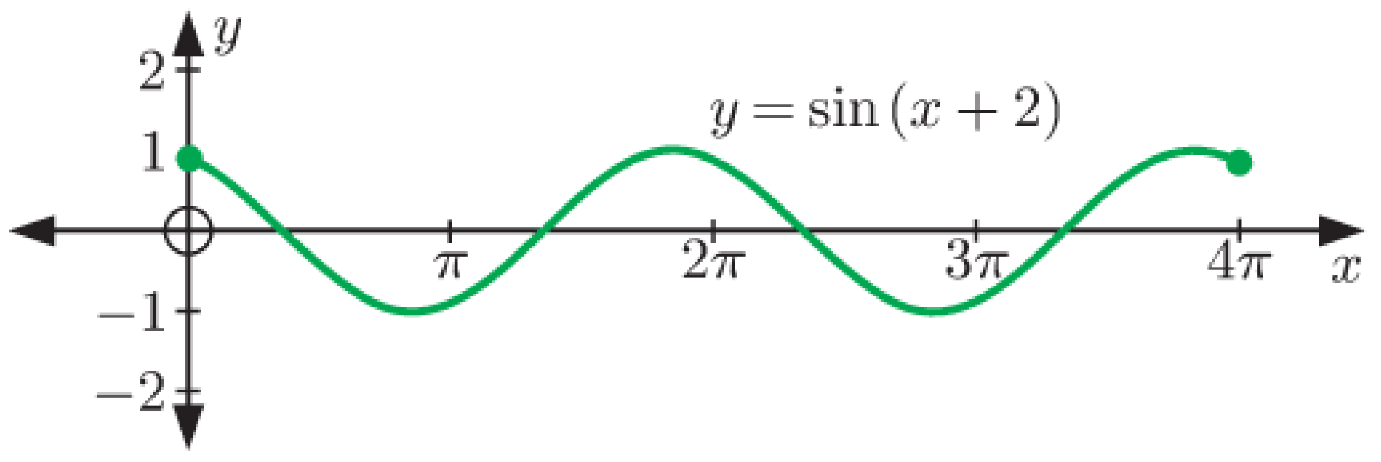

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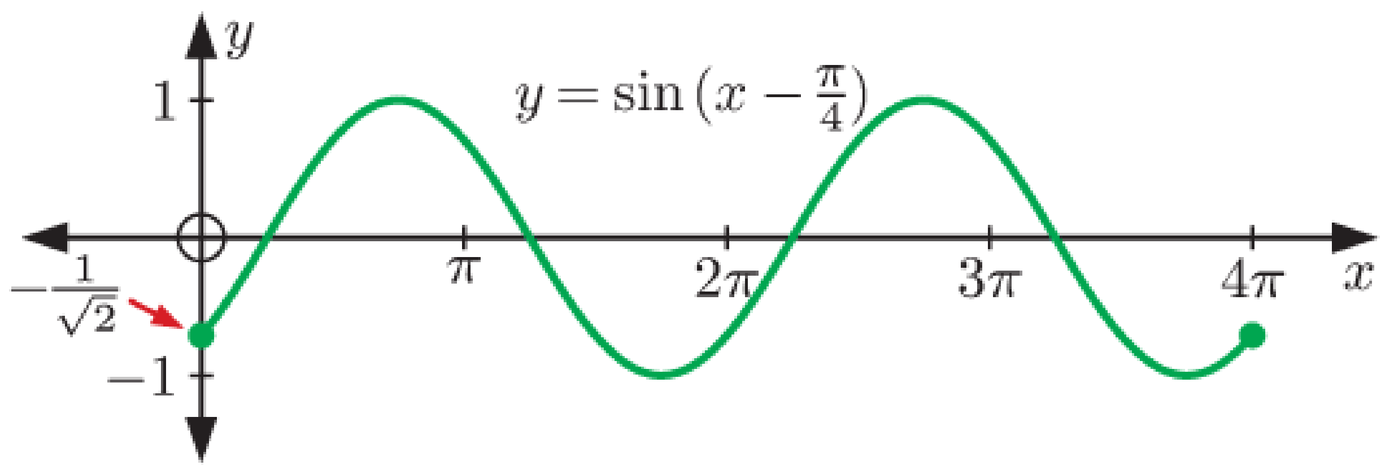

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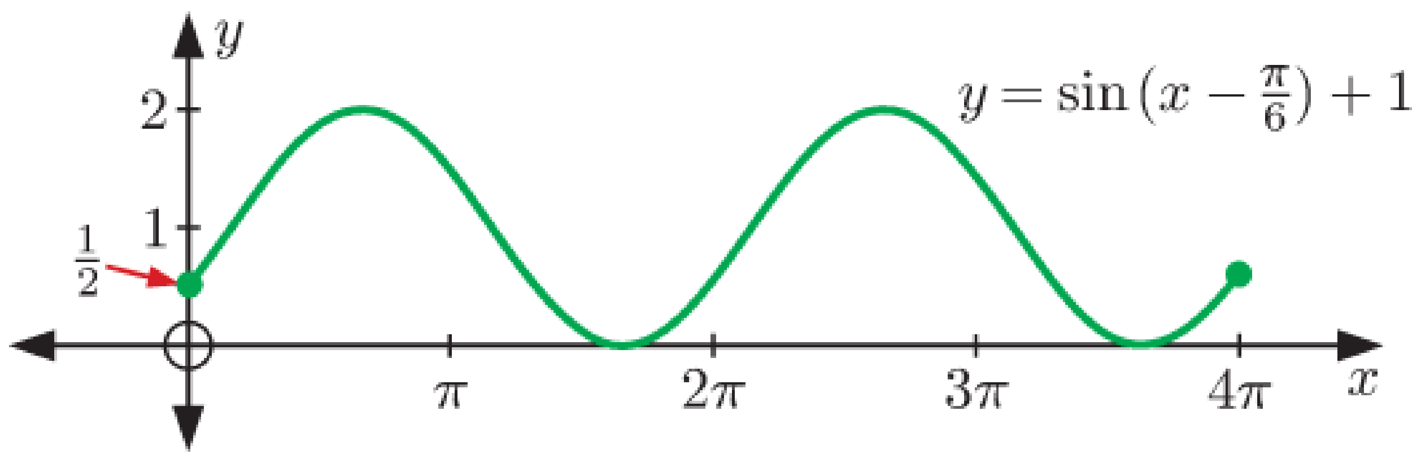

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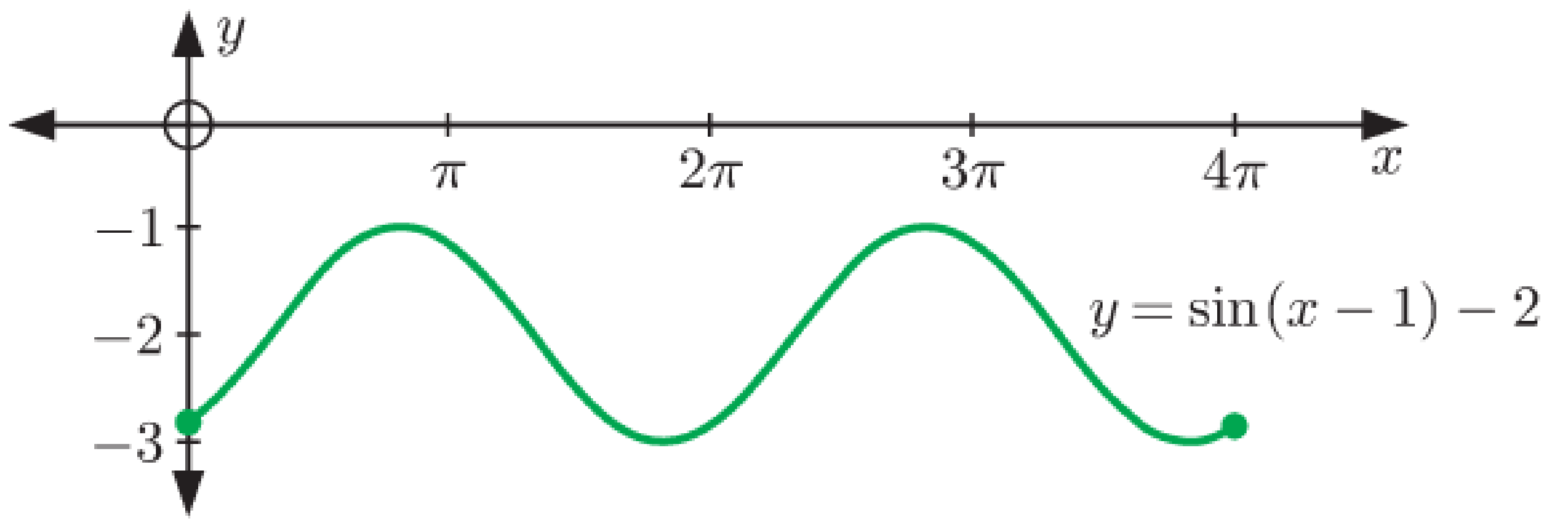
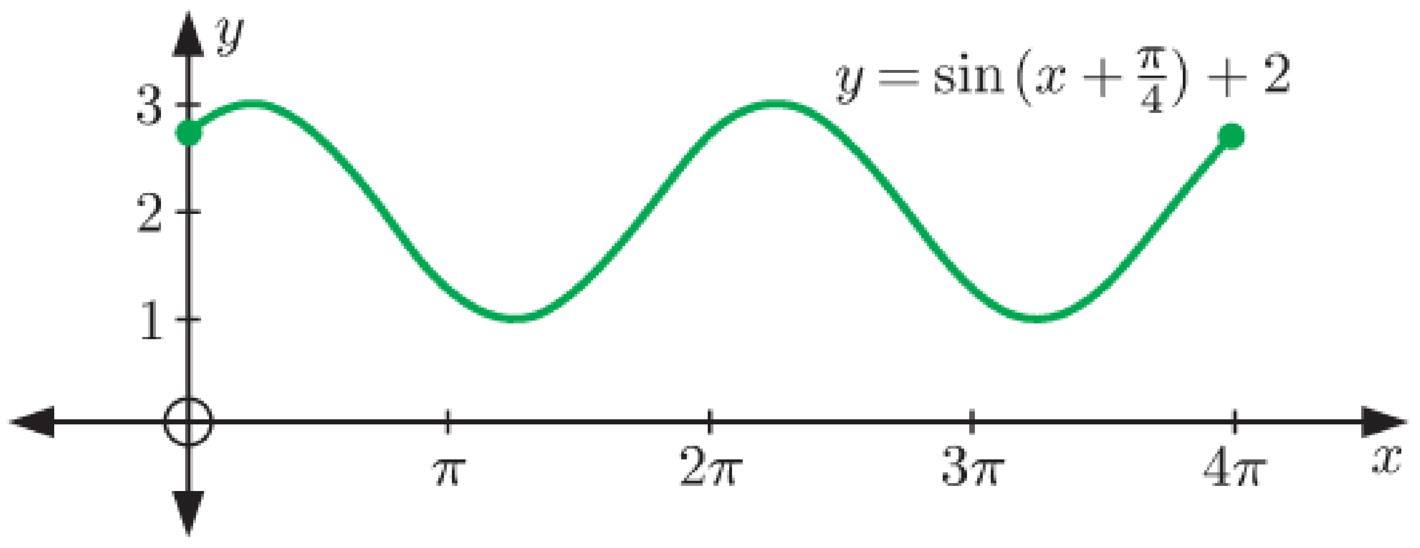


i



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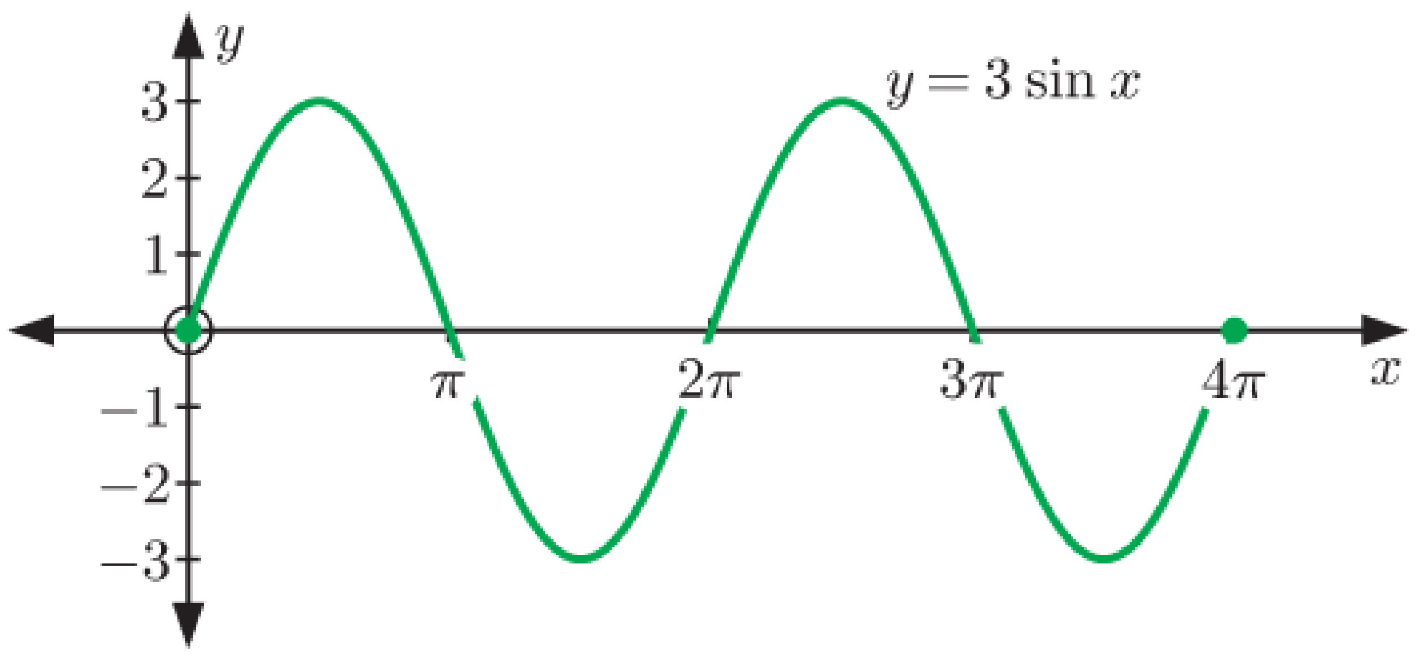

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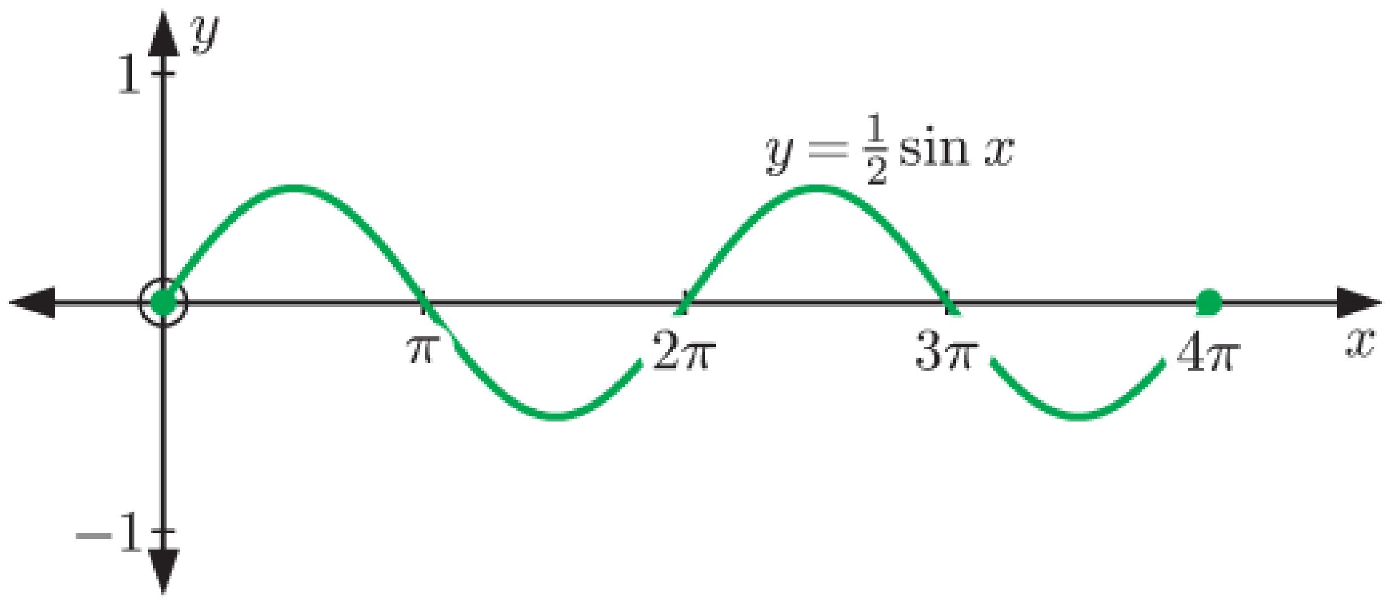
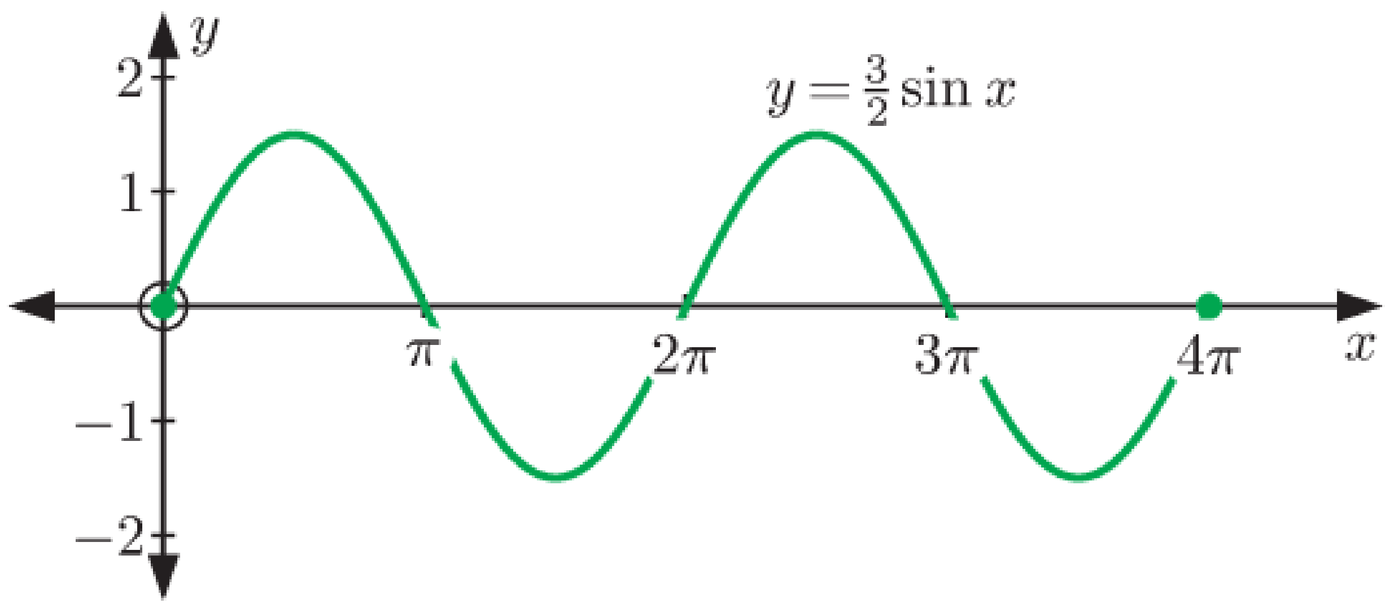
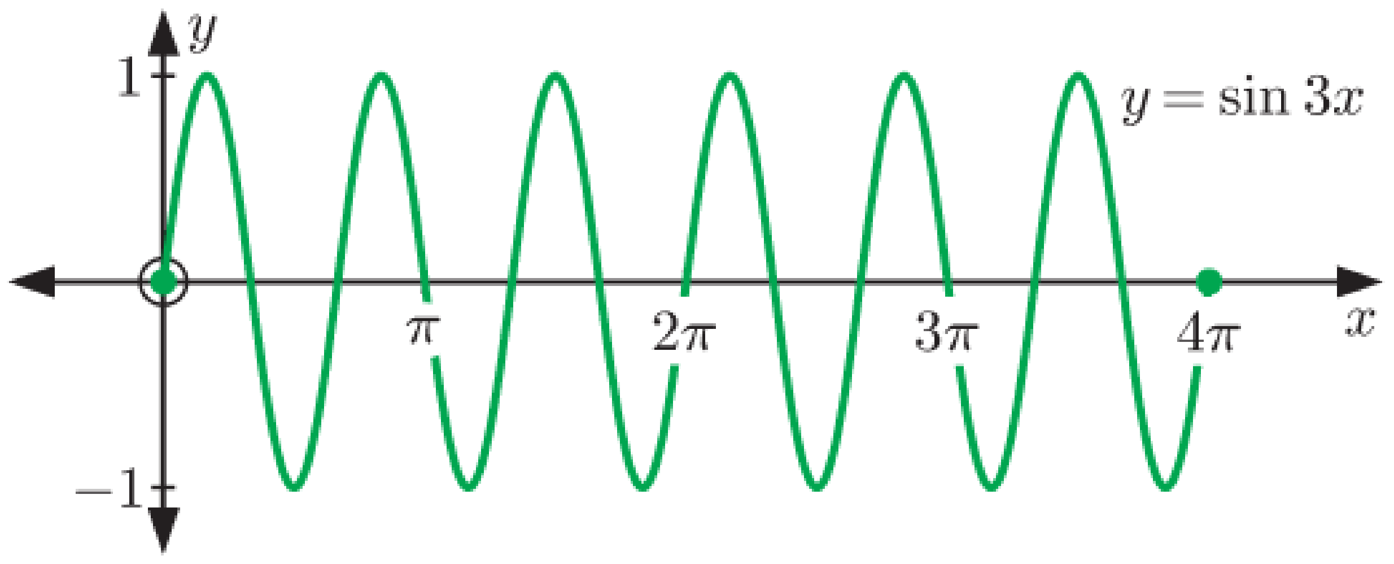


l



m



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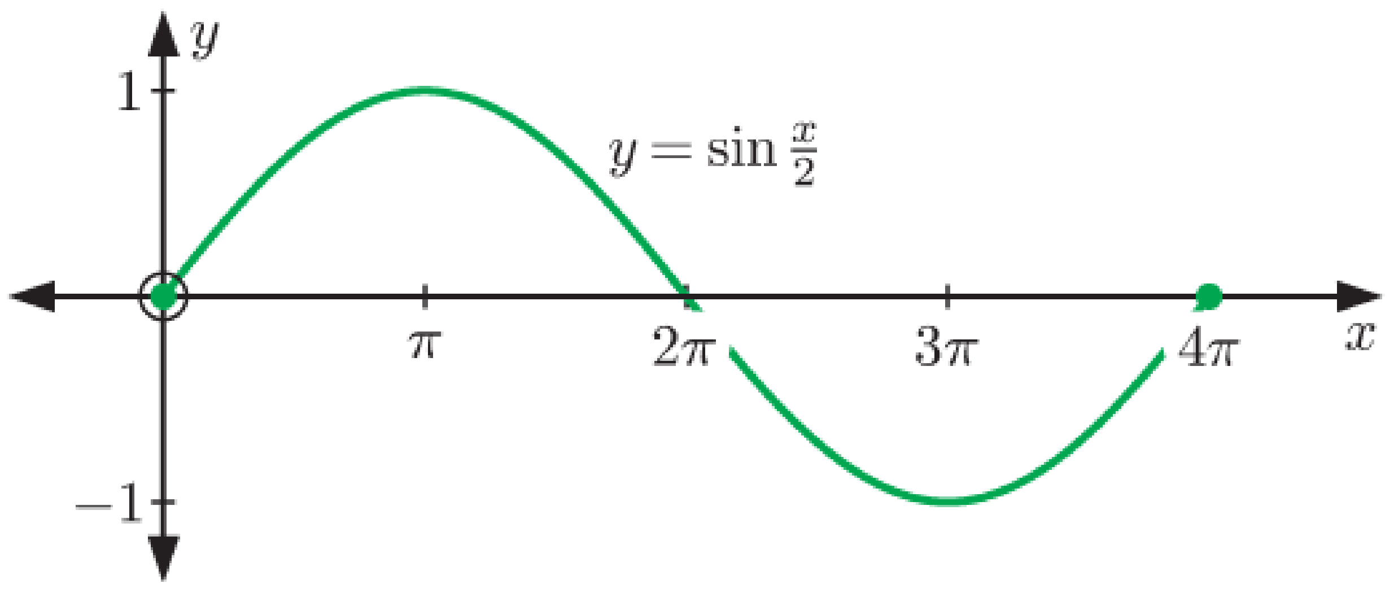
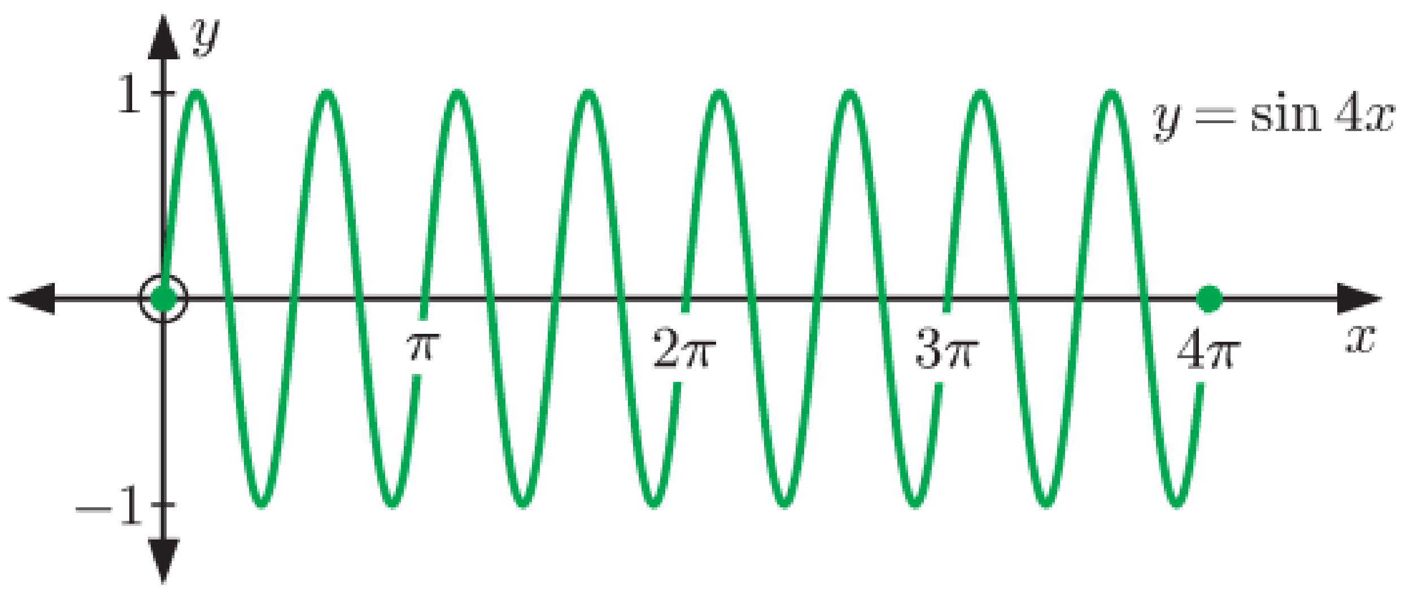
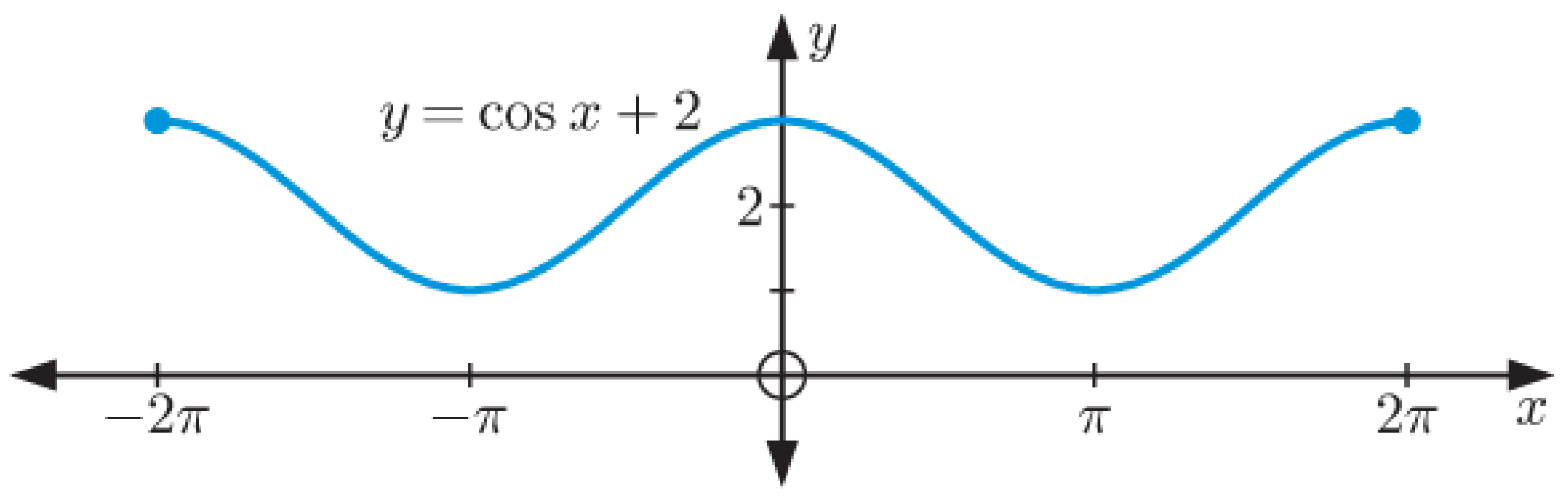


o



9 **a**



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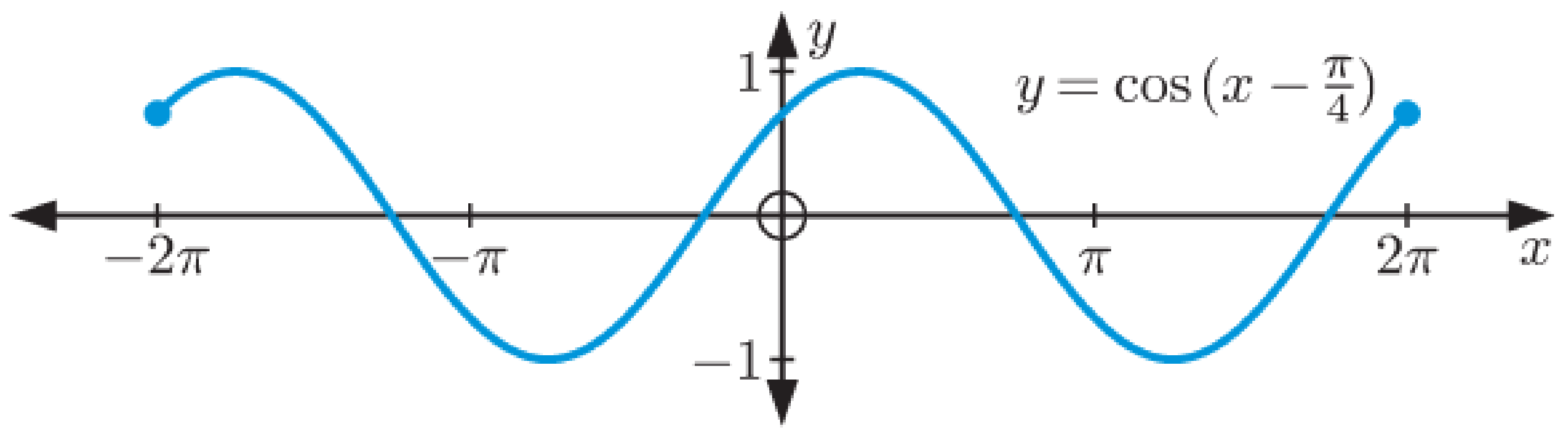

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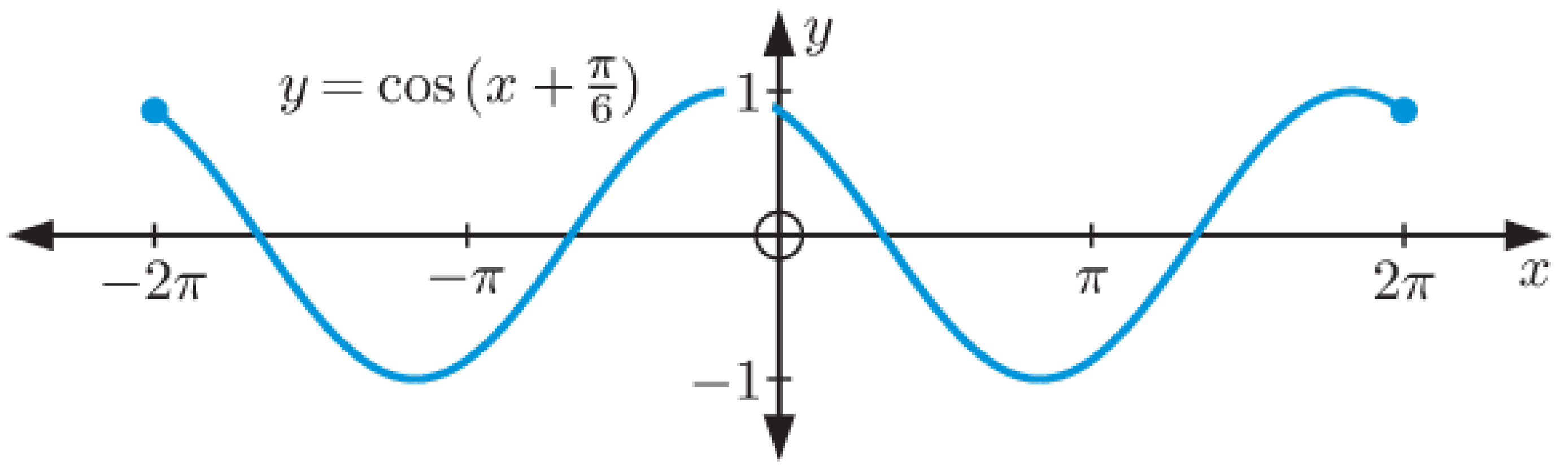

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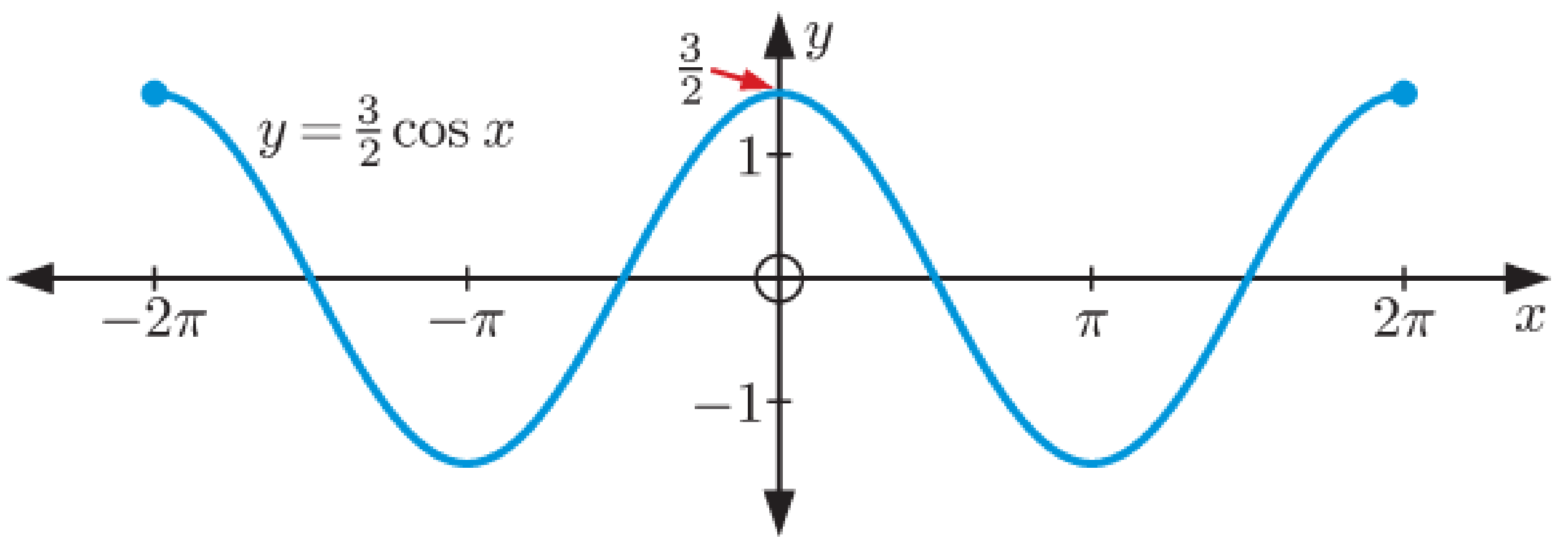
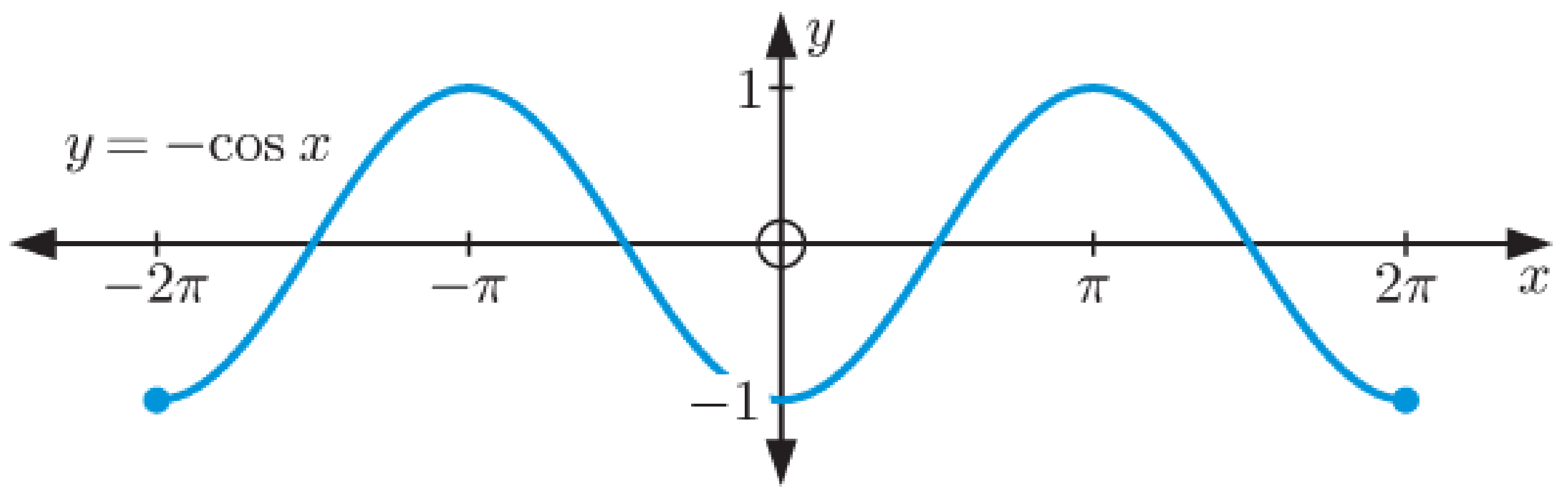


e



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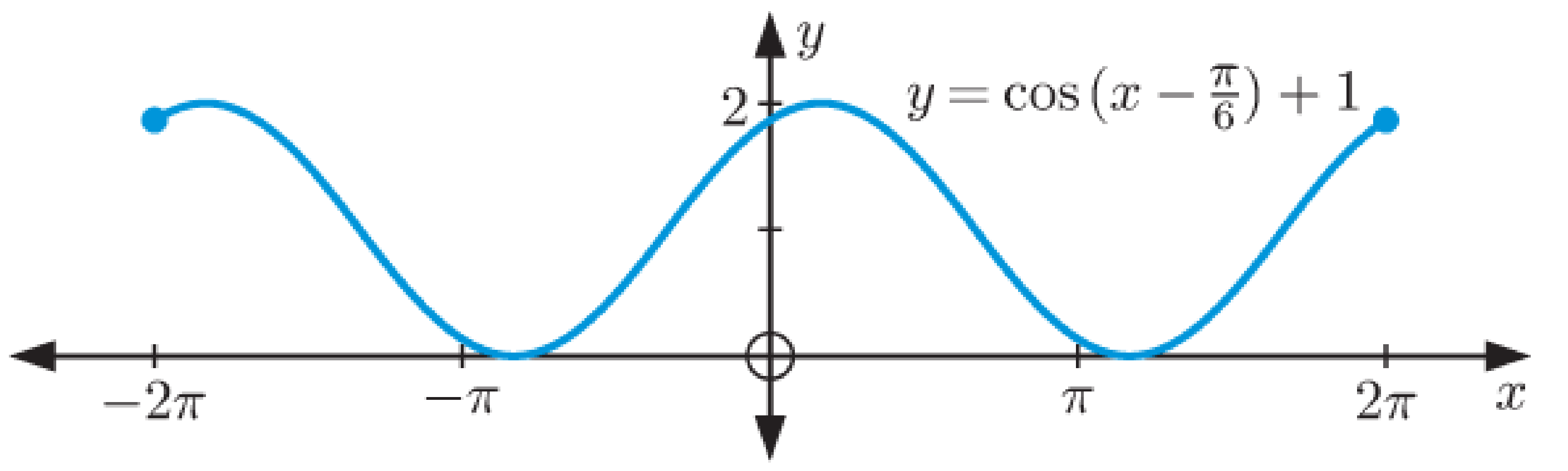
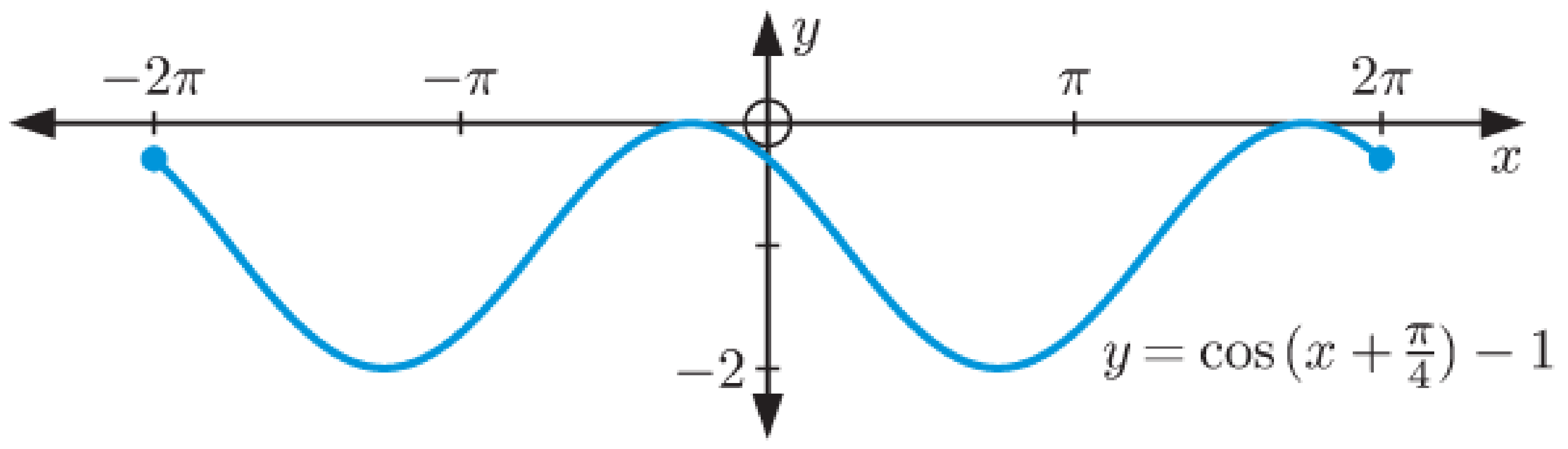
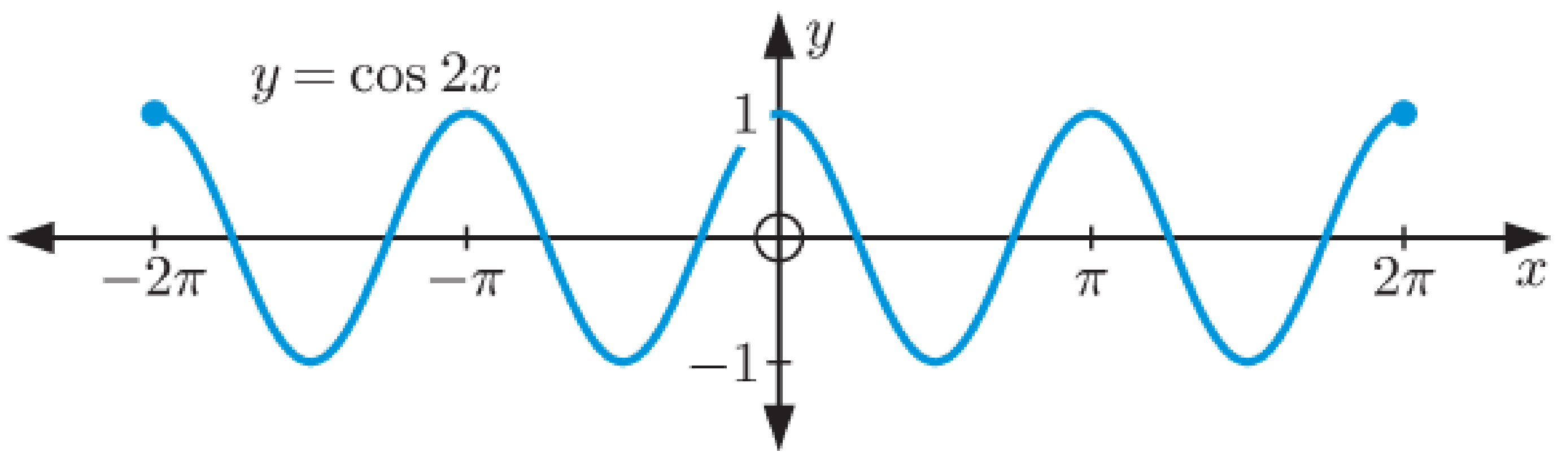


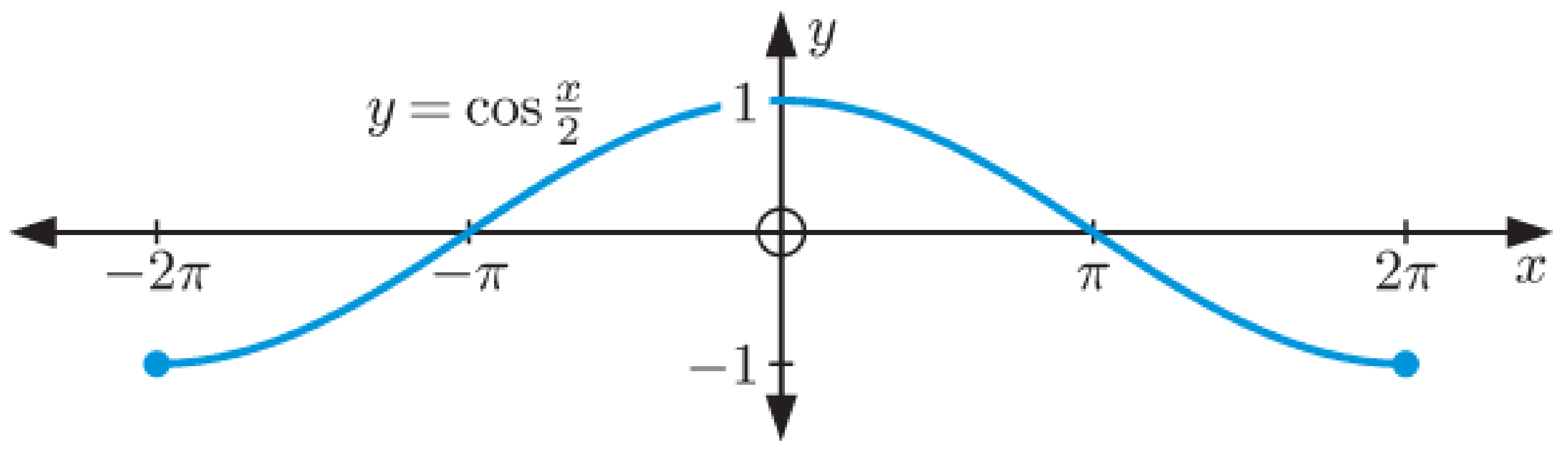
g



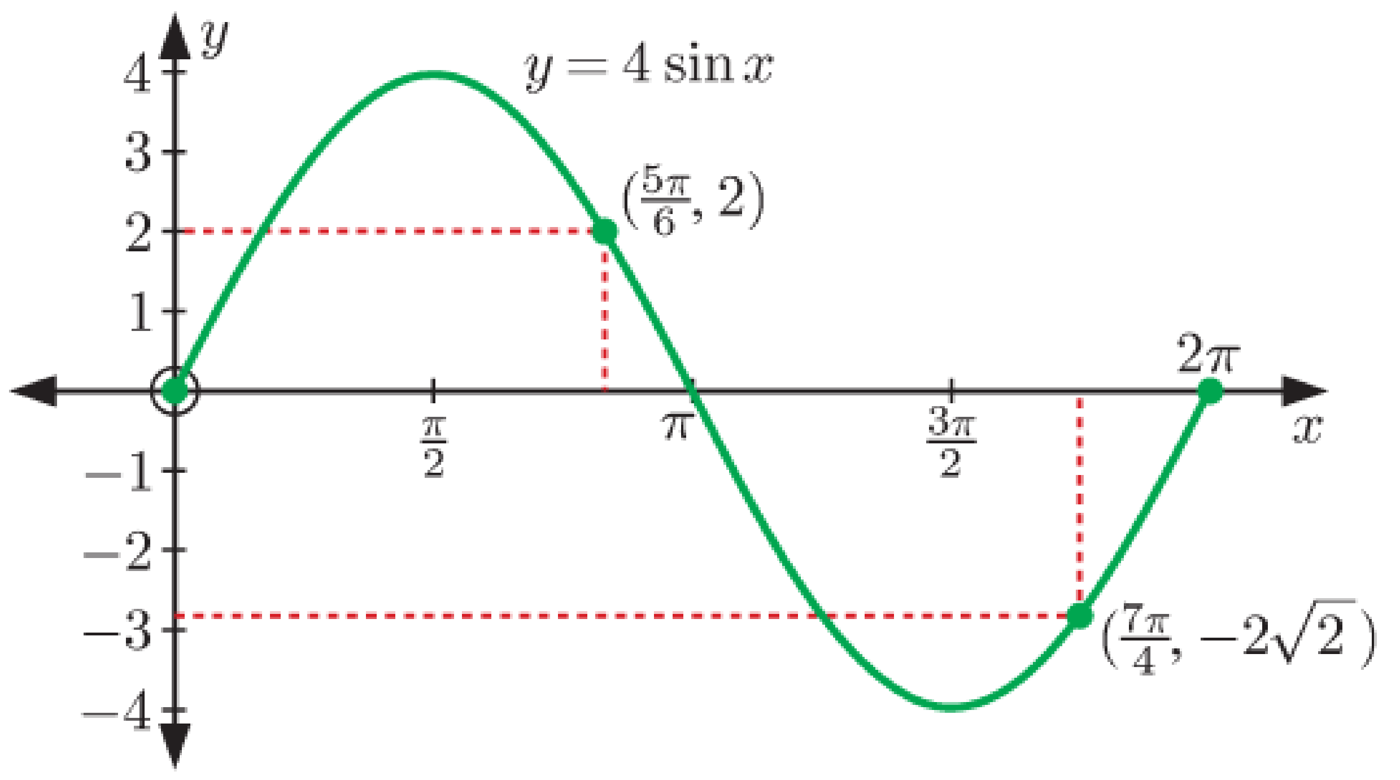
h



i



10 **a**



b **i** $y = 2$ **ii** $y = -2\sqrt{2} \approx -2.83$

11 **a** $d > 3$ **b** $d < -3$ **c** $-3 < d < 3$

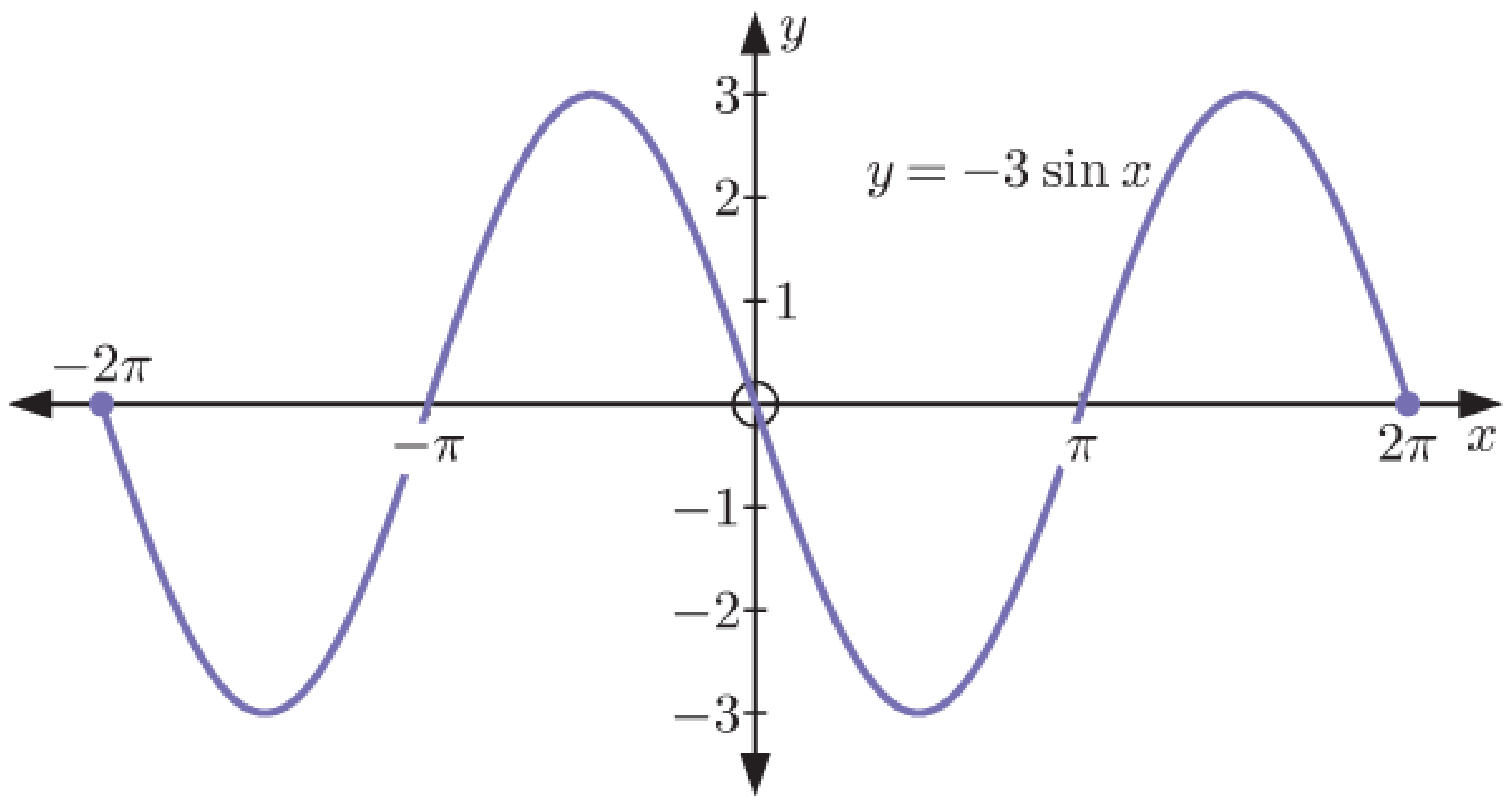
12 **a** A horizontal stretch with scale factor $\frac{1}{3}$, then a vertical stretch with scale factor 2.

b A vertical stretch with scale factor 2, then a reflection in the x-axis.

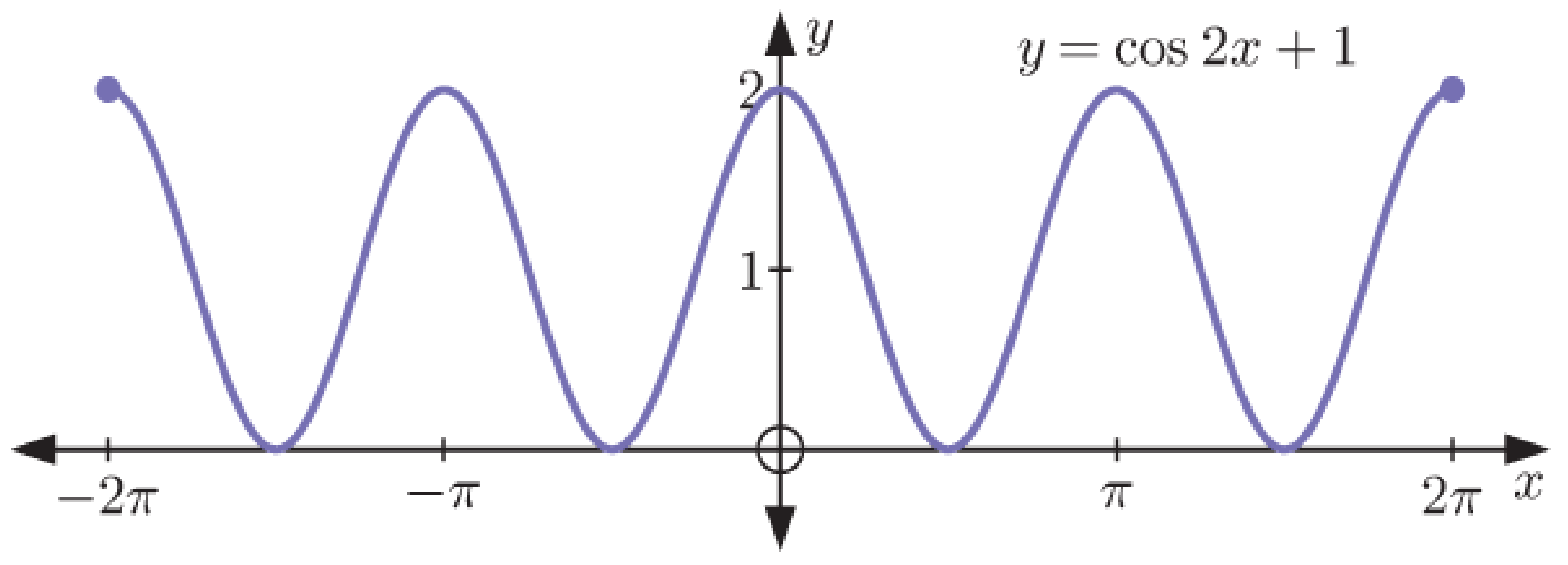
c A vertical stretch with scale factor 3, then a translation 5 units downwards.

d A horizontal stretch with scale factor $\frac{1}{2}$, then a translation $\frac{\pi}{6}$ units left.

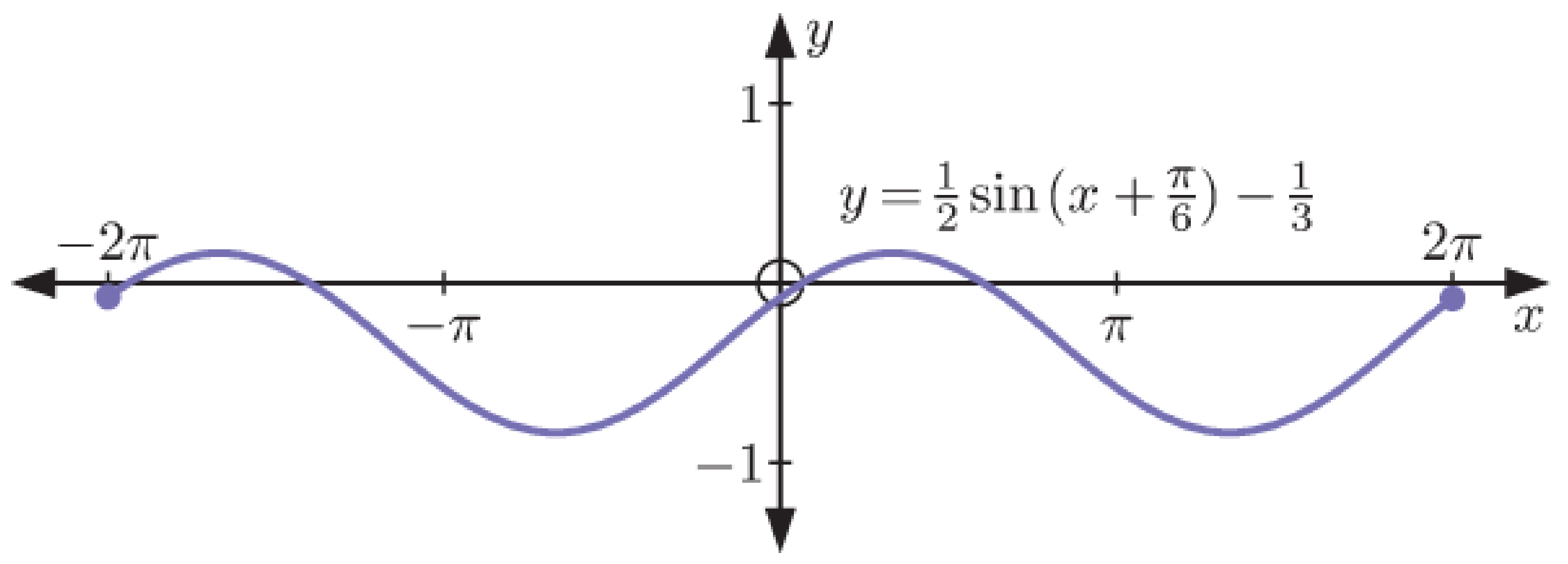
13 **a**



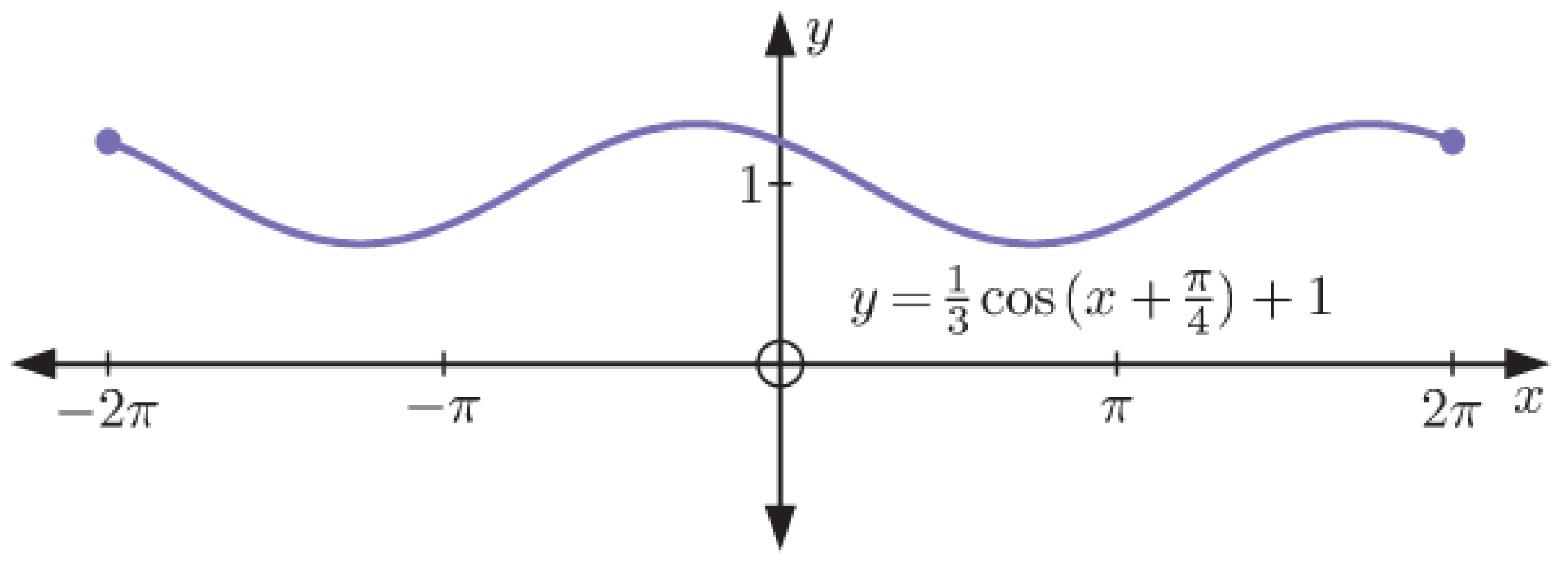
b



c



d



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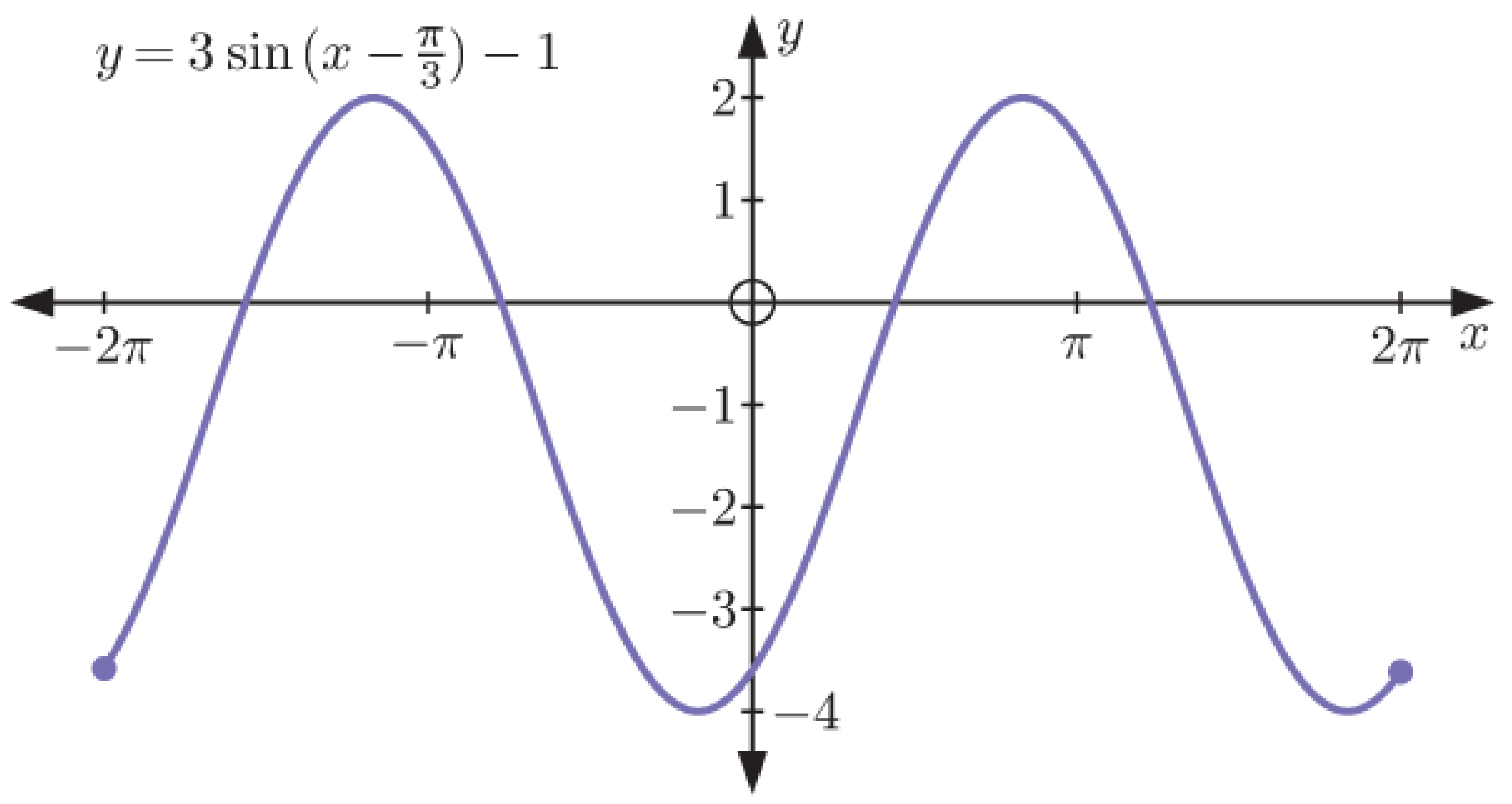
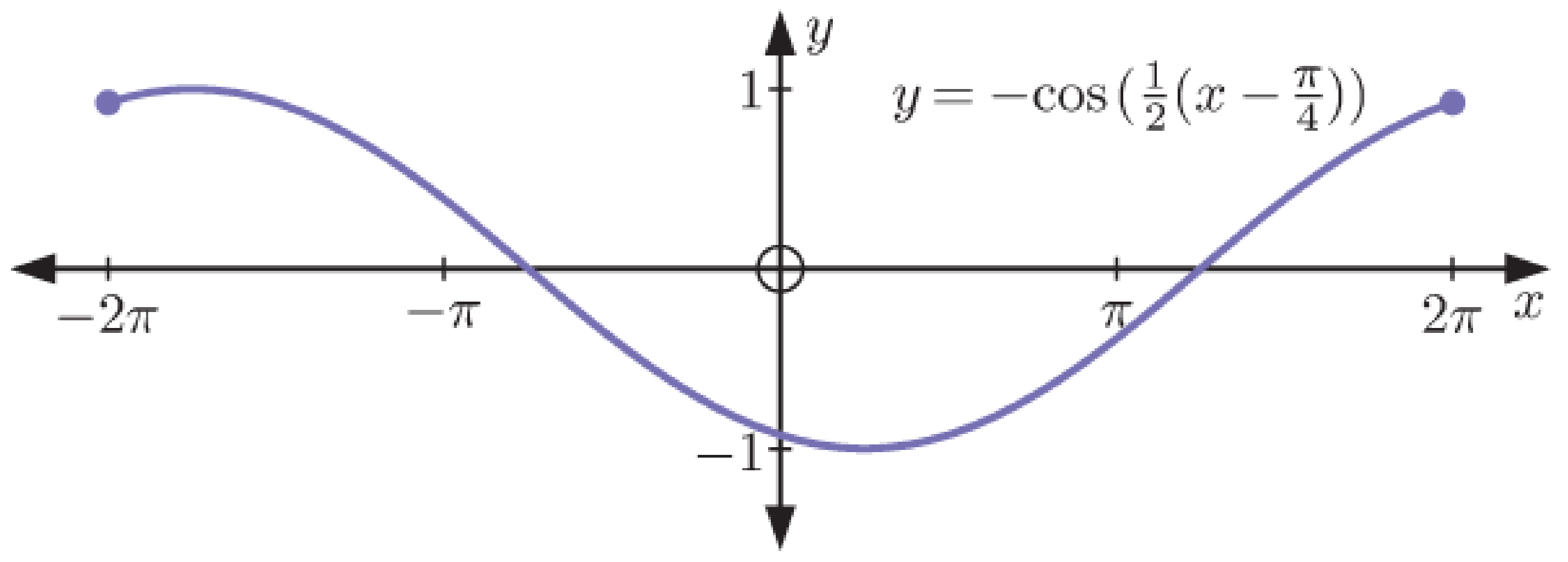


f



14 **a** b, c, d (provided the function has x-intercepts)
b d **c** d

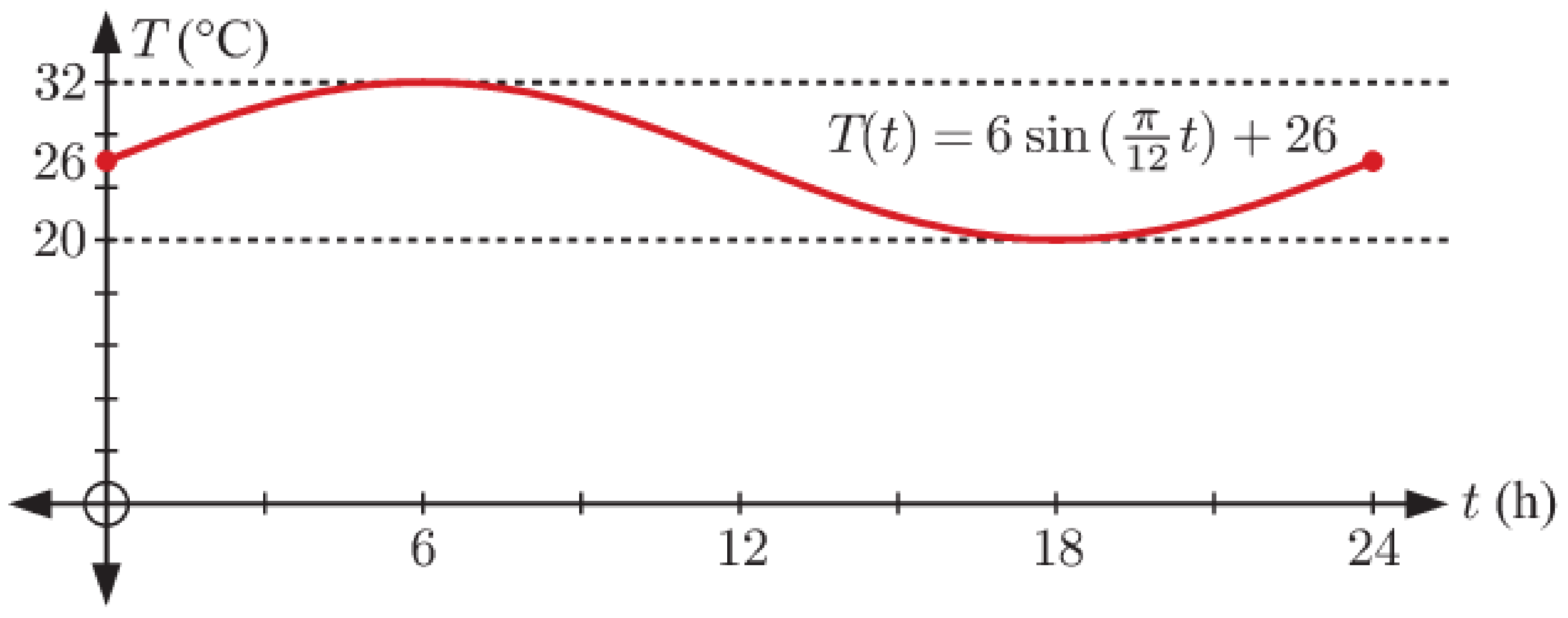
15 **a** $a = 4,\ d = 1$ **b** $a = -2,\ d = 3$ **c** $a = \frac{1}{3},\ d = \frac{4}{3}$

16 **a** $y = \sin x - 2$ **b** $y = \sin 3x$ **c** $y = \sin\left(x + \frac{\pi}{2}\right)$
d $y = 2\sin x + 1$ **e** $y = 4\sin\frac{x}{2} - 1$ **f** $y = 6\sin\frac{2\pi x}{5}$

17 **a** $y = 2\cos 2x$ **b** $y = \cos\frac{x}{2} + 2$ **c** $y = -5\cos\frac{\pi x}{3}$

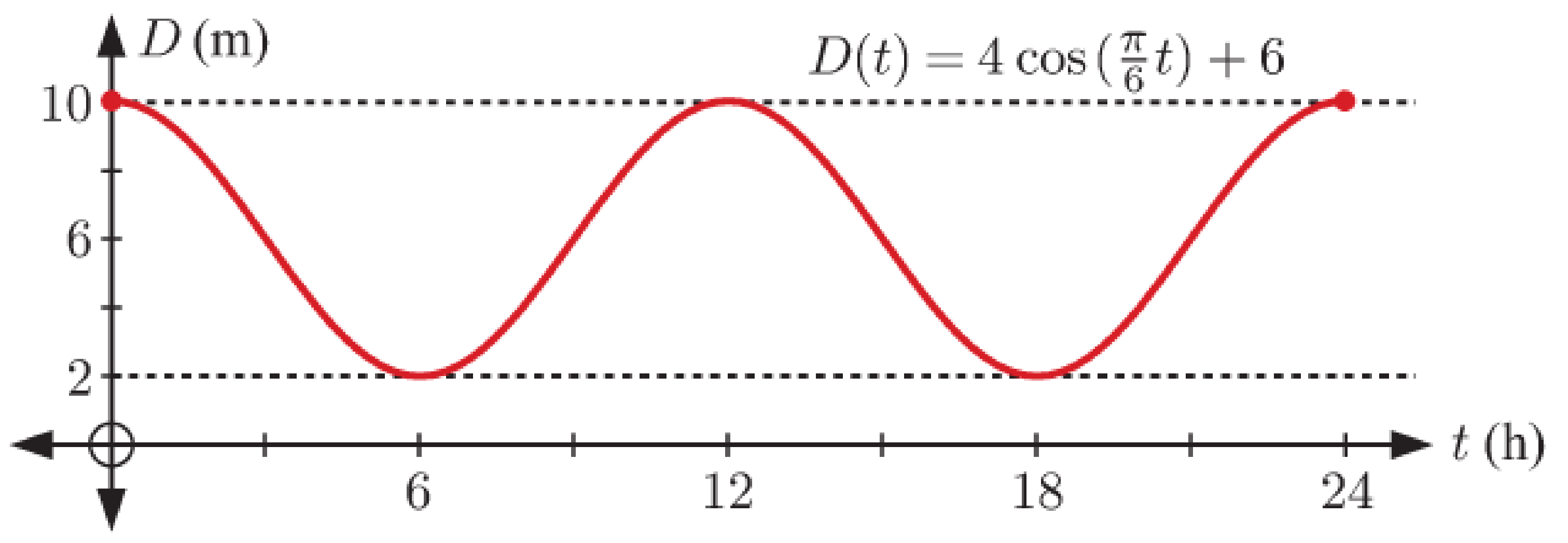
EXERCISE 17D

1 **a**



b **i** 26°C **ii** 29°C **c** 32°C, at 6 pm

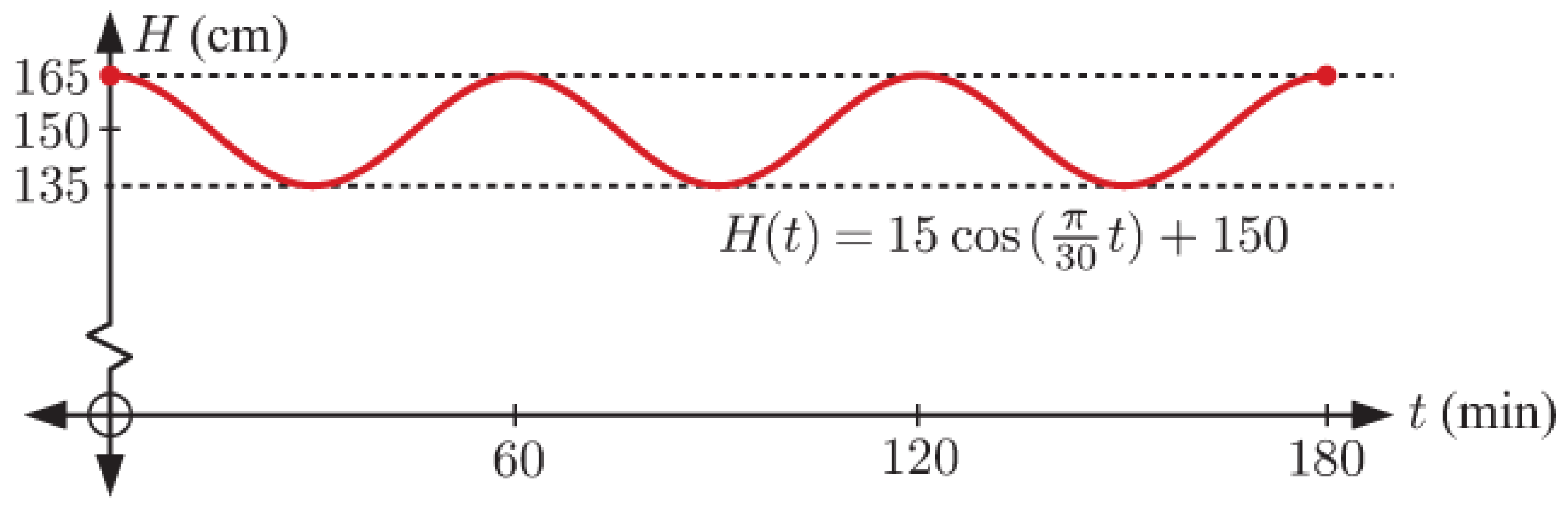
2 **a**



b highest = 10 m, at midnight, midday, and midnight the next day
lowest = 2 m, at 6 am and 6 pm

c no (water height is 4 m)

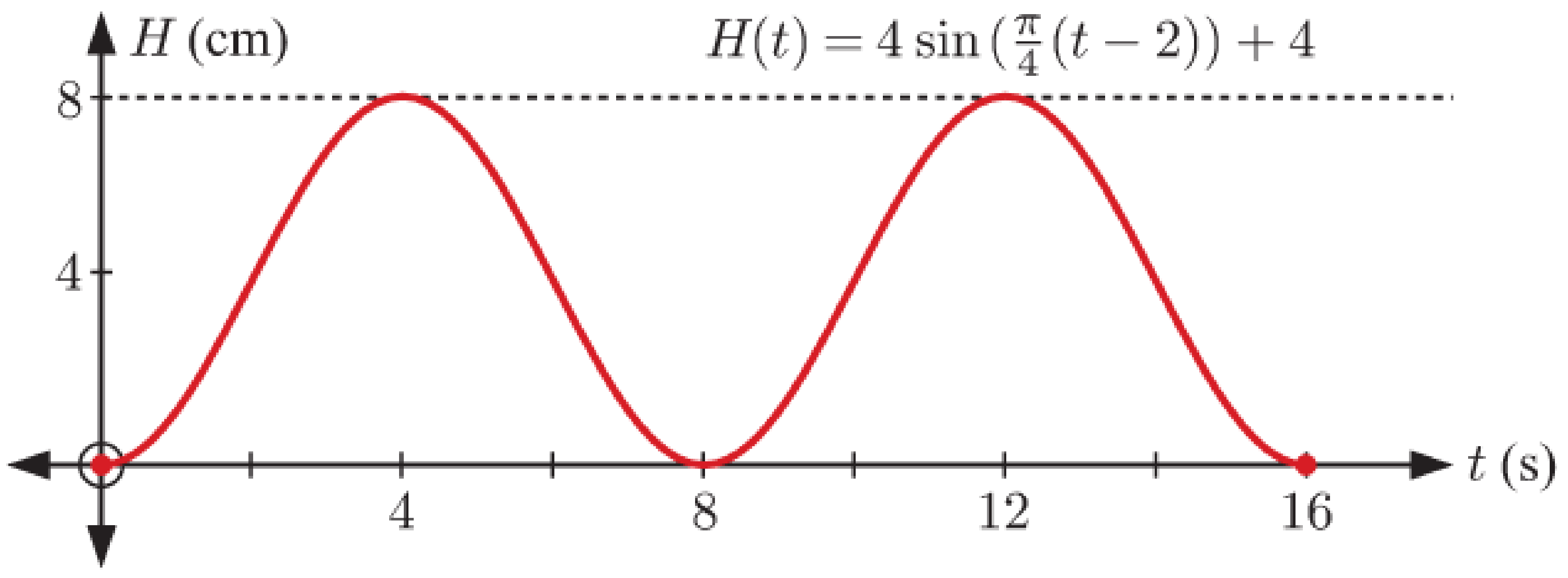
3 **a**



b 15 cm

c **i** ≈ 160.0 cm **ii** ≈ 138.9 cm **iii** ≈ 158.8 cm
iv ≈ 138.9 cm

4 **a**



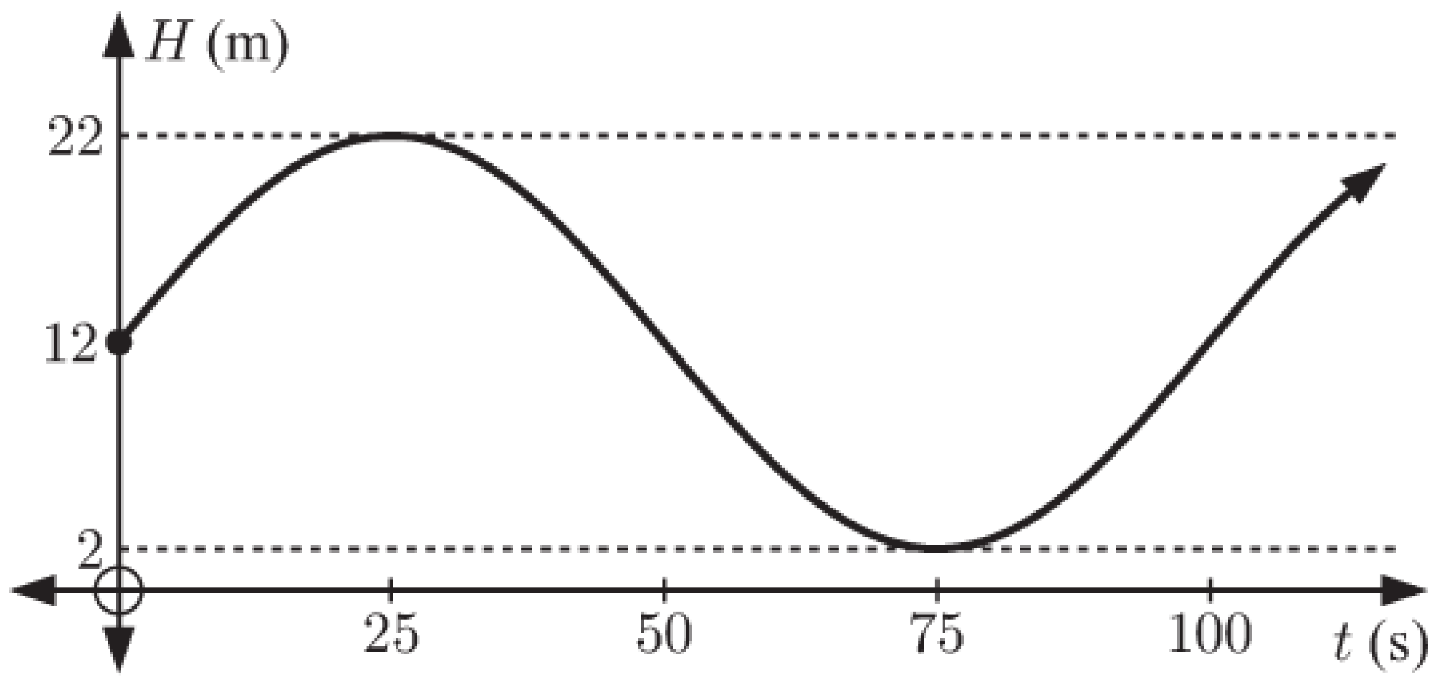
b 4 cm

c no (ball diameter is 4.28 cm, gate height is ≈ 3.07 cm)

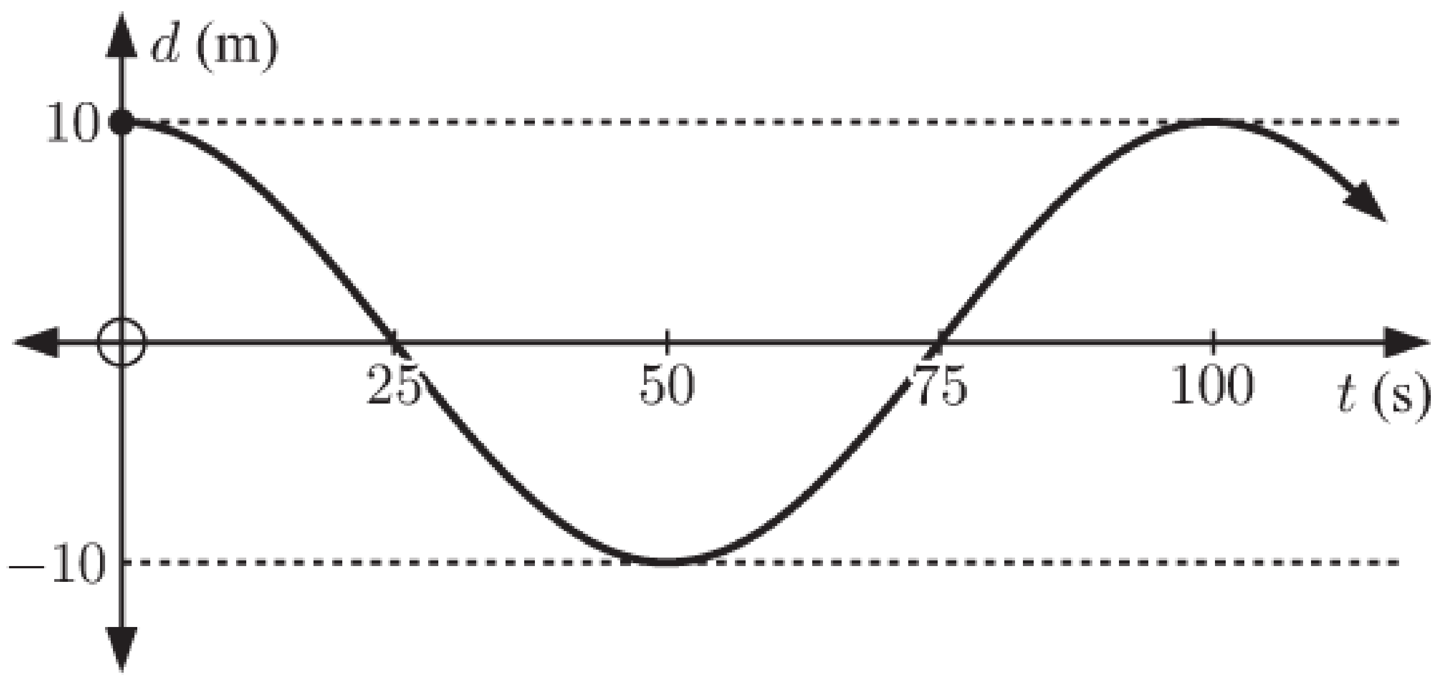
5 $T(t) = 5.2\sin\left(\frac{\pi}{12}(t - 8)\right) + 10.6$ °C

6 $H(t) = 0.6\cos\left(\frac{5\pi}{31}(t - 1.5)\right) + 0.76$ m

7 **a**



b



c Both graphs are periodic with an amplitude of 10 m and a period of 100 s. The graphs differ by a horizontal translation of 25 s and the principal axis is also translated by 12 m.

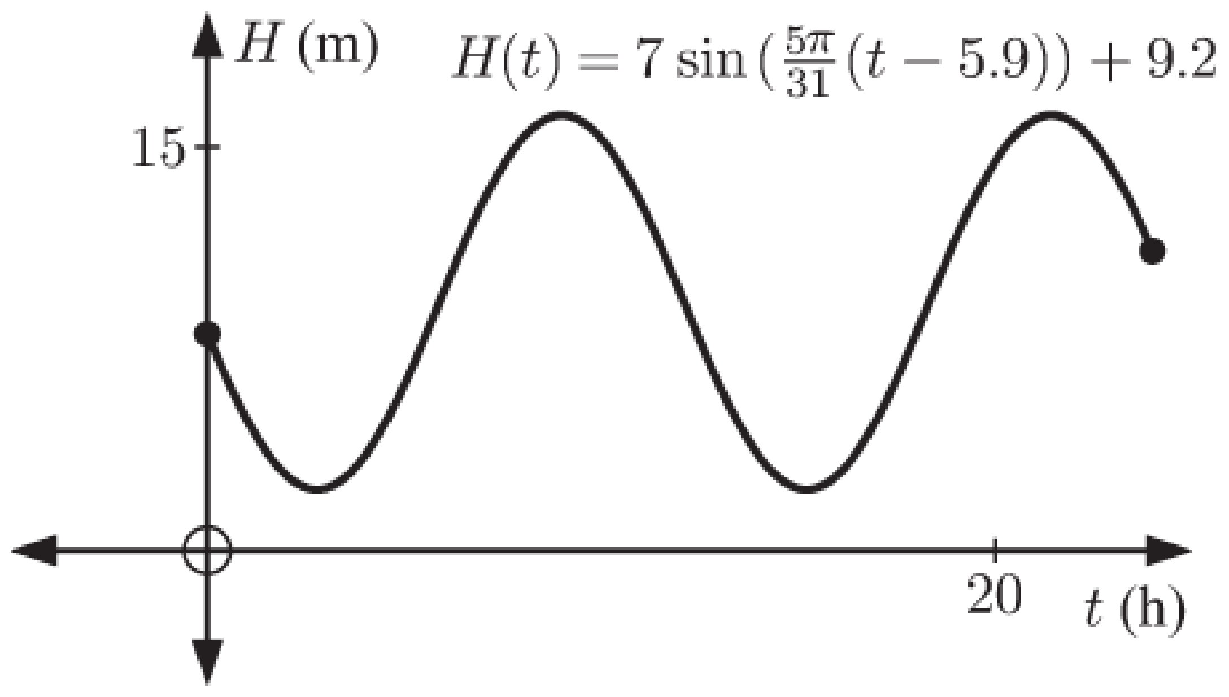
d **i** $H(t) = 10\sin\left(\frac{\pi}{50}t\right) + 12$ m

ii $d(t) = 10\sin\left(\frac{\pi}{50}(t + 25)\right)$ m

Note: The function of horizontal displacement of the light will be different depending on how the coordinate system is defined.

8 **a** $H(t) = 7\sin\left(\frac{5\pi}{31}(t - 5.9)\right) + 9.2$ m

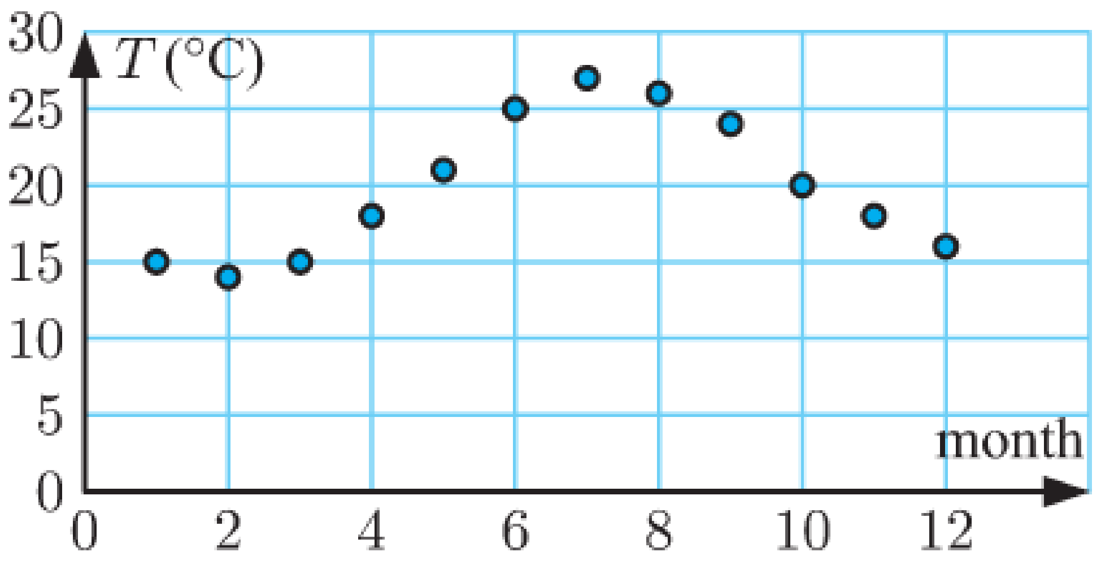
b



9 **a** $H(t) = 6\cos\left(\frac{\pi}{6}t\right)$ **b** $d(t) = 12\sin 2\pi t$

EXERCISE 17E

1 **a**



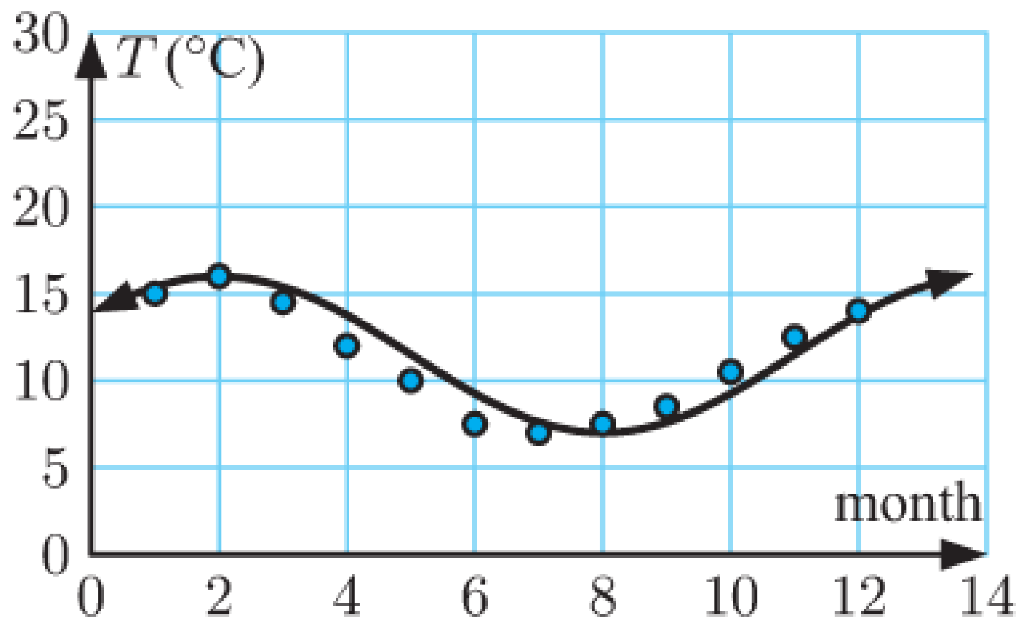
b The data appears to be periodic.

c **i** $b \approx \frac{\pi}{6}$ **ii** $a \approx 6.5$ **iii** $d \approx 20.5$ **iv** $c \approx 4.5$

d Using technology, $T \approx 6.15\sin(0.575t - 2.69) + 20.4$. Our model was a reasonable fit.

2 **a** $T \approx 4.5\cos\left(\frac{\pi}{6}(t-2)\right) + 11.5$

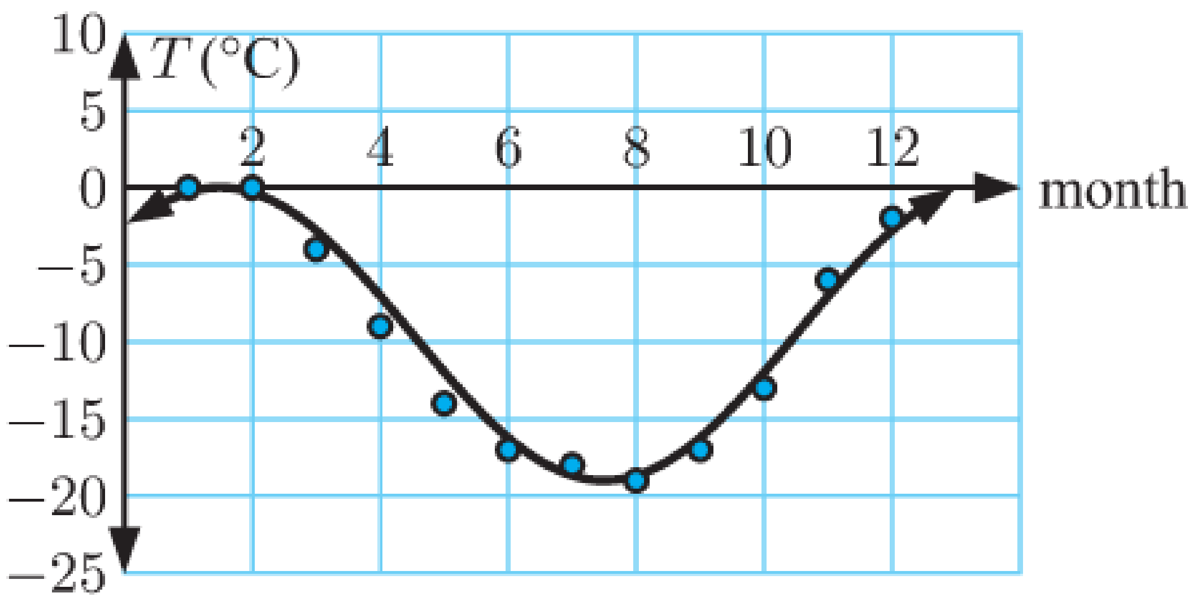
b



c Using technology, $T \approx 4.29\cos(0.533t - 0.805) + 11.2$.

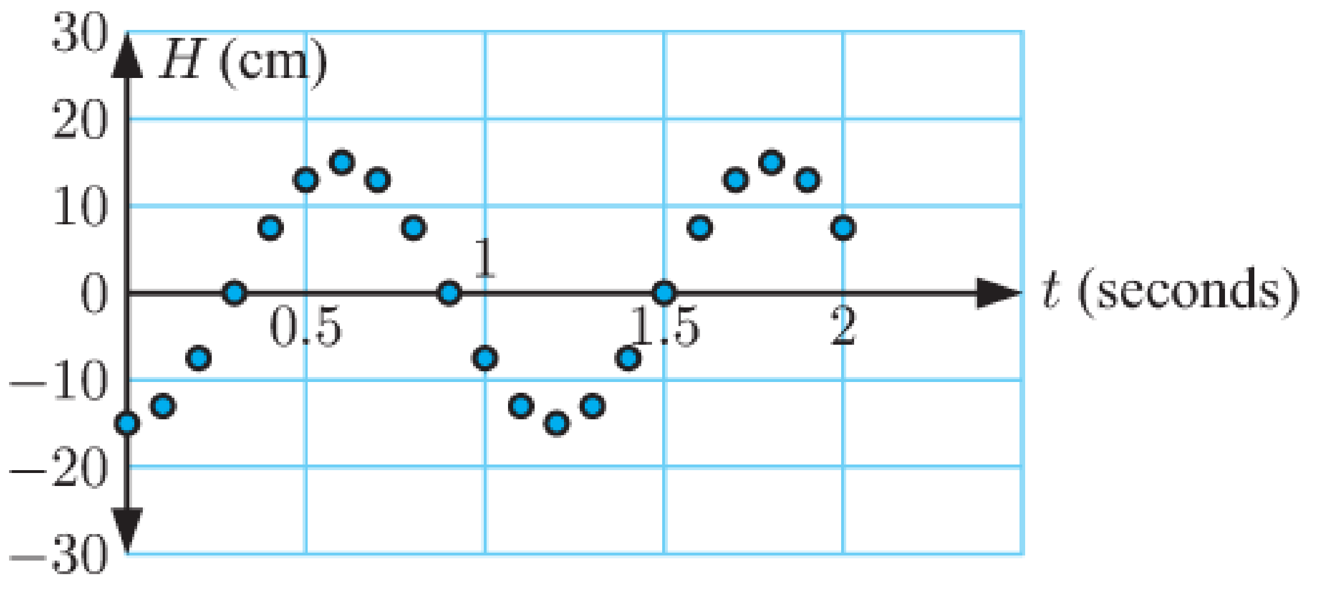
3 **a** $T \approx 9.5\sin\left(\frac{\pi}{6}(t - 10.5)\right) - 9.5$

b



c The model is a reasonable fit, but not perfect.

4 **a**



b $H \approx 15.0\sin(5.24t - 1.57) + 0.000\,170$ **c** ≈ 14.5 cm

d The spring will not oscillate indefinitely at the same rate.

EXERCISE 17F

1 **a** A horizontal translation $\frac{\pi}{2}$ units to the right.

b A vertical stretch with scale factor 4.

c A horizontal stretch with scale factor $\frac{2}{\pi}$.

d A horizontal stretch with scale factor $\frac{1}{2}$, then a translation 1 unit downwards.

e A vertical stretch with scale factor $\frac{1}{2}$, then a reflection in the x-axis.

f A translation 2 units upwards.

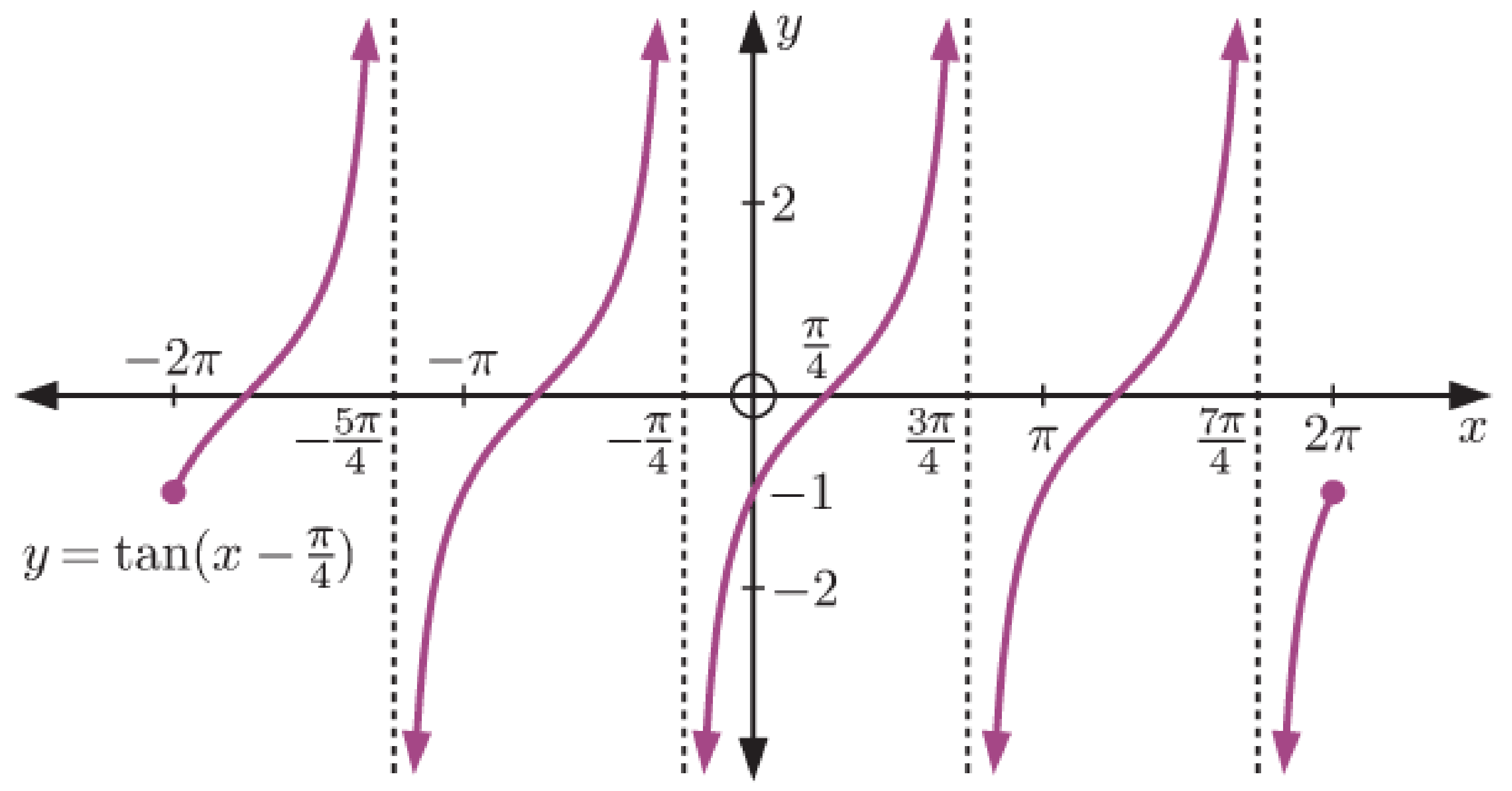
2 **a** $\frac{\pi}{3}$ **b** 4π **c** 1 **d** 2 **e** $\frac{3\pi}{2}$ **f** $\frac{\pi}{n}$

3 **a** **i** $\frac{k\pi}{2}$, $k \in \mathbb{Z}$ **ii** $x = \frac{\pi}{4} + \frac{k\pi}{2}$, $k \in \mathbb{Z}$

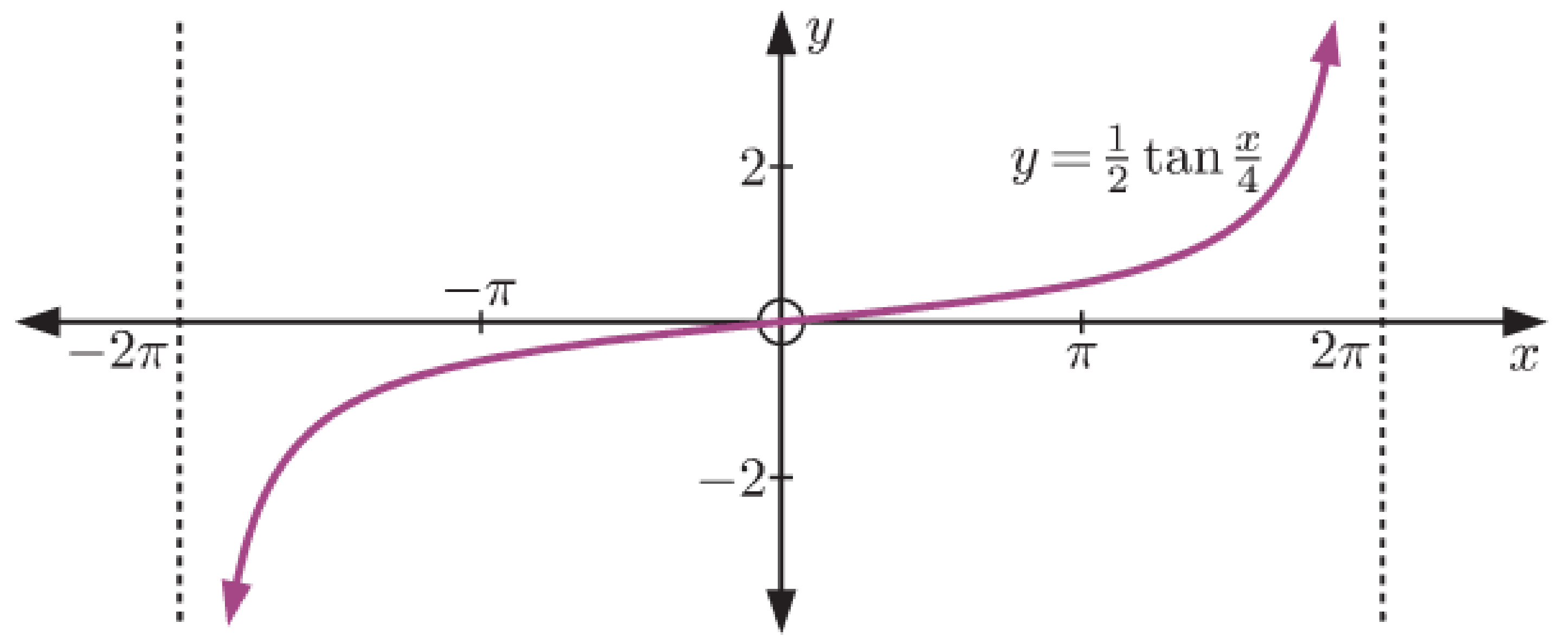
b **i** $\frac{2\pi}{3} + k\pi$, $k \in \mathbb{Z}$ **ii** $x = \frac{\pi}{6} + k\pi$, $k \in \mathbb{Z}$

c **i** $\frac{\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$ **ii** $x = \frac{7\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$

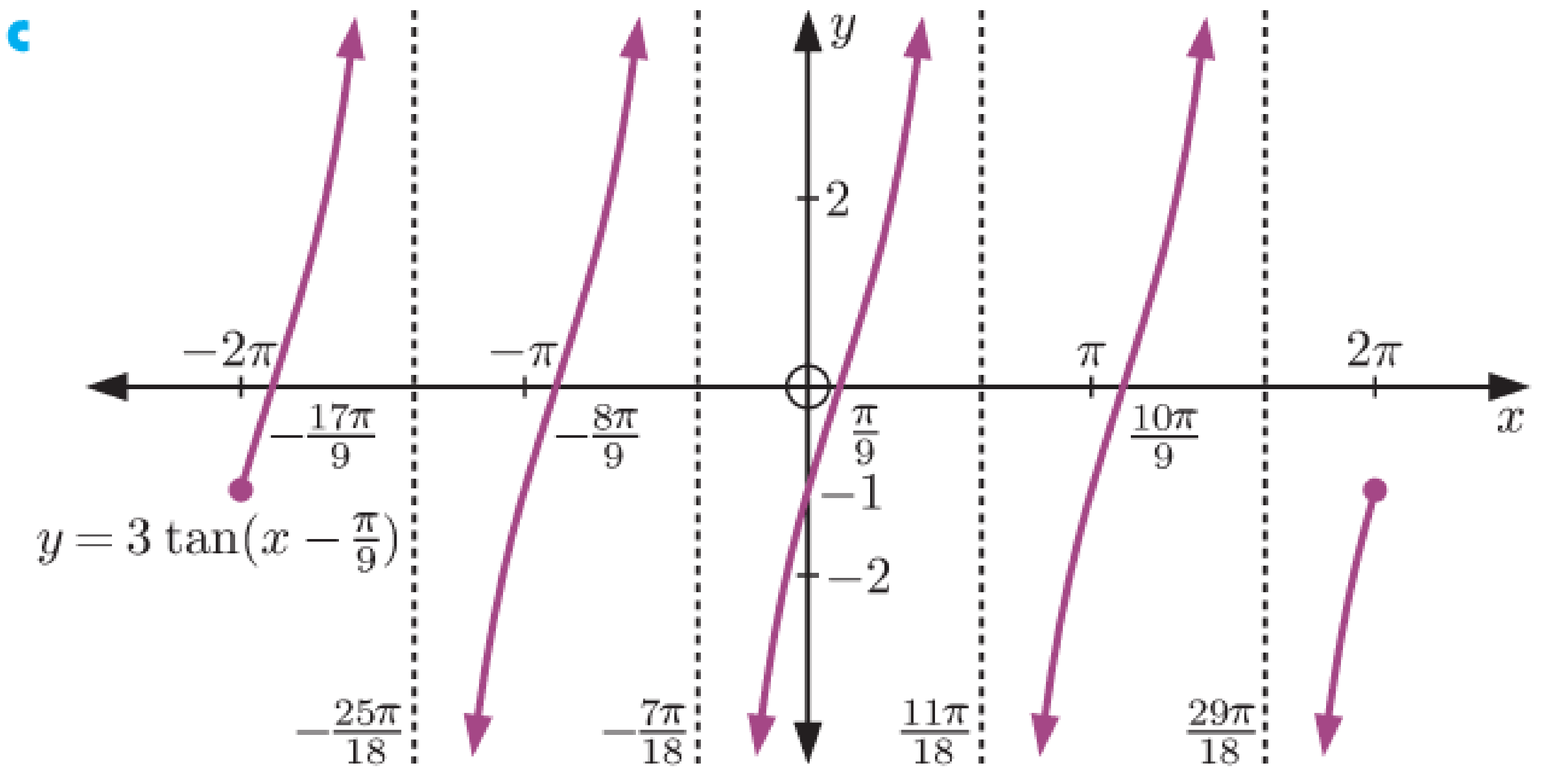
4 **a**



b



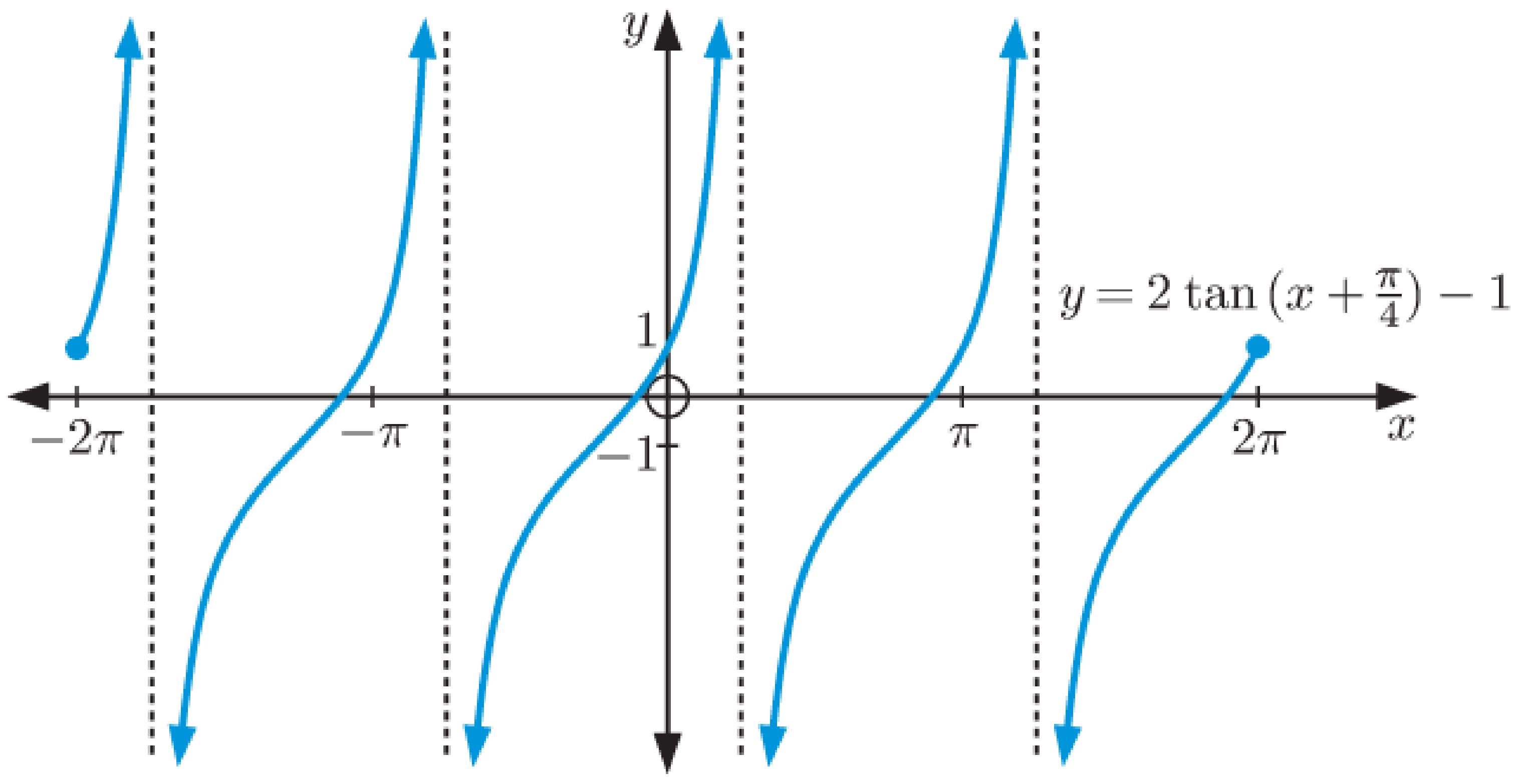
c



5 $p = \frac{1}{2}$, $q = 1$ **6** $a = \frac{3}{2}$, $b = -\frac{2\pi}{15} + \frac{2k\pi}{3}$, $k \in \mathbb{Z}$

7 **a** A vertical stretch with scale factor 2, then a translation $\frac{\pi}{4}$ units left and 1 unit downwards.

b



8 **Hint:** The function is undefined when $b(x - c) = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.

9 **a** **i** $(f \circ g)(x) = \tan\left(2x - \frac{\pi}{2}\right)$

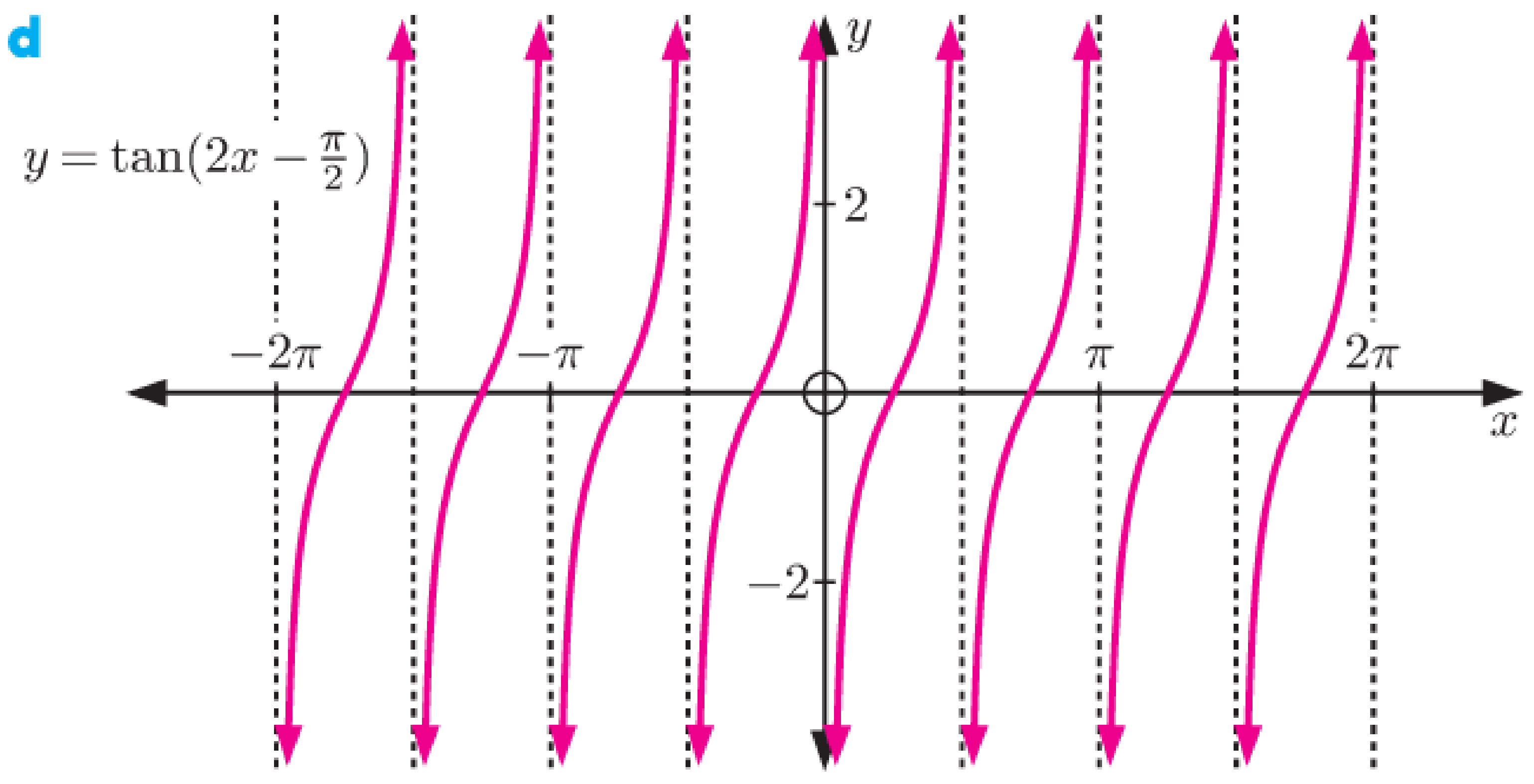
ii $(g \circ f)(x) = 2\tan x - \frac{\pi}{2}$

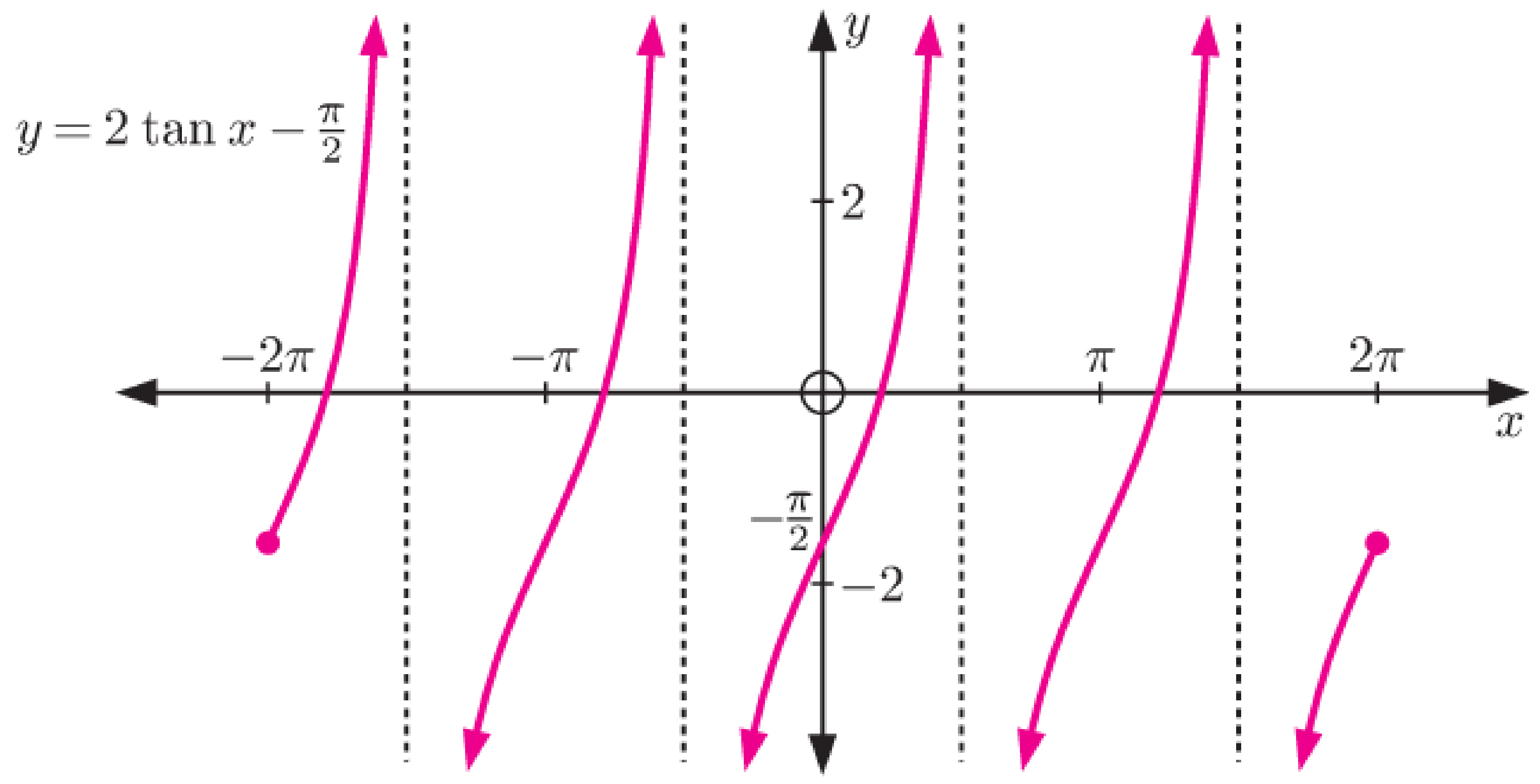
b **i** $\frac{1}{\sqrt{3}}$ **ii** $-\frac{\pi}{2}$

c **i** period $\frac{\pi}{2}$, vertical asymptotes $x = \dfrac{k\pi}{2}$, $k \in \mathbb{Z}$

ii period π, vertical asymptotes $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

d





EXERCISE 17G.1

1 **a** $x \approx 0.3, 2.8, 6.6, 9.1, 12.9$ **b** $x \approx 5.9, 9.8, 12.2$
c $x \approx 0.3, 2.8$ **d** $x \approx 3.8, 5.6$

2 **a** $x \approx 1.2, 5.1, 7.4$ **b** $x \approx 4.4, 8.2, 10.7$
c $x \approx 5.2$ **d** $x \approx 2.5, 3.8$

3 **a** $x \approx 0.4, 1.2, 3.5, 4.3, 6.7, 7.5, 9.8, 10.6, 13.0, 13.7$
b $x \approx 1.7, 3.0, 4.9, 6.1, 8.0, 9.3, 11.1, 12.4, 14.3, 15.6$
c $x \approx 3.2, 4.6$ **d** $x \approx 1.6, 3.1, 4.8, 6.2$

4 **a** $x \approx 1.1, 4.2, 7.4$ **b** $x \approx 2.2, 5.3$
c $x \approx 1.3, 4.4$ **d** $x \approx 2.0$

EXERCISE 17G.2

1 **a** $x \approx 0.446, 2.70, 6.73, 8.98$
b $x \approx 2.52, 3.76, 8.80, 10.0$
c $x \approx 0.588, 3.73, 6.87, 10.0$

2 **a** $x \approx -0.644, 0.644$ **b** $x \approx -4.56, -1.42, 1.72, 4.87$
c $x \approx -2.76, -0.384, 3.53$

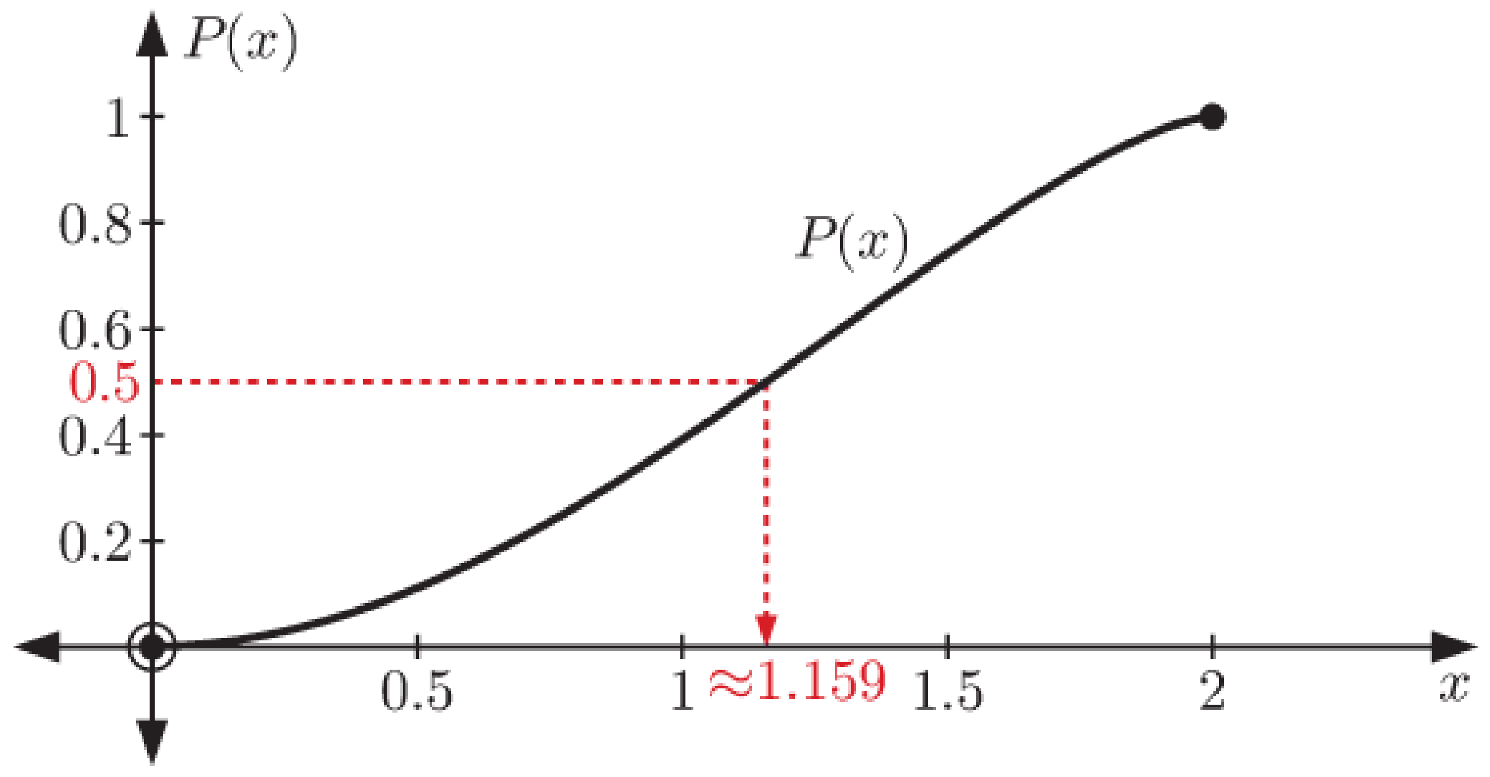
3 **a** $x \approx 1.08, 4.35$ **b** $x \approx 0.666, 2.48$

4 $x \approx -0.951, 0.234, 5.98$

5 **a** **Note:** The function $P(x)$ can be written in many different ways.

$$P(x) = 1 + \frac{x^2 - 2}{\pi} \arccos \frac{x}{2} - \frac{x}{2\pi}\sqrt{4 - x^2}$$

b



c $x \approx 1.159$

EXERCISE 17G.3

1 **a** $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ **b** $x = \frac{5\pi}{4}$ or $\frac{7\pi}{4}$ **c** $x = \frac{\pi}{6}$ or $\frac{7\pi}{6}$
d $x = \frac{3\pi}{2}$ **e** $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ **f** $x = 0, \pi$, or 2π

2 **a** $x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$ **b** $x = \pi$ **c** $x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$

3 **a** $x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$, or $\frac{10\pi}{3}$ **b** $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$, or $\frac{11\pi}{4}$
c $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$, or $\frac{13\pi}{4}$

4 **a** $x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$
b $x = -\frac{5\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}$, or $\frac{5\pi}{4}$
c $x = -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}$, or $\frac{7\pi}{4}$

5 **a** $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$, or $\frac{11\pi}{6}$ **b** $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$
c $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$

6 **a** $0 \leqslant 2x \leqslant 4\pi$ **b** $0 \leqslant \dfrac{x}{4} \leqslant \frac{\pi}{2}$
c $\frac{\pi}{2} \leqslant x + \frac{\pi}{2} \leqslant \frac{5\pi}{2}$ **d** $-\frac{\pi}{6} \leqslant x - \frac{\pi}{6} \leqslant \frac{11\pi}{6}$
e $-\frac{\pi}{2} \leqslant 2\left(x - \frac{\pi}{4}\right) \leqslant \frac{7\pi}{2}$ **f** $-2\pi \leqslant -x \leqslant 0$

7 **a** $-3\pi \leqslant 3x \leqslant 3\pi$ **b** $-\frac{\pi}{4} \leqslant \dfrac{x}{4} \leqslant \frac{\pi}{4}$
c $-\frac{3\pi}{2} \leqslant x - \frac{\pi}{2} \leqslant \frac{\pi}{2}$ **d** $-\frac{3\pi}{2} \leqslant 2x + \frac{\pi}{2} \leqslant \frac{5\pi}{2}$
e $-2\pi \leqslant -2x \leqslant 2\pi$ **f** $0 \leqslant \pi - x \leqslant 2\pi$

8 **a** $x = \frac{\pi}{3}, \frac{5\pi}{3}$, or $\frac{7\pi}{3}$
b $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$, or $\frac{17\pi}{6}$
c $x = 0, \frac{4\pi}{3}$, or 2π

9 **a** $x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}$, or $\frac{23\pi}{12}$
b $x = \frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}$, or $\frac{35\pi}{18}$
c $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}$, or $\frac{5\pi}{3}$ **d** $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$
e $x = \frac{4\pi}{3}$ **f** $x = \frac{3\pi}{4}$

10 **a** $x = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \frac{14\pi}{9}$, $\frac{16\pi}{9}$, or $\frac{17\pi}{9}$
b $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, or $\frac{7\pi}{4}$ **c** $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

11 **a** $x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$ **b** $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}$, or $\frac{7\pi}{4}$
c $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}$, or $\frac{5\pi}{3}$

12 **a** $x = -\frac{5\pi}{3}, -\pi, \frac{\pi}{3}$, or π **b** $x = 0, \frac{3\pi}{2}$, or 2π
c $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, or π **d** $x = 0, \frac{\pi}{6}, \pi, \frac{7\pi}{6}$, or 2π

13 **a** $b = \frac{1}{3}$, $d = 2$ **b** $a = -2$, $b = \frac{1}{2}$
c $b = 4$, $d = -1$ **d** $b = \frac{1}{2}$, $d = -4$

14 $x = \frac{\pi}{3}$ or $\frac{4\pi}{3}$
a $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$
b $x = \frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12}, \frac{5\pi}{6}, \frac{13\pi}{12}, \frac{4\pi}{3}, \frac{19\pi}{12}$, or $\frac{11\pi}{6}$
c $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$, or $\frac{5\pi}{3}$

15 **a** $x = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi$, or 2π **b** $x = \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}$, or $\frac{5\pi}{3}$
c $x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$, or 2π **d** $x = \frac{7\pi}{6}, \frac{3\pi}{2}$, or $\frac{11\pi}{6}$

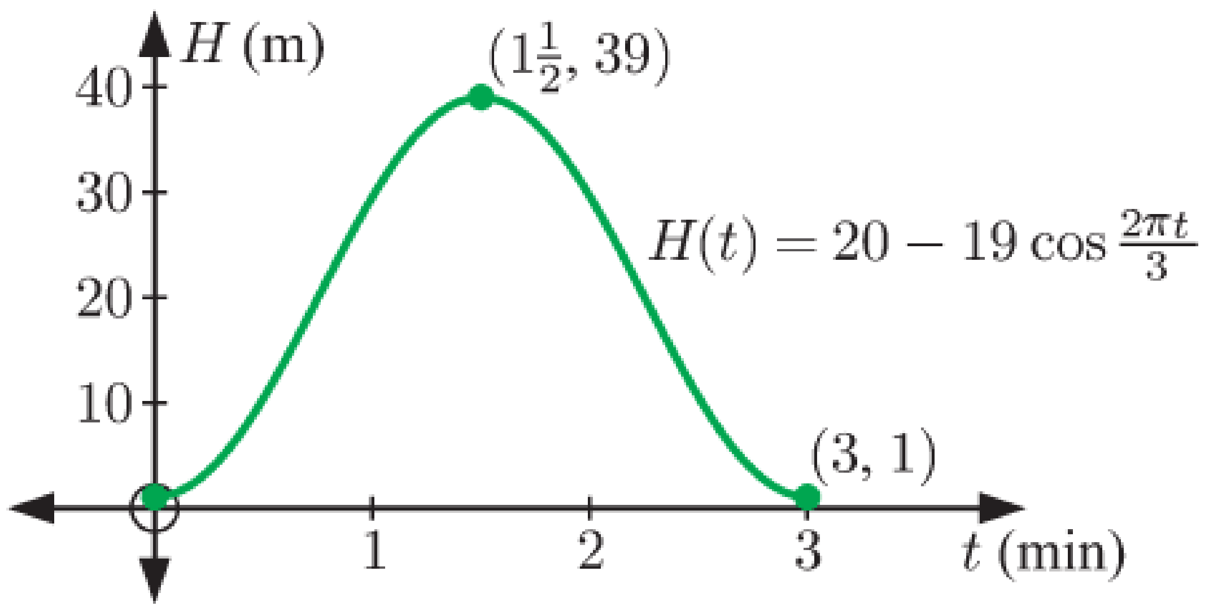
EXERCISE 17H

1 **a** **i** 7500 grasshoppers **ii** $\approx$ 10 300 grasshoppers
b 10 500 grasshoppers, when $t = 4$ weeks
c **i** at $t = 1\frac{1}{3}$ weeks and $6\frac{2}{3}$ weeks
ii at $t = 9\frac{1}{3}$ weeks

d $2.51 \leqslant t \leqslant 5.49$

2 **a** 1 m above ground **b** at $t = 1\frac{1}{2}$ min **c** 3 min

d



e $0.570 \leqslant t \leqslant 2.43$ min

3 **a** 400 water buffalo

b **i** 577 water buffalo **ii** 400 water buffalo

c 650, which is the maximum population.

d 150, after 3 years **e** $t \approx 0.262$ years

4 **a** **i** true **ii** true **b** 116.8 cents L^{-1}

c on the 5th, 11th, 19th, and 25th days

d 98.6 cents L^{-1} on the 1st and 15th days

5 **a** $H(t) = 3\cos\left(\frac{\pi}{2}t\right) + 4$ **b** $t \approx 1.46$ s

REVIEW SET 17A

1 **a** not periodic **b** periodic

2 **a** minimum $= 0$, maximum $= 2$

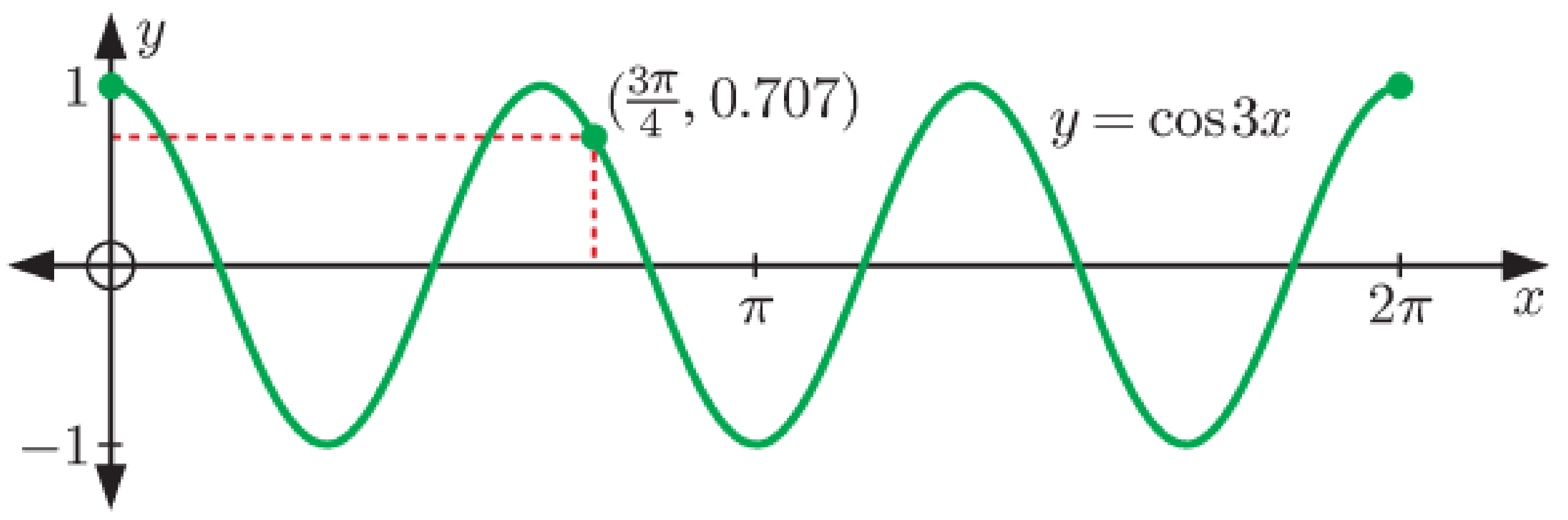
b minimum $= -2$, maximum $= 2$

3 **a** 10π **b** $\frac{\pi}{2}$ **c** 4π **d** $\frac{\pi}{3}$

4

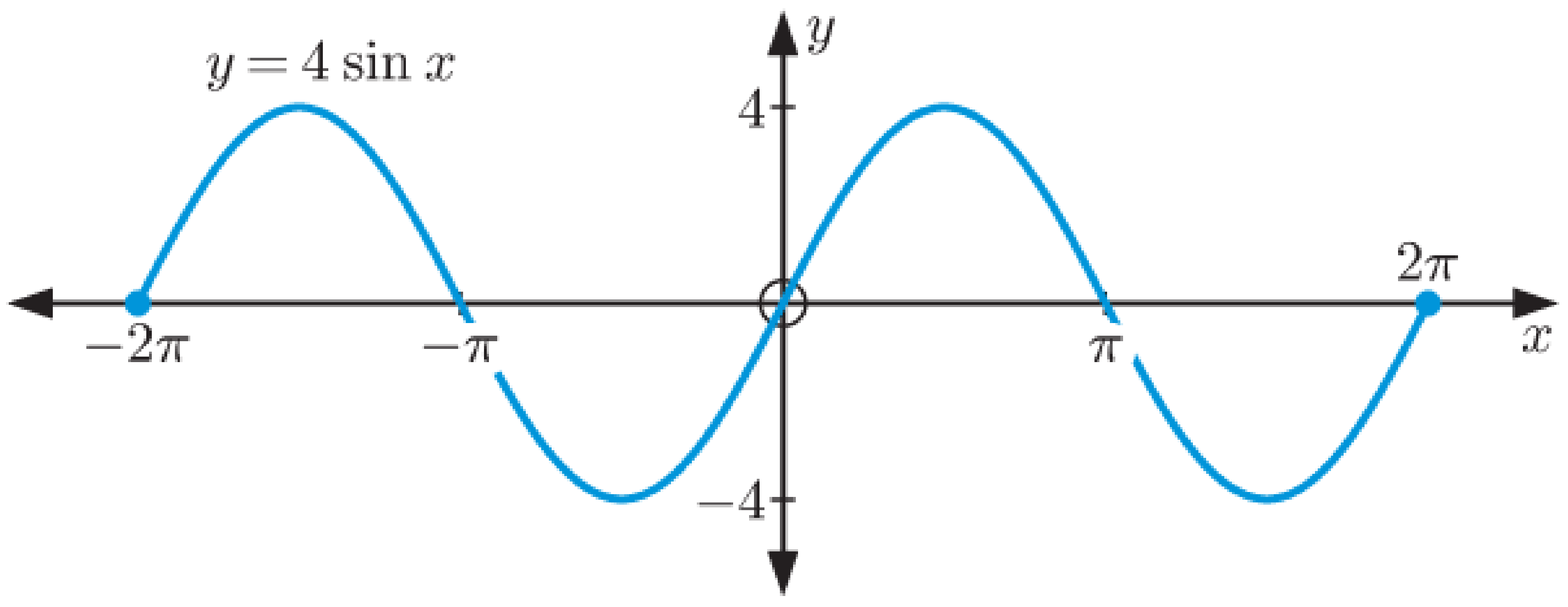
Function	*Period*	*Amplitude*	*Range*
$y = -3\sin\frac{x}{4} + 1$	8π	3	$-2 \leqslant y \leqslant 4$
$y = 3\cos\pi x$	2	3	$-3 \leqslant y \leqslant 3$

5 **a**

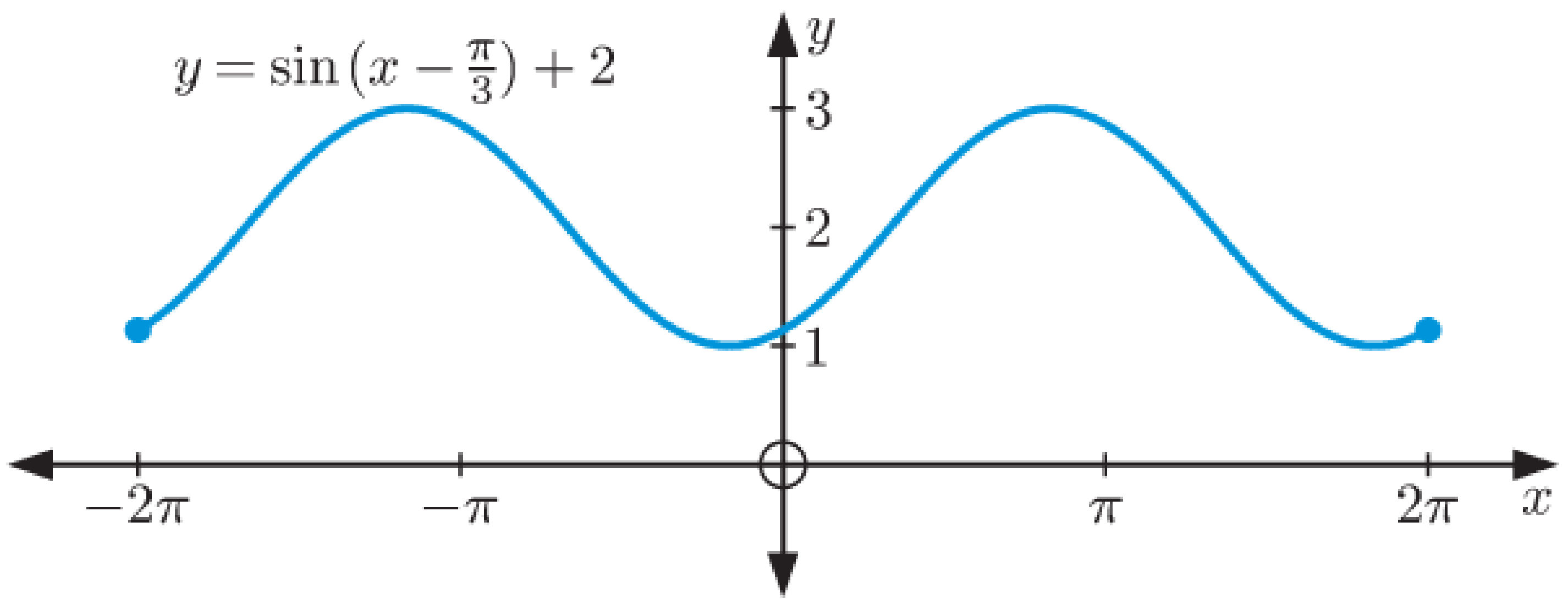


b $y = \frac{1}{\sqrt{2}} \approx 0.707$

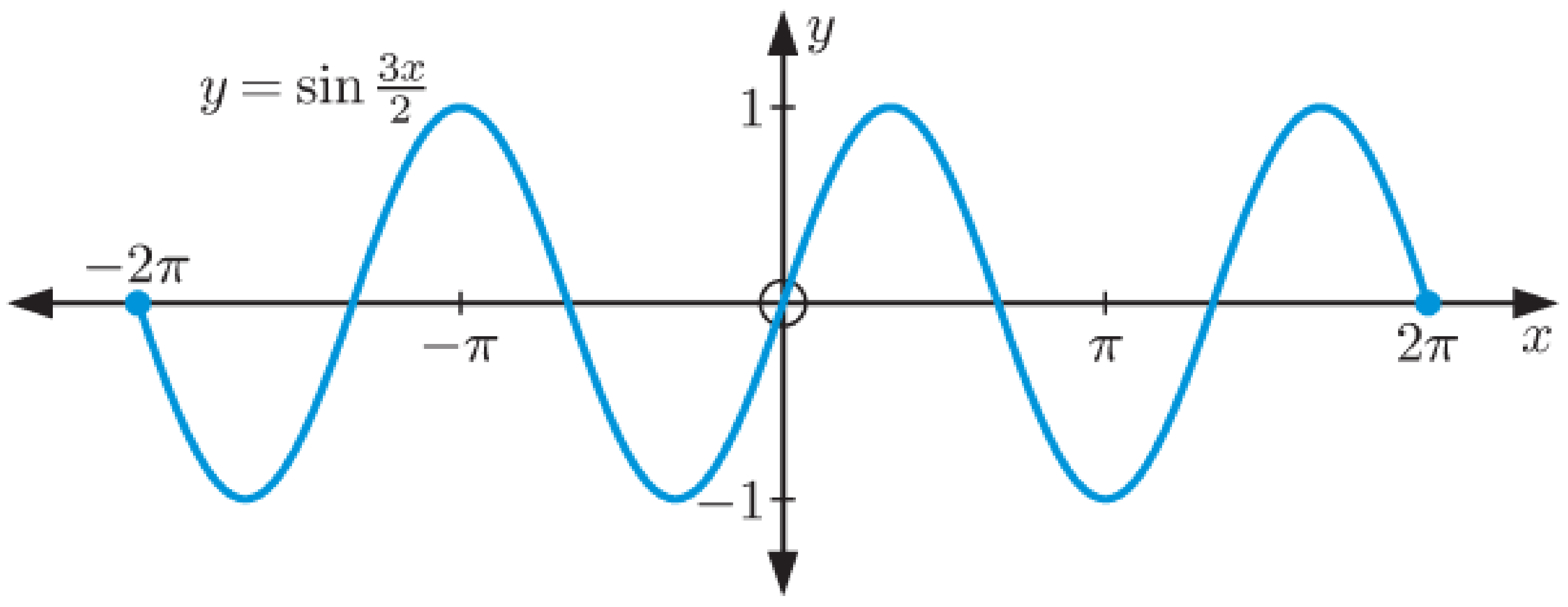
6 **a**



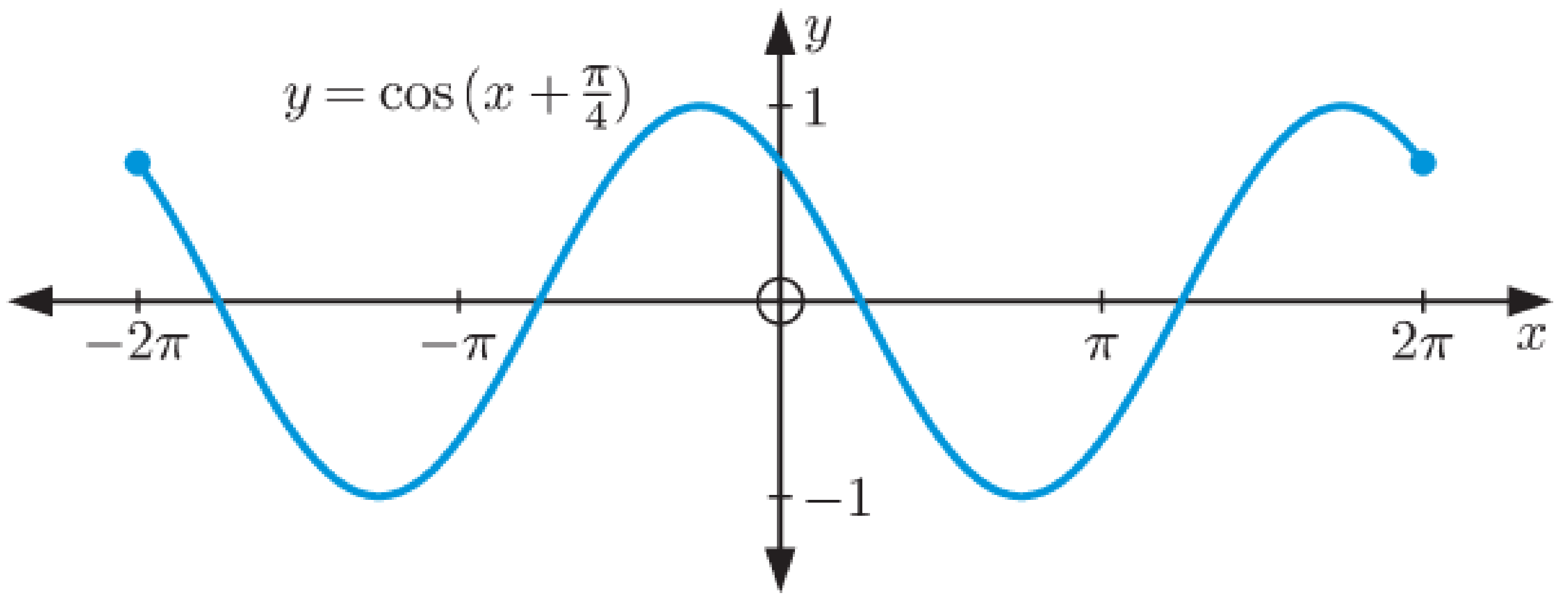
b



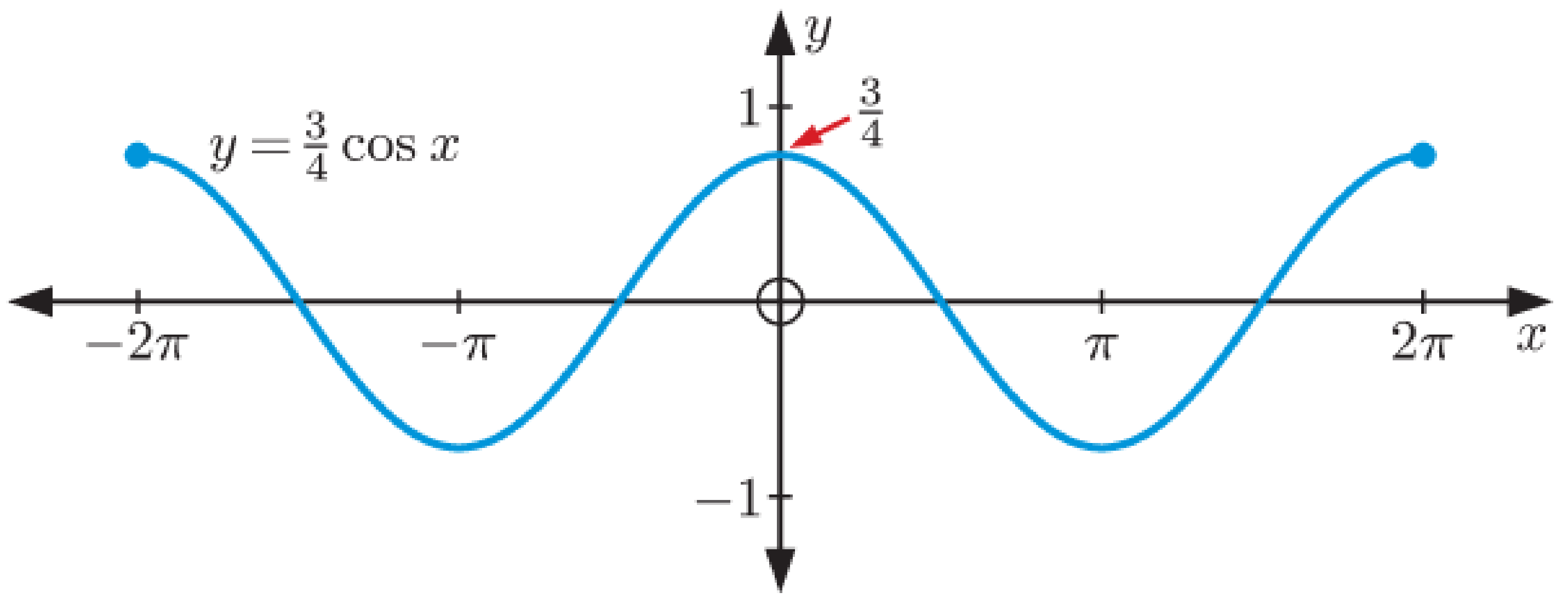
c



d



e



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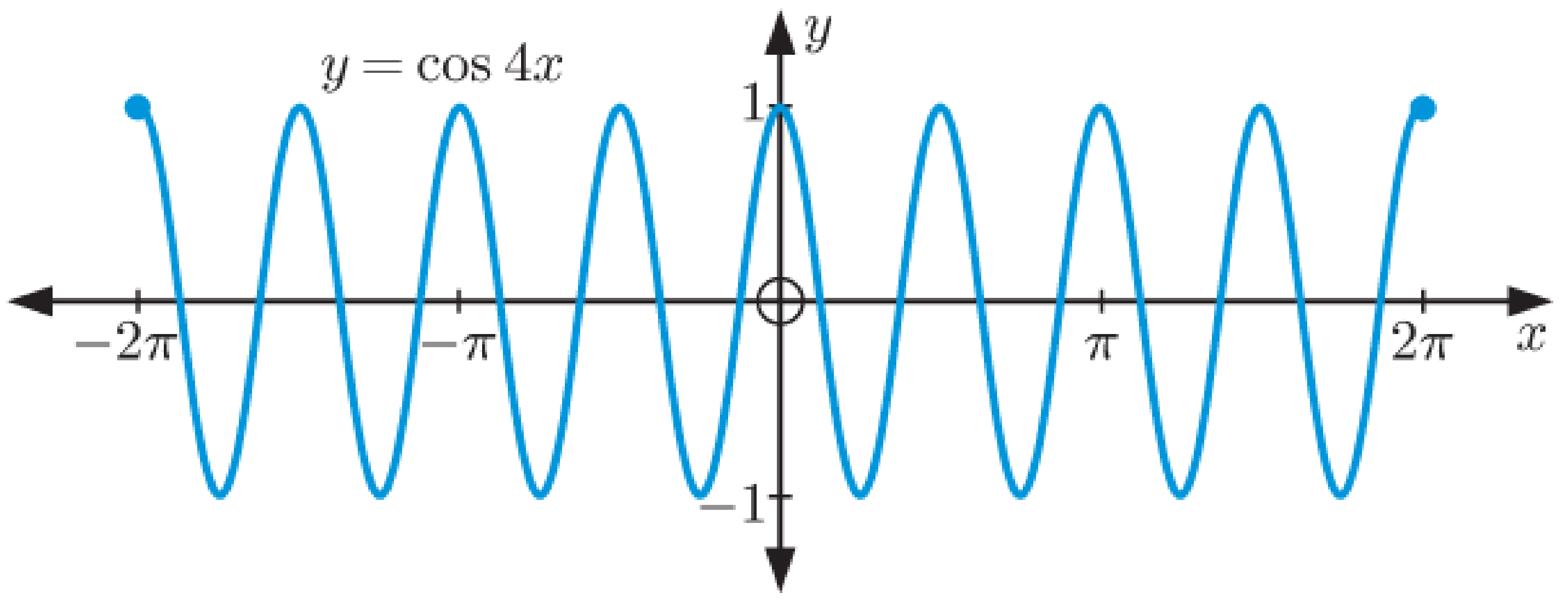


7 **a** A vertical stretch with scale factor 3, then a horizontal stretch with scale factor $\frac{1}{2}$.

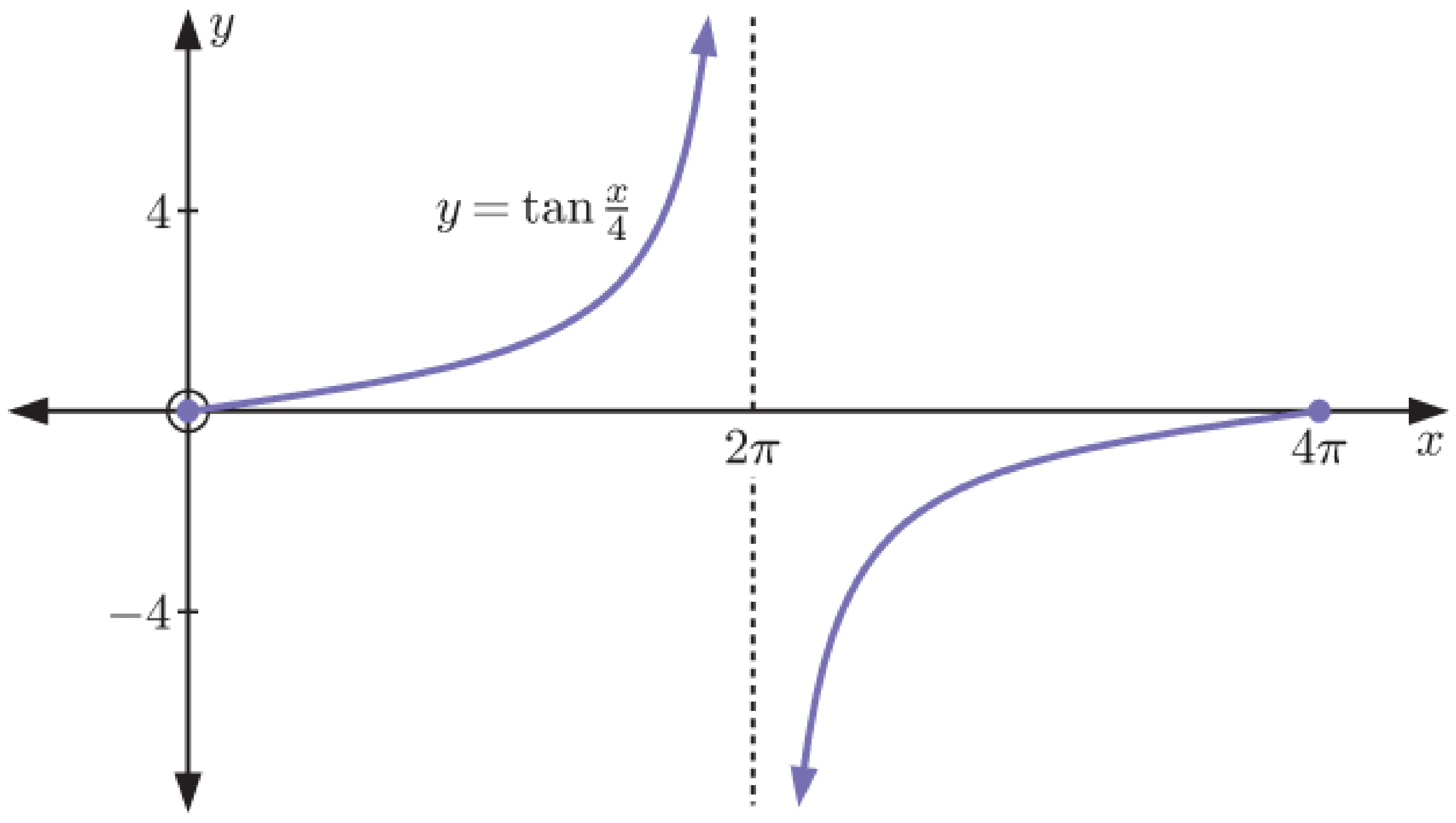
b A translation $\frac{\pi}{3}$ units right and 1 unit downwards.

c A reflection in the x-axis, then a horizontal stretch with scale factor $\frac{1}{2}$.

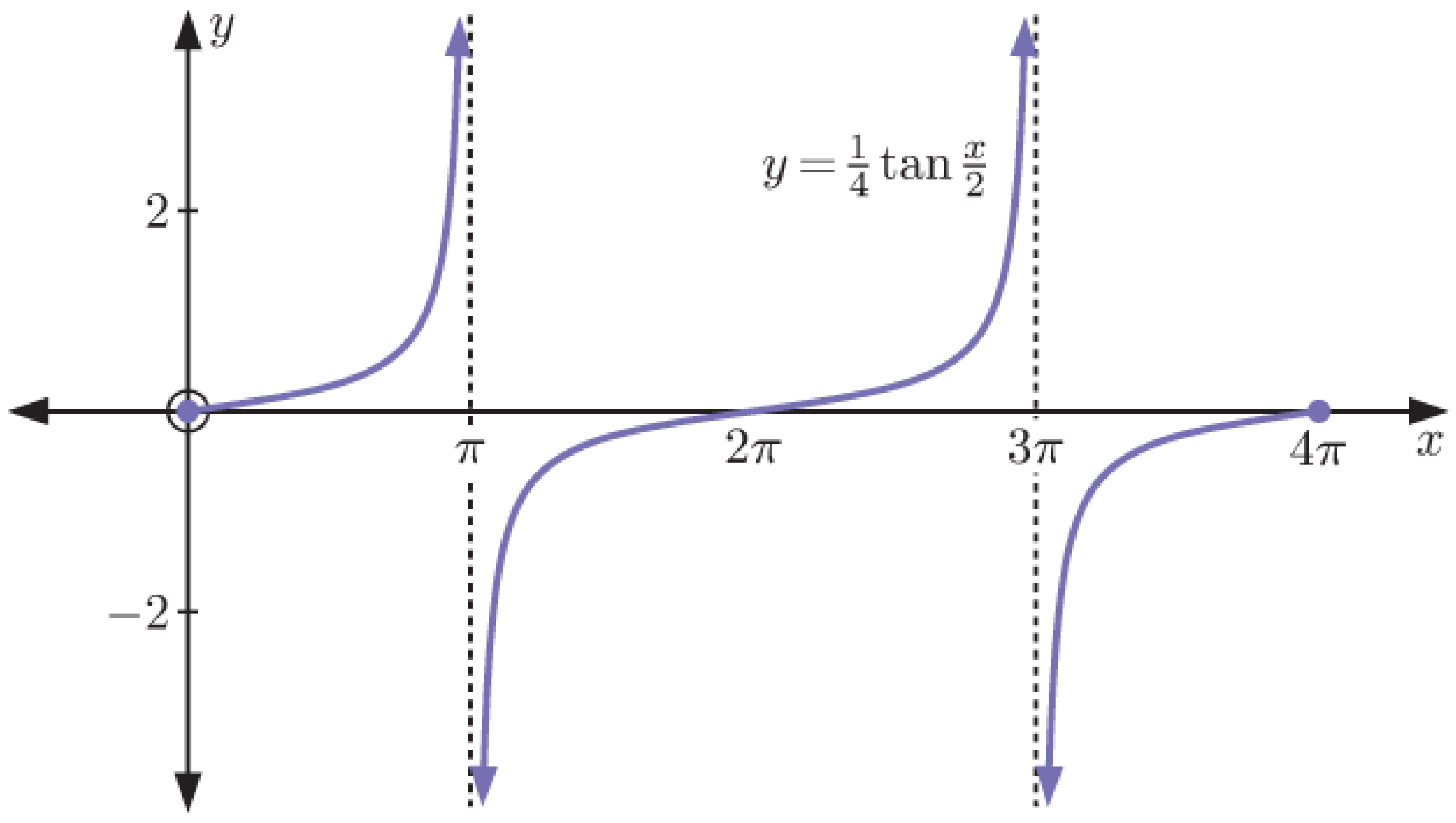
d A vertical stretch with scale factor 2, then a horizontal stretch with scale factor 2, then a translation $\frac{\pi}{2}$ units right and $\frac{1}{2}$ unit upwards.

8 **a** $y = -4\cos 2x$ **b** $y = \cos\frac{\pi x}{4} + 2$

9 **a**



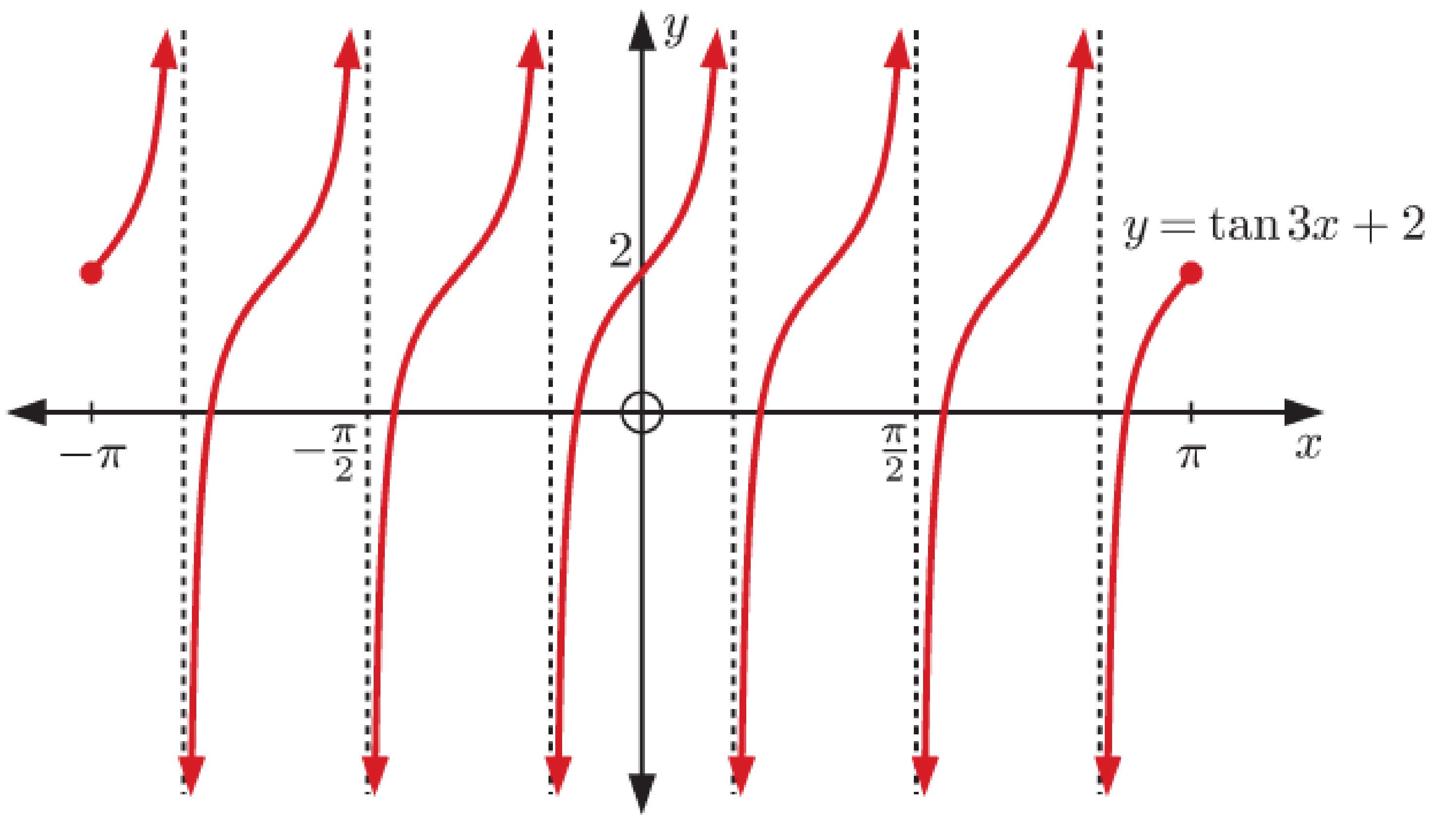
b



10 **a** A horizontal stretch with scale factor $\frac{1}{3}$, then a vertical translation 2 units upwards.

b $\frac{\pi}{3}$

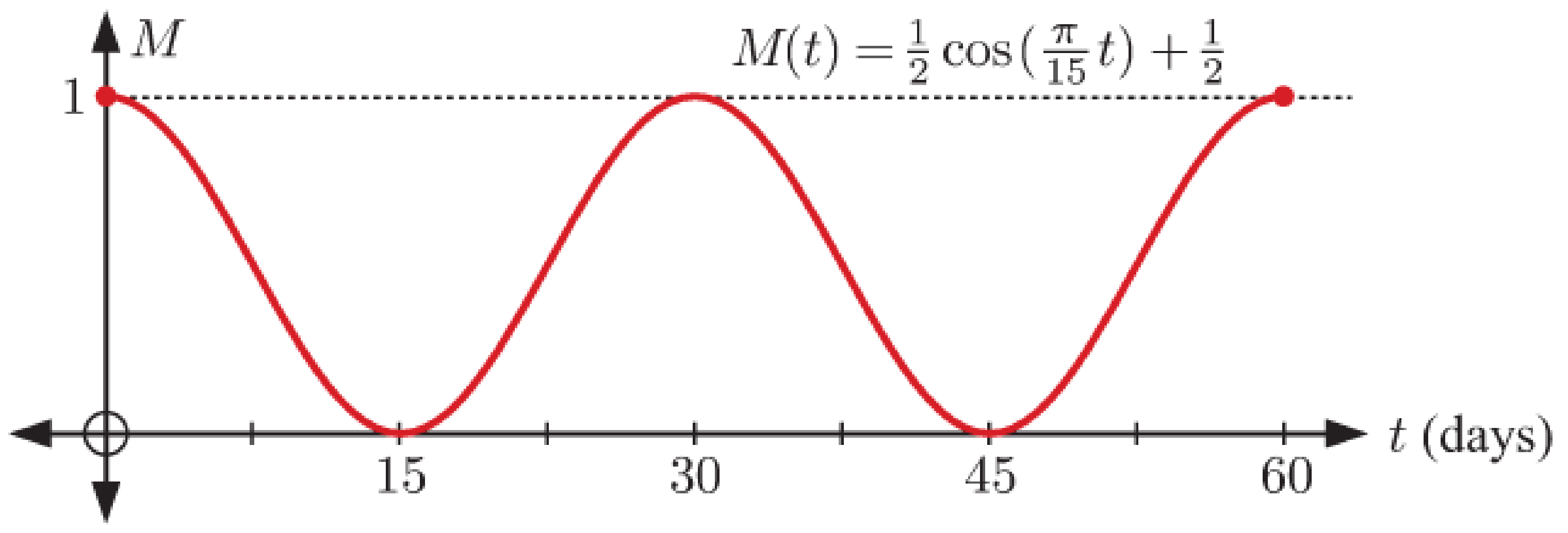
c



11 **a** $a = 7,\ b = \frac{\pi}{8},\ c = 1,\ d = 10$

b $g(x) = 14\sin\left(\frac{\pi}{8}(x-3)\right) + 14$

12 **a**

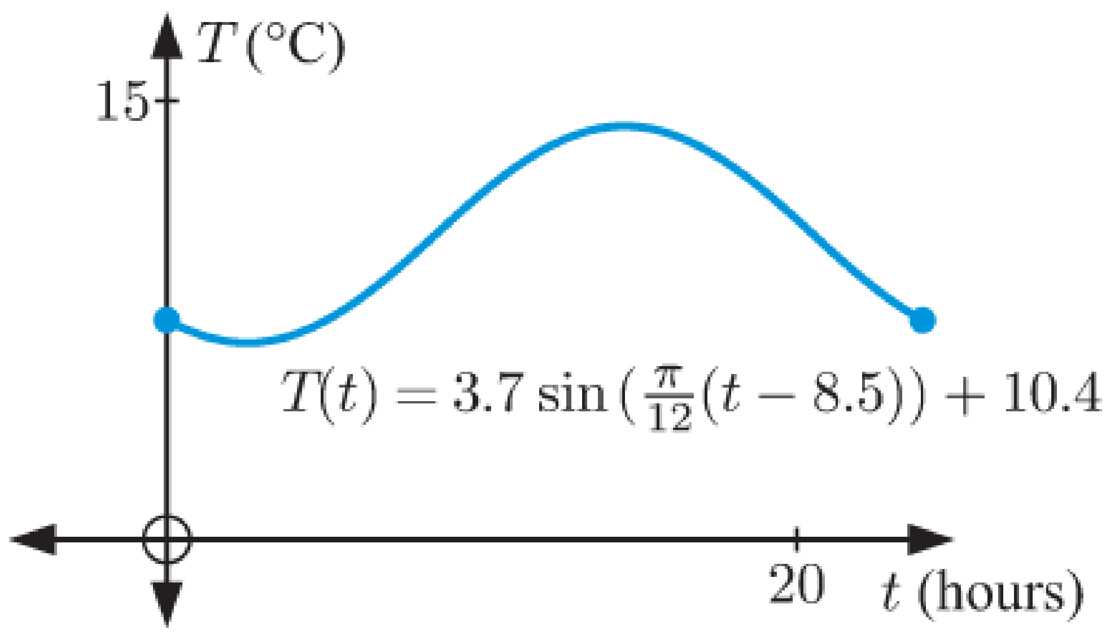


b **i** 0.75 **ii** 0.25 **iii** ≈ 0.835 **iv** ≈ 0.165

c once every 30 days **d** January 16, February 15

13 **a** $T(t) = 3.7\sin\left(\frac{\pi}{12}(t - 8.5)\right) + 10.4$ °C

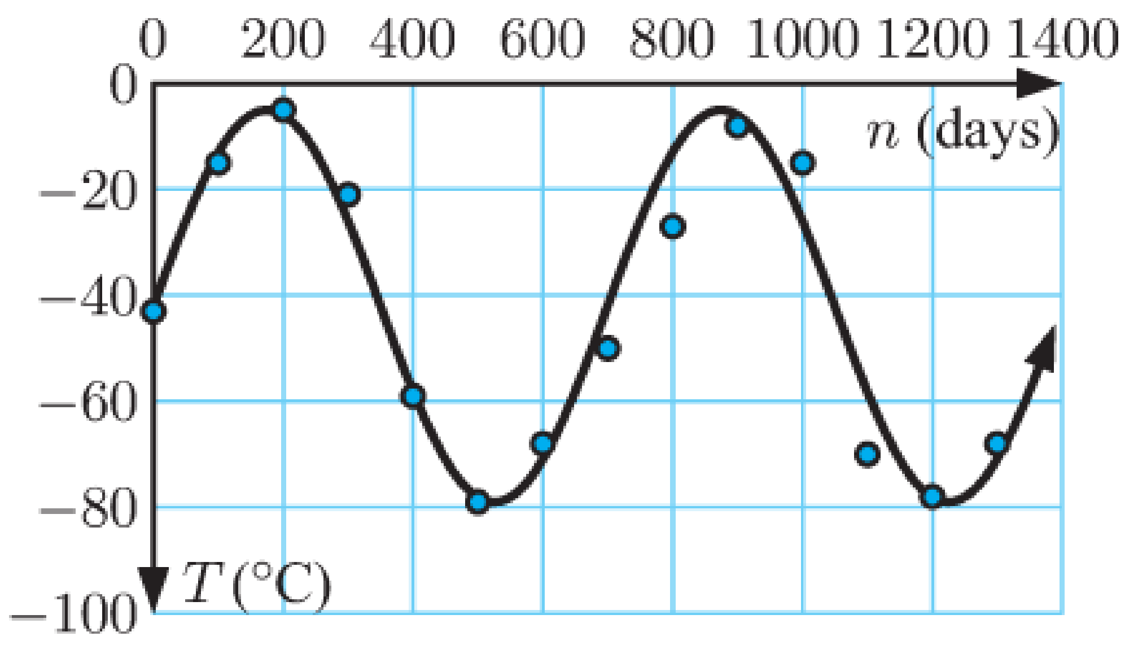
b



14 **a** maximum: -5°C, minimum: -79°C

b ≈ 700 Mars days **c** $T \approx 37\sin(0.008\,98n) - 42$

d



e Using technology,
$T \approx 36.5\sin(0.009\,01x - 0.0903) - 43.2$.
Our model fits the data well.

15 **a** $x \approx 2.0,\ 4.3,\ 8.3,\ 10.6$ **b** $x \approx 0.5,\ 5.8,\ 6.7,\ 12.1$

16 **a** $x \approx 0.392,\ 2.75,\ 6.68$ **b** $x \approx 5.42$

17 **a** $x \approx 1.12,\ 5.17,\ 7.40$ **b** $x \approx 0.184,\ 4.62$

18 **a** $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$ **b** $x = \frac{\pi}{4}$ or $\frac{7\pi}{4}$

c $x = \frac{\pi}{3},\ \frac{2\pi}{3},\ \frac{4\pi}{3}$, or $\frac{5\pi}{3}$

19 **a** $x = \frac{\pi}{8},\ \frac{3\pi}{8},\ \frac{5\pi}{8},\ \frac{7\pi}{8},\ \frac{9\pi}{8},\ \frac{11\pi}{8},\ \frac{13\pi}{8}$, or $\frac{15\pi}{8}$

b $x = \frac{3\pi}{2}$ **c** $x = \frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{7\pi}{6}$, or $\frac{11\pi}{6}$

20 **a** $x = 0,\ \frac{3\pi}{2},\ 2\pi,\ \frac{7\pi}{2}$, or 4π **b** $x = \frac{\pi}{6},\ \frac{2\pi}{3},\ \frac{7\pi}{6}$, or $\frac{5\pi}{3}$

21 **a** 5000 beetles **b** smallest 3000, largest 7000

c $0.5 < t < 2.5$ and $6.5 < t \leqslant 8$

REVIEW SET 17B

1 **a** The function repeats itself over and over in a horizontal direction, in intervals of length 8 units.

b **i** 8 **ii** 5 **iii** -1

2 **a** A translation $\frac{\pi}{3}$ units right and 1 unit upwards.

b A horizontal stretch with scale factor $\frac{1}{3}$.

3 **a** 6π **b** $\frac{\pi}{4}$

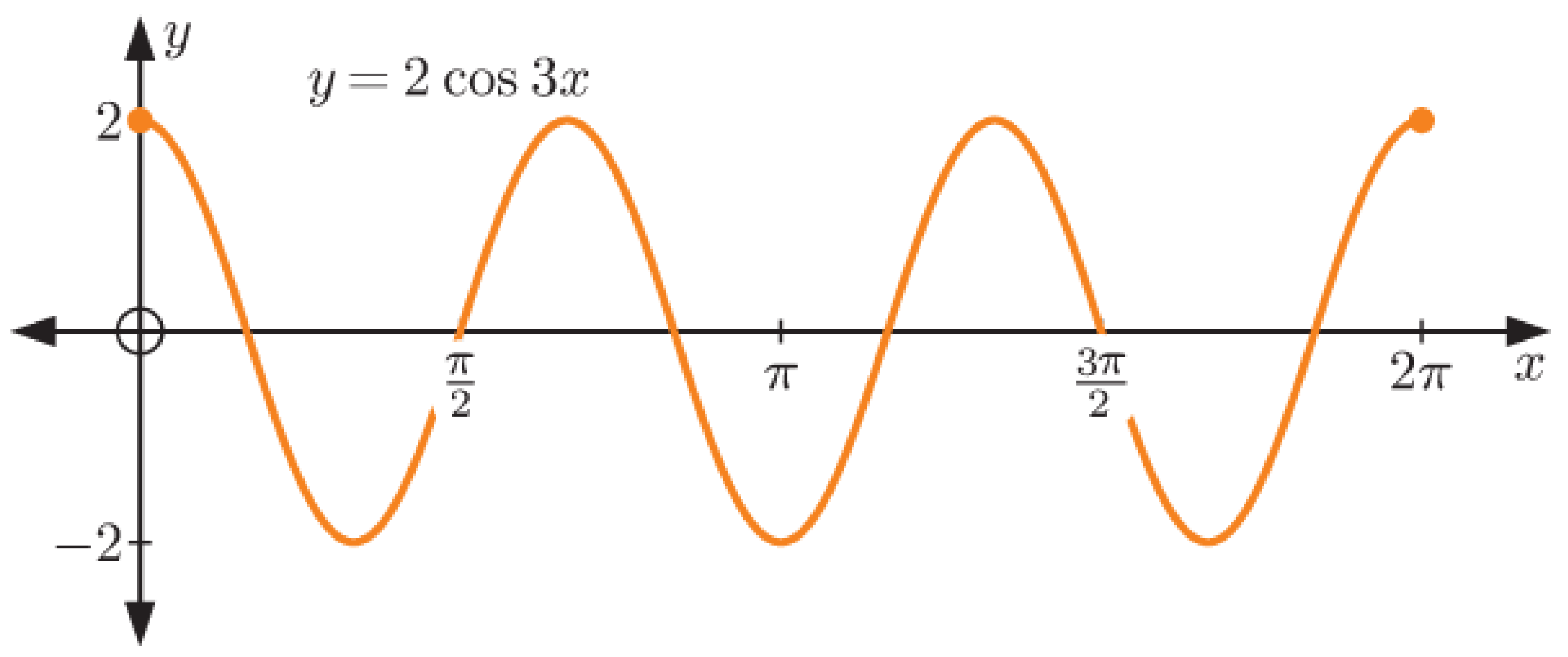
4 **a** $b = \frac{1}{3}$ **b** $b = 24$ **c** $b = \frac{2\pi}{9}$

5 **a** minimum $= -8$, maximum $= 2$

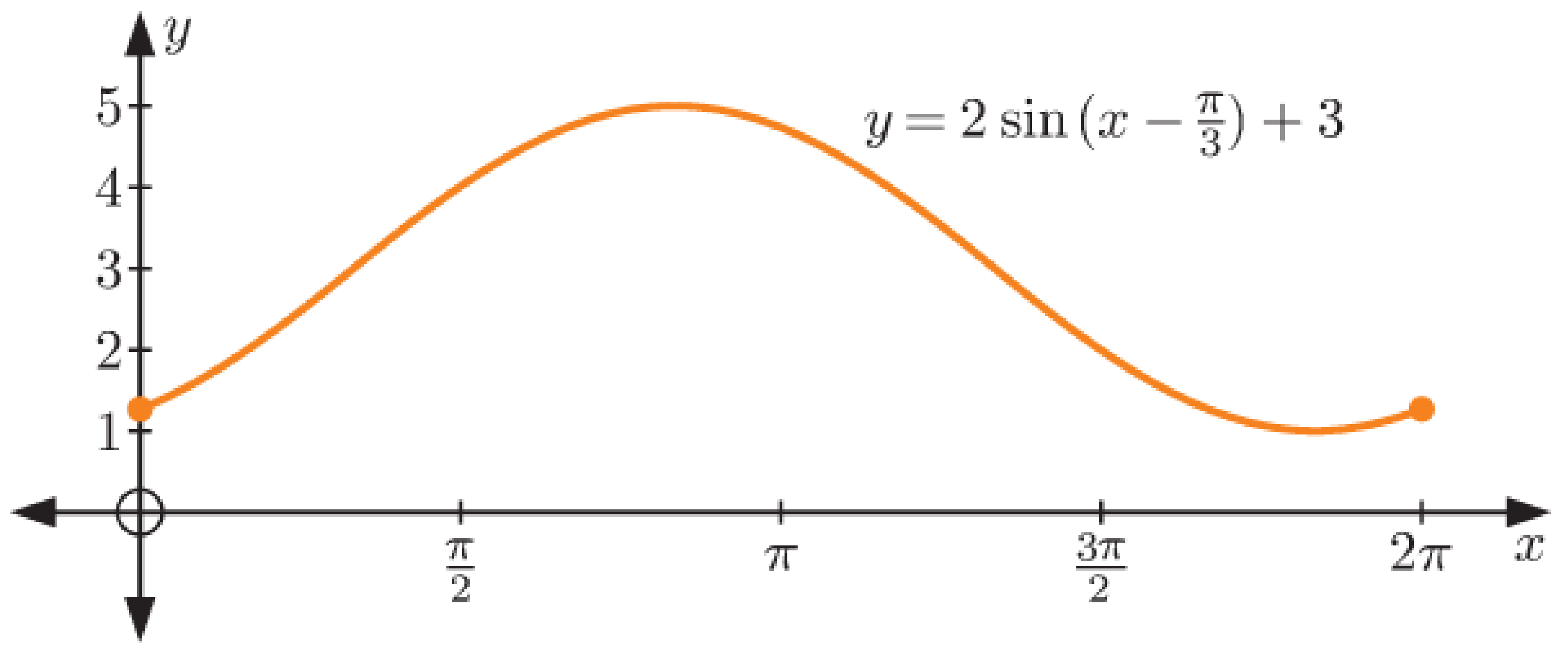
b minimum $= \frac{2}{3}$, maximum $= 1\frac{1}{3}$

6 **a** $y = 5$ **b** $y = -4$

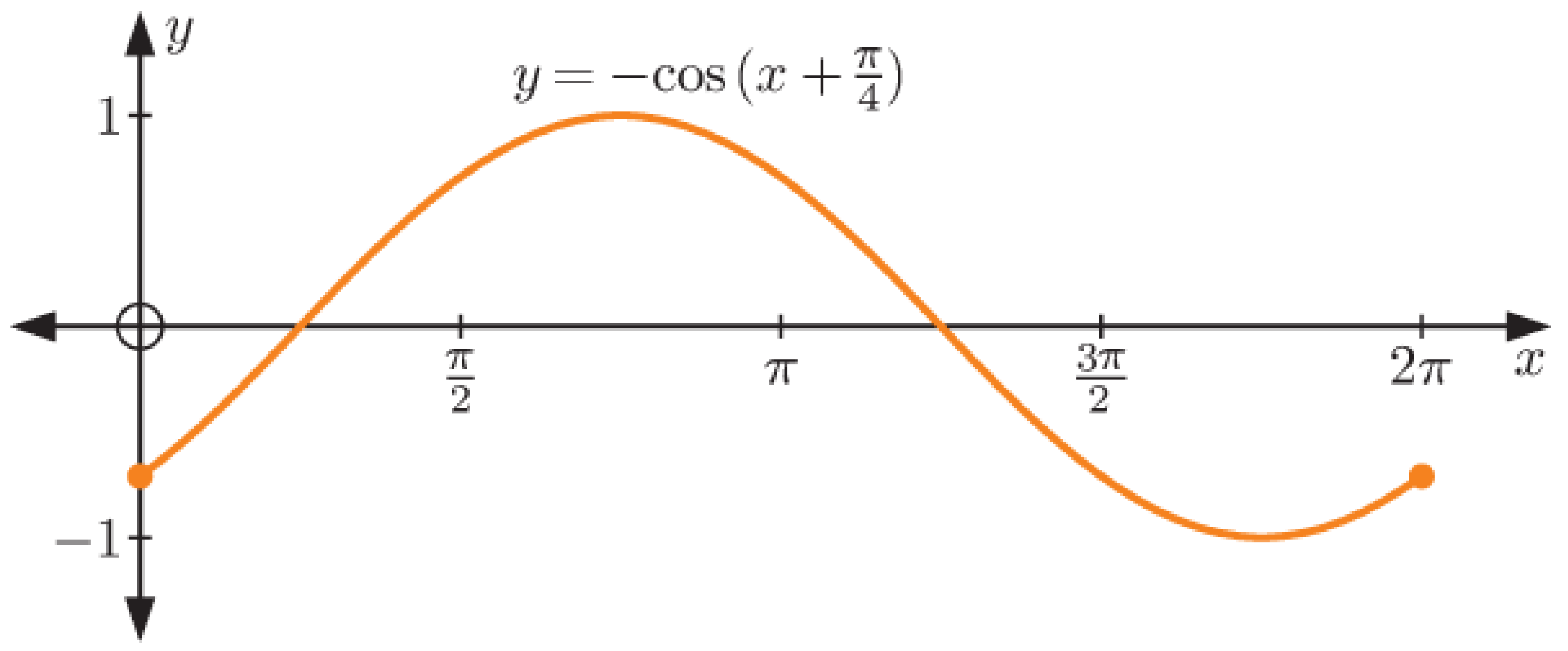
7 **a**



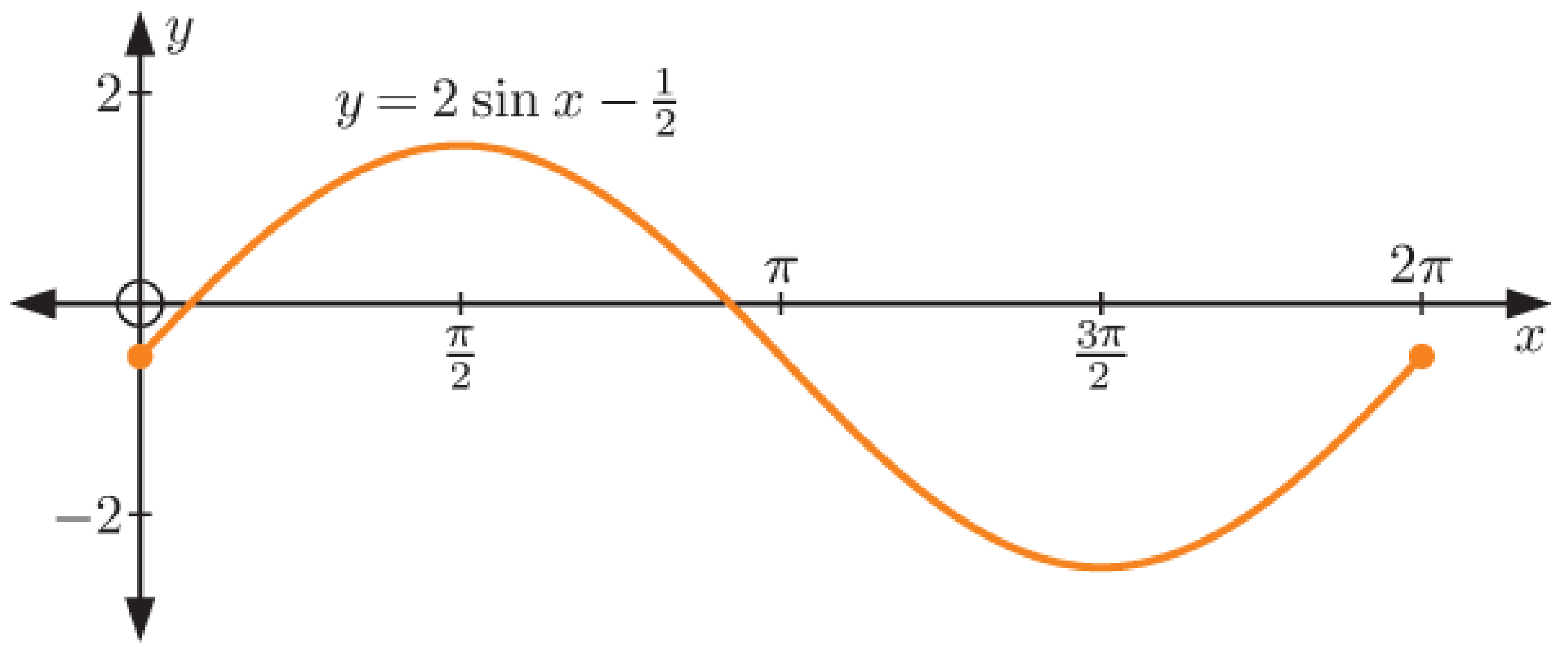
b



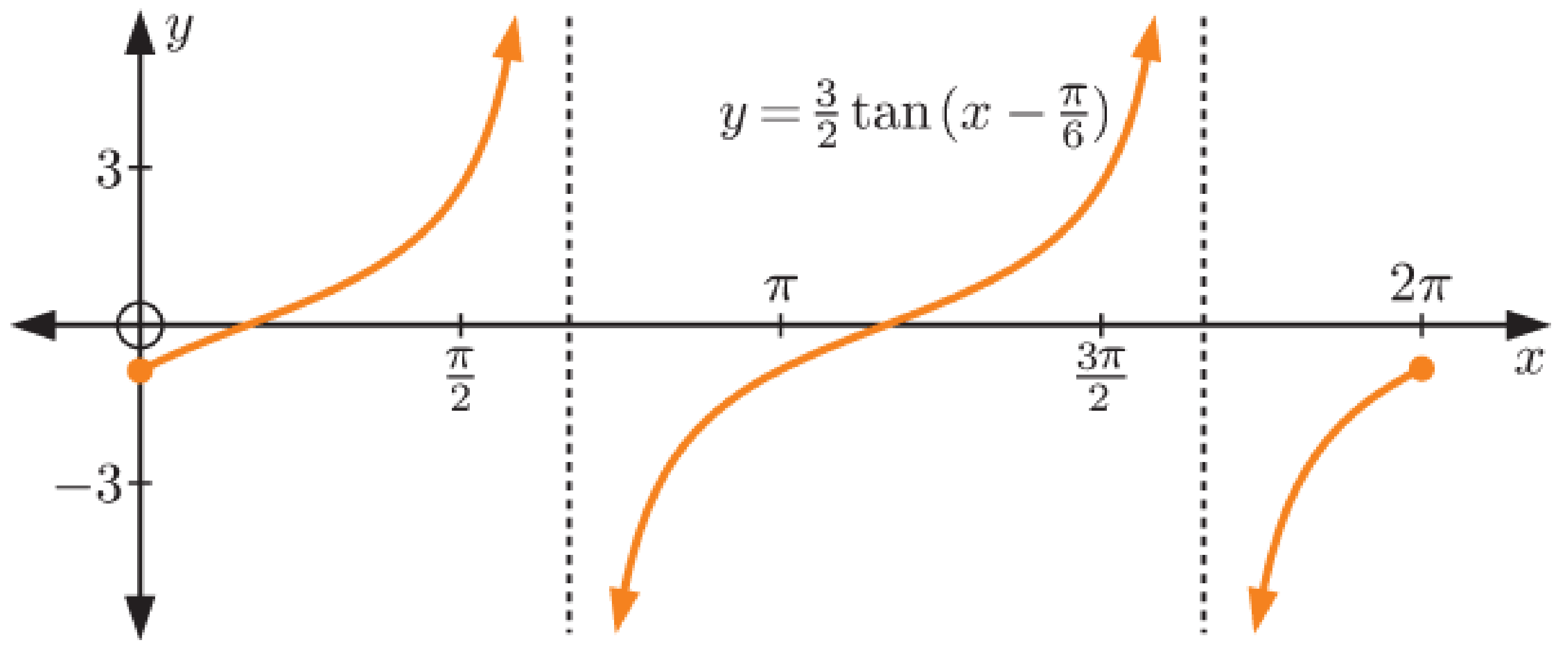
c



d



e



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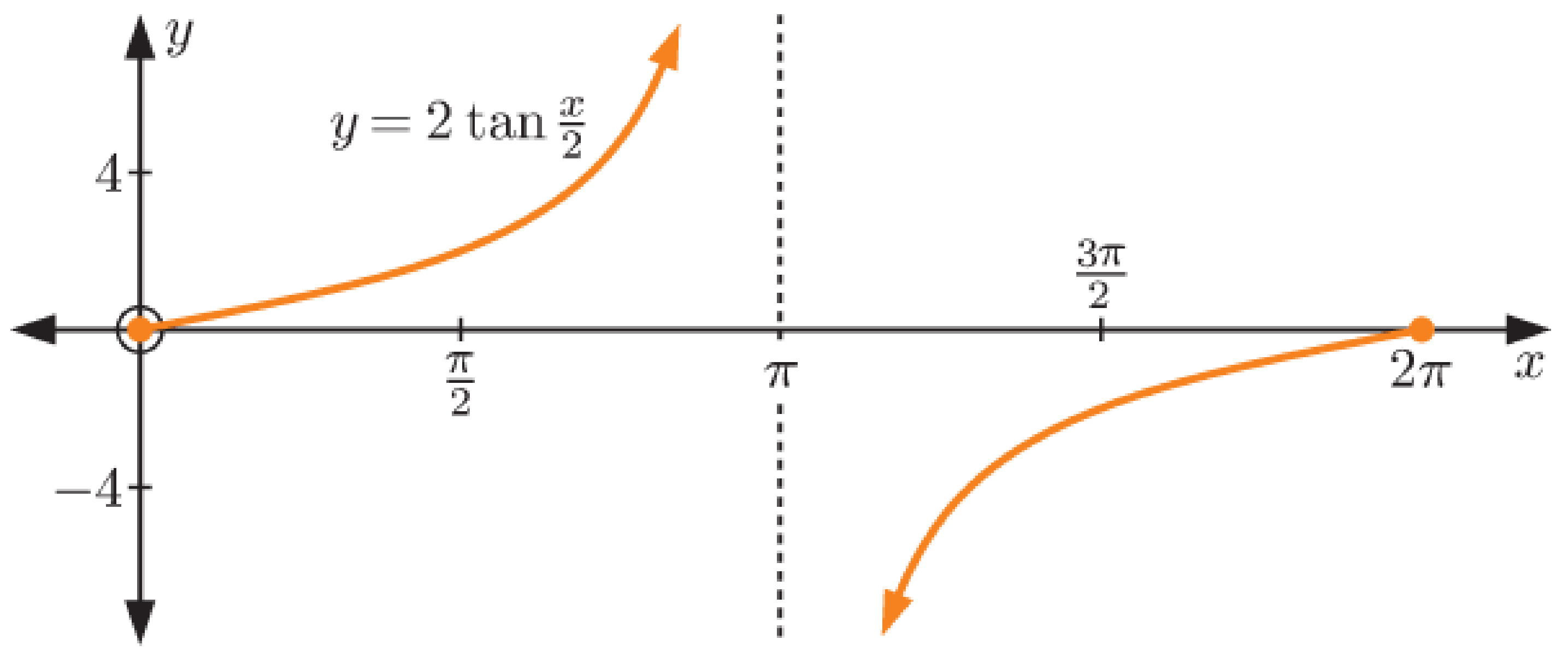
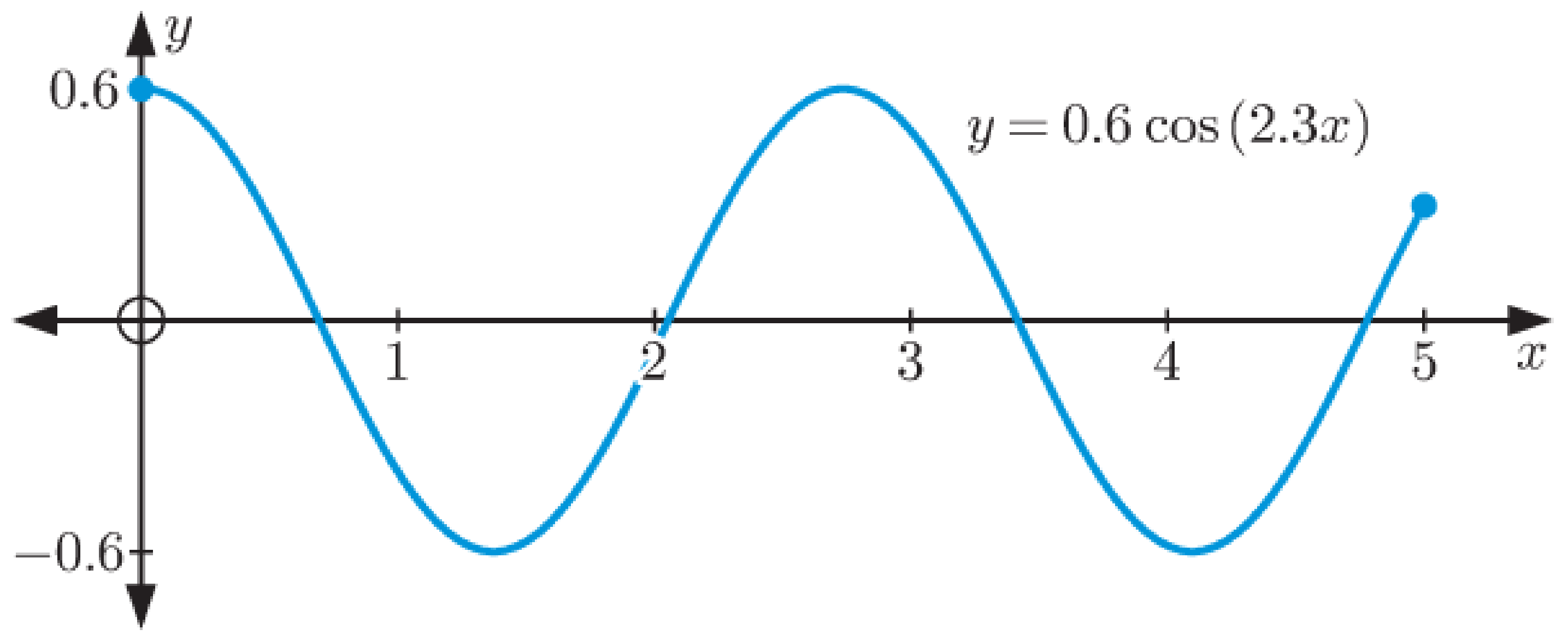


8 **a** $y = 4\sin x + 6$ **b** $y = 4\cos\left(x - \frac{\pi}{2}\right) + 6$

9

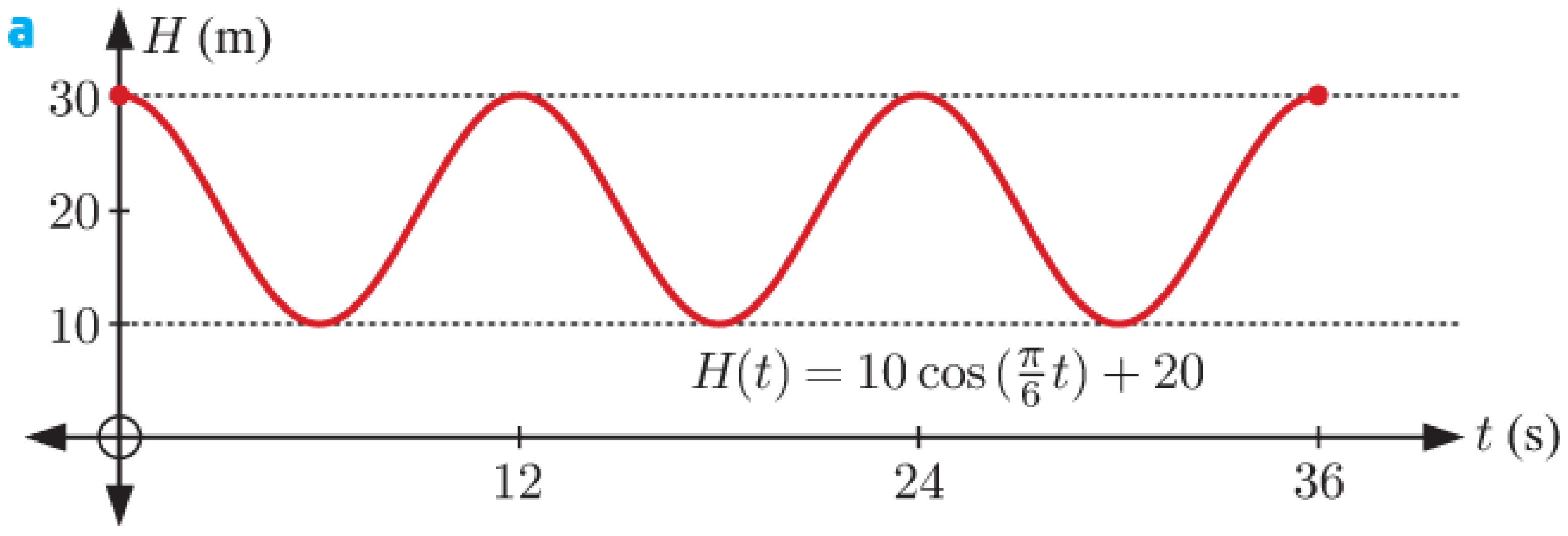


10 $a = \frac{3}{2}$, $b = -\frac{1}{2}$

11 **a** A reflection in the x-axis, then a horizontal stretch with scale factor $\frac{1}{2}$.

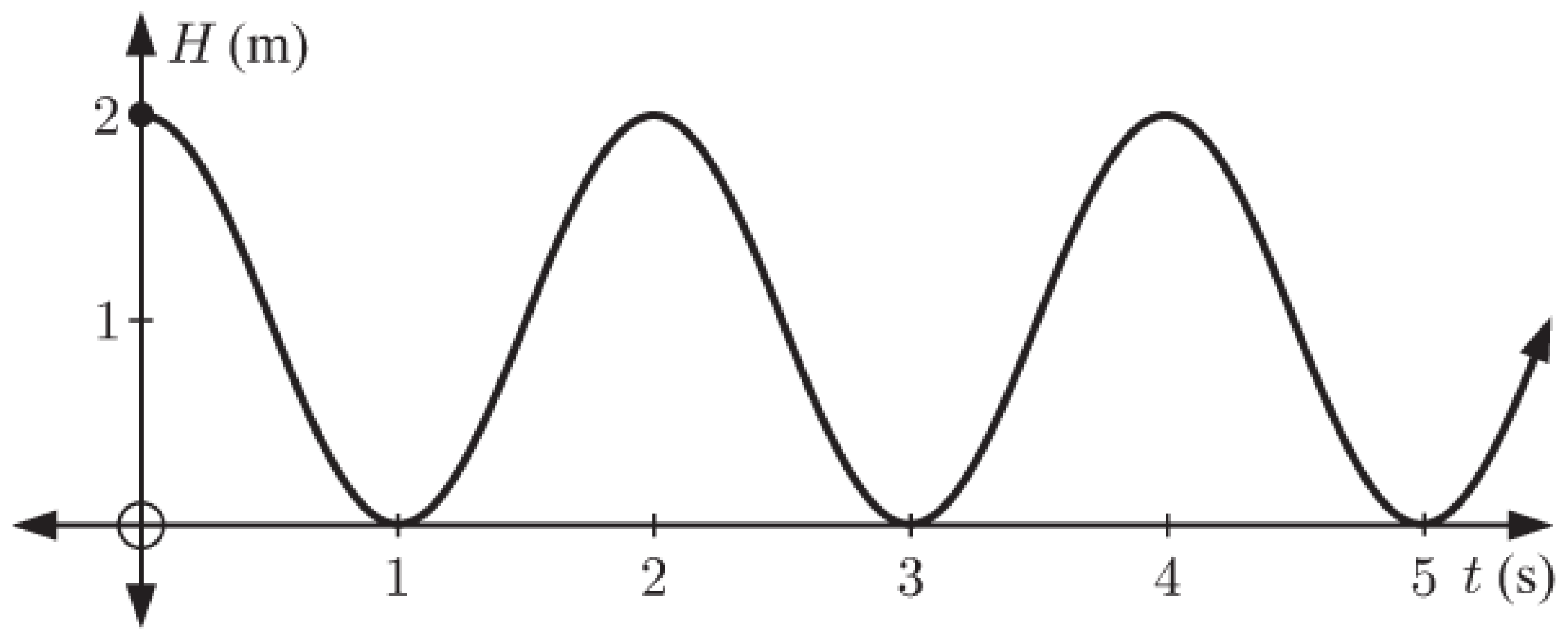
b A vertical stretch with scale factor 2, then a horizontal stretch with scale factor 2, then a translation $\frac{\pi}{2}$ units right and $\frac{1}{2}$ unit upwards.

12 **a**



b 20 m **c** 10 m **d** 12 seconds

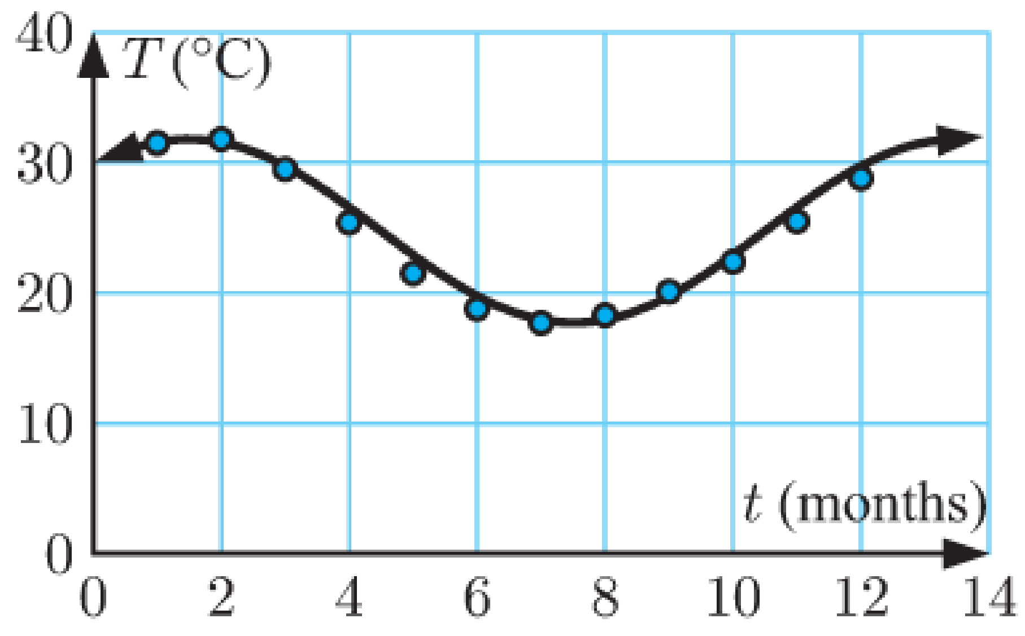
13 **a**



b $H(t) = \sin(\pi(t - 1.5)) + 1$

14 **a** $a \approx 7.05$, $b \approx \frac{\pi}{6}$, $c \approx 10.5$, $d \approx 24.75$

b



c Using technology, $T \approx 7.20\sin(0.488t - 1.08) + 24.7$. The model fits reasonably well but not perfectly.

15 **a** $x \approx -6.1, -3.4$ **b** $x \approx 0.8$

16 **a** $x \approx 1.27, 5.02$ **b** $x \approx 1.09, 2.05$

17 **a** $x \approx 1.33, 4.47, 7.61$ **b** $x \approx 5.30$

c $x \approx 2.83, 5.97, 9.11$

18 **a** $x = \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}$, or $\frac{17\pi}{9}$ **b** $x = \frac{5\pi}{3}$

c $x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}$, or $\frac{19\pi}{12}$

19 **a** $x = 0, \pi$, or 2π **b** $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

20 **a** $x = -\frac{\pi}{2}$ or $\frac{\pi}{2}$ **b** $x = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}$, or $\frac{5\pi}{6}$

c $x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}$, or $\frac{2\pi}{3}$

21 **a** $\frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9}$ **b** $\frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}$

22 **a** 28 milligrams per m^3 **b** 8:00 am Monday

INDEX